Lecture 21 - supplementary

- Norms of Vectors
- Distances
- Angles

In [1]:
```
import numpy as np
import pandas as pd
import scipy.stats as stats
import matplotlib.pyplot as plt
%matplotlib inline
plt.style.use('bmh')
```

Norm of Vectors

In [2]:
```
def plotvec(*argv):
    colors=['b','k','r','g','c','m']
    xmin=0
    xmax=-1000000
    ymin=0
    ymax=-1000000
    origin=[0,0]
    plt.figure()
    for e in enumerate(argv):
        i=e[0]
        arg=e[1]
        plt.quiver(*origin,*arg,angles='xy',scale_units='xy',scale=1,
                   color=colors[i%len(colors)])
        xmin=min(xmin,arg[0])
        xmax=max(xmax,arg[0])
        ymin=min(ymin,arg[1])
        ymax=max(ymax,arg[1])
    plt.xlim(min(-1, xmin-1), max(1,xmax+1))
    plt.ylim(min(-1,ymin-1),max(1,ymax+1))
```

Euclidean Norm

Euclidean Norm

The **Euclidean norm** of an n-vector $\mathbf{x}$, denoted $\|\mathbf{x}\|$, is the square-root of the inner product of the vector with itself, i.e.

$$\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2}$$

The Euclidean norm of the vector $\mathbf{x}$ is sometimes written with a subscript 2, as $\|\mathbf{x}\|_2$. (The subscript 2 indicates that the entries of $\mathbf{x}$ are raised to the second power.)

The Euclidean norm of the vector is also known as the **L2-norm.**

Things we can do with Euclidean norm (or L2-norm):

- Compute *length* of a vector: $\text{length}(\mathbf{x}) = \|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$
- Compute *distance* between vectors: $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$
- Relate inner products to norm, **Cauchy-Schwarz inequality**: $|\mathbf{x}^T\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$

Properties of Norm. Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors and β a scalar

1. Nonnegative Homogeneity: $\|\beta\mathbf{x}\| = |\beta| \times \|\mathbf{x}\|$
2. Nonnegativity: $\|\mathbf{x}\| \geq 0$
3. Definiteness: $\|\mathbf{x}\| = 0$ only if $\mathbf{x} = \mathbf{0}$

Triangle Inequality: $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

where

$$\begin{aligned}
\|\mathbf{x} + \mathbf{y}\|^2 &= (\mathbf{x} + \mathbf{y})^T(\mathbf{x} + \mathbf{y}) \\
&= \mathbf{x}^T\mathbf{x} + \mathbf{x}^T\mathbf{y} + \mathbf{y}^T\mathbf{x} + \mathbf{y}^T\mathbf{y} \\
&= \|\mathbf{x}\|^2 + 2\mathbf{x}^T\mathbf{y} + \|\mathbf{y}\|^2
\end{aligned}$$

Let's understand this property (**Triangle Inequality**) better using the virtual whiteboard.

Distances

Euclidean Distance

Euclidean Distance

We already know how to compute **Euclidean distance** between vectors, $\mathbf{x}$ and $\mathbf{y}$, in an Euclidean geometry:

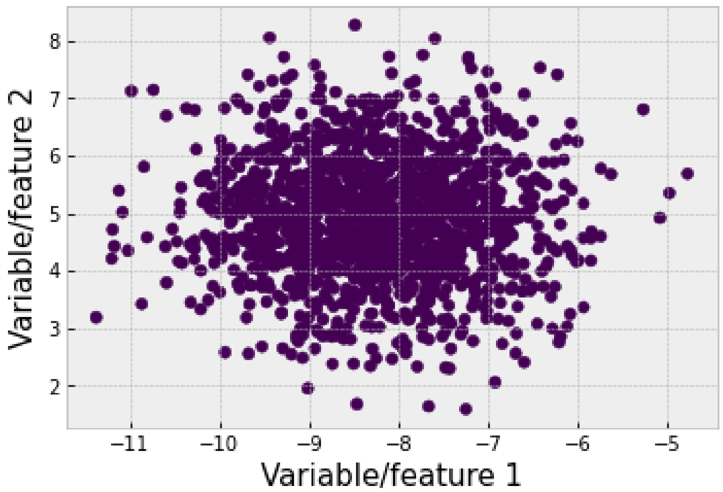
$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$

The Euclidean distance corresponds to the shortest line that connects the two vectors $\mathbf{x}$ and $\mathbf{y}$.

