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Advanced Calculus and Complex Analysis

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Evaluation of double integration:Cartesian coordinates	**Solving Tip!**
A double integral is evaluated by repeated single variable integration, integrate with respect to one variable treating the other variable as constant. **Order of integration:** **Case 1:** If the region $R = \{(x, y)\,\vert a \le x \le b,\ c \le y \le d\}$ where a, b, c, d are constants, then $$\int\int_R f(x,y)dxdy = \int_a^b \left[\int_c^d f(x,y)dx\right] dy$$ or $$= \int_a^b \left[\int_c^d f(x,y)dy\right] dx$$	Here the limits of x and y are constants, the order of integration is immaterial.
Case 2: If the region $R = \{(x, y)\,\vert a \le x \le b,\ g(x) \le y \le h(x)\}$ where a, b are constants, then $$\int\int_R f(x,y)dxdy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)dy\right] dx$$	Here the limits for x are constants and the limits for y are functions of x, so we integrate first with respect to y and then integrate with respect to x.
Case 3: If the region $R = \{(x, y)\,\vert g(y) \le x \le h(y),\ c \le y \le d,\}$ where c, d are constants, then $$\int\int_R f(x,y)dxdy = \int_c^d \left[\int_{g(y)}^{h(y)} f(x,y)dx\right] dy$$	Here the limits for y are constants and the limits for x are functions of y, so we integrate first with respect to x and then integrate with respect to y.

Evaluation of double integration:Cartesian coordinates	**Solving Tip!**
Working Procedure:	

Step1:	Write I=the given integral.
Step2:	See the limits of the integral and decide the case from order of integration.
Step3:	Perform the inside integration according to the order.
Step4:	**Upper limit minus lower limit substitution:**Put a minus between the two brackets. Fill up in the first bracket as a substitution of the upper limit where ever the integrated variable occurred and Do the same for the lower limit in the second bracket.
Step5:	Double integral reduced to a single integral. Integrate it!
Step6:	Perform the step4 that, Upper limit minus lower limit substitution
Step7:	Simplification

Evaluation of double integration:Cartesian coordinates	Solving Tip!
1.Evaluate $\boldsymbol{\int_0^1 \int_0^1 dxdy}$	
Solution:	
Let $I = \int_0^1 \int_0^1 dxdy$	
$= \int_0^1 \left[\int_0^1 dx \right] dy$	Since limits are constants, we can take integration first with respect to any variable and then integrate with respect to the rest of the variable.
$= \int_0^1 [x]_0^1 \, dy$	We taken the first integration with respect to x
$= \int_0^1 [(1) - (0)] \, dy$	Instead of x, put the upper limit substitution minus put the lower limit substitution
$= \int_0^1 [1] \, dy$	simplification
$= \int_0^1 dy$	Integrate with respect to y
$= [y]_0^1$	
$= [(1) - (0)]$	upper limit and lower limit substitution
$= 1$	simplification

Evaluation of double integration:Cartesian coordinates	Solving Tip!
2.Evaluate $\int_0^1 \int_0^2 x(x+y)dxdy$	
Solution:	
Let $I = \int_0^1 \int_0^2 x(x+y)dxdy$	
$= \int_0^1 \left[\int_0^2 (x^2+xy)dx \right] dy$	Since limits are constants, we can take integration first with respect to any variable and then integrate with respect to the rest of the variable.
$= \int_0^1 \left[\frac{x^3}{3} + \frac{x^2}{2}y \right]_0^2 dy$	We taken the first integration with respect to x
$= \int_0^1 \left[\left(\frac{2^3}{3} + \frac{2^2}{2}y \right) - \left(\frac{0^3}{3} + \frac{0^2}{2}y \right) \right] dy$	Instead of x, put the upper limit substitution minus put the lower limit substitution
$= \int_0^1 \left[\frac{8}{3} + 2y \right] dy$	simplification
	Integrate with respect to y
$= \left[\frac{8}{3}.y + 2.\frac{y^2}{2} \right]_0^1$	
$= \left[\left(\frac{8}{3}.1 + 1 \right) - \left(\frac{8}{3}.0 + 0^2 \right) \right]$	upper limit and lower limit substitution
$= \frac{11}{3}$	simplification

Evaluation of double integration:Cartesian coordinates	Solving Tip!
3.Evaluate $\int_0^1\int_0^1 \frac{dxdy}{\sqrt{1-x^2}\sqrt{1-y^2}}$	
Solution:	
Let $I = \int_0^1\int_0^1 \frac{dxdy}{\sqrt{1-x^2}\sqrt{1-y^2}}$	
$= \int_0^1 \frac{1}{\sqrt{1-y^2}}\left[\int_0^1 \frac{1}{\sqrt{1-x^2}}dx\right] dy$	Since limits are constants, we can take integration first with respect to any variable and then integrate with respect to the rest of the variable.
$= \int_0^1 \frac{1}{\sqrt{1-y^2}}\left[\sin^{-1} x\right]_0^1 dy$	We taken the first integration with respect to x
$= \int_0^1 \frac{1}{\sqrt{1-y^2}}\left[\left(\sin^{-1} 1\right) - \left(\sin^{-1} 0\right)\right] dy$	Instead of x, put the upper limit substitution minus put the lower limit substitution
$= \int_0^1 \frac{1}{\sqrt{1-y^2}}\left[\frac{\pi}{2} - 0\right] dy$	simplification
$= \frac{\pi}{2}\int_0^1 \frac{1}{\sqrt{1-y^2}}dy$	Take out the constant term
	Integrate with respect to y
$= \frac{\pi}{2}\left[\sin^{-1} y\right]_0^1$	
$= \frac{\pi}{2}\left[\left(\sin^{-1} 1\right) - \left(\sin^{-1} 0\right)\right]$	upper limit and lower limit substitution
$= \frac{\pi}{2}\frac{\pi}{2}$	simplification
$= \frac{\pi^2}{4}$	simplification
Note:	
We have an another method $I = \int_0^1 \frac{dx}{\sqrt{1-x^2}} . \int_0^1 \frac{dy}{\sqrt{1-y^2}}$	If the limits are constants and integrand is variable separable.

Evaluation of double integration:Cartesian coordinates	**Solving Tip!**
4.Evaluate $\int\limits_0^1 \int\limits_0^{x^2} (x^2+y^2)\,dy\,dx$	
Solution:	
Let $I = \int\limits_0^1 \int\limits_0^{x^2} (x^2+y^2)dydx$	
$= \int\limits_0^1 \left[\int\limits_0^{x^2} (x^2+y^2)dy \right] dx$	Since one of the limit is variable(Here in x), we can take integration first with respect to the absent variable(i.e, y) and then integrate with respect to the rest of the variable(i.e., x).
$= \int\limits_0^1 \left[x^2y + \frac{y^3}{3} \right]_0^{x^2} dx$	Integrate, first with respect to y
$= \int\limits_0^1 \left[\left(x^2x^2 + \frac{(x^2)^3}{3} \right) - \left(x^2(0) + \frac{(0)^3}{3} \right) \right] dx$	$\int\limits_a^b x^n dx = \left[\frac{x^{n+1}}{n+1} \right]_a^b$ Instead of x, put the upper limit substitution minus put the lower limit substitution
$= \int\limits_0^1 \left[x^4 + \frac{x^6}{3} \right] dx$	
$= \left[\frac{x^5}{5} + \frac{x^7}{21} \right]_0^1$	Integrate with respect to x
$= \left[\left(\frac{1}{5} + \frac{1}{21} \right) - \left(\frac{0}{5} + \frac{0}{21} \right) \right]$	Instead of x, put the upper limit substitution minus put the lower limit substitution
$= \frac{1}{5} + \frac{1}{21}$	simplification
$= \frac{21+5}{105}$	
$= \frac{26}{105}$	

Evaluation of double integration: Cartesian coordinates	**Solving Tip!**
LEARNING TIME EXCERCISE	
1. Evaluate $\int_0^1 \int_1^2 dx dy$	
Solution:	
Let $I = \int_0^1 \int_1^2 dx dy$	
$= \int_0^1 \left[\quad \right] dy$	Since limits are constants, we can take integration first with respect to any variable and then integrate with respect to the rest of the variable.
$= \int_0^1 \quad dy$	We taken the first integration with respect to x
$= \int_0^1 [(2) - (1)] dy$	Instead of x, put the upper limit substitution minus put the lower limit substitution
$= \int_0^1 \quad dy$	simplification
$= \int_0^1 dy$	Integrate with respect to y
$= [\ ]_0^1$	
$= [(\ ) - (\ )]$	upper limit and lower limit substitution
$= 1$	simplification

Evaluation of double integration:Cartesian coordinates	Solving Tip!
2.Evaluate $\int_0^1 \int_1^2 xydxdy$	
Solution:	
Let $I = \int_0^1 \int_1^2 xydxdy$	
$= \int_0^1 y \left[\quad\quad \right] dy$	Since limits are constants, we can take integration first with respect to any variable and then integrate with respect to the rest of the variable.
$= \int_0^1 y \left[\quad \right]_1^2 dy$	We taken the first integration with respect to x
$= \int_0^1 y \left[\left(\frac{2^2}{2} \right) - \left(\frac{1}{2} \right) \right] dy$	Instead of x, put the upper limit substitution minus put the lower limit substitution
$= \int_0^1 \quad\quad dy$	simplification
$= \frac{3}{2} \int_0^1 ydy$	Take out the constant and integrate with respect to y
$= \frac{3}{2} \left[\quad \right]_0^1$	
$= \frac{3}{2} \left[\left(\frac{}{2} \right) - \left(\frac{}{2} \right) \right]$	upper limit and lower limit substitution
$= \frac{3}{4}$	simplification

Evaluation of double integration:Cartesian coordinates	**Solving Tip!**
3.Evaluate $\int\limits_0^1 \int\limits_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$	
Solution:	
Let $I = \int\limits_0^1 \int\limits_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$	
$= \int\limits_0^1 \left[\int\limits_x^{\sqrt{x}} (x^2+y^2)\,dy\right] dx$	Since one of the limit is variable(Here in x), we can take integration first with respect to the absent variable(i.e, y) and then integrate with respect to the rest of the variable(i.e., x).
$= \int\limits_0^1 \left[x^2y + \frac{y^3}{3}\right]_x^{\sqrt{x}} dx$	Integrate, first with respect to y
$= \int\limits_0^1 \left[\left(x^{5/2} + \frac{x^{3/2}}{3}\right) - \left(x^3 + \frac{x^3}{3}\right)\right] dx$	$\int\limits_a^b x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b$ Instead of x, put the upper limit substitution minus put the lower limit substitution
$= \int\limits_0^1 \left[x^{5/2} + \frac{x^{3/2}}{3} - \frac{4x^3}{3}\right] dx$	
$= \left[\frac{x^{7/2}}{7/2} + \frac{x^{5/2}}{3\times 5/2} - \frac{4}{3}\frac{x^4}{4}\right]_0^1$	Integrate with respect to x
$= \left[\left(\frac{2}{7} + \frac{2}{15}\right) - \left(\frac{1}{3}\right)\right]$	Instead of x, put the upper limit substitution minus put the lower limit substitution
$= \frac{30+14-35}{105}$	simplification
$= \frac{44-35}{105}$	
$= \frac{9}{105} = \frac{3}{35}$	

