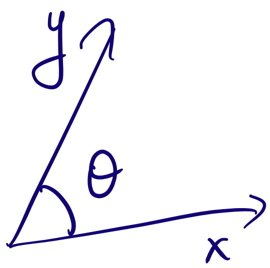
Lecture 22

- Vector projections
- Span
- Bases
- Gram-Schmidt Process

$$x^T y = x \cdot y = \|x\| \, \|y\| \cos\theta$$



Vector correlation:

$$\cos(\theta) = \frac{x^T y}{\|x\| \, \|y\|} = r$$

vector correlation

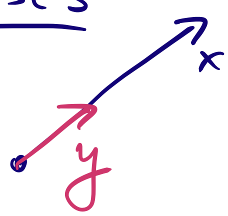
$$x^T y = \sum_{i=1}^{N} x_i y_i$$

$$[x_1 \ x_2 \ x_3] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$r = \frac{\sum_i x_i y_i}{\sqrt{\sum_i x_i^2} \sqrt{\sum_i y_i^2}}$$

<u>extreme cases</u>

(1)



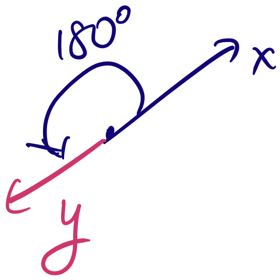
$$\vec{y} = c \cdot \vec{x}$$

$$c > 0$$

$$\theta = \angle(x, y) = 0^\circ$$

$$r = \cos(0^\circ) = 1$$

(2)



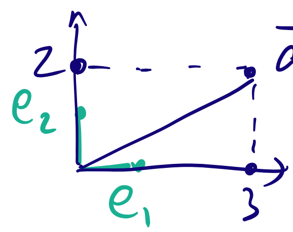
$$\vec{y} = c \cdot \vec{x}$$

$$c < 0$$

$$\theta = \angle(x, y) = 180°$$

$$r = \cos(180°) = -1$$

$\begin{bmatrix}0\\1\end{bmatrix}$



$\begin{bmatrix}1\\0\end{bmatrix}$

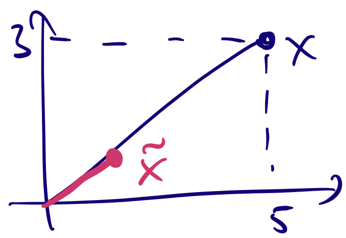
$$\vec{a} = \begin{bmatrix}3\\2\end{bmatrix} = 3 \cdot \vec{e}_1 + 2 \cdot \vec{e}_2$$

Any vector x with a norm of 1 is a unit vector

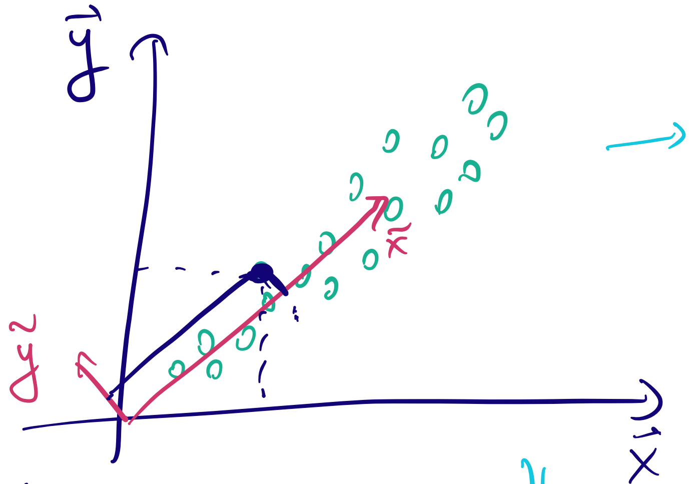
$$\vec{x} = \begin{bmatrix}5\\3\end{bmatrix} \Rightarrow \tilde{x} = \frac{\vec{x}}{\|\vec{x}\|}$$

$$\|\tilde{x}\| = 1$$

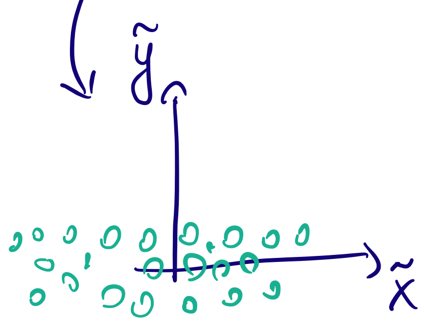
$$\|\tilde{x}\| = \left\| \frac{\vec{x}}{\|\vec{x}\|} \right\| = \frac{\|\vec{x}\|}{\|\vec{x}\|} = 1$$



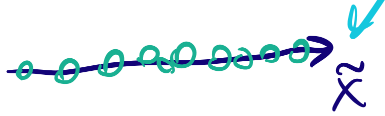
Why project?



