Unit·2 Tutorial·2

① ~~If~~ $F = x^2\vec{i} + y^2\vec{j}$. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ from $(0,0)$ to $(1,1)$ along the path $y = x$.

$dr = dx\,\vec{i} + dy\,\vec{j}$

$F\cdot dr = x^2 dx + y^2 dy$

$y = x$

$dy = dx \Rightarrow F\cdot dr = \int_0^1 x^2 dx + x^2 dx$

$= 2\int_0^1 x^2 dx$

$= \frac{2}{3}$ //.

② Show that $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\,\vec{k}$ is a conservative field.

Field is conservative if $\Delta \times F = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d/dx & \frac{d}{dy} & d/dz \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix} = 0\hat{i} - \hat{j}\,[-6x^2z + 6x^2z] + \hat{k}\,[4x - 4x]$$

$= \underline{\underline{0}}$

③ Find the work done in moving a particle in the force field $\vec{A} = 3x^2\vec{i} + (2xz - y)\vec{j} - z\vec{k}$ from $t=0$ to $t=1$ along the curve $x = 2t^2 \quad y = t \quad z = 4t^3$.

$\vec{A} = 3\cdot 4t^4\vec{i} + (2\cdot 2t^2\cdot 4t^3 - t)\vec{j} - 4t^3\vec{k}$

$= 12t^4\,\vec{i} + (16t^5 - t)\vec{j} - 4t^3\vec{k}$

$\int_0^1 A\cdot dr = [12t^4\cdot 4t\,dt]\,\vec{i} + (16t^5 - t)dt\,\vec{j} - 4t^3\cdot 12t^2 dt\,\vec{k}$

$= 48t^5 dt\,\vec{i} + (16t^5 - t)dt\,\vec{j} - 48t^5 dt\,\vec{k}$

$= [8t^6]\,\hat{i} + \left(\frac{16}{6}t^6 - \frac{t^2}{2}\right)\vec{j} - 8t^6\,\hat{k}$

$= 8 + 16/6 - 1/2 - 8 = 13/6$ //.

④ Using Green's theorem evaluate $\int_C (2xy - x^2)\,dx + (x^2+y^2)\,dy$ where C is the closed curve of the region bounded by $y = x^2$ & $y^2 = x$

$$\iint_R \left(\frac{dN}{dx} - \frac{dM}{dy}\right) dx\,dy = \oint_C (M\,dx + N\,dy)$$

$$M = 2xy - x^2 \qquad N = x^2 + y^2.$$

$$\iint \frac{d}{dx}(x^2+y^2) - \frac{d}{dy}(2xy - x^2).$$

$$= 2x - 2x + 0 = [0]_0^1 = 0.$$

⑤ Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ where $\vec{A} = z\hat{i} + x\vec{j} - 3y^2z\vec{k}$ and s is surface of the cylinder $x^2+y^2 = 16$ included in the first octand below $z = 0$ & $z = 5$.

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So

Vector normal to s is given by $\nabla(x^2+y^2) = 2x\vec{i} + 2y\vec{j}$

$\hat{n}$ = a unit vector normal to surface.

$$\Rightarrow \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{(2x)^2 + (2y)^2}} = \frac{x\vec{i} + y\vec{j}}{\sqrt{x^2+y^2}} \qquad \text{as } x^2+y^2 = 16$$

$$\therefore \Rightarrow \frac{x\vec{i} + y\vec{j}}{4}$$

$$\iint_S \vec{A}\,\hat{n}\,ds = \iint_R \vec{A}\,\hat{n}\,\frac{dy\,dz}{|\vec{i}\cdot\hat{n}|}$$

Region R is $y = 0$ $y = 4$ & $z = 0$ & $z = 5$.

$$\vec{i}\cdot\hat{n} = \vec{i}\left(\frac{1}{4}x\vec{i} + \frac{1}{4}\vec{j}\right) = \frac{1}{4}x.$$

$$\vec{A}\,\hat{n} = (z\vec{i} + x\vec{j} - 3y^2z\vec{k})\left(\frac{1}{4}x\,\vec{i} + \frac{1}{4}\vec{j}\right)$$

$$= \frac{1}{4}xz + \frac{1}{4}xy = \frac{1}{4}x(y+z).$$

$$\iint_S \vec{A}\cdot\vec{n}\,ds = \iint_R \vec{A}\,\hat{n}\,\frac{dy\cdot dz}{|\vec{i}\cdot\hat{n}|} = \iint_R \frac{1}{4}x(y+z)\,\frac{dy\,dz}{\frac{1}{4}x}.$$

$$= \int_0^5 \int_0^4 (y+z)\,dy\,dz = \int_0^5 \left[\frac{y^2}{2} + zy \right]_0^4$$

$$= \int_0^5 (8+4z)\,dz = \left[8z + 2z^2 \right]_0^5$$

$$= 40 + 50$$

$$= \underline{\underline{90}}.$$

(6) Show that $\vec{F} = (2xy+z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ is a conservative field. Find the scalar potential and the work done in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix} = 0\vec{i} - \vec{j}(2x-2x) + \hat{k}(2x-2x) = 0$$

