Module – 5 Complex Integration

Cauchy's integral formulae – Problems – Taylor's expansions with simple problems – Laurent's expansions with simple problems – Singularities – Types of Poles and Residues – Cauchy's residue theorem (without proof) – Contour integration: Unit circle, semicircular contour – Application of Contour integration in Engineering.

Cauchy's Integral Theorem

If $f(z)$ is analytic at every point of the region R bounded by a simple closed curve C and if $f'(z)$ is continuous at all points inside and on C, then $\int_C f(z)\,dz = 0$

Cauchy's integral formula for n^{th} derivative

If $f(z)$ is analytic inside and on a simple closed curve C and $z = a$ is any interior point of the region R enclosed by C, then $f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$

$$(i.e.)\quad \boxed{\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(a)}$$

Taylor's series

If $f(z)$ is analytic inside a circle C with centre at a then Taylor's series about $z = a$ is

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots$$

Laurent's series

If C_1, C_2 are two concentric circles with centre at z = a and radii r_1 and r_2 ($r_1 < r_2$) and if f(z) is analytic inside and on the circles and within the annular region between C_1 and C_2, then for any z in the annular region, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n},$$

where $a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{(z-a)^{n+1}}dz$ and $b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{(z-a)^{-n+1}}dz$

Cauchy's Residue theorem

If f(z) is analytic inside a closed curve C except at a finite number of isolated singular points $a_1, a_2, \dots a_n$ inside C, then

$\int_C f(z)dz = 2\pi i \times$(sum of the residues of f(z) at these singular points).

Contour Integration

Type I:

$$\boxed{\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta}$$

$$\text{Let } z = e^{i\theta}, \quad dz = ie^{i\theta} d\theta = iz d\theta \;\Rightarrow d\theta = \frac{dz}{iz}$$

Then we have

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right); \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\cos 2\theta = \text{Real part of } z^2 \;; \qquad \cos n\theta = \text{Real part of } z^n$$

$$\sin 2\theta = \text{Im part of } z^2; \qquad \sin n\theta = \text{Im part of } z^n$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} = \text{Real part of} \left[\frac{1+z^2}{2}\right];$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} = \text{Real part of} \left[\frac{1-z^2}{2}\right]$$

$\therefore$

$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \int_C f(z)\, dz$, where C is $|z| = 1$ and solve by known method.

Type II:

$$\boxed{\int_{-\infty}^{\infty} f(x)\, dx}$$

Using Cauchy's integral formula, find $\int_C \frac{z+4}{z^2+2z+5}\, dz$**, where C is** $|z+1-i| = 2$

Solution:

$|\, z + 1 - i \,| = 2$

$|x + iy + 1 - i| = 2$

$|(x+1) + i(y-1)| = 2, \qquad \sqrt{(x+1)^2 + (y-1)^2} = 2$

Squaring on both sides,

$$(x+1)^2+(y-1)^2=4$$

This is equation of circle with centre $(-1,1)$ and radius 2.

$$z^2+2z+5=0$$

$$z=\frac{-2\pm\sqrt{4-4(1)(5)}}{2(1)}=\frac{-2\pm 4i}{2}=-1\pm 2i$$

$$\int_C \frac{z+4}{z^2+2z+5}\,dz=\int_C \frac{z+4}{\left[z-(-1+2i)\right]\left[z-(-1-2i)\right]}\,dz$$

Here –1+2*i* lies inside the circle c and –1–2*i* lies outside the circle c.

Let $a=-1+2i$

By Cauchy's integral formula, $f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$

Substituting for *a*, $f(-1+2i)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-(-1+2i)}\,dz$ ......(1)

Comparing equation (1) with given problem,

$$f(z)=\frac{z+4}{z-(-1-2i)}$$

$$f(-1+2i)=\frac{-1+2i+4}{-1+2i-(-1-2i)}=\frac{2i+3}{-1+2i+1+2i}=\frac{2i+3}{4i}$$

Substituting for $f(-1+2i)$ in (1)

$$\frac{2i+3}{4i}=\frac{1}{2\pi i}\int_C \frac{z+4}{z^2+2z+5}\,dz$$

Cross multiplying

$$\int_C \frac{z+4}{z^2+2z+5}\,dz=\frac{(2i+3)(2\pi i)}{4i}=\frac{\pi}{2}(3+2i)$$

