

Laplace Transform of Periodic function

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-sT}}$$

where T is period of the function $f(t)$

function is periodic if $f(x+T) = f(x)$
 $T \rightarrow$ least +ve integer

Eg $f(x) = \sin x$
 $f(x+\pi) = \sin(x+\pi) = -\sin x$

$$f(x+2\pi) = \sin x$$

$$f(x+4\pi) = \sin x$$

Q Find the Laplace Transform of the function

$$f(t) = \begin{cases} \sin \omega t & 0 \leq t \leq \frac{\pi}{\omega} \\ 0 & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$$

Ans $f(t)$ is periodic function
with period $T = \frac{2\pi}{\omega}$

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-sT}}$$

$$= \frac{\int_0^{\frac{2\pi}{\omega}} e^{-st} f(t) dt}{1 - e^{-s \cdot \frac{2\pi}{\omega}}}$$

$$= \frac{\int_0^{\pi/\omega} e^{-st} \sin \omega t \cdot dt}{1 - e^{-\frac{52\pi}{\omega}}} + \frac{\int_{\pi/\omega}^{2\pi/\omega} e^{-st} \cdot 0 \cdot dt}{1 - e^{-\frac{52\pi}{\omega}}}$$

$$\left[e^{ax} \sin bx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) \right]$$

$$= \left[\frac{e^{-st}}{s^2 + \omega^2} (-s \sin \omega t - \omega \cos \omega t) \right]_0^{\pi/\omega}$$

$$= \frac{1 - e^{-2s\pi/\omega}}{1 - e^{-2s\pi/\omega}}$$

$$= \left[\frac{e^{-s\pi/\omega}}{s^2 + \omega^2} (0 - \omega(-1)) \right] - \left[\frac{1}{s^2 + \omega^2} (0 - \omega(1)) \right]$$

$$= \frac{1 - e^{-2s\pi/\omega}}{1 - e^{-2s\pi/\omega}}$$

$$\left[\begin{aligned} e^a \cdot e^b &= e^{a+b} \\ (e^a)^2 &= e^a \cdot e^a = e^{2a} \end{aligned} \right]$$

$$= \frac{\omega e^{-s\pi/\omega} + \omega}{s^2 + \omega^2}$$

$$= \frac{1 - (e^{-\pi s/\omega})^2}{1 - (e^{-\pi s/\omega})^2}$$

$$= \omega (1 + e^{-2\pi s/\omega})$$

$$(1 + e^{-\pi s/\omega}) (1 - e^{-\pi s/\omega}) (s^2 + \omega^2)$$

$$= \frac{\omega}{1 - e^{-\pi s/\omega}} \cdot \frac{1}{s^2 + \omega^2}$$

Q Find the L.T. of the periodic function

$$f(t) = \begin{cases} t & 0 \leq t \leq a \\ 2a - t & a \leq t \leq 2a \end{cases}$$

$$\text{and } f(t+2a) = f(t)$$

Ans $f(t)$ is periodic $T = 2a$

$$L[f(t)] = \int_0^T e^{-st} f(t) dt$$

$$= \frac{1 - e^{-sT}}{1 - e^{-sT}}$$

$$= \int_0^{2a} \frac{e^{-st} f(t) dt}{1 - e^{-2as}}$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} \cdot t \, dt + \int_a^{2a} e^{-st} (2a-t) \, dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left(t \cdot \frac{e^{-st}}{-s} \right) - \left(1 \cdot \frac{e^{-st}}{s^2} \right) + 0 \right]_0^a + \left[(2a-t) \frac{e^{-st}}{-s} \right] - (-1) \cdot \frac{e^{-st}}{s^2} \Bigg|_a^{2a}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \left[\frac{-a \cdot e^{-sa}}{s} - \frac{e^{-sa}}{s^2} \right] - \left(0 - \frac{1}{s^2} \right) \right\}$$

$$+ \left[\left(0 + \frac{e^{-sa}}{s^2} \right) - \left(-a \cdot \frac{e^{-st}}{s} + \frac{e^{-sa}}{s^2} \right) \right]$$

$$= \frac{1}{1-e^{-2as}} \left[-a \frac{e^{-sa}}{s} - \frac{e^{-sa}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{a \cancel{e^{-sa}}}{s} - \frac{\cancel{e^{-sa}}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{-ae^{-sa} + 1 + e^{-2as}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2as}} \frac{(1-e^{-as})^2}{s^2}$$

