Burkhard Heer · Alfred Maußner

Dynamic General Equilibrium Modeling

Computational Methods and Applications

Third Edition

Springer

Contents

Part I Representative Agent Models

1 Basic Models ... 3
- 1.1 Introduction ... 3
- 1.2 The Deterministic Finite Horizon Ramsey Model ... 4
  - 1.2.1 The Ramsey Problem ... 4
  - 1.2.2 The Karush-Kuhn-Tucker Theorem ... 7
- 1.3 The Deterministic Infinite Horizon Ramsey Model ... 9
  - 1.3.1 Recursive Utility ... 10
  - 1.3.2 Euler Equations ... 11
  - 1.3.3 Dynamic Programming ... 13
  - 1.3.4 The Saddle Path ... 17
  - 1.3.5 Models with Analytical Solution ... 21
- 1.4 The Stochastic Ramsey Model ... 25
  - 1.4.1 Stochastic Output ... 25
  - 1.4.2 Stochastic Euler Equations ... 27
  - 1.4.3 Stochastic Dynamic Programming ... 28
- 1.5 Labor Supply, Growth, and the Decentralized Economy ... 31
  - 1.5.1 Substitution of Leisure ... 31
  - 1.5.2 Growth and Restrictions on Technology and Preferences ... 32
  - 1.5.3 Parameterizations of Utility and Important Elasticities ... 39
  - 1.5.4 The Decentralized Economy ... 44
- 1.6 Model Calibration and Evaluation ... 47
  - 1.6.1 The Benchmark Business Cycle Model ... 47
  - 1.6.2 Calibration ... 51

1.6.3 Model Evaluation 56
1.7 Numerical Solution Methods 63
1.7.1 Overview 63
1.7.2 Accuracy of Solutions 66
Appendices 68
A.1 Solution to Example 1.3.1 68
A.2 Restrictions on Technology and Preferences 70
Problems 75

2 Perturbation Methods: Framework and Tools 79
2.1 Introduction 79
2.2 Order of Approximation 80
2.3 Tools 81
2.3.1 A Brief List 81
2.3.2 Application to the Deterministic Ramsey Model 82
2.4 The Stochastic Linear-Quadratic Model 88
2.4.1 The Model 89
2.4.2 Policy Functions 89
2.4.3 Certainty Equivalence 91
2.5 A Canonical DSGE Model 92
2.5.1 Example 92
2.5.2 Generalization 93
2.6 More Tools and First Results 96
2.6.1 Computer Algebra versus Paper and Pencil 96
2.6.2 Derivatives of Composite Functions and Tensor Notation 98
2.6.3 Derivatives of Composite Functions and Matrix Chain Rules 103
2.6.4 Computation of Partial Derivatives 106
Appendices 111
A.3 Solution of the Stochastic LQ Problem 111
A.4 Third-Order Effects 113
Problems 115

3 Perturbation Methods: Solutions 119
3.1 Introduction 119
3.2 First-Order Solution 120
3.2.1 First-Order Policy Functions 120
3.2.2 BA Model 120
3.2.3 System Reduction 123

3.2.4 Digression: Solving Separately for the Deterministic and Stochastic Components 126
3.3 Second-Order Solution 131
3.3.1 Second-Order Policy Functions...................131
3.3.2 Coefficients of the State Variables 132
3.3.3 Coefficients of the Perturbation Parameter 135
3.4 Third-Order Solution 139
3.4.1 Third-Order Policy Functions 139
3.4.2 Coefficients of the State Variables 139
3.4.3 Coefficients of the State-Dependent Uncertainty... 142
3.4.4 Coefficients of the Perturbation Parameter 143
3.5 Implementation....................................... 144
Appendices 146
A.5 Coefficients of the State-Dependent Uncertainty... 146
A.6 Coefficients of the Perturbation Parameter 152

4 Perturbation Methods: Model Evaluation and Applications . 155
4.1 Introduction 155
4.2 Second Moments.. 155
4.2.1 Analytic Second Moments: Time Domain 156
4.2.2 Digression: Unconditional Means 158
4.2.3 Analytical Second Moments: Frequency Domain .. 159
4.2.4 Second Moments: Monte-Carlo Approach......... 165
4.3 Impulse Responses 174
4.4 The Benchmark Business Cycle Model 176
4.5 Time-to-Build Model 181
4.6 A New Keynesian Model 186
4.6.1 The Monopolistically Competitive Economy........187
4.6.2 Price Staggering 199
4.6.3 Wage Staggering................................ 202
4.6.4 Nominal Frictions and Interest Rate Shocks 204
4.6.5 Habits and Adjustment Costs 206
Appendices 217
A.7 Derivation of the Demand Function 217
A.8 Price Phillips Curve 219
A.9 Wage Phillips Curve............................. 223
Problems 225