Using Cauchy's integral formula, evaluate $\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)\ (z-1)}dz$, where C is $|z|=3$

Solution:

We know that, Cauchy's integral formula is $f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$

$(i.e)\ 2\pi i\ f(a)=\int_C \frac{f(z)}{z-a}\,dz$

Given: $\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)}dz$ Here, $f(z)=\sin \pi z^2+\cos \pi z^2$

The points $a_1=1, a_2=2$ lies inside $|z|=3$

Now, $\frac{1}{(z-1)(z-2)}=\frac{-1}{(z-1)}+\frac{1}{(z-2)}$ (by Partial fraction method)

$$\therefore \int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)}dz=-\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)}dz+\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-2)}dz$$
$$=-2\pi i\ f(1)+2\pi i\ f(2)$$

$$f(z)=\sin \pi z^2+\cos \pi z^2$$
$$f(1)=\sin \pi+\cos \pi=-1 \text{ and } f(2)=\sin 4\pi+\cos 4\pi=1$$

$$\therefore \int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)}dz=-2\pi i(-1)+2\pi i(1)=4\pi i$$

Using Cauchy's integral formula, evaluate $\int_C \frac{1}{(z-2)\ (z+1)^2}dz$, where C is $|z|=\frac{3}{2}$

Solution:

Here $z=-1$ is a pole lies inside the circle
$z=2$ is a pole lies out side the circle

$$\therefore \int_C \frac{dz}{(z+1)^2(z-2)}=\int \frac{\frac{1}{z-2}}{(z+1)^2}\,dz$$

Here $f(z)=\frac{1}{z-2}$, $f'(z)=-\frac{1}{(z-2)^2}$

Hence by Cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}}dz=\frac{2\pi i}{n!}f^n(a)$$

$$\int_C \frac{dz}{(z+1)^2(z-2)} = \int_C \frac{\frac{1}{z-2}}{[z-(-1)]^2}\,dz = \frac{2\pi i}{1!} f'(-1)$$

$$= 2\pi i\left[\frac{-1}{(-1-2)^2}\right] \quad \left(\because f'(z) = \frac{-1}{(z-2)^2}\right) = 2\pi i\left[\frac{-1}{9}\right]$$

$$\int_C \frac{1}{(z-2)\ (z+1)^2}dz = \frac{-2}{9}\pi i.$$

Using Cauchy's integral formula, evaluate $\int_c \frac{z}{z^2+1}dz$ where C is $|z+i|=1$.

Solution:

Consider the curve

$|z+i|=1 \Rightarrow |x+iy+i|=1$

$|x+i(y+1)|=1 \Rightarrow x^2+(y+1)^2=1$

Which is a circle with centre (0,–1) and radius 1

The poles are obtained by $z^2+1=0$

$\Rightarrow z=i$ is a simple pole which lies outside C.

$z=-i$ is a simple pole which lies inside C.

$$\int_c \frac{z}{z^2+1}dz = \int_c \frac{z}{(z+i)(z-i)}dz = \int_c \frac{\frac{z}{(z-i)}}{(z+i)} = 2\pi i f(-i)\ldots(1)$$

$$f(z) = \frac{z}{(z-i)},\ f(-i) = \frac{-i}{(-i-i)} = \frac{-i}{-2i} = \frac{1}{2}$$

$$(1) \Rightarrow \int_c \frac{z}{z^2+1}dz = 2\pi i f(-i) = 2\pi i\left(\frac{1}{2}\right) = \pi i$$

Expand $f(z)=\log(1+z)$ in Taylor's series about $z=0$

Solution: Let $f(z)=\log(1+z)$ $f(0)=\log 1=0$

$$f'(z) = \frac{1}{1+z} \quad f'(0) = \frac{1}{1+0} = 1$$

$$f''(z) = \frac{-1}{(1+z)^2} \qquad f''(0) = -1$$

$$f'''(z)=\frac{2}{(1+z)^3} \quad f'''(0)=2$$

$$f^{iv}(z)=\frac{-6}{(1+z)^4} \qquad f^{iv}(0)=-6$$

$$\log(1+z)= f(0)+\frac{f'(0)}{1!}z+\frac{f''(0)}{2!}z^2+\ldots= z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\ldots.$$