→ mean subtraction

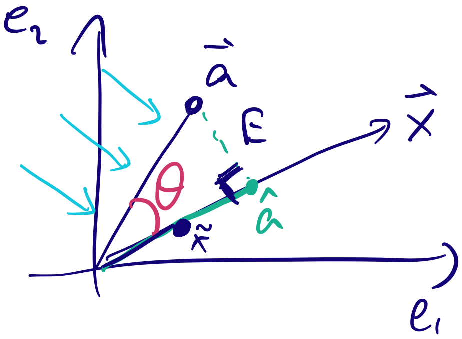


① uncorrelated data for feature extraction



② Dimensionality Reduction

Projection



$\hat{a} = \text{proj}_x \vec{a}$

$\equiv$ projection of $\vec{a}$ onto $\vec{x}$

$$\cos(\theta) = \frac{x^T a}{\|x\| \|a\|}$$

$$\|\hat{a}\| = \|a\| \cdot \cos(\theta) = \|a\| \frac{x^T a}{\|x\| \|a\|}$$

$$\tilde{x} = \frac{\vec{x}}{\|\vec{x}\|}$$

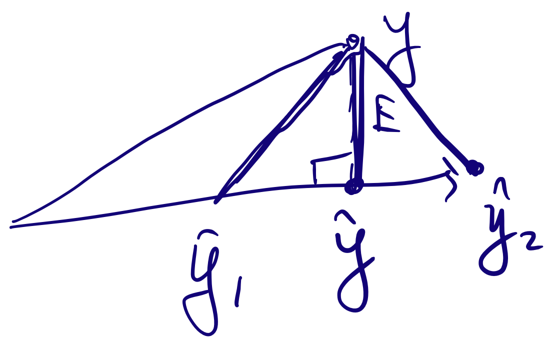
$$\Rightarrow \hat{a} = \|\hat{a}\| \tilde{x} = \cancel{\|a\|} \frac{x^T a}{\|x\| \cancel{\|a\|}} \cdot \frac{x}{\|x\|}$$

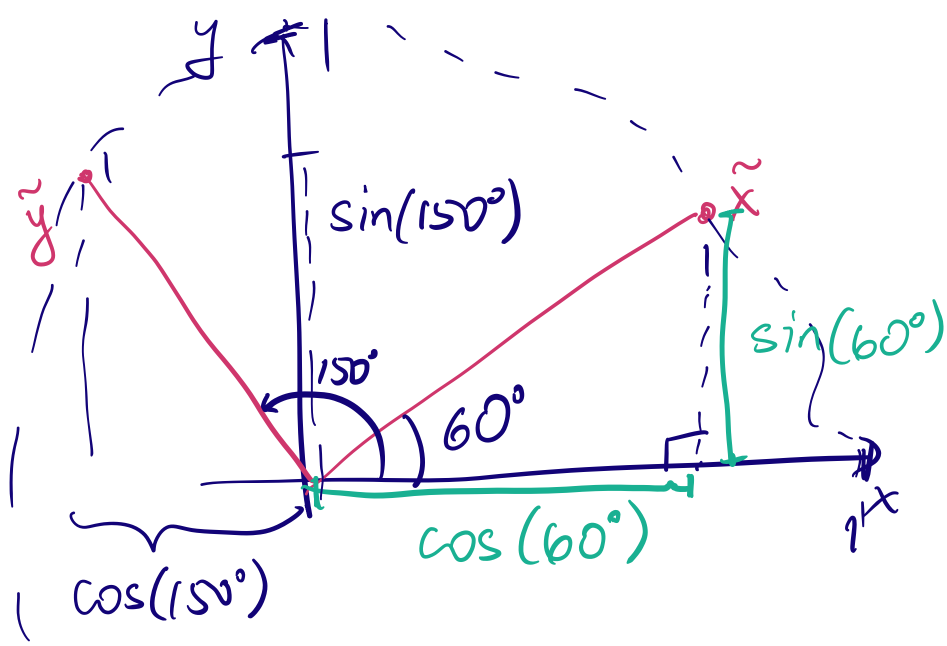
$$\boxed{\hat{a} = \text{proj}_x \vec{a} = \frac{x^T a}{\|x\|} \cdot \frac{x}{\|x\|}}$$

$$= \underbrace{\tilde{x}^T a}_{\text{displacement along the direction } \tilde{x}} \cdot \tilde{x}$$

$$\|E\| = \|y\| \cdot \sin\theta$$

The perpendicular direction of projection is the one that minimizes the error of projection