Hence its conservative.

$$\vec{F} = \nabla\phi = (2xy+z^3)\vec{i} - x^2\vec{j} + 3xz^2\vec{k}$$

$$\frac{d\phi}{dx}\vec{i} + \frac{d\phi}{dy}\vec{j} + \frac{d\phi}{dz}\vec{k}$$

$$\Rightarrow \phi = x^2y + z^3x + f_1.$$

$$\phi = x^2y + f_2 \qquad\qquad \phi = x^2y + z^3x + f.$$

$$\phi = xz^3 + f_3$$

$$\text{work done} = \int_{1,-2,1}^{3,1,4} \vec{F}\cdot d\vec{r} = \int_{1,-2,1}^{3,1,4} d\phi. = \left[\phi\right]_{1,-2,1}^{3,1,4}$$

$$= 9 + 64\times 3 - (-2+1)$$

$$= 202 \ \underline{\text{unit}}$$

(7) $\iint \left(\frac{dN}{dx} - \frac{dN}{dy}\right) dx\,dy$. Verify Green's theorem in the plane for $\int (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of region bounded by (a) $y = \sqrt{x}$; $y = x^2$ & (b) $x = 0 ; y = 0 \ x + y = 1$

$\iint \left(\frac{dN}{dx} - \frac{dM}{dy}\right) dx\,dy$.

$\int (3x^2 - 8y^2)dx + (4y - 6xy)dy$

$\frac{dM}{dy} = -16y \quad \frac{dN}{dx} = -6y$.

$\frac{dN}{dx} - \frac{dM}{dy} = -6y + 16y = 10y$.

$\int_0^1 \int_0^{1-x} 10y\,dy\,dx$

$\int_0^1 [5y^2]_0^{1-x} dx$

$\int_0^1 5[x^2 + 1 - 2x]\,dx$

$5\left[\frac{x^3}{3} - x^2 + x\right]_0^1$

$= 5/3$

$\int_0^1 \int_{x^2}^{\sqrt{x}} 10y\,dy\,dx$.

$\int_0^1 [5y^2]_{x^2}^{\sqrt{x}}$

$\int_0^1 [5x - 5x^4]$

$\left[\frac{5}{2}x^2 - x^5\right]_0^1 = \frac{5}{2} - 1$

$= 3/2$

(4) Find $\int_c x^2 + y^2\,dx - 2xy\,dy$ and the curve C is the rectangle in XY plane bounded by $x = 0$, $x = a$, $y = b$, $y = 0$.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ d/dx & d/dy & d/dz \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = 0\vec{i} - \vec{j}0 + \vec{k}(-2y - (2y))$$

$= -4y\vec{k}$

$\int_0^b \int_0^a -4y\,dx\,dy$.

$= \left(-4a\,y^2/2\right)_0^b = \frac{-4ab^2}{2} = \underline{\underline{-2ab^2}}$

1 & 10 ques is wrongly printed in tutorial.

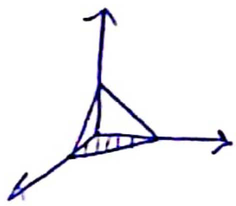
10 Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ where $\vec{A} = (x+y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}$ and S is the surface of the plane $2x+y+2z=6$ in the first octant.

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} \quad \text{where } \phi = 2x+y+2z-6.$$

$$\hat{n} = \frac{2\vec{i}+\vec{j}+2\vec{k}}{\sqrt{4+1+4}} = \frac{2}{3}\vec{i}+\frac{1}{3}\vec{j}+\frac{2}{3}\vec{k}$$

$$\vec{k}\cdot\hat{n} = \frac{2}{3}.$$

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\vec{k}\cdot\hat{n}|}$$



$$\vec{A}\cdot\hat{n} = [(x+y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}]\cdot\left[\frac{2}{3}\vec{i}+\frac{1}{3}\vec{j}+\frac{2}{3}\vec{k}\right]$$

$$= \frac{2}{3}(x+y^2) - \frac{2}{3}x + \frac{4}{3}yz = \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6-2x-y}{2}\right)$$

$$= \frac{4}{3}y(3-x).$$

Hence $\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{\vec{k}\cdot\hat{n}}$.

$$= \iint_R \frac{4}{3}y(3-x)\cdot\frac{3}{2}\,dx\,dy.$$

$$= \int_0^3\int_0^{6-2x} 2y(3-x)\,dy\,dx = \int_0^3 2(3-x)\left[\frac{y^2}{2}\right]_0^{6-2x}$$

$$= \int_0^3 (3-x)(6-2x)^2\,dx = 4\int_0^3 (3-x)^3\,dx.$$

$$= 4\left[\frac{(3-x)^4}{4(-1)}\right]_0^3 = -(0-81) = 81.$$

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8 Verify Green's theorem in the plane for $\int (x^2 - 2xy)\,dx + (x^2y + 3)\,dy$ where C is the boundary of region bounded by $y^2 = 8x$, $x = 2$.

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