Find the Taylor's series expansion of f(z) = $\frac{z}{(z+1)(z-3)}$, in the region $|z|<1$

Solution:

Splitting f(z) into partial fractions, we have

$$f(z)=\frac{z}{(z+1)(z-3)}=\frac{A}{(z+1)}+\frac{B}{(z-3)}$$

$$\Rightarrow z=A(z-3)+B(z+1)$$

$$\text{put } z=-1, \text{ we get } A=\frac{1}{4}$$

$$\text{put } z=3, \text{ we get } B=\frac{3}{4}$$

$$\therefore \quad f(z)=\frac{1}{4}\left(\frac{1}{z+1}\right)+\frac{3}{4}\left(\frac{1}{z-3}\right)=\frac{1}{4}\left(\frac{1}{1+z}\right)+\frac{3}{4}\left(\frac{1}{-3}\right)\left(\frac{1}{1-\frac{z}{3}}\right)$$

$$=\frac{1}{4}\left[(1+z)^{-1}-\left(1-\frac{z}{3}\right)^{-1}\right]$$

$$=\frac{1}{4}\left[\left(1-z+z^2-\ldots.\right)-\left(1+\frac{z}{3}+\frac{z^2}{9}+\ldots\right)\right]$$

$$=\frac{1}{4}\left[\left((-1)-\frac{1}{3}\right)z+\left((-1)^2-\left(\frac{1}{3}\right)^2\right)z^2+\ldots\right]$$

$$\therefore f(z)=\frac{1}{4}\sum_{n=1}^{\infty}\left((-1)^n-\left(\frac{1}{3}\right)^n\right)z^n$$

Obtain Taylor's Series to represent the function $f(z)=\frac{z^2-1}{(z+2)(z+3)}$ in the region | z | < 2

Solution:

$$f(z)=\frac{z^2-1}{(z+2)(z+3)}=\frac{z^2-1}{z^2+5z+6}$$

Since the degree of the numerator and denominator are same we have to divide and apply partial fractions.

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{-5z-7}{z^2+5z+6}=1+\frac{-5z-7}{(z+3)(z+2)}$$

$$|z|<2 \quad \Rightarrow \frac{|z|}{2}<1 \text{ and } \therefore \frac{|z|}{3}<1$$

Consider

$$\frac{-5z-7}{(z+3)(z+2)}=\frac{3}{z+2}-\frac{8}{z+3}=\frac{3}{2\left(1+\frac{z}{2}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}=\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2}-....\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-...\right)$$

$$\therefore \frac{z^2-1}{z^2+5z+6}=1+\frac{-5z-7}{z^2+5z+6}=1+\frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2}-....\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-...\right)$$

Find the Laurent's series expansion of $\frac{1}{(z-2)(z-1)}$ valid in the regions $|z|>2$ and $0<|z-1|<1$

Solution:

$$f(z)=\frac{1}{(z-2)(z-1)}=\frac{A}{(z-1)}+\frac{B}{(z-2)}=\frac{A(z-2)+B(z-1)}{(z-2)(z-1)}$$

$$\Rightarrow 1=A(z-2)+B(z-1)$$

Put $z=1,\ A=-1$

$z=2,\ B=1$

$$\therefore f(z)=\frac{-1}{(z-1)}+\frac{1}{(z-2)}$$

Region 1:

$$|z| > 2 \Rightarrow 2 < |z|$$

$$\Rightarrow \left|\frac{2}{z}\right| < 1$$

$$f(z) = \frac{-1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$= -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\ldots\right) + \frac{1}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\ldots\right)$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

$$= -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$$

Region 2:

Put $z-1=t \Rightarrow z=1+t$

$$0<|z-1|<1 \Rightarrow 0<|t|<1$$

$$\Rightarrow |t|<1$$

$$f(z) = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

$$= \frac{-1}{t} + \frac{1}{t-1}$$

$$= \frac{-1}{t} + \frac{1}{-(1-t)}$$

$$= \frac{-1}{t} - (1-t)^{-1}$$

$$= \frac{-1}{t} - \left(1+t+t^2+\ldots\right)$$

$$=\frac{-1}{(z-1)}-\left(1+(z-1)+(z-1)^2+\ldots\right)$$

$$=\frac{-1}{(z-1)}-\sum_{n=0}^{\infty}(z-1)^n$$

Expand f(z) = $\frac{z^2-1}{z^2+5z+6}$ in a Laurent's series expansion for $|z|>3$ and $2<|z|<3$

