Lecture 21 – supplementary

- Norms
- Distances
- Angles

Euclidean norm

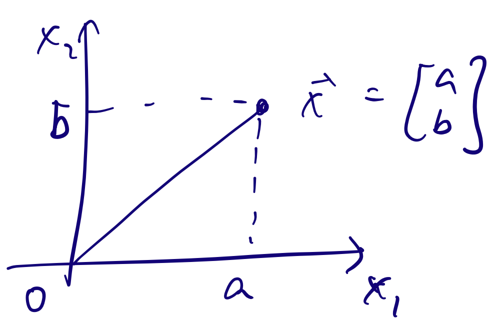
Length of a vector $\vec{x}$: $\|x\|$

Euclidean norm: $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \dots + x_N^2}$

$$= \sqrt{x^T x}$$

$$\|x\|^2 = x_1^2 + x_2^2 = [x_1 \quad x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x^T x$$

$$\|x\| = \sqrt{x \cdot x}$$



$$\|x\| = \sqrt{a^2 + b^2}$$

- Euclidean Norm is also called the L_2-norm

- L_p-norm:

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \dots + |x_N|^p\right)^{1/p}$$

$p=0$: # elements in x that are non-zero

$p=1$: L_1-norm: $\|x\|_1 = |x_1| + |x_2| + \dots |x_N|$

$p=2$: L_2-norm = Euclidean norm.

Properties of the norm:

① Non-negative homogeneity

$$\|\beta \cdot x\| = |\beta| \cdot \|x\|$$

② Non-negativity

$$\|x\| \geq 0$$

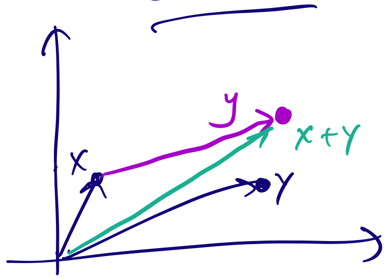
③ Definiteness: $\|x\| = 0$ only if $x = 0$

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Triangle Inequality

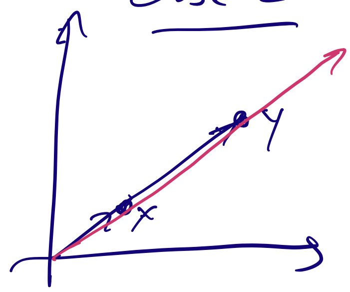
$$\|x+y\| \le \|x\| + \|y\|$$

case 1



$$\|x+y\| < \|x\| + \|y\|$$

case 2



$$y = \alpha \cdot x$$

$$\alpha \geq 0$$

$$\|x+y\| = \|x\| + \|y\|$$

$$\|x+y\|^2 = \left(\sqrt{(x+y)^T(x+y)}\right)^2$$

$$= (x+y)^T(x+y)$$

$$= (x^T+y^T)(x+y)$$

$$= x^Tx + y^Tx + x^Ty + y^Ty$$

$$= \|x\|^2 + 2\cdot x^Ty + \|y\|^2$$

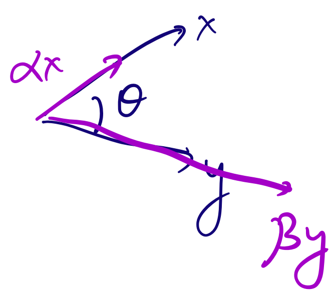
$$\|x+y\| = \sqrt{\|x\|^2 + 2x^Ty + \|y\|^2}$$

$$\leq \|x\| + \|y\|$$

Distances between 2 vectors

$$d(x,y) = \|x-y\|$$

Angle between 2 vectors :



$$\theta = \angle(x,y) = \arccos\left(\frac{x^T y}{\|x\| \cdot \|y\|}\right)$$

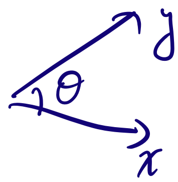
$$\angle(\alpha x, \beta y) = \angle(x,y)$$

$$\alpha > 0, \ \beta > 0$$

Cosine Distance :

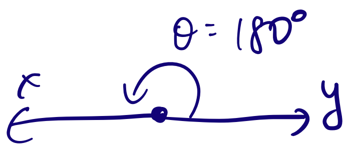
$$d_c(x,y) = 1 - \cos(\theta) = 1 - \frac{x^T y}{\|x\| \|y\|}$$

$$-1 \le \cos(\theta) \le 1$$



θ small $\Rightarrow \cos(\theta) \xrightarrow{\theta \to 0} 1$

$$d_c(x,y) = 1 - \cos(\theta) \xrightarrow{\theta \to 0} 0$$



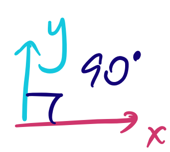
$\Rightarrow \cos(\theta) = \cos(180^\circ) = -1$

$\Rightarrow d_c(x,y) = 2$

$$0 \le d_c(x,y) \le 2$$

Let $\vec{x} \neq \vec{0}$, $\vec{y} \neq \vec{0}$, $\theta = \angle(x,y)$

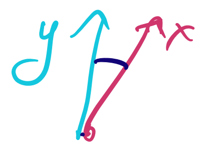
(1) $\theta = \frac{\pi}{2} = 90^\circ$



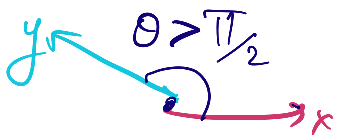
x is orthogonal to y.

$x \perp y \iff \theta = 90^\circ$

② $\theta < \frac{\pi}{2}$: acute angle : $\underline{x^T y > 0}$

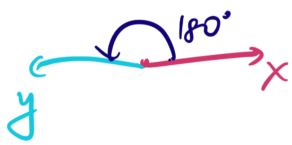


③ $\theta > \frac{\pi}{2}$: obtuse angle : $x^T y < 0$



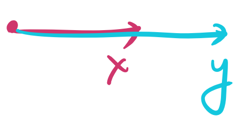
④ $\theta = \pi = 180°$: $\vec{x}$ and $\vec{y}$ are anti-aligned

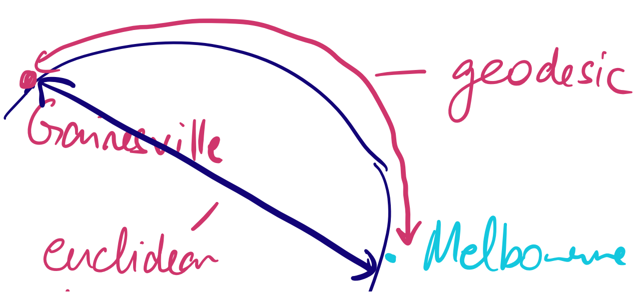
$$x^T y = -\|x\|\,\|y\|$$



⑤ $\theta = 0 \Rightarrow$ $\vec{x}$ and $\vec{y}$ are aligned

$$x^T y = \|x\| \cdot \|y\|$$





distance