What is Gauss Law?

According to the Gauss law, **the total flux linked with a closed surface is $1/\varepsilon_0$ times the charge enclosed by the closed surface.**

$$\oint \vec{E}.\vec{ds} = \frac{1}{\epsilon_0}q$$

For example, a point charge q is placed inside a cube of edge 'a'. Now, as per **Gauss law**, the flux through each face of the cube is $q/6\varepsilon_0$.

The electric field is the basic concept of knowing about electricity. Generally, the electric field of the surface is calculated by applying Coulomb's law, but to calculate the electric field distribution in a closed surface, we need to understand the concept of Gauss law. It explains the electric charge enclosed in a closed or the electric charge present in the enclosed closed surface.

Gauss Law Formula

As per the Gauss theorem, the total charge enclosed in a closed surface is proportional to the total flux enclosed by the surface. Therefore, if ϕ is total flux and ϵ_0 is electric constant, the total electric charge Q enclosed by the surface is;

$Q = \phi\, \epsilon_0$

The **Gauss law** formula is expressed by;

$\boldsymbol{\phi = Q/\varepsilon_0}$

Where,

Q = total charge within the given surface,

ε_0 = the electric constant.

The Gauss Theorem

The net flux through a closed surface is directly proportional to the net charge in the volume enclosed by the closed surface.

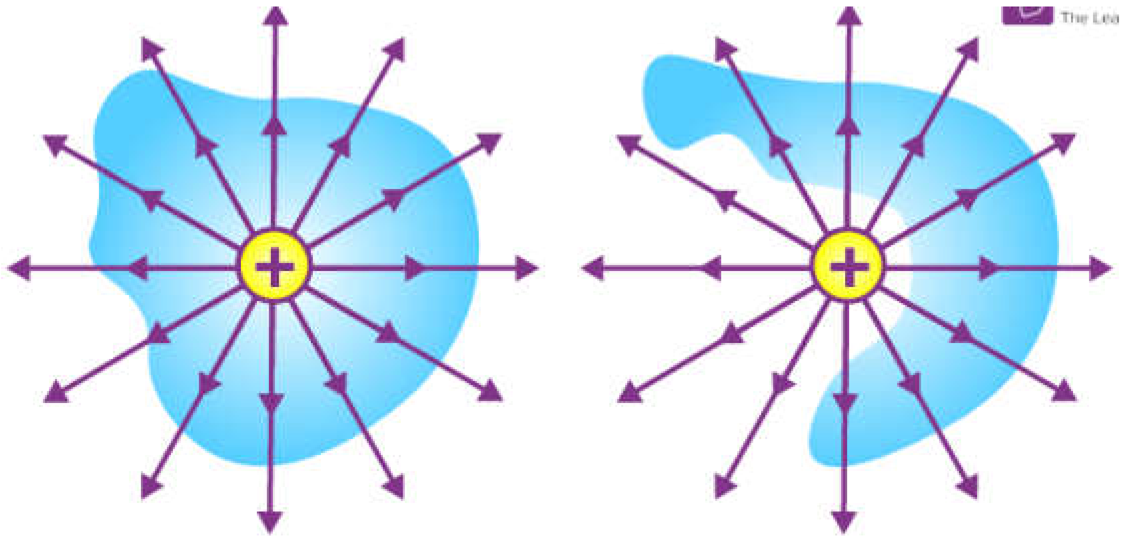
$\Phi = \rightarrow E.d \rightarrow A = q_{net}/\varepsilon_0$

In simple words, the **Gauss theorem** relates the 'flow' of electric field lines (flux) to the charges within the enclosed surface. If no charges are enclosed by a surface, then the net electric flux remains zero.

This means that the number of electric field lines entering the surface equals the field lines leaving the surface.

The Gauss theorem statement also gives an important corollary:

The electric flux from any closed surface is only due to the sources (positive charges) and sinks (negative charges) of the electric fields enclosed by the surface. Any charges outside the surface do not contribute to the electric flux. Also, only electric charges can act as sources or sinks of electric fields. Changing magnetic fields, for example, cannot act as sources or sinks of electric fields.



Gauss Law in Magnetism

The net flux for the surface on the left is non-zero as it encloses a net charge. The net flux for the surface on the right is zero since it does not enclose any charge.

