UNIT - 3   ODE: Ordered Differential Equations:

$$\frac{d^2y}{dx^2} = 5 \qquad \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0$$

$$\frac{d}{dx} = D, \quad \frac{d^2}{dx^2} = D^2$$

$$\therefore \quad D^2y + 4Dy + 3y = 0$$

$Y = CF + PI$ → Particular Integral.

CF ↓ Complementary Function

→ Auxillary Equation

Replace D by m and remove Y

~~$(D^2 - 7D + 12)Y = 0$~~

i.e $(D^2 - 7D + 12)\,Y = 0$

$$m^2 - 7m + 12 = 0$$

$A\vec{E}$

If the roots are $m = 1, 2$

CF is $= Ae^{x} + Be^{2x}$

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If $m = 1, 1$

$$CF = e^{x}(A + Bx)$$

or $e^{x}(Ax + B)$

If $m = 2 \pm 3i$

$\alpha = 2, 3$

$$CF = e^{2x}(A \cos 3x + B \sin 3x)$$

$$\underset{\text{(general)}}{CF} = e^{2x}(A \cos \beta x + B \sin \beta x)$$

CF → Find a root and classify its root (Complimentary function).

~~(1) When roots are~~

Particular Integral:

Type (I) : e^{ax}

Type (II) : $\sin ax$ (or) $\cos ax$

Type (III) : x^{m}

Type (IV) : $e^{ax} \sin ax$ (or) $e^{ax} \cos ax$
(Type I and II Combined)

Type V : $e^{ax} x^m$ (Type II and III)

Type VI : $e^{ax} x^m \sin ax$ (Type I, II and III)

or

$e^{ax} x^m \cos ax$

Qns,

1) Solve $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$

2) Solve $(D^2 + 3D + 2)y = 0$

3) $(D^2 + 4)y = 0$

Soln:

1) $(D^2 - 7D + 12)y = 0$

AE $m^2 - 7m + 12 = 0$

$m = 3, 4$

$CF = Ae^{3x} + Be^{4x}$

(or)

$C_1 e^{3x} + C_2 e^{4x}$

$PI = 0$

$\therefore y = CF + PI = Ae^{3x} + Be^{4x}$

Q) Solve

$$(D^2+3D+2)y = e^{-2x}$$

Q) $(D^2+6D+9)y = 3e^{4x}$

Q) $(D^2+9)y = e^{-2x}$

Q) $(D^2+2D+1)y = e^{-x}+3$

Q) $(D^2+3D+2)y = \sin x$

Q) $(D^2+6D+8)y = \cos^2 x$

(1) AE $m^2+3m+2=0$

$m=-2,-1$

$$CF = Ae^{-2x} + Be^{-x}$$

$$PI = \frac{1}{D^2+3D+2} e^{-2x}$$

$a=-2$ Replace D by a

$$= \frac{1}{(-2)^2+3(-2)+2} e^{-2x}$$

$$= \frac{1}{0} e^{-2x}$$

We got Den $=0$ ∴ we have to do $x = \frac{1}{\phi'(D)} e^{ax}$

until den mei 0 nahi jata

$$= x \cdot \frac{1}{2D+3} \cdot e^{-2x}$$

$$= x \cdot \frac{1}{-1} e^{-2x}$$

$$= -x e^{-2x}$$

$\therefore$ PI $= -x e^{-2x}$

$CF = Ae^{-2x} + Be^{-2x}$

$\therefore$ Final ans:

$$y = Ae^{-2x} + Be^{-2x} - xe^{-2x}$$

(2) $AE = D^2 + 6D + 9 = 0$

$m = 3, 3$

$CF = Ae^{2x} + Be^{3x}$

$$PF = \frac{1}{D^2 + 6D + 9} \cdot 3e^{4x}$$

$$= \frac{1}{16 + 24 + 9} \cdot 3e^{4x}$$

$$= \frac{1}{49} 3e^{4x}$$

$\therefore y = CF + PF$

(3) $(D^2+9)y = e^{-2x}$

$m = -3i, -3i$

$CF = e^{-2x}(A\cos 3x + B\sin 3x)$

$PF = \frac{1}{D^2+9} \cdot e^{-2x}$

$= \frac{1}{18} e^{-2x}$

(4) $PI = \frac{1}{D^2+2D+1} \cdot e^{-x} + 3 \cdot \frac{1}{D^2+2D+1} e^{0x}$

$= x \cdot \frac{1}{2D+2} \cdot e^{-x} + 3$

$= \frac{x^2}{2} e^{-x} + 3$

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(5) $PI = \left(\frac{1}{D^2 + 3D + 2}\right) y = 1 \sin x$