Evaluation of double integration:Cartesian coordinates	**Solving Tip!**
4.Evaluate $\int\limits_0^1 \int\limits_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$	
Solution:	
Let $I = \int\limits_0^1 \int\limits_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$	
$= \int\limits_0^1 \left[\int\limits_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2} \right] dx$	Since one of the limit is variable(Here in x), we can take integration first with respect to the absent variable(i.e, y) and then integrate with respect to the rest of the variable(i.e., x).
$= \int\limits_0^1 \left[\int\limits_0^{\sqrt{1+x^2}} \frac{dy}{y^2+\left(\sqrt{x^2+1}\right)^2} \right] dx$	$\int \frac{dx}{\sqrt{x^2+a^2}} = \frac{1}{a}\tan^{-1}\frac{x}{a}$
$= \int\limits_0^1 \left[\frac{1}{\sqrt{x^2+1}} \tan^{-1} \frac{y}{\sqrt{x^2+1}} \right]_0^{\sqrt{1+x^2}} dx$	Integrate, first with respect to y
$= \int\limits_0^1 \frac{1}{\sqrt{x^2+1}} [\tan^{-1}(1) - \tan^{-1}(0)] dx$	
$= \frac{\pi}{4} \int\limits_0^1 \frac{1}{\sqrt{x^2+1}} dx$	Instead of x, put the upper limit substitution minus put the lower limit substitution
$= \frac{\pi}{4} \left[\log\left(x+\sqrt{x^2+1}\right)\right]_0^1$	Integrate with respect to x. $\int\limits_a^b \frac{1}{\sqrt{x^2+1}} dx = \frac{\pi}{4}\left[\log\left(x+\sqrt{x^2+1}\right)\right]_a^b$
$= \frac{\pi}{4}\left[\left(\log\left(1+\sqrt{2}\right)\right) - \left(\log(1)\right)\right]$	Instead of x, put the upper limit substitution minus put the lower limit substitution
$= \frac{\pi}{4}\log\left(1+\sqrt{2}\right)$	simplification

Problems based on Evaluation of double integration:Polar coordinates	**Solving Tip!**
If $f(r,\theta)$ is defined over the region R in polar coordinates, then the double integral of $f(r,\theta)$ over R is $$\iint_R f(r,\theta)drd\theta = \int_{\theta_1}^{\theta_2}\int_{f_1(\theta)}^{f_2(\theta)} f(r,\theta)drd\theta$$ **Formulae:** 1. $\int_0^{\pi/2} \cos^n\theta d\theta = \frac{n-1}{n}.\frac{n-3}{n-2}\cdots\frac{2}{3}.1$ if $n$is odd and $n \geq 3$ 2. $\int_0^{\pi/2} \cos^n\theta d\theta = \frac{n-1}{n}.\frac{n-3}{n-2}\cdots\frac{1}{2}.\frac{\pi}{2}$ if $n$is even and $n \geq 3$ 3. $\int_0^{\pi/2} \sin^n\theta d\theta = \frac{n-1}{n}.\frac{n-3}{n-2}\cdots\frac{2}{3}.1$ if $n$is odd and $n \geq 3$ 4. $\int_0^{\pi/2} \sin^n\theta d\theta = \frac{n-1}{n}.\frac{n-3}{n-2}\cdots\frac{1}{2}.\frac{\pi}{2}$ if $n$is even and $n \geq 3$ 5. $\int [f(x)]^n.f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$	

Problems based on Evaluation of double integration:Polar coordinates	**Solving Tip!**
Working Procedure:	

Step1:	Write I=the given integral.
Step2:	See the limits of the integral and decide the case from order of integration.
Step3:	Perform the inside integration according to the order.
Step4:	**Upper limit minus lower limit substitution:**Put a minus between the two brackets. Fill up in the first bracket as a substitution of the upper limit where ever the integrated variable occurred and Do the same for the lower limit in the second bracket.
Step5:	Double integral reduced to a single integral. Integrate it!
Step6:	Perform the step4 that, Upper limit minus lower limit substitution
Step7:	Simplification

Problems based on Evaluation of double integration:Polar coordinates	**Solving Tip!**
1. Evaluate $\int_{-\pi/2}^{\pi/2}\int_{0}^{2\cos\theta} r^2 dr d\theta$	
Let $I = \int_{-\pi/2}^{\pi/2}\int_{0}^{2\cos\theta} r^2 dr d\theta$	
$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta$	$\int_l^u x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_l^u$
$= \int_{-\pi/2}^{\pi/2} \left[\left(\frac{2^3\cos^3\theta}{3}\right) - \left(\frac{0}{3}\right)\right] d\theta$	upper limit minus lower limit substitution
$= \int_{-\pi/2}^{\pi/2} \left[\frac{8\cos^3\theta}{3} - 0\right] d\theta$	
$= \frac{8}{3}\int_{-\pi/2}^{\pi/2} \cos^3\theta d\theta$	Integrate with respect to θ
$= \frac{8}{3}.2\int_{0}^{\pi/2} \cos^3\theta d\theta \because \cos\theta$ is an even funtion.	
$= \frac{16}{3}.\frac{2}{3}.1 = \frac{32}{9}$	For even function, $\int_{-a}^{a} f(x)dx = 2.\int_0^a f(x)dx$

Problems based on Evaluation of double integration:Polar coordinates	**Solving Tip!**
2. Evaluate $\int_0^{\pi/2}\int_0^{\cos\theta} r^2 dr d\theta$	
Let $I = \int_0^{\pi/2}\int_0^{\cos\theta} r^2 dr d\theta$	
$= \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^{\cos\theta} d\theta$	$\int_l^u x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_l^u$
$= \int_0^{\pi/2} \left[\left(\frac{\cos^3\theta}{3}\right) - \left(\frac{0}{3}\right)\right] d\theta$	upper limit minus lower limit substitution
$= \int_0^{\pi/2} \left[\frac{\cos^3\theta}{3} - 0\right] d\theta$	
$= \frac{1}{3}\int_0^{\pi/2} \cos^3\theta d\theta$	$\int_0^{\pi/2} \cos^n\theta d\theta = \frac{n-1}{n}.\frac{n-3}{n-2}\cdots\frac{2}{3}.1$ if $n$is odd.
$= \frac{1}{3}.\frac{2}{3}.1 = \frac{2}{9}$	

Problems based on Evaluation of double integration:Polar coordinates	**Solving Tip!**
3. Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\sin\theta\, dr d\theta$.	
Solution:	
$\therefore\ I = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r\sin\theta\, dr d\theta$	First integrate with respect to r
$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)}\sin\theta\, d\theta$	$\int_l^u x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_l^u$
$= \int_0^{\pi}\left[\frac{a^2(1+\cos\theta)^2}{2}\right]\sin\theta\, d\theta$	
$= \frac{-a^2}{2}\int_0^{\pi}(1+\cos\theta)^2 .-\sin\theta\, d\theta$	$\int_l^u [f(x)]^n f'(x)dx = \left[\frac{[f(x)]^{n+1}}{n+1}\right]_l^u$
$= \frac{-a^2}{2}\left[\frac{(1+\cos\theta)^3}{3}\right]_0^{\pi}$	
$= \frac{-a^2}{2}\left[\left(\frac{(1+\cos\pi)^3}{3}\right)-\left(\frac{(1+\cos 0)^3}{3}\right)\right]$	
$= \frac{-a^2}{2}\left[\left(\frac{(1+(-1))^3}{3}\right)-\left(\frac{(1+1)^3}{3}\right)\right]$	
$= \frac{-a^2}{2}\left[(0)-\frac{2^3}{3}\right]$	
$= \frac{-a^2}{2}\left[-\frac{8}{3}\right]$	
$= \frac{4a^2}{3}$	

Evaluation of double integration:Polar coordinates	**Solving Tip!**
LEARNING TIME EXCERCISE	
1. Evaluate $\int_0^{\pi/2}\int_0^{2\cos\theta} r^2 dr d\theta$	
Let $I = \int_0^{\pi/2}\int_0^{2\cos\theta} r^2 dr d\theta$	
$= \int_0^{\pi/2} \Big[\quad \Big]_0^{2\cos\theta} d\theta$	$\int_l^u x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_l^u$
$= \int_0^{\pi/2} \Big[\Big(\qquad \Big) - \Big(\quad \Big)\Big] d\theta$	upper limit minus lower limit substitution
$= \int_0^{\pi/2} \Big[\qquad \Big] d\theta$	
$= \frac{8}{3}\int_0^{\pi/2} \quad \theta d\theta$	Integrate with respect to θ
$= \frac{8}{3}.$	
$= \frac{16}{9}$	

Evaluation of double integration:Polar coordinates	**Solving Tip!**
2. Evaluate $\int_0^{\pi/2}\int_0^{\pi/2} \sin(\theta+\phi)d\theta d\phi$	
Let $I = \int_0^{\pi/2}\int_0^{\pi/2} \sin(\theta+\phi)d\theta d\phi$	
$= \int_0^{\pi/2} [\qquad\qquad]_0^{\pi/2} d\phi$	$\int_l^u \sin(ax+b)dx = \left[\frac{-\cos(ax+b)}{a}\right]_l^u$
$= \int_0^{\pi/2} \left[\left(\qquad\qquad\right) - (\qquad\qquad)\right] d\theta$	upper limit minus lower limit substitution $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$
$= \int_0^{\pi/2} [\qquad\qquad] d\phi$	
$= [\qquad\qquad]_0^{\pi/2}$	Integrate with respect to θ
$= \left[\left(\qquad\qquad\right) - (\qquad\qquad)\right]$	
$= 0 + 1 + 1 - 0$ $= 2$	

Evaluation of double integration:Polar coordinates	**Solving Tip!**
3. Evaluate $\displaystyle\int_{0}^{\pi/2}\int_{a\cos\theta}^{2a\cos\theta} r^2\,dr\,d\theta$	
Let $I = \displaystyle\int_{0}^{\pi/2}\int_{a\cos\theta}^{2a\cos\theta} r^2\,dr\,d\theta$	
$= \displaystyle\int_{0}^{\pi/2}\Big[\qquad\Big]_{a\cos\theta}^{2a\cos\theta} d\theta$	$\displaystyle\int_{l}^{u} x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_{l}^{u}$
$= \displaystyle\frac{1}{3}\int_{0}^{\pi/2}[(\qquad\qquad) - (\qquad\qquad)]\,d\theta$	upper limit minus lower limit substitution
$= \displaystyle\frac{1}{3}\int_{0}^{\pi/2}[\qquad\qquad]\,d\theta$	
$= \displaystyle\frac{7a^3}{3}\int_{0}^{\pi/2}[\qquad]\,d\theta$	Integrate with respect to θ
$= \displaystyle\frac{7a^3}{3}.$	$\displaystyle\int_{0}^{\pi/2}\sin^n\theta\,d\theta = \frac{n-1}{n}.\frac{n-3}{n-2}\cdots\frac{2}{3}.1$ if n is odd and $n \geq 3$
$= \displaystyle\frac{14a^3}{9}$	

Evaluation of double integration: Change of order	Solving Tip!

Change of order of integration

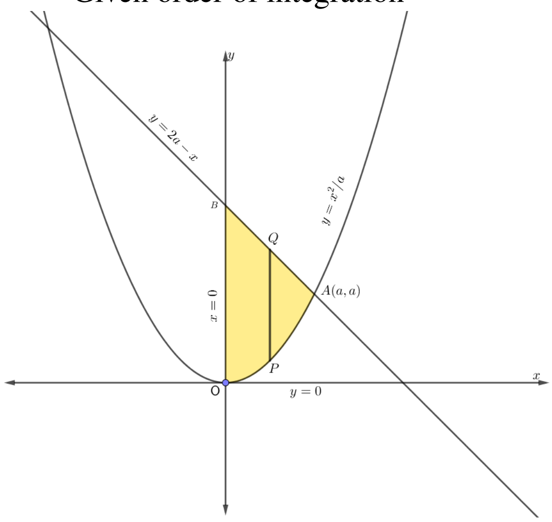
We solve the double integration according to the limits i.e., by order of integration. This may sometimes be difficult to evaluate. But change in the order of integration may makes the evaluation be easy.

