Gauss Divergence Theorem.

If $\vec{F}$ is a vector point function, finite and differentiable in a region, R bounded by a closed surface, S, then the surface integral of the normal component of $\vec{F}$ taken over S is equal to the integral of divergence of $\vec{F}$ taken over "V".

$$\iint_S \underbrace{\vec{F}.\hat{n}}_{\text{Normal Component of } \vec{F}.} ds = \iiint_V \nabla . \vec{F} . dv.$$

① Verify the Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$.

Solution:-

$$\nabla . \vec{F} = \left[\vec{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right]\left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right).$$

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(4xz) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y \Rightarrow (4z - y).$$

$$\iiint_V \nabla\cdot\vec{F}\, dv = \int_0^1\int_0^1\int_0^1 (4z - y)\, dx\, dy\, dz.$$

$$\Rightarrow \int_0^1\int_0^1 \left[4zx - yx\right]_0^1 dy\, dz.$$

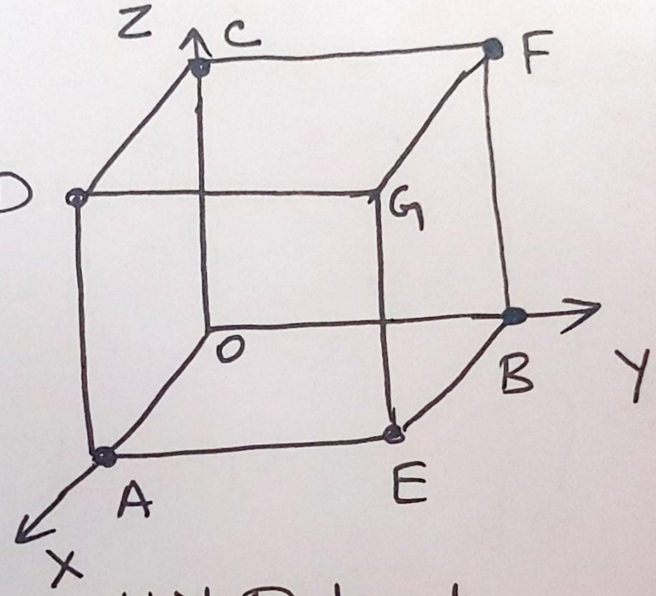
$$= \int_0^1\int_0^1 \left(\left[4z - y\right] - 0\right) dy\, dz$$

$$= \int_0^1 \left[4zy - \frac{y^2}{2}\right]_0^1 dz \Rightarrow \int_0^1 \left[\left(4z - \frac{1}{2}\right) - 0\right] dz$$

$$= \left[\frac{4z^2}{2} - \frac{1}{2}z\right]_0^1 = \left[2z^2 - \frac{1}{2}z\right]_0^1 = \left[\left(2 - \frac{1}{2}\right) - 0\right]$$

$$= 3/2. \Rightarrow \text{R.H.S}.$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}.$$



Surface	Face	Unit Outward Normal Vector.
S1.	AEGD	$\vec{i}$
S2	OBFC	$-\vec{i}$
S3	EBFG	$\vec{j}$
S4	OADC	$-\vec{j}$
S5	DGFC	$\vec{k}$
S6	OAEB	$-\vec{k}$.

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Evaluation of $\iint_{S1} \vec{F}.\hat{n}\,ds + \iint_{S2} \vec{F}.\hat{n}.ds.$

$$\Rightarrow \iint_{\substack{AEGD \\ x=1}} \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right)\vec{i}.\,dy\,dz.$$

$$+ \iint_{\substack{OBFC \\ x=0}} \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right)\left(-\vec{i}\right)dy\,dz.$$

$$\Rightarrow \int_0^1\int_0^1 4xz\,dy\,dz + \int_0^1\int_0^1 (-4xz)\,dy\,dz.$$

$$= \int_0^1\int_0^1 4z\,dy\,dz + \int_0^1\int_0^1 -4(0)z\,dy\,dz.$$

$$= \int_0^1 \left[4zy\right]_0^1 dz. = \int_0^1 \left(4z(1) - 4z(0)\right)dz$$

$$= \int_0^1 4z\,dz = \left[\frac{4z^2}{2}\right]_0^1 = \frac{4(1)^2}{2} - 0.$$

$= 2$ //.

Evaluation of $\iint\limits_{S3} \vec{F}.\hat{n}ds + \iint\limits_{S4} \vec{F}.\hat{n}ds$. (5)

$$= \iint\limits_{\substack{EBFG \\ y=1}} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})(\vec{j})\,dx\,dz$$

$$+ \iint\limits_{\substack{OADC \\ y=0}} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})(-\vec{j})\,dx\,dz.$$

$$= \int_0^1\int_0^1 -y^2\,dx\,dz + \int_0^1\int_0^1 +y^2\,dx\,dz.$$

$$= \int_0^1\int_0^1 -(1)\,dx\,dz + 0. \Rightarrow -\int_0^1 [x]_0^1\,dz.$$

$$= -\int_0^1 1\,dz = -[z]_0^1 = \underline{\underline{-1}}$$

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Evaluation of $\iint_{S5} \vec{F}\cdot\hat{n}\,ds + \iint_{S6} \vec{F}\cdot\hat{n}\,ds$.

$$\Rightarrow \iint_{DGFC} \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right)\left(\vec{k}\right) dx\,dy$$

$$+ \iint_{OAEB} \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right)\left(-\vec{k}\right) dx\,dy.$$

$$= \int_0^1\int_0^1 yz\,dx\,dy + \int_0^1\int_0^1 -yz\,dx\,dy.$$

$z=1$. $\quad z=0$.

$$= \int_0^1\int_0^1 y\,dx\,dy + 0.$$

$$= \int_0^1 \left[xy\right]_0^1 dy = \int_0^1 \left[(1)y - (0)y\right] dy$$

$$= \int_0^1 y\,dy \Rightarrow \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$$

⑦.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 2 + (-1) + \frac{1}{2}$$

$$= 1 + \frac{1}{2} = \frac{3}{2} \Rightarrow L.H.S$$

R.H.S = LHS.

$$\iiint_V \nabla\cdot\vec{F}\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds.$$

$$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\cdot dv.$$

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