5 Weighted Residuals Methods 231
5.1 Introduction 231
5.2 Analytical Framework 232
5.2.1 Motivation 232
5.2.2 Residual, Test, and Weight Function 236
5.2.3 Common Test Functions 239
5.2.4 Spectral and Finite Element Functions 241
5.2.5 Illustration 241
5.2.6 General Procedure 243
5.3 Implementation 244
5.3.1 State Space 244
5.3.2 Basis Functions 246
5.3.3 Residual Function 248
5.3.4 Projection and Solution 251
5.3.5 Accuracy 255
5.4 The Deterministic Growth Model 256
5.5 The Benchmark Business Cycle Model 260
5.6 The Benchmark Search and Matching Model 264
5.6.1 Motivation 264
5.6.2 The Model 265
5.6.3 Galerkin Solution 270
5.6.4 Results 272
5.7 Disaster Risk Models 275
5.7.1 Motivation 275
5.7.2 The Benchmark Business Cycle Model with Disaster Risk 276
5.7.3 Generalized Expected Utility 285
5.7.4 Adjustment Costs of Capital 291
5.7.5 Variable Disaster Size and Conditional Disaster Probability 294
5.7.6 The Full Model 299
Problems 305

6 Simulation-Based Methods 311
6.1 Introduction 311
6.2 Extended Path Method 312
6.2.1 Motivation 312
6.2.2 The General Algorithm 319
6.2.3 Application: The Benchmark Business Cycle Model 321
6.2.4 Application: The Model of a Small Open Economy 323

6.2.5 Conclusion 335
6.3 Simulation and Function Approximation 336
6.3.1 Motivation 336
6.3.2 The General Algorithm 341
6.3.3 Application: The Benchmark Business Cycle Model 345
6.3.4 Application: The Limited Participation Model of Money 349
6.3.5 Conclusion 360
Problems 362

7 Discrete State Space Value Function Iteration 369
7.1 Introduction 369
7.2 Solution of Deterministic Models 371
7.3 Solution of Stochastic Models 383
7.3.1 Framework 383
7.3.2 Approximations of Conditional Expectations 383
7.3.3 Basic Algorithm 385
7.3.4 Initialization 386
7.3.5 Interpolation 387
7.3.6 Acceleration 388
7.3.7 Value Function Iteration and Linear Programming 391
7.3.8 Evaluation 393
7.4 Further Applications 397
7.4.1 Nonnegative Investment 398
7.4.2 The Benchmark Model 401
Problems 405

Part II Heterogenous Agent Models

8 Computation of Stationary Distributions 413
8.1 Introduction 413
8.2 Easy Aggregation and Gorman Preferences 415
8.2.1 A Numerical Example 415
8.2.2 Gorman Preferences 425
8.3 A Simple Heterogeneous Agent Model with Aggregate Certainty 429
8.4 The Stationary Equilibrium of a Heterogeneous Agent Economy 436
8.4.1 Discretization of the Distribution Function 442
8.4.2 Discretization of the Density Function 449

8.4.3 Monte-Carlo Simulation ... 451
8.4.4 Function Approximation ... 453
8.5 The Risk-Free Rate ... 456
8.5.1 The Exchange Economy ... 457
8.5.2 Computation ... 459
8.5.3 Results ... 462
8.6 Heterogeneous Productivity and Income Distribution ... 463
8.6.1 Empirical Facts on the Income and Wealth Distribution and Income Dynamics ... 464
8.6.2 The Model ... 470
8.6.3 Computation ... 477
8.6.4 Results ... 479
Problems ... 481

9 Dynamics of the Distribution Function ... 485
9.1 Introduction ... 485
9.2 Motivation ... 486
9.3 Transition Dynamics ... 489
9.3.1 Partial Information ... 490
9.3.2 Guessing a Finite Time Path for the Factor Prices ... 500
9.4 Aggregate Uncertainty: The Krusell-Smith Algorithm ... 505
9.4.1 The Economy ... 505
9.4.2 Computation ... 508
9.4.3 Calibration and Numerical Results ... 513
9.5 Applications ... 517
9.5.1 Costs of Business Cycles with Indivisibilities and Liquidity Constraints ... 518
9.5.2 Income Distribution and the Business Cycle ... 526
Problems ... 541

