Lecture 24:

- Matrix Operations
- Systems of Linear Equations
- Gauss - Jordan Elimination

---

1 or 2 Recitations – extra credit

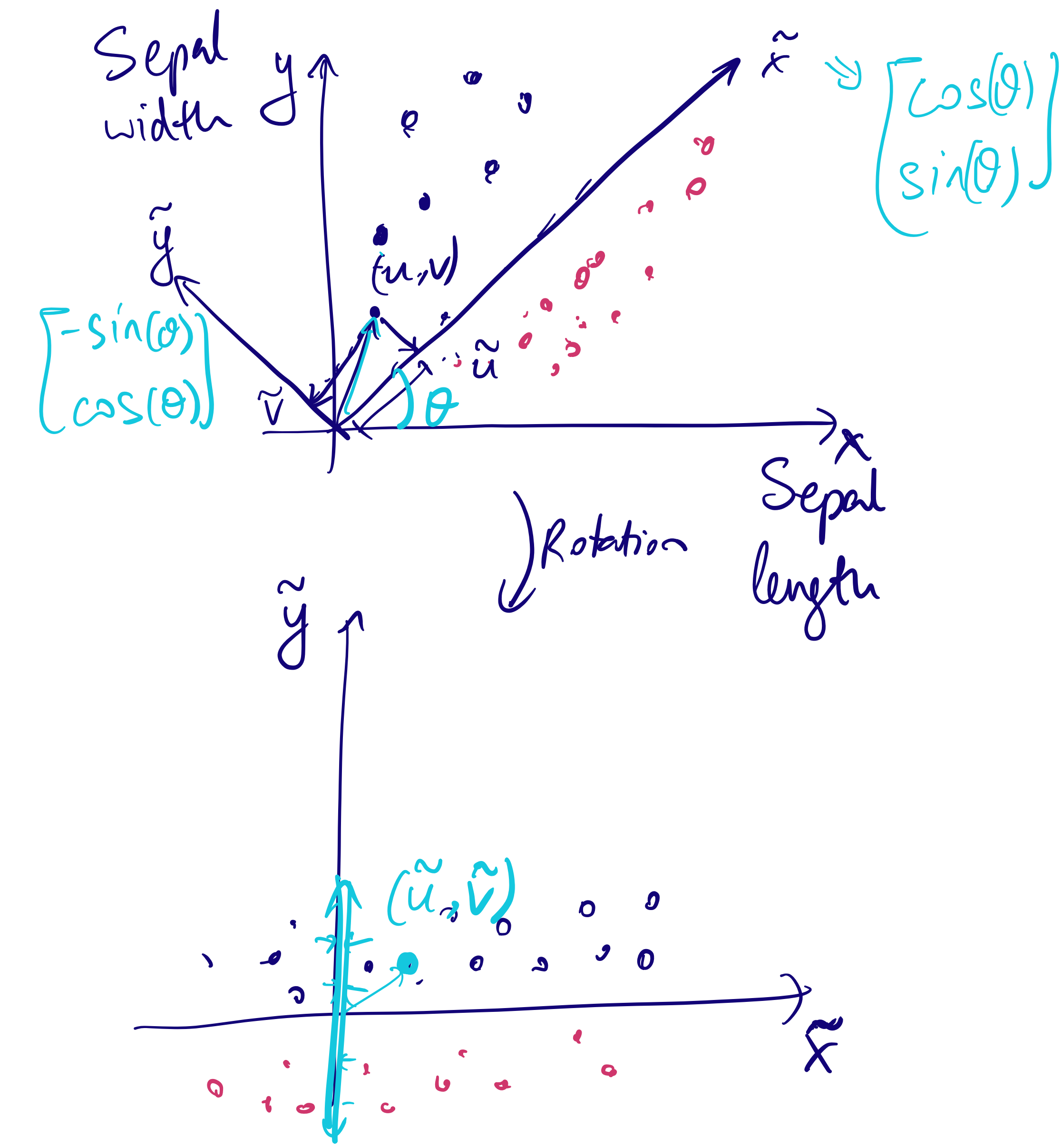
- Recitation next Tuesday

1 SA

1 HW

1 Discussion – extra credit

Exam 3



Inner product :

$$x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad y = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$x \cdot y = x^T y = \underset{1\times 3}{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$$

$$= 1\times 0 + 1\times 1 + 0\times 1$$

$$= 1$$

1×1

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \quad 2\times 2$$

$$B = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 1 & -3 \end{bmatrix} \quad 2\times 3$$

$$B^T @ A \Rightarrow \underline{3\times 2}$$

3×2 $\quad$ 2×2

$$B^T = \begin{bmatrix} 2 & 3 \\ -1 & 1 \\ 1 & -3 \end{bmatrix} \quad 3\times 2$$

$$A @ B \Rightarrow \underline{2\times 3}$$

2×2 $\quad$ 2×3

$A @ B^T \Rightarrow$ not valid

$\underset{2\times 2}{A} \quad @ \quad \underset{3\times 2}{B^T}$

$2 \neq 3$

$A@B = AB$

$$= \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 3 & 1 & -3 \end{bmatrix}$$