⇒ **Note:** The Gauss law is only a restatement of the Coulombs law. If you apply the Gauss theorem to a point charge enclosed by a sphere, you will get back Coulomb's law easily.

Applications of Gauss Law

1. In the case of a charged ring of radius R on its axis at a distance x from the centre of the ring.

$$E = \frac{1}{4\pi\epsilon_0}\frac{qx}{(R^2+x^2)^{3/2}}$$

. At the centre, $x = 0$ and $E = 0$.

2. In the case of an infinite line of charge, at a distance, 'r'. $E = (1/4 \times \pi r\varepsilon_0)\ (2\pi/r) = \lambda/2\pi r\varepsilon_0$. Where λ is the linear charge density.

3. The intensity of the electric field near a plane sheet of charge is $E = \sigma/2\varepsilon_0 K$, where σ = surface charge density.

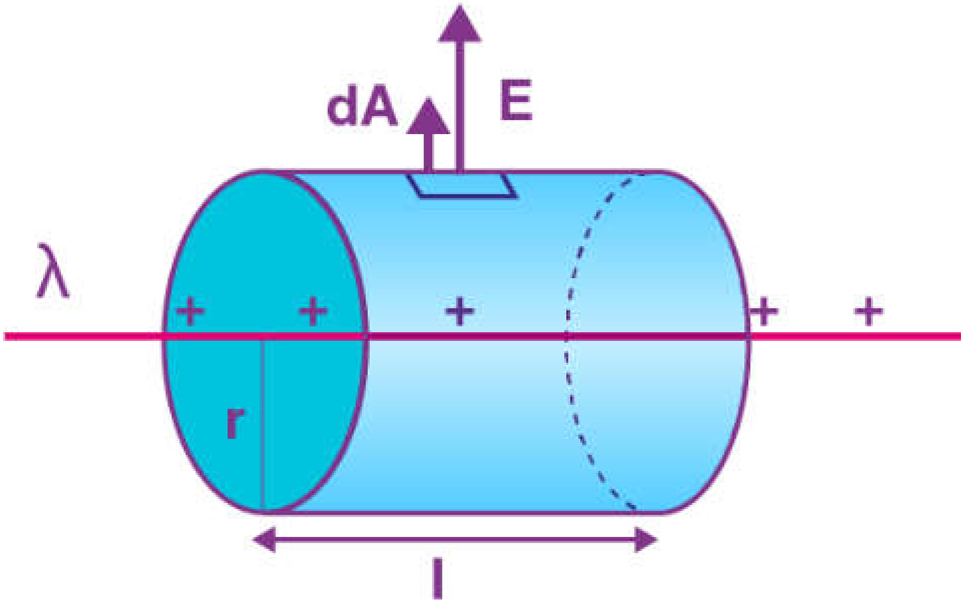
4. The intensity of the electric field near a plane charged conductor $E = \sigma/K\varepsilon_0$ in a medium of dielectric constant K. If the dielectric medium is air, then $E_{air} = \sigma/\varepsilon_0$.

5. The field between two parallel plates of a condenser is $E = \sigma/\varepsilon_0$, where σ is the surface charge density.

Electric Field due to Infinite Wire – Gauss Law Application

Consider an infinitely long line of charge with the charge per unit length being λ. We can take advantage of the cylindrical symmetry of this situation. By symmetry, The electric fields all point radially away from the line of charge, and there is no component parallel to the line of charge.

We can use a cylinder (with an arbitrary radius (r) and length (l)) centred on the line of charge as our Gaussian surface.



Applications of Gauss Law – Electric Field due to Infinite Wire

As you can see in the above diagram, the electric field is perpendicular to the curved surface of the cylinder. Thus, the angle between the electric field and area vector is zero and $\cos\theta = 1$

The top and bottom surfaces of the cylinder lie parallel to the electric field. Thus the angle between the area vector and the electric field is 90 degrees, and $\cos\theta = 0$.

