UNIT-1

RANDOM VARIABLES

Random variables:

A real valued function defined on the outcome of a probability experiment is called a random variable.

Ex: (i) Tossing fair coin
(ii) Throwing a fair die.

Ex:
1) Tossing a coin
$S = \{H, T\}$
2) Tossing 2 coins simultaneously
$S = \{HH, HT, TH, TT\}$

A Random variable is the rule that assigns a numerical value to each possible outcome of an experiment.

i) Discrete Random variable.
ii) Continuous Random variable.

i) Discrete Random Variables: (countable)

A random variable whose set of possible values is either finite or countably infinite is called discrete random variable.

Ex.: i) The number of student in a class
ii) The number of traffic accidents.
iii) The number of Telephone calls.

Probability mass function: (pmf) $P(x)$

If X is a discrete random variable $P(x) = P[X = x]$ is called ~~the~~ probability mass function, provided $P(x_i)$ satisfy the following conditions.

i) $P(x) \geq 0$ (or) $0 \leq P(x) \leq 1$

ii) $\sum P(x) = 1$

* Cumulative distribution function (cdf) (discrete R.v)

The cumulative distribution function $F(x)$ of a discrete random variable X with probability distribution $p(x)$ is given by

$$F(x) = P(X \le x) = \sum_{x_j \le x} p_j$$

* Mean (or) Expected value of a Discrete random Variable X.

$E(X)$ (or) ($\bar{x}$)

mean $E(x) = \sum x \cdot P(x)$

* Variance :

Variance $Var(x) = E(x^2) - [E(x)]^2$

$E(x^2) = \sum x_i^2 p(x)$

Properties of distribution fn. :-

i) $F(-\infty) = 0$

ii) $F(\infty) = 1$

iii) $F(x) \ge 0$

iv) $P(x_1 \le X \le x_2) = F(x_2) - F(x_1)$.

v) $\frac{d}{dx} F(x) = f(x)$

Results :-

i) $P(X \le \infty) = 1$

ii) $P(X \le -\infty) = 0$

iii) $P(X > x) = 1 - P(X \le x)$

iv) $P(X \le x) = 1 - P(X > x)$

Formula :-

i) mean $E(x) = \sum x_i P(x_i)$

ii) $E(x^2) = \sum x_i^2 p(x_i)$

iii) Variance $Var(x) = E(x^2) - [E(x)]^2$

iv) $E(ax+b) = a\,E(x) + b$

v) $Var[ax \pm b] = a^2\, Var[x]$

v) Standard deviation(S.D) $= \sqrt{Var(x)} = \sigma$.

vi) $P(A|B) = \frac{P(A \cap B)}{P(B)}$

1. A random variable x has the following probability distribution

x :	-2	-1	0	1	2	3
$P(x)$:	0.1	K	0.2	$2K$	0.3	$3K$

a) Find K b) Evaluate $P(x<2)$ and $P(-2<x<2)$

c) Find the cdf of x d) Evaluate the mean of x

soln:-

a) <u>To find K</u>

W.k.T $\Sigma P(x) = 1$

$0.1 + K + 0.2 + 2k + 0.3 + 3k = 1$

$6K + 0.6 = 1 \Rightarrow 6k = 1 - 0.6 = 0.4$

$K = \frac{0.4}{6} = \frac{4}{60} = \frac{1}{15}$ $\therefore \boxed{K = 1/15}$

The probability distribution is

x :	-2	-1	0	1	2	3
$P(X=x)$:	$1/10$	$1/15$	$1/5$	$\frac{2}{15}$	$3/10$	$1/5$

b) <u>To find $P(X<2)$ & $P(-2<x<2)$</u>

$P(X<2) = P(X=1) + P(X=0) + P(X=-1) + P(X=-2)$

$P(X<2) = \frac{2}{15} + \frac{1}{5} + \frac{1}{15} + \frac{1}{10} = \frac{4+6+2+3}{30} = \frac{15}{30} = \frac{1}{2}$

$P(-2<x<2) = P(X=-1) + P(X=0) + P(X=1)$

$= \frac{1}{15} + \frac{1}{5} + \frac{2}{15} = \frac{1+3+2}{15} = \frac{6}{15} = \frac{2}{5}$

c) <u>To find cdf of X :- $F(x)$</u>

x	$P(x)$	$F(x)$
-2	$\frac{1}{10}$	$\frac{1}{10}$
-1	$\frac{1}{15}$	$\frac{1}{10}+\frac{1}{15} = \frac{3+2}{30} = \frac{5}{30} = \frac{1}{6}$
0	$\frac{1}{5}$	$\frac{1}{6}+\frac{1}{5} = \frac{6+5}{30} = \frac{11}{30}$
1	$\frac{2}{15}$	$\frac{11}{30}+\frac{2}{15} = \frac{11+4}{30} = \frac{15}{30} = \frac{1}{2}$
2	$\frac{3}{10}$	$\frac{1}{2}+\frac{3}{10} = \frac{8}{10} = \frac{4}{5}$
3	$\frac{1}{5}$	$\frac{4}{5}+\frac{1}{5} = 1$

