Continuation of Vector/Matrix Review

Note Title

LECTURE 04

★ Range & Nullspace (or Kernel)

<u>Def.</u> $A \in \mathbb{R}^{m \times n}$.

range$(A) := \{ \mathbf{y} \in \mathbb{R}^m \mid \mathbf{y} = A\mathbf{x},\ \mathbf{x} \in \mathbb{R}^n \}$

often written as Ran(A) or Im(A).

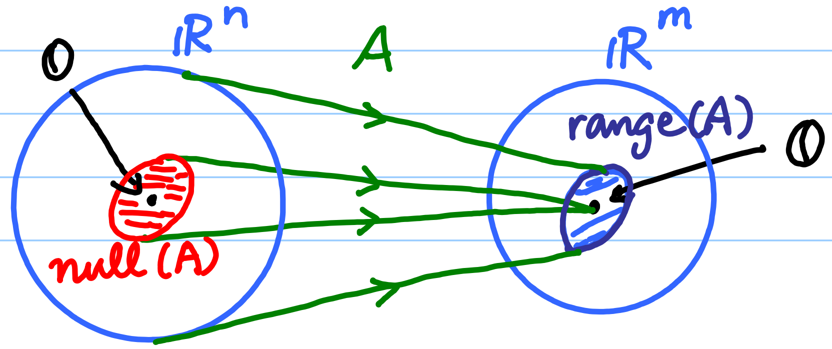
This is also called the image

column space of A.

null$(A) := \{ \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0} \}$

is called the nullspace (or kernel) of A

Ker(A)



<u>Thm</u> range$(A) = $ span$\{ \mathbf{a}_1, \cdots, \mathbf{a}_n \}$

= a set of all possible linear combi. of $\{ \mathbf{a}_1, \cdots, \mathbf{a}_n \}$

(Proof) Need to show two things

(1) range$(A) \subset$ span$\{ \mathbf{a}_1, \cdots, \mathbf{a}_n \}$

(2) span$\{ \mathbf{a}_1, \cdots, \mathbf{a}_n \} \subset$ range(A)

Now, (1) is easy since any $\mathbf{y} \in$ range(A) by definition, $\exists\, \mathbf{x} \in \mathbb{R}^n$ s.t. $\mathbf{y} = A\mathbf{x}$.

This is a lin. combi. of col vectors of A

So, $\mathbf{y} \in$ span$\{ \mathbf{a}_1, \cdots, \mathbf{a}_n \}$.

(2) Take any $y \in \operatorname{span}\{a_1, \cdots, a_n\}$.
By definition, $\exists \{x_1, \cdots, x_n\}$ s.t.
$$y = x_1 a_1 + \cdots + x_n a_n = A\mathbf{x} \in \operatorname{range}(A)$$
by setting $\mathbf{x} = (x_1, \cdots, x_n)^T$ //

☆ Linear Independence, Bases

Def. The vectors $\{a_1, \cdots, a_n\}$, $a_j \in \mathbb{R}^m$ are called linearly independent if
$$\sum_{j=1}^{n} x_j a_j = \mathbf{0} \iff x_j = 0, \quad 1 \le j \le n$$

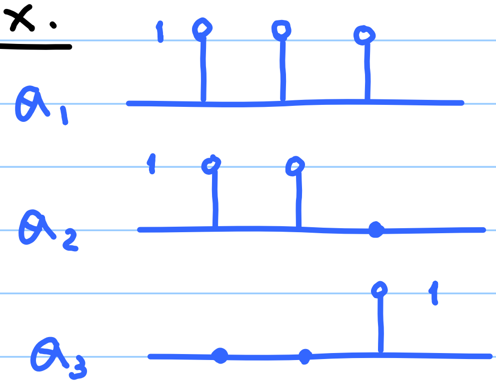
A set of m linearly independent vectors in $\mathbb{R}^m$ is called a basis in $\mathbb{R}^m$. $\Rightarrow$ a matrix representation of a basis in $\mathbb{R}^m$ is an $m \times m$ matrix. Note that any vector in $\mathbb{R}^m$ can be written as a lin. combi. of the m basis vectors in $\mathbb{R}^m$

Def. The dimension of $\operatorname{span}\{a_1, \cdots, a_n\}$ is the maximal number of linearly independent vectors among $\{a_1, \cdots, a_n\}$

i.e., if $\exists j$, $a_j = x_1 a_1 + \cdots + x_{j-1} a_{j-1} + x_{j+1} a_{j+1} + \cdots + x_n a_n$

then such a_j is useless in some sense (or more precisely, it is redundant).

Ex.



In $\mathbb{R}^3$, these are linearly dependent.

$$a_1 = a_2 + a_3$$

So $\dim \operatorname{span}\{a_1, a_2, a_3\} = 2$.

We cannot write any vector in $\mathbb{R}^3$ by a lin. combi. of $\{a_2, a_3\}$.
Only a certain subset of vectors in $\mathbb{R}^3$ can be written as a lin. combi of $\{a_2, a_3\}$ (no control on the first and second entries of a vector in $\mathbb{R}^3$.)

✱ Rank

Def. The column rank of A

$$:= \dim(\operatorname{range}(A))$$
$$= \#\text{ of linearly indep. col. vec's of } A.$$

The row rank of A

$$:= \dim(\operatorname{range}(A^T))$$
$$= \#\text{ of linearly indep. row vec's of } A.$$

$$\operatorname{rank}(A) := \dim(\operatorname{range}(A))$$

$A \in \mathbb{R}^{m \times n}$ is said to be of full rank if $\operatorname{rank}(A) = \min(m, n)$.