Solution:

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{-5z-7}{z^2+5z+6}=1+\frac{-5z-7}{(z+3)(z+2)}$$

Consider $\frac{-5z-7}{(z+3)(z+2)}$

$$\frac{-5z-7}{(z+3)(z+2)}=\frac{A}{z+2}+\frac{B}{z+3}=\frac{A(z+3)+B(z+2)}{(z+3)(z+2)}$$

$$-5z-7=A(z+3)+B(z+2)$$

Put $z=-2$ then $A=3$

Put $z=-3$ then $B=-8$

Substituting we get, $\frac{-5z-7}{(z+3)(z+2)}=\frac{3}{z+2}-\frac{8}{z+3}$

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{3}{z+2}-\frac{8}{z+3}$$

(i) **Given** $|z|>3 \quad \Rightarrow \quad \frac{3}{|z|}<1$

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{3}{z+2}-\frac{8}{z+3}=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$=1+\frac{3}{z}\left(1-\frac{2}{z}+\frac{4}{z^2}-\ldots\right)-\frac{8}{z}\left(1-\frac{3}{z}+\frac{9}{z^2}-\ldots\right)$$

(ii) **Given** $2<|z|<3 \Rightarrow \quad \frac{2}{|z|}<1$ and $\frac{|z|}{3}<1$

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{3}{z+2}-\frac{8}{z+3}=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{z}\left(1-\frac{2}{z}+\frac{4}{z^2}-...\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-...\right)$$

Obtain the Laurent's series expansion for the function f(z) = $\mathbf{\frac{4z}{\left(z^2-1\right)(z-4)}}$ **in**

$|z-1|>4$ **and** $2<|z-1|<3$

Solution:

Put $z-1=u \Rightarrow z=u+1$

Now, $f(z)=\dfrac{4z}{\left(z^2-1\right)(z-4)}=\dfrac{4z}{(z-1)(z+1)(z-4)}$

Hence $f(u)=\dfrac{4(u+1)}{u(u+2)(u-3)}$

$$\frac{4(u+1)}{u(u+2)(u-3)}=\frac{A}{u}+\frac{B}{u+2}+\frac{C}{u-3}=\frac{A(u+2)(u-3)+Bu(u-3)+Cu(u+2)}{u(u+2)(u-3)}$$

$$4(u+1)=A(u+2)(u-3)+Bu(u-3)+Cu(u+2)$$

Put $u=0$ then $A=\dfrac{-2}{3}$

Put $u=-2$ then $B=\dfrac{-2}{5}$

Put $u=3$ then $C=\dfrac{16}{15}$

$$f(u)=\frac{4(u+1)}{u(u+2)(u-3)}=\frac{-2/3}{u}+\frac{-2/5}{u+2}+\frac{16/15}{u-3}$$

(i) $|u|>4 \quad \Rightarrow \quad \dfrac{4}{|u|}<1$

$$f(u)=\frac{-2/3}{u}-\frac{2/5}{u+2}+\frac{16/15}{u-3}$$

$$f(u) = -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u\left(1+\frac{2}{u}\right)}\right) + \frac{16}{15}\left(\frac{1}{u\left(1-\frac{3}{u}\right)}\right)$$

$$= -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u}\right)\left(1+\frac{2}{u}\right)^{-1} + \frac{16}{15}\left(\frac{1}{u}\right)\left(1-\frac{3}{u}\right)^{-1}$$

$$= \frac{1}{u}\left[-\frac{2}{3} - \frac{2}{5}\left(1-\frac{2}{u}+\frac{4}{u^2}-....\right) + \frac{16}{15}\left(1+\frac{3}{u}+\frac{9}{u^2}+....\right)\right]$$

$$\therefore f(z) = \frac{1}{(z-1)}\left[-\frac{2}{3} - \frac{2}{5}\left(1-\frac{2}{(z-1)}+\frac{4}{(z-1)^2}-....\right) + \frac{16}{15}\left(1+\frac{3}{(z-1)}+\frac{9}{(z-1)^2}+....\right)\right]$$

