USEFUL BASIC RESULTS

▪ TRIGONOMETRY ▪

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta = 1 + \tan^2\theta \qquad \operatorname{cosec}^2\theta = 1 + \cot^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \text{ or } 2\cos^2\theta - 1$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta \Rightarrow \sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \Rightarrow \cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$$

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$
$\sin(90° - \theta) = \cos\theta$	$\cos(90° - \theta) = \sin\theta$	$\tan(90° - \theta) = \cot\theta$
$\sin(90° + \theta) = \cos\theta$	$\text{oos}(90° + \theta) = -\sin\theta$	$\tan(90° + \theta) = -\cot\theta$
$\sin(180° - \theta) = \sin\theta$	$\cos(180° - \theta) = -\cos\theta$	$\tan(180° - \theta) = -\tan\theta$
$\sin(180° + \theta) = -\sin\theta$	$\cos(180° + \theta) = -\cos\theta$	$\tan(180° + \theta) = \tan\theta$
$\sin(270° - \theta) = -\cos\theta$	$\cos(270° - \theta) = -\sin\theta$	$\tan(270° - \theta) = \cot\theta$
$\sin(270° + \theta) = -\cos\theta$	$\cos(270° + \theta) = \sin\theta$	$\tan(270° + \theta) = -\cot\theta$
$\sin(360° - \theta) = -\sin\theta$	$\cos(360° - \theta) = \cos\theta$	$\tan(360° - \theta) = -\tan\theta$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Formulae : Converting product into sum

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$$

$$2 \cos A \sin B = \sin (A + B) - \sin (A - B)$$

$$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$$

$$2 \sin A \sin B = \cos (A - B) - \cos (A + B)$$

$$\sin^{-1} (\sin \theta) = \theta$$

$$\cos^{-1} (\cos \theta) = \theta$$

$$\tan^{-1} (\tan \theta) = \theta$$

Formulae : Converting sum or difference into product

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$$

$$\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$$

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$$

$$\cos C - \cos D = -2 \sin \frac{C+D}{2} \sin \frac{C-D}{2}$$

Limits

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

$$\lim_{x \to 0} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$$

Hyperbolic Functions

$$\sin h\, x = \frac{e^x - e^{-x}}{2} \Rightarrow \sin h\, 0 = 0$$

$$\cos h\, x = \frac{e^x + e^{-x}}{2} \Rightarrow \cos h\, 0 = 1$$

$$\tan h\, x = \frac{\sin h\, x}{\cos h\, x}$$

$$= \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\cos h^2 x - \sin h^2 x = 1$$

■ DIFFERENTIAL CALCULUS ■

$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(\log_e x) = \frac{1}{x}$
$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\cos x) = -\sin x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$
$\frac{d}{dx}(a^x) = a^x \log_e a$	$\frac{d}{dx}(e^x) = e^x$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$
$\frac{d}{dx}(\sin h\, x) = \cos h\, x$	$\frac{d}{dx}(\cos h\, x) = \sin h\, x$
$\frac{d}{dx}(\tan h\, x) = \sec h^2\, x$	$\frac{d}{dx}(\cot h\, x) = -\text{cosec}\, h^2\, x$
$\frac{d}{dx}(\sec h\, x) = \text{sec}h\, x \tan h\, x$	$\frac{d}{dx}(\text{cosec}\, h\, x) = -\text{cosec}hx \coth x$

Rules of Differentiation

1. Product Rule

If $y = uv$, where u and v are functions of x, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

2. Quotient Rule

If $y = \frac{\text{Nr}}{\text{Dr}}$, where Nr and Dr are functions of 'x' then

$$\frac{dy}{dx} = \frac{\text{Dr}\frac{d}{dx}(\text{Nr}) - \text{Nr}\frac{d}{dx}(\text{Dr})}{(\text{Dr})^2}$$

Chapterwise Important Results

Chapter 1

Matrices

1. **Linear Dependence of Vectors :** A system of vectors $X_1, X_2, \ldots, X_n$ is linearly dependent if at least one of the vectors is a linear combination of the remaining vectors of the system otherwise it is called as linearly independent.

   i.e., a system of vectors $X_1, X_2, \ldots\ldots, X_n$ is said to be linearly dependent if there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ at least one of which is non zero, such that $\lambda_1 X_1 + \lambda_2 X_2 + \ldots\ldots + \lambda_n X_n = 0$ holds true otherwise the given vectors are linearly independent.