In [3]:
```
from sklearn.datasets import make_blobs

n_samples = 1500
X, T = make_blobs(n_samples=n_samples,centers=1,cluster_std=1)

plt.scatter(X[:,0],X[:,1],c=T)
plt.xlabel('Variable/feature 1',size=15)
plt.ylabel('Variable/feature 2',size=15);
```



In [4]:
```
X.shape # NxD
```

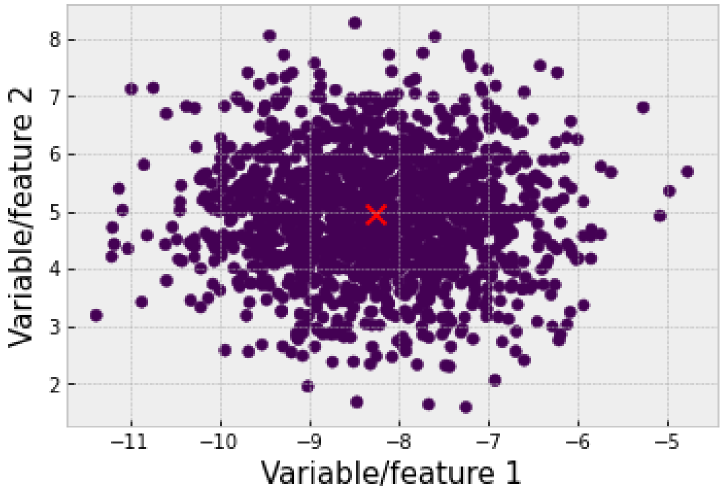
Out[4]:
```
(1500, 2)
```

In [5]:
```
mu = np.mean(X, axis=0)

mu
```

Out[5]:
```
array([-8.26901522,  4.94757234])
```

In [6]:
```python
plt.scatter(X[:,0],X[:,1],c=T)
plt.scatter(mu[0],mu[1],s=100,marker='x',c='r')
plt.xlabel('Variable/feature 1',size=15)
plt.ylabel('Variable/feature 2',size=15);
```



In [7]:
```python
X[0,:] - mu
```

Out[7]:
```
array([-0.6721042 ,  1.17276898])
```

In [9]:
```python
# euclidean distance between the first sample and the mean
np.sqrt(np.sum((X[0,:] - mu)**2))
```

Out[9]:
```
1.3517067542671104
```

In [10]:
```python
X - mu
```

Out[10]:
```
array([[-0.6721042 ,  1.17276898],
       [-0.37863589, -1.98451559],
       [ 0.00260459, -0.28653816],
       ...,
       [-2.06812809, -0.48654278],
       [-0.10885856,  0.56076306],
       [-1.16558745, -0.19023004]])
```

In [14]:
```
# Nx1 distance vector, one for each sample
distances = np.sqrt(np.sum((X - mu)**2, axis=1))

distances.shape
```

Out[14]:
```
(1500,)
```

In [15]:
```
distances
```

Out[15]:
```
array([1.35170675, 2.02031366, 0.28655   , ..., 2.12458882, 0.57123147,
       1.18100871])
```

In [16]:
```
from scipy.spatial.distance import cdist
```

In [18]:
```
?cdist
```

In [19]:
```
cdist(X,mu)
```

```
---------------------------------------------------------------------------
ValueError                                Traceback (most recent call last)
<ipython-input-19-f6ad91531173> in <module>
----> 1 cdist(X,mu)

~/opt/anaconda3/lib/python3.8/site-packages/scipy/spatial/distance.py in cdi
st(XA, XB, metric, *args, **kwargs)
   2742         raise ValueError('XA must be a 2-dimensional array.')
   2743     if len(sB) != 2:
-> 2744         raise ValueError('XB must be a 2-dimensional array.')
   2745     if s[1] != sB[1]:
   2746         raise ValueError('XA and XB must have the same number of col
umns '

ValueError: XB must be a 2-dimensional array.
```

In [20]:
```
mu.shape
```

Out[20]:
```
(2,)
```

In [21]:
```
mu[np.newaxis].shape
```

Out[21]:
```
(1, 2)
```

In [22]:
```
cdist(X,mu[np.newaxis])
```

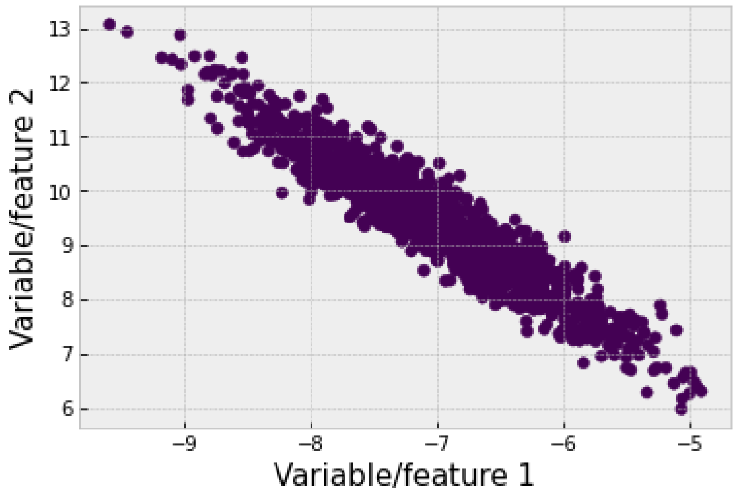
Out[22]:
```
array([[1.35170675],
       [2.02031366],
       [0.28655   ],
       ...,
       [2.12458882],
       [0.57123147],
       [1.18100871]])
```