(ii) $2 < |u| < 3 \Rightarrow \quad \frac{2}{|u|} < 1 \text{ and } \frac{|u|}{3} < 1$

$$f(u) = -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u\left(1+\frac{2}{u}\right)}\right) + \frac{16}{15}\left(\frac{1}{-3\left(1-\frac{u}{3}\right)}\right)$$

$$= -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u}\right)\left(1+\frac{2}{u}\right)^{-1} - \frac{16}{45}\left(1-\frac{u}{3}\right)^{-1}$$

$$= \frac{1}{u}\left[-\frac{2}{3} - \frac{2}{5}\left(1-\frac{2}{u}+\frac{4}{u^2}-....\right) - \frac{16}{45}\left(1+\frac{u}{3}+\frac{u^2}{9}+....\right)\right]$$

$$\therefore f(z) = \frac{1}{(z-1)}\left[-\frac{2}{3} - \frac{2}{5}\left(1-\frac{2}{(z-1)}+\frac{4}{(z-1)^2}-....\right) - \frac{16}{45}\left(1+\frac{(z-1)}{3}+\frac{(z-1)^2}{9}+....\right)\right]$$

Find the Laurent's series expansion of $f(z) = \frac{7z-2}{z(z-2)(z+1)}$ **in 1 < | z + 1| < 3**

Solution:

The singular points are z = 0, z = 2, z = –1

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$\Rightarrow$ 7z – 2 = A(z – 2) (z + 1) + B z (z + 1) + C z (z – 2)

Put z = 0, –2 = A(–2) $\Rightarrow$ A = 1

z = 2, 14 – 2 = B 2(2 + 1) $\Rightarrow$ B = 2

$z = -1, \quad -7 - 2 = C(-1)(-1-2) \Rightarrow C = -3$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

Put t = z + 1 $\Rightarrow$ z = t – 1

$\therefore 1 < |t| < 3$

$1 < |t| \Rightarrow \left|\frac{1}{t}\right| < 1$ and $\left|\frac{t}{3}\right| < 1$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{t-1} + \frac{2}{t-3} - \frac{3}{t}$$

$$= \frac{1}{t\left(1-\frac{1}{t}\right)} + \frac{2}{(-3)\left(1-\frac{t}{3}\right)} - \frac{3}{t}$$

$$= \frac{1}{t}\left(1-\frac{1}{t}\right)^{-1} - \frac{2}{3}\left(1-\frac{t}{3}\right)^{-1} - \frac{3}{t}$$

$$= \frac{1}{t}\left[1+\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\ldots\right] - \frac{2}{3}\left[1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\left(\frac{t}{3}\right)^3+\ldots\right] - \frac{3}{t}$$

$$= -\frac{2}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \ldots - \frac{2}{3}\left[1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\left(\frac{t}{3}\right)^3+\ldots\right]$$

$$= -2(z+1)^{-1} + (z+1)^{-2} + (z+1)^{-3} + \ldots - \frac{2}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\left(\frac{z+1}{3}\right)^3+\ldots\right]$$

Evaluate $\int_C \frac{z\ dz}{(z-1)(z-2)^2}$, where C is the circle $|z-2| = \frac{1}{2}$ by Cauchy Residue theorem.

Solution:

The poles are obtained by $(z-1)(z-2)^2 = 0$

$\Rightarrow$ z = 1 is a simple pole and z = 2 is a pole of order 2.

C is the circle $|z-2| = \frac{1}{2}$

Here z = 1 lies outside C and z = 2 lies inside C.

Residue at *z*=2: (Pole of order 2)

$$\text{Res } f(z) = \underset{z\to 2}{lt}\frac{d}{dz}(z-2)^2\frac{z}{(z-1)(z-2)^2} = \underset{z\to 2}{lt}\frac{z-1-z}{(z-1)^2} = -1$$

By Cauchy Residue theorem,

$$\int_C \frac{z\,dz}{(z-1)(z-2)^2} = 2\pi i(-1) = -2\pi i$$

Using Cauchy's residue theorem evaluate $\int_C \frac{3z^2+z-1}{(z^2-1)(z-3)}dz$ **, where C is** $|z| = 2$

Solution:

