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ECE478 Financial Signal Processing
Problem Set III: Stochastic Calculus
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Notation here follows Shreve, *Stochastic Calculus for Finance, vols. I & II,* Springer, 2004.

Theoretical Problems

Throughout, assume all indicated stochastic processes are adapted to the appropriate filtration.

1. Let $\mathcal{F}_t$ be the filtration generated by a Wiener process $W(t)$. Let $R(t)$ be the interest rate process used to define the discount process $D(t)$. Assume there exists a unique risk-neutral measure, leading to the Wiener process $\tilde{W}(t)$ with respect to $\tilde{P}$. If $V(T)$ is a random variable that is $\mathcal{F}_T$-measurable, and $V(t)$ is defined via:

$$V(t) = \frac{1}{D(t)} \tilde{E}\left(D(T) V(T) | \mathcal{F}_t\right)$$

then $D(t) V(t)$ is a martingale. **Note:** Throughout this problem, assume that $V(t)$ is continuous a.s., and $V(T) > 0$ a.s.

(a) Show that $V(t) > 0$ a.s. (from its definition above).

(b) Show that there exists an adapted process $\tilde{\Gamma}(t)$ such that:

$$dV(t) = R(t) V(t) dt + \frac{\tilde{\Gamma}(t)}{D(t)} d\tilde{W}(t)$$

Hint: Start with a formula for $d(D(t) V(t))$ as per the martingale representation theorem, then as you expand this out recognize that $dV dt = 0$.

(c) Show that there exists an adapted process $\sigma(t)$ such that we can write:

$$dV(t) = R(t) V(t) dt + \sigma(t) V(t) d\tilde{W}(t)$$

By the way, $\sigma(t)$ can be random and in particular it is fine if the formula for $\sigma(t)$ you derived involves $V(t)$. **Remark:** This may seem obvious, but for it to work we have to use $V(t) > 0$ (specifically, can't be 0). In particular, show that yields $\sigma > 0$ a.s. The point of this problem is that every strictly positive asset is a generalized geometric Brownian motion.

2. Let $X(t), Y(t)$ be Itô processes given by:

$$\begin{aligned} dX(t) &= a(t) dt + b(t) dW(t) \\ dY(t) &= c(t) dt + d(t) dW(t) \end{aligned}$$

where a, b, c, d are adapted processes. Let $V(t) = X(t) e^{Y(t)}$. Obtain an SDE satisfied by $V(t)$, simplified so it has the above form (i.e., in the form of an Itô process). Note that $X(t), Y(t)$, but not $dX(t)$ or $dY(t)$, can appear in your final expression for $dV(t)$.

3. Let $\{W_1(t), W_2(t)\}$ be two-dimensional Wiener processes, and:

$$\begin{aligned} dX(t) &= \alpha_1(t)\,dt + \sigma_{11}(t)\,dW_1(t) \\ dY(t) &= \alpha_2(t)\,dt + \sigma_{21}(t)\,dW_1(t) + \sigma_{22}(t)\,dW_2(t) \end{aligned}$$

Assume $\sigma_{11}(t), \sigma_{22}(t)$ are positive a.s., and $\sigma_{21}(t) \geq 0$ a.s.

(a) Show that $dXdY = \rho(t)\,dt$. Specifically, find ρ. It represents the correlation between the processes.

(b) Find the constraint on $a(t), b(t)$ (adapted to the filtration) such that $dW'(t) = a(t)\,dW_1(t) + b(t)\,dW_2(t)$ with $W'(0) = 0$ results in a Wiener process.

(c) Now define $W_2'(t)$ a Wiener process so that we can write:

$$\begin{aligned} dX(t) &= \alpha_1(t)\,dt + \sigma_{11}(t)\,dW_1(t) \\ dY(t) &= \alpha_2(t)\,dt + \sigma_{22}^{'}(t)\,dW_2^{'}(t) \end{aligned}$$

Find $\sigma_{22}^{'}(t)$, and $\rho_0(t)$ the correlation between $W_1(t), W_2^{'}(t)$, i.e.:

$$dW_1(t)\,dW_2^{'}(t) = \rho_0(t)\,dt$$

4. Consider a general Itô process:

$$dV(t) = \alpha(t, V(t))\,dt + \sigma(t, V(t))\,dW(t)$$

and assume $V(t) > 0$ a.s. Find the SDE satisfied by $X = \log V$. [You will need this below!]

