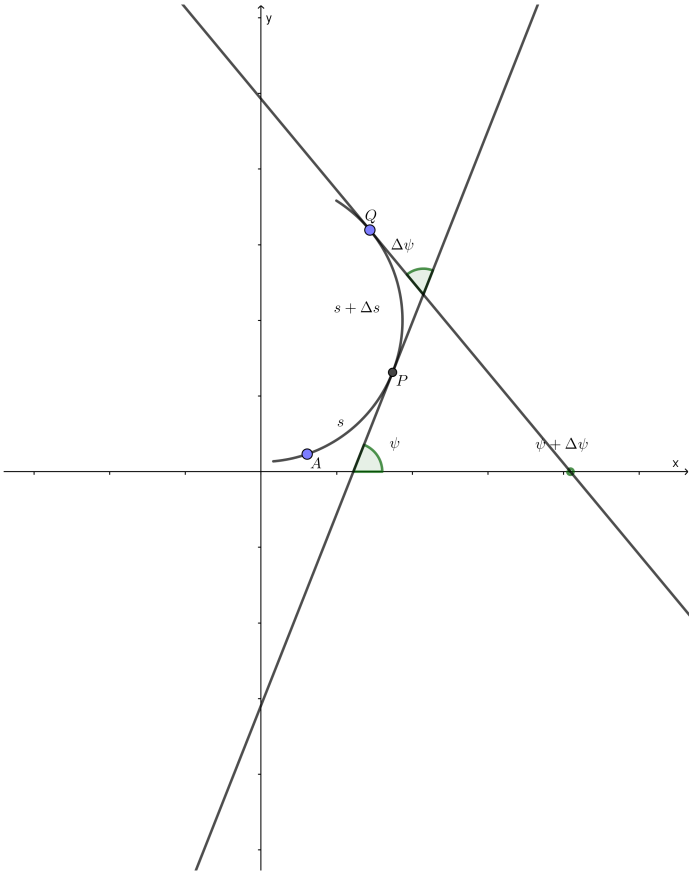
UNIT IV

Applications of Differential Calculus and Beta Gamma functions

Measure of curvature



Let ϕ be a curve that does not intersect itself and having tangents at each point. Let A be a fixed point on the curve from which arc length is measured. Let P and Q be neighbouring points on the curve so that $AP = s$ and $AQ = s + \Delta s$.

$\therefore$ Length of arc $PQ = \Delta s$

Let the tangents at P and Q make angles ψ and $\psi + \Delta\psi$ respectively with the positive direction of $x-$axis.

$\therefore$ $\Delta\psi$ is the angle between the tangents at P and Q. Precisely, $\Delta\psi$ is the angle through which the tangent turns from P to Q as P moves along the arc through the distance Δs.

The angle $\Delta\psi$ is called the angle of contigence of the arc PQ or the total curvature of the arc PQ. The ratio $\dfrac{\Delta\psi}{\Delta s}$ is called the average curvature of the arc PQ. The curvature of the curve at P is defined as $\lim\limits_{\Delta s\to\, 0} \dfrac{\Delta\psi}{\Delta s} = \dfrac{d\psi}{ds}$ and it is denoted by the greek letter kappa κ. Thus $\kappa = \dfrac{d\psi}{ds}$.

Note:

1. s and ψ are called the intrinsic coordinates of P and $f(s,\psi)=0$ is called the intrinsic equation of the curve. 2. Since the difference in angles and difference in arc length are $|\Delta\psi|$ and $|\Delta s|$, we have $\kappa = \left|\dfrac{d\psi}{ds}\right|$, so curvature is positive quantity.

Theorem: The curvature of a circle at any point is a constant and is equal to the reciprocal of the radius of the circle.

Note: If $r \longrightarrow \infty$, the curvature tends to zero. i.e., when radius $r \longrightarrow \infty$, the circle approaches a straight line.

Hence the curvature of a straight line is zero at any of its points. In other words, the straight line does not bend at any point.

Definition-Radius of curvature

If the curvature at a point P on a curve is k, then $\dfrac{1}{k}$ is called the radius of curvature at P (if $k \neq 0$). Radius of curvature is denoted by ρ.

Thus $\rho = \dfrac{1}{\kappa} = \dfrac{ds}{d\psi}$

Note:

From the definition of curvature it is obvious that we should know the intrinsic equation of the curve. This is not easy in many cases. Generally equation of a curve is given in Cartesian or polar coordinates. So, we shall derive formula for radius of curvature for Cartesian equation of a given curve.

Points to ponder:

1. When calculating ρ only positive value should be taken. i.e., numerical value of ρ is taken as radius of curvature, since it cannot be negative. If $y_2 > 0$, the curve is concave up and if $y_2 < 0$ then it is concave down or convex up at the point.

2. At a point of inflexion i.e., when $y_2 = 0$, the curvature is defined as zero.

3. If the equation of the curve is $x = f(y)$, then $\rho = \dfrac{\left[1 + \left[\dfrac{dx}{dy}\right]^2\right]^{3/2}}{\dfrac{d^2x}{dy^2}} = \dfrac{(1 + x_1^2)}{x_2}$, if $x_2 \neq 0$

where $x_1 = \dfrac{dx}{dy}$ and $x_2 = \dfrac{d^2x}{dy^2}$

4. If at a point $\dfrac{dy}{dx} = \infty$, $\rho = \dfrac{[1 + y_1^2]^{3/2}}{y_2}$ cannot be used. i.e., if the tangent is parallel to y-axis, then $\dfrac{dx}{dy} = 0$ So, we use $\rho = \dfrac{(1 + x_1^2)}{x_2}$ in such cases.

