Unit 2 tutorial 1

Part A

① Find the directional derivative of $\phi = 2xy + 5yz + zx$ at the point (1,2,3) in the direction of $3\hat{i} - 5\hat{j} + 4\hat{k}$.

$(2y+z)\hat{i} + (2x+5z)\hat{j} + (5y+x)\hat{k}$

$(7\hat{i} + 17\hat{j} + 11\hat{k}) \cdot \dfrac{(3\hat{i} - 5\hat{j} + 4\hat{k})}{\sqrt{9+25+16}}$

$= \dfrac{21 - 85 + 44}{5\sqrt{2}} = \dfrac{-20}{5\sqrt{2}} = -2\sqrt{2}$.

② Find the angle of intersection at the point (2,-1,2) of the surfaces $x^2+y^2+z^2=9$ & $z=x^2+y^2-3$

$\theta = \cos^{-1}\left|\dfrac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\cdot|\nabla\phi_2|}\right|$

$\nabla\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k} = 4\hat{i} - 2\hat{j} + 4\hat{k}$

$\nabla\phi_2 = 2x\hat{i} + 2y\hat{j} - 1\hat{k} = 4\hat{i} - 2\hat{j} - \hat{k}$

$\cancel{\theta = \dfrac{4+4-4}{2\sqrt{3}\cdot 2\sqrt{2}}}$ $\quad \theta = \cos^{-1} \dfrac{16+4-4}{6 \times \sqrt{21}} = \cos^{-1}\left[\dfrac{8}{3\sqrt{21}}\right]$

③ Find the angle between the normals to the surface $xy = z^2$ at the points (-2,-2,2) & (1, 9, -3).

$\nabla\phi = \dfrac{d(xy - z^2)}{dx \cdot dy \cdot dz} = y\hat{i} + x\hat{j} - 2z\hat{k}$

$= -2\hat{i} - 2\hat{j} - 4\hat{k} = \nabla\phi_1$

$\nabla\phi_2 = 9\hat{i} + \hat{j} + 6\hat{k}$.

$\dfrac{\nabla\phi \cdot \nabla\phi_2}{|\nabla\phi_1|\cdot|\nabla\phi_2|} = \dfrac{-18-2-24}{2\sqrt{6}\cdot\sqrt{118}} = \dfrac{-22}{\sqrt{118 \times 6}} = \dfrac{-11}{\sqrt{177}}$

(4) The temperature of points in space is given by $T(x,y,z) = x^2+y^2-z$. A mosquito located at $(1,1,2)$ desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move?

$$\nabla \phi = \frac{dF}{dx}\hat{i} + \frac{dF}{dy}\hat{j} + \frac{dF}{dz}\hat{k}$$

$$= 2x\hat{i} + 2z\hat{j} - \hat{k}$$

$$\nabla T \cdot = \underline{2\vec{i} + 2\vec{j}} - \vec{k}.$$

(5) If $\vec{F} = (x+y+1)\hat{i} + \hat{j} - (x+y)\vec{k}$ show that $\vec{F}$. curl $\vec{F} = 0$.

$$F \cdot (\nabla \times F) = [F \cdot \nabla \cdot F] = 0 \qquad \text{(Triple product).}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d/dx & d/dy & d/dz \\ x+y+1 & 1 & -(x+y) \end{vmatrix} = -1\hat{i} - \hat{j}(-1) + \hat{k}(-1)$$

$$= -\hat{i} + \hat{j} - \hat{k} = \nabla \times F$$

$$F \cdot (\nabla \times F) = [(x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}] \cdot (-\hat{i} + \hat{j} - \hat{k})$$

$$= -x - y - 1 + 1 + x + y = 0.$$

$$\therefore \quad F \cdot (\nabla \times F) = 0.$$

(6)

i) Prove that $\vec{F} = (2x+yz)\hat{i} + (4y+xz)\hat{j} - (6z - xy)\vec{k}$ is solenoidal as well as irrotational. Also find the scalar potential of $\vec{F}$.

Solenoidal $= \nabla \cdot \vec{F} = 0$

$$= 2 + 4 - 6 = 0.$$

Irrotational $= \nabla \times F = 0$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d/dx & d/dy & d/dz \\ 2x+yz & 4y+xz & -(6z - xy) \end{vmatrix}$$

$(-x+x)\hat{i} - (-y+y)\hat{j} + \vec{k}(z-z) = 0.$

ϕ = irrotational

$\vec{F} = \nabla\phi$

$\vec{F} = \sum \hat{i} \frac{\partial\phi}{dx}$

$\Rightarrow 2x + yz = \frac{d\phi}{dx} \quad = x^2 + yzx + c_1$

$\Rightarrow 4y + xz = \frac{d\phi}{dy} \quad = 2y^2 + xyz + c_2$

$\Rightarrow 6z - xy = \frac{d\phi}{dz} \quad = 3z^2 - xyz + c_2$

$\therefore \quad x^2 + 2y^2 + 3z^2 + xyz + c = \phi$

6 Q) (ii) Find the unit normal to the surface $x^4 - 3xyz + z^2 + 1 = 0$ at the point $(1,1,2)$.

$(4x^3 - 3yz)\hat{i} - 3xz\hat{j} + (-3xy + 2z)\hat{k}$

$(4-3)\hat{i} - 3\hat{j}\ (-3+2)\hat{k} \quad = \hat{i} - 3\hat{j} - \hat{k} = N.$

unit vector $= \frac{\vec{N}}{|\vec{N}|} = \frac{\hat{i} - 3\hat{j} - \hat{k}}{\sqrt{11}}$