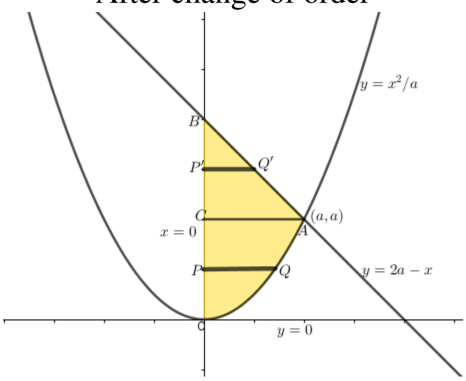
For doing this we have to identify the region R of integration from the limits of the given double integral. Sometimes this region may split into two regions when we change the order of integration and hence the given double integral will be the sum of two double integrals.

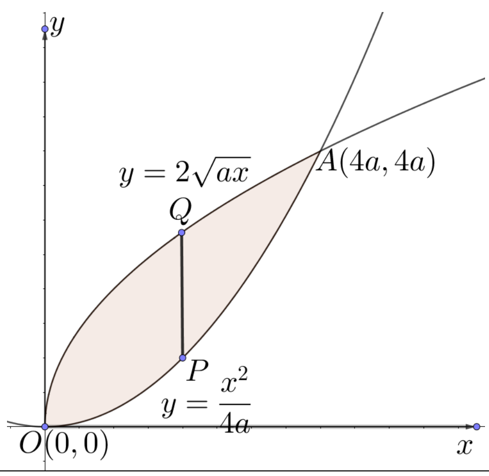
Working Procedure:

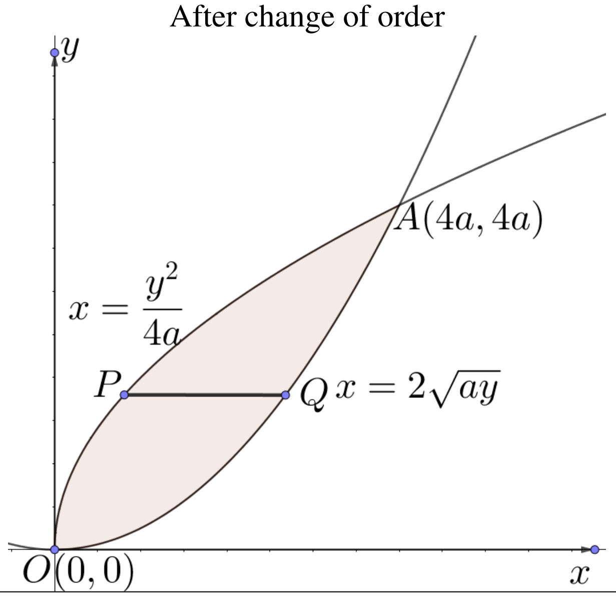
Step1:	Write I=the given integral and Identify the boundaries of the region.
Step2:	Draw the geometrical representation of I and label it as 'Before the change of order'. In the region, draw a strip PQ which is parallel to the corresponding first integration's with respect to variable's axis and the end values of the PQ's are the curve's equation representation (in terms of that variable) of where the ends lie on the curve. i.e., the limits of inside integration is end values of PQ and outside integral's limits are the values in which PQ covers the region from where to where.
Step3:	Find the point of intersections.
Step4:	Draw the same diagram without strip and label it as 'After change the order'

Evaluation of double integration: Change of order	Solving Tip!
Step5: In this region, change the position of the strip as vertical(horizontal) when it is horizontal(vertical) before.	
Step6: Write I= that mathematical representation for this diagram. i.e., inside integral limits are the end values of the strip and outside integral limits are the values from where to where we move the strip to cover the region.	
Step7: Evaluate this integral by follow the steps which mentioned in either cartesian or polar coordinates	

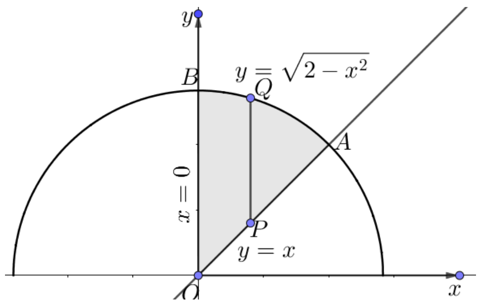
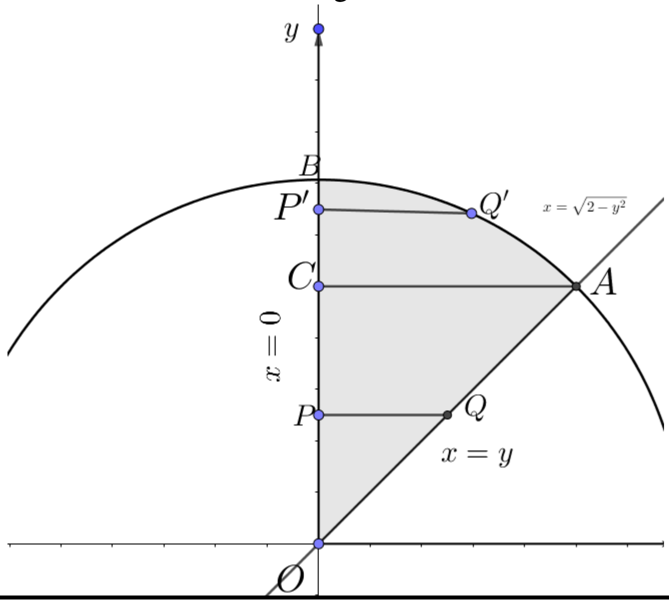
Evaluation of double integration: Change of order	**Solving Tip!**
1. Change the order of integration and evaluate $\boldsymbol{\int_0^a \int_{x^2/a}^{2a-x} xy\,dx\,dy.}$	
Solution:	
Let $I = \int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx$	
The region of integration is bounded by the lines $x = 0, x = a, ay = x^2, y = 2a - x.$	Identify the boundaries of the region
In the given integral, first integrate with respect to y and then with respect to x. After changing the order we have to first integrate with respect to x, then with respect to y.	Draw the graph for “order of integration”
Given order of integration 	In the given integral, according to the limits, first integration with respect to y. $\therefore$ we draw the strip PQ parallel to y axis
To find A: Solve $y = x^2/a - - - (1)$ and $y = 2a - x - - - (2)$	Find the point of intersections
$\therefore \dfrac{x^2}{a} = 2a - x$	Use (1) in (2)
$\Rightarrow x^2 = 2a^2 - ax$	
$\Rightarrow x^2 + ax - 2a^2 = 0$	It is quadratic. We use $x = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$
$\Rightarrow x = \dfrac{-a \pm \sqrt{a^2 - 4(1)(-2a^2)}}{2(1)}$	Here $A = 1; B = a; C = -2a^2$
$= \dfrac{-a \pm \sqrt{a^2 + 8a^2}}{2}$	Simplification

Evaluation of double integration: Change of order	Solving Tip!
$= \frac{-a \pm \sqrt{9a^2}}{2} = \frac{-a \pm 3a}{2} = \frac{-4a}{2}, \frac{2a}{2} = a$	Here the area lies on I quadrant.
Substitute x value in (2), we get	
$(2) \Rightarrow\ y = 2a - a = a$	
$\therefore\ A(a, a)$	
To find B:Solve $x = 0 - - - (3)$ and $y = 2a - x - - - (4)$	Find the point of intersection
$\therefore\ (4) \Rightarrow\ y = 2a - 0 = a$	Use (3) in (4)
$\therefore\ B(0, 2a)$	
After change of order 	After changing the order, first integrate with respect to x. So draw the strip parallel to x-axis. Here Q traverses on the two curves so that we divide the integral as two parts.
$\therefore\ I = \iint\limits_{OAB} xydxdy$	
$= \iint\limits_{OAC} xydxdy + \iint\limits_{CAB} xydxdy$	Since one end of the strip traverses on two curves, the region OAB splits in to two regions OAC and CAB.
$= \int_0^a \int_0^{\sqrt{ay}} xydxdy + \int_a^{2a} \int_0^{2a-y} xydxdy$	For OAC, the strip PQ and for the CAB, the strip PQ'.
$= \int_0^a y \left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy + \int_a^{2a} y \left[\frac{x^2}{2}\right]_0^{2a-y} dy$	
$= \int_0^a y \left[\frac{ay}{2}\right] dy + \int_a^{2a} y \left[\frac{(2a-y)^2}{2}\right] dy$	
$= \frac{a}{2} \int_0^a y^2 dy + \frac{1}{2} \int_a^{2a} y \left[4a^2 - 4ay + y^2\right] dy$	
$= \frac{a}{2} \left[\frac{y^3}{3}\right]_0^a + \frac{1}{2} \int_a^{2a} \left[4a^2y - 4ay^2 + y^3\right] dy$	

Evaluation of double integration: Change of order	Solving Tip!
$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a + \frac{1}{2}\left[4a^2\frac{y^2}{2} - 4a\frac{y^3}{3} + \frac{y^4}{4}\right]_a^{2a}$ $= \frac{a}{2}\left[\frac{a^3}{3}\right] + \frac{1}{2}\left[\left(8a^4 - \frac{32a^4}{3} + \frac{16a^4}{4}\right) - \left(2a^4 - \frac{4a^3}{3} + \frac{a^4}{4}\right)\right]$ $= \frac{a^4}{6} + \frac{1}{2}\left[\frac{5a^4}{12}\right]$ $= \frac{a^4}{6} + \frac{5a^4}{24} = \frac{3a^4}{8}$	
2. Change the order of integration in I=$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dydx$ and hence evaluate. **Solution:** Let $I = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dydx$	
The region of integration is bounded by the curves $x^2 = 4ay$ and $y^2 = 4ax$.	Identify the boundaries of the region
In the given integral, first integrate with respect to y and then with respect to x. After changing the order we have to first integrate with respect to x, then with respect to y.	Draw the graph for “order of integration”
Given order of integration 	In the given integral, according to the limits, first integration with respect to y. ∴ we draw the strip PQ parallel to y axis

Evaluation of double integration: Change of order	**Solving Tip!**
To find the intersection points: Solve $y = x^2/4a - - - (1)$ and $y = 2\sqrt{ax} - - - (2)$	Find the point of intersections
$\therefore\ 2\sqrt{ax} = \dfrac{x^2}{4a}$	Use (2) in (1)
$\Rightarrow\ 4ax = \dfrac{x^4}{16a^2}$	Squaring on both sides
$\Rightarrow\ 64a^3x = x^4$	Simplification
$\Rightarrow\ x^4 - 64a^3x = 0$	Simplification
$\Rightarrow\ x(x^3 - 64a^3) = 0$	Simplification
$\Rightarrow$ either $x = 0$ or $(x^3 - 64a^3) = 0$	Simplification
For $x = 0, \therefore (1) \Rightarrow\ y = (0)^2/4a = 0$	Substitution
$\therefore O(0,0)$	One of the point of intersection
For $(x^3 - 64a^3) = 0, \Rightarrow\ x^3 = 64a^3$	Simplification
$\Rightarrow\ x^3 = (4a)^3$	Simplification
$\Rightarrow\ x = 4a$	Simplification
$\therefore (1) \Rightarrow\ y = (4a)^2/4a = 4a$	Substitution
$\therefore A(4a, 4a)$	Simplification
After change of order 	After changing the order, first integrate with respect to x. So draw the strip parallel to x-axis. Here the ends P and Q are representing the lower and upper limits of the inside integral and the outside integral limits are the values which the strip PQ moves from where to where for cover the region.