Problems	**Solving Tip!!!**
1. Find the radius of curvature at the point $\left(\frac{1}{4}, \frac{1}{4}\right)$ on $\sqrt{x} + \sqrt{y} = 1$. **Solution** The given curve is $\sqrt{x} + \sqrt{y} = 1$ Differentiating with respect to x, we get $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}.\frac{dy}{dx} = 0$ $\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} = \frac{\sqrt{x}-1}{\sqrt{x}} = 1 - \frac{1}{\sqrt{x}} = 1 - x^{-1/2}$ $\frac{d^2y}{dx^2} = -\left(-\frac{1}{2}.x^{-3/2}\right) = \frac{1}{2x^{3/2}}$ At the point $\left(\frac{1}{4}, \frac{1}{4}\right)$, $y_1 = \frac{dy}{dx} = 1 - \frac{1}{\sqrt{1/4}} = 1 - 2 = -1$ $y_2 = \frac{d^2y}{dx^2} = \frac{1}{2(1/4)^{3/2}} = \frac{4^{3/2}}{2} = \frac{2^3}{2} = 4$ $\therefore$ the radius of curvature is $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$ $= \frac{(1+1)^{3/2}}{4} = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$ 2. Find the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on $x^3 + y^3 = 3axy$. **Solution** The given curve is $x^3 + y^3 = 3axy - - - (1)$ Differentiating with respect to x, we get $3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y.1\right]$	

Problems	**Solving Tip!!!**
$\frac{dy}{dx}\left[y^2 - ax\right] = ay - x^2 \Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} - - - (2)$ Differentiating (2) with respect to (2) we get $\frac{d^2y}{dx^2} = \frac{(y^2 - ax)\left(a\frac{dy}{dx} - 2x\right) - (ay - x^2)\left(2y\frac{dy}{dx} - a\right)}{(y^2 - ax)^2}$ At the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, $y_1 = \frac{dy}{dx} = \frac{a.\frac{3a}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - a\frac{3a}{2}} = -1$ $y_2 = \frac{d^2y}{dx^2}$ $= \frac{\left(\left(\frac{9a^2}{4}\right) - a.\frac{3a}{2}\right)(-a - 3a) - \left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)(-3a - a)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$ $= \frac{(-4a)\left[\frac{3a^2}{4} - \left(-\frac{3a^2}{4}\right)\right]}{\left(\frac{3a^2}{4}\right)^2}$ $= \frac{(-4a).2.\left[\frac{3a^2}{4}\right]}{\left(\frac{3a^2}{4}\right)^2} = \frac{-8a}{\left(\frac{3a^2}{4}\right)} = -\frac{32}{3a}$ $\therefore$ the radius of curvature $\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$ $= \frac{2\sqrt{2}}{-32/3a} = -\frac{2\sqrt{2}.3a}{32} = -\frac{3a}{8\sqrt{2}}$ Since ρ is positive, $\rho = \frac{3a}{8\sqrt{2}}$	

Problems	Solving Tip!!!
3. Find the radius of curvature of the curve $xy^2 = a^3 - x^3$ at $(a, 0)$.	
Solution	
The given curve is $xy^2 = a^3 - x^3$	
Differentiating with respect to x, we get,	
$$x.2y\frac{dy}{dx} + y^2 = -3x^2$$ $$\frac{dy}{dx} = -\frac{3x^2 + y^2}{2xy}$$	
At the point $(a, 0)$, $\frac{dy}{dx} = \infty$	$\frac{\text{any thing}}{0} = \infty$
so, we use $\rho = \frac{(1 + x_1^2)^{3/2}}{x_2}$	
Now, $\frac{dx}{dy} = -\frac{2xy}{3x^2 + y^2}$	
$$\frac{d^2x}{dy^2} = -2\frac{\left[(3x^2 + y^2)\left(x.1 + y\frac{dx}{dy}\right) - (xy)\left(3.2x\frac{dx}{dy} + 2y\right)\right]}{(3x^2 + y^2)^2}$$	
At the point $(a, 0)$,	
$$x_1 = \frac{dx}{dy} = 0$$ $$x_2 = \frac{d^2x}{dy^2} = -2\frac{3a^2.a}{(3a^2)^2} = -\frac{2}{3a}$$ $$\therefore \rho = \frac{(1 + x_1^2)^{3/2}}{x_2} = \frac{(1 + 0)^{3/2}}{-2/3a} = -3a/2$$	
Since ρ is positive,	
$$\rho = \frac{3a}{2}$$	

Radius of Curvature – Polar coordinates

Polar form of radius of curvature is

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2}$$

where $r_1 = \dfrac{dr}{d\theta}, r_2 = \dfrac{d^2r}{d\theta^2}$

1. Show that the radius of curvature of $r = a(1 + \cos\theta)$ is $\dfrac{2\sqrt{2ar}}{3}$

Solution

Given $r = a(1 + \cos\theta)$

$\Rightarrow r_1 = -a\sin\theta$ and $r_2 = -a\cos\theta$

$\therefore r^2 + r_1^2 = 2a^2(1 + \cos\theta)$

$r^2 + 2r_1^2 - rr_2 = 3a^2(1 + \cos\theta)$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2}$$

$$= \frac{2\sqrt{2}a^3(1 + \cos\theta)^{3/2}}{3a^2(1 + \cos\theta)}$$

$$= \frac{2\sqrt{2}a^3(1 + \cos\theta)^{3/2}}{3a^2(1 + \cos\theta)}$$

$$= \frac{2\sqrt{2}\sqrt{a}(a(1 + \cos\theta))^{1/2}}{3} = \frac{2\sqrt{2ar}}{3}$$

2. Show that the radius of curvature at the point (r, θ) of the curve $r^2 \cos 2\theta = a^2$ is $\dfrac{r^3}{a^2}$

Solution:

Given $r^2 \cos 2\theta = a^2 \Rightarrow r^2 = a^2 \sec 2\theta$

$\Rightarrow 2rr_1 = 2a^2 \sec 2\theta \tan 2\theta$

$\Rightarrow r_1 = r \tan 2\theta$

and $r_2 = 2r \sec^2 2\theta + r \tan^2 2\theta$ $(\because r_1 = r \tan 2\theta)$

Now

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2}$$

$$= \frac{(r^2 + r^2 tan^2 2\theta)^{3/2}}{2r^2 \tan^2 2\theta + r^2 - r^2(2\sec^2 2\theta + \tan^2 2\theta)}$$

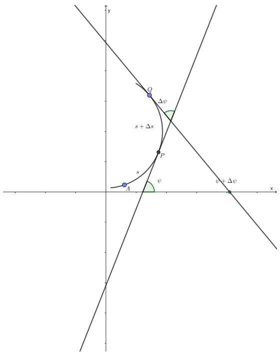
$$= \frac{r^3 \sec^3 2\theta}{r^2 \sec^2 2\theta}$$

$$= r\frac{r^2}{a^2} = \frac{r^3}{a^2}$$

Centre of curvature and Circle of curvature

Let Γ be a simple curve having tangent at each point. At any point P on this curve we can draw a circle having the same curvature at P as the curve Γ.

This circle is called the circle of curvature and its centre is called the centre of curvature and its radius is the radius of curvature at p.



Draw the normal at P to the curve Γ in the direction of the concavity of the curve (which is positive direction of the normal) and cut off a segment $PC = \rho$, the radius of curvature of Γ at P. The point C is called the centre of curvature of the given curve at P. The circle with centre C and radius P (passing through P) is called the circle of curvature of the given curve at P.