Thus, the electric flux is only due to the curved surface

According to Gauss Law,

$\Phi = \rightarrow E.d \rightarrow A$

$\Phi = \Phi_{curved} + \Phi_{top} + \Phi_{bottom}$

$\Phi = \rightarrow E\ .\ d \rightarrow A = \int E\ .\ dA \cos 0 + \int E\ .\ dA \cos 90° + \int E\ .\ dA \cos 90°$

$\Phi = \int E\ .\ dA \times 1$

Due to radial symmetry, the curved surface is equidistant from the line of charge, and the electric field on the surface has a constant magnitude throughout.

$\Phi = \int E\ .\ dA = E \int dA = E\ .\ 2\pi rl$

The net charge enclosed by the surface is:

$q_{net} = \lambda.l$

Using Gauss theorem,

$\Phi = E \times 2\pi rl = q_{net}/\varepsilon_0 = \lambda l/\varepsilon_0$

$E \times 2\pi rl = \lambda l/\varepsilon_0$

$E = \lambda/2\pi r\varepsilon_0$

Problems on Gauss Law

Problem 1: A uniform electric field of magnitude E = 100 N/C exists in the space in the X-direction. Using the Gauss theorem calculate the flux of this field through a plane square area of edge 10 cm placed in the Y-Z plane. Take the normal along the positive X-axis to be positive.

Solution:

The flux $\Phi = \int E.\cos\theta \, ds$.

As the normal to the area points along the electric field, $\theta = 0$.

Also, E is uniform so, $\Phi = E.\Delta S = (100 \text{ N/C})(0.10\text{m})^2 = 1 \text{ N-m}^2$.

Problem 2: A large plane charge sheet having surface charge density $\sigma = 2.0 \times 10^{-6}$ C-m^{-2} lies in the X-Y plane. Find the flux of the electric field through a circular area of radius 1 cm lying completely in the region where x, y, and z are all positive and with its normal, making an angle of 60^0 with the Z-axis.

Solution:

The electric field near the plane charge sheet is $E = \sigma/2\varepsilon_0$ in the direction away from the sheet. At the given area, the field is along the Z-axis.

The area = $\pi r^2 = 3.14 \times 1 \text{ cm}^2 = 3.14 \times 10^{-4} \text{ m}^2$.

The angle between the normal to the area and the field is 60^0.

Hence, according to Gauss theorem, the flux

$= \vec{E}.\Delta\vec{S}$

$= E.\Delta S \cos\theta$

$= \sigma/2\varepsilon_0 \times pr^2 \cos 60°$

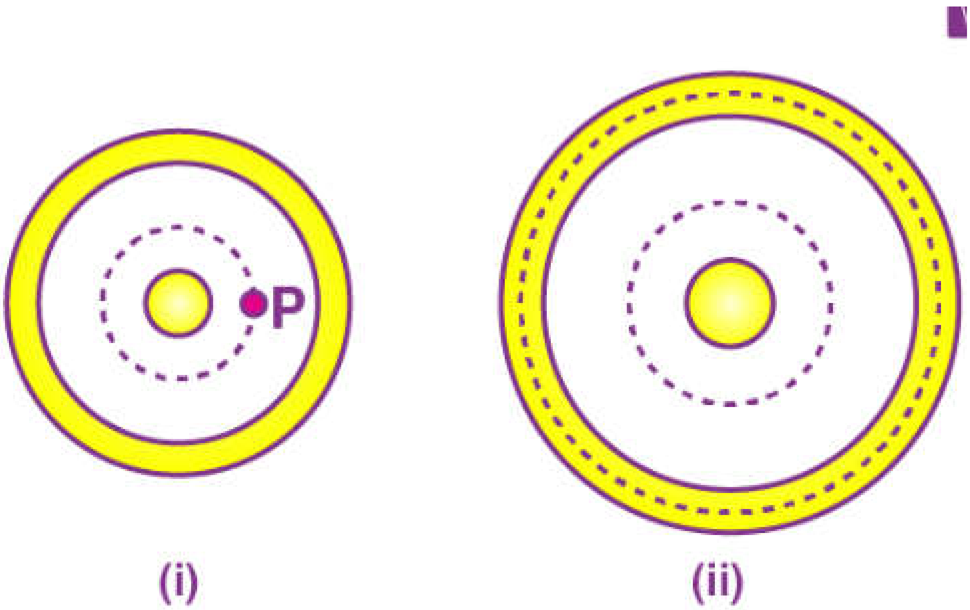
$= \frac{2.0\times10^{-6}C/m^2}{2\times8.85\times10^{-12}C^2/N-m^2} \times (3.14 \times 10^{-4} m^2)\frac{1}{2}$

$= 17.5 \text{ N-m}^2\text{C}^{-1}$.