Evaluation of double integration: Change of order	**Solving Tip!**
$\therefore\ I = \int_{0}^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dxdy$	Integral representation for change of order
$= \int_{0}^{4a}\left[\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dx\right] dy$	
$= \int_{0}^{4a} [x]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$	
$= \int_{0}^{4a}\left[(2\sqrt{ay}) - \left(\frac{y^2}{4a}\right)\right] dy$	
$= \left[2\sqrt{a}\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a}$	
$= \left[\left(2\sqrt{a}\frac{(4a)^{3/2}}{3/2} - \frac{(4a)^3}{12a}\right) - \left(2\sqrt{a}\frac{(0)}{3/2} - \frac{(0)^3}{12a}\right)\right]$	$\sqrt{a}(a^{3/2}) = a^{1/2}.a^{3/2} = a^{\frac{1}{2}+\frac{3}{2}}$
$= \left[\frac{4}{3}.(4)^{3/2}a^2 - \frac{16a^2}{3}\right]$	$= a^{\frac{4}{2}} = a^2$
$= \left[\frac{4}{3}.(2)^3a^2 - \frac{16a^2}{3}\right]$	$(4)^{3/2} = (\sqrt{4})^3 = 2^3 = 8$
$= \left[\frac{32a^2}{3} - \frac{16a^2}{3}\right]$	
$= \frac{16a^2}{3}$	
3. Evaluate $\boldsymbol{\int_{0}^{1}\int_{x}^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dydx}$ **by changing the order of integration.**	
Solution:	
Let $I = \int_{0}^{1}\int_{x}^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dydx$	
The region of integration is bounded by $x = 0,\ x = 1$ and $y = x, y = \sqrt{2-x^2}$	

Evaluation of double integration: Change of order	**Solving Tip!**
$y=\sqrt{2-x^2} \Rightarrow \; y^2 = 2-x^2 \Rightarrow \; x^2+y^2=2$	
Which is the circle with center $(0,0)$ and radius $\sqrt{2}$	
In the given integral, first integrate with respect to y and then with respect to x. After changing the order we have to first integrate with respect to x, then with respect to y.	Draw the graph for "order of integration"
Given order of integration 	In the given integral, according to the limits, first integration with respect to y. $\therefore$ we draw the strip PQ parallel to y axis
The region of integration is OAB.	
To find A, solve $y=x---(1)$ and $x^2+y^2=2---(2)$	
$(2) \Rightarrow \; x^2+x^2=2$	Using (1) in (2)
$2x^2=2 \Rightarrow \; x=\pm 1$	Since the point A is in I quadrant.
Use x value in (1),$y=1$	
$\therefore A(1,1)$	
And from the diagram, B is $(0,\sqrt{2})$	
After change of order 	After changing the order, first integrate with respect to x. So draw the strip parallel to x-axis. Here Q traverses on the two curves so that we divide the integral as two parts.

Evaluation of double integration: Change of order	**Solving Tip!**
$\therefore\ I = \iint\limits_{OAC} \frac{x}{\sqrt{x^2+y^2}}dxdy + \iint\limits_{CAB} \frac{x}{\sqrt{x^2+y^2}}dxdy$	Integral representation for change of order.
$= \int_0^1\int_0^y \frac{x}{\sqrt{x^2+y^2}}dxdy + \int_1^{\sqrt{2}}\int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}dxdy$	
$= \frac{1}{2}\int_0^1\int_0^y (x^2+y^2)^{-1/2}.2xdxdy + \frac{1}{2}\int_1^{\sqrt{2}}\int_0^{\sqrt{2-y^2}} (x^2+y^2)^{-1/2}.2xdxdy$	
$= \frac{1}{2}\int_0^1 \left[\frac{(x^2+y^2)^{1/2}}{1/2}\right]_0^y dy + \frac{1}{2}\int_1^{\sqrt{2}} \left[\frac{(x^2+y^2)^{1/2}}{1/2}\right]_0^{\sqrt{2-y^2}} dy$	
$= \frac{1}{2}\int_0^1 \left[\left(\frac{(y^2+y^2)^{1/2}}{1/2}\right) - \left(\frac{(0+y^2)^{1/2}}{1/2}\right)\right] dy$	
$+\frac{1}{2}\int_1^{\sqrt{2}} \left[\left(\frac{\left(\left(\sqrt{2-y^2}\right)^2+y^2\right)^{1/2}}{1/2}\right) - \left(\frac{(0+y^2)^{1/2}}{1/2}\right)\right] dy$	
$= \frac{1}{2}\int_0^1 \left[\frac{(2y^2)^{1/2}}{1/2} - \frac{y}{1/2}\right] dy$	
$+\frac{1}{2}\int_1^{\sqrt{2}} \left[\frac{(2-y^2+y^2)^{1/2}}{1/2} - \frac{y}{1/2}\right] dy$	
$= \int_0^1 \left[(2y^2)^{1/2} - y\right] dy + \int_1^{\sqrt{2}} \left[2^{1/2} - y\right] dy$	
$= \int_0^1 \left[(\sqrt{2}-1)y\right] dy + \int_1^{\sqrt{2}} \left[\sqrt{2} - y\right] dy$	
$= \left[(\sqrt{2}-1)\frac{y^2}{2}\right]_0^1 + \left[\sqrt{2}y - \frac{y^2}{2}\right]_1^{\sqrt{2}}$	
$= \left[\left((\sqrt{2}-1)\frac{1}{2} - 0\right)\right] + \left[\left(\sqrt{2}\sqrt{2} - \frac{2}{2}\right) - \left(\sqrt{2}.1 - \frac{1}{2}\right)\right]$	
$= \frac{\sqrt{2}-1}{2} + 2 - 1 - \left(\frac{2\sqrt{2}-1}{2}\right)$	
$= \frac{\sqrt{2}-1+2-2\sqrt{2}+1}{2} = \frac{2-\sqrt{2}}{2}$	

Evaluation of double integration: Change of order	**Solving Tip!**
LEARNING TIME EXCERCISE	
1. Change the order of integration and evaluate $\boldsymbol{\int_0^1 \int_{x^2}^{2-x} xydxdy.}$	
Solution:	
Let $I = \int_0^1 \int_{x^2}^{2-x} xydydx$	
The region of integration is bounded by the lines $x = 0, x = a, ay = x^2, y = 2a - x.$	Identify the boundaries of the region
In the given integral, first integrate with respect to y and then with respect to x. After changing the order we have to first integrate with respect to x, then with respect to y.	Draw the graph for "order of integration"
Given order of integration	In the given integral, according to the limits, first integration with respect to y. $\therefore$ we draw the strip PQ parallel to y axis
To find A: Solve $y = x^2 - - - (1)$ and $y = 2 - x - - - (2)$	Find the point of intersections
$\therefore\ x^2 = 2 - x$	Use (1) in (2)
$\Rightarrow\ x^2 = 2 - x$	
$\Rightarrow\ x^2 + x - 2 = 0$	It is quadratic. We use $x = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$
$\Rightarrow\ x = \dfrac{-1 \pm \sqrt{1 - 4(1)(-2)}}{2(1)}$	Here $A = 1; B = a; C = -2a^2$

Evaluation of double integration: Change of order	**Solving Tip!**
$= \frac{-1 \pm 3}{2} = \frac{-4}{2}, \frac{2}{2} = 1$	Here the area lies on I quadrant.
Substitute x value in (2), we get	
$(2) \Rightarrow \; y = 2 - 1 = 1$	
$\therefore \; A(1,1)$	
To find B:Solve $x = 0 - - - (3)$ and $y = 2 - x - - - (4)$	Find the point of intersection
$\therefore \; (4) \Rightarrow \; y = 2 - 0 = 2$	Use (3) in (4)
$\therefore \; B(0,2)$	
After change of order	After changing the order, first integrate with respect to x. So draw the strip parallel to x-axis. Here Q traverses on the two curves so that we divide the integral as two parts.
$\therefore \; I = \iint\limits_{OAB} xydxdy$	
$= \iint\limits_{OAC} xydxdy + \iint\limits_{CAB} xydxdy$	Since one end of the strip traverses on two curves, the region OAB splits in to two regions OAC and CAB.
$= \int\limits_0^1 \int\limits_0^{\sqrt{y}} xydxdy + \int\limits_1^2 \int\limits_0^{2-y} xydxdy$	For OAC, the strip PQ and for the CAB, the strip PQ'.
$= \int\limits_0^1 y \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int\limits_1^2 y \left[\frac{x^2}{2}\right]_0^{2-y} dy$	
$= \int\limits_0^1 y \left[\frac{y}{2}\right] dy + \int\limits_1^2 y \left[\frac{(2-y)^2}{2}\right] dy$	
$= \frac{1}{2} \int\limits_0^1 y^2 dy + \frac{1}{2} \int\limits_1^2 y \left[4 - 4y + y^2\right] dy$	
$= \frac{1}{2} \left[\quad \right]_0^1 + \frac{1}{2} \int\limits_1^2 [\quad] \, dy$	

Evaluation of double integration: Change of order	Solving Tip!
$= \frac{1}{2}\Big[\quad\Big]_0^1 + \frac{1}{2}\Big[\qquad\qquad\qquad\Big]_1^2$ $= \frac{1}{2}\Big[\quad\Big] + \frac{1}{2}\Big[\Big(\qquad\qquad\Big) - \Big(\qquad\qquad\Big)\Big]$ $= \frac{3}{8}$ **2. Change the order of integration** $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2 dx dy}{\sqrt{y^2 - a^2x^2}}$	Draw the geometrical representation for the given integral Draw the geometrical representation for the change of order Write the modified integral.

Tutorial-1

TUTORIAL PROBLEMS	Solving Tip!
1.Evaluate $\int_0^3\int_0^2 xy(x+y)dxdy$	
Solution:	
Let $I = \int_0^3\int_0^2 xy(x+y)dxdy$	Since limits are constants, we can take integration first with respect to inside variable and then integrate with respect to the rest of the variable.
$= \int_0^3 \left[\int_0^2 x^2y + xy^2dx\right] dy$	We taken the first integration with respect to x
$= \int_0^3 \left[\quad \right]_0^2 dy$	$\int_a^b x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b$
$= \int_0^3 \left[\left(\quad \right) - \left(\quad \right)\right] dy$	Instead of x, put the upper limit substitution minus put the lower limit substitution
$= \int_0^3 \quad dy$	simplification
$= \left[\quad \right]_0^3$	Integrate with respect to y
$= \left[\left(\quad \right) - \left(\quad \right)\right]$	Instead of x, put the upper limit substitution minus put the lower limit substitution
$=$	simplification
$= 30$	

TUTORIAL PROBLEMS	**Solving Tip!**
2.Evaluate $\int_{0}^{1}\int_{1}^{2}(x^2y+y^2+6)dxdy$	

Solution:

Let $I = \int_{0}^{1}\int_{1}^{2}(x^2y+y^2+6)dxdy$

$$= \int_{0}^{1}\left[\int_{1}^{2}(x^2y+y^2+6)dx\right]dy$$

$$= \int_{0}^{1}\left[\frac{x^3}{3}y+xy^2+6x\right]_1^2 dy$$

$$= \int_{0}^{1}\left[\left(\frac{2^3}{3}y+2y^2+6(2)\right)-\left(\frac{1^3}{3}y+1.y^2+6(1)\right)\right]dy$$

$$= \int_{0}^{1}\left[\frac{7}{3}y+y^2+6\right]dy$$

$$= \left[\qquad\qquad\right]_0^1$$

$$= \left[\left(\qquad\right)-\left(\qquad\right)\right]$$

$$=$$

$$= \frac{15}{2}$$

Tutorial Problems	**Rough work!**
3.Evaluate $\int_1^2 \int_3^4 (xy + e^y)\,dx\,dy$ **Solution:**	Answer $\dfrac{21}{4} + e^2 - e$

TUTORIAL PROBLEMS	**Solving Tip!**
4.Evaluate $\int_0^1 \int_0^{x^2} (x^2+y^2)\,dy\,dx$	
Solution:	
Let $I =$	
$=$	Since one of the limit is variable, we can take integration first with respect to the absent variable and then integrate with respect to the rest of the variable.
$= \int_0^1 [\quad]_0^{x^2} dx$	Integrate, first with respect to y
$= \int_0^1 \left[\quad\right]_0^2 dy$	$\int_a^b x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b$
$= \int_0^1 \left[\left(\quad\right) - \left(\quad\right)\right] dy$	Instead of x, put the upper limit substitution minus put the lower limit substitution
$= \int_0^1 \quad dx$	simplification
$= \left[\quad\right]_0^1$	Integrate with respect to y
$= \left[\left(\quad\right) - \left(\quad\right)\right]$	Instead of x, put the upper limit substitution minus put the lower limit substitution
$=$	simplification
$= \frac{26}{105}$	