Note:

1. From the definition of circle of curvature it follows that at the given point, the curvature of the curve and curvature of the circle are same

2.It is quite possible that the circle of curvature at a point crosses the curve as in the figure just as a tangent line crosses the curve at the point of inflexion.

Problems	**Solving Tip!**
1. Find the circle of curvature at $(3, 4)$ on $xy = 12$. **Solution** The given curve is $xy = 12 \Rightarrow y = \frac{12}{x}$ $\frac{dy}{dx} = -\frac{12}{x^2}$ and $\frac{d^2y}{dx^2} = \frac{24}{x^3}$ At the point $(3, 4)$, $y_1 = \frac{dy}{dx} = -\frac{4}{3}$ $y_2 = \frac{d^2y}{dx^2} = \frac{8}{9}$ The centre of curvature $(\overline{x}, \overline{y})$ is given by $\overline{x} = x - \frac{y_1(1+y_1^2)}{y_2} = 3 + \frac{4}{3}\frac{25}{8} = \frac{43}{6}$ $\overline{y} = y + \frac{(1+y_1^2)}{y_2} = 4 + \frac{1+\frac{16}{9}}{\frac{8}{9}} = \frac{57}{8}$ $\therefore$ the centre of curvature is $(\overline{x}, \overline{y})$=$\left(\frac{43}{6}, \frac{57}{8}\right)$ and the radius of curvature is $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{125}{24}$ $\therefore$ the equation of the circle of curvature at $(3, 4)$ is $(x - \overline{x})^2 + (y - \overline{y})^2 = \rho^2$ $\Rightarrow (x - \frac{43}{6})^2 + (y - \frac{57}{8})^2 = \left(\frac{125}{24}\right)^2$	

Problems	**Solving Tip!!!**
2. Find the circle of curvature at $(\frac{a}{4}, \frac{a}{4})$ on $\sqrt{x}+\sqrt{y}=\sqrt{a}$ **Solution** The given curve is $\sqrt{x}+\sqrt{y}=\sqrt{a} \Rightarrow \sqrt{y}=\sqrt{a}-\sqrt{x}$ $\frac{1}{2\sqrt{y}}\frac{dy}{dx}=-\frac{1}{2\sqrt{x}} \Rightarrow \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}=-\frac{(\sqrt{a}-\sqrt{x})}{\sqrt{x}}=-\frac{\sqrt{a}}{\sqrt{x}}+1$ and $\frac{d^2y}{dx^2}=\frac{\sqrt{a}}{2x\sqrt{x}}$ At the point $(\frac{a}{4}, \frac{a}{4})$, $y_1=\frac{dy}{dx}=-1$ $y_2=\frac{d^2y}{dx^2}=\frac{4}{\sqrt{a}}$ The centre of curvature $(\overline{x}, \overline{y})$ is given by $\overline{x}=x-\frac{y_1(1+y_1^2)}{y_2}=\frac{a}{4}+\frac{a}{4}(1+1)=\frac{3a}{4}$ $\overline{y}=y+\frac{(1+y_1^2)}{y_2}=\frac{a}{4}+\frac{1+1}{\frac{4}{a}}=\frac{3a}{4}$ $\therefore$ the centre of curvature is $(\overline{x}, \overline{y})$=$\left(\frac{3a}{4}, \frac{3a}{4}\right)$ and the radius of curvature is $\rho=\frac{(1+y_1^2)^{3/2}}{y_2}=\frac{a}{\sqrt{2}}$ $\therefore$ the equation of the circle of curvature at $(3,4)$ is $(x-\overline{x})^2+(y-\overline{y})^2=\rho^2$ $\Rightarrow \left(x-\frac{3a}{4}\right)^2+\left(y-\frac{3a}{4}\right)^2=\left(\frac{a}{\sqrt{2}}\right)^2$	

Problems	**Solving Tip!!!**
3. Find the circle of curvature at $(\frac{a}{4}, \frac{a}{4})$ on $\sqrt{x}+\sqrt{y}=\sqrt{a}$ **Solution** The given curve is $\sqrt{x}+\sqrt{y}=\sqrt{a} \Rightarrow \sqrt{y}=\sqrt{a}-\sqrt{x}$ $\frac{1}{2\sqrt{y}}\frac{dy}{dx}=-\frac{1}{2\sqrt{x}} \Rightarrow \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}=-\frac{(\sqrt{a}-\sqrt{x})}{\sqrt{x}}=-\frac{\sqrt{a}}{\sqrt{x}}+1$ and $\frac{d^2y}{dx^2}=\frac{\sqrt{a}}{2x\sqrt{x}}$ At the point $(\frac{a}{4}, \frac{a}{4})$, $y_1=\frac{dy}{dx}=-1$ $y_2=\frac{d^2y}{dx^2}=\frac{4}{\sqrt{a}}$ The centre of curvature $(\overline{x}, \overline{y})$ is given by $\overline{x}=x-\frac{y_1(1+y_1^2)}{y_2}=\frac{a}{4}+\frac{a}{4}(1+1)=\frac{3a}{4}$ $\overline{y}=y+\frac{(1+y_1^2)}{y_2}=\frac{a}{4}+\frac{1+1}{\frac{4}{a}}=\frac{3a}{4}$ $\therefore$ the centre of curvature is $(\overline{x}, \overline{y})$=$\left(\frac{3a}{4}, \frac{3a}{4}\right)$ and the radius of curvature is $\rho=\frac{(1+y_1^2)^{3/2}}{y_2}=\frac{a}{\sqrt{2}}$ $\therefore$ the equation of the circle of curvature at $(3,4)$ is $(x-\overline{x})^2+(y-\overline{y})^2=\rho^2$ $\Rightarrow \left(x-\frac{3a}{4}\right)^2+\left(y-\frac{3a}{4}\right)^2=\left(\frac{a}{\sqrt{2}}\right)^2$	

Problems	Solving Tip!!!
4. Find the circle of curvature at the point P on the curve $y = e^x$ where the curve crosses the y-axis. **Solution** The given curve is $y = e^x$ $\dfrac{dy}{dx} = e^x$ and $\dfrac{d^2y}{dx^2} = e^x$ Also given P is the point on the Y-axis where the curve crosses it. Equation of y-axis is $x = 0$. $\therefore\ y = e^0 = 1$ $\therefore\ P$ is $(0, 1)$ At P, $y_1 = \dfrac{dy}{dx} = 1$ and $y_2 = \dfrac{d^2y}{dx^2} = 1$ At P, the coordinates of the centre of curvature is given by $\overline{x} = x - \dfrac{y_1(1+y_1^2)}{y_2} = 0 - \dfrac{1(1+1)}{1} = -2$ $\overline{y} = y + \dfrac{(1+y_1^2)}{y_2} = 1 + \dfrac{1+1}{1} = 3$ $\therefore$ the centre of curvature is $(\overline{x}, \overline{y})$=$(-2, 3)$ and the radius of curvature is $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$ $= \dfrac{(1+(1)^2)^{3/2}}{1} = 2\sqrt{2}$ $\therefore$ the equation of the circle of curvature at $(3, 4)$ is $(x-\overline{x})^2 + (y-\overline{y})^2 = \rho^2$ $\Rightarrow (x+2)^2 + (y-3)^2 = 8$	

Evolute

The locus of centre of curvature of a given curve Γ is called the evolute of the curve. The given curve Γ is called involute of the evolute.