$|z| = 2$ is the equation of the circle with centre at origin and radius 2.

$$(z^2-1)(z-3) = 0$$

$$(z^2-1) = 0, \quad (z-3) = 0$$

$$z^2 = 1, \quad z = 3$$

$$z = \pm 1, \quad z = 3$$

$z = 1, -1$ lies inside the circle and $z = 3$ lies outside the circle

Residue at $z = 1$ is

$$= Lt_{z\to 1}\left((z-1)\frac{3z^2+z-1}{(z+1)(z-1)(z-3)}\right)$$

$$= Lt_{z\to 1}\left(\frac{3z^2+z-1}{(z+1)(z-3)}\right) = -\frac{3}{4}$$

Residue at $z = -1$ is

$$= Lt_{z\to -1}\left((z+1)\frac{3z^2+z-1}{(z+1)(z-1)(z-3)}\right)$$

$$= Lt_{z\to -1}\left(\frac{3z^2+z-1}{(z-1)(z-3)}\right) = \frac{1}{8}$$

By Cauchy's Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\left(\text{Sum of the Residues of } \mathrm{f}(\mathrm{z}) \text{ at each of its poles which lies inside } C\right)$$

$$\therefore \int_C \frac{3z^2+z-1}{\left(z^2-1\right)(z-3)}dz = 2\pi i\left(\frac{1}{8}-\frac{3}{4}\right) = -\frac{5\pi i}{4}$$

Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$, where C is $|z-i|=2$ using Cauchy's residue theorem

Solution:

Let $f(z)=\frac{z-1}{(z+1)^2(z-2)}$

poles of $f(z)$ are z = - 1 (pole of order 2) and z = 2 (simple pole)

Given: $|z-i|=2$

$$|x+iy-i|=2 \Rightarrow |x+i(y-1)|=2$$

Squaring on both sides $\sqrt{x^2+(y-1)^2}=2 \Rightarrow x^2+(y-1)^2=4$

This is equation of circle with centre $(0,1)$ and radius 2

Hence, The pole $z=2$ lies outside C and $z=-1$ lies inside C

Residue of f(z) at $z=-1$

$$= Lt_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left((z+1)^2\frac{(z-1)}{(z+1)^2(z-2)}\right)$$

$$= Lt_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left(\frac{(z-1)}{(z-2)}\right) = Lt_{z\to -1}\left(\frac{(z-2)(1)-(z-1)(1)}{(z-2)^2}\right)$$

$$= Lt_{z\to -1}\left(\frac{-1}{(z-2)^2}\right) = -\frac{1}{9}$$

By Cauchy's Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\left(\text{Sum of the Residues of } f(z) \text{ at each of its poles which lies inside } C\right)$$

$$\therefore \int_C \frac{(z-1)}{(z+1)^2(z-2)}\,dz = 2\pi i\left(0-\frac{1}{9}\right) = -\frac{2\pi i}{9}$$

Using Cauchy's residue theorem, find $\displaystyle\int_C \frac{z+1}{(z-3)(z-1)}\,dz$ **, where C is** $|z| = 2$

Solution:

The singular points are given by (z – 1) (z – 3) = 0 $\Rightarrow$ z = 1 , 3

Given C is | z | = 2

If z = 1 then | z | = | 1 | = 1 < 2

If z = 3 then | z | = | 3 | = 3 > 2

$$\int_C f(z)\,dz = 2\pi i\left(\text{Sum of the Residues of } f(z) \text{ at each of its poles which lies inside } C\right)$$

Residue at z=1:

$$\text{Res}\Big|_{z=1} = \lim_{z\to 1}(z-1)\,f(z) = \lim_{z\to 1}(z-1)\frac{z+1}{(z-3)(z-1)} = -1$$

$$\therefore \int_C \frac{z+1}{(z-3)(z-1)}\,dz = 2\pi i(-1) = -2\pi i$$

Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ **by using Contour integration.**

Solution:

Consider the unit circle | z | = 1 as contour C.