1st row second column

$$= \begin{bmatrix} 1\times 2 + 3\times 3 & 1\times -1 + 3\times 1 & 1\times 1 + 3\times -3 \\ (2\times 2)+(-2\times 3) & 2\times -1 + -2\times 1 & 2\times 1 + (-3)(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 2 & -8 \\ -2 & -4 & 8 \end{bmatrix}$$

$B @ A \Rightarrow$ not valid

$AB \neq BA \Rightarrow$ matrix multiplication is not commutative

$\cancel{A^T B = B^T A}$ ✗

$\cancel{AB = B^T A}$ ✗

$(AB)^T = B^T A^T$

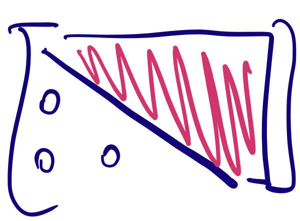
$$I \cdot B = \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 2 & -1 & 1 \\ 3 & 1 & -3 \end{bmatrix}}$$

$$= \begin{bmatrix} 2 & -1 & 1 \\ 3 & 1 & -3 \end{bmatrix}$$

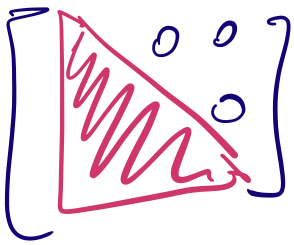
$$B \cdot I = \underset{2\times 3}{\begin{bmatrix} 2 & -1 & 1 \\ 3 & 1 & -3 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$ $\Rightarrow$ not valid

$$I^T = I$$

$\Downarrow$ rotation matrix with $\theta = 0$



Upper diagonal



Lower diagonal

$$A = [a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_{n-1}]$$

$\Downarrow$

$$\|A\|^2 = \|a_0\|^2 + \|a_1\|^2 + \cdots \|a_{n-1}\|^2$$

$$\|A\| = \sqrt{\|a_0\|^2 + \|a_1\|^2 + \cdots \|a_{n-1}\|^2}$$

---

distance b/w two matrices A & B:

$$\|A - B\|$$

Linear Equations:

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$f: \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longrightarrow \alpha \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$f: \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \longrightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\underset{[4\times 1]}{x} \longrightarrow \underbrace{\underset{[3\times 4]}{A}\, x}_{[3\times 1]}$$

$$f(\alpha x + \beta y) = \alpha \cdot f(x) + \beta \cdot f(y)$$

then f is linear!

$$f(x) = A \cdot x$$

$\Rightarrow$ linear ?

$$f(\alpha x + \beta y) = A(\alpha x + \beta y)$$
$$= A \cdot \alpha x + A \cdot \beta y$$
$$= \alpha A x + \beta A y$$
$$= \alpha f(x) + \beta f(y) \checkmark$$

↳ Linear function

$$f(x) = Ax + b$$

↳ Superposition property ?

$\underline{b = 0} \Rightarrow$ linear ✓

$b \neq 0 \Rightarrow$ affine function

$$-I = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$b = f(x) \Rightarrow Ax = b$$

find x

$$f : x \longrightarrow \underline{Ax}$$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 30 \\ 18 \\ 2 \end{bmatrix}$$

find x s.t. $Ax = b$.

$\hookrightarrow Ax = b$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 30 \\ 18 \\ 2 \end{bmatrix}$$

(3×3) (3×1) (3×1)

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\begin{cases} 3\cdot x_1 = 30 \\ 1\cdot x_1 + 2\cdot x_2 = 18 \\ x_2 - x_3 = 2 \end{cases}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -8 \\ 0 & 8 & -7 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \\ 8 \\ 7 \end{bmatrix}$$

4×3 3×1 4×1

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^4$$

$\Downarrow$ 4 equations

$\underline{3}$ unknowns

If A is $m \times n$

① More equations than unknowns:

$m > n$

over-determined

e.g. $\dim(A) = 4 \times 3$

② Less equations than unknowns:

under-determined

$m < n$

e.g. $\dim(A) = 2 \times 3$

③ Same # of equations as unknowns

$m = n$

unique solution

(square)

<u>Augmented</u> <u>Matrix</u>

$Ax = b$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\left[\begin{array}{c|c} A & b \end{array}\right] \qquad b = \begin{bmatrix} 30 \\ 18 \\ 2 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} 3 & 0 & 0 & 30 \\ 1 & 2 & 0 & 18 \\ 0 & 1 & -1 & 2 \end{array}\right] \begin{array}{l} R_1 \leftarrow \\ R_2 \\ R_3 \end{array}$$

(1) Swap rows

(2) Scaling rows

$R_1 \leftarrow \frac{1}{3} R_1$

(3) Pivot

$R_1 \leftarrow R_1 - 3R_2$

$$\begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$