$$\frac{dz}{du} = \frac{2}{3} \cdot \frac{1}{\sqrt{z+a^2}}$$

$$\therefore \sqrt{z+a^2}\, dz = \frac{2}{3} \cdot du.$$

Integrating, we get

$$\frac{(z+a^2)^{3/2}}{3/2} = \frac{2}{3} u + b$$

$$\frac{2}{3} (z+a^2)^{3/2} = \frac{2}{3} (x+ay) + b$$

Taking $b = \frac{2}{3} b$.

$$(z+a^2)^3 = (x+ay+b)^2$$

which is the complete integral.

— x —

Solve : (i) $p(1+q) = qz$

(ii) $z^2(p^2+q^2+1) = 1$

— x —

Type iv Seperable equations

$$f(x,p) = \phi(y,q)$$

Let $f(x,p) = \phi(y,q) = a$

$\Rightarrow p = f(x,a) \; ; \; q = \phi(y,a)$

$dz = p\,dx + q\,dy$

$dz = f(x,a)\,dx + \phi(y,a)\,dy$

Integrating, we get

$$z = \int f(x,a)\,dx + \phi(y,a)\,dy$$

which is the complete integral.

—x—

Example:

Solve: (i) $p - q = x^2 + y^2$

Solution: Let $p - q = x^2 + y^2$

$$p - x^2 = q + y^2 = a$$

$p - x^2 = a \Rightarrow p = a + x^2$

$q + y^2 = a \Rightarrow q = a - y^2$

$\therefore dz = p\,dx + q\,dy$

$\Rightarrow dz = (a + x^2)\,dx + (a - y^2)\,dy$

Integrating, we get

$$z = \int (a + x^2)\,dx + \int (a - y^2)\,dy$$

$$z = ax + \frac{x^3}{3} + ay - \frac{y^3}{3} + b \quad \text{———} \; (1)$$

which is the complete integral

Singular Integral:

Diff ① partially w.r.t 'a' & 'b' we get

$0 = x + y$

$0 = 1$

$\therefore$ Therefore there is no singular integral.

General solution:-

Put $b = f(a)$ in ①

Then $z = ax + \frac{x^3}{3} + ay - \frac{y^3}{3} + f(a)$ ——— ②

Diff ② partially w.r.t. 'a' and eliminating 'a'

$0 = x + y + f'(a)$ ——— ③

Eliminating 'a' between ② & ③ we get the general solution:

HW Solve (i) $p^2 + q^2 = x^2 + y^2$

(ii) $pq = xy$.

Example:-

Solve: $p^2 + q^2 = x + y$

Solution: $p^2 - x = x - q^2 = a$

$\therefore p^2 - x = a \Rightarrow p^2 = a + x \Rightarrow p = \sqrt{a + x}$

$x - q^2 = a \Rightarrow q^2 = x - a \Rightarrow q = \sqrt{x - a}$

$\therefore dz = p\,dx + q\,dy$

$= \sqrt{x + a}\,dx + \sqrt{x - a}\,dy$

Integrating, we get

$z = \frac{(x+a)^{3/2}}{3/2} + \frac{(x-a)^{3/2}}{3/2} + b.$ which is the complete integral.

Lagrange's Linear Equations

Equations of the form $Pp + Qq = R$ are called Lagrange's linear equations, where P, Q, R are the function of x, y & z

Solution of $Pp + Qq = R$.

The subsidiary equations are

(i) Method of grouping
(ii) Method of multipliers

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{Q}.$$

By solving the subsidiary equations, we get

$u(x,y) = C_1$ & $v(x,y) = C_2$.