2. Let A be a square matrix. If $|A| \neq 0$ then the given vectors are linearly independent. If $|A| = 0$ it is linearly dependent.

3. **Characteristic equations, Eigenvalues, Eigenvectors and Cayley-Hamilton Theorem.**

   (*i*) Let A be the given square matrix

   (*ii*) Compute $|A - \lambda I| = 0$ where 'I' is the unit matrix.

   (*iii*) $|A - \lambda I| = 0$ is called the characteristic equation (in terms of λ).

   (*iv*) Solve the characteristic equation, values of λ are called as Eigenvalues.

   (*v*) For each Eigenvalue find the corresponding Eigen vectors X which satisfies the equations. $(A - \lambda I) X = 0$

   (*vi*) In the characteristic equation if we put $\lambda = A$ and let I be the unit matrix (if A satisfies its own characteristic equation then Cayley theorem is proved) then the answer is a null matrix which implies Cayley theorem is verified.

4. **The matrix of the quadratic form**

   $$X'AX = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x^3 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{31}x_3x_1$$

is given by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The elements of A is obtained as follows

In the I Row : a_{11} = Coefficient of ${x_1}^2$

$$a_{12} = \frac{1}{2} \text{ coefficient of } x_1 x_2$$

$$a_{13} = \frac{1}{2} \text{ coefficient of } x_1 x_3$$

II Row : $$a_{21} = \frac{1}{2} \text{ coefficient of } x_1 x_2$$

$$a_{22} = \text{Coefficient of } x_2^2$$

$$a_{23} = \frac{1}{2} \text{ coefficient of } x_2 x_3$$

III Row : $$a_{31} = \frac{1}{2} \text{ coefficient of } x_1 x_3$$

$$a_{32} = \frac{1}{2} \text{ coefficient of } x_2 x_3$$

$$a_{33} = \text{Coefficient of } x_3^2$$

5. Reduction of Q.F. to Canonical form

(i) Find the characteristic equation, Eigenvalues, Eigenvectors of A.

(ii) Corresponding to a Eigenvector find the normalised Eigenvectors

(iii) Find the normalised modal matrix "P" [Each column corresponds to normalised Eigenvector]

(iv) $P' = (P)^T$

(v) Orthogonal transformation : $X = PY$

(vi) $Y' (P' AP) Y$ gives the required quadratic form where

$$Y' = (y_1 \;\; y_2 \;\; y_3) \text{ and } Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

6. Diagonalisation of a Matrix :

(i) For the matrix A find $|A - \lambda I| = 0$

(ii) Find the Eigenvalues, Eigenvectors of A.

(iii) The Eigen vectors corresponding to Eigenvalues are written coulmnwise and it is denoted by B.

(iv) Find $B^{-1} = \dfrac{\text{Adj } B}{|B|}$.

(v) $D = B^{-1} AB$

D is called the diagonal matrix.

(vi) $B^{-1} A^n B = D^n$

7. Nature of a Quadratic Form

Let X′AX be the given quadratic form in the variables $x_1, x_2, \ldots, x_n$.

$$i.e., \quad X'AX = d_1 x_1^2 + d_2 x_2^2 + \ldots \ldots + d_n x_n^2 \qquad \ldots (1)$$

Let the rank of A be r.

Then X′AX contains only 'r' terms.

The number of positive terms in (1) is called the ***index*** of the quadratic form and it is denoted by 's'.

The difference between the number of positive terms and the negative terms is called the ***signature*** of the quadratic form.

i.e., signature = Number of positive terms − Number of negative terms

= s − (total no. of terms – positive terms)

= s − (rank of A − s)

= $s - (r - s)$

∴ signature = $2s - r$,

where s - number of positive terms

r - rank of A

Differential Calculus of Several Variables

1. If $u = f(x, y)$ is a function of x and y where $x = f(t)$, $y = g(t)$ then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

In differential form

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

du is called the total differential of u.