4) **To find mean :-**

mean $E(x) = \sum x_i P(x_i)$

$$= \left(-2 \times \frac{1}{10}\right) + \left(-1 \times \frac{1}{15}\right) + \left(0 \times \frac{1}{5}\right) + \left(1 \times \frac{2}{15}\right) + \left(2 \times \frac{3}{10}\right) + \left(3 \times \frac{1}{5}\right)$$

$$= -\frac{1}{5} - \frac{1}{15} + 0 + \frac{2}{15} + \frac{\cancel{6}^{3}}{\cancel{10}_{5}} + \frac{3}{5}$$

$$= \frac{-3-1+2+9+9}{15} = \frac{16}{15}$$

HW

1. The discrete random variable X has the probability distribution given by

x :	0	1	2	3	4
$P(x)$:	K	$3K$	$5K$	$7K$	$9K$

Find (i) K (ii) mean (iii) variance (iv) $Var(3x-4)$

v) $P(0<x<3 \mid x>1)$.

2) A random variable X has the following probability distribution

x :	0	1	2	3	4	5	6	7
$P(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find a) The value of k (b) $P(1.5 < X < 4.5 / X > 2)$

and c) The smallest value of λ for which $P(X \le \lambda) > \frac{1}{2}$

d) Evaluate $P(X<6)$, $P(X \geq 6)$

Soln:-

a) Find K:

W.K.T $\Sigma P(x) = 1$

$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$10k^2 + 9k - 1 = 0$ $\left(\because \text{The quadratic form } x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \right)$

$k = \frac{-9 \pm \sqrt{81 - 4(10)(-1)}}{2(10)}$

$k = \frac{-9 \pm \sqrt{121}}{20} = \frac{-9 \pm 11}{20} = \frac{2}{20}, \frac{-20}{20}$ $\;\; \frac{1}{10}, -1$

$k = 1/10$ (or) -1

Since $P(X) \geq 0$ the value $k = -1$ is not permissible.

$\boxed{k = 1/10}$

x	0	1	2	3	4	5	6	7
$P(x)$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

To Find

b) $P(1.5 < x < 4.5) / X > 2)$

$P(A/B) = \frac{P(A \cap B)}{P(B)}$

$P(1.5 < X < 4.5 / X > 2) = \frac{\{P(X=2) + P(X=3) + P(X=4)\} \cap \{P(X=3) + P(X=4) + P(X=5) + P(X=6) + P(X=7)\}}{P(X>2)}$

$= \frac{P(X=3) + P(X=4)}{P(X=3) + P(X=4) + P(X=5) + P(X=6) + P(X=7)}$

$= \frac{\frac{2}{10} + \frac{3}{10}}{\frac{2}{10} + \frac{3}{10} + \frac{1}{100} + \frac{2}{100} + \frac{17}{100}} = \frac{\frac{5}{10}}{\frac{20+30+1+2+17}{100}}$

$P(1.5<x<4.5/x>2) = \dfrac{\frac{5}{10}}{\frac{7\cancel{0}}{10\cancel{0}}} = \dfrac{5}{7}$

c) To Find the smallest value of λ which $P(X \le \lambda) > \frac{1}{2}$.

by trails $P(X \le 0) = 0$

$P(X \le 1) = 0 + \frac{1}{10} = \frac{1}{10}$

$P(X \le 2) = \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$

$P(X \le 3) = \frac{3}{10} + \frac{2}{10} = \frac{5}{10} = \frac{1}{2}$.

$P(X \le 4) = \frac{1}{2} + \frac{3}{10} = \frac{8}{10}$.

$\therefore$ The smallest value of λ satisfying the condition $P(X \le \lambda) > \frac{1}{2}$ is $\underline{4}$

$\therefore \boxed{\lambda = 4}$

d) To Find $P(X<6)$ & $P(X \ge 6)$

$P(x \ge 6) = P(X=6) + P(X=7) = \frac{2}{100} + \frac{17}{100} = \frac{19}{100}$

$P(X<6) = 1 - P(X \ge 6)$ $\quad (\because P(X<x) = 1 - P(X \ge x))$

$P(X<6) = 1 - \frac{19}{100} = \frac{81}{100}$

HW

1. A discrete R.V X has the following probability distribution

x:	0	1	2	3	4	5	6	7	8
$P(x)$:	a	3a	5a	7a	9a	11a	13a	15a	17a

Find the value of a, $P(x<3)$, Variance and distribution function of x.

