Lecture 27

$$\left[\begin{array}{ccc|c} 1 & -4 & 0 & 0 \\ 0 & 1 & 5 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

consistent

$$\left[\begin{array}{ccc|c} 1 & -4 & 0 & 0 \\ 0 & 1 & 5 & 5 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

inconsistent

$$0 \cdot x + 0 \cdot y + 0 \cdot z = 1$$

leading entries

$$\left[\begin{array}{ccc|c} 1 & -4 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

consistent

$n = m$
dependent
$\therefore$ infinite # of solutions

Determinant of A

$\rightarrow$ talk about this soon.

Inverse of A

$$Ax = b$$

$$\underbrace{A^{-1}A}\, x = A^{-1}b$$

$$I \; \underline{x} = A^{-1}b$$

$$\boxed{\left[\begin{array}{c|c} A & I \end{array}\right]}$$

Augmented Matrix to find inverse

$\xrightarrow{\text{Gauss-Jordan elimination}} A^{-1}$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Inverse

$$\left[\begin{array}{c|c} A & I \end{array}\right]$$

$$= \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

Scale
$R_1 \leftarrow \frac{1}{3} \cdot R_1$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

Pivot
$R_2 - R_1$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 2 & 0 & -1/3 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

Scale
$R_2 \leftarrow \frac{1}{2} \cdot R_2$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & -1/6 & 1/2 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

Pivot + Scale
$R_3 \leftarrow -(R_3 - R_2)$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & -1/6 & 1/2 & 0 \\ 0 & 0 & 1 & -1/6 & 1/2 & -1 \end{array}\right] \Rightarrow \text{rref!}$$

$$\underbrace{\qquad\qquad}_{I} \qquad \underbrace{\qquad\qquad}_{A^{-1}}$$

$x = A^{-1} b$

$m = n$ & independent

$Ax = b$

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 30 \\ 18 \\ 2 \end{bmatrix}$$

$$x = A^{-1} b$$

$$= \begin{bmatrix} 1/3 & 0 & 0 \\ -1/6 & 1/2 & 0 \\ -1/6 & 1/2 & -1 \end{bmatrix} \begin{bmatrix} 30 \\ 18 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 4 \\ 2 \end{bmatrix}$$

Matrix Inverse : 2x2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{array}{c} R_1 \\ R_2 \end{array}\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \quad R_1 \leftarrow \frac{1}{a} R_1$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right] \quad R_2 \leftarrow R_2 - c \cdot R_1$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right] \quad R_2 \leftarrow \frac{a}{ad-bc} R_2$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] R_1 \leftarrow R_1 - R_2 \cdot \frac{b}{a}$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\underbrace{\phantom{\begin{matrix}1 & 0\end{matrix}}}_{I} \qquad \underbrace{\phantom{\begin{matrix}\frac{d}{ad-bc} & \frac{-b}{ad-bc}\end{matrix}}}_{A^{-1}}$$

$$A^{-1} = \frac{1}{\underbrace{ad-bc}_{\det(A)}} \overbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}^{\text{Adjugate matrix } \mathrm{Adj}(A)}$$

If $\underbrace{\boxed{ad-bc}}_{\Downarrow\ \det(A)} = 0 \Rightarrow$ no solution

If $\det(A) \neq 0 \Rightarrow$ columns of A are linearly independent

$\therefore$ unique sol^n.

In an orthogonal matrix, the vector elements are orthonormal to each other

$$A^TA = \overset{n \times n}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}} \Rightarrow I$$

$$\boxed{a_i^T a_i = 1}$$

$$a_i^T a_j = 0 \text{ for } i \neq j$$

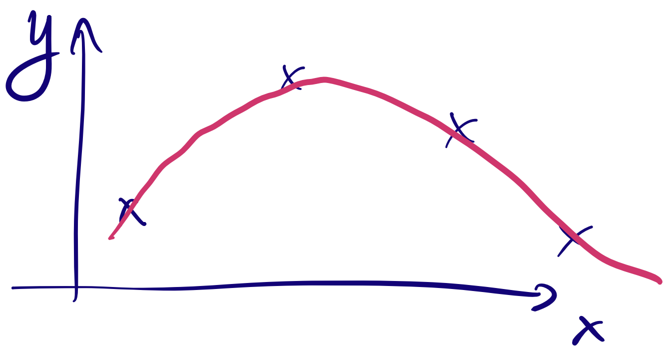
For orthogonal matrices,

$$A^T = A^{-1}$$

$$\nexists \; A^{-1}A = I$$

$$A = \underbrace{\begin{bmatrix} \begin{bmatrix} a_1 \\ \\ \\ \end{bmatrix} & \begin{bmatrix} a_2 \\ \\ \\ \end{bmatrix} & \begin{bmatrix} a_3 \\ \\ \\ \end{bmatrix} \end{bmatrix}}_{n \times n = 3 \times 3} \Rightarrow \text{square matrix}$$