2. If $u = f(x_1, x_2, \ldots, x_n)$ where $x_1, x_2, \ldots, x_n$ are all functions of 't' then

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial u}{\partial x_2} \cdot \frac{dx_2}{dt} + \ldots + \frac{\partial u}{\partial x_n} \cdot \frac{dx_n}{dt}$$

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \ldots + \frac{\partial u}{\partial x_n} dx_n$$

3. If $u = f(x_1, x_2, \ldots, x_n)$ where $x_1, x_2, \ldots, x_n$ are all functions of several independent variables $t_1, t_2, \ldots, t_n$ then

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_1} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_1} + \ldots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_1}$$

$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_2} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_2} + \ldots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_2}$$

4. Let $u = f(x, y) = c$ be a given implicit function of x and y. Then

$$\frac{dy}{dx} = -\left(\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right)$$

5. **Taylors Theorem**

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f + \frac{1}{2!}\left(h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x\, \partial y} + k^2\frac{\partial^2}{\partial y^2}\right) f + \ldots$$

Putting $x = a$ and $y = b$, we get

Corollary 1 :

$$f(a+h, b+k) = f(a, b) + [\, h f x(a, b) + k f y(a, b) + \frac{1}{2!}[h^2 f_{xx}(a, b) + 2\, hk f_{xy}(a, b) + k^2 f_{yy}(a, b)] + \ldots$$

Corollary 2 : Putting $a + h = x$ and $b + k = y$ so that $h = x - a$ and $k = y - b$, we get

$$f(x, y) = f(a, b) + [\,(x-a) f_x (a, b) + (y-b) f_y (a, b)\,]$$
$$+ \frac{1}{2\,!}\ [\,(x-a)^2 f_{xx} (a, b) + 2\,(x-a)\,(y-b) f_{xy} (a, b)$$
$$+ (y-b)^2 f_{yy} (a, b)\,] + \ldots\ldots$$

This formula is used to expand $f(x, y)$ in the neighbourhood of (a, b).

Corollary 3 : Putting $a = 0,\ b = 0$ in corollary 2, we get

$$f(x, y) = f(0, 0) + [\,x f x (0, 0) + y f y (0, 0)]$$
$$+ \frac{1}{2\,!}\ [x^2 f_{xx} (0, 0) + 2\,xy f_{xy} (0, 0) + y^2 f_{yy} (0, 0)] + \ldots\ldots$$

This formula is used to expand $f(x, y)$ in powers of x and y [or to expand $f(x, y)$ in the neighbourhood of origin (0, 0).]

6. Rule to find maxima and minima of a function of two variables.

Let $f(x, y)$ be the given function. To find the maximum and minimum values of $f(x, y)$ we have to follow the following rules.

Rule 1 : Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$

From the following equations find x and y.

$$\frac{\partial f}{\partial x} = 0\ ;\ \frac{\partial f}{\partial y} = 0$$

These values of x and y gives the points at which maxima or minima exists. Let the points be (a_1, b_1), (a_2, b_2), etc.

Rule 2 : Find $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$, $\dfrac{\partial^2 f}{\partial x\,\partial y}$

Rule 3 : Calculate the value of $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x\,\partial y}\right)^2$ at each of the points found in **Rule 1.**

Rule 4 : If $\dfrac{\partial^2 f}{\partial x^2} < 0$ or $\dfrac{\partial^2 f}{\partial y^2} < 0$ and $\dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x\,\partial y}\right)^2 > 0$,

then 'f' has a **maximum** and the corresponding value of 'f' is called the **maximum value.**

Rule 5 : If $\dfrac{\partial^2 f}{\partial x^2} > 0$ or $\dfrac{\partial^2 f}{\partial y^2} > 0$ and $\dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x\,\partial y}\right)^2 > 0$,

then 'f' has a minimum and the corresponding value of 'f' is called the minimum value.

Rule 6 : If $\dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x\,\partial y}\right)^2 < 0$, then we have neither a maximum nor a minimum. Such a point is called a **saddle point.**

Rule 7 : If $\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \, \partial y}\right)^2 = 0$

further investigation is required.

7. Constrained Maxima and Minima

Let $u = f(x, y, z)$ be a given function for which the extremum values to be determined subject to the condition (constraint).

$$g(x, y, z) = 0$$

Form the auxiliary function $F(x, y, z)$ given by

$$F(x, y, z) = f(x, y, z) + \lambda g(x, y, z)$$

Find
$$\frac{\partial F}{\partial x} = 0$$
$$\frac{\partial F}{\partial y} = 0$$
$$\frac{\partial F}{\partial z} = 0$$
$$\frac{\partial F}{\partial \lambda} = 0$$

Using these equations we can find x, y, z for which $f(x, y, z)$ can have a conditional extremum.