Procedure:

Step1: Write the given curve as $y = f(x) - - - -(1)$

Step2: Find the centre of curvature $(\overline{x}, \overline{y})$ are given by

$$\overline{x} = x - \frac{y_1(1+y_1^2)}{y_2} - - - -(2)$$

$$\overline{y} = y + \frac{(1+y_1^2)}{y_2} - - - (3)$$

Step3: Eliminating x, y using $(1), (2)$ and (3), we get the relation in $\overline{x}, \overline{y}$,

Replacing $\overline{x}, \overline{y}$, we get the equation of locus of $(\overline{x}, \overline{y})$ which is the evolute of the given curve

The process of elimination of x and y would become simpler if the point (x, y) is taken in terms of a parameter t.

1. Find the equation of the evolute of the parabola $y^2 = 4ax$.

Solution

Given $y^2 = 4ax - - - (1)$

Let $P(at^2, 2at)$ be any point on the parabola.

Differentiating with respect to x,

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$\frac{d^2y}{dx^2} = \frac{-2a}{y^2}.\frac{dy}{dx} = \frac{-4a^2}{y^3}$$

At the point $(at^2, 2at)$,

$$y_1 = \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{-4a^2}{(2at)^3} = \frac{-1}{2at^3}$$

$\therefore$ The centre of curvature $(\overline{x}, \overline{y})$ at P is given by,

$$\overline{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= at^2 - \frac{\frac{1}{t}\left(1+\frac{1}{t^2}\right)}{\frac{-1}{2at^3}}$$

$\overline{x} = at^2 + 2a(1+t^2)$

$= at^2 + 2a + 2at^2$

$= 3at^2 + 2a - - - -(2)$

$\overline{y} = y + \dfrac{1+y_1^2}{y_2}$

$= 2at + \dfrac{1+\dfrac{1}{t^2}}{-\dfrac{1}{2at^3}}$

$= 2at - \left(\dfrac{2at^3}{1}\right)\left[1+\dfrac{1}{t^2}\right] = 2at - 2at^3\left[1+\dfrac{1}{t^2}\right]$

$\overline{y} = 2at - \left[2at^3 + 2at^3\dfrac{1}{t^2}\right]$

$= 2at - [2at^3 + 2at]$

$= 2at - 2at^3 + 2at$

$\overline{y} = -2at^3 - - - (4)$

Eliminating t from (2) and (3)

$(2) \Rightarrow 3at^2 = \overline{x} - 2a$

$$\Rightarrow t^2 = \frac{\overline{x} - 2a}{3a}$$

$$\Rightarrow t = \left[\frac{\overline{x} - 2a}{3a}\right]^{1/2}$$

Substitute (3), we get

$$(3) \Rightarrow \overline{y} = -2a.\left[\left[\frac{\overline{x} - 2a}{3a}\right]^{1/2}\right]^3$$

Squaring on both sides,

$$\Rightarrow \overline{y}^2 = \frac{4a^2}{27a^3}\left[\overline{x} - 2a\right]^3$$

$$\Rightarrow \overline{y}^2 = \frac{4}{27a}\left[\overline{x} - 2a\right]^3$$

$$\Rightarrow 27a\overline{y}^2 = 4\left(\overline{x} - 2a\right)^3$$

$\therefore$ the locus of $(\overline{x}, \overline{y})$ is $27ay^2 = 4\left(x - 2a\right)^3$, which is the equation of the evolute of the parabola.

2. Find the equation of the evolute of the parabola $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

Solution

Let $P(a\cos\theta, a\sin\theta)$ be any point on the ellipse.

Given $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 - - - (1)$

Differentiating (1) with respect to x,

$$2\frac{x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 4a$$

$$\frac{2y}{b^2}\frac{dy}{dx} = -2\frac{x}{a^2}$$

$$\frac{dy}{dx} = -2\frac{x}{a^2}\frac{b^2}{2y}$$

$$= -\frac{b^2}{a^2}\frac{x}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\frac{y.1 - x.\dfrac{dy}{dx}}{y^2}$$

At the point $(a\cos\theta, a\sin\theta)$,

$$y_1 = \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{a\cos\theta}{b\sin\theta} = -\frac{b}{a}\frac{\cos\theta}{\sin\theta}$$

$$y_2 = \frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left[\frac{b\sin\theta - a\cos\theta\left[\dfrac{-b}{a}\dfrac{\cos\theta}{\sin\theta}\right]}{b^2\sin^2\theta}\right]$$

$$= -\frac{b^2}{a^2}\left[\frac{b\sin\theta + \dfrac{b\cos^2\theta}{\sin\theta}}{b^2\sin^2\theta}\right]$$

$$= -\frac{b^2}{a^2}\left[\frac{\dfrac{b\sin^2\theta + b\cos^2\theta}{\sin\theta}}{b^2\sin^2\theta}\right]$$

$$= -\frac{b^2}{a^2}\left[\frac{b\,(\sin^2\theta + \cos^2\theta)}{b^2\sin^3\theta}\right]$$

$$= -\frac{b}{a^2}\frac{1}{\sin^3\theta}$$

$\therefore$ The centre of curvature $(\overline{x}, \overline{y})$ at P is given by,

$$\overline{x} = x - \frac{y_1(1+y_1^2)}{y_2} \quad - - - (2)$$

$$= a\cos\theta - \frac{\left(\dfrac{-b}{a}\dfrac{\cos\theta}{\sin\theta}\right)\left[1+\dfrac{b^2}{a^2}\dfrac{\cos^2\theta}{\sin^2\theta}\right]}{-\dfrac{b}{a^2}.\dfrac{1}{\sin^3\theta}}$$