Ans: $a = 1/81$; $P(X<3) = 1/9$

distribution function of X.

$F(0) = 1/81$ $F(1) = 4/81$, $F(2) = 9/81$, $F(3) = \frac{16}{81}$, $F(4) = \frac{25}{81}$

$F(5) = \frac{36}{81}$ $F(6) = \frac{49}{81}$ $F(7) = \frac{64}{81}$; $F(8) = 1$

3) If the probability mass function of a R.V X is given by $P(X=r) = kr^3$; $r=1,2,3,4$ find (i) The value of k
(ii) $P(\frac{1}{2} < X < \frac{5}{2} / X>1)$ (iii) The mean and variance of X
(iv) The distribution function of X.

Soln:-

given $P(X=r) = kr^3$

The probability distribution is given by

X :	1	2	3	4
$P(x)$:	k	$8k$	$27k$	$64k$

(i) To Find k :-

We know that $\sum P(x) = 1$

$$k + 8k + 27k + 64k = 1$$

$$100k = 1$$

$$\boxed{k = \frac{1}{100}}$$

(ii) To Find $P(\frac{1}{2} < X < \frac{5}{2} / X>1)$

$$P(\frac{1}{2} < X < \frac{5}{2} / X>1) = \frac{P(\frac{1}{2} < X < \frac{5}{2} \cap X>1)}{P(X>1)} \quad \left(\because P(A/B) = \frac{P(A\cap B)}{P(B)}\right)$$

$$= \frac{P(X=2)}{P(X=2)+P(X=3)+P(X=4)}$$

$$= \frac{8k}{8k+27k+64k} = \frac{8k}{99k}$$

$$P(\frac{1}{2} < X < \frac{5}{2} / X>1) = \frac{8}{99}$$

X :	1	2	3	4
$P(X)$:	$\frac{1}{100}$	$\frac{8}{100}$	$\frac{27}{100}$	$\frac{64}{100}$

iii) <u>To find mean & Variance of x :</u>

$$E(X) = \sum_i x_i P(x_i) = 1\left(\frac{1}{100}\right) + 2\left(\frac{8}{100}\right) + 3\left(\frac{27}{100}\right) + 4\left(\frac{64}{100}\right)$$

$$= \frac{1+16+81+256}{100} = \frac{354}{100} = \underline{3.54}$$

$$E(x^2) = \sum x_i^2 P(x_i) = 1\left(\frac{1}{100}\right) + 4\left(\frac{8}{100}\right) + 9\left(\frac{27}{100}\right) + 16\left(\frac{64}{100}\right)$$

$$= \frac{1+32+243+1024}{100} = \frac{1300}{100}$$

$$E(x^2) = \underline{13}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= 13 - (3.54)^2 = 13 - 12.5316 = \underline{0.4684}$$

$\therefore$ mean(μ) = $\underline{3.54}$ & variance = $\underline{0.4684}$

iv) <u>To Find the distribution fn. of x :</u>

x	$P(x)$	$F(x) = P(X \le x)$
1	$1/100$	$F(1) = \frac{1}{100}$
2	$\frac{8}{100}$	$F(2) = \frac{1}{100} + \frac{8}{100} = \frac{9}{100}$
3	$\frac{27}{100}$	$F(3) = \frac{9}{100} + \frac{27}{100} = \frac{36}{100}$
4	$\frac{64}{100}$	$F(4) = \frac{36}{100} + \frac{64}{100} = \frac{100}{100} = \underline{1}$

4. If the random variables X takes the values 1, 2, 3 and 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$ Find the probability distribution and cumulative distribution function of X.

