Anjali Agrawal Tutorial-sheet 3 RA1811003010096

① $\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta$

$2\int_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta$

$= \frac{2\cdot 2^3}{3}\int_0^{\pi/2}\cos^3\theta\,d\theta$

$= \frac{2}{3}\cdot\frac{2}{3}\cdot 8 = \frac{32}{9}$

② $\int_0^{\pi/2}\int_0^{\infty}\frac{r\,dr\,d\theta}{(r^2+a^2)^2}$ $r^2+a^2=t$

$2r\,dr = dt$

$\int_0^{\pi/2}\int_{a^2}^{\infty}\frac{dt}{2t^2}\,d\theta = \int_0^{\pi/2}\left[-\frac{1}{2}\frac{1}{t}\right]_{a^2}^{\infty} d\theta$

$= \int_0^{\pi/2}\frac{1}{2}\frac{1}{a^2}\,d\theta$

$= \frac{1}{2a^2}\cdot\frac{\pi}{2} = \frac{\pi}{4a^2}$

③ Chang into polar co-ordinates $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$

$x = r\cos\theta$ $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$ $\int_0^{\pi/2}\int_0^{\infty} re^{-r^2}\,dr\,d\theta$

④ Evaluate $\int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^4\,dr\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\frac{a^5}{5}\,d\phi\,d\theta = \int_0^{2\pi}\frac{a^5}{5}\cdot\pi\,d\theta$

$= \frac{2a^5\pi^2}{5}$

$$\int_0^{\log a}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx.$$

$$\int_0^{\log a}\int_0^{x} e^{2(x+y)}\,dx\,dy = \int_0^{\log a}\left[\frac{e^{2(x+y)}}{2}\right]_0 dy - \int_0^{\log a}\int_0^{x} e^{(x+y)}\,dx\cdot dy$$

$$\int_0^{\log a}. \qquad = \frac{1}{2}\int_0^{\log a} e^{4x}\,dy - \frac{e^{2x}}{2}\,dx\,dy - e^{2x}dy + e^{x}\,dy$$

$$= \frac{1}{8}\left[e^{4(\log a)}\right] - \frac{3}{4}e^{2\log a} + e^{\log a}$$

$$= \frac{a^4}{8} - \frac{3a^2}{4} + a.$$

Find $\iiint_V \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$ where v is region of space bounded by co-ordinate planes and the sphere $x^2+y^2+z^2=1$

$$8\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} \sin^{-1}\frac{\sqrt{1-x^2-y^2}}{1-x^2-y^2}\,dy\,dx = \int_0^1\int_0^{\sqrt{1-x^2}} \sin^{-1}1\,dy\,dx$$

$$= \int_0^1 \pi/_2 \cdot \sqrt{1-x^2}\,dx$$

$$= \frac{\pi}{2}\cdot\left(\frac{1}{2}\sin^{-1}x + \frac{x}{2}\sqrt{1-x^2}\right)_0^1$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\cdot \pi/_2\right]$$

$$= \frac{\pi^2}{8}.$$

7 Find the volume of the tetrahedron bounded by the planes $x=0, y=0, z=0$ $x+y+z=5$.

$$\int_0^5\int_0^{5-y}\int_0^{5-x-y} dz\cdot dx\cdot dy$$

$$\int_0^5\int_0^{5-y} 5-x-y\,dx\,dy = \int_0^5 \left(5x-\frac{x^2}{2}-yx\right)_0^{5-y} dy\,d.$$

$$\int_0^5 25-5y-\frac{(25+y^2-10y)}{2}-5y+y^2\,dy$$

$$=\left[25y-\frac{5y^2}{2}-\frac{25}{2}y-y^3/6+\frac{10y^2}{4}-\frac{5y^2}{2}+y^3/3\right]_0^5$$

$$125-\frac{125}{2}-\frac{125}{2}-\frac{125}{6}+\frac{250}{4}-\frac{125}{2}+\frac{125}{3}$$

$$=-\frac{125}{6}+\frac{250}{6}=\frac{125}{6}\ //\text{Ans}.$$

(8) Find the volume of the tetrahedron bounded by the coordinate planes and $\frac{x}{a}+y/b+z/c=1$

$$=\int_0^a\int_0^{b(1-x/a)}\int_0^{c(1-x/a-y/b)} dz\,dy\,dx. = c\int_0^a\int_0^{b(1-x/a)} \left(1-\frac{x}{a}-y/b\right)dy\,dx$$

$$c\int_0^a \left[(1-x/a)y-y^2/2b\right]_0^{b(1-x/a)} dx$$

$$c\int_0^a b(1-x/a)^2-b^2/2b\,(1-x/a)^2\Big]dx$$

$$\frac{bc}{2}\int_0^a (1-x/a)^2dx=\frac{bc}{2}\left[\frac{(1-x/a)^3}{3}\times(-a)\right] = \frac{abc}{6}.$$

(9) Find the volume bounded by the cylinder $x^2+y^2=4$ & the planes $y+z=4$ & $z=0$.

Z varies as $z=0$ to $z=4-y$

x & y varies as circle.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{4-y} dz\,dy\,dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\,dy\,dx.$$

$$=\int_{-2}^{2}\left[4y-\frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$\int_{-2}^{2} 4(\sqrt{4-x^2}) + 4\sqrt{4-x^2} - \frac{1}{2}\left[(4-x^2) - (4-x^2)\right] dx.$$

$$\int_{-2}^{2} 8\sqrt{4-x^2}\, dx.$$

$$= 16\int_0^2 \sqrt{4-x^2}\, dx = 16\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]$$

$$= 16\left[0 + 2\pi/2 + 0\right]$$

$$= \underline{\underline{16\pi}}$$

10) Find the volume of sphere $x^2+y^2+z^2=a^2$ using triple integration

$$8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx.$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy \cdot dx$$

$$= 8\int_0^a \left(\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right)_0^{\sqrt{a^2-x^2}}$$

$$= 2\pi\int_0^a a^2-x^2\, dx$$

$$= 2\pi\left[a^2x - \frac{x^3}{3}\right] = \underline{\frac{4\pi}{3}a^3}$$