Tutorial Problems	**Rough work!**
5.Evaluate $\int_0^1 \int_0^y e^{x/y} dx dy$ **Solution:**	Answer $\frac{1}{2}(e-1)$

Tutorial Problems	**Solving Tip!**
6.Evaluate $\int_0^{\pi/2}\int_0^{\sin\theta} r\,dr\,d\theta$	
Solution:	
Let $I =$	Since limits are in polar, we can take integration first with respect to r and then integrate with respect to θ.
$= \int_0^{\pi/2} [\qquad]\, d\theta$	Integrate, first with respect to r
$= \int_0^{\pi/2} \left[\qquad\right]_0^2 d\theta$	$\int_a^b x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b$
$= \int_0^{\pi/2} \left[\left(\qquad\right) - \left(\qquad\right)\right] d\theta$	Instead of x, put the upper limit substitution minus put the lower limit substitution
$= \int_0^{\pi/2} \qquad d\theta$	Integrate with respect to θ
$=$	$\int \sin^n \theta = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}\cdots\frac{2}{3}.1 \text{ when n is odd} \\ \frac{n-1}{n}.\frac{n-3}{n-2}\cdots\frac{1}{2}.\frac{\pi}{2} \text{ when n is even} \end{cases}$
$=$	Simplification
$=$	
$= \frac{\pi}{8}$	

Tutorial Problems	**Solving Tip!**
7.Evaluate $\int_{0}^{\pi}\int_{0}^{a(1+\cos\theta)} r\,dr\,d\theta$ **Solution:**	Answer $\frac{3\pi a^2}{4}$

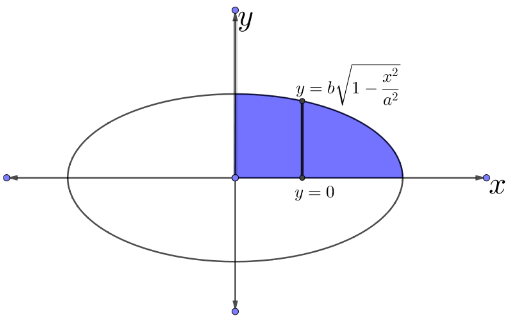
Tutorial Problems	**Solving Tip!**
8. Change the order of integration for $\int_0^a \int_y^a dxdy$ **Solution:** The region of integration is bounded by	Answer $\int_0^a \int_0^x dydx$

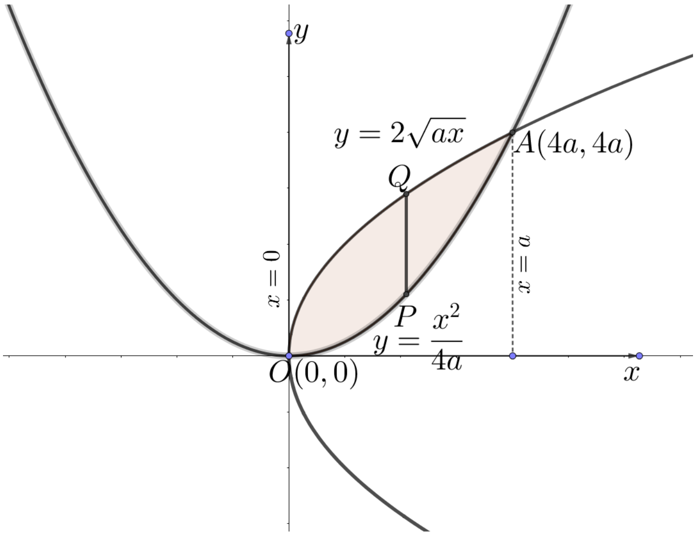
Tutorial Problems	**Solving Tip!**
9. Change the order of integration and hence evaluate $\int\limits_{0}^{1}\int\limits_{x}^{\sqrt{x}} xydydx$ **Solution:**	Answer $1/24$

Tutorial Problems	**Solving Tip!**
10. Change the order of integration and hence evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)dxdy$**.** **Solution:**	Answer $241/60$

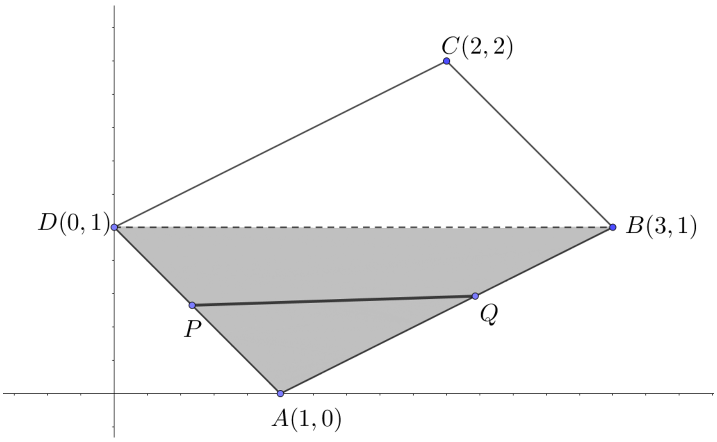
Problems based on Area as Double integral	**Solving Tip!**
## Area as Double integral Double integrals are used to obtain the area of bounded plane regions. The area A of a bounded region R in Cartesian coordinates is $A = \iint\limits_R dxdy$ **Working Procedure:**	

Step1:	Draw a diagram for the required area.
Step2:	Decide the strip for the order of integration.
Step3:	Write I=the double integral representation.
Step4:	Evaluate this integral by follow the steps which mentioned in either Cartesian or polar coordinates

Problems based on Area as Double integral	**Solving Tip!**
1. Find the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, using double integration. **Solution:**	
By the symmetry of the curve, the area of the ellipse is 	**Step1 and Step2**
A=$4\times$ area of the first quadrant $=4\int\limits_0^a\int\limits_0^{b\sqrt{1-x^2/a^2}}dydx$ $=4\int\limits_0^a[y]_0^{b\sqrt{1-x^2/a^2}}\,dx$ $=4\int\limits_0^a\left[b\sqrt{1-\frac{x^2}{a^2}}\right]dx$ $=4\int\limits_0^a\left[b\sqrt{\frac{a^2-x^2}{a^2}}\right]dx$ $=\frac{4b}{a}\int\limits_0^a\left[\sqrt{a^2-x^2}\right]dx$	**Step3 and Step4** $\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$
$=\frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$	Upper limit minus lower limit substitution
$=\frac{4b}{a}\left[0+\frac{a^2}{2}\sin^{-1}(1)\right]=\frac{4b}{a}\left[\frac{a^2}{2}.\frac{\pi}{2}\right]=\pi ab.$	Simplification

Problems based on Area as Double integral(Cartesian)	**Solving Tip!**
2. Find the area enclosed by the curves $\boldsymbol{y^2 = 4ax}$ and $\boldsymbol{x^2 = 4ay}$.	
The region is bounded by the curves $y^2 = 4ax - - - (1)$ $x^2 = 4ay - - - (2)$	Identify the region
	Draw a diagram according to the given limits
To find the intersection points	Solve the corresponding curves' equation
Solve (1) and (2) $\therefore\ \frac{x^4}{16a^2} = 4ax$ $\Rightarrow\ x^4 = 64a^3x$ $\Rightarrow\ x^4 - 64a^3x = 0$ $\Rightarrow\ x(x^3 - 64a^3) = 0$ $\Rightarrow\ x = 0$ or $(x^3 - 64a^3) = 0$ $\Rightarrow\ x = 0$ or $x^3 = 64a^3$ $\Rightarrow\ x = 0$ or $x^3 = (4a)^3$ $\Rightarrow\ x = 0$ or $x = 4a$ $\therefore$ when $x = 0$, $(1) \Rightarrow y = \frac{x^2}{4a} = \frac{0}{4a} = 0$	

Problems based on Area as Double integral(Cartesian)	**Solving Tip!**
when $x = 4a$, $(1) \Rightarrow y = \dfrac{x^2}{4a} = \dfrac{(4a)^2}{4a} = 4a$	
$\therefore$ Area $= \int\limits_0^{4a} \int\limits_{x^2/4a}^{2\sqrt{ax}} dydx$	Write the mathematical representation for the above diagram
$= \int\limits_0^{4a} [y]_{x^2/4a}^{2\sqrt{ax}} dx$	
$= \int\limits_0^{4a} [(2\sqrt{ax}) - (x^2/4a)]\, dx$	
$= \int\limits_0^{4a} \left[2\sqrt{a}x^{1/2} - \dfrac{1}{4a}x^2\right] dx$	
$= \left[2\sqrt{a}\dfrac{x^{3/2}}{3/2} - \dfrac{1}{4a}\dfrac{x^3}{3}\right]_0^{4a}$	
$= \left[\left(2\sqrt{a}\dfrac{(4a)^{3/2}}{3/2} - \dfrac{1}{4a}\dfrac{(4a)^3}{3}\right) - \left(2\sqrt{a}\dfrac{0}{3/2} - \dfrac{1}{4a}\dfrac{0}{3}\right)\right]$	$(4a)^{3/2} = (4a)^{1+\frac{1}{2}} = 4a\,(4a)^{1/2}$
$= \left[\left(2\sqrt{a}\dfrac{(4a)(4a)^{1/2}}{3/2} - \dfrac{1}{4a}\dfrac{4a.(4a)^2}{3}\right)\right]$	
$= \left[\dfrac{32a^2}{3} - \dfrac{16a^2}{3}\right] = \dfrac{16a^2}{3}$	

Problems based on Area as Double integral(Cartesian)	**Solving Tip!**
LEARNING TIME EXCERCISE	
1. Find the area of the parallelogram whose vertices are A(0,1), B(3,1), C(2,2), D(0,1)	
Solution:	
	
From the diagram,	
Required Area $= 2 \times$ (**area of triangle ABD**)	
First we find the equations of AB and AD	Use the line joining between the points formule
Equation of AB joining A(1,0) and B(3,1) is	
	$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$
$y = \frac{1}{2}(x-1) - - - (1)$	
Equation of AD joining A(1,0) and D(0,1) is	
$y = -(x+1) - - - (2)$	

Problems based on Area as Double integral(Cartesian)	**Solving Tip!**
Take a strip PQ parallel to x-axis with P on (2) and Q on (1) $\therefore\ (1) \Rightarrow\ x = -y+1$ and $(1) \Rightarrow\ x = 2y+1$ $\therefore$ **Required Area** $= 2 \times$ (**area of triangle ABD**) $= 2 \iint\limits_{ABD} dxdy$ $= 2 \int \int dxdy$ $= 3$ sq.units	See diagram Write the corresponding limits

Problems based on Area as Double integral in Polar	**Solving Tip!**

Area as Double integral in Polar

Double integrals are used to obtain the area of bounded plane regions. The area A of a bounded region R in polar coordinates is $A = \iint\limits_R r dr d\theta$

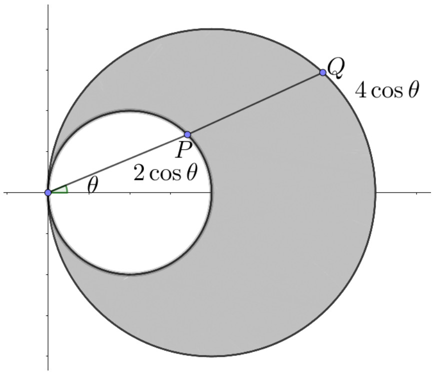
Working Procedure:

Step1:	Draw a diagram for the required area.
Step2:	Draw the radial strip according to the end points give the limits for inside integral and rotate the strip from the x axis make the angles. They are the outside integral limits.
Step3:	Write I=the double integral representation.
Step4:	Evaluate this integral by follow the steps which mentioned in the polar coordinates

Problems based on Area as Double integral(Polar)	**Solving Tip!**

1. Find the area between $r = 2\cos\theta$ and $r = 4\cos\theta$

Solution:



From the diagram, r varies from $2\cos\theta$ to $4\cos\theta$ and

θ varies from $\dfrac{-\pi}{2}$ to $\dfrac{\pi}{2}$

$$\therefore \textbf{Area} = \int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{2}\right]_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[\left(\frac{4^2\cos^2\theta}{2}\right) - \left(\frac{2^2\cos^2\theta}{2}\right)\right] d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[8\cos^2\theta - 2\cos^2\theta\right] d\theta = 6\int_{-\pi/2}^{\pi/2}\cos^2\theta\, d\theta$$

$$= 6.2\int_{0}^{\pi/2}\cos^2\theta\, d\theta \qquad \because \cos^2\theta \text{ is an even function.}$$

$$= 12\frac{1}{2}\frac{\pi}{2}$$

$$= 3\pi$$

Solving Tip!