Mahalanobis Distance

Suppose you have data that looks like this:

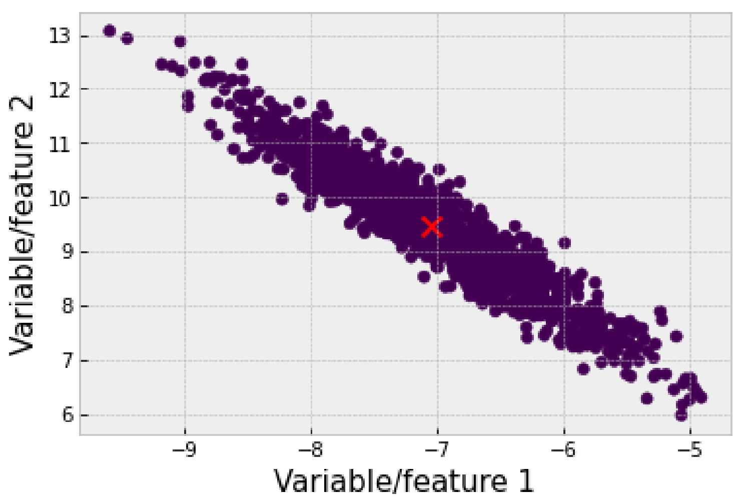
In [23]:
```
X = np.dot(X, [[0.60834549, -0.63667341], [-0.40887718, 0.85253229]])

plt.scatter(X[:,0],X[:,1],c=T)
plt.xlabel('Variable/feature 1',size=15)
plt.ylabel('Variable/feature 2',size=15);
```



In [24]:
```
mu = np.mean(X, axis=0)

plt.scatter(X[:,0],X[:,1],c=T)
plt.scatter(mu[0],mu[1],s=100,marker='x',c='r')
plt.xlabel('Variable/feature 1',size=15)
plt.ylabel('Variable/feature 2',size=15);
```



We could use the information that the variables are linearly dependent, but using the covariance matrix!

Mahalanobis Distance

The **Mahalanobis distance** between vectors, $\mathbf{x}$ and $\mathbf{y}$ is defined as:

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(\mathbf{x}-\mathbf{y})^T \mathbf{K}^{-1} (\mathbf{x}-\mathbf{y})}$$

where $K = \text{cov}(\mathbf{x}, \mathbf{y})$, and K^{-1} is the inverse of the covariance matrix. **We will define inverse of matrices later.**

In [26]:
```python
Mdist = cdist(X, mu[np.newaxis], metric='mahalanobis')

Mdist
```

Out[26]:
```
array([[1.30593429],
       [1.967803  ],
       [0.27726316],
       ...,
       [2.1452191 ],
       [0.5508525 ],
       [1.19174529]])
```

In [27]:
```python
Edist = cdist(X, mu[np.newaxis], metric='euclidean')

Edist
```

Out[27]:
```
array([[1.68156537],
       [1.56283896],
       [0.27310643],
       ...,
       [1.3911795 ],
       [0.62204882],
       [0.85723235]])
```

Angle between Vectors

The angle between two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$\theta = \angle(\mathbf{x}, \mathbf{y}) = \arccos\left(\frac{\mathbf{x}^T\mathbf{y}}{||\mathbf{x}||\,||\mathbf{y}||}\right)$$

where $\arccos$ denotes the inverse cosine, normalized to lie in the interval $[0, \pi]$. (The default angle unit is radians; 360° is 2π radians.)

In other words, we define θ as the unique number between 0 and π that satisfies

$$\mathbf{x}^T\mathbf{y} = ||\mathbf{x}||\,||\mathbf{y}||\cos\theta$$

Cosine Distance

The **cosine distance** between vectors, $\mathbf{x}$ and $\mathbf{y}$ is defined as:

$$d(\mathbf{x}, \mathbf{y}) = 1 - \cos(\theta) = 1 - \frac{\mathbf{x}^T\mathbf{y}}{||\mathbf{x}||\,||\mathbf{y}||}$$

where $\theta = \angle(\mathbf{x}, \mathbf{y})$ is the angle between $\mathbf{x}$ and $\mathbf{y}$.