Simulating Stochastic Differential Equations

We are going to be viewing continuous-time deterministic functions and stochastic processes in discretized time. For simplicity, we will take equally spaced time intervals, say $t = n\delta$, $0 \leq n \leq N$, with final time $T = N\delta$. (Careful, including $t = 0$, this is $N + 1$ points). We will use the notation:

$$x[n] = x(n\delta) = x_{n\delta}$$

Consider a stochastic differential equation of the form:

$$dX(t) = \beta(t, X(t))\,dt + \gamma(t, X(t))\,dW(t)$$

where $\beta(\cdot), \gamma(\cdot)$ are deterministic functions, and as usual $W(t)$ is a Wiener process. The parameter δ controls the variance of the increment dW; specifically, in the discretization, let us use the notation:

$$d_W[n] = W[n+1] - W[n]$$

where $\{d_W[n]\}$ are iid $N(0, \delta)$. The SDE actually means:

$$X(t) = X(0) + \int_0^t \beta(u, X(u))\,du + \int_0^t \gamma(u, X(u))\,dW(u)$$

In discretized form, with $t = n\delta$ this becomes:

$$X[n] = X[0] + \sum_{m=0}^{n-1} \beta(m\delta, X[m])\,\delta + \sum_{m=0}^{n-1} \gamma(m\delta, X[m])\, d_W[m]$$

If a unit of time for us represents one year, and with the daycount convention of 260 days per year, let's take $\delta = 1/260$ and $N = 520$, which means each "step" represents one day and the total time span two years.
In some cases, the object $X(t)$ is necessarily positive, in which case the above simulation technique can fail: each d_W can be arbitrarily large negative, which can drive the value to negative. In some cases, there is a mathematical guarantee the exact solution to the SDE is positive (e.g., geometric Brownian motion or the CIR interest rate model); the problem is not with the SDE, but with our simplistic simulation. One way to get around this is to define $Y = \log X$, derive the SDE involving dY, and simulate that. [It is not the only technique, nor always the best, but it is the only one we will explore]

1. Start with basic geometric Brownian motion:

$$dS_t = \alpha S_t dt + \sigma S_t dW_t$$

with $S(0) = 1$, $\alpha = 0.1$, and σ to be varied as $0.05, 0.1, 0.3$. Also assume a constant underlying interest rate $r = 0.05$. Under the risk-neutral measure, dS_t satisfies a modified SDE involving dt and $d\tilde{W}_t$. Also, we consider a European call option with $V(T) = (S(T) - K)^+$, and let us set the strike price $K = e^{\alpha T} S(0)$.

   (a) Geometric Brownian motion is theoretically always positive. As you go through your simulations, it is possible that you will generate negative S at some point. Given the size of $S(0)$ and δ, it is highly unlikely, but nevertheless include in your code the ability to detect if $S[n] \leq 0$ at any time step. If this happens, reject the path: stop continuing with that path; if you need to create say M paths, keep going until you get M "valid" paths, but also count how many "bad" paths came up along the way. Report out if you ever get a "bad" path, and how often it occurs. In this case, it should happen rarely, and hopefully never!

   (b) Write the modified SDE involving $d\tilde{W}_t$. Specifically, calculate the coefficients that appear in this SDE from the specific values of α, σ, r provided.

   (c) Generate 1000 paths of $S[n]$. [Using the actual, not risk-neutral, model] **Note:** This may take too long or crash your computer if it isn't "powerful" enough; start with say 100 paths (useful to debug your code, too), and then crank it up a bit at a time. If you can reach 1000 great, if not fewer is fine. Don't go over 1000, though. **Note:** Repeat this for each value of σ.

   (d) Use a Monte Carlo approach to estimate $E(S[N/2])$ and $E(S[N])$ directly. [Yes, the actual values, not using the risk-neutral measure] Compare to the theoretical values for $E(S)$. **Note:** Again, do these for each value of σ.

   (e) We now want to connect this to the Black-Scholes model. Use the Black-Scholes model to obtain a formula for $V[N/2]$ in terms of $S[N/2]$, and graph it at a function of $S[N/2]$, specifically three curves (one for each of the three values of σ we are considering).

