(X) (2) Find the eqn of the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Soln: The parametric form of the ellipse are $x = a\cos\theta$; $y = b\sin\theta$.

$\frac{dx}{d\theta} = -a\sin\theta$ | $\frac{dy}{d\theta} = b\cos\theta$.

$$\frac{dy}{dx} = \frac{b\cos\theta}{-a\sin\theta} = \boxed{-\frac{b}{a}\cot\theta = Y_1}$$

$$Y_2 = -\frac{b}{a}(-\operatorname{cosec}^2\theta) \cdot \frac{d\theta}{dx}.$$

$$= \frac{b}{a}\operatorname{cosec}^2\theta \cdot \frac{1}{\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{b}{a}\operatorname{cosec}^2\theta \cdot \frac{1}{-a\sin\theta}$$

$$= -\frac{b}{\cancel{a}}\operatorname{cosec}^2\theta \cdot \frac{1}{\cancel{a}}\operatorname{cosec}\theta$$

$$\boxed{Y_2 = -\frac{b}{a^2}\operatorname{cosec}^3\theta}$$

$$\bar{X} = x - \frac{Y_1(1+Y_1^2)}{Y_2}$$

$$= a\cos\theta - \frac{\left(-\frac{b}{a}\cot\theta\right)\left(1+\frac{b^2}{a^2}\cot^2\theta\right)}{-\frac{b}{a^2}\operatorname{cosec}^3\theta}.$$

$$= a\cos\theta - \frac{\frac{b}{a}\frac{\cos\theta}{\sin\theta}\left[\frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2\sin^2\theta}\right]}{\frac{b}{a^2}\cdot\frac{1}{\sin^3\theta}}$$

$$= a\cos\theta - \frac{\frac{b}{a^3}\frac{\cos\theta}{\sin^3\theta}\left[a^2\sin^2\theta + b^2\cos^2\theta\right]}{\left(\frac{b}{a^2\sin^3\theta}\right)}$$

$$= a\cos\theta - \frac{\cancel{b}}{a^{\cancel{3}}a}\frac{\cos\theta}{\sin^{\cancel{3}}\theta}(a^2\sin^2\theta + b^2\cos^2\theta) \times \frac{\cancel{a^2}\cancel{\sin^3\theta}}{\cancel{b}}$$

$$= a\cos\theta - \frac{\cancel{a^2}}{\cancel{a}}\cos\theta\sin^2\theta - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos\theta\left[1 - \sin^2\theta\right] - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos\theta(\cos^2\theta) - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos^3\theta - \frac{b^2}{a}\cos^3\theta$$

$$\overline{X} = \left(\frac{a^2-b^2}{a}\right)\cos^3\theta \Rightarrow a\overline{X} = (a^2-b^2)\cos^3\theta \quad \text{--- (1)}$$

$$\overline{Y} = y + \left(\frac{1+y_1^2}{y_2}\right)$$

$$= b\sin\theta + \left(\frac{1+\frac{b^2}{a^2}\cot^2\theta}{-\frac{b}{a^2}\operatorname{cosec}^3\theta}\right)$$

$$= b\sin\theta + \frac{\left(\frac{a^2+b^2\cot^2\theta}{a^2}\right)}{\left(-\frac{b}{a^2}\operatorname{cosec}^2\theta\right)}$$

$$= b\sin\theta + \frac{a^2 + b^2 \cdot \frac{\cos^2\theta}{\sin^2\theta}}{-b\,\mathrm{cosec}^3\theta}$$

$$= b\sin\theta - \frac{\left(\frac{a^2\sin^2\theta + b^2\cos^2\theta}{\sin^2\theta}\right)}{b\frac{1}{\sin^3\theta}}$$

$$= b\sin\theta - \left[\frac{(a^2\sin^2\theta + b^2\cos^2\theta)}{b\sin^2\theta} \times \sin^3\theta\right]$$

$$= b\sin\theta - \left[\frac{a^2}{b}\sin^3\theta + \frac{b^2}{b}\sin\theta\cos^2\theta\right]$$

$$= b\sin\theta - a^2/b\,\sin^3\theta - b\sin\theta\cos^2\theta$$

$$= b\sin\theta\left[1-\cos^2\theta\right] - a^2/b\,\sin^3\theta$$

$$= b\sin^3\theta - a^2/b\,\sin^3\theta$$

$$= \frac{b^2\sin^3\theta - a^2\sin^3\theta}{b}$$

$$\bar{Y} = \left(\frac{b^2-a^2}{b}\right)\sin^3\theta$$

$$b\bar{y} = (b^2-a^2)\sin^3\theta$$

$$b\bar{y} = (a^2-b^2)\sin^3\theta \quad — (2)$$

taking $2/3^{th}$ root on both sides of (1) & (2)

$$(1) \Rightarrow (a\bar{x})^{2/3} = (a^2-b^2)^{2/3}(\cos^3\theta)^{2/3} \quad — (3)$$

$$(2) \Rightarrow (b\bar{y})^{2/3} = (a^2-b^2)^{2/3}(\sin^3\theta)^{2/3} \quad — (4)$$

Envelope.

Find the envelope of the family of St. Line.

$$y = mx + \frac{1}{m}$$

$$y = \frac{m^2 x + 1}{m}.$$

$$my = m^2 x + 1$$

$$\boxed{m^2 x - my + 1 = 0}$$

$$\boxed{Ax^2 + Bx + C = 0}$$

$$A = x \ ; \ B = -y \ ; \ C = 1.$$

Equation of envelope is

$$B^2 - 4AC = 0.$$

$$(-y)^2 - 4x(1) = 0$$

$$y^2 - 4x = 0$$

$$y^2 = 4x$$

(2) Find the envelope of $y = mx + \sqrt{a^2m^2 + b^2}$

$$y^2 = m^2x^2 + a^2m^2 + b^2$$

$$y^2 - m^2x^2 - a^2m^2 - b^2 = 0$$

$$y - mx = \sqrt{a^2m^2 + b^2}$$

Squaring on both sides.

$$(y - mx)^2 = a^2m^2 + b^2$$

$$y^2 + m^2x^2 - 2mxy - a^2m^2 - b^2 = 0$$

$$m^2(x^2 - a^2) - 2xym + (y^2 - b^2) = 0$$

$A = x^2 - a^2$; $B = -2xy$ $C = y^2 - b^2$

$$B^2 - 4AC = 0$$

$$4x^2y^2 - 4(x^2 - a^2)(y^2 - b^2) = 0$$

$$4x^2y^2 - 4[x^2y^2 - x^2b^2 - a^2y^2 + a^2b^2] = 0$$

$$\cancel{x^2y^2} - \cancel{x^2y^2} + x^2b^2 + a^2y^2 - a^2b^2 = 0$$

$$x^2b^2 + a^2y^2 = a^2b^2$$

$\div a^2b^2$ on both sides.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$