- If the two vectors have a small angle between them, the cosine of that angle will be close to 1, then they are said to be **similar**. Therefore their difference will approach 0.

- The angle is a **symmetric**: $\angle(\mathbf{x}, \mathbf{y}) = \angle(\mathbf{y}, \mathbf{x})$.
- The angle is not affected by scaling each of the vectors by a positive scalar: $\angle(\alpha\mathbf{x}, \beta\mathbf{y}) = \angle(\mathbf{x}, \mathbf{y})$

For example, the the angle between the vectors $\mathbf{x} = [1, 2, 1]^T$ and $\mathbf{y} = [2, 0, 3]^T$ is

$$\arccos\left(\frac{5}{\sqrt{6}\sqrt{13}}\right) \simeq \arccos(0.5661) \simeq 0.9689 \simeq 55.52^\circ$$

In [ ]:

```

```

In [ ]:

```

```

In [ ]:

```

```

In [ ]:

```

```

In [ ]:

```

```

- If the angle is $\pi/2 = 90^\circ$, i.e., $\vec{\mathbf{x}}^T \vec{\mathbf{y}} = 0$, the vectors are said to be **orthogonal.**

We write $\vec{\mathbf{x}} \perp \vec{\mathbf{y}}$ if $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ are orthogonal.

- If the angle is zero, which means $\vec{\mathbf{x}}^T \vec{\mathbf{y}} = ||\vec{\mathbf{x}}|| ||\vec{\mathbf{y}}||$, the vectors are **aligned.** Each vector is a positive multiple of the other.
- If the angle is $\pi = 180^\circ$, which means $\vec{\mathbf{x}}^T \vec{\mathbf{y}} = -||\vec{\mathbf{x}}|| ||\vec{\mathbf{y}}||$, the vectors are **anti-aligned.** Each vector is a negative multiple of the other.

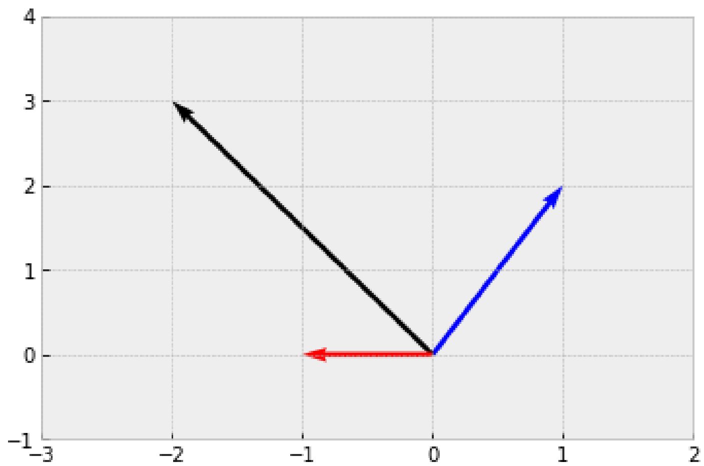
- If $\angle(\vec{\mathbf{x}}, \vec{\mathbf{y}}) < \pi/2 = 90^\circ$, the vectors are said to make an **acute angle.** This is the same as $\vec{\mathbf{x}}^T \vec{\mathbf{y}} > 0$, i.e., the vectors have positive inner product.
- If $\angle(\vec{\mathbf{x}}, \vec{\mathbf{y}}) > \pi/2 = 90^\circ$, the vectors are said to make an **obtuse angle.** This is the same as $\vec{\mathbf{x}}^T \vec{\mathbf{y}} < 0$, i.e., the vectors have negative inner product.

Other Applications of Angles between Vectors

1. We can calculate the "spherical" distances. This type of distance is also commonly referred to as **geodesic distances** (shortest path on a curved surface).
2. Document similarity via angles. If n-vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ represent the word counts for two documents, their angle $\angle(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ can be used as a measure of document dissimilarity.
3. Clustering.

In [28]:
```python
a=np.array([[1,2]])
b=np.array([[-2,3]])
c=np.array([[-1,0]])

plotvec(a[0,:],b[0,:], c[0,:])
```



In [29]:
```python
print('Euclidean distance:', cdist(a,b))
print('Cosine distance:', cdist(a,b,metric = 'cosine'))
```

```
Euclidean distance: [[3.16227766]]
Cosine distance: [[0.50386106]]
```

In [30]:
```python
print('Euclidean distance:', cdist(a,c))
print('Cosine distance:', cdist(a,c,metric = 'cosine'))
```

```
Euclidean distance: [[2.82842712]]
Cosine distance: [[1.4472136]]
```

In [ ]:
```python

```