8. Jacobians

(i) The Jacobian of u, v w.r.t. 'x' and 'y' is given by

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

(ii)
$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

(iii) $\frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)} = 1$

(iv)
$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \times \frac{\partial(r, s)}{\partial(x, y)}$$

Applications of Differential Calculus

1. **The radius of curvature of the curve $y = f(x)$ is given by**

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \qquad \text{(in Cartesian form)}$$

when $\frac{dy}{dx} = \infty$, then $$\rho = \frac{\left\{1 + \left(\frac{dx}{dy}\right)\right\}^{\frac{3}{2}}}{\frac{d^2x}{dy^2}} \qquad \text{(in Cartesian form)}$$

2. **The radius of curvature of the curve $x = f(\theta)$ and $y = g(\theta)$ is given by**

$$\rho = \frac{\{f'^2 + g'^2\}^{\frac{3}{2}}}{f'g'' - g'f''} \qquad \text{(in Parametric form)}$$

where $$f' = \frac{df}{d\theta} \qquad f'' = \frac{d^2f}{d\theta^2}$$

$$g' = \frac{dg}{d\theta} \qquad g'' = \frac{d^2g}{d\theta^2}.$$

3. **The radius of curvature of the curve $r = f(\theta)$ is given by**

$$r = \frac{\left\{r^2 + r_1^2\right\}^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2} \qquad \textbf{(Polar form)}$$

where $$r_1 = \frac{dr}{d\theta}$$

$$r_2 = \frac{d^2r}{d\theta^2}$$

4. **The centre of curvature $\left(\overline{X}, \overline{Y}\right)$ is given by**

$$\overline{X} = x - \frac{\frac{dy}{dx}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}}$$

$$\overline{Y} = y - \left\{\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right\}$$

5. The Equation of circle of curvature whose centre is $(\overline{X}, \overline{Y})$ and radius is equal to the radius of curvature is given by

$$(x-\overline{X})^2 + (y-\overline{Y})^2 = \rho^2$$

6. **The parametric equation of some standard curves**

Curve	**Parametric form**
(i) $y^2 = 4ax$	(Parabola) $x = at^2$, $y = 2at$
(ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	(Ellipse) $x = a\cos\theta$, $y = b\sin\theta$
(iii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	(Hyperbola) $x = a\sec\theta$, $y = b\tan\theta$
(iv) $x^{2/3} + y^{2/3} = a^{2/3}$	$x = a\cos^3\theta$, $y = a\sin^3\theta$
(v) $xy = c^2$	(Rectangular hyperbola) $x = ct$, $y = \frac{c}{t}$

EVOLUTE

The locus 'C' of the centre of curvature for a curve 'C' is called the **Evolute.**

To find the Evolute of a given curve

1. Find the centre of curvature $(\overline{X}, \overline{Y})$ of the given curve.

2. Eliminate the parameter from $\overline{X}$ and $\overline{Y}$.

3. The locus of $\overline{X}$ and $\overline{Y}$ is called the evolute of the given curve.

ENVELOPES

I. ***To find the envelope of the given family of curves $F(x, y, \alpha) = 0$***

Let the

(i) Given family of curves be $F(x, y, \alpha) = 0$ ... (1)

(ii) Partially differentiating (1) w.r.t.. 'α' we get

$$\frac{\partial F(x, y, \alpha)}{\partial \alpha} = 0 \quad \text{... (2)}$$

From (1) and (2) eliminate the parameter 'α' The elliminant is called the envelope of the given family of curves.

II. When the parameter 'α' in $F(x, y, \alpha) = 0$ is in quadratic of the form $A\alpha^2 + B\alpha + C = 0$ then the envelope is given by $B^2 - 4AC = 0$.

Sequences and Series

1. (a) $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots$

   (b) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$

   (c) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

2. $(1+x)^{\frac{p}{q}} = 1 + \dfrac{p}{q}x + \dfrac{\frac{p}{q}\left(\frac{p}{q}-1\right)}{2!}x^2 + \dfrac{\frac{p}{q}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}-2\right)}{3!}x^3 + \ldots$

3. $\lim\limits_{n\to\infty}(u_n + v_n) = \lim\limits_{n\to\infty} u_n + \lim\limits_{n\to\infty} v_n$

4. $\lim\limits_{n\to\infty}(u_n - v_n) = \lim\limits_{n\to\infty} u_n - \lim\limits_{n\to\infty} v_n$

5. $\lim\limits_{n\to\infty} C\,u_n = C\lim\limits_{n\to\infty} u_n$, C is a constant