Soln:-

X is a discrete random variable.

given $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$

let $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = K \rightarrow (1)$

from (1) $(2P(X=1)=K \Rightarrow P(X=1)=K/2$; $3P(X=2)=K$ $P(X=3)=K$; $5P(X=4)=K \Rightarrow$ $P(X=2)=K/3$; $P(X=4)=\frac{K}{5})$

x :	1	2	3	4
$P(x)$:	$\frac{K}{2}$	$\frac{K}{3}$	K	$\frac{K}{5}$

We know that $\sum P(x_i)=1$ $\quad \frac{K}{2}+\frac{K}{3}+K+\frac{K}{5}=1$

$$\frac{15K+10K+30K+6K}{30}=1$$

$$\frac{61K}{30}=1 \Rightarrow \boxed{K=\frac{30}{61}}$$

The Probability distribution is

x :	1	2	3	4
$P(x)$:	$\frac{15}{61}$	$\frac{10}{31}$	$\frac{30}{61}$	$\frac{6}{61}$

To Find cumulative distribution function of x:

x	$P(x)$	$F(x)$
1	$\frac{15}{61}$	$F(1)=\frac{15}{61}$
2	$\frac{10}{61}$	$F(2)=\frac{15}{61}+\frac{10}{61}=\frac{25}{61}$
3	$\frac{30}{61}$	$F(3)=\frac{25}{61}+\frac{30}{61}=\frac{55}{61}$
4	$\frac{6}{61}$	$F(4)=\frac{55}{61}+\frac{6}{61}=\frac{61}{61}=1$

5. The probability function of an infinite discrete distribution is given by $P(X=j) = \frac{1}{2^j}$; $j = 1, 2, 3 \ldots \infty$. Find the mean and variance of the distribution. Also find $P(x \text{ is even})$, $P[X \geq 5]$ and $P[x \text{ is divisible by } 3]$.

Soln:- given $P(X=j) = \frac{1}{2^j}$; $j = 1, 2 \ldots \infty$

(i) To find Mean & Variance :-

(Formula.
$(1-x)^{-2} = 1+2x+3x^2+\ldots$
$(1-x)^{-3} = \frac{1}{2}(1\cdot2+2\cdot3x+3\cdot4x^2+\ldots)$
$(1-x)^{-1} = 1+x+x^2+x^3+\ldots$)

$$\text{Mean} = \sum_{j=1}^{\infty} x_j P(x_j)$$

$$= 1\cdot\frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + \ldots$$

$$= \frac{1}{2}\left[1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \ldots\right]$$

$$E(X) = \frac{1}{2}\left[1-\tfrac{1}{2}\right]^{-2} = \frac{1}{2}\left[\frac{1}{2}\right]^{-2} = \frac{1}{2}\times 4 = \underline{2}$$

$$\text{Var}[X] = E[X^2] - [E(X)]^2$$

$$E[X^2] = \sum_{j=1}^{\infty} x_j^2 P(x_j) = \sum_{j=1}^{\infty} (x_j)(x_j+1)P(x_j) - \sum_{j=1}^{\infty} x_j P(x_j)$$

$$E(X^2) = \left[(1)(2)\left(\frac{1}{2}\right) + (2)(3)\left(\frac{1}{2}\right)^2 + (3)(4)\left(\tfrac{1}{2}\right)^3 + \ldots\right] - 2$$

$$= \left(\frac{1}{2}\right)\left[(1)(2) + 2\cdot3\left(\frac{1}{2}\right) + 3\cdot4\left(\frac{1}{2}\right)^2 + \ldots\right] - 2$$

$$= \left(1-\tfrac{1}{2}\right)^{-3} - 2$$

$$= \left(\frac{1}{2}\right)^{-3} - 2 = 2^3 - 2 = 8-2 = \underline{6}$$

$\left((1-x)^{-3} = \frac{1}{2}(1\cdot2+2\cdot3x+3\cdot4x^2+\ldots)\right)$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 6 - (2)^2 = 6-4 = \underline{2}$$

$\therefore$ mean $= 2$

$\text{Var}(X) = 2$

ii) To Find P(x is even) :-

$P(x \text{ is even}) = P[x=2] + P[x=4] + P[x=6] + \cdots\cdots$

$P(x \text{ is even}) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \cdots\cdots$ $\quad P(x=j) = \frac{1}{2^j}$

$= \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots\cdots$ $\quad [\because (1-x)^{-1} = 1 + x + x^2 + \cdots$ $\quad (x + x^2 + x^3 + \cdots = (1-x)^{-1} - 1)$

$= (1 - 1/4)^{-1} - 1$

$= \left(\frac{3}{4}\right)^{-1} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$

iii) To Find $P[x \geq 5]$:

$P[x \geq 5] = P[x=5] + P[x=6] + P[x=7] + \cdots$

$= \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^7 + \cdots\cdots$ $\quad ((1-x)^{-1} = 1 + x + x^2 + \cdots \infty)$

$= \left(\frac{1}{2}\right)^5 \left[1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \cdots\right]$