Put $\dfrac{x}{r} = \cos\theta$ in $r = 2\cos\theta$

$\Rightarrow\ r = 2\dfrac{x}{r}$

$\Rightarrow\ r^2 = 2x \quad \because x^2 + y^2 = r^2$

$\Rightarrow\ x^2 + y^2 = 2x \Rightarrow$

$x^2 + y^2 - 2x = 0$ which is a circle

with center (1,0) and radius=1

Similarly for $r = 4\cos\theta$

then we have a circle

$x^2 + y^2 - 4x = 0$ with center (2,0)

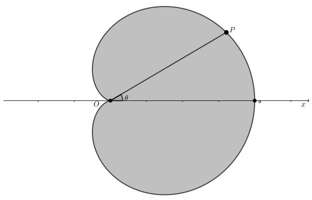
and radius=2.

Problems based on Area as Double integral(Polar)	**Solving Tip!**

2. Find the area of the cardioid $r = a(1+\cos\theta)$, using double integral. Solution:

It is symmetric about the initial line Ox.

$\therefore$ Area$= 2 \times$ (area above the initial line)



Take a radial strip OP.

So θ varies from 0 to π

r varies from 0 to $a(1+\cos\theta)$

$$\therefore \text{ Area} = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= 2\int_0^{\pi}\left[\left(\frac{(a(1+\cos\theta))^2}{2}\right) - (0)\right] d\theta$$

$$= a^2\int_0^{\pi}\left[1+\cos^2\theta + 2\cos\theta\right] d\theta$$

Solving Tip: $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$= a^2\int_0^{\pi}\left[1+\frac{1+\cos 2\theta}{2} + 2\cos\theta\right] d\theta$$

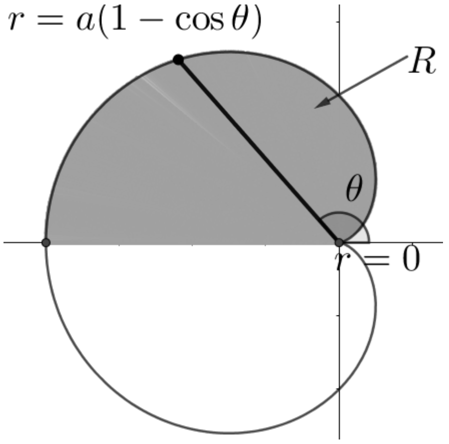
$$= a^2\left[\theta + \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + 2\sin\theta\right]_0^{\pi} d\theta$$

$$= a^2\left[\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} + 2\sin\theta\right]_0^{\pi} d\theta$$

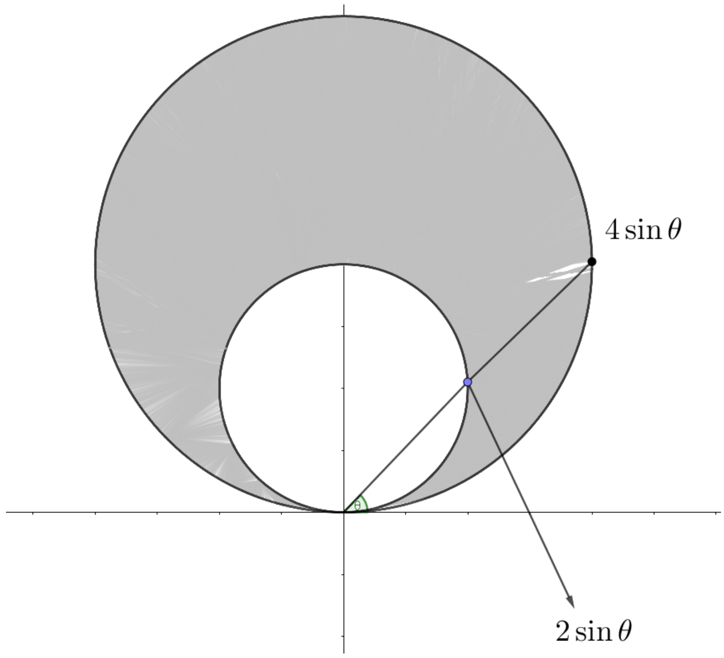
$$= a^2\left[\frac{3\theta}{2} + \frac{\sin 2\theta}{4} + 2\sin\theta\right]_0^{\pi} d\theta$$

$$= a^2\left[\left(\frac{3\pi}{2} + 0 + 0\right) - (0)\right]$$

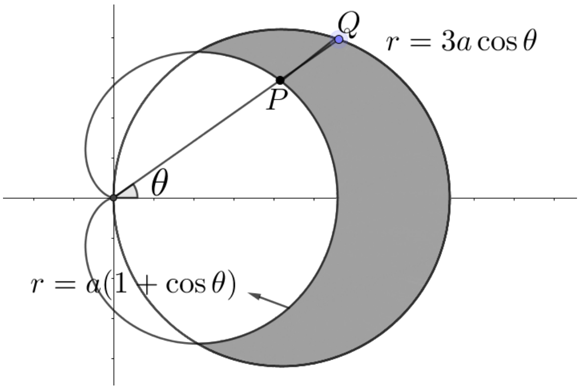
$$= \frac{3\pi a^2}{2}$$

Problems based on Area as Double integral(Polar)	**Solving Tip!**
3. Evaluate $\iint r\sin\theta\,dr\,d\theta$ over the cardioid $r = a(1-\cos\theta)$ above the initial line.	
Solution:	
	
To integrate first with respect to r, the limits are from $r=0$ to $r=a(1-\cos\theta)$ and to cover the region of integration R, θ varies from 0 to π	
$\therefore$ **Required Area** $= \iint\limits_R r\sin\theta\,dr\,d\theta$	
$= \int_0^{\pi}\int_0^{a(1-\cos\theta)} r\sin\theta\,dr\,d\theta$	
$= \int_0^{\pi}\left[\int_0^{a(1-\cos\theta)} r\,dr\right]\sin\theta\,d\theta$	First integration with respect to r
$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)}\sin\theta\,d\theta$	
$= \int_0^{\pi}\left[\left(\frac{(a(1-\cos\theta))^2}{2}\right)-\left(\frac{(0)^2}{2}\right)\right]\sin\theta\,d\theta$	
$= \frac{a^2}{2}\int_0^{\pi}(1-\cos\theta)^2.\sin\theta\,d\theta$	$\int_l^u [f(x)]^n f'(x)dx = \left[\frac{[f(x)]^{n+1}}{n+1}\right]_l^u$
$= \frac{a^2}{2}\left[\left(\frac{(1-\cos\pi)^3}{3}\right)-\left(\frac{(1-\cos 0)^3}{3}\right)\right]$	
$= \frac{a^2}{2}\left[\left(\frac{(1-(-1))^3}{3}\right)-\left(\frac{(1-1)^3}{3}\right)\right]$	
$= \frac{a^2}{2}\left[\frac{(2)^3}{3}\right] = \frac{a^2}{2}\left[\frac{8}{3}\right] = \frac{4a^2}{3}$	

Problems based on Area as Double integral(Polar)	**Solving Tip!**
4. Evaluate $\iint\limits_{A} r^3 dr d\theta$, where A is the region between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.	
Solution:	



Let $I = \int\limits_{0}^{\pi} \int\limits_{2\sin\theta}^{4\sin\theta} r^3 dr d\theta$	First integration with respect to r
$= \int\limits_{0}^{\pi} \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta$	
$= \int\limits_{0}^{\pi} \left[\left(\frac{4^4\sin^4\theta}{4}\right) - \left(\frac{2^4\sin^4\theta}{4}\right)\right] d\theta$	Upper limit minus lower limit substitution
$= \int\limits_{0}^{\pi} \left[64\sin^4\theta - 4\sin^4\theta\right] d\theta = 60\int\limits_{0}^{\pi} \sin^4\theta d\theta$	Simplification
$= 120\int\limits_{0}^{\pi/2} \sin^4\theta d\theta$ since $f(\theta+\pi) = f(\theta)$	Integration with respect to θ
$= 120\frac{3}{4}\frac{1}{2}\frac{\pi}{2}$	$\int\limits_{0}^{\pi/2} \sin^n\theta d\theta = \frac{n-1}{n}.\frac{n-3}{n-2}\cdots\frac{2}{3}.1$ if $n$is odd and $n \geq 3$
$= \frac{45\pi}{2}$	

Problems based on Area as Double integral(Cartesian)	**Solving Tip!**
LEARNING TIME EXCERCISE	
1. Find the area which is inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1+\cos\theta)$.	
Solution:	
	Eliminating r from $r = 3a\cos\theta$ and $r = a(1+\cos\theta)$
	$3a\cos\theta = a(1+\cos\theta)$ $\Rightarrow 3\cos\theta = 1+\cos\theta$ $\Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2}$ $\Rightarrow \theta = \frac{-\pi}{3}$ or $\frac{\pi}{3}$ $\therefore \theta$ varies from $\frac{-\pi}{3}$ to $\frac{\pi}{3}$
	Ans.πa^2

Problems based on Evaluation by transforming a double integration into polar coordinates	Solving Tip!

Transforming a double integration into polar coordinates

The given integral $\iint\limits_R f(x,y)dxdy$. The transformation to polar coordinates, by putting $x = r\cos\theta, y = r\sin\theta$ and

$x^2 + y^2 = r^2, \;\; dxdy = rdrd\theta$

$\therefore \; \iint\limits_R f(x,y)dxdy = \iint\limits_R f(r,\theta)rdrd\theta$

1. Transform into polar coordinate and evaluate
$\int\limits_0^2 \int\limits_0^{\sqrt{2x-x^2}} \dfrac{x}{x^2+y^2}dydx.$

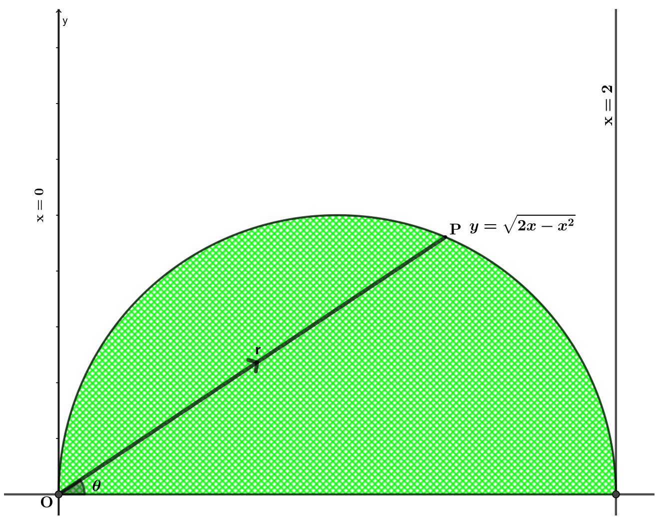
Solution:

The region of integration is bounded by $y = 0$,

$y = \sqrt{2x - x^2} \Rightarrow y^2 = 2x - x^2 \Rightarrow x^2 + y^2 - 2x = 0$

which is the circle with centre $(1, 0)$ and radius $r = 1$

and the $x = 0, \; x = 2$



General eqn. of the circle

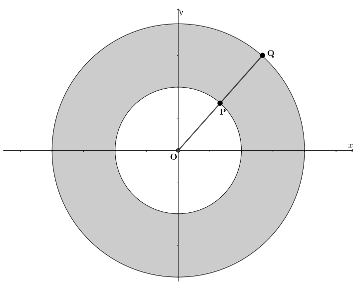
$x^2 + y^2 + 2fx + 2gy + d = 0$ with centre $(-f, -g)$ and radius $r = \sqrt{f^2 + g^2 - d}$

Draw the diagram. It is a upper semi circle. From the diagram, OP as a radial strip with the ends $r = 0$ and the another end point lies on the circle

$$\begin{aligned} & x^2 + y^2 - 2x = 0 \\ \Rightarrow\; & (x^2 + y^2) - 2x = 0 \\ \Rightarrow\; & r^2 - 2r\cos\theta = 0 \\ \Rightarrow\; & r(r - 2\cos\theta) = 0 \\ \Rightarrow\; & r = 0, r = 2\cos\theta \end{aligned}$$

and which moves for cover the region from $\theta = 0$ and $\theta = \pi/2$

Problems based on Evaluation by transforming a double integration into polar coordinates	**Solving Tip!**
To change a polar coordinates, put $x = r\cos\theta, y = r\sin\theta$ and $x^2 + y^2 = r^2, \;\; dxdy = rdrd\theta$	
$$\begin{aligned} & x^2 + y^2 - 2x = 0 \\ & \Rightarrow (x^2 + y^2) - 2x = 0 \\ & \Rightarrow \; r^2 - 2r\cos\theta = 0 \\ & \Rightarrow \; r(r - 2\cos\theta) = 0 \\ & \Rightarrow \; r = 0, r = 2\cos\theta \end{aligned}$$	
Take a radial strip **OP**. $\therefore$ the limits of double integration:	
For r: $\;\; r = 0$ to $r = 2\cos\theta$ For θ: $\;\; \theta = 0$ to $\theta = \pi/2$	
$$\therefore \; I = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \frac{\cancel{r}\cos\theta}{\cancel{r^2}} \cancel{r} drd\theta$$	
$$= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \cos\theta drd\theta$$	
$$= \int_0^{\frac{\pi}{2}} \cos\theta \, [r]_0^{2\cos\theta} \, d\theta$$	
$$= \int_0^{\frac{\pi}{2}} \cos\theta \, [(2\cos\theta) - (0)] \, d\theta$$	
$$= \int_0^{\frac{\pi}{2}} \cos\theta \, [2\cos\theta] \, d\theta$$	
$$= 2\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta$$	
$$= 2\int_0^{\frac{\pi}{2}} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta$$	$\cos^2\theta = \dfrac{1 + \cos 2\theta}{2}$

Problems based on Evaluation by transforming a double integration into polar coordinates	**Solving Tip!**
$= 2.\frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta$	
$= \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$	
$= \left[\left(\frac{\pi}{2} + \frac{\sin \cancel{2}\frac{\pi}{\cancel{2}}}{2}\right) - (0+0)\right]$	$\sin\pi = 0$ and $\sin 0 = 0$
$= \frac{\pi}{2}$	
2. By transforming into polar coordinates evaluate $\iint \frac{x^2y^2}{x^2+y^2}dxdy$ **over the annular region between the circles** $x^2+y^2=4$ **and** $x^2+y^2=16$.	
Solution:	
To change to polar coordinates, put put $x = r\cos\theta, y = r\sin\theta$ and $x^2+y^2=r^2,\;\; dxdy = rdrd\theta$	
$\therefore$ given $x^2+y^2=4 \Rightarrow r^2=4 \Rightarrow r=2$	
and $x^2+y^2=16 \Rightarrow r^2=16 \Rightarrow r=4$	
	
Take a radial strip OPQ, for the annular region r varies from 2 to 4 and θ varies from 0 to 2π.	
$\therefore\;\; I = \iint \frac{x^2y^2}{x^2+y^2}dxdy$	
$= \int_0^{2\pi}\int_2^4 \frac{(r\cos\theta)^2(r\sin\theta)^2}{r^2} rdrd\theta$	conversion step

Problems based on Evaluation by transforming a double integration into polar coordinates	**Solving Tip!**
$= \int_0^{2\pi}\int_2^4 r^3(\cos\theta\sin\theta)^2 dr d\theta$	
$= \int_0^{2\pi}\int_2^4 r^3 \left(\frac{\sin 2\theta}{2}\right)^2 dr d\theta$	$\sin 2\theta = 2\sin\theta\cos\theta$
$= \int_0^{2\pi}\int_2^4 r^3 \left(\frac{\sin^2 2\theta}{4}\right) dr d\theta$	
$= \frac{1}{4}\int_0^{2\pi} \sin^2 2\theta \left[\frac{r^4}{4}\right]_2^4 d\theta$	Integrate w. r. to r
$= \frac{1}{4}\int_0^{2\pi} \sin^2 2\theta \left[\left(\frac{4^4}{4}\right) - \left(\frac{2^4}{4}\right)\right] d\theta$	limits substitution
$= \frac{1}{4}\int_0^{2\pi} \sin^2 2\theta \left[4^3 - 2^2\right] dd\theta$	
$= \frac{1}{4}\int_0^{2\pi} \sin^2 2\theta \left[64 - 4\right] d\theta$	
$= \frac{1}{4}\int_0^{2\pi} \sin^2 2\theta \left[60\right] d\theta$	
$= \frac{1}{4}\left[60\right]\int_0^{2\pi} \sin^2 2\theta d\theta$	
$= 15\int_0^{2\pi} \left[\frac{1-\cos 4\theta}{2}\right] d\theta$	$\sin^2\theta = \frac{1-\cos 2\theta}{2}$
$= \frac{15}{2}\int_0^{2\pi} (1-\cos 4\theta) d\theta$	
$= \frac{15}{2}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi}$	
$= \frac{15}{2}\left[\left(2\pi - \frac{\sin 8\pi}{4}\right) - \left(0 - \frac{\sin 0}{4}\right)\right]$	
$= \frac{15}{\cancel{2}}.\cancel{2}\pi = 15\pi$	

Tutorial-2

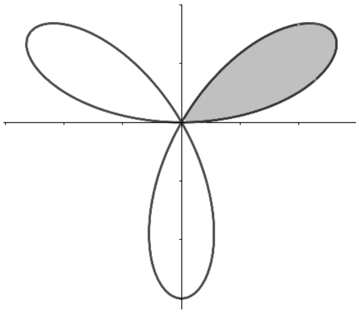
TUTORIAL PROBLEMS	Solving Tip!
1. Find the area of the circle $x^2 + y^2 = a^2$ **Solution**	Draw a diagram

TUTORIAL PROBLEMS	**Solving Tip!**
2. Find by double integration, the area lying between the parabola $y = 4x - x^2$ and the line $y = x$. **Solution**	Draw a diagram Ans.9/2

TUTORIAL PROBLEMS	**Solving Tip!**
3. Find by, double integration, the area of one loop of the lemniscate $r^2 = a^2 \cos 2\theta$. **Solution**	Draw a diagram Ans. $\frac{a^2}{2}$

TUTORIAL PROBLEMS	**Solving Tip!**
4. Evaluate $\iint\limits_A r^3 dr d\theta$, where A is the area between circles $r = 2\sin\theta$ and $r = 4\sin\theta$ **Solution**	Draw a diagram Ans. $45\pi/2$

TUTORIAL PROBLEMS	Solving Tip!
5. Find the area of the cardioid $r = a(1 - \cos\theta)$ **Solution**	Draw a diagram Ans. $\dfrac{3\pi a^2}{2}$

TUTORIAL PROBLEMS	Solving Tip!
6. Find the area of a loop of the curve in the first octant $r = a\sin 3\theta$ **Solution** 	Draw a diagram $\pi a^2/12$

TUTORIAL PROBLEMS	Solving Tip!
7. Using double integral find the area enclosed by the curves $y = 2x^2$ and $y^2 = 4x$. **Solution**	Draw a diagram Ans. $\frac{2}{3}$

TUTORIAL PROBLEMS	Solving Tip!
8. Find the smaller of the areas bounded by $y = 2 - x$ and $x^2 + y^2 = 4$ using double integral. **Solution**	Draw a diagram Ans.$\pi - 2$

TUTORIAL PROBLEMS	Solving Tip!
9. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by changing to polar coordinates. **Solution**	Ans.$\pi/4$

TUTORIAL PROBLEMS	Solving Tip!
10. Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy$ **by changing into polar coordinates.** **Solution**	Ans.$\pi a^4/8$

Triple integration in Cartesian coordinates	Solving Tip!
Triple integration **Working Procedure:** **If all the limits are constants** i.e., $\int_{z_l}^{z_u}\int_{y_l}^{y_u}\int_{x_l}^{x_u} f(x,y,z)dxdydz$	For a constant limits!!!
Step1: Write I =given integral **Step2:** $I=\int_{z_l}^{z_u}\int_{y_l}^{y_u}\left[\int_{x_l}^{x_u} f(x,y,z)dx\right]dydz$ First integrate with respect to inside integral variable and take upper limit and lower limit substitutions and then simplification reduces to the integral as double integral. **Step3:** $I=\int_{z_l}^{z_u}\left[\int_{y_l}^{y_u} g(y,z)dy\right]dz$ Do this integral by solving the double integral as before.	
If the limits are in variables, suppose $\int_{z_l}^{z_u}\int_{h_l(z)}^{h_u(z)}\int_{g_l(y,z)}^{g_u(y,z)} f(x,y,z)dxdydz$	For a variable limits!!!
Step1: Write I =given integral **Step2:** $I=\int_{z_l}^{z_u}\int_{h_l(z)}^{h_u(z)}\left[\int_{g_l(y,z)}^{g_u(y,z)} f(x,y,z)dx\right]dydz$ First integrate with respect to the absent variable and take upper limit and lower limit substitutions and then simplification reduces to the integral as double integral with variable limits. **Step3:** $I=\int_{z_l}^{z_u}\left[\int_{h_l(z)}^{h_u(z)} g(y,z)dy\right]dz$ Do this integral by solving the double integral with varibale limits as before.	

