① Find the radius of curvature for the curve $xy^2 = a^3 - x^3$ at $(a,0)$.

Given: $xy^2 = a^3 - x^3$.

diff w.r.t 'x' we get

$$x \cdot 2y \cdot \frac{dy}{dx} + y^2(1) = 0 - 3x^2.$$

$$\frac{dy}{dx} = \frac{-3x^2 - y^2}{2xy}$$

$$\frac{dy}{dx}_{(a,0)} = \frac{-3a^2 - 0^2}{0} = \infty,$$ so we can find $\frac{dx}{dy}$.

$$\Rightarrow \frac{dx}{dy} = \frac{0}{-3a^2} = 0.$$

$$\Rightarrow \boxed{\frac{dx}{dy}_{(a,0)} = 0} \quad \text{—— ①} \qquad \Rightarrow \frac{dx}{dy} = \frac{-2xy}{(3x^2+y^2)}$$

diff again w.r.t 'y':

$$\frac{d^2x}{dy^2} = \frac{-2\left[(3x^2+y^2)\left(x+y\frac{dx}{dy}\right) - xy\left(6x+2y\right)\right]}{(3x^2+y^2)^2}.$$

$$\frac{d^2x}{dy^2} = \frac{-2\left[(3x^2+y^2)\left(x+y\frac{dx}{dy}\right) - 6x^2y - 2xy^2\right]}{(3x^2+y^2)^2}$$

$$\frac{d^2x}{dy^2}_{(a,0)} = \frac{-2\left[(3a^2+0)(a+0) - 0 - 0\right]}{(3a^2+0)^2} = \frac{-2\cancel{6}a^3}{3\cancel{9}a^4}$$

$$\boxed{\frac{d^2x}{dy^2}_{(a,0)} = \frac{-2}{3a}}$$

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}} = \frac{\left[1+(0)^2\right]^{3/2}}{\left(-2/3a\right)}$$

$$\underline{\rho = -\frac{3a}{2}}$$

Since radius of curvature cannot be negative,

$$\Rightarrow \boxed{\rho = \frac{3a}{2}}$$

(2) Find the circle of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$. at $(1/4, 1/4)$.

Given: $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

diff wrt 'x' we get,

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0.$$

$$\frac{dy}{dx} = -\frac{\cancel{2}\sqrt{y}}{\cancel{2}\sqrt{x}} = -\frac{\sqrt{y}}{\sqrt{x}}.$$

$$\frac{dy}{dx}_{(1/4, 1/4)} = \frac{-\sqrt{1/4}}{\sqrt{+1/4}} = -1$$

$$\left(\frac{dy}{dx}\right)_{(1/4,1/4)} = -1$$

$$\frac{d^2y}{dx^2} = -\left[\frac{\sqrt{x}\cdot\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx} - \sqrt{y}\cdot\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}\right]$$

$$\frac{d^2y}{dx^2}_{(1/4,1/4)} = -\left[\frac{\sqrt{1/4}\cdot\frac{1}{2\sqrt{1/4}}(-1) - \sqrt{1/4}\cdot\frac{1}{2\sqrt{1/4}}}{(\sqrt{1/4})^2}\right]$$

$$= -\left[\frac{-1/2 - 1/2}{1/4}\right] = -\left(\frac{-1}{1/4}\right) = 4$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2}_{(1/4,1/4)} = 4.}$$

$$\bar{x} = x - \frac{y'(1+y'^2)}{y''} = \frac{1}{4} - \frac{(-1)[1+(-1)^2]}{4}$$

$$= \frac{1}{4} + \frac{2}{4} = \frac{3}{4}.$$

$$\boxed{\bar{x} = 3/4}$$

$$\bar{y} = y + \frac{1+y'^2}{y''}$$

$$= \frac{1}{4} + \frac{1+(-1)^2}{4}$$

$$= \frac{1}{4} + \frac{2}{4} = \frac{3}{4}$$

$$\boxed{\bar{y} = \frac{3}{4}}$$

$$\therefore \rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} = \frac{\left[1+(-1)^2\right]^{3/2}}{4}$$

$$= \frac{(2)^{3/2}}{4} = \frac{(8)^{1/2}}{4} = \frac{\cancel{2}\sqrt{2}}{\cancel{4}2} = \frac{\cancel{\sqrt{2}}}{\cancel{2}\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \boxed{\rho = \frac{1}{\sqrt{2}}}$$

$\therefore$ Equation of circle of curvature is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$\Rightarrow \left(x-\tfrac{3}{4}\right)^2 + \left(y-\tfrac{3}{4}\right)^2 = \left(\tfrac{1}{\sqrt{2}}\right)^2$$

$$\Rightarrow \left(x-\tfrac{3}{4}\right)^2 + \left(y-\tfrac{3}{4}\right)^2 = \frac{1}{2}.$$

(3) Test the convergence of the series $\sum \frac{n^p}{\sqrt{n+1}+\sqrt{n}}$.

Soln:

Let $u_n = \frac{n^p}{\sqrt{n+1}+\sqrt{n}}$

$$= \frac{n^p}{\sqrt{n}\left[\sqrt{1+\frac{1}{n}}+1\right]} = \frac{1}{n^{-p} \cdot n^{\frac{1}{2}}\left[\sqrt{1+\frac{1}{2}}+1\right]}$$

$$u_n = \frac{1}{n^{-p+\frac{1}{2}}\left[\sqrt{1+\frac{1}{n}}+1\right]}$$

Let $v_n = \left(\frac{1}{n^{-p+\frac{1}{2}}}\right)$.

$$\frac{u_n}{v_n} = \frac{\frac{\cancel{1}}{n^{-p+\frac{1}{2}}\left[\sqrt{1+\frac{1}{n}}+1\right]}}{\left(\cancel{1/n^{-p+\frac{1}{2}}}\right)} =$$

$$\frac{u_n}{v_n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2} \text{ (finite)}$$

$\therefore$ Both $\sum u_n$ & $\sum v_n$ converges or diverges together.

But $\sum v_n = \frac{1}{n^{-p+\frac{1}{2}}}$.

(i) Σv_n is converges if $-p+\frac{1}{2} > 1$

or $-p > +\frac{1}{2}$

$\Rightarrow \underline{p < -\frac{1}{2}}$, v_n is converges.

(ii) Σv_n is diverges, if $-p+\frac{1}{2} < 1$.

or $-p < +\frac{1}{2}$

$\Rightarrow p \geq -\frac{1}{2}$, v_n is converges.

Along v_n, Σu_n is ~~convergent~~ or divergers

Comparing the values of 'p':