$$= a\cos\theta + \frac{a^2}{b}\frac{\sin^3\theta}{1}\left[\frac{-b}{a}\frac{\cos\theta}{\sin\theta}\right]\left(1+\frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$= a\cos\theta - a\cos\theta\sin^2\theta\left(1+\frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$= a\cos\theta - a\cos\theta\sin^2\theta - a\cos\theta\sin^2\theta\frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}$$

$$= a\cos\theta(1-\sin^2\theta) - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos^3\theta - \frac{b^2}{a}\cos^3\theta$$

$$\overline{x} = \frac{a^2 - b^2}{a}\cos^3\theta$$

$$\overline{y} = y + \frac{1 + y_1^2}{y_2}$$

$$= b\sin\theta + \frac{\left[1 + \frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right]}{-\frac{b}{a^2}.\frac{1}{\sin^3\theta}}$$

$$= b\sin\theta - \frac{a^2\sin^3\theta}{b}\left[1 + \frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right]$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - \frac{a^2}{b}\sin^3\theta\frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\sin\theta\cos^2\theta$$

$$= b\sin\theta - b\sin\theta\cos^2\theta - \frac{a^2}{b}\sin^3\theta$$

$$= b\sin\theta(1 - \cos^2\theta) - \frac{a^2}{b}\sin^3\theta$$

$$= b\sin^3\theta - \frac{a^2}{b}\sin^3\theta$$

$$\overline{y} = -\frac{a^2 - b^2}{b}\sin^3\theta - - - (3)$$

Eliminating θ from (2) and (3)

$(2) \Rightarrow \dfrac{a\overline{x}}{a^2-b^2} = \cos^3\theta$

$\Rightarrow \cos\theta = \left[\dfrac{a\overline{x}}{a^2-b^2}\right]^{1/3}$

$(3) \Rightarrow \sin\theta = \left[\dfrac{-b\overline{y}}{a^2-b^2}\right]^{1/3}$

We know that $\cos^2\theta + \sin^2\theta = 1$,

$\Rightarrow \left[\dfrac{a\overline{x}}{a^2-b^2}\right]^{2/3} + \left[\dfrac{-b\overline{y}}{a^2-b^2}\right]^{2/3} = 1$

$\Rightarrow \left[\dfrac{a\overline{x}}{a^2-b^2}\right]^{2/3} + \left[\dfrac{b\overline{y}}{a^2-b^2}\right]^{2/3} = 1$

$\Rightarrow (a\overline{x})^{2/3} + (b\overline{y})^{2/3} = (a^2-b^2)^{2/3}$

$\therefore$ the locus of $(\overline{x}, \overline{y})$ is $(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$, which is the equation of the evolute of the given ellipse.

3. Find the evolute of the rectangular hyperbola $xy = c^2$ Solution: Given $xy = c^2 - - - -(1)$

Let $P(ct, \frac{c}{t})$ be any point on (1)

$(1) \Rightarrow y = \frac{c^2}{x}$

$\therefore \frac{dy}{dx} = -\frac{c^2}{x^2}$

and

$\frac{d^2y}{dx^2} = \frac{2c^2}{x^3}$

At the point $(ct, \frac{c}{t})$

$y_1 = \frac{dy}{dx} = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2}$

and $y_2 = \frac{d^2y}{dx^2} = \frac{2c^2}{c^3t^3} = \frac{2}{ct^3}$

The centre of curvature $(\overline{x}, \overline{y})$ at P is given by

$\overline{x} = x - \frac{y_1(1 + y_1^2)^{3/2}}{y_2}$

$= ct - \frac{-\frac{1}{t^2}\left(1 + \frac{1}{t^4}\right)}{\frac{2}{ct^3}}$

$= ct + \frac{ct}{2}\left(1 + \frac{1}{t^4}\right)$

$= \frac{3ct}{2} + \frac{c}{2t^3} - - - (2)$

$\overline{y} = y + \frac{1 + y_1^2}{y_2}$

$= \frac{c}{t} + \frac{1 + \frac{1}{t^4}}{\frac{2}{ct^3}}$

$$= \frac{c}{t} + \frac{c}{2t}\,(t^4+1)$$

$$= \frac{3c}{2t} + \frac{ct^4}{2t}$$

$$= \frac{c}{2t}(3+t^4)$$

$$= \frac{c}{2t^3}(3t^2+t^6) - - - (3)$$

$$\therefore \overline{x} + \overline{y} = \frac{c}{2t^3}\,[3t^4+1+3t^2+t^6]$$

$$= \frac{c}{2t^3}\,[1+3t^2+3t^4+t^6]$$

$$= \frac{c}{2t^3}\,[1+t^2]^3$$

$$= \frac{c}{2}\left[\frac{1+t^2}{t}\right]^3$$

$$(\overline{x}+\overline{y})^{1/3} = \left(\frac{c}{2}\right)^{1/3}\left[\frac{1+t^2}{t}\right] - - - - (4)$$

$$\therefore \overline{x} - \overline{y} = \frac{c}{2t^3}\,[3t^4+1-3t^2-t^6]$$

$$= \frac{c}{2t^3}\,[1-3t^2+3t^4-t^6]$$

$$= \frac{c}{2t^3}\,[1-t^2]^3$$

$$= \frac{c}{2}\left[\frac{1-t^2}{t}\right]^3$$

$$(\overline{x}-\overline{y})^{1/3} = \left(\frac{c}{2}\right)^{1/3}\left[\frac{1-t^2}{t}\right] - - - (5)$$

Eliminating t from (4) and (5), we get the equation of the evolute

$$(\overline{x}+\overline{y})^{2/3}-(\overline{x}-\overline{y})^{2/3}=\left(\frac{c}{2}\right)^{2/3}\left[\left[\frac{1+t^2}{t}\right]-\left[\frac{1-t^2}{t}\right]^2\right]$$

$$=\left(\frac{c}{2}\right)^{2/3}\left[\left[\frac{1+t^2}{t}\right]-\left[\frac{1-t^2}{t}\right]^2\right]$$

$$=\left(\frac{c}{2}\right)^{2/3}\left[\frac{\left(1+t^2\right)^2-\left(1-t^2\right)^2}{t^2}\right]$$

$$=\left(\frac{c}{2}\right)^{2/3}\left[\frac{4t^2}{t^2}\right]$$

$$=\left(\frac{c}{2}\right)^{2/3}[4]$$

$$=\left(\frac{c}{2}\right)^{2/3}2^2$$

$$=\left(c^{2/3}\right)2^{-2/3}2^2$$

$$=\left(c^{2/3}\right)2^{2-2/3}$$

$$=\left(c^{2/3}\right)2^{4/3}$$

$$\Rightarrow (\overline{x}+\overline{y})^{2/3}-(\overline{x}-\overline{y})^{2/3}=(4c)^{2/3}$$

$\therefore$ the locus of $(\overline{x},\overline{y})$ is

$$(x+y)^{2/3}-(x-y)^{2/3}=(4c)^{2/3}$$

which is the equation of evolute.