The solution of the Lagrange's Linear equation is

$\phi(u, v) = 0$

Example: Solve $xp + yq = z$

Solution: This is of the form $Pp + Qq = R$, where

$P = x$; $Q = y$; $R = z$

The subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Taking first two ratios

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating, we get

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$\Rightarrow \log x = \log y + \log c$

$\Rightarrow \log x - \log y = \log c$

$\Rightarrow \log\left(\frac{x}{y}\right) = \log c$

$\Rightarrow \boxed{\frac{x}{y} = C_1} \Rightarrow$

Taking the second & third ratios

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, we get

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

$\log y = \log z + \log d$

$\log y - \log z = \log d$

$\log\left(\frac{y}{z}\right) = \log d.$

$\Rightarrow \boxed{\frac{y}{z} = d}$

$\therefore$ The solution is

$$\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$$

Solve: $x^2 p + y^2 q = z^2$

Solution:- This is of the form $Pp + Qq = R$

$$P = x^2 \; ; \; Q = y^2 \; ; \; R = z^2$$

The subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

Taking first two ratios

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Integrating,

$$\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$

$$\Rightarrow \frac{x^{-1}}{-1} = \frac{y^{-1}}{-1} + c$$

$$-\frac{1}{x} = -\frac{1}{y} + c$$

$$\Rightarrow \boxed{\frac{1}{y} - \frac{1}{x} = c} \Rightarrow \boxed{u = c}$$

Taking second & third ratios

$$\frac{dy}{y^2} = \frac{dz}{z^2}$$

Integrating

$$\int \frac{dy}{y^2} = \int \frac{dz}{z^2}$$

$$-\frac{1}{y} = -\frac{1}{z} + d$$

$$\Rightarrow \boxed{\frac{1}{z} - \frac{1}{y} = d.} \; \boxed{v = d}$$

$\therefore$ The solution is $\phi\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{y}\right) = 0$

Example: Solve $z(x-y) = x^2 p - y^2 q$

Solution:- This is of the form $Pp + Qq = R$

where $P = x^2 \; ; \; Q = -y^2 \; ; \; R = z(x-y)$

The subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{z(x-y)}$$

Taking first two ratios

$$\frac{dx}{x^2} = \frac{dy}{-y^2}$$

Integrating, we get

$$\int \frac{dx}{x^2} = -\int \frac{dy}{y^2}$$

$$-\frac{1}{x} = \frac{1}{y} + c_1$$

$$\Rightarrow c = \frac{1}{x} + \frac{1}{y} \Rightarrow$$

$$\boxed{\frac{1}{x} + \frac{1}{y} = c}$$

$$\frac{dx+dy}{x^2-y^2} = \frac{dz}{z(x+y)}$$

$$\Rightarrow \frac{dx+dy}{\cancel{(x+y)}(x+y)} = \frac{dz}{z\cancel{(x+y)}}$$

Integrating, we get

$$\int \frac{d(x+y)}{x+y} = \int \frac{dz}{z}$$

$$\log(x+y) = \log z + \log d$$

$$\log(x+y) - \log z = \log d$$

$$\cancel{\log}\left(\frac{x+y}{z}\right) = \cancel{\log} d \Rightarrow \frac{x+y}{z} = d$$

$\therefore$ The solution is $\phi\left(\frac{1}{x}+\frac{1}{y}, \frac{x+y}{z}\right) = 0$

Example:-

Solve $y^2p - xyq = x(z-2y)$

Solution:- This is of the form $Pp + Qq = R$

where $P = y^2$; $Q = -xy$; $R = x(z-2y)$

The subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Taking first two ratios

$$\frac{dx}{y^2} = \frac{dy}{-xy} \Rightarrow \frac{dx}{y} = \frac{dy}{-x}$$

$$\Rightarrow y\,dy = -x\,dx$$

$\Rightarrow$ ~~$y\,dy + x\,dx = 0$~~

Integrating, we get

$$\frac{y^2}{2} = -\frac{x^2}{2} + c \Rightarrow \frac{x^2+y^2}{\cancel{2}} = c$$

$$\Rightarrow \boxed{x^2+y^2 = c}$$

Taking the last two ratios

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\frac{dy}{-y} = \frac{dz}{z-2y}$$