10 Overlapping Generations Models with Perfect Foresight ... 543
10.1 Introduction ... 543
10.2 The Steady State in OLG Models ... 545
10.2.1 An Elementary Model ... 545
10.2.2 Computational Methods ... 550
10.2.3 Direct Computation ... 552
10.2.4 Computation of the Policy Functions ... 556
10.3 The Laffer Curve ... 561
10.4 The Transition Path ... 579
10.4.1 A Stylized 6-Period OLG Model ... 580

10.4.2 Computation of the Transition Path 581
10.5 The Demographic Transition 589
10.5.1 The Model 590
10.5.2 Calibration 597
10.5.3 Computation 598
10.5.4 Results 609
10.6 Conclusion 614
Appendices 618
A.10 Derivation of Aggregate Bequests in (10.29) 618
Problems 620

11 OLG Models with Uncertainty 627
11.1 Introduction 627
11.2 Overlapping Generations Models with Individual Uncertainty 628
11.2.1 The Model 630
11.2.2 Computation of the Stationary Equilibrium 642
11.2.3 Multi-Dimensional Individual State Space 663
11.3 Overlapping Generations with Aggregate Uncertainty 672
11.3.1 Perturbation Methods 674
11.3.2 The OLG Model with Quarterly Periods 675
11.3.3 Business Cycle Dynamics of Aggregates and Inequality 685
11.3.4 The Krusell-Smith Algorithm and Overlapping Generations 693
Appendices 715
A.11 Derivation of the Stationary Dynamic Program of the Household 715
A.12 First-Order Conditions of the Stationary Dynamic Program (11.13) 718
A.13 Derivation of the Parameters of the AR(1)-Process with Annual Periods 720
Problems 721

Part III Numerical Methods

12 Linear Algebra 727
12.1 Introduction 727
12.2 Complex Numbers 727
12.3 Vectors 729

12.4 Norms 729
12.5 Linear Independence 730
12.6 Matrices 730
12.7 Linear and Quadratic Forms 735
12.8 Eigenvalues and Eigenvectors 737
12.9 Matrix Factorization 738
12.9.1 Jordan Factorization 738
12.9.2 Schur Factorization 740
12.9.3 QZ Factorization 740
12.9.4 LU and Cholesky Factorization 741
12.9.5 QR Factorization 742
12.9.6 Singular Value Decomposition 743

13 Function Approximation 747
13.1 Introduction 747
13.2 Function Spaces 748
13.3 Taylor's Theorem 749
13.4 Implicit Function Theorem 752
13.5 Lagrange Interpolation 753
13.5.1 Polynomials and the Weierstrass Approximation Theorem 753
13.5.2 Lagrange Interpolating Polynomial 754
13.5.3 Drawbacks 756
13.6 Spline Interpolation 758
13.6.1 Linear Splines 759
13.6.2 Cubic Splines 760
13.7 Orthogonal Polynomials 762
13.7.1 Orthogonality in Euclidean Space 762
13.7.2 Orthogonality in Function Spaces 763
13.7.3 Orthogonal Interpolation 765
13.7.4 Families of Orthogonal Polynomials 766
13.8 Chebyshev Polynomials 766
13.8.1 Definition 766
13.8.2 Zeros and Extrema 769
13.8.3 Orthogonality 770
13.8.4 Chebyshev Regression 771
13.8.5 Chebyshev Evaluation 774
13.8.6 Examples 775
13.9 Multivariate Extensions 777
13.9.1 Tensor Product and Complete Polynomials 777

13.9.2 Multidimensional Splines 779
13.9.3 Multidimensional Chebyshev Regression 781
13.9.4 The Smolyak Polynomial 783
13.9.5 Neural Networks 788

14 Differentiation and Integration 791
14.1 Introduction 791
14.2 Differentiation 792
14.2.1 First-Order Derivatives 792
14.2.2 Second-Order Derivatives 796
14.3 Numerical Integration 798
14.3.1 Newton-Cotes Formulas 798
14.3.2 Gaussian Formulas 799
14.3.3 Monomial Integration Formula 801
14.4 Approximation of Expectations 803
14.4.1 Expectation of a Function of Gaussian Random Variables 803
14.4.2 Gauss-Hermite Integration 804
14.4.3 Monomial Rules for Expectations 806

