The Cooper Union
Department of Electrical Engineering
ECE478 Financial Signal Processing
Exam II
December 13, 2022

Time: 50 minutes. **Closed book, closed notes. No calculators.**

Some notes and results:
Assume W_t denotes a **standard** Wiener process.
Geometric Brownian motion S_t satisfies:

$$dS_t = \alpha S_t dt + \sigma S_t dW_t$$

Black-Scholes-Merton (BSM): formula: Assuming an underlying asset S_t that is geometric Brownian motion, the price at time t of a European option with strike price K is $c(t, S_t)$ where $c(t, x)$ is an explicit function involving K, σ and the underlying interest rate r.

1. **[16 pts.]** Consider the function appearing in the BSM formula, $c(t, x)$. Using the notation $c_t = \partial c / \partial t$, etc., associate each of the following "greeks" with c_t, c_{tt}, c_x, c_{xx}, c_{tx}, c_σ (the derivative with respect to volatility)

   (a) delta

   (b) gamma

   (c) vega

   (d) theta

2. **[12 pts.]** Let:

$$\begin{aligned} dX(t) &= a(t)\,dt + b(t)\,dW(t) \\ dY(t) &= c(t)\,dt + d(t)\,dW(t) \end{aligned}$$

   Write the SDE satisfied by:

$$V = X/Y$$

3. **[24 pts.]** Let:

$$d\tilde{W}_t = \theta_t dt + dW_t \tag{1}$$

where θ_t is adapted to the filtration associated with W_t. Let:

$$X(t) = -\frac{1}{2}\int_0^t \theta^2(u)\,du - \int_0^t \theta(u)\,dW(u)$$

and:

$$Z(t) = \exp(X(t))$$

Define the random variable $Z = Z(T)$. We can use Z as a Radon-Nikodym derivative, and under the probability measure $\tilde{P}$ so generated, $Z(t)$ is the associated Radon-Nikodym process, and $\tilde{W}_t$ is a standard Wiener process.

(a) This is called ______ Theorem.

(b) One property of the standard Wiener process prescribes $d\tilde{W}d\tilde{W}$. What should it be? Compute it using equation (1) to confirm it is what it should be.

(c) The above tells us that the quadratic variation $\left[\tilde{W}, \tilde{W}\right](t)$ is what?

(d) Write the SDE satisfied by $X(t)$.

(e) Part of the proof requires expanding $d\left(\tilde{W}Z\right)$ and conclude that $\tilde{W}Z$ is a martingale under the original measure $\mathbb{P}$. What exactly must be true about $d\left(\tilde{W}Z\right)$ for it to be a martingale (under $\mathbb{P}$)? **NOTE:** Do not actually try to obtain an expression for $d\left(\tilde{W}Z\right)$. Just say what we are hoping to happen.

(f) Concluding that $d\left(\tilde{W}Z\right)$ is a martingale by observing its SDE is based on the ______ theorem. **Hint: Two words, not someone's name.**

(g) Using the Radon-Nikodym theorem, we conclude that $\tilde{W}_t$ is a martingale under the measure $\tilde{\mathbb{P}}$. This combined with the formula for $\left[\tilde{W}, \tilde{W}\right](t)$ allows us to conclude that $\tilde{W}_t$ is a standard Brownian motion under $\tilde{\mathbb{P}}$. THIS result is called _________ Theorem.

(h) In finance theory, $\tilde{\mathbb{P}}$ is called the ______ measure.

4. **[16 pts.]** Consider an Itô integral:

$$X(t) = \int_0^t \Delta(u)\, dW(u)$$

where $\Delta(t)$ is adapted to the filtration generated by W_t. Also assume $\{t_j\}$ as listed below references a partition.

   (a) The Itô integral is defined as the limit of which of the following:

$$\begin{array}{ll} 1: & \sum \Delta(t_j)\left[W(t_j) - W(t_{j-1})\right] \\ 2: & \sum \Delta(t_j)\left[W(t_{j+1}) - W(t_j)\right] \\ 3: & \text{Either one- you get the same result either way.} \end{array}$$

   (b) Prove that $X(t)$ is a martingale. **Hint:** For $t < T$, write $X(T) = X(t) +$ sum of several terms.

5. **[6 pts.]** Let W_1, W_2 be independent Wiener processes and:

$$\begin{aligned} dS_1 &= \alpha_1 S_1 dt + \sigma_{11} S_1 dW_1 \\ dS_2 &= \alpha_2 S_2 dt + \sigma_{21} S_2 dW_1 + \sigma_{22} S_2 dW_2 \end{aligned}$$

   Find $dS_1 dS_2$.

6. **[6 pts.]** Let W_1, W_2 be independent Wiener processes generating a filtration $\mathcal{F}_t$, and $\alpha(t), \beta(t)$ are adapted processes with $\alpha > 0, \beta > 0$ a.s. Let:

$$d\tilde{W} = \alpha dW_1 + \beta dW_2$$

   with $W(0) = 0$. I will tell you that $\tilde{W}$ is a continuous martingale (you don't need to verify). We need one other property to conclude that $\tilde{W}$ is itself a Wiener process, and that imposes a condition on α, β. What else do we need (express it in terms of $d\tilde{W}$, not an integral), and derive the condition on α, β.

7. **[20 pts.]** Short answer.

   (a) A market is complete iff for every security a replicating portfolio *(exists/ is unique)*.

   (b) The definition of a risk-neutral measure is one under which the _______ of every security is _______.

   (c) The *(First / Second)* Fundamental Theorem of Asset Pricing is that the market model is complete iff the risk-neutral measure *(exists / is unique)*.

   (d) The *other* Fundamental Theorem of Asset Pricing (the one that is not part **b)** is that the risk-neutral measure *(exists / is unique)* iff the market model satisfies the condition that _______.