# MAXWELL'S EQUATIONS



## I → Gauss's Law in Electrostatics :-

$$\oint \vec{D} \cdot \vec{ds} = \int_v \rho_v \, dv \Rightarrow \text{Integral Form.}$$

Applying Divergence Theorem, $\longrightarrow \iint_S$ to $\iiint_V$.

$$\oint \vec{D} \, \vec{ds} = \int_v (\nabla \cdot \vec{D}) \, dv.$$

$$\Rightarrow \boxed{\nabla \cdot \vec{D} = \rho_v} \Rightarrow \text{Point Form (or) Differential Form.}$$

The Divergence of Electric Flux density is equal to Volume charge density, $\rho_v$.

## II → Gauss's Law in Magnetostatics :-

Total Magnetic Flux through any closed surface = 0.

$$\oint_S \vec{B} \cdot \vec{ds} = 0 \Rightarrow \text{Integral Form.}$$

Applying Divergence Theorem,

$$\oint \vec{B} \cdot \vec{ds} = \int_v (\nabla \cdot \vec{B}) \cdot dv = 0.$$

$$\boxed{\nabla \cdot \vec{B} = 0} \Rightarrow \text{Point Form (or) Differential Form.}$$

In a closed surface, the Magnetic flux entering is equal to the Magnetic flux leaving, therefore the net divergence is zero.

## III → Maxwells III - Eqn ⇒ Faraday's Law :-

$$\oint \vec{E}\cdot\vec{dl} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot\vec{ds} \Rightarrow \text{Integral Form.}$$

Applying Stoke's Theorem { Relating Line Integral ($\int$) to Surface Integral ($\iint$) & Vice Versa }

$$\oint_L \vec{E}.\vec{dl} = \iint_S (\nabla\times\vec{E})\,\vec{ds} = -\int \frac{\partial\vec{B}}{\partial t}\,\vec{ds}.$$

$$\Rightarrow \boxed{\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}} \Rightarrow \text{Differential Form.}$$

## IV → Maxwell's IV Eqn ⇒ Modified Ampere's Law :-

$$\oint \vec{H}.\vec{dl} = \iint_S \left(\vec{J_c} + \frac{\partial\vec{D}}{\partial t}\right)\vec{ds} \Rightarrow \text{Integral Form.}$$

Applying Stoke's Theorem,

$$\oint \vec{H}.\vec{dl} = \iint_S (\nabla\times\vec{H}).\vec{ds} = \iint_S \Big(\underbrace{J_c}_{\text{Conduction Current}} + \underbrace{\frac{\partial\vec{D}}{\partial t}}_{\text{Displacement Current.}}\Big)\vec{ds}$$

$$\boxed{\nabla\times\vec{H} = \vec{J_c} + \frac{\partial\vec{D}}{\partial t}} \Rightarrow \text{Differential Form.}$$

The Magnetomotive force around a closed path is equal to the sum of the Conduction Current & displacement current by the path.

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Three Constitutive relations in Field Theory Analysis ⇒

E-Field ← $\vec{D} = \epsilon_e \vec{E}$

H-Field ← $\vec{B} = \mu\vec{H}$

$\vec{J} = \sigma\vec{E}$

# Electromagnetic Wave Equation



① Wave Eqn in E-Field :-

Maxwell's Eqn from Faraday's Law, $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

w.k.t. $\vec{B} = \mu \vec{H}$.

$$\Rightarrow \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \longrightarrow (1)$$

Taking Curl on both sides,

$$\nabla \times \nabla \times \vec{E} = -\mu \, \nabla \times \frac{\partial H}{\partial t} \longrightarrow (2)$$

According to the Identity,

$$\nabla \times \nabla \times E = (\nabla . E) . \nabla - (\nabla . \nabla) E \Rightarrow (\nabla . E) \nabla - \nabla^2 E$$

Since, there is no electric field within the Conductor, $\nabla . E = 0$.

$$\Rightarrow \nabla \times \nabla \times E = -\nabla^2 E . \longrightarrow (3)$$

Maxwells Eqn from Ampere's Law, $\nabla \times \vec{H} = \vec{J}_c + \frac{\partial \vec{D}}{\partial t}$

$$\nabla \times \vec{H} = \sigma E + \varepsilon_e \frac{\partial E}{\partial t} .$$

Differentiating w.r.t $t$.

$$\nabla \times \frac{\partial H}{\partial t} = \sigma \frac{\partial E}{\partial t} + \varepsilon_e \frac{\partial^2 E}{\partial t^2} . \longrightarrow (4)$$

substituting (3) & (4) in (2) we get,

$$-\nabla^2 E = -\mu \left[ \sigma \frac{\partial E}{\partial t} + \varepsilon_e \frac{\partial^2 E}{\partial t^2} \right]$$

$$\boxed{\nabla^2 E - \mu \sigma \frac{\partial E}{\partial t} - \mu \varepsilon_e \frac{\partial^2 E}{\partial t^2} = 0}$$ Wave Equation in terms of E-Field.