4. Show that the evolute of the cycloid $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ is another cycloid.

Solution

Let P $'\theta'$ be any point on the cycloid.

Given $x = a(\theta - \sin\theta)$

$$\therefore \frac{dx}{d\theta} = a(1 - \cos\theta) = 2a\sin^2\frac{\theta}{2}$$

$$\therefore \frac{dy}{d\theta} = a\sin\theta = 2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\therefore \frac{dy}{dx} = \frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2a\sin^2\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

and $\dfrac{d^2y}{dx^2} = \dfrac{d}{d\theta}\left(\cot\dfrac{\theta}{2}\right)\dfrac{d\theta}{dx}$

$$= -\text{cosec}^2\frac{\theta}{2}\frac{1}{2}\frac{1}{2a\sin^2\frac{\theta}{2}}$$

$$= \frac{-\text{cosec}^4\frac{\theta}{2}}{4a}$$

$\therefore y_1 = \cot\dfrac{\theta}{2}$ and $y_2 = -\dfrac{\text{cosec}^4\frac{\theta}{2}}{4a}$

$\therefore$ The centre of curvature $(\overline{x}, \overline{y})$ at θ is given by

$$\overline{x} = x - \frac{y_1\left(1+y_1^2\right)}{y_2}$$

$$= a(\theta - \sin\theta) - \cot\frac{\theta}{2}\frac{\left(1+\cot^2\frac{\theta}{2}\right)}{\frac{-\text{cosec}^4\frac{\theta}{2}}{4a}}$$

$$= a(\theta - \sin\theta) + \frac{4a\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\frac{\text{cosec}^2\frac{\theta}{2}}{\text{cosec}^4\frac{\theta}{2}}$$

$$= a(\theta - \sin\theta) + 4a\cos\frac{\theta}{2}\sin\theta 2$$

$$= a(\theta - \sin\theta) + 2a\sin\theta$$

$$= a\theta + a\sin\theta$$

$$\overline{x} = a(\theta + \sin\theta) - - - (2)$$

and $\overline{y} = y + \dfrac{1+y_1^2}{y_2}$

$$= a(1-\cos\theta) + \frac{1+\cot^2\dfrac{\theta}{2}}{\dfrac{-\text{cosec}^4\dfrac{\theta}{2}}{4a}}$$

$$= a(1-\cos\theta) - 4a\frac{\text{cosec}^2\dfrac{\theta}{2}}{-\text{cosec}^4\dfrac{\theta}{2}}$$

$$= a(1-\cos\theta) - 4a\sin^2\frac{\theta}{2}$$

$$= a\ 2\sin^2\frac{\theta}{2} - 4a\sin^2\frac{\theta}{2}$$

$$= 2a\sin^2\frac{\theta}{2} - 4a\sin^2\frac{\theta}{2}$$

$$= -2a\sin^2\frac{\theta}{2}$$

$$\overline{y} = -a(1-\cos\theta) - - - (3)$$

Eliminating θ from (2) and (3) is very difficult.

$\therefore$ the locus of $(\overline{x}, \overline{y})$ is given by the parametric equation

$$x = a(\theta+\sin\theta),\, y = -a(1-\cos\theta)$$

which is another cycloid.

Envelope

Consider the system of straight lines $y = mx + \dfrac{1}{m} - - - (1)$, where m is a parameter. For different values of m, we have different straight lines and so (1) represents a family of straight lines. Each member of this family touches the curve $y^2 = 4x$. This curve is called the envelope of the family of lines.

Definition:

Let $f(x, y, \alpha) = 0$ be a single parameter family of curves, where α is the parameter. The envelope of this family of curves is a curve which touches every member of the family.

Methods for finding envelope of single parameter

1.

Step1: Write the given curves $f(x, y, \alpha) = 0 - - - (1)$

Step2: Find $\dfrac{\partial}{\partial \alpha}(f(x, y, \alpha)) = 0 - - - (2)$

Step3: Eliminate α from (1) and (2). The eliminant, if exists, is an equation of x and y. It is the envelope of the family (1)

2. From (1) and (2) solve for x and y in terms of α. It will give the parametric representation of the envelope

3.If the equation of the family of curves(1) can be written in the form $A\alpha^2 + B\alpha + C = 0$, where A, B and C are functions of x and y, then $B^2 - 4AC = 0$ is an equation of envelope.

Note:

1. A point $P(a, b)$ is a singular point of a curve

$f(x, y, \alpha) = 0 - - - (1)$ (α is fixed) if it satisfies (1) and $\dfrac{\partial f}{\partial x} - - - (2)$ and $\dfrac{\partial f}{\partial y} - - - (3)$. P is said to be an ordinary point if atleast one of (2) and (3) is not satisfied.

2. The characteristic points of the family of curves $f(x, y, \alpha) = 0 - - - (1)$ are those ordinary points of the family where the equations $f(x, y, \alpha) = 0$, $\dfrac{\partial f(x, y, \alpha)}{\partial \alpha} = 0$ simultaneously hold. Characteristic points are isolated on each curve. In fact, the envelope of a family of curves $f(x, y, \alpha) = 0$, α is a parameter, is the locus of their isolated characteristic points.

3. Not every singular parameter family has envelope. For example, the family is concentric circles $x^2 + y^2 = \alpha^2$ has no envelope, as there is no characteristic point.

Problem	**Solving Tip!!!**
1.Find the envelope of the family of straight lines $y = mx + \dfrac{1}{m}$ **Solution:** $y = mx + \dfrac{1}{m}$ $\Rightarrow y = \dfrac{m^2x+1}{m}$ $\Rightarrow my = m^2x + 1$ $\Rightarrow m^2x - my + 1 = 0$ $\Rightarrow xm^2 - ym + 1 = 0,$ which is the quadratic in the parameter m Here $A = x$, $B = -y$ $C = 1$ $\therefore$ the envelope is $B^2 - 4AC = 0$ $\Rightarrow y^2 - 4x = 0$ $\Rightarrow y^2 = 4x$ **2.Find the envelope of the straight lines represented by** $x \cos\alpha + y \sin\alpha = a \sec\alpha$, **where** α **is the parameter.** **Solution:** Given $x \cos\alpha + y \sin\alpha = a \sec\alpha$ dividing by $\cos\alpha$ $\Rightarrow a \tan^2\alpha - y \tan\alpha + (a - x) = 0$ which is quadratic in $\tan\alpha$ Here $A = a$, $B = -y$ and $C = a - x$ $\therefore$ the envelope is $B^2 - 4AC = 0$ $\Rightarrow y^2 - 4a(a - x) = 0,$ which is the envelope.	