$= \left(\frac{1}{2}\right)^5 \left[\left(1 - \frac{1}{2}\right)^{-1}\right]$

$= \left(\frac{1}{2}\right)^5 \left[\frac{1}{2}\right]^{-1} = \left(\frac{1}{2}\right)^5 \times 2 = \frac{1}{2^4} = \frac{1}{16}$

iv) To Find P[x is divisible by 3 (or) P(x) is multiple of 3]

$= P[x=3] + P[x=6] + P[x=9] + \cdots\cdots$

$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^9 + \cdots\cdots$

$= \left(\frac{1}{8}\right) + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \cdots\cdots$ $\quad$ (Formula $(1-x)^{-1} = 1 + x + x^2 + \cdots$) $\quad (\because x + x^2 + x^3 + \cdots = (1-x)^{-1} - 1)$

$= \left(1 - \frac{1}{8}\right)^{-1} - 1$

$= \left(\frac{7}{8}\right)^{-1} - 1 = \frac{8}{7} - 1 = \frac{1}{7}$

Continuous Random Variables:

Defn: A random variable x is said to be continuous if it takes all possible values between certain limits say from 'a' to real number 'b'.

Ex: height, weight, time.

Probability density function:- (P.d.f)

For a continuous random variable X, a probability density function such that

i) $f(x) \geq 0$

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Cumulative distribution functions (c.d.f):-

The cumulative distribution function of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx \quad \text{for } -\infty < x < \infty.$$

Mean (or) Expected value of a C.R.V 'x':-

Let X be a continuous random variable with p.d.f 'f' is defined in $(-\infty, \infty)$ then expected value of X is defined as

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx. \qquad (E(x) \overset{\text{mean}}{=} \bar{x} = \mu.)$$

Variance of C.R.V of 'x':-

$$Var(x) = E(x^2) - [E(x)]^2$$

[Note: Area $=$ $P[a \leq x \leq b] = \int_a^b f(x)\,dx$

$f(x) = \frac{d}{dx} F(x) = F'(x)$]

Note: If x is continuous R.V then
$P[x_1 \leq x \leq x_2] = P[x_1 < x \leq x_2] = P[x_1 \leq x < x_2] = P[x_1 < x < x_2] = P$

1. A continuous R.V x that can assume any value between $x=2$ and $x=5$ has a density fn given by $f(x) = k(1+x)$. Find $P[X<4]$.

Soln:- given $f(x) = k(1+x)$ & $x=2$ & $x=5$.

We know that $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_2^5 k(1+x)\,dx = 1$$

$$k\left[x+\frac{x^2}{2}\right]_2^5 = 1$$

$$k\left(5+\frac{25}{2}-2-\frac{\cancel{4}^{2}}{\cancel{2}}\right] = 1$$

$$k\left(1+\frac{25}{2}\right) = 1$$

$$k\left(\frac{27}{2}\right) = 1 \qquad \therefore \boxed{k=\frac{2}{27}}$$

(ii) To find $P[X<4]$; $\therefore f(x) = \frac{2}{27}(1+x)$ & $x=2$ & $x=5$)

$$P[X<4] = \int_2^4 \frac{2}{27}(1+x)\,dx$$

$$= \frac{2}{27}\left[x+\frac{x^2}{2}\right]_2^4$$

$$= \frac{2}{27}\left[4+\frac{\cancel{16}^{8}}{\cancel{2}}-2-\frac{\cancel{4}^{2}}{\cancel{2}}\right]$$

$$P[X<4] = \frac{16}{27}$$

2) The p.d.f of a r.v x is $f(x) = kx$, $0<x<1$, Find k and $P(X>0.5)$.

Soln:- given $f(x) = kx$; $0<x<1$

(i) To find k :-

We know that $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 kx\,dx = 1$$

$$k\left[\frac{x^2}{2}\right]_0^1 = 1 \Rightarrow \frac{k}{2} = 1 \Rightarrow \boxed{k=2}$$

$\therefore f(x) = 2x$; $0<x<1$

(ii) To find $P[X > 0.5]$:-

$$P[X > 0.5] = \int_{0.5}^{1} 2x\, dx$$

$$= 2\left[\frac{x^2}{2}\right]_{0.5}^{1} = 1 - 0.25 = \underline{\underline{0.75}}$$

3) A continuous RV X has a p.d.f $f(x) = Kx^2 e^{-x}$; $x > 0$. Find K, mean and variance.

Soln :-

Given $f(x) = Kx^2 e^{-x}$; $x > 0$.