$$z\,dy - 2y\,dy = -y\,dz$$

$$z\,dy + y\,dz - 2y\,dy = 0$$

Integrating, we get

$$yz - \cancel{2}\frac{y^2}{\cancel{2}} = d$$

The soln is $\phi(x^2+y^2, yz-y^2) = 0$

Solve $x(y-z)p + y(z-x)q = z(x-y)$

Solution: This is of the form $Pp + Qq = R$ where

$P = x(y-z)$; $Q = y(z-x)$; $R = z(x-y)$

The subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

Choosing the multipliers 1, 1, 1, each ratio is equal to

$$\frac{dx+dy+dz}{xy-xz+yz-yx+zx-zy} = \frac{dx+dy+dz}{0}$$

$\Rightarrow \quad dx+dy+dz = 0$

Integrating, we get

$\boxed{x+y+z = c.}$ $\Rightarrow$

Choosing the multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, each ratio is equal to

$$\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y-z+z-x+x-y} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$\Rightarrow \quad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

Integrating, we get

$\log x + \log y + \log z = \log d$

$\log(xyz) = \log d.$

$\Rightarrow \quad \boxed{xyz = d.}$

$\therefore$ The solution is $\phi(x+y+z, xyz) = 0$

① Solve $x(\cancel{z}^2-y^2)p+y(x^2-z^2)q=z(x^2-y^2)$

(Hint (Multipliers, x,y,z ; $\frac{1}{x},\frac{1}{y},\frac{1}{z}$) : Ans: $\phi(x^2+y^2+z^2, xyz)=0$)

② Solve $(mz-ny)p+(nx-lz)q=ly-mx$

(Hint: Multipliers l,m,n ; x,y,z : Ans: $\phi(lx+my+nz, x^2+y^2+z^2)=0$)

③ Solve $(3z-4y)p+(4x-2z)q=2y-3x$

(Hint: Multipliers $\cancel{x,y,z}$; $2,3,4$: Ans: $\phi(x^2+y^2+z^2, 2x+3y+4z)=0$

—x—

Example: Solve $x^2p+y^2q=z(x+y)$

Solution: This is of the form $Pp+Qq=R$

where $P=x^2$; $Q=y^2$; $R=z(x+y)$

The subsidiary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$=\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{z(x+y)}$$

Taking first two ratios

$$\frac{dx}{x^2}=\frac{dy}{y^2}$$

Integrating, we get

$$-\frac{1}{x}=-\frac{1}{y}+c$$

$$\Rightarrow \boxed{\frac{1}{y}-\frac{1}{x}=c}$$

$$\frac{dx-dy}{x^2-y^2}=\frac{dz}{z(x+y)}$$

$$\frac{dx-dy}{\cancel{(x+y)}(x-y)}=\frac{dz}{z\cancel{(x+y)}}$$

Integrating

$$\log(x-y)=\log z+\log d$$

$$\log(x-y)-\log z=\log d$$

$$\cancel{\log}\left(\frac{x-y}{z}\right)=\cancel{\log}\, d$$

$$\boxed{\left(\frac{x-y}{z}\right)=d}$$

∴ The solution is $\phi\left(\frac{1}{y}-\frac{1}{x}, \frac{x-y}{z}\right)=0$

Solve $\frac{y^2 z}{x} p + xzq = y^2$

<u>Solution:-</u> This is of the form $Pp + Qq = R$

where $P = \frac{y^2 z}{x}$; $Q = xz$; $R = y^2$

The subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2} \Rightarrow \frac{x\,dx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}$$

Taking the first two ratios

$$\frac{x\,dx}{y^2 \cancel{z}} = \frac{dy}{x\cancel{z}}$$

$$x^2\,dx = y^2\,dy$$

Integrating, we get

$$\frac{x^3}{3} = \frac{y^3}{3} + c$$

$$\Rightarrow \boxed{x^3 - y^3 = c.}$$

Taking the first & the last ratios

$$\frac{x\,dx}{\cancel{y^2} z} = \frac{dz}{\cancel{y^2}}$$

$$x\,dx = z\,dz$$

Integrating, we get

$$\frac{x^2}{2} = \frac{z^2}{2} + d$$

$$\boxed{x^2 - z^2 = d.}$$

$\therefore$ The solution is $\phi\left(x^3 - y^3,\ x^2 - z^2\right) = 0.$