15 Nonlinear Equations and Optimization 811
15.1 Introduction 811
15.2 Stopping Criteria for Iterative Algorithms 812
15.3 Nonlinear Equations 815
15.3.1 Single Equations 815
15.3.2 Multiple Equations 818
15.4 Numerical Optimization 829
15.4.1 Golden Section Search 829
15.4.2 Gauss-Newton Method 831
15.4.3 Quasi-Newton 835
15.4.4 Genetic Search Algorithms 838

16 Difference Equations and Stochastic Processes 847
16.1 Introduction 847
16.2 Difference Equations 848
16.2.1 Linear Difference Equations 848
16.2.2 Nonlinear Difference Equations 851
16.2.3 Boundary Value Problems and Shooting 854
16.3 Stochastic Processes 856
16.3.1 Univariate Processes 856

16.3.2 Trends 858
16.3.3 Multivariate Processes 858
16.4 Markov Processes 859
16.4.1 The First-Order Autoregressive Process 859
16.4.2 Markov Chains 860
16.5 Linear Filters 866
16.5.1 Definitions 866
16.5.2 The HP-Filter 867

Bibliography 870

List of Figures

1.1 Boundedness of the Capital Stock 12
1.2 Phase Diagram of the Infinite-Horizon Ramsey Model 18
1.3 No Path Leaves the Region A_2 19
1.4 Convergence of the Capital Stock in the Infinite-Horizon Ramsey Model 21
1.5 Stationary Distribution of the Capital Stock in the Stochastic Infinite-Horizon Ramsey Model 30
1.6 Risk Aversion 40
1.7 Impulse Responses in the Benchmark Model 57
1.8 Impulse Responses from an Estimated VAR 58
1.9 Productivity Shock in the Benchmark Business Cycle Model . 62

2.1 Eigenvalues of J 84
2.2 Approximate Time Path of the Capital Stock in the Deterministic Growth Model 86
2.3 Policy Function for Consumption in the Deterministic Growth Model 88

4.1 Third-Order Approximate Policy Function for Capital 167
4.2 Impulse Responses in the Time-to-Build Model 185
4.3 Structure of the NK Model 188
4.4 Impulse Responses in the Monopolistically Competitive Economy 198
4.5 Interest Rate Shock and Nominal Rigidities 205
4.6 Interest Rate Shock and Real Frictions 211
4.7 TFP Shock in the NK Model 213
4.8 Government Spending Shock in the NK Model 214

5.1 Approximations of e^{-t} 242
5.2 Ergodic Set of the Benchmark Business Cycle Model from a Second-Order Solution 246
5.3 Euler Equation Residuals: Deterministic Growth Model 260
5.4 Policy Function for the Value of Employment 273
5.5 Distribution of Unemployment in the Search and Matching Model 273
5.6 Simulated Time Path of Unemployment 274

6.1 Example Solutions of the Finite-Horizon Ramsey Model 314
6.2 Approximate Time Paths of the Capital Stock in the Deterministic Growth Model 316
6.3 Simulated Time Path of the Stochastic Growth Model 318
6.4 Ergodic Set of the Benchmark Business Cycle Model from the Extended Path Simulation 324
6.5 Impulse Responses to a Productivity Shock in the Small Open Economy Model 332
6.6 Impulse Responses to a World Interest Rate Shock in the Small Open Economy Model 333
6.7 Impulse Response to a Money Supply Shock in the Limited Participation Model 358

7.1 VI versus LP 398
7.2 Policy Function for Consumption in the Stochastic Growth Model with Nonnegative Investment 401

8.1 Dynamics of Aggregate Capital Stock K_t 420
8.2 Dynamics of the Gini Coefficient of Wealth 423
8.3 Dynamics of the Gini Coefficient of Market Income 424
8.4 Savings Function 441
8.5 Convergence of the Distribution Mean 447
8.6 Convergence of the Capital Stock 448
8.7 Invariant Density Function of Wealth 449
8.8 Invariant Density Function, Employed Worker 451
8.9 Next-Period Assets of the Employed Worker 460
8.10 Next-Period Assets of the Unemployed Worker 460
8.11 Savings in the Exchange Economy 461
8.12 Stationary Distribution Function 462
8.13 Lorenz Curve of US Earnings, Income, and Wealth in 1992 465