Problem 3: A charge of 4×10^{-8} C is distributed uniformly on the surface of a sphere of radius 1 cm. It is covered by a concentric, hollow conducting sphere of radius 5 cm.

- Find the electric field at a point 2 cm away from the centre.
- A charge of 6×10^{-8}C is placed on the hollow sphere. Find the surface charge density on the outer surface of the hollow sphere.

Solution:



(a) Let us consider figure (i).

Suppose we have to find the field at point P. Draw a concentric spherical surface through P. All the points on this surface are equivalent; by symmetry, the field at all these points will be equal in magnitude and radial in direction.

The flux through this surface

$$= \oint \vec{E} . \vec{dS}$$
$$= \oint EdS = E \oint dS$$
$= 4\pi x^2 E$.

where $x = 2$ cm $= 2 \times 10^{-2}$ m.

From Gauss law, this flux is equal to the charge q contained inside the surface divided by ε_0. Thus,

$\Rightarrow 4\pi x^2 E = q/\varepsilon_0$ or, $E = q/4\pi\varepsilon_0 x^2$

$= (9 \times 10^9) \times [(4 \times 10^{-8})/(4 \times 10^{-4})] = 9 \times 10^5$ N C^{-1}.

(b) Let us consider figure (ii).

Take the Gaussian surface through the material of the hollow sphere. As the electric field in a conducting material is zero, the flux

$$\oint \vec{E} . \vec{dS}$$
through this Gaussian surface is zero.

Using Gauss law, the total charge enclosed must be zero. Hence, the charge on the inner surface of the hollow sphere is 4×10^{-8}C.

But the total charge given to this hollow sphere is 6×10^{-8} C. Hence, the charge on the outer surface will be 10×10^{-8}C.

Solved Questions on Gauss Law

How is **Gauss law** Related to **Coulomb's law**?

One of the fundamental relationships between the two laws is that **Gauss's law** can be used to derive **Coulomb's law** and vice versa. We can further say that Coulomb's law is equivalent to Gauss's law meaning they are almost the same thing. While this relation is discussed extensively in electrodynamics we will look at a derivation with the help of an example.

Let's take a point charge q. Now, if we apply Coulomb's law, the electric field generated is given by:

$$E = kq/r^2$$

where k=1 /4πε0. If we take the sphere of the radius (r) that is centred on charge q. Now for the surface S of this sphere, we will have:

$$\int_S \vec{E}.\vec{ds} = \int_S \frac{kq}{r^2} ds = \frac{kq}{r^2} \int_S ds = \frac{kq}{r^2}(4\pi r^2) = 4\pi kq = \frac{q}{\epsilon_0}$$

At the end of the equation, we can see that it refers to Gauss law. All in all, we can determine the relation between Gauss law and Coulomb's law by deducing the spherical symmetry of the electric field and by performing the integration.

How do we choose an appropriate Gaussian Surface for different cases?

In order to choose an appropriate Gaussian Surface, we have to take into account the state that the ratio of charge and the dielectric constant is given by a (two-dimensional) surface integral over the electric field symmetry of the charge distribution. There are three different cases that we will need to know.

- Spherical, when the charge distribution is spherically symmetric.
- Cylindrical, when the charge distribution is cylindrically symmetric.
- Pillbox, when the charge distribution has translational symmetry along a plane.

We can choose the size of the surface depending on where we want to calculate the field. Gauss theorem is helpful for finding a field when there is a certain symmetry as it tells us how the field is directed.

How is electric flux related to Gauss law?

When we talk about the relation between electric flux and Gauss law, the law states that the net electric flux in a closed surface will be zero if the volume that is defined by the surface contains a net charge.

To establish the relation, we will first take a look at the Gauss law.

If we take Gauss's law, it is represented as:

$\Phi_E = Q/\varepsilon_o$

Here,

- Φ_E = electric flux through a closed surface S enclosing any volume V.
- Q = total charge enclosed within V,
- ε_o = electric constant.