Envelope of two parameter family of curves

1. Let $f(x, y, \alpha, \beta) = 0 - - - (1)$

be a two parameter family of curves, where α and β are parameters such that $\phi(\alpha, \beta) = 0 - - - (2)$

Find β in terms of α from (2) and substitute in (1) and thus the problem is reduced to one parameter family and proceed as above.

2. The following method is more convenient in many cases. For a fixed point (x,y) of the envelope treating β as a function of α differentiate (1) and (2) with respect to α and eliminate α and β.

The eliminate α and β

The eliminant gives the envelope.

1. Find the envelope of the family of straight lines $\frac{x}{a} + \frac{y}{a} = 1$, where $ab = c^2$, a, b are parameters.

Solution:

Given $\frac{x}{a} + \frac{y}{b} = 1 - - - (1)$

and $ab = c^2 - - - -(2)$

$(2) \Rightarrow b = \frac{c^2}{a}$

Substituting in (1), $\frac{x}{a} + \frac{y}{\frac{c^2}{a}} = 1$

$\Rightarrow \frac{x}{a} + \frac{1}{c^2}ay = 1$

$\Rightarrow c^2x + a^2y = c^2a$

$\Rightarrow a^2y - c^2a + c^2x = 0$

which is quadratic

Here $A = y$, $B = -c^2$ and $C = c^2x$

$\therefore$ the envelope is $B^2 - 4AC = 0$

$\Rightarrow\ c^4 - 4yc^2x = 0 \Rightarrow\ 4yx = c^2$

2. Find the envelope of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a^n+b^n=c^n$, a and b are the parameters and c is a constant.

Solution:

Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1---(1)$

$a^n+b^n=c^n$

Differentiating (1) with respect to a (treating b as a function of a)

$$\therefore x^2\left(-\frac{2}{a^3}\right)+\left(\frac{-2}{b^3}\right)\frac{db}{da}=0$$

$$\frac{db}{da}=-\frac{b^3}{a^3}\frac{x^2}{y^2}---(3)$$

and $na^{n-1}+nb^{n-1}\frac{db}{da}=0$

$$\Rightarrow \frac{db}{da}=-\frac{a^{n-1}}{b^{n-1}}---(4)$$

From (3) and (4), $-\frac{b^3x^2}{a^3y^2}=-\frac{a^{n-1}}{b^{n-1}}$

$$\frac{x^2}{a^{n+2}}=\frac{y^2}{b^{n+2}}$$

$$\frac{\frac{x^2}{a^2}}{a^n}=\frac{\frac{y^2}{b^2}}{b^n}$$

$$\frac{\frac{x^2}{a^2}}{a^n}=\frac{\frac{y^2}{b^2}}{b^n}=\frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}}{a^n+b^n}$$

$\because$ each ratio$=\frac{\text{sum of numerator}}{\text{sum of denominator}}$ and using (1)&(2)

$$\frac{\frac{x^2}{a^2}}{a^n} = \frac{\frac{y^2}{b^2}}{b^n} = \frac{1}{c^n}$$ $$\Rightarrow \frac{x^2}{a^{n+2}} = \frac{1}{c^n}$$ $\Rightarrow a^{n+2} = c^n x^2 \Rightarrow a = (c^n x^2)^{1/(n+2)}$ similarly$b = (c^n y^2)^{1/(n+2)}$ Substituting in (2), we get $(c^n x^2)^{n/(n+2)} + (c^n y^2)^{n/(n+2)} = c^n$ $$c^{\frac{n^2}{n+2}}\left[x^{\frac{2n}{n+2}} + y^{\frac{2n}{n+2}}\right] = c^n$$ $$x^{\frac{2n}{n+2}} + y^{\frac{2n}{n+2}} = \frac{c^n}{c^{\frac{n^2}{n+2}}}$$ $$x^{\frac{2n}{n+2}} + y^{\frac{2n}{n+2}} = c^{n-\frac{n^2}{n+2}}$$ $$x^{\frac{2n}{n+2}} + y^{\frac{2n}{n+2}} = c^{\frac{n^2+2n-n^2}{n+2}}$$ $$x^{\frac{2n}{n+2}} + y^{\frac{2n}{n+2}} = c^{\frac{2n}{n+2}}$$ which is the required envelope	In R.H.S, take Dr. to Nr.

The Gamma and Beta functions

Improper Integrals

Consider the integral $\int_a^b f(x)dx$ in $a \leq x \leq b$

The integral is called an improper integral if

(i) either $a \longrightarrow -\infty$ or $b \longrightarrow +\infty$ or both, in which case it is called an improper integral of first kind.

(ii) f(x) is unbounded at one or more points of the interval of integration, in which case it is called an improper integral of second kind.

Integral which is both first and second kind is called an improper integral of third kind

Example:

Improper integrals of first kind	$\int_\infty^0 e^x dx,\ \int_\infty^0 \cos hx dx,\ \int_{-\infty}^\infty \frac{dx}{1+x^2}$
Improper integrals of second kind	$\int_{-1}^1 \frac{1}{\sqrt{x}} dx,\ \int_1^7 \frac{dx}{x^2-4} dx$

The Gamma Function

The Gamma function is defined as the definite integral

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \text{for } n > 0$$

The improper integral can be proved to be convergent for $n > 0$

Hence the integral is a function of n.

Recurrence formula for $\Gamma(n)$	
By definition,	
$\Gamma(n) = \int\limits_0^\infty e^{-x}x^{n-1}dx \text{for } n > 0$	
$\therefore \Gamma(n+1) = \int\limits_0^\infty e^{-x}x^{n}dx \text{for } n > -1$	$\because n+1 > 0 \Rightarrow n > -1$
$= \int\limits_0^\infty x^n e^{-x}dx$	$\because \dfrac{d}{dx}\left(-e^{-x}\right) = e^{-x}$
$= \int\limits_0^\infty x^n d\left(-e^{-x}\right)$	$\because d\left(-e^{-x}\right) = e^{-x}dx$
$= \left[-e^{-x}x^n\right]_0^\infty + \int\limits_0^\infty nx^{n-1}e^{-x}dx$	$\int\limits_a^b udv = [uv]_a^b - v\int\limits_a^b du$
$= 0 + n\int\limits_0^\infty x^{n-1}e^{-x}dx$	For $n > 0$, both x^n e^{-x} vanishes at both limits
$= n\Gamma(n)$	By definition.
Cor 1:	
$\Gamma(n+1) = n!$, if n is a positive integer.	
Cor 2:	
$\Gamma(n+a) = (n+a-1)(n+a-2)\cdots a\Gamma(a)$, where n is a positive integer.	