Triple integration in Cartesian coordinates	Solving Tip!
1. Evaluate $\int_0^1\int_0^2\int_0^3 x^2yz\,dx\,dy\,dz$	
Solution:	
Let $I = \int_0^1\int_0^2\left[\int_0^3 x^2dx\right]yz\,dy\,dz$	**Step1**
$= \int_0^1\int_0^2\left[\frac{x^3}{3}\right]_0^3 yz\,dy\,dz$	**Step2**
$= \int_0^1\int_0^2\left[\left(\frac{3^3}{3}\right)-\left(\frac{0}{3}\right)\right]yz\,dy\,dz$	**Upper limit minus lower limit substitutions**
$= \int_0^1\int_0^2[9-0]\,yz\,dy\,dz$	Simplifications
$= 9\int_0^1\left[\int_0^2 y\,dy\right]z\,dz$	
$= 9\int_0^1\left[\frac{y^2}{2}\right]_0^2 z\,dz$	
$= 9\int_0^1\left[\left(\frac{2^2}{2}\right)-\left(\frac{0^2}{2}\right)\right]z\,dz$	
$= 9\int_0^1[(2)-(0)]\,z\,dz$	
$= 18\int_0^1 z\,dz$	
$= 18\left[\frac{z^2}{2}\right]_0^1$	
$= 18\left[\left(\frac{1}{2}\right)-(0)\right]$	
$= 18\left[\frac{1}{2}\right]$	
$= 9$	

Triple integration in Cartesian coordinates	**Solving Tip!**
2. Evaluate $\int_0^1\int_{y^2}^1\int_0^{1-x} xdzdxdy$	
Solution:	
$I=\int_0^1\int_{y^2}^1\int_0^{1-x} xdzdxdy$	**Step1**
$=\int_0^1\int_{y^2}^1 x\left[\int_0^{1-x} dz\right]dxdy$	First integration with respect to the absent limit
$=\int_0^1\int_{y^2}^1 x\,[z]_0^{1-x}\,dxdy$	Upper limit minus lower limit substitutions
$=\int_0^1\int_{y^2}^1 x\,[1-x]\,dxdy$	Simplification
$=\int_0^1\int_{y^2}^1 (x-x^2)dxdy$	Integrate with respect to absent variable from the limit i.e., x
$=\int_0^1\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{y^2}^1 dy$	Upper limit minus lower limit substitutions
$=\int_0^1\left[\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{y^4}{2}-\frac{y^6}{3}\right)\right]dy$	Simplification and integrate with respect to y
$=\left[\frac{1}{6}y-\left(\frac{y^5}{10}-\frac{y^7}{21}\right)\right]_0^1$	
$=\left[\frac{1}{6}-\frac{1}{10}+\frac{1}{21}\right]=\frac{24}{210}=\frac{4}{35}$	

Triple integration in Cartesian coordinates	**Solving Tip!**
3. Show that $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{\sqrt{1-x^2-y^2-z^2}} = \frac{\pi^2}{8}$	
Solution:	
Let $I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{\sqrt{1-x^2-y^2-z^2}}$	
$= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{\sqrt{\left(\sqrt{1-x^2-y^2}\right)^2 - z^2}}$	$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$
$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}} dydx$	
$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\left(\sin^{-1}(1)\right) - \left(\sin^{-1}(0)\right)\right] dydx$	
$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\frac{\pi}{2} - 0\right] dydx$	
$= \frac{\pi}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} dydx$	
$= \frac{\pi}{2} \int_0^1 [y]_0^{\sqrt{1-x^2}} dx$	
$= \frac{\pi}{2} \int_0^1 \left[\left(\sqrt{1-x^2}\right) - (0)\right] dx$	
$= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2} dx$	
$= \frac{\pi}{2} \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_0^1$	$\int \sqrt{a^2-x^2} dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$
$= \frac{\pi}{2} \left[\left(0 + \frac{1}{2}\sin^{-1}(1)\right) - \left(0 + \frac{1}{2}\sin^{-1}(0)\right)\right]$	
$= \frac{\pi}{2} \left[\frac{1}{2}\left(\frac{\pi}{2}\right) - (0)\right]$	
$= \frac{\pi^2}{8}$	

Triple integration in Cartesian coordinates	**Solving Tip!**
LEARNING TIME EXCERCISE	
1. Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dx\,dy\,dz$	
Solution:	
	$e^{\log_e x} = x$
$= \frac{5}{8}$	

Triple integration in Cartesian coordinates	Solving Tip!
2. Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dx\,dy\,dz$ **Solution:** $= \frac{1}{9}(24\log 2 - 19)$	$e^{\log_e x} = x$

Triple integration in Cartesian coordinates	Solving Tip!
3. Show that $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dzdydx}{\sqrt{a^2-x^2-y^2-z^2}} = \frac{\pi^2 a^2}{8}$ **Solution:** Let $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dzdydx}{\sqrt{a^2-x^2-y^2-z^2}}$ $= \frac{\pi^2 a^2}{8}$	$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

Volume as Triple integral	Solving Tip!

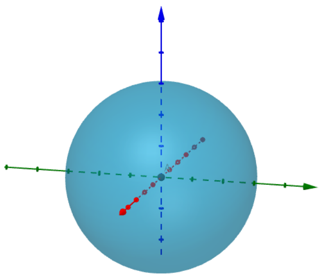
Volume as Triple Integral

If V is the volume enclosed by the region D, then volume

$V = \iiint\limits_D dxdydz$

Working Procedure:

Step1:	Draw the diagram and decide the first octant if the diagram is symmetric about the coordinate planes.
Step2:	Find the limits for the variables.
Step3:	Solve the triple integral.

Volume as Triple integral	Solving Tip!
1.Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$, using triple integral. **Solution:** 	From the diagram, 4 parts in the top and 4 parts in the bottom, totally 8. We have to evaluate the volume in the first octant and multiply by 8 for getting the whole volume.
By the symmetry, Volume $= 8 \times$ (volume of the sphere in I-octant)	
$= 8 \times \left[\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdydx \right]$	For the I-octant, for the z, from the 0 to the sphere surface $x^2 + y^2 + z^2 = a^2$
$= 8 \times \left[\int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sqrt{a^2-x^2-y^2}\right] dydx \right]$	After integrating with respect to z, it will be reduced to double integral. Now the region is circle $x^2 + y^2 = a^2 \Rightarrow \ y = \sqrt{a^2 - x^2}$
$= 8 \times \left[\int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sqrt{\left(\sqrt{a^2-x^2}\right)^2 - y^2}\right] dydx \right]$	$\therefore \ y = 0$ and $y = \sqrt{a^2 - x^2}$
$= 8 \int_0^a \left[\frac{y}{2}\sqrt{\left(\sqrt{a^2-x^2}\right)^2 - y^2} + \frac{\left(\sqrt{a^2-x^2}\right)^2}{2} \sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right) \right]_0^{\sqrt{a^2-x^2}} dx$	After integrating it,
$= 8 \int_0^a \left[\left(0 + \frac{\left(\sqrt{a^2-x^2}\right)^2}{2} \sin^{-1}(1)\right) - \left(0 + \frac{\left(\sqrt{a^2-x^2}\right)^2}{2} \sin^{-1}(0)\right) \right] dx$	now $x^2 = a^2 \Rightarrow \ x = +a$
$= 8\frac{\pi}{4} \int_0^a (a^2 - x^2)\, dx = 8\frac{\pi}{4} \left(a^2 x - \frac{x^3}{3}\right)_0^a$ $= 2\pi \left[\left(a^3 - \frac{a^3}{3}\right) - \left(0 - \frac{0}{3}\right) \right]_0^a$ $= 2\pi \left[\frac{3a^3 - a^3}{3}\right] = \frac{4}{3}\pi a^3$	since in the I octant. Hence $x = 0$ and $x = a$ is the outside integral limits.

Volume as Triple integral	**Solving Tip!**

LEARNING TIME EXCERCISE

1. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ **by triple integral. Solution:**

Since the ellipsoid is symmetric about the coordinate planes, the volume of the ellipsoid $= 8\times$ volume in the first octant.

Volume of the first octant is bounded by the planes $x=0, y=0, z=0$ and the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

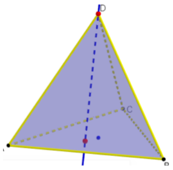
$$\therefore \text{Volume} = 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dzdydx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}dydx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \frac{c}{b}\left[\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\right]dydx$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y}{2}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\right.$$

$$\left. + \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\sin^{-1}\frac{y}{b\sqrt{1-\frac{x^2}{a^2}}}\right]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

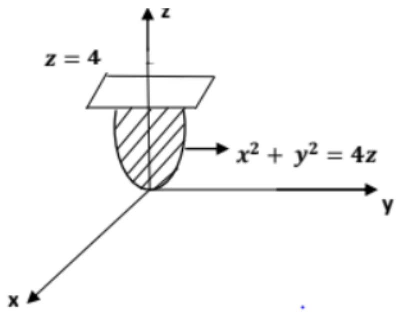
Volume as Triple integral	Solving Tip!
$= 2\pi bc\left[a - \frac{a}{3}\right] = \frac{4}{3}\pi abc$ **2. Find the volume of the tetrahedron bounded by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and the coordinate planes.** **Solution:**  The region of integration is bounded by $x = 0, y = 0, z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, Volume $= \iiint\limits_{v} dzdydx$ $= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dzdydx$ $=$ $=$	

Volume as Triple integral	Solving Tip!
$= c\int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \left[\left(\left(1-\frac{x}{a}\right)-\frac{y}{b}\right)\right] dydx$ $= c\int_0^a \left[\left(\left(1-\frac{x}{a}\right)y-\frac{y^2}{2b}\right)\right]_0^{b\left(1-\frac{x}{a}\right)} dx$ $= c\int_0^a \left[\left(b\left(1-\frac{x}{a}\right)^2-\frac{b^2\left(1-\frac{x}{a}\right)^2}{2b}\right)-(0)\right] dx$ $= \frac{bc}{2}\int_0^a \left[\left(1-\frac{x}{a}\right)^2\right] dx$ $= \frac{bc}{2}\left[\frac{1}{-1/a}\frac{\left(1-\frac{x}{a}\right)^3}{3}\right]_0^a$ $= \frac{bc}{2}\left[-\frac{a}{3}\left(1-\frac{x}{a}\right)^3\right]_0^a$ $= \frac{bc}{2}\left[\left(-\frac{a}{3}\left(1-\frac{a}{a}\right)^3\right)-\left(-\frac{a}{3}\left(1-0\right)^3\right)\right]$ $= \frac{bc}{2}\left[\frac{a}{3}\right]$ $= \frac{abc}{6}$	

Tutorial-3

TUTORIAL PROBLEMS	**Rough Work!**
1. Find the volume of the portion of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ **which lies in the first octant using triple integral.** **Solution:**	Ans. $\frac{\pi abc}{6}$

TUTORIAL PROBLEMS	**Rough Work!**
2. Find the volume bounded by the xy-plane, the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$ using triple integral. **Solution:**	Ans. $\frac{1}{3\pi}(9\pi - 4)$

TUTORIAL PROBLEMS	Rough Work!
3. Find the volume of the paraboloid $x^2 + y^2 = 4z$ **cut off** $z = 4$. **Solution:**  Put $x = r\cos\theta, y = r\sin\theta, z = z \therefore\ x^2 + y^2 = r^2$ $\therefore$ Paraboloid:$r^2 = 4z$ and plane $z = 4$. Hence the limits: $\frac{r^2}{4} \leq z \leq 4$ $0 \leq r \leq 4$ $0 \leq \theta \leq \frac{\pi}{2}$ $\therefore\ V = 4 \int_0^{\pi/2} \int_0^{4} \int_{r^2/4}^{4} r dr d\theta$	We have four parts. We consider the positive octant. Ans.32π

TUTORIAL PROBLEMS	**Rough Work!**
4. Find the volume cut off from the sphere $x^2 + y^2 + z^2 = a^2$ **by the cone** $x^2 + y^2 = z^2$. **Solution:**	Ans. $\dfrac{2\pi a^3}{3}(2-\sqrt{2})$

TUTORIAL PROBLEMS	Rough Work!
5. Find the volume in the positive octant bounded by the plane $x + 2y + 3z = 4$ and the coordinate planes. **Solution:**	Ans. $\frac{16}{9}$

TUTORIAL PROBLEMS	Rough Work!
6. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ in the positive octant. **Solution:**	Ans. $\dfrac{\pi a^3}{6}$

TUTORIAL PROBLEMS	**Rough Work!**
7. Find the volume of the tetrahedron bounded by the plane $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$ **and the coordinate planes.** **Solution:**	Ans.4

TUTORIAL PROBLEMS	Rough Work!
8. Find the volume of the cylinder $x^2 + y^2 = 4$ bounded by the plane $z = 0$ and the surface $z = x^2 + y^2 + 2$. **Solution:**	Ans.16π

TUTORIAL PROBLEMS	**Rough Work!**
9. Evaluate $\iiint_V dxdydz$, **where V is the volume enclosed by the cylinder** $x^2 + y^2 = 1$ **bounded by the planes** $z = 0$**,** $z = 2 - x$**.** **Solution:**	Ans.$2\pi - \frac{4}{3}$

TUTORIAL PROBLEMS	Rough Work!
10. Find the volume of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. **Solution:**	Ans.8π