Lecture 26

Examples:

Example 1:

$$\begin{cases} x + 2y = 1 \\ 2x - y = 7 \end{cases}$$

$$\Updownarrow$$

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

Augmented matrix:

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & -1 & 7 \end{array}\right]$$

① Pivot $R_2 \leftarrow R_2 - 2R_1$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -5 & 5 \end{array}\right]$$

② Scale $R_2 \leftarrow \frac{-1}{5} \cdot R_2$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \end{array}\right] \Rightarrow \text{r.e.f.}$$

③ Pivot $R_1 \leftarrow R_1 - 2R_2$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right]$$

identity matrix — solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Example 2 :

$$\begin{cases} 3x + 4y = 4 \\ 6x - 2y = 3 \end{cases}$$

$$\Updownarrow$$

$$\begin{bmatrix} 3 & 4 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Augmented matrix :

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left[\begin{array}{cc|c} 3 & 4 & 4 \\ 6 & -2 & 3 \end{array}\right]$$

① Scale $R_1 \leftarrow \frac{1}{3} R_1$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 4/3 & 4/3 \\ 6 & -2 & 3 \end{array}\right]$$

② Pivot $R_2 \leftarrow R_2 - 6R_1$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 4/3 & 4/3 \\ 0 & -10 & -5 \end{array}\right]$$

③ Scale $R_2 \leftarrow \frac{-1}{10} \cdot R_2$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 4/3 & 4/3 \\ 0 & 1 & 1/2 \end{array}\right] \Rightarrow \text{ref}$$

④ Pivot $R_1 \leftarrow R_1 - \frac{4}{3} R_2$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2/3 \\ 0 & 1 & 1/2 \end{array}\right]$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/2 \end{bmatrix}$$

Example 3:

$$\begin{cases} -x + y = -1 \\ y - z = 6 \\ x + z = -1 \end{cases}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ -1 \end{bmatrix}$$

Augmented Matrix:

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \left[\begin{array}{ccc|c} -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 6 \\ 1 & 0 & 1 & -1 \end{array}\right]$$

① Swap $R_3 \leftrightarrow R_1$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 6 \\ -1 & 1 & 0 & -1 \end{array}\right]$$

② Pivot $R_3 \leftarrow R_3 + R_1$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 6 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

③ Pivot $R_3 \leftarrow R_3 - R_2$ + Scale $R_3 \leftarrow \frac{1}{2} R_3$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 1 & -4 \end{array}\right] \Rightarrow \text{r.e.f.}$$

④ Pivot $R_2 \leftarrow R_2 + R_3$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

⑤ Pivot $R_1 \leftarrow R_1 - R_3$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$$

Example 4:

$$\begin{cases} x + 2y - z = 9 \\ 2x - y + 3z = -2 \\ 3x - 3y - 4z = 1 \end{cases}$$

$$\Updownarrow$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 3 & -3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \\ 1 \end{bmatrix}$$

Augmented Matrix:

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 2 & -1 & 3 & -2 \\ 3 & -3 & -4 & 1 \end{array}\right]$$

① Pivot $R_2 \leftarrow R_2 - 2R_1$

+ Scale $R_2 \leftarrow \frac{-1}{5} R_2$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 0 & 1 & -1 & 4 \\ 3 & -3 & -4 & 1 \end{array}\right]$$

② Pivot $R_3 \leftarrow R_3 - 3R_1$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 0 & 1 & -1 & 4 \\ 0 & -9 & -1 & -26 \end{array}\right]$$

③ Pivot $R_3 \leftarrow R_3 + 9 \cdot R_2$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -10 & 10 \end{array}\right]$$

④ Scale $R_3 \leftarrow \frac{-1}{10} \cdot R_3$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow \text{ref}$$

⑤ Pivot $R_2 \leftarrow R_2 + R_3$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

⑥ Pivot $R_1 \leftarrow R_1 - 2R_2 + R_3$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

Example 5:

$$\begin{cases} 3x + y + 3z = 1 \\ x + 2y - z = 2 \\ 2x - y + 4z = 4 \end{cases}$$