Meanwhile, the electric flux Φ_E can now be defined as a surface integral of the electric field. It is given as:

$\Phi_E = \iint E \, . \, dA$

Here,

- E = electric field.
- dA = vector representing an infinitesimal element of area of the surface.

Notably, flux is considered as an integral of the electric field. This relation or form of Gauss's law is known as the integral form.

What is the differential form of the Gauss theorem?

The differential form of Gauss law relates the electric field to the charge distribution at a particular point in space. To elaborate, as per the law, the divergence of the electric field (E) will be equal to the volume charge density (p) at a particular point. It is represented as:

$\Delta E = \rho/\varepsilon_o$

Here,

ε_o = permittivity of free space.

How to find the electric field using Gauss law?

Normally, the Gauss law is used to determine the electric field of charge distributions with symmetry. There are several steps involved in solving the problem of the electric field with this law. They are as follows:

1. First, we have to identify the spatial symmetry of the charge distribution.
2. The next step involves choosing a correct Gaussian surface with the same symmetry as the charge distribution. Its consequences should also be identified.
3. Evaluate the integral $\Phi_s E$ over the Gaussian surface and then calculate the flux through the surface.
4. Find the amount of charge enclosed by the Gaussian surface.
5. Evaluate the electric field of the charge distribution.

However, students have to keep in mind the three types of symmetry in order to determine the electric field. The types of symmetry are:

- Spherical symmetry
- Cylindrical symmetry
- Planar symmetry

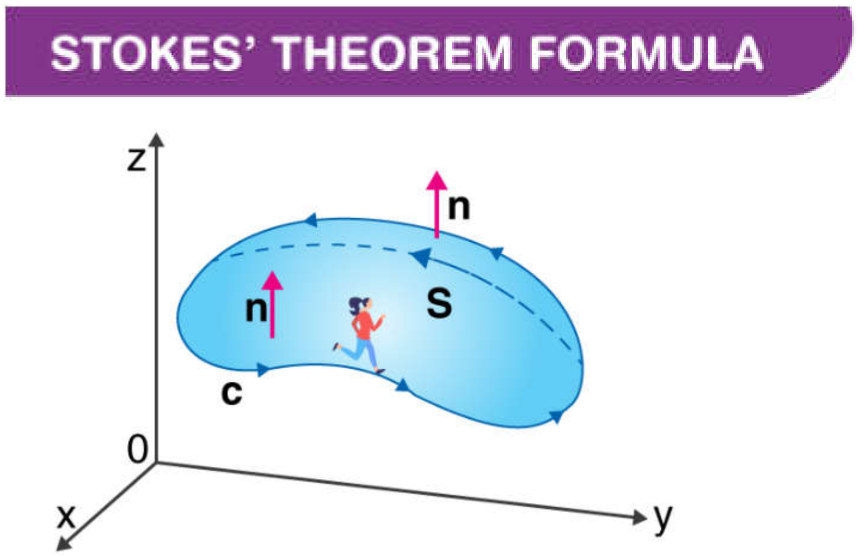
Calculations of inappropriate coordinate systems are to be performed along with the correct Gaussian surface for the particular symmetry.

Stokes Theorem

Stokes Theorem (also known as Generalized Stoke's Theorem) is a declaration about the integration of differential forms on manifolds, which both generalizes and simplifies several theorems from vector calculus. As per this theorem, a line integral is related to a surface integral of vector fields. Learn the stokes law here in detail with formula and proof.

Stokes' Theorem Formula

The Stoke's theorem states that "the surface integral of the curl of a function over a surface bounded by a closed surface is equal to the line integral of the particular vector function around that surface."