Rotation

$$\tilde{x} = \begin{bmatrix} \cos(60^\circ) \\ \sin(60^\circ) \end{bmatrix} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} \Rightarrow \|\tilde{x}\| = 1$$

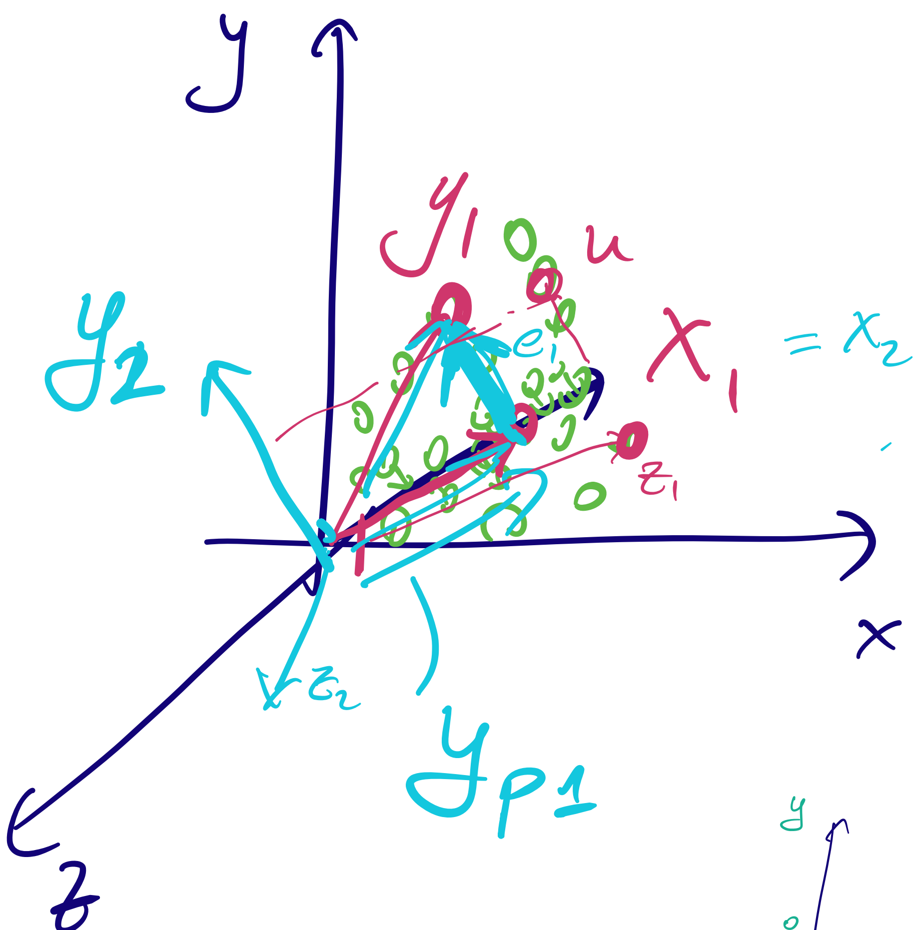
$$\tilde{y} = \begin{bmatrix} \cos(150^\circ) \\ \sin(150^\circ) \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix} \Rightarrow \|\tilde{y}\| = 1$$

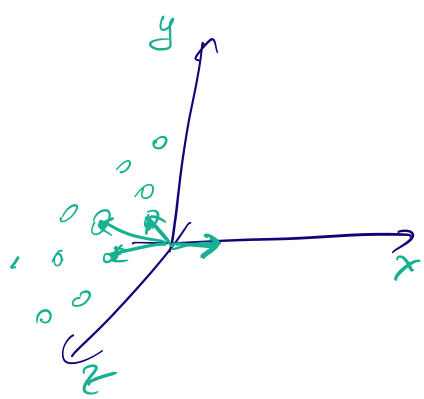
If $\|\tilde{x}\| = 1$, $\|\tilde{y}\| = 1$ and $x^T y = 0$,