(i) To find K :- We know that $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_{0}^{\infty} K x^2 e^{-x} dx = 1 \qquad \left(\because \int_0^{\infty} x^{n-1} e^{-x} dx = \Gamma n \quad \Gamma n+1 = n!\right)$$

$$K \cdot 2! = 1 \qquad \left(\int_0^{\infty} x^{3-1} e^{-x} dx = \Gamma 3 = 2!\right)$$

$$\boxed{K = \frac{1}{2}}$$

$$\therefore f(x) = \frac{1}{2} x^2 e^{-x}; \ x > 0$$

(ii) To find mean:

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$E(X) = \int_0^{\infty} x\left(\frac{1}{2} x^2 e^{-x}\right) dx \qquad \left(\because \int_0^{\infty} x^{4-1} e^{-x} dx = \Gamma 4 \quad \Gamma 4 = 3!\right)$$

$$= \frac{1}{2} \int_0^{\infty} x^3 e^{-x} dx$$

$$= \frac{1}{2} \times 3! = \frac{1}{\cancel{2}} \times 1 \times \cancel{2} \times 3 = \underline{\underline{3}}$$

$\therefore$ Mean $E(X) = 3$

ii) To find Variance :-

$$Var[X] = E[X^2] - [E[X]]^2$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_{0}^{\infty} x^2 \left(\frac{1}{2} x^2 e^{-x}\right) dx$$

$$= \frac{1}{2}\int_{0}^{\infty} x^4 e^{-x}\, dx \qquad \left(\int_0^{\infty} x^{5-1} e^{-x} dx = \Gamma 5 \right.$$

$$\left. \Gamma 5 = 4! \right)$$

$$= \frac{1}{2} \times 4! = \frac{1}{2} \times 1 \times 2 \times 3 \times 4$$

$$E[X^2] = \underline{\underline{12}}$$

$$Var[X] = E[X^2] - (E[X])^2$$

$$= 12 - 3^2 = 12 - 9 = \underline{\underline{3}}$$

$\therefore$ Mean $E[X] = \underline{\underline{3}}$ & Variance $Var[X] = \underline{\underline{3}}$

4) A Continuous R.V X has a p.d.f $f(x) = 3x^2$, $0 < x < 1$. Find a & b such that

(i) $P[X \le a] = P[X > a]$ (ii) $P[X > b] = 0.05$

Soln:- $P[X \le a] = P[X > a] = \frac{1}{2}$ $\quad (\because P[X > a] = 1 - P[X \le a]$

i) $P[X \le a] = \int_0^a 3x^2 dx = \frac{1}{2}$ $\quad 2P[X \le a] = 1$

$\quad P[X \le a] = \frac{1}{2})$

$$\Rightarrow 3\left[\frac{x^3}{3}\right]_0^a = \frac{1}{2}$$

$$a^3 = \frac{1}{2} \qquad \boxed{a = \left(\frac{1}{2}\right)^{\frac{1}{3}}}$$

(ii) $P[X > b] = 0.05$

$$\int_b^1 f(x)\,dx = 0.05$$

$$\int_b^1 3x^2 dx = 0.05$$

$$3\left[\frac{x^3}{3}\right]_b^1 = 0.05$$

$$1^3 - b^3 = 0.05$$

$$b^3 = 1 - 0.05 = 0.95$$

$$\boxed{b = (0.95)^{\frac{1}{3}}}$$

HW

4) In a continuous R.V the p.d.f is given by $f(x) = kx(2-x)$, $0 < x < 2$. Find k, mean, variance and the distribution fn.

Ans: $k = 3/4$; $E[X] = 1$; $V(X) = \frac{1}{5}$; $F(x) = 0$ when $x < 0$

$F(x) = \frac{1}{4}[3x^2 - x^3]$ when $0 \le x \le 2$ & $F(x) = 1$;

5) The diameter of an electric cable X is a continuous r.v with p.d.f $f(x) = kx(1-x)$; $0 \le x \le 1$.

i) Find the value of k.

ii) c.d.f of X.

iii) the value of a such that $P[X < a] = 2P[X > a]$

iv) $P[X \le 1/2 / 1/3 < X < \frac{2}{3}]$.

Soln:- given $f(x) = kx(1-x)$; $0 \le x \le 1$.