Type (II) $\sin ax$ or $\cos ax$

$PI = \frac{1}{\phi(D)} \sin ax$

Replace D^2 by $-a^2$

Here $a = 1$

Replace D^2 by -1

$$= \frac{1}{-1 + 3D + 2} \cdot \sin x$$

$$= \frac{1}{3D + 1} \sin x$$

$$= \frac{(3D - 1)}{(3D+1)(3D-1)} \sin x$$

$$= \frac{3D - 1}{9D^2 - 1} \sin x$$

$$= \frac{3D - 1}{-9 - 1} \sin x$$

$$= -\frac{1}{10}\left[3D(\sin x) - \sin x\right]$$

$$= -\frac{1}{10}\left[3\cos x - \sin x\right]$$

⑥ $(D^2+6D+8)y = \cos^2 x$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{e^{0x}}{2} + \frac{\cos 2x}{2}$$

$m = 4, 2$

$CF = Ae^{2x} + Be^{4x}$

$$PI = \frac{1}{D^2+6D+8} \cdot \frac{e^{0x}}{2} + \frac{1}{D^2+6D+8} \cdot \frac{\cos 2x}{2}$$

$$= \frac{1}{16} + \cancel{\frac{1}{10}} \cancel{\frac{\cos 2x}{2}}$$

solving

$a = 2$

D^2 by -1

$$= \frac{1}{-1+6(2)+8} \cdot \frac{\cos 2x}{2}$$

$$= \frac{1}{19} \frac{\cos 2x}{2}$$

$$= \frac{\cos 2x}{38}$$

$$PI = \frac{1}{16} + \frac{\cos 2x}{38}$$

$$\therefore Y = Ae^{2x} + Be^{4x} + \frac{1}{16} + \frac{\cos 2x}{38}$$

Type ③ $PI = \frac{1}{\phi(D)} x^n$

Convert to $(1+x)^{-1}$, $(1-x)^{-1}$, $(1-x)^{-2}$, $(1+x)^{-2}$,

Q) Type ①: $(D^2+2D+1)\,y = e^{-x} + 3e^{0x}$ |

If $\phi D = 0$ then replace it by $x \cdot \frac{1}{\phi'(D)} e^{ax}$

$$PI = \frac{1}{D^2+2D+1} \cdot e^{-x} + 3\frac{1}{D^2+2D+1} e^{0x}$$

$$= \cancel{\frac{1}{5} e^{-x}} \cancel{+ \frac{3}{1}}$$

$$= \frac{x}{2D+2} e^{-x} + \frac{3}{D^2+2D+1} e^{0x} \;|$$

$$= \frac{x^2 \cdot e^{-x}}{2} + 3$$

(2) $(D^2+3D+2)y = \sin x$

$$PI = \frac{1}{D^2+3D+2} \cdot \sin x$$

$$= \frac{1}{-1+3D+2} \sin x$$

$$= \frac{1}{3D+1} \sin x$$

$$= \frac{(3D-1)}{(3D+1)(3D-1)} \sin x$$

$$= \frac{3D-1}{9D^2-1} \sin x$$

D^2 by -1

$$= \frac{3D-1}{-10} \sin x$$

$$= \frac{-1}{10} (3D\sin x - \sin x)$$

$$\therefore \frac{-1}{10} (3\cos x - \sin x)$$

Type ③ Numericals:

Remember Binomial expansion

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \dots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

$(1-x)^{-2} = 1 + 2x - 3x^2 + 4x^3 \dots$

④ $(D^2 + 3D + 2)y = x^2 + 1$

$$PI = \frac{1}{D^2 + 3D + 2}(x^2+1)$$

$$= \frac{1}{2\left[1 + \frac{D^2 + 3D}{2}\right]}(x^2+1)$$

Expansion:

$$= \frac{1}{2} \cdot \left[1 + \frac{D^2 + 3D}{2}\right]^{-1}(x^2+1)$$

$$= \frac{1}{2}\left[1 - \left(\frac{D^2+3D}{2}\right) + \left(\frac{D^2+3D}{2}\right)^2\right] x^2 + 1$$

$$= \frac{1}{2}\left[1 - \frac{D^2}{2} - \frac{3D}{2} + \frac{D^4}{4} + \frac{9D^2}{4} + 3D^3\right](x^2+1)$$

$$= \frac{1}{2}\left[x^2 + 1 - \frac{3D(x^2+1)}{2} + \frac{7}{4}D^2 + 3D^3 + \frac{D^4}{4}\right](x^2+1)$$

$$= \frac{1}{2}\left[x^2+1 - \frac{3D(x^2+1)}{2} + \frac{7}{4}\frac{D^2(x^2+1)}{2} + 3D^3(x^2+1)\right]$$