Put $z = e^{i\theta}$, then $\dfrac{1}{z} = e^{-i\theta}$

$$\therefore d\theta = \frac{dz}{iz},\ \sin\theta = \frac{z-\frac{1}{z}}{2i} = \frac{z^2-1}{2iz}$$

$$\therefore I=\int_C \frac{\frac{dz}{iz}}{13+5\frac{(z^2-1)}{2iz}}=\int_C \frac{\frac{dz}{iz}}{\frac{26iz+5z^2-5}{2iz}}=2\int_C \frac{dz}{5z^2+26iz-5}$$

Let $f(z)=\dfrac{1}{5z^2+26iz-5}$ $\qquad \therefore I=2\int_C f(z)dz$

The poles of f(z) are given by $5z^2 + 26iz - 5 = 0$

$$z=\frac{-26i\pm\sqrt{(26i)^2-4\ .\ 5(-5)}}{10}=\frac{-26i\pm\sqrt{-676+100}}{10}=\frac{-26i\pm\sqrt{-576}}{10}=\frac{-26i\pm 24i}{10}$$

$$z=-\frac{i}{5}\ ,\ -5i$$

which are simple poles.

Now $5z^2+26iz-5=5\left(z+\frac{i}{5}\right)(z+5i)$

Since $\left|\frac{-i}{5}\right|=\frac{1}{5}<1,$ the pole $z=\frac{-i}{5}$ lies inside C

and $|-5i|=5>1,$ $\therefore$ the pole $z=-5i$ lies outside C.

Now $R\left(-\frac{i}{5}\right)=\lim_{z\to\frac{-i}{5}}\left(z+\frac{i}{5}\right)f(z)=\lim_{z\to\frac{-i}{5}}\left(z+\frac{i}{5}\right)\frac{1}{5\left(z+\frac{i}{5}\right)(z+5i)}=\lim_{z\to\frac{-i}{5}}\frac{1}{5(z+5i)}$

$$=\lim_{z\to\frac{-i}{5}}\frac{1}{5\left(-\frac{i}{5}+5i\right)}=\frac{1}{24i}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz=2\pi i\left(\frac{1}{24i}\right)=\frac{\pi}{12}$$

$$\therefore I=2.\frac{\pi}{12}=\frac{\pi}{6}$$

Evaluate $\displaystyle\int_0^{2\pi}\frac{d\theta}{13+12\cos\theta}$ by using Contour integration.

Solution:

Consider the unit circle | z | = 1 as contour C.

Put $z=e^{i\theta}$, then $\frac{1}{z}=e^{-i\theta}$

$$\therefore d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{z^2+1}{2z}$$

$$\therefore I = \int_C \frac{\frac{dz}{iz}}{13+12\frac{(z^2+1)}{2z}} = \int_C \frac{dz}{\frac{iz(13z+6z^2+6)}{z}} = \int_C \frac{dz}{i(6z^2+13z+6)} = \frac{1}{i6}\int_C \frac{dz}{(z^2+\frac{13}{6}z+1)}$$

$$Let \quad f(z) = \int_C \frac{dz}{(z^2+\frac{13}{6}z+1)} \qquad \therefore I = \frac{1}{6i}\int_C f(z)dz$$

The poles of f(z) are given by $z^2 + \frac{13}{6}z + 1 = 0$

By solving we get $z = -\frac{2}{3}, \quad -\frac{3}{2}$

which are simple poles.

Now $z^2 + \frac{13}{6}z + 1 = \left(z+\frac{2}{3}\right)\left(z+\frac{3}{2}\right)$

Since $\left|\frac{-2}{3}\right| = \frac{2}{3} < 1,$ the pole $z = \frac{-2}{3}$ lies inside C

and $\left|\frac{-3}{2}\right| = 1.5 > 1, \therefore$ the pole $z = \frac{-3}{2}$ lies outside C.

$$Now \quad R\left(-\frac{2}{3}\right) = \lim_{z\to\frac{-2}{3}}\left(z+\frac{2}{3}\right)f(z) = \lim_{z\to\frac{-2}{3}}\left(z+\frac{2}{3}\right)\frac{1}{\left(z+\frac{2}{3}\right)\left(z+\frac{3}{2}\right)} = \lim_{z\to\frac{-2}{3}}\frac{1}{\left(z+\frac{3}{2}\right)}$$

$$= \lim_{z\to\frac{-2}{3}}\frac{1}{\left(z+\frac{3}{2}\right)} = \frac{6}{5}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\left(\frac{6}{5}\right) = \frac{12\pi i}{5}, \qquad \therefore I = \frac{1}{6i}\times\left(\frac{12\pi i}{5}\right) = \frac{2\pi}{5}.$$

Evaluate $\int_0^{2\pi}\frac{\cos 3\theta\, d\theta}{5-4\cos\theta}$ by using Contour integration

Solution:

Consider the unit circle | z | = 1 as contour C.