6. $\lim\limits_{n\to\infty}(u_n \cdot v_n) = \lim\limits_{n\to\infty} u_n \cdot \lim\limits_{n\to\infty} v_n$

7. $\lim\limits_{n\to\infty}\dfrac{u_n}{v_n} = \dfrac{\lim\limits_{n\to\infty} u_n}{\lim\limits_{n\to\infty} v_n}$

8. $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e \quad (e = 2.71)$

9. $\lim_{n \to \infty} \left(1 + \frac{2}{n}\right)^n = e^2$

10. $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n+1} = e$

11. $\lim_{n \to \infty} \left(1 + \frac{1}{n+1}\right)^n = e$

12. $\log\left(\frac{m}{n}\right) = \log m - \log n$

13. $\log m^n = n \log m$

14. $\log mn = \log m + \log n$

15. $\lim_{n \to \infty} \frac{K}{n} = 0$ (K is a constant)

$\lim_{n \to \infty} \frac{K}{n^2} = 0$

16. $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

17. (a) $n! = n(n-1)(n-2)\ldots 1$

(b) $(n+1)! = (n+1)n(n-1)(n-2)\ldots 1$

18. (a) $\log \infty = \infty$

(b) $\log 1 = 0$

(c) $\log_e e = 1$

1. COMPARISON TEST (Only for series with positive terms)

Consider the two positive term series

$$\sum u_n = u_1 + u_2 + \ldots.. + u_n + \ldots\ldots$$

$$\sum v_n = v_1 + v_2 + \ldots.. + v_n + \ldots\ldots$$

(i) If $u_n \leq v_n$ for every n and $\sum v_n$ converges then $\sum u_n$ also converges.

(ii) If $u_n \geq v_n$ for every n and $\sum v_n$ converges then $\sum u_n$ also converges.

(iii) ***Limit form*** :

If $\mathop{\text{Lt}}_{n \to q} \frac{u_n}{v_n}$ = a finite quantity then the series $\sum u_n$ and $\sum v_n$ either both converge or both diverge together.

2. INTEGRAL TEST

Theorem:

Let $\sum u_n = u_1 + u_2 + \ldots.. + u_n + \ldots\ldots \quad \ldots(1)$

be a series with positive and decreasing terms.

$$u_1 \geq u_2 \geq u_3 \geq \ldots\ldots$$

Let f be a non-negative decreasing function in $[1, \infty)$ such that

$$f(1) = u_1, f(2) = u_2, f(3) = u_3, \ldots f(n) = u_n \qquad \ldots (2)$$

Then the improper integral

$$\int_1^\infty f(x)\, dx \text{ and the series } \sum_{n=1}^{\infty} u_n \qquad \ldots (3)$$

are both finite (in this case $\sum u_n$ is convergent) or both infinite (in this case $\sum u_n$ is divergent).

3. D' ALEMBERT'S RATIO TEST

In a series with positive terms $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$

if (i) $\lim\limits_{n \to \infty} \dfrac{u_{n+1}}{u_n} < 1$, *then the series is convergent.*

(ii) $\lim\limits_{n \to \infty} \dfrac{u_{n+1}}{u_n} > 1$, *then the series is divergent.*

(iii) $\lim\limits_{n \to \infty} \dfrac{u_{n+1}}{u_n} = 1$, *the test fails and in this case we can use comparison test.*

4. LEIBNITZ THEOREM

If in the alternating series

$$u_1 - u_2 + u_3 - u_4 + u_5 - \ldots\ldots\ldots \quad (u_n > 0) \qquad \ldots (1$$

the terms are such that

(i) $u_1 > u_2 > u_3 > \ldots\ldots\ldots\ldots$ **(numerically)**

and *(ii)* $\lim\limits_{n \to \infty} u_n = 0$

then the given series is convergent.

Note : If $\lim\limits_{n \to \infty} u_n \neq 0$, then the series is **oscillatory.**

❑ ABSOLUTE CONVERGENCE

The alternating series $\sum\limits_{n=1}^{\infty} u_n$ is said to be absolutely convergent if the series $\sum\limits_{n=1}^{\infty} |u_n|$ is convergent.

❑ CONDITIONAL CONVERGENCE

Let $\quad u_n = u_1 - u_2 + u_3 - u_4 + \ldots.$

If $\sum u_n$ is convergent while $\sum |u_n|$ is divergent then $\sum u_n$ is said to be **conditionally convergent.**