7. Prove that $\text{div}(r^n\vec{r}) = (n+3)r^n$. Deduce that $r^n\vec{r}$ is solenoidal if and only if $n = -3$.

$\text{div}\,\vec{r} = \frac{dx}{dx} + \frac{dy}{dy} + \frac{\partial z}{\partial z} = 1+1+1 = 3.$

$\text{div}(r^n\vec{r}) = \nabla r^n \vec{r} + r^n \text{div}\,\vec{r} \qquad [\text{div}\,\phi\vec{v} = \phi\,\text{div}\,\vec{v} + \nabla\phi\cdot\vec{v}]$

$\text{div}\,\vec{r} = 3 \qquad \nabla r^n = n\,r^{n-2}\,\vec{r}$

$\text{div}\,r^n\vec{r} = n(r^{n-2}\,\vec{r}\cdot\vec{r}) + r^n\cdot 3.$

$= n\cdot r^{n-2} r^2 + 3r^n = nr^n + 3r^n$

$= r^n(n+3)$

$\Rightarrow n = -3$. for it to be 0.

8i) If $\vec{A}$ and $\vec{B}$ are irrotational prove that $\vec{A}\times\vec{B}$ is solenoidal

For irrotational vector $\vec{A}$ & $\vec{B}$ so that.

$$\text{curl } \vec{A} = \text{curl} \vec{B} = 0 .$$

& for $\vec{A}\times\vec{B}$ to be solenoidal.

$$\text{div}(\vec{A}\times\vec{B}) = 0 .$$

$$\text{div}(\vec{A}\times\vec{B}) = \vec{b}\,\text{curl } \vec{a} - \vec{a}\,\text{curl } \vec{b} = 0 .$$

$$\therefore \text{curl } \vec{a} = \text{curl } \vec{b} = 0 .$$

(ii) Show that the vector

$$\vec{F} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$$ is irrotational and find its scalar potential.

for irrotational. $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d/dx & d/dy & d/dz \\ A_1 & A_2 & A_3 \end{vmatrix} = 0 . \Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d/dx & d/dy & d/dz \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$

$$(-1+1)\hat{i} + (3z^2 - 3z^2) - \hat{j} + \hat{k}(6x - 6x) = \vec{0} .$$

$$\vec{F} = \frac{d\phi}{dx}\hat{i} + \frac{d\phi}{dy}\hat{j} + \frac{d\phi}{dz}\hat{j} .$$

$$\therefore \quad \phi = 3x^2y + z^3x + k_1$$

$$\phi = 3x^2y - zy + k_2$$

$$\phi = xz^3 - yz + k_3$$

$$\therefore \phi = \underline{3x^2y + xz^3 - yz + K} .$$

(9) If $\nabla\phi = (y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j} + (6z^3 - 3x^2yz^2)\vec{k}$ find ϕ.

$F = \nabla\phi = \frac{d\phi}{dx}\hat{i} + \frac{d\phi}{dy}\hat{j} + \frac{d\phi}{dz}\hat{k}$

$\therefore\ \phi = y^2x - x^2yz^3 + K_1$

$\phi = 3y + xy^2 - x^2yz^3 + K_2$

$\phi = z^4 \cdot \frac{3}{2} - x^2yz^3 + K_3$

$\therefore\ \phi = xy^2 + 3y + \frac{3}{2}z^4 - x^2yz^3 + K.$

(10) Find the values of a & b so that the surfaces $ax^3 - by^2z = (a+3)x^2$ and $4x^2y - z^3 = 11$ may cut orthogonally at $(2,-1,-3)$.

$3ax^2\hat{i} - (a+3)2x\hat{i}] - 2byz\hat{j} - by^2\hat{k} = \phi_1$

$8xy\hat{i} + 4x^2\hat{j} - 3z^2\hat{k} = \phi_2.$

$(12a\hat{i} - 4a\hat{i} - 12\hat{i}) - 6b\hat{j} - b\hat{k} = \phi_1$

$-16\hat{i} + 16\hat{j} - 27\hat{k} = \phi_2$

$\phi_1 \cdot \phi_2 = 0.$

$-(8a-12)16 - 6b \times 16 + 27b = 0$ —— (1) $= 128a + 69b = 192$

for finding orthogonally → two planes should intersect at $(2,-1,-3)$

$\therefore\ 8a + 3b = 4(a+3)$

$4a + 3b = 12$ —— (2)

solving eqn (1) & (2).

eqn (1) − (2) × 23

$128a + 69b = 192$

$-\ 92a + 69b = 276$

$36a = -84$

$a = -7/3 = \underline{-2.33}$ Ans.

Substitute in (2).

$b = \frac{12 + \frac{28}{3}}{3} = 4 + 3.11 = \underline{7.11}$ Ans