then $\tilde{x}$ & $\tilde{y}$ are said to be

$\boxed{\text{Orthonormal}}$

Gram-Schmidt Process





Linear dependence

$a_0, \ldots \; a_{k-1}$ are linearly dependent

if $\exists$ non-zero constants $\beta_0, \ldots \beta_{k-1}$

there exists

Such that :

$$\beta_0 a_0 + \beta_1 a_1 + \dots \beta_{k-1} \cdot a_{k-1} = 0$$

$$a_0 = -\left(\frac{\beta_1}{\beta_0} a_1 + \dots + \frac{\beta_{k-1}}{\beta_0} a_{k-1}\right)$$

$\mathbb{R}^3$: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

Assume : that they are linearly dependent:

$$\beta_0 \begin{bmatrix}1\\0\\0\end{bmatrix} + \beta_1 \begin{bmatrix}0\\1\\0\end{bmatrix} + \beta_2 \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\beta_0 \neq 0$
$\beta_1 \neq 0$
$\beta_2 \neq 0$

However $\begin{bmatrix}\beta_0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\\beta_1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\beta_2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\Rightarrow$ $\beta_0 = 0$
$\beta_1 = 0$
$\beta_2 = 0$
$\Rightarrow$ Goes against our assumptions for linear dependence.

$\Rightarrow$ Thus, these 3 vectors are in fact linearly independent

Gram-Schmidt Process

$\Rightarrow$ produces a set of orthonormal vectors.

$$\boxed{\text{Orthogonal vectors are linearly independent}}$$

Proof:

Assume that 2 orthogonal vectors $\vec{u}, \vec{v}$ are not linearly independent.

$$\boxed{u^T v = \vec{u} \cdot \vec{v} = 0}$$

$$\boxed{\begin{array}{l} \exists \{\beta_0, \beta_1\} \neq \{0, 0\} \text{ s.t.:} \\ \beta_0 \cdot \vec{u} + \beta_1 \cdot \vec{v} = \vec{0} \end{array}}$$

Left-multiply by $\vec{u}^T$ on both sides:

$$\beta_0 \cdot \vec{u}^T\vec{u} + \beta_1 \underbrace{\vec{u}^T\vec{v}}_{=0} = \underbrace{\vec{u}^T\vec{0}}_{0}$$

$$\Leftrightarrow \beta_0 \cdot \vec{u}^T\vec{u} = 0$$

$$\Leftrightarrow \beta_0 \cdot \|\vec{u}\|^2 = 0$$

Either $\beta_0 = 0$ or $\|\vec{u}\|^2 = 0$

- However, the norm of $\|\vec{u}\|^2$ will only be 0 iff $\vec{u} = \vec{0}$

So, $\beta_0 = 0$ $\Rightarrow$ However, $\beta_0 \neq 0$

Thus. $\vec{u}$ & $\vec{v}$ are linearly independent!

$\boxed{\text{So, orthogonal vectors are linearly independent!}}$

$$\boxed{\text{orthogonality} \longrightarrow \text{linear independence}}$$

$$\cancel{\longleftarrow}$$

$$u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Assume that they are linearly dependent.

$\exists \{\beta_1, \beta_2\} \neq \{0, 0\}$:

$$\beta_1 \vec{u} + \beta_2 \vec{v} = \vec{0}$$

$$\Leftrightarrow \beta_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \beta_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} \beta_1 \\ \beta_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta_2 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \left.\begin{array}{l} \beta_1 = 0 \\ \beta_1 + \beta_2 = 0 \\ \beta_2 = 0 \end{array}\right\}$$

violates the condition $\beta_1 \neq 0, \beta_2 \neq 0$

$\Rightarrow \vec{u}$ & $\vec{v}$ are linearly independent

$\vec{u}$ & $\vec{v}$ are linearly independent.

But are they orthogonal?:

$$\vec{u}^T \vec{v} = [1\ 1\ 0]\begin{bmatrix}0\\1\\1\end{bmatrix} = 1\cdot 0 + 1\cdot 1 + 0\cdot 1$$

$$= 1 \neq 0$$

$\therefore$ They are <u>not</u> orthogonal

<u>Identity matrix</u>

$$I_k = \left.\begin{bmatrix}1 & & 0\\ & 1 & \\ 0 & & 1 \\ & & & \ddots\end{bmatrix}\right\} k \text{ rows}$$

$$I_2 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$I_3 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$