Polynomial Interpolation :



$$A \cdot \underline{c} = y \Rightarrow \text{System of linear equations}$$

$$c = A^{-1} y$$

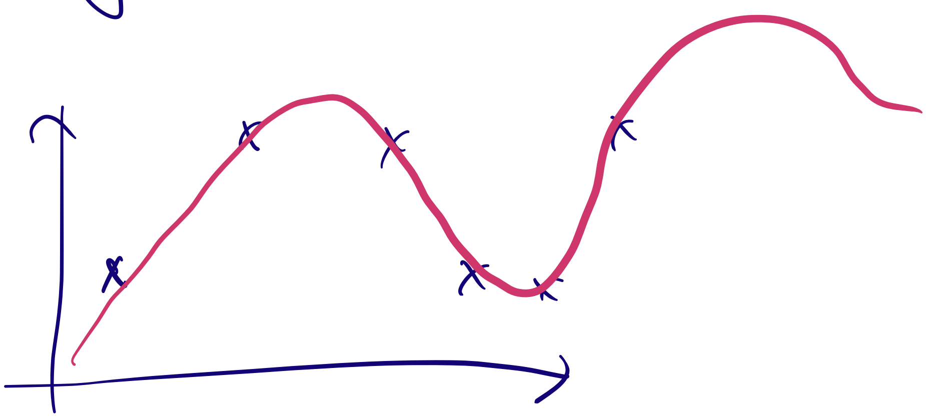
$$A = \begin{bmatrix} x_0^0 & x_0^1 & x_0^2 & x_0^3 & \cdots \\ x_1^0 & x_1^1 & x_1^2 & x_1^3 & \cdots \\ \vdots & & & & \end{bmatrix}$$

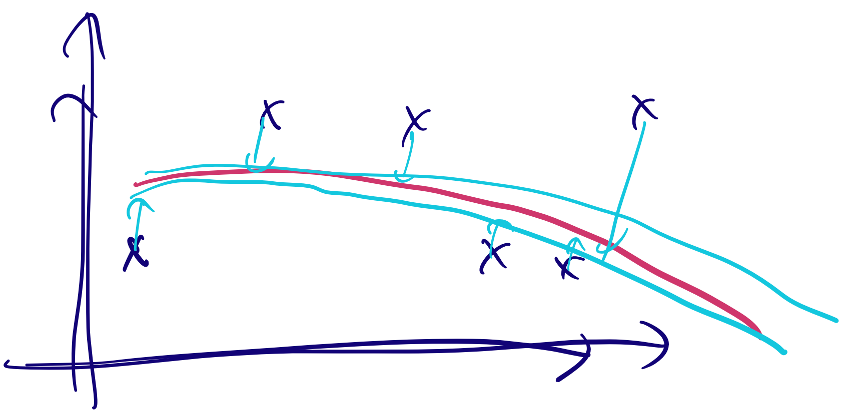
$$y = c_0 \cdot x^0 + c_1 x^1 + c_2 x^2 + \cdots$$

$$\begin{cases} y_1 = c_0 \cdot x_1^0 + c_1 x_1^1 + \cdots \\ y_2 = c_0 \cdot x_2^0 + c_1 x_2^1 + \cdots \\ \vdots \end{cases}$$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} x_1^0 & x_1^1 & x_1^2 & \cdots \\ x_2^0 & x_2^1 & \cdots & \\ \vdots & & & \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \end{bmatrix}}_{c}$$

$$Ac = y$$





$$Ax = b$$

$$\boxed{r = Ax - b} \Rightarrow \|r\|_2^2$$

$$\boxed{Ax = b}$$

$$\|Ax - b\|_2^2 = r_0^2 + r_1^2 + \dots + r_{n-1}^2$$

$$b_0 - A_0 x \qquad b_1 - A_1 x \qquad b_{n-1} - A_{n-1} x$$

$$A = \begin{bmatrix} A_0 \\ \hline A_1 \\ \hline \vdots \\ A_n \end{bmatrix}$$

$$(Ax - b)^T (Ax - b) = \|Ax - b\|^2 = f(x)$$

minimize

$$\nabla f(x) = 2A^T(Ax - b) = 0$$

$$\hookrightarrow A^T A x - A^T b = 0$$

$$\hookrightarrow \underline{A^T A x = A^T b}$$

Note: if A is orthogonal,

$A^T A = I$,

$x = A^{-1} b$

$$\boxed{x = (A^T A)^{-1} A^T b}$$

pseudo-inverse

Tall A

over-determined $\hookrightarrow$ # of equations > # of unknowns

$$\underset{m \times n}{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}} \qquad \boxed{m > n}$$

$\Rightarrow \text{rank}(A) \leq \min(m, n)$

$\Rightarrow \text{rank}(A) = n$

$$\underset{n \times \cancel{m}}{A^T} \; \underset{\cancel{m} \times n}{A} \Rightarrow n \times n$$

$$A^+ = \text{pseudo-inverse } (A)$$

$$= (A^T A)^{-1} A^T$$

Wide A

underdetermined system of equation

$\therefore$ # of unknowns > # of equation

$[\quad A \quad]_{m \times n}$ $\boxed{m < n}$

$$\text{pseudo-inverse}(A) = A^T \underbrace{(A A^T)^{-1}}$$

$$\underset{m \times n}{A} \; \underset{n \times m}{A^T} = m \times m$$

$\text{rank}(A) = m$

$$\underset{m \times n}{\begin{bmatrix} \quad A \quad \end{bmatrix}} \underset{n \times m}{\begin{bmatrix} A \\ \\ \end{bmatrix}} \Rightarrow m \times m \; [\quad]$$

$A : \underline{4} \times \underline{2}$

$$A^{+} = \underbrace{(A^T A)^{-1}}_{2 \times 2} \underbrace{A^T}_{2 \times 4} \Rightarrow A^{+} : 2 \times 4$$