$\oint_C \vec{F}.\vec{dr} = \iint_S(\nabla \times \vec{F}).\vec{dS}$

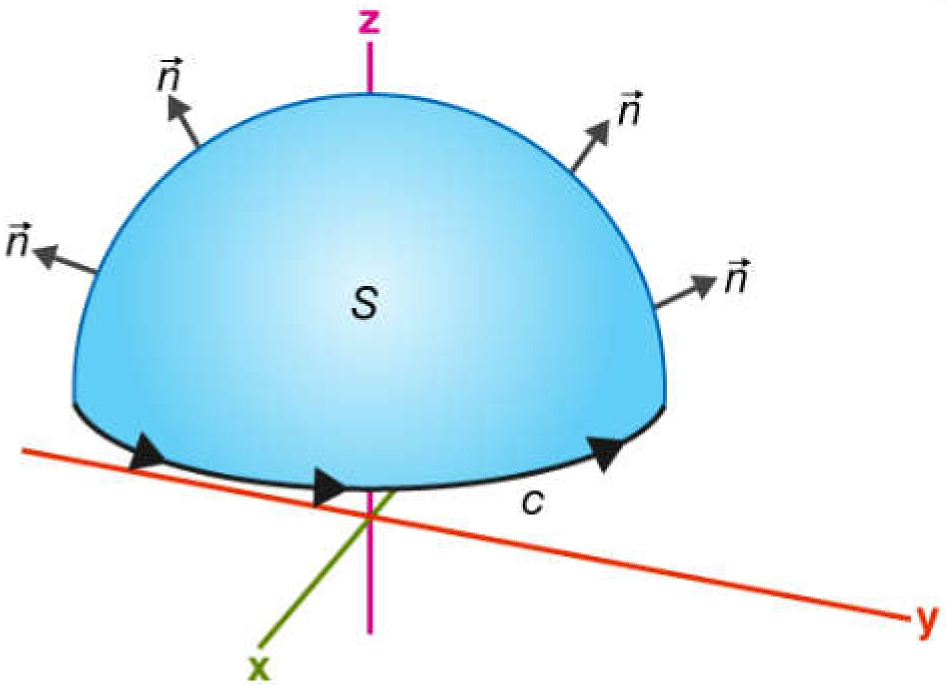
Where,

C = A closed curve.

S = Any surface bounded by C.

F = A vector field whose components have continuous derivatives in an open region of R^3 containing S.

This classical declaration, along with the classical divergence theorem, fundamental theorem of calculus, and Green's theorem are exceptional cases of the general formulation specified above.



This means that:

If you walk in the positive direction around C with your head pointing in the direction of n, the surface will always be on your left.

S is an oriented smooth surface bounded by a simple, closed smooth-boundary curve C with positive orientation.

Stokes Theorem Statement

The line integral around S (the boundary curve) of F's tangential component is equal to the surface integral of the normal component of the curl of F.

The positively oriented boundary curve of the oriented surface S will be ∂S.

Thus, stokes theorem can also be expressed as:

$$\int\int_S curl\vec{F}.\, d\vec{S} = \int_{\partial S} \vec{F}.\, d\vec{r}$$

Gauss Divergence theorem

The Gauss divergence theorem states that the vector's outward flux through a closed surface is equal to the volume integral of the divergence over the area within the surface. Put differently, the sum of all sources subtracted by the sum of every sink results in the net flow of an area.

Gauss divergence theorem is a result that describes the flow of a vector field by a surface to the behaviour of the vector field within the surface.

Stokes Theorem Applications

Stokes' theorem provides a relationship between line integrals and surface integrals. Based on our convenience, one can compute one integral in terms of the other. Stokes' theorem is also used in evaluating the curl of a vector field. Stokes' theorem and the generalized form of this theorem are fundamental in determining the line integral of some particular curve and evaluating a bounded surface's curl. Generally, this theorem is used in physics, particularly in electromagnetism.

6.6 POISSON'S AND LAPLACE'S EQUATIONS

In preceding sections, we have found capacitance by first assuming a known charge distribution on the conductors and then finding the potential difference in terms of the assumed charge. An alternate approach would be to start with known potentials on each conductor, and then work backward to find the charge in terms of the known potential difference. The capacitance in either case is found by the ratio Q/V.

The first objective in the latter approach is thus to find the potential function between conductors, given values of potential on the boundaries, along with possible volume charge densities in the region of interest. The mathematical tools that enable this to happen are Poisson's and Laplace's equations, to be explored in the remainder of this chapter. Problems involving one to three dimensions can be solved either analytically or numerically. Laplace's and Poisson's equations, when compared to other methods, are probably the most widely useful because many problems in engineering practice involve devices in which applied potential differences are known, and in which constant potentials occur at the boundaries.