<u>Thm</u>. $A \in \mathbb{R}^{m \times n}$, $m \geq n$ is of full rank
$\iff \forall \mathbb{x}, \mathbb{y} \in \mathbb{R}^n$, $\mathbb{x} \neq \mathbb{y}$,
$A\mathbb{x} \neq A\mathbb{y}$.

(Proof) [$\Rightarrow$] If $\text{rank}(A) = n$, i.e., full rank,
then $\{\mathbb{a}_1, \cdots, \mathbb{a}_n\}$ are lin. indep.
So, they form a basis of range(A).
This means that $\forall \mathbb{b} \in \text{range}(A)$,
There exists a unique ... $\exists! \; \mathbb{x} \in \mathbb{R}^n$ s.t. $\mathbb{b} = \sum_{j=1}^{n} x_j \mathbb{a}_j$.

[$\Leftarrow$] Suppose A is not of full rank.
Then $\{\mathbb{a}_1, \cdots, \mathbb{a}_n\}$ are linearly dep.
i.e., $\exists \; \mathbb{c} \in \mathbb{R}^n$, $\mathbb{c} \neq \mathbb{0}$ s.t.

$$\sum_{j=1}^{n} c_j \mathbb{a}_j = \mathbb{0}, \text{ i.e., } A\mathbb{c} = \mathbb{0}.$$

Then set $\mathbb{y} = \mathbb{x} + \mathbb{c} \neq \mathbb{x}$.
But $A\mathbb{y} = A(\mathbb{x} + \mathbb{c}) = A\mathbb{x} + \underbrace{A\mathbb{c}}_{= \mathbb{0}}$
$= A\mathbb{x}$ contradiction!
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☆ <u>Inverse</u>

<u>Def</u>. A is said to be **nonsingular** or **invertible** $\iff$ A is <u>square</u> and of <u>full rank</u>

Hence $A \in \mathbb{R}^{m \times m}$: nonsingular
$\Rightarrow \{\mathbb{a}_1, \cdots, \mathbb{a}_m\}$ form a basis of $\mathbb{R}^m$

That means: the canonical basis vector $e_j \in \mathbb{R}^m$ can also be written as a lin. combi. of $\{a_1, \cdots, a_m\}$

$$e_j = \sum_{i=1}^{m} z_{ij} a_i \quad , \exists z_{ij} , \ 1 \le i \le m$$

$$\Rightarrow \quad e_j = A z_j , \quad z_j = (z_{1j}, \cdots, z_{mj})^T$$

So, $[e_1 | e_2 | \cdots | e_m] = A \, [z_1 | z_2 | \cdots | z_m]$

$$\Leftrightarrow \qquad \underbrace{I}_{m \times m \text{ identity matrix}} = A Z$$

Such matrix $Z \in \mathbb{R}^{m \times m}$ is called the **inverse** of A and written as A^{-1}.

Any nonsingular matrix has a unique inverse, and $A A^{-1} = A^{-1} A = I$.

<u>Thm</u> (<u>Equivalences of a nonsingular matrix</u>)

For $A \in \mathbb{R}^{m \times m}$, the following statements are equivalent:

(a) A has an inverse A^{-1}
(b) $\mathrm{rank}(A) = m$
(c) $\mathrm{range}(A) = \mathbb{R}^m$
(d) $\mathrm{null}(A) = \{0\}$
(e) 0 is not an eigenvalue of A
(f) 0 is not a singular value of A
(g) $\det(A) \neq 0$.

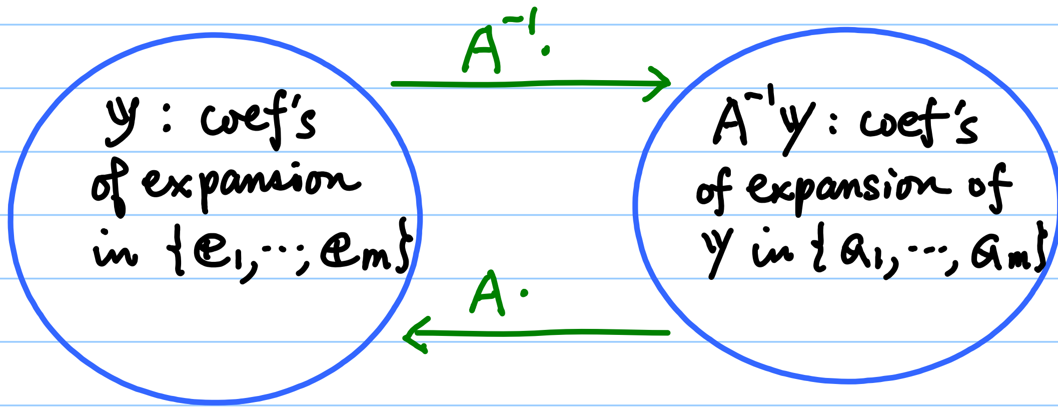
★ $\underline{\text{Matrix}^{-1} \overset{\text{times}}{\times} \text{vector}}$

$\mathbf{y} = A\mathbf{x}$, A: nonsingular

$\Rightarrow \mathbf{x} = A^{-1}\mathbf{y}$.

This means that $A^{-1}\mathbf{y}$ represents an expansion coefficients of $\mathbf{y}$ in the basis of col's of A.

So, Multiplication by A^{-1} is a change of basis operation!



Note: $\mathbf{y} = I^{-1}\mathbf{y}$.