(f) Now take the first 10 paths you generated, $S^{(i)}[n]$, $1 \leq i \leq 10$. For each $S^{(i)}[N/2]$, you can compute $V^{(i)}[N/2]$ from the Black-Scholes formula. On the other hand, you can use a Monte Carlo approach to compute $V^{(i)}[N/2]$ using the martingale property of the discounted stock price. So, for each i, grow 1000 **[or fewer if your computer complains!]** paths from $N/2$ to N *using the risk-neutral model* and average to estimate $V^{(i)}[N/2]$. Compare these estimated values with the exact values (via BSM) in these cases. Report the results how you see fit: a table; you could superimpose two scatter plots ($S^{(i)}[N/2]$ versus actual $V^{(i)}[N/2]$, and $S^{(i)}[N/2]$ versus actual $V^{(i)}[N/2]$). **Comments:** Careful. First, you need to grow NEW paths, emanating from time $t = T/2$, towards $t = T$, taking **actual** $S^{(i)}[N/2]$ (generated from the **actual** SDE) as an initial condition, but you are computing $V^{(i)}[N/2]$ as an expectation by virtue of the martingale property, which means you must *extend* time from $N/2$ to N using the *risk-neutral SDE!*

2. **Cox-Ingersoll-Ross Interest Rate Model**

$$dR(t) = (a - \beta R(t))\,dt + \sigma\sqrt{R(t)}dW(t)$$

with α, β, σ positive, and let $R(0) = r > 0$. This model for interest rate $R(t)$ guarantees $R(t) > 0$. This model is typically used for short term interest rates, which tend to exhibit volatility. Although no closed form solution exists, it is possible to determine certain properties for it. In particular the mean and variance of $R(t)$ are given by:

$$\begin{aligned} E(R(t)) &= e^{-\beta t}r + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) \longrightarrow \frac{\alpha}{\beta} \\ var(R(t)) &= \frac{\sigma^2}{\beta}r\left(e^{-\beta t} - e^{-2\beta t}\right) + \frac{\alpha\sigma^2}{2\beta^2}\left(1 - 2e^{-\beta t} - e^{-2\beta t}\right) \longrightarrow \frac{\alpha\sigma^2}{2\beta^2} \end{aligned}$$

The simple simulation approach we are taking here is likely going to cause "bad" paths (where $R[n] \leq 0$ occurs along the way). The approach we will take is to define $X(t) = \log R(t)$, and compare simulation of X with direct simulation of R. In this problem, set $\beta = 1$, $\alpha = 0.10\beta$, $r = 0.05$, $\sigma = 0.5$.

(a) Derive the SDE satisfied by $X(t)$.

(b) Try to generate 1000 paths for $R(t)$ over a span $0 \leq t \leq 10$, using $\delta = 0.01$. In your code, trap the condition that $R \leq 0$ (if this occurs, your code should display an exception, should halt that one path, but should continue computing the other paths). Do not repeat until you get 1000 paths. Instead, run your simulation 1000 times, and report out how many "valid" paths you get.

(c) From the set of "valid" paths, compute the sample mean and variance at times $t = 1, 2, \cdots, 10$.

(d) Graph the first 10 "valid" paths for $R(t)$, and maybe one or two of the "bad" paths, just to see what they look like.

(e) Now generate 1000 paths of $X(t)$, and from that form $R(t)$. graph 10 paths for $R(t)$ obtained this way. Also, compute the sample mean and variance of $R(t)$ at times $t = 1, 2, \cdots, 10$ using this alternate approach.

(f) Graph the theoretical $E(R(t))$ as a curve for $0 \leq t \leq 10$, superimpose markers at $t = 1, 2, \cdots, 10$ for the values found from each of the simulation (use different marker symbols for each simulation method). In your graph, also draw a line at the asymptotic limit. Also, display a table of the absolute error between the estimated $E(R(t))$ and theoretical values for each of these two simulation approaches, for $t = 1, 2, \cdots, 10$.

(g) Repeat the above for $var(R(t))$. In your graph, also draw a line at the asymptotic limit.

Remark: The problem with just keeping "good paths" is not just that we end up with fewer paths, but the statistical properties of the paths we keep may be skewed. For example, paths that start to go down are more likely to cross into negative territory, so the paths we keep may be more likely to start upwards. An issue we are not directly addressing is, aside from the possibility of crossing into the negative, is whether the solution to the *discrete* form of the SDE really "matches" (statistically) that of the underlying continuous model. Thus, for example, although the log approach never seems to generate an "invalid" path, it is not clear if it is a "good" method.