Augmented Matrix:

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left[\begin{array}{ccc|c} 3 & 1 & 3 & 1 \\ 1 & 2 & -1 & 2 \\ 2 & -1 & 4 & 4 \end{array}\right]$$

① Swap $R_1 \leftrightarrow R_2$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 3 & 1 & 3 & 1 \\ 2 & -1 & 4 & 4 \end{array}\right]$$

② Pivot $R_2 \leftarrow R_2 - 3R_1$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -5 & 6 & -5 \\ 2 & -1 & 4 & 4 \end{array}\right]$$

③ Scale $R_2 \leftarrow R_2/-5$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & -6/5 & 1 \\ 2 & -1 & 4 & 4 \end{bmatrix}$$

④ Pivot $R_3 \leftarrow R_3 - 2 \cdot R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & -6/5 & 1 \\ 0 & -5 & 6 & 0 \end{bmatrix}$$

⑤ Pivot $R_3 \leftarrow R_3 + 5 \cdot R_2$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & -6/5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix} \rightarrow 0x + 0y + 0z = 5$$

$\not\Rightarrow$ no solution!

because this is impossible.

Example 6:

a. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$ (I) $\quad$ sol^n $= \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$

b. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\rightarrow 0 = 1 \Rightarrow$ no solution

c. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\rightarrow 0 = 0 \Rightarrow$ consistent but infinite solutions.

Let $z = t$ for some constant t.

$\begin{cases} x + 2t = 3 \Rightarrow x = 3 - 2t \\ y - t = 5 \Rightarrow y = 5 + t \end{cases}$

$\rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3-2t \\ 5+t \\ t \end{bmatrix}$

d. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

e. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\rightarrow$ inconsistent $\Rightarrow$ no solution

f. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\rightarrow$ infinite # sol^n

Let $z = t$;

$x = 2$

$y = 3$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ t \end{bmatrix}$

d. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \to \\ \to \end{array}$

infinite # of sol^{n}s.

Let $y = s$ and $z = t$:

$x = 4 - 2s - 3t$.

Example 7:

t	s
0	5
1	23
2	37

We want to find $\{a, b, c\}$ such that $S = at^2 + bt + c$.

We are given 3 "data points".

$$\begin{cases} 5 = a(0)^2 + b(0) + c \\ 23 = a(1)^2 + b(1) + c \\ 37 = a(2)^2 + b(2) + c \end{cases}$$

$$\Leftrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 23 \\ 37 \end{bmatrix}$$

Augmented Matrix:

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left[\begin{array}{ccc|c} 0 & 0 & 1 & 5 \\ 1 & 1 & 1 & 23 \\ 4 & 2 & 1 & 37 \end{array}\right]$$

① Swap $R_1 \longleftrightarrow R_3$

$$\Rightarrow \left[\begin{array}{ccc|c} 4 & 2 & 1 & 37 \\ 1 & 1 & 1 & 23 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

② Scale $R_1 \longleftarrow R_1/4$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1/2 & 1/4 & 37/4 \\ 1 & 1 & 1 & 23 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

③ Pivot $R_2 \longleftarrow R_2 - R_1$ & Scale $R_2 \longleftarrow 2R_2$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1/2 & 1/4 & 37/4 \\ 0 & 1 & 3/2 & 27.5 \\ 0 & 0 & 1 & 5 \end{array}\right] \Rightarrow \text{r.e.f.}$$

(4) Pivot $R_2 \leftarrow R_2 - \frac{3}{2} \cdot R_3$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1/2 & 1/4 & 37/4 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

(5) Pivot $R_1 \leftarrow R_1 - \frac{1}{2} \cdot R_2 - \frac{1}{4} \cdot R_3$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2 \\ 20 \\ 5 \end{bmatrix}$$

$\Rightarrow s = -2t^2 + 20t + 5$

$\Rightarrow$ When $t = 8$, $s = -2(8)^2 + 20(8) + 5$

$= -128 + 160 + 5$

$= 37$

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