(2) wave Eqn in Magnetic field:-

The Maxwell's equation from Ampere's Law is.,

$$\nabla \times \vec{H} = \vec{J}_c + \frac{\partial \vec{D}}{\partial t}.$$

$$\Rightarrow \nabla \times \vec{H} = \sigma E + \varepsilon \frac{\partial E}{\partial t}.$$

Taking Curl on both sides.,

$$\nabla \times \nabla \times H = \sigma (\nabla \times E) + \varepsilon \left[ \nabla \times \frac{\partial E}{\partial t} \right]. \rightarrow (1)$$

According to the Identity.,

$$\nabla \times \nabla \times H = (\nabla . H) . \nabla - (\nabla . \nabla) H \Rightarrow \nabla . (\nabla . H) - \nabla^2 H$$

From Gauss Law, w.k.t $\nabla . B = 0$.

$$\mu [\nabla . H] = 0.$$

$$\nabla . H = 0$$

$$\Rightarrow \nabla \times \nabla \times H = -\nabla^2 H. \rightarrow (2)$$

w.k.t. Maxwells Equation from Faraday's Law is

$$\nabla \times \vec{E} = -\mu \frac{\partial H}{\partial t} \rightarrow (3)$$

Differentiating w.r.t "t".

$$\nabla \times \frac{\partial E}{\partial t} = -\mu \frac{\partial^2 H}{\partial t^2} \rightarrow (4)$$

Substituting (2), (3), (4) in (1) we get.,

$$-\nabla^2 H = -\sigma \mu \frac{\partial H}{\partial t} - \mu \varepsilon \frac{\partial^2 H}{\partial t^2}.$$

$$\boxed{\nabla^2 H - \sigma \mu \frac{\partial H}{\partial t} - \mu \varepsilon \frac{\partial^2 H}{\partial t^2} = 0}$$ Wave Equation in terms of H-Field.

## Wave Equation in Free Space :-

$\Rightarrow \sigma = 0 \; ; \; \rho = 0.$

Conductivity $= 0$      charge density $= 0$.

The wave Equation's Can be written as.

$$\nabla^2 E - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0 \; ; \; \nabla^2 H - \mu\varepsilon \frac{\partial^2 H}{\partial t^2} = 0.$$

For Free space    $\mu = \mu_0 \mu_r \Rightarrow \mu = \mu_0$ $\boxed{\mu_r = 1}$

$\varepsilon = \varepsilon_0 \varepsilon_r \Rightarrow \varepsilon = \varepsilon_0$ $\boxed{\varepsilon_r = 1}$

$$\boxed{\nabla^2 E - \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} = 0 \; ; \; \nabla^2 H - \mu_0 \varepsilon_0 \frac{\partial^2 H}{\partial t^2} = 0}$$

w.k.t. $\dfrac{1}{\sqrt{\mu_0 \varepsilon_0}} = V_0 = 3 \times 10^8 \, m/sec$. ie., $\mu_0 \varepsilon_0 = \dfrac{1}{V_0^2}$

$$\boxed{\begin{aligned} \nabla^2 E - \frac{1}{V_0^2} \frac{\partial^2 E}{\partial t^2} &= 0 \\ \nabla^2 H - \frac{1}{V_0^2} \frac{\partial^2 H}{\partial t^2} &= 0. \end{aligned}}$$

Wave Equations in Free Space.