9.1 Value Function of the Employed Worker 496

9.2 Savings of the Workers . . . 496
9.3 Dynamics of the Density Function over Time . . . 497
9.4 Convergence of the Aggregate Capital Stock . . . 498
9.5 The Dynamics of the Density Function . . . 504
9.6 Goodness of Fit for Stationary Density . . . 504
9.7 Distribution Function in Period $T = 3,000$ . . . 515
9.8 Prediction Errors without Updating . . . 516
9.9 Time Path of the Aggregate Capital Stock . . . 517
9.10 Consumption Functions in the Storage Economy . . . 521
9.11 Savings Functions in the Storage Economy . . . 521
9.12 Invariant Density Functions in the Storage Economy . . . 523
9.13 Consumption in the Economy with Intermediation . . . 523
9.14 Savings Functions in the Economy with Intermediation . . . 524
9.15 Invariant Density Functions in the Economy with Intermediation Technology . . . 524
9.16 Lorenz Curve of Income . . . 538
9.17 Lorenz Curve of Wealth . . . 540

10.1 Wealth-Age Profile in the Standard OLG Model . . . 554
10.2 Labor-Supply-Age Profile in the Standard OLG Model . . . 555
10.3 Survival Probabilities ϕ^s in Benchmark Equilibrium . . . 569
10.4 Productivity-Age Profile $\bar{y}^s$ . . . 570
10.5 Wealth-Age Profile Approximation . . . 573
10.6 Wealth-Age Profile with Age-Dependent Productivities . . . 575
10.7 Labor-Supply-Age Profile in the Economy with Age-Dependent Productivities . . . 575
10.8 Consumption-Age Profile in the Economy with Age-Dependent Productivities . . . 576
10.9 Laffer Curves . . . 576
10.10 Wealth-Age and Labor-Supply-Age Profiles in the New and in the Old Steady State . . . 583
10.11 Wealth-Age and Labor-Age Profiles in the Old Steady State and for the Household Born in Period $t = -2$ . . . 585
10.12 Transition from the Old to the New Steady State . . . 586
10.13 Survival Probabilities in the Years 2014 and 2100 . . . 591
10.14 US Population Growth Rate 1950-2100 (annual %) . . . 591
10.15 Stationary Age Distribution, Initial and Final Steady State . . . 600
10.16 Decline of the Labor Force Share during the Transition . . . 600
10.17 Increase in the Old-Age Dependency Ratio During the Transition . . . 602

10.18 Convergence of Transition Path $\tilde{K}$ 607
10.19 Individual Wealth $\tilde{a}_t^{s,j}$ in the Initial and Final Steady State ... 610
10.20 Individual Labor Supply $l_t^{s,j}$, Initial and Final Steady State 611
10.21 Convergence of Capital per Working Hour, $\tilde{K}_t = K_t/(A_t L_t)$, and Aggregate Labor $\tilde{L}_t$ 612
10.22 Convergence of Factor Prices w_t and r_t 613
10.23 Convergence of Government Variables $\widetilde{tr}_t$ and τ_t^p 613

11.1 Measure μ^s of the s-Year-Old Cohort 645
11.2 Labor Supply of the Low- and High-Skilled Workers with Idiosyncratic Productivity θ_4 650
11.3 Wealth-Age Profile in the Stochastic OLG Model 656
11.4 Consumption-Age Profile in the Stochastic OLG Model 657
11.5 Labor-Supply-Age Profile in the Stochastic OLG Model 658
11.6 Lorenz Curve of US and Model Earnings 659
11.7 Lorenz Curve of US and Model Wealth 659
11.8 Policy Functions as a Function of Wealth 668
11.9 Policy Functions as a Function of Accumulated Average Earnings 669
11.10 Cumulative Distribution Functions 671
11.11 Steady-State Age Profiles of Capital, Consumption, and Working Hours in the OLG Model of Example 11.3.1 683
11.12 Impulse Responses to a Technology Shock in the OLG Model of Example 11.3.1 686
11.13 Impulse Responses to a Government Demand Shock in the OLG Model of Example 11.3.1 687
11.14 Nonstochastic Steady-State Distribution of $\tilde{k}$ (case 1) 703
11.15 Nonstochastic Steady-State Age Profiles 710
11.16 Simulation Results 711

