SRM Institute of Science and Technology

Tiruchirappalli Campus

Cycle Test - II . NOV 2022 . Set B

Calculus and Linear Algebra

Answer Key

Part - A

1. a 2. b 3. c 4. d 5. 2

6. b 7. c 8. c 9. c 10. c

11. $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$ — (1)

$= (y+z)e^{t} + (x+z)(-e^{-t}) + (x+y)\left(-\frac{1}{t^2}\right)$ — (1)

$= \left(e^{-t}+\frac{1}{t}\right)e^{t} + \left(e^{t}+\frac{1}{t}\right)\left(-e^{-t}\right) + \left(e^{t}+e^{-t}\right)\left(-\frac{1}{t^2}\right)$ — (1)

$= 1 + \frac{e^{t}}{t} - 1 - \frac{e^{-t}}{t} - \frac{e^{t}}{t^2} - \frac{e^{-t}}{t^2}$ — (1)

$= -\frac{e^{t}-e^{-t}}{t^2}$

12. $f = \tan u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$ — (1)

$\therefore f(tx, ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = t^{1/2} f(x,y)$

$\therefore$ f is the homogeneous function of degree $\frac{1}{2}$. — (1)

$\therefore$ by Euler's theorem

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{1}{2}f$ — (1)

i.e., $x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = \frac{1}{2}\tan u$

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\cdot\frac{\tan u}{\sec^2 u}$

$= \frac{\sin 2u}{4}$ — (1)

13. $J\left(\frac{x,y,z}{r,\theta,z}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$ — (1)

$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$ — (1)

$= 1\left(r\cos^2\theta + r\sin^2\theta\right)$ — (1)

$= r\left(\cos^2\theta + \sin^2\theta\right)$

(1)

$= r$

14. The given equation can be written as

$(D^2 - 4D - 12)y = 5e^{6x}$ — (1)

The A.E is $m^2 - 4m - 12 = 0$

$\Rightarrow (m-6)(m+2) = 0$

$\Rightarrow m_1 = 6,\ m_2 = 2$

$\therefore CF = Ae^{6x} + Be^{2x}$ — (1)

$P.I = \frac{1}{D^2-4D-12}5e^{6x} = \frac{5}{36-24-12}e^{6x} = \frac{5}{0}e^{6x}$

$= \frac{x}{2D-4}5e^{6x} = \frac{5xe^{6x}}{8}$ — (1)

$\therefore$ The solution is $CF + PI$ — (1)

i.e., $y = Ae^{6x} + Be^{2x} + \frac{5}{8}xe^{6x}$

The given diff...

15. The A.E is $m^2+9=0$

$m^2=-9 \Rightarrow m=\pm 3i$ — (1)

The CF = $A\cos 3x + B\sin 3x$ — (1)

$PI = \frac{1}{D^2+9}\cos 3x$

$= \frac{1}{-9+9}\cos 3x = \frac{1}{0}\cos 3x$

$= \frac{x}{2D}\cos 3x = \frac{x}{2}\int \cos 3x\,dx$

$= \frac{x}{6}\sin 3x$ — (1)

∴ The solution is $y = CF + PI$

i.e., $y = A\cos 3x + B\sin 3x + \frac{x\sin 3x}{6}$ — (1)

16. Put $x = e^z \Rightarrow z = \log x$, $xD = D'$, $x^2D^2 = D'(D'-1)$, $D' = \frac{d}{dz}$

The equation becomes

$[D'(D'-1) + D' + 1]y = \sin z$

$(D'^2+1)y = \sin z$ — (1)

The AE is $m^2+1=0$

$\Rightarrow m = \pm i$

∴ $CF = A\cos z + B\sin z$ — (1)

$PI = \frac{1}{D'^2+1}\sin z = \frac{1}{-1+1}\sin z = \frac{1}{0}\sin z$

$= \frac{z}{2D'}\sin z = \frac{z}{2}\int \sin z = -\frac{z\cos z}{2}$ — (1)

The solution is $y = CF + PI$

$y = A\cos z + B\sin z - \frac{z\cos z}{2}$

$= A\cos(\log x) + B\sin(\log x) - \frac{\log x\cos(\log x)}{2}$

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17 (i) $f(x,y) = x^2y^2 + 2x^2y + 3xy^2$; $f(2,-1) = 2$