$$= \frac{1}{2}\left[x^2 + 1 - 3x + \frac{7}{2}\right]$$

$$= \frac{1}{2}\left[x^2 - 3x + \frac{9}{2}\right]$$

Type 4 : Combination of Type ① and Type ②

Formula (For Type ②)

$$PI = \frac{1}{\phi(D)} x \cdot f(x)$$

$$= x \cdot \frac{1}{\phi(D)} f(x) - \frac{\phi'(D)}{(\phi(D))^2} f(x)$$

① $(D^2 + 4D + 4)y = e^{3x}\sin x$

First eliminate Type ①, i.e e^{3x}

D should be replaced by $D+a$

i.e $D+3$

$$\therefore PI = e^{3x} \cdot \frac{1}{(D+3)^2 + 4(D+3) + 4} \sin x$$

Now eliminate Type ②;

$$PI = e^{3x} \cdot \frac{1}{D^2 + 6D + 9 + 4D + 12 + 4} \cdot \sin x$$

$$= e^{3x} \cdot \frac{1}{\cancel{D^2} - 1 + 10D + 25} \sin x$$

$$= e^{3x} \cdot \frac{1}{10D + 24} \sin x$$

$$= e^{3x} \cdot \frac{(10D-24)}{(10D+24)(10D-24)} \sin x$$

$$= e^{3x} \cdot \frac{10D-24}{(100D^2-576)} \sin x$$

$$= \cancel{\frac{-1}{-676} e^{3x}} \quad e^{3x} \left(\frac{10D-24}{-676} \right) \sin x$$

$$PI = \frac{e^{3x}}{-676} (10\cos x - 24\sin x)$$

Type 5 $\quad e^{ax} x^n$

(2) $(D^2+D+1)\,y = x^2 e^{-x}$

$$PI = \frac{1}{D^2+D+1} x^2 e^{-x}$$

Eliminate Type (1)

Replace D by $(D-1)$

$$PI = e^{-x} \frac{1}{(D-1)^2+(D-1)+1} x^2$$

$$= e^{-x} \frac{1}{D^2-2D+1+D\cancel{-1}\cancel{+1}} x^2$$

Type-5

(3) $(D^2+9)y = (x^2+1)e^{3x}$

$$= \frac{1}{(D^2+9)} e^{3x} \cdot (1+x^2)$$

Replace D by (D - 3)

$$= \frac{1}{(D^2-6D+9+9)} e^{3x}(1+x^2)$$

$$= \frac{e^{3x}}{(D^2-6D+18)}(1+x^2)$$

$$= \frac{e^{3x}}{18\left[1+\frac{D^2-6D}{18}\right]}(1+x^2)$$

$$= \frac{1}{18} e^{3x}\left[1+\left(\frac{D^2-6D}{18}\right)\right]^{-1}(1+x^2)$$

$$= \frac{1}{18} e^{3x}\left[1-\left(\frac{D^2-6D}{18}\right)+\left(\frac{D^2-6D}{18}\right)^2\right](1+x^2)$$

$$= \frac{1}{18} e^{3x}\left[1-\frac{D^2-6D}{18}+\frac{D^4-12D^3+36D^2}{324}\right](1+x^2)$$

$$= \frac{1}{18} e^{3x}\left[\frac{324-18D^2-108D+D^4-12D^3+36D^2}{324}\right](1+x^2)$$

Type-6

4. $(D^2-1)y = xe^x \sin x$

(Refer formula from behind)

$$PI = \frac{1}{D^2-1} e^x x \sin x$$

$$= e^x \frac{1}{(D+1)^2-1} x \cdot \sin x$$

$$= e^x \frac{1}{D^2+2D+1-1} \cdot x \sin x$$

$$= e^x \cdot \frac{1}{D^2+2D} \cdot x\sin x$$

use formula:

$$= e^x \left(x \cdot \frac{1}{D^2+2D} \sin x - \frac{2D+2}{(D^2+2D)^2} \sin x \right)$$

$$= e^x \left(x \cdot \frac{1}{2D-1} \sin x - \frac{2D+2}{D^4+4D^2+4D^3} \sin x \right)$$

$$= e^x \left(x \cdot \frac{2D+1}{(2D-1)(2D+1)} \sin x - \right.$$

$$\left. \frac{2D+2}{D^2(D^2+4D+4)} \sin x \right)$$

$$= e^x \left(x \cdot \frac{(2D+1)\sin x}{4D-1} + \frac{2D+2}{4D+4-1} \sin x \right)$$

$$= e^x \left(x \, \frac{2D+1}{-5} \sin x + \frac{(2D+2)(4D-3)}{(4D+3)(4D-3)} \sin x \right)$$

$$= e^x \left(-\frac{x}{5} (2D\sin x + \sin x) + \left(\frac{8D^2+8D-6D-6}{16D^2-9} \sin x \right) \right.$$

$$= e^x \left(\frac{-x}{5} (2\cos x + \sin x) + \left(\frac{8D^2+2D-6}{-25} \right) \sin x \right)$$

$$= e^{x}\left(-\frac{x}{5}(2\cos x + \sin x) + \frac{8D^{2}(\sin x) + 2D\sin x - 6\sin x}{-25}\right.$$