(i) To find k :-

We know that $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_0^1 kx(1-x)\, dx = 1$$

$$k \int_0^1 (x - x^2)\, dx = 1$$

$$k \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$k \left[\frac{1}{2} - \frac{1}{3}\right] = 1$$

$$k \left[\frac{3-2}{6}\right] = 1 \Rightarrow \boxed{k = 6}$$

$$k \left[\frac{1}{6}\right] = 1$$

$\therefore f(x) = 6x(1-x)$; $0 \le x \le 1$

ii) To find cumulative distribution fn. of x :-

(i) $F(x) = 0$; when $x < 0$

(ii)

$$F(x) = \int_{-\infty}^{x} f(x)\,dx.$$

$$F(x) = \int_{0}^{x} 6\,(x(1-x))\,dx$$

$$= 6\int_{0}^{x} x - x^2\,dx$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^x = 6\left[\frac{3x^2-2x^3}{6}\right]_0^x$$

$$F(x) = 3x^2 - 2x^3 \quad \text{when} \quad 0 \le x \le 1$$

$$F(x) = 1 \quad \text{when} \quad x \geq 1$$

iii) $P[x<a] = 2[P[x>a)]$

We know that

$$P[x<a] = 1/2 = P[x>a]$$

$$P[x<a] = 1/2$$

$$\int_0^a f(x)\,dx = 1/2$$

$$6\int_0^a x(1-x)\,dx = 1/2$$

$$6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^a = 1/2$$

$$6\left[\frac{3x^2-2x^3}{6}\right]_0^a = 1/2$$

$$3a^2 - 2a^3 = 1/2$$

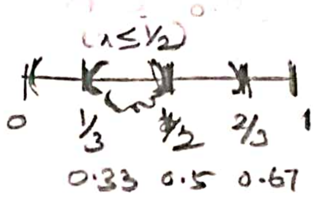
$$6a^2 - 4a^3 - 1 = 0$$

$$4a^3 - 6a^2 + 1 = 0$$

$$\therefore a = 1/2$$

ii) To find $P[x \leq 1/2 \,/\, 1/3 < x < 2/3]$:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$



$$= \frac{P[(x \leq 1/2) \cap (1/3 < x < 2/3)]}{P(1/3 < x < 2/3)}$$

$$= \frac{P[1/3 < x < 1/2]}{P[1/3 < x < 2/3]}$$

$$P[1/3 < x < 1/2] = \int_{1/3}^{1/2} f(x)\,dx \qquad (f(x) = 6(x - x^2) \text{ in } 0 \leq x \leq 1$$

$$= \int_{1/3}^{1/2} 6(x - x^2)\,dx$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{1/3}^{1/2}$$

$$= \cancel{6}\left[\frac{3x^2 - 2x^3}{\cancel{6}}\right]_{1/3}^{1/2}$$

$$= 3(1/4) - 2(1/8) - 3(1/9) + 2\left(\frac{1}{27}\right)$$

$$= \left(\frac{3-1}{4}\right) - \frac{1}{3} + \frac{2}{27}$$

$$= 1/2 - \frac{1}{3} + \frac{2}{27}$$

$$= \frac{27 - 18 + 4}{54} = \underline{\underline{\frac{13}{54}}}$$

$$P[1/3 < x < 2/3] = \int_{1/3}^{2/3} 6(x - x^2)\,dx$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{1/3}^{2/3} = \cancel{6}\left[\frac{3x^2 - 2x^3}{\cancel{6}}\right]_{1/3}^{2/3}$$

$$= 3(4/9) - 2(8/27) - 3(1/9) + 2\left(\frac{1}{27}\right)$$

$$= \frac{36 - 16 - 9 + 2}{27} = 13/27$$

$$\therefore P[x \leq 1/2 \,/\, 1/3 < x < 2/3] = \frac{13/54}{13/27} = \frac{13}{54} \times \frac{27}{13} = \underline{\underline{1/2}}$$

6 If the density function of a continuous random variable X is given by

$$f(x) = \begin{cases} ax & 0 \le x \le 1 \\ a & 1 \le x \le 2 \\ 3a - \cancel{2a}x & 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of a and b) find the c.d.f of X.

c) $P[X > 1.5]$.

Soln :- a) To find a :-

since $f(x)$ is p.d.f $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^3 f(x)\,dx = 1$$

$$\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx = 1$$

$$a\left[\frac{x^2}{2}\right]_0^1 + a[x]_1^2 + \left[3ax - a\frac{x^2}{2}\right]_2^3 = 1$$

$$\frac{a}{2} + a(2-1) + 3a(3) - a(9/2) - 6a + a(\cancel{4}/2) = 1$$

$$\frac{a}{2} + a + 9a - \frac{9a}{2} - 6a + 2a = 1$$

$$a + 9a - 6a + 2a + \left(\frac{-9a + a}{2}\right) = 1$$

$$a + 9a - 6a + 2a - 4a = 1$$

$$2a = 1 \Rightarrow \boxed{a = 1/2}$$

$$\therefore f(x) = \begin{cases} \frac{x}{2} & ; 0 \le x \le 1 \\ 1/2 & ; 1 \le x \le 2 \\ 3/2 - \frac{x}{2} & ; 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

b) To find the c.d.f of x :-

$$F(x) = P[X \le x] = \int_{-\infty}^{x} f(x)\,dx$$

(i) If $x < 0$ then $F(x) = 0$

(ii) If $0 \le x \le 1$ then $F(x) = 0 + \int_0^x \frac{x}{2}\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{4}$