## Electromagnetic wave Equation in Phasor Form:-

In phasor Form, $\vec{E} = \vec{E}_0 e^{j\omega t}$.

$$\frac{\partial E}{\partial t} = j\omega \vec{E}_0 e^{j\omega t}.$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = j\omega E_0 e^{j\omega t} (j\omega).$$

$$= (j\omega)^2 . E_0 . e^{j\omega t}.$$

$$\boxed{\frac{\partial \vec{E}}{\partial t} = j\omega \vec{E}} \quad \& \quad \boxed{\frac{\partial^2 \vec{E}}{\partial t^2} = (j\omega)^2 \vec{E}}.$$

$$\boxed{\begin{aligned} \therefore \nabla^2 E &= \left[ j\omega\mu(\sigma + j\omega\epsilon) \right] \vec{E} \\ \nabla^2 H &= \left[ j\omega\mu(\sigma + j\omega\epsilon) \right] \vec{H}. \end{aligned}}$$

# WAVE PARAMETERS:-

## 1. Propagation Constant :-

The phasor form of EM wave equation is.,

$$\nabla^2 E = \left[j\omega\varepsilon_e(\sigma + j\omega\varepsilon_e)\right]\vec{E} \quad \rightarrow (1)$$

$$\nabla^2 H = \left[j\omega\mu(\sigma + j\omega\varepsilon_e)\right]\vec{H} \quad \rightarrow (2)$$

propagation Constant, $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon_e) \rightarrow (3)$

$$\gamma = \sqrt{j\omega\mu\sigma - \omega^2\mu\varepsilon_e}$$

$$\boxed{\gamma = \alpha + j\beta}$$

propagation Constant. Attenuation Constant Phase Constant.

$$(\alpha + j\beta)^2 = -\omega^2\mu\varepsilon_e + j\omega\mu\sigma \quad \rightarrow (4)$$

$$\alpha^2 + 2j\alpha\beta - \beta^2 = -\omega^2\mu\varepsilon_e + j\omega\mu\sigma.$$

Equating Real parts, $\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon \rightarrow (5)$

Taking Modulus of $(\alpha + j\beta)^2 \Rightarrow \{eq^n (4)\}$.

$$|\alpha + j\beta|^2 = |-\omega^2\mu\varepsilon + j\omega\mu\sigma|$$

$$\alpha^2 + \beta^2 = \sqrt{\omega^4\mu^2\varepsilon^2 + \omega^2\mu^2\sigma^2} \quad \rightarrow (6).$$

Adding (5) & (6)

$$2\alpha^2 = -\omega^2\mu\varepsilon_e + \sqrt{\omega^4\mu^2\varepsilon^2 + \omega^2\mu^2\sigma^2}.$$

⑧

$$2\alpha^2 = -\omega^2\mu\varepsilon + \sqrt{\omega^4\mu^2\varepsilon^2\left(1+\frac{\sigma^2}{\omega^2\varepsilon^2}\right)}$$

$$\alpha^2 = \frac{-\omega^2\mu\varepsilon}{2} + \frac{\omega^2\mu\varepsilon}{2}\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}.$$

$$\alpha^2 = \frac{\omega^2\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right]$$

$$\boxed{\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right]}}$$

⑥ − ⑤ we get .,

$$\alpha^2+\beta^2-\alpha^2+\beta^2 = \sqrt{\omega^4\mu^2\varepsilon^2+\omega^2\mu^2\sigma^2} - (-\omega^2\mu\varepsilon)$$

$$2\beta^2 = \sqrt{\omega^4\mu^2\varepsilon^2+\omega^2\mu^2\sigma^2} + \omega^2\mu\varepsilon.$$

$$= \omega\mu\sqrt{\omega^2\varepsilon^2+\sigma^2} + \omega^2\mu\varepsilon$$

$$\beta^2 = \frac{\omega\mu}{2}\sqrt{\omega^2\varepsilon^2+\sigma^2} + \frac{\omega^2\mu\varepsilon}{2}$$

$$\beta = \sqrt{\frac{\omega\mu}{2}\sqrt{\omega^2\varepsilon^2+\sigma^2} + \frac{\omega^2\mu\varepsilon}{2}}$$

$$\beta = \sqrt{\frac{\omega^2\mu\varepsilon_e}{2}\left[\frac{2}{\omega^2\mu\varepsilon_e}\,\frac{\omega\mu}{2}\sqrt{\omega^2\varepsilon_e^2+\sigma^2}+1\right]}$$ 9

$$= \sqrt{\frac{\omega^2\mu\varepsilon_e}{2}\left[\frac{1}{\omega\varepsilon_e}\sqrt{\omega^2\varepsilon_e^2+\sigma^2}+1\right]}.$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon_e}{2}\left[\sqrt{\frac{\omega^2\varepsilon_e^2}{\omega^2\varepsilon_e^2}+\frac{\sigma^2}{\omega^2\varepsilon_e^2}}+1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon_e}{2}\left\{\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon_e^2}}\right)+1\right\}}$$

$$\boxed{\beta = \omega\sqrt{\frac{\mu\varepsilon_e}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon_e^2}}+1\right)}}$$