Obtaining Poisson's equation is exceedingly simple, for from the point form of Gauss's law,

$$\nabla \cdot \mathbf{D} = \rho_v \tag{21}$$

the definition of **D**,

$$\mathbf{D} = \epsilon \mathbf{E} \tag{22}$$

and the gradient relationship,

$$\mathbf{E} = -\nabla V \tag{23}$$

by substitution we have

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \mathbf{E}) = -\nabla \cdot (\epsilon \nabla V) = \rho_v$$

or

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\epsilon} \tag{24}$$

for a homogeneous region in which ϵ is constant.

Equation (24) is *Poisson's equation*, but the "double ∇" operation must be interpreted and expanded, at least in rectangular coordinates, before the equation can be useful. In rectangular coordinates,

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

and therefore

$$\nabla \cdot \nabla V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$
$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{25}$$

Usually the operation $\nabla \cdot \nabla$ is abbreviated ∇^2 (and pronounced "del squared"), a good reminder of the second-order partial derivatives appearing in Eq. (5), and we have

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\epsilon}} \tag{26}$$

in rectangular coordinates.

If $\rho_v = 0$, indicating zero *volume* charge density, but allowing point charges, line charge, and surface charge density to exist at singular locations as sources of the field, then

$$\boxed{\nabla^2 V = 0} \tag{27}$$

which is *Laplace's equation*. The ∇^2 operation is called the *Laplacian of V*.

In rectangular coordinates Laplace's equation is

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{(rectangular)}} \tag{28}$$

and the form of $\nabla^2 V$ in cylindrical and spherical coordinates may be obtained by using the expressions for the divergence and gradient already obtained in those coordinate systems. For reference, the Laplacian in cylindrical coordinates is

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \quad \text{(cylindrical)}} \tag{29}$$

and in spherical coordinates is

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \quad \text{(spherical)}} \tag{30}$$

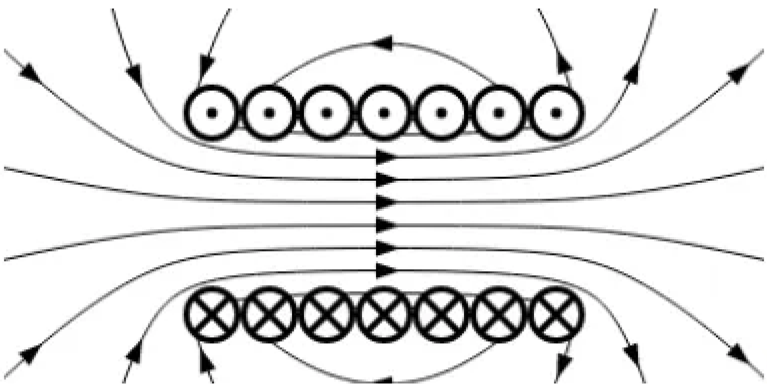
These equations may be expanded by taking the indicated partial derivatives, but it is usually more helpful to have them in the forms just given; furthermore, it is much easier to expand them later if necessary than it is to put the broken pieces back together again.

Laplace's equation is all-embracing, for, applying as it does wherever volume charge density is zero, it states that every conceivable configuration of electrodes or conductors produces a field for which $\nabla^2 V = 0$. All these fields are different, with different potential values and different spatial rates of change, yet for each of them $\nabla^2 V = 0$. Because *every* field (if $\rho_v = 0$) satisfies Laplace's equation, how can we expect to reverse the procedure and use Laplace's equation to find one specific field in which we happen to have an interest? Obviously, more information is required, and we shall find that we must solve Laplace's equation subject to certain *boundary conditions*.

Every physical problem must contain at least one conducting boundary and usually contains two or more. The potentials on these boundaries are assigned values, perhaps V_0, V_1, ..., or perhaps numerical values. These definite equipotential surfaces will provide the boundary conditions for the type of problem to be solved. In other types of problems, the boundary conditions take the form of specified values of E (alternatively, a surface charge density, ρ_S) on an enclosing surface, or a mixture of known values of V and E.