12.1 Gaussian Plane 728

13.1 Linear Interpolation 755
13.2 Polynomial Approximation of the Runge Function on an Equally Spaced Grid 757
13.3 Polynomial Approximation of the Runge Function on Chebyshev Zeros 757
13.4 Monomials on [0,1.2] 758
13.5 Spline Interpolation 759
13.6 Spline Interpolation of the Runge Function 762

13.7 Orthogonal Vectors 762
13.8 Orthogonal Projection in Euclidean Space 763
13.9 Chebyshev Polynomials T_1 through T_5........ 768
13.10 Weight Function of the Chebyshev Polynomials. 770
13.11 Approximation of the Runge Function with Chebyshev Polynomials 776
13.12 Approximation of a Kinked Function with Chebyshev Polynomials 777
13.13 Rectangular Grid 779
13.14 Tensor and Smolyak Grid 785
13.15 Neural Networks 788

15.1 Bisection Method 816
15.2 Modified Newton-Raphson Method 817
15.3 Secant Method 818
15.4 Gauss-Seidel Iterations 820
15.5 Dogleg Step 827
15.6 Golden Section Search 830
15.7 Stochastic Universal Sampling 841

16.1 Topological Conjugacy 853
16.2 Local Stable and Unstable Manifolds 854

List of Tables

1.1 Calibration of the Benchmark Business Cycle Model 56
1.2 Business Cycles Statistics from the Benchmark Model 61

2.1 Code List for Equation (2.40) 108
2.2 Computation of Derivatives of Example 1.6.1 110

4.1 Euler Equation Residuals: Benchmark Business Cycle Model 178
4.2 Second Moments: German Data 179
4.3 Second Moments: Benchmark Business Cycle Model....... 180
4.4 Second Moments: Time-to-Build Model 186
4.5 Calibration of the NK Model 196
4.6 Second Moments: NK Model............................. 215

5.1 Weighted Residuals Solution of the Deterministic Growth Model 259
5.2 Euler Equation Residuals of the Galerkin Solution of the Benchmark Business Cycle Model 263
5.3 Second Moments from the Benchmark Business Cycle Model: Perturbation versus Galerkin Solution 264
5.4 Calibration of the Search and Matching Model 270
5.5 Data on Global Real Returns 276
5.6 Calibration of the Benchmark Model with Disaster Risk.... 278
5.7 Annualized Real Returns in the Benchmark Model 284
5.8 Annualized Real Returns with GEU....................... 290
5.9 Annualized Real Returns with GEU and Adjustment Costs of Capital 294
5.10 Model Calibration with Variable Disaster Size 297
5.11 Annualized Real Returns with Variable Disaster Size....... 298

5.12 Baseline Calibration of the Full Disaster Risk Model 302
5.13 Annualized Real Returns in the Full Disaster Risk Model ... 303

6.1 Second Moments from the Benchmark Business Cycle Model: Extended Path Solution 324
6.2 Second Moments from the Small Open Economy Model ... 335
6.3 Successful generalized stochastic simulation (GSS) Solutions of the Stochastic Growth Model 341
6.4 GSS Solutions of the Benchmark Business Cycle Model 348
6.5 Second Moments from the Benchmark Business Cycle Model: GSS Solutions 349
6.6 Calibration of the Limited Participation Model 355
6.7 Second Moments from the Limited Participation Model 359

7.1 Value Function Iteration in the Deterministic Growth Model: Runtime 380
7.2 Value Function Iteration in the Deterministic Growth Model: Accuracy 382
7.3 Value Function Iteration in the Stochastic Growth Model: Runtime 394
7.4 Value Function Iteration in the Stochastic Growth Model: Accuracy 396
7.5 VI Solution of the Benchmark Business Cycle Model 404

8.1 Statistics for the Computation of the Invariant Distribution 452
8.2 Credit Limit and Interest Rate 463
8.3 Results of Tax Reform Policies 479

9.1 Calibration of Employment Rates 532
9.2 Correlation of Income Shares and Output 539

10.1 Computation of the Steady State of the OLG Model 558
10.2 Calibration of the Large-Scale OLG Model 571
10.3 Computation of Laffer Curves: Runtime 574
10.4 Computation of the Transition Path: Runtime 588

11.1 Calibration of OLG Model with Idiosyncratic Uncertainty .. 639
11.2 Comparison of Runtime and Accuracy 644
11.3 Comparison of Runtime 672
11.4 Second Moments of the OLG Model of Example 11.3.1 689
11.5 Comparison of Second Moments Across Studies 691

