CS181 Practice Questions: POMDPs and Mixture Models

1. **Belief State Methods**

   In belief state methods, the agent has to maintain a probability distribution over the current state of the world.

   $$b(s_t) = P(s_t | o_1, \ldots, o_t,\ a_1, \ldots, a_{t-1})$$

   Express $b(s_t)$ as a function of $b(s_{t-1})$.

$$\begin{aligned}
b(s_t) &= P(s_t \mid o_1, \ldots, o_t, a_1, \ldots, a_{t-1}) \\
&\propto P(o_t \mid o_1, \ldots, o_{t-1}, a_1, \ldots, a_{t-1}, s_t) P(s_t \mid o_1, \ldots, o_{t-1}, a_1, \ldots, a_{t-1}) \\
&= P(o_t \mid s_t) \sum_{s_{t-1}} P(s_t, s_{t-1} \mid o_1, \ldots, o_{t-1}, a_1, \ldots, a_{t-1}) \\
&= P(o_t \mid s_t) \sum_{s_{t-1}} P(s_t \mid s_{t-1}, a_{t-1}) P(s_{t-1} \mid o_1, \ldots, o_{t-1}, a_1, \ldots, a_{t-1}) \\
&= P(o_t \mid s_t) \sum_{s_{t-1}} P(s_t \mid s_{t-1}, a_{t-1}) b(s_{t-1})
\end{aligned}$$

2. **POMDPs vs MDPs**

Why can we not use the same approaches to solve an MDP as a POMDP? That is, why is the POMDP case so much harder to compute an optimal solution for?

In an MDP we can assume that, while computing the optimal policy, we did not need to remember all of the past actions because the actions were Markovian. However, when we are computing the optimal policy in a POMDP, the optimal thing to do now may depend upon the entire history of actions/observations up until that point.

3. **Belief States**

Show that the set of belief states is uncountably infinite in a finite state space.

Recall that a belief state is just a probability distribution over the states, i.e. a vector $\boldsymbol{b}$ where $b_i = P(s_i)$. Suppose we have n states (clearly the result will still hold if the number of states is infinite). Then, each element in the belief space is a vector in the n-simplex (i.e. the n dimensional vectors whose elements are all non-negative and whose components sum to 1). But clearly the set of all such vectors is uncountably infinite. To see this, just consider the two dimensional case, where the simplex is a line segment of slope -1 from $(1, 0)$ to $(0, 1)$. The set of all points on this line is uncountably infinite.

4. **Bounds (Bishop 9.17)**

Show that as a consequence of the constraint $0 \leq p(x_n|\mu_k) \leq 1$ for the discrete variable x_n, the incomplete-data log likelihood function for a mixture of Bernoulli distributions is bounded above, and hence that there are no singularities for which the likelihood goes to infinity.

The result follows from Bishop Eq. (9.51). The largest value that the argument to the logarithm on the r.h.s. of Eq. (9.51) can have is 1, since $\forall n, k : 0 \leq p(x_n|\mu_k) \leq 1$, $0 \leq \pi_k \leq 1$ and $\sum_k^K \pi_k = 1$. Thus the maximum value for $ln p(X|\mu, \pi)$ equals 0.

5. **Gaussian (Bishop 9.3)**

> Consider a Gaussian mixture model in which the marginal distribution $p(z)$ for the latent variable is given by Bishop Eq. (9.10), and the conditional distribution $p(x|z)$ for the observed variable is given by Eq. (9.11). Show that the marginal distribution $p(x)$, obtained by summing $p(z)p(x|z)$ over all possible values of z, is a Gaussian mixture of the form Eq. (9.7).

From Bishop Eqs. (9.10) and (9.11), we have

$$p(x) = \sum p(x|z)p(z) = \sum_z \prod_{k=1}^{K} (\pi_k N(x|\mu_k, \Sigma_k))^{z_k}$$

We use the 1-of-K representation for z:

$$= \sum_z \prod_{k=1}^{K} (\pi_k N(x|\mu_k, \Sigma_k))^{I_{kj}} = \sum_{j=1}^{K} \pi_j N(x|\mu_j, \Sigma_j)$$

Note: $I_{kj} = 1$ when $k = j$ and 0 otherwise.

6. **EM and Mixture Models (Bishop 9.8)**

Show that if we maximize (9.40) with respect to μ_k while keeping the responsibilities $\gamma(znk)$ fixed, we obtain the closed form solution given by (9.17). (Hint use Bishop Equation 2.43)

Following the hint, we write the r.h.s. of Bishop Eq. (9.40) as

$$-\frac{1}{2}\sum_{n=1}^{N}\sum_{j=1}^{K}\gamma(z_{nj})(x_n-\mu_j)^T\Sigma^{-1}(x_n-\mu_j)+c$$

c represents terms independent of mu_j. We take derivative wrt μ_k:

$$-\sum_{n=1}^{N}\gamma(z_{nk})(\Sigma^{-1}\mu_k-\Sigma^{-1}x_n)$$

Set this to zero and rearrange to get Eq. (9.17).

7. Gaussian Mixture Models

> Suppose we have a Gaussian mixture model where the covariance matrices Σ_k are all constrained to have a common value, call it Σ. Derive the EM condition for maximizing the likelihood function with respect to Σ under such a model.

In this case, we have that the expected complete-data log likelihood function is

$$\mathbb{E}_Z[\ln p(X, Z \mid \mu, \Sigma, \pi)] = \sum_{n=1}^{N} \sum_{k=1}^{K} \gamma(z_{n,k})(\ln \pi_k + \ln(\boldsymbol{x}_n \mid \mu_k, \Sigma))$$

We can differentiate this with respect to Σ^{-1}, which gives us

$$\frac{N}{2}\Sigma - \frac{1}{2} \sum_{n=1}^{N} \sum_{k=1}^{K} \gamma(z_{n,k})(\boldsymbol{x}_n - \mu_k)(\boldsymbol{x}_n - \mu_k)^{\mathsf{T}}$$

Now, we can set this equal to 0 to see that

$$\Sigma = \frac{1}{N} \sum_{n=1}^{N} \sum_{k=1}^{K} \gamma(z_{n,k})(\boldsymbol{x}_n - \mu_k)(\boldsymbol{x}_n - \mu_k)^{\mathsf{T}}$$