$Put\ z = e^{i\theta}, \ then\ \frac{1}{z} = e^{-i\theta}$

$$\therefore d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{z^2+1}{2z}$$

cos3θ=R.P. of $e^{i3\theta}$ =R.P. of $(e^{i\theta})^3$ =R.P. of z^3

$$\therefore \qquad I = \int_C \frac{R.P. of\ z^3 \frac{dz}{iz}}{5-4\frac{(z^2+1)}{2z}} = R.P.\ of\ \int_C \frac{z^3 dz}{\frac{iz(5z-2z^2-2)}{z}}$$

$$= R.P.\ of \int_C \frac{z^3 dz}{i(-2z^2+5z-2)}$$

$$= R.P.\ of \int_C \frac{z^3 dz}{-i(2z^2-5z+2)}$$

$$= R.P. of\ \frac{-1}{2i}\int_C \frac{z^3 dz}{(2z-1)(z-2)}$$

Let $\int_C f(z)dz = \int_C \frac{z^3 dz}{(2z-1)(z-2)}$ $\qquad \therefore I = R.P.$ of $\frac{-1}{2i}\int_C f(z)dz$

The poles of f(z) are given by

$(2z-1)(z-2)=0$

$z=\frac{1}{2},\ z=2$

$z=\frac{1}{2},\ z=2$ (simple poles)

$z=\frac{1}{2}$ is a pole lies inside c.

$z=2$ is a pole lies outside c.

$$\text{Now} \quad \operatorname{Res}\left(z=\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right) f(z) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{z^3}{\left(z-\frac{1}{2}\right)\ (z-2)} = \frac{-1}{12}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i \ \left(\frac{-1}{12}\right) = \frac{-\pi i}{6}$$

$$\therefore I = R.P.of\ \frac{-1}{2i}\ .\frac{-\pi i}{6} = R.P.of\ \frac{\pi}{12} = \frac{\pi}{12}$$

Evaluate $\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta + p^2}$ **, | p | < 1**

Solution: Let $z = e^{i\theta}$, $dz = i e^{i\theta} d\theta \;\Rightarrow\; d\theta = \frac{dz}{iz}$, $\sin\theta = \frac{z^2-1}{2iz}$

$$\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = \int_C \frac{\left(dz/iz\right)}{1-2p\left(\frac{z^2-1}{2iz}\right)+p^2} \quad , C \text{ is } |z|=1$$

$$= \int_C \frac{dz}{iz - p(z^2-1) + izp^2} = -\int_C \frac{dz}{pz^2 - iz(p^2+1) - p} = -\frac{1}{p}\int_C \frac{dz}{z^2 - iz\left(p+\frac{1}{p}\right) - 1}$$

$$\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = -\frac{1}{p}\int_C \frac{dz}{(z-ip)\left(z-\frac{i}{p}\right)} \quad \text{........(1)}$$

The poles are given by $z = ip \;\&\; z = \frac{i}{p}$

| z | = | i p | = p < 1. $\therefore$ z= ip lies inside C and $z = \frac{i}{p}$ lies outside C.

$$\therefore \left[\text{Res of } f(z)\right]_{z=ip} = \underset{z\to ip}{\text{Lt}} (z-ip)\left[\frac{1}{(z-ip)\left(z-\frac{i}{p}\right)}\right] = \underset{z\to ip}{\text{Lt}} \left(\frac{1}{z-\frac{i}{p}}\right) = \frac{1}{i\left(p-\frac{1}{p}\right)} = \frac{ip}{1-p^2}$$

By Cauchy Residue Theorem $\int_C \frac{dz}{(z-ip)\left(z-\frac{i}{p}\right)} = 2\pi i\left(\frac{ip}{1-p^2}\right) = \frac{-2\pi p}{1-p^2}$

From (1) $\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = -\frac{1}{p}\left(-\frac{2\pi p}{1-p^2}\right) = \frac{2\pi}{1-p^2}$