$$= \frac{1}{(1+e^{-as})(1-e^{-as})} \cdot \frac{(1-e^{-as})^2}{s^2}$$

$$= \frac{1-e^{-as}}{1+e^{-as}}$$

$$= \frac{1}{1+e^{-as}} \cdot \frac{1}{s^2}$$

$$= \frac{1}{s} \left[\frac{1 \cdot \frac{e^{-as}}{2} \cdot \frac{e^{-as}}{2}}{1 + e^{-\frac{as}{2}} \cdot e^{-\frac{as}{2}}} \right]$$

$$\frac{1}{s^2} \left[\frac{e^{\frac{as}{2}} - e^{-\frac{as}{2}}}{e^{\frac{as}{2}} + e^{-\frac{as}{2}}} \right]$$

$$= \frac{1}{s^2} \tanh \left(\frac{as}{2} \right)$$

8 $f(t) = \begin{cases} 1, & \text{when } 0 < t \leq a_1 \\ -1, & \text{when } a_1 < t \leq a \end{cases}$

Ans $\frac{1}{s} \tanh \frac{sa}{4}$

~~Ans T=a~~

9 $f(t) = \begin{cases} E, & 0 < t < a_1 \\ -E, & a_1 < t < a \end{cases}$

Ans $\frac{E}{s} \tanh \frac{as}{4}$

Initial And Final Value Theorem

Statement of IVT

Let $L[f(t)] = F(s) \times f'(t)$ exists,

Then

$$\lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

Statement of FVT

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

Q Find IVT of $f(t) = a \cdot e^{-bt}$

IVT

$$\lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

LHS $\lim_{t \rightarrow 0^+} f(t)$

Let $f(t) = a e^{-bt}$

$$\lim_{t \rightarrow 0} f(t) = \lim_{t \rightarrow 0} [a e^{-bt}] = a e^0 = a$$

RHS $\lim_{s \rightarrow \infty} sF(s)$

$$F(s) = L[f(t)]$$

$$= L[a e^{-bt}]$$

$$= a L[e^{-bt}]$$

$$= a \cdot \frac{1}{s+b}$$

$$sF(s) = s \cdot \frac{a}{s+b}$$

$$\begin{aligned}
 \lim_{s \rightarrow \infty} sf(s) &= \lim_{s \rightarrow \infty} \left[a \left(\frac{s}{s+b} \right) \right] \\
 &= \lim_{s \rightarrow \infty} \left[a \left(\frac{s}{s(1+bs)} \right) \right] \\
 &= \lim_{s \rightarrow \infty} \left[a \cdot \frac{1}{1+bs} \right] \\
 &= a
 \end{aligned}$$

$$\underline{\underline{\text{LHS} = \text{RHS} = a}}$$

Q FVT of $a e^{-bt}$

FVT

LHS let $f(t) = a e^{-bt}$
 \lim

Q $f(t) = e^{-t} \cdot \sin t$

IVT

$$\lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sf(s)$$

LHS

Let $f(t) = e^{-t} \cdot \sin t$

$$\lim_{t \rightarrow 0} f(t) = \lim_{t \rightarrow 0} [e^{-t} \cdot \sin t]$$

$$= e^0 \cdot \sin 0$$

$$= 1 \cdot 0 = 0$$

RHS

$$f(s) = L[f(t)]$$

$$f(s) = L[e^{-t} \cdot \sin t]$$

$$= \left[\frac{1}{s^2 + 1} \right]_{s \rightarrow s+1}$$

$$f(s) = \frac{1}{(s+1)^2 + 1}$$

$$sf(s) = \frac{s}{(s+1)^2 + 1}$$

$$\lim_{s \rightarrow \infty} sf(s) = \lim_{s \rightarrow \infty} \left[\frac{s}{(s+1)^2 + 1} \right]$$

$$= \lim_{s \rightarrow \infty} \left[\frac{s}{s^2(1+1/s)^2 + 1} \right]$$

$$= \lim_{s \rightarrow \infty} \left[\frac{s}{s^2(1+1/s)^2 + 1/s^2} \right]$$

$$= \lim_{s \rightarrow \infty} \frac{1}{s(1+1/s)^2 + 1/s^2}$$

$$= 0$$

$$\underline{\underline{LHS = RHS}}$$

$$Q \quad f(t) = 1 + e^{-t}(\sin t + \cos t)$$

$$Q \quad f(t) = e^{-t}(t+2)^2$$