The Beta functions

The Beta function denoted by B(m,n) with parameters m,n is defined as

$$B(m,n) = \int_0^1 = x^{m-1}(1-x)^{m-1}dx \text{ for } m,n > 0$$

The improper integral can be proved to be convergent for $m,n > 0$

Property of Beta function

1.B(m,n)=B(n,m)

Other form of Beta function

1.$B(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy$

2.$B(m,n) = 2\int_0^\infty \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$

Relation between Beta and Gamma Functions

$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$\Gamma(m) = \int_0^\infty e^{-x}x^{m-1}dx$

put $x = t^2$

$\Gamma(m) = 2\int_0^\infty e^{-t^2}t^{2m-1}dt$

$\Gamma(n) = 2\int_0^\infty e^{-t^2}t^{2n-1}dt$

i.e.,$\Gamma(m) = 2\int_0^\infty e^{-x^2}x^{2m-1}dx$

$\Gamma(n) = 2\int_0^\infty e^{-y^2}y^{2n-1}dy$

$$\Gamma(m)\Gamma(n) = 4\left(\int_0^\infty e^{-x^2} x^{2m-1} dx\right)\left(\int_0^\infty e^{-y^2} y^{2n-1} dy\right)$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dxdy$$

$$= 4\iint_A e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dxdy$$

where A is the area of integration, namely the first quadrant. Changing to polar coordinates

$x = r\cos\theta,\ y = r\sin\theta$

$x^2 + y^2 = r^2$ and $dxdy = rdrd\theta$

The limits for r are 0 to ∞ and limits for 0 to $\pi/2$

$$\therefore \Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^{\pi/2} e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} rdrd\theta$$

$$= \left(2\int_0^\infty e^{-r^2} r^{2(m+n)-1} dr\right)\left(2\int_0^{\pi/2} \cos^{2m-1}\theta(r\sin^{2n-1}\theta d\theta\right)$$

$$= \Gamma(m+n)B(m,n)$$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Problems

1. Evaluate $\int\limits_0^\infty e^{-x^2}dx$ **and prove** $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Solution: Put $x^2 = t$ $2xdx = dt \Rightarrow dx = \dfrac{dt}{2\sqrt{t}}$

$$\therefore \int\limits_0^\infty e^{-x^2}dx = \int\limits_0^\infty e^{-t}.\frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int\limits_0^\infty e^{-t}t^{-1/2}dt$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\sqrt{\pi}$$

2.Evaluate $\int\limits_0^1 x^6(1-x)^9dx$

Solution:

$$\int\limits_0^1 x^6(1-x)^9dx = B(7,10)$$

$$= \frac{\Gamma(7)\Gamma(10)}{\Gamma(7)}$$

$$= \frac{6!9!}{16!}$$

3.Evaluate $\int\limits_0^{\pi/2} \sin^6 x\cos^{10}\theta d\theta$

Solution:

$$\int\limits_0^{\pi/2} \sin^6 x\cos^{10}\theta d\theta = \frac{1}{2}B\left(\frac{7}{2},\frac{11}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{7}{2}\right).\Gamma\left(\frac{11}{2}\right)}{\Gamma(9)}$$

$$= \frac{1}{2}.\frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right).\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{8!}$$

$$= \frac{1}{512}.\frac{(225)(63)}{8!}\pi$$

4.Evaluate $I = \int\limits_0^{\pi/2} \sqrt{\tan\theta} d\theta$

Solution

Ans. $\frac{\pi}{\sqrt{2}}$

5. Prove that $\frac{B(m,n+1)}{n} = \frac{B(m+1,n)}{m} = \frac{B(m,n)}{m+n}$

Proof

$$\frac{B(m,n+1)}{n} = \frac{\Gamma(m).\Gamma(n+1)}{n\Gamma(m+n+1) ----(1)}$$

$$= \frac{\frac{\Gamma(m+1)}{m}.n\Gamma(n)}{n\Gamma(m+n+1)} \because \Gamma(n+1) = n\Gamma(n)$$

$$= \frac{\Gamma(m+1).\Gamma(n)}{m\Gamma(m+n+1)}$$

$$= \frac{B(m+1,n)}{m}$$

$$\frac{B(m,n+1)}{n} = \frac{\Gamma(m).n\Gamma(n)}{n(m+n)\Gamma(m+n) ----(1)}$$

$$= \frac{\Gamma(m).\Gamma(n)}{(m+n)\Gamma(m+n)} \because \Gamma(n+1) = n\Gamma(n)$$

$$= \frac{B(m,n)}{m}$$

6. Evaluate $\dfrac{\int\limits_0^{\pi/2} \sin^6\theta\cos^8\theta d\theta}{\int\limits_0^{\pi/2}\sin^6\theta d\theta \int\limits_0^{\pi/2}\cos^8\theta d\theta}$

Solution:

$$\frac{\int\limits_0^{1} \sin^6\theta\cos^8\theta d\theta}{\int\limits_0^{\pi/2}\sin^6\theta d\theta + \int\limits_0^{\pi/2}\cos^8\theta d\theta}$$

$$= \frac{\frac{1}{2}B\left(\frac{7}{2},\frac{9}{2}\right)}{\frac{1}{2}B\left(\frac{7}{2},\frac{1}{2}\right)+\frac{1}{2}B\left(\frac{1}{2},\frac{9}{2}\right)}$$

$$= \frac{\dfrac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{9}{2}\right)}{\Gamma(8)}}{\dfrac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(4)}+\dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{9}{2}\right)}{\Gamma(4)}}$$

$$= \frac{\dfrac{\Gamma\left(\frac{9}{2}\right)}{\Gamma(8)}}{\Gamma\left(\frac{1}{2}\right)\left[\dfrac{1}{\Gamma(4)}+\dfrac{\frac{7}{2}}{\Gamma(5)}\right]} = \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{7!\Gamma\left(\frac{1}{2}\right)\left[\frac{1}{6}+\frac{7}{2\times 24}\right]}$$

$$= \frac{105\times 48}{16\times 7!\times 15} = \frac{1}{240}$$