~~(iii) If $1 \le x \le 2$ then $F(x) = \int_{-\infty}^{x} f\,dx$~~

iii) If $1 \le x \le 2$ then $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$$= \int_0^1 \frac{x}{2}\,dx + \int_1^x \frac{1}{2}\,dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \frac{1}{2}\left[x\right]_1^x$$

$$= \frac{1}{4} + \frac{1}{2}[x-1] = \frac{1}{4} + \frac{1}{2}x - \frac{1}{2}$$

$$F(x) = \frac{1}{2}x - \frac{1}{4}$$

iv) If $2 \le x \le 3$ then $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$$= \int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{1}{2}\,dx + \int_2^x \left(\frac{3}{2} - \frac{x}{2}\right)dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \frac{1}{2}\left[x\right]_1^2 + \left[\frac{3}{2}x - \frac{x^2}{4}\right]_2^x$$

$$= \frac{1}{4} - 0 + \frac{1}{2}[2-1] + \frac{3}{2}x - \frac{x^2}{4} - \frac{3}{2}(2) + \frac{4}{4}^{1}$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{3}{2}x - \frac{x^2}{4} - 3 + 1$$

$$= \frac{1}{4} + \frac{1}{2} - 2 + \frac{3}{2}x - \frac{x^2}{4} = \frac{3}{2}x - \frac{x^2}{4} + \left(\frac{1+2-8}{4}\right)$$

$$F(x) = \frac{3}{2}x - \frac{x^2}{4} - \frac{5}{4}$$

v) If $x > 3$ then $F(x) = \int_{-\infty}^{x} f(x)\,dx \Rightarrow F(x) = 1$

$$F(x) = \int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{1}{2}\,dx + \int_2^3 \left(\frac{3}{2} - \frac{x}{2}\right)dx + \int_3^x f(x)\,dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \frac{1}{2}\left[x\right]_1^2 + \left[\frac{3}{2}x - \frac{x^2}{4}\right]_2^3 + 0$$

$$= \frac{1}{4} - 0 + \frac{1}{2}(2-1) + \frac{3}{2}(3) - \frac{9}{4} - \frac{3}{2}(2) + \frac{4}{4}$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{9}{2} - \frac{9}{4} - 3 + 1 = \frac{1+2+18-9}{4} - 2$$

$$= \frac{12}{4} - 2 = 3 - 2 = 1$$

If $x > 3$ then $F(x) = 1$

c) $P[x > 1.5] = \int_{1.5}^{3} f(x)\,dx$

$$= \int_{1.5}^{2} \frac{1}{2}\,dx + \int_{2}^{3} \left(\frac{3}{2} - \frac{x}{2}\right) dx$$

$$= \frac{1}{2}[x]_{1.5}^{2} + \left[\frac{3}{2}x - \frac{x^2}{4}\right]_2^3$$

$$= \frac{1}{2}[2-1.5] + \frac{3}{2}\cdot 3 - \frac{9}{4} - \frac{3}{2}(2) + \frac{4}{4}$$

$$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{9}{2} - \frac{9}{4} - 3 + 1$$

$$= \frac{1}{4} + \frac{18-9}{4} - 2$$

$$= \frac{10}{4} - 2 = \frac{10-8}{4} = \frac{2}{4} = \frac{1}{2}$$

HW

1) x is a continuous random variable with p.d.f given by $f(x) = Kx$ in $0 \le x \le 2$ $= 2K$ in $2 \le x \le 4$ and $= 6K - Kx$ in $4 \le x \le 6$. Find the value of K and $F(x)$.

Ans

$$f(x) = \begin{cases} Kx & 0 \le x \le 2 \\ 2K & 2 \le x \le 4 \\ 6K - Kx & 4 \le x \le 6 \end{cases} \qquad K = 1/8,\ F'$$

$F(x) = 0$ when $x < 0$

$F(x) = \frac{x^2}{16}$ when $0 \le x \le 2$

$F(x) = \frac{1}{4}(x-1)$ when $2 \le x \le 4$

$F(x) = -\frac{1}{16}(20 - 12x + x^2)$ when $4 \le x \le 6$

$F(x) = 1$ when $x \ge 6$