11.6 Calibration of the OLG Model with Individual and Aggregate Uncertainty 697
11.7 Runtime: Krusell-Smith Algorithm and OLG Models 704
11.8 Cyclical Behavior of the Income Distribution 713

13.1 Tabulated Values of the Sine and Cosine Function 767

16.1 Iterative Computation of the Ergodic Distribution 862
16.2 Simulation of a Markov Chain 863

Acronyms

AD	automatic differentiation	106
AI	Artificial Intelligence	vii
AR(1)	first-order autoregressive	52
AR(2)	second-order autoregressive	473
CAS	computer algebra system	97
CES	constant elasticity of substitution	23
CPU	central processing unit	63
CRRA	constant relative risk aversion	43
DARE	discrete algebraic Riccati equation	90
DGE	dynamic general equilibrium	vi
DSGE	dynamic stochastic general equilibrium	xi
etc	and so forth	13
FOC	first-order conditions	4
FT	Fourier transform	160
GA	genetic algorithm	838
GDP	gross domestic product	54
GEU	generalized expected utility	276
GSS	generalized stochastic simulation	xxxii
HP	Hodrick-Prescott	59
IES	intertemporal elasticity of substitution	41
iid	independently and identically distributed	89
IRF	impulse response function	174
KKT	Karush-Kuhn-Tucker	7
LAPACK	linear algebra package	135
lhs	left-hand side	9
LP	linear programming	391
LQ	linear-quadratic	88

NIPA	national product and income accounts	54
NK	New Keynesian	187
OLG	overlapping generations	vii
OLS	ordinary least squares	498
PEA	parameterized expectations approach	234
rhs	right-hand side	9
SD	symbolic differentiation	106
s.t.	subject to	7
TAS	time-additive separable	10
TFP	total factor productivity	48
VAR(1)	first-order vector autoregressive	94
VI	value function iteration	x
wrt	with respect to	237

List of Symbols

$\mathbb{Z}$	set of all integers
$\mathbb{R}$	real line
$\mathbb{R}_+$	non-negative real numbers, i.e., $x \in \mathbb{R}$ and $x \geq 0$
$\mathbb{R}_{++}$	positive real numbers, i.e., $x \in \mathbb{R}$ and $x > 0$
$\mathbb{R}^n$	Euclidean n-space
$\mathbb{C}^n$	complex n-space
C^n	class of functions having n continuous derivatives
f' or $f^{(1)}$	first derivative of a single valued function of a single argument
f'' or $f^{(2)}$	second derivative of a single valued function of a single argument
$f^{(n)}$	nth order derivative of a singe valued function of a single argument
f_i or $D_i f$ or f_{x_i}	first partial derivative of a single valued function with respect to its ith argument
f_{ij} or $D_i D_j f$ or $f_{x_i x_j}$	second partial derivative of a single valued function with respect to argument i and j (in this order)
$A = (a_{ij})$	n by m matrix A with typical element a_{ij}
$A^{-1} = (a^{ij})$	the inverse of matrix A with typical element a^{ij}
A', A^T	the transpose of the matrix $A = (a_{ij})$ with elements $A' = (a_{ji})$
$J(\bar{\mathbf{x}})$	the Jacobian matrix of the vector valued function $\mathbf{f}(\bar{\mathbf{x}})$ at the point $\bar{\mathbf{x}}$
$H(\bar{\mathbf{x}})$	the Hesse matrix of the single valued function $f(\bar{\mathbf{x}})$ at the point $\bar{\mathbf{x}}$

$\nabla f(\mathbf{x})$	the gradient of f at $\mathbf{x}$, that is, the row vector of partial derivatives $\partial f(\mathbf{x})/\partial x_i$
$\|\mathbf{x}\|_2$	the Euclidian norm (length) of the vector $\mathbf{x} \in \mathbb{R}^n$, which is given by $\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$
$\mathrm{tr} A$	the trace of the square matrix A, i.e., the sum of its diagonal elements
$\det A$	the determinant of the square matrix A
$\epsilon \sim N(\mu, \sigma^2)$	the random variable ϵ is normally distributed with mean μ and variance σ^2
$\forall$	for all
$\exists$	exists
!	factorial, i.e., $n! = 1 \times 2 \times \cdots \times n$.
$\binom{n}{k}$	binomial coefficient
$x = \underset{x}{\operatorname{argmin}} f(x)$	the value x that minimizes the function $f(x)$.