Variable Coefficients:

Cauchy – Homogenous Equation (Euler's type)

~~Legendre's~~ type

Legendre's

Cauchy Homogenous Linear Equation:

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2}\frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = F(x)$$

Where $a_0, a_1, a_2 \dots\dots$ are constant

$F(x)$ is a fn of x

Let $x = e^{z}$, $z = \log x$

$\frac{d}{dx} = D$, $\frac{d}{dz} = D'$

$\frac{d^2}{dx^2} = D^2$

Let us assume $xD = D'$

$x^2 D^2 = D'(D'-1)$

Q) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 4\sin(\log x)$

$(x^2D^2 + xD + 1)\, y = 4\sin(\log x)$

Let $x = e^z$ $\quad xD = D'$

$\log x = z$ $\quad x^2D^2 = D'(D'-1)$

$\frac{d}{dx} = D$, $\frac{d}{dz} = D'$

Now:

$(D'(D'-1) + D' + 1)y = 4\sin z$

$(D'^2 \cancel{-D'} \cancel{+D'} + 1)\, y = 4\sin z$

$(D'^2 + 1)\; y = 4\sin z$

Now AE is :

Replace D' by m

$m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

CF: $A\cos z + B\sin z$

$$PI = 4 \cdot \frac{1}{D'^2+1} \sin z$$

$$= 4 \cdot \frac{z}{2D'} \sin z$$

$$= \frac{4z}{2} \int \sin z \, dz$$

$$PI = -2z\cos z$$

$$\therefore y = -2z\cos z$$

$$\therefore y = A\cos(\log z) + B\sin(\log x)$$
$$- 2 \log x \cos z$$

(2) $$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$$

Legendre's Type :

$$(ax+b)^n \frac{d^n y}{dx^n} + P_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}}$$

$$+ \ldots\ldots P_n y = F(x)$$

Let $(ax+b) = e^z$

$$z = \log(ax+b)$$

$$(ax+b)D = aD'$$

$$(ax+b)^2 D^2 = a^2 D'(D'-1)$$

Q) Solve $(2x+5)^2 \frac{d^2y}{dx^2} - 6(2x+5)\frac{dy}{dx} + 8y = 0$

Let $2x+5 = e^z$

$$z = \log(2x+5)$$

$$(2x+5)D = 2D'$$

$$(2x+5)^2 D^2 = 4D'(D'-1)$$

$$\left[4D'(D'-1) - 6(2D') + 8\right]Y = 0$$

$$(4D'^2 - 4D' - 12D' + 8)Y = 0$$

$$(4D'^2 - 16D' + 8)Y = 0$$

$$m = \frac{16 \pm \sqrt{(16)^2 - 4(4)(8)}}{2(4)}$$

$$m = 2 \pm \sqrt{2}$$

$$CF = Ae^{2+2\sqrt{2}} + Be^{2-\sqrt{2}}$$

$$PI = 0$$

$$\therefore Y = A(2x+5)^{2+\sqrt{2}} + B(2x+5)^{2-\sqrt{2}}$$

(2) $$(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + Y = 4\cos(\log(1+x))$$

Let $(1+x) = e^z$ $\qquad (1+x)^2 \frac{d^2y}{dx^2} = D'(D'-1)$

$z = \log(1+x)$ $\qquad (1+x)\frac{dy}{dx} = D'$

$\cancel{(1+x)D = D'}$

$\cancel{(1+x)^2 D^2 = D'^2}$

$\cancel{[D'^2 + D']y = 4\cos z}$

$\cancel{y[D'^2 + D'] - 4\cos z = 0}$

$\cancel{[D'(D'+1)]y - 4\cos z = 0}$

$\therefore [D'(D'-1) + D'] + y = 4\cos z$

$(D'^2 + 1)y = 4\cos z$

$D' = \pm i$

$CF = A\cos z + B\sin z$

$PI = \cancel{4\cos z}\ \dfrac{4}{(D'^2+1)} \cos z$

$PI = \dfrac{4z}{2D'} \cos z$

$= \dfrac{4z}{2D} \cos z$

~~replace~~ D' by -1

$= -2z\sin z$