Before using Laplace's equation or Poisson's equation in several examples, we must state that if our answer satisfies Laplace's equation and also satisfies the boundary conditions, then it is the only possible answer. This is a statement of the Uniqueness Theorem, the proof of which is presented in Appendix D.

What is Ampere's Circuital Law?



Ampere's Circuital Law states the relationship between the current and the magnetic field created by it.

This law states that the integral of magnetic field density (B) along an imaginary closed path is equal to the product of current enclosed by the path and permeability of the medium.

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I$$

It alternatively says, the integral of magnetic field intensity (H) along an imaginary closed path is equal to the current enclosed by the path.

$$\oint \overrightarrow{B} \cdot \overrightarrow{dl} = \mu_0 \, I$$

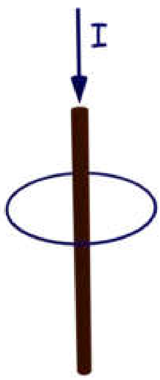
$$\Rightarrow \oint \frac{\overrightarrow{B}}{\mu_0} \cdot \overrightarrow{dl} = I$$

$$\Rightarrow \oint \overrightarrow{H} \cdot \overrightarrow{dl} = I$$

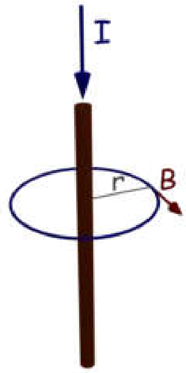
$$\left[\because \overrightarrow{H} = \frac{\overrightarrow{B}}{\mu_0}\right]$$

Let us take an electrical conductor, carrying a current of I ampere, downward as shown in the figure below.

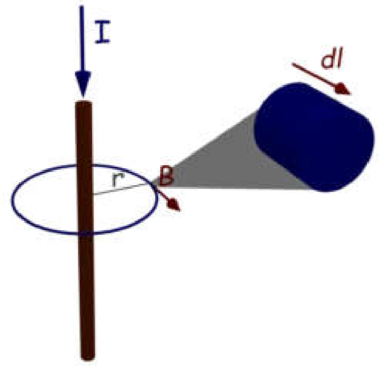
Let us take an imaginary loop around the conductor. We also call this loop as amperian loop.



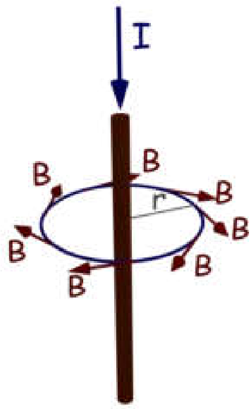
Let us also imagine the radius of the loop is r and the flux density created at any point on the loop due to current through the conductor is B.



Let us consider an infinitesimal length dl of the amperian loop at the same point.



At each point on the amperian loop, the value of B is constant since the perpendicular distance of that point from the axis of conductor is fixed, but the direction will be along the tangent on the loop at that point.



The close integral of the magnetic field density B along the amperian loop, will be,

$$\oint B \cdot dl \qquad [\,dot\ product]$$

$$[\because Direction\ of\ B\ \&\ dl\ is\ same$$
$$at\ each\ point\ on\ the\ loop.]$$

$$= B \oint dl = B \cdot (2\pi r)$$

Now, according to **Ampere's Circuital Law**

$$\oint B \cdot dl = \mu_0 \cdot I$$

Therefore,

$$2\pi r\, B = \mu_0\, I$$
$$\Rightarrow \frac{B}{\mu_0} = \frac{I}{2\pi r}$$
$$\Rightarrow H = \frac{I}{2\pi r}$$

Instead of one current carrying conductor, there are N number of conductors carrying same current I, enclosed by the path, then

$$H = \frac{NI}{2\pi r}$$

Gauss Divergence theorem Significances

In vector calculus, divergence theorem is also known as Gauss's theorem. It relates the flux of a vector field through the closed surface to the divergence of the field in the volume enclosed.

The Gauss divergence theorem states that the vector's outward flux through a closed surface is equal to the volume integral of the divergence over the area within the surface. The sum of all sources subtracted by the sum of every sink will result in the net flow of an area. Gauss divergence theorem is the result that describes the flow of a vector field by a surface to the behaviour of the vector field within it.