List of Programs

Fortran
BM_EP.f90 323
BM_VI.f90 403
DGM_VI.f90 379
Differentiation.f90
CDHesse 797
CDJac 796
IVDenF.f90 451
Optimization.f90
GaussNewton 834
GSearch1 845
GSearch2 845
GSS 831
SGM_NNI_VI.f90 400
SGM_VI.f90 . . . . . 390, 396, 405
SOE_EP.f90 334

GAUSS
AK280_perturb.g 678, 680, 684, 690
AK60_direct.g 553
AK60_proj.g 560
AK60_value.g 557
AK70_prog_pen.g 664, 666, 722
AK70_stoch_inc.g 644, 652
AK70_stock_inc.g 643
BM_pert.g 56
CoRRAM_1.src

SolveModel . . . 144
CoRRAM_2.src
HPFilter . . . 869
Impulse1 . . . 175
Impulse2 . . . 175
CoRRAM_3.src
Bisec . . . 815
CDHesse . . . 797
CDJac . . . 796
costs_cycles.g . . . 520
Demo_trans.g . . . 591, 598, 605, 609
DGM_VI.g . . . 379
DGM_WRM.g . . . 257
dynamics_income_distrib.g . . . 531, 532, 534
Function.src
Cheb_coef . . . 773
Cheb_eval . . . 775
CSpline2_coef . . . 781
CSpline2_eval . . . 781
CSpline_coef . . . 257, 761
CSpline_eval . . . 257, 761
Find . . . 760
LSpline_coef . . . 760
LSpline_eval . . . 760
GetPar.g . . . 54
Gorman.g . . . 419
IVDenF.g . . . 450, 503
equivec . . . 442, 863
IVDisF.g . . . 444
IVdisF.g . . . 443
IVExpF.g . . . 455
IVMonteCarlo.g . . . 452
Krusell_Smith_algo.g . . . 514
Laffer.g . . . 569, 573, 574, 621
Laffer_p.g . . . 574, 578
NLEQ.src
Fixp1 . . . 821
FixvMN1 . . . 821
FixvMN2 . . . 821
MNRStep . . . 826

OLG6_trans.g 582, 585, 587, 588
OLG_Krusell_Smith.g 698, 708
Ramsey1.g 85
Ramsey2.g 314
Ramsey3.g 315
Ramsey4.g 318
Risk_free_rate.g 459
SGM_NNI_VI.g 400
SGM_VI.g 390, 405
GSS 440, 831
Markov_AR1_R 866
Markov_AR1_T 865
SGM_VI_MT.g 395
SOE.g 331
SVar.g 59
Tax_reform.g 477
transition_guess.g 503
transition_part.g 493–495, 503

Julia
AK60_value.jl 558
AK70_prog_pen.jl 664
AK70_stock_inc.jl 643

MATLAB®
BM_CGC.m 262
BM_CGT.m 262
BM_CGT_Eqs.m 262
BM_EP.m 323
BM_GSS.m 347
BM_GSS_NN.m 347
BM_pert.m 56, 176
CDHesse.m 797
CDHesseRE.m 797
CDJac.m 796
ChebBase.m 787
Cubic.m 145
Der_BM.m 109
DGM.m 87
DGM_VI.m 379

DR_V1.m 280
DR_V2.m 289
DR_V3.m 293
DR_V4.m 297
DR_V5.m 297
DR_V6.m 302
DRV1_Eqs_Smolyak.m 282
DRV2_Eqs_Smolyak.m 293
DRV_V3.m 308
Example_Valder.m 109
GetPar.m 54, 178
GH_NW.m 806
GH_quad.m 806
GSearch1.m 340, 845
Impulse1.m 175
Impulse2.m 175, 308
Laffer.m 569, 573, 574, 621
Laffer_p.m 574, 578
Linear.m 145
LP_GSS_LR.m 356
Mon_quad.m 809
NK_Model.m 197
Num_Stab_Approx.m 345
Quadratic.m 145
Ramsey2.m 314
Ramsey3.m 315
Search_CGT.m 272, 275, 306
Search_CGT_Eqs.m 272
SGM_GSS.m 340
SGM_LP.m 397
SGM_VI.m 390, 405
SOE_EP.m 334
SolveModel.m 144
SVar.m 59
TTB.m 184

Python

AK60_value.py 557
AK70_prog_pen.py 664, 667
AK70_stock_inc.py 643

IVDenF.py 450
OLG_Krusell_Smith.py 698, 708
transition_part.py 493