$f_x = 2xy^2 + 4xy + 3y^2$; $f_x(2,-1) = -1$ — (1)

$f_{xx} = 2y^2 + 4y$; $f_{xx}(2,-1) = -2$ — (1)

$f_{xxx} = 0$; $f_{xxx}(2,-1) = 0$ — (1)

$f_y = 2x^2y + 2x^2 + 6xy$; $f_y(2,-1) = -12$ — (1)

$f_{yy} = 2x^2 + 6x$; $f_{yy}(2,-1) = 20$ — (1)

$f_{yyy} = 0$; $f_{yyy}(2,-1) = 0$ — (1)

$f_{xy} = 4xy + 4x + 6y$; $f_{xy}(2,-1) = -6$ — (1)

$f_{xxy} = 4y + 4$; $f_{xxy}(2,-1) = 0$ — (1)

$f_{xyy} = 4x + 6$; $f_{xyy}(2,-1) = 14$ — (1)

The Taylor's series of $f(x,y)$ at (a,b) is

$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)$ — (1)

$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$

$+ \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] \cdots$

Here $(a,b) = (2,-1)$

$= 2 + (x-2)(-1) + (y+1)(-12) + \frac{1}{2}\left[(x-2)^2(-2) + 2(x-2)(y+1)(-6) + (y+1)^2(20)\right]$

$+ \frac{1}{6}\left[(x-2)^3(0) + 3(x-2)^2(y+1)(0) + 3(x-2)(y+1)^2(14) + (y+1)^3(0)\right]$ — (1)

$= 2 - (x-2) - (y+1)(12) + \frac{1}{2}\left[-2(x-2)^2 - 12(x-2)(y+1) + 20(y+1)^2\right] + \frac{1}{6}\left[42(x-2)(y+1)^2\right]$ — (1)

Let x, y, z be a point on the sphere

$x^2+y^2+z^2 = 24$

$\therefore$ The distance from the point (1, 2, -1) to the sphere is

$$d = \sqrt{(x-1)^2+(y-2)^2+(z+1)^2} \quad \text{—(1)}$$

Now $f = d^2 = (x-1)^2+(y-2)^2+(z+1)^2$

The problem is

Min & Max $f = (x-1)^2+(y-2)^2+(z+1)^2$ — (1)

subject to $\phi = x^2+y^2+z^2-24 = 0$

Now $F = f + \lambda\phi$ — (1)

$= (x-1)^2+(y-2)^2+(z+1)^2+\lambda(x^2+y^2+z^2-24)$

for stationary values

$F_x = 0 \Rightarrow 2(x-1)+\lambda(2x) = 0$ —— 1

$F_y = 0 \Rightarrow 2(y-2)+\lambda(2y) = 0$ —— 2 (4)

$F_z = 0 \Rightarrow 2(z+1)+\lambda(2z) = 0$ —— 3

$F_\lambda = 0 \Rightarrow x^2+y^2+z^2-24 = 0$ —— 4

From 1, 2 & 3 we get

$x = \frac{1}{\lambda+1}$; $y = \frac{2}{\lambda+1}$; $z = -\frac{1}{\lambda+1}$ →(1)

Sub in 4 we get

$$\frac{1}{(\lambda+1)^2}+\frac{4}{(\lambda+1)^2}+\frac{1}{(\lambda+1)^2} = 24$$

$\Rightarrow (\lambda+1)^2 = \frac{6}{24} = \frac{1}{4}$

$\therefore \lambda+1 = \pm\frac{1}{2}$

$\Rightarrow \lambda = -\frac{1}{2}$ or $-\frac{3}{2}$ — (1)

$\therefore$ If $\lambda = -\frac{1}{2}$, then

$x = 2$; $y = 4$; $z = -2$ — (1)

If $\lambda = -\frac{3}{2}$, then

$x = -2$, $y = -4$, $z = 2$ — (1)

$\therefore d = \sqrt{1+4+1} = \sqrt{6}$ — (1)

$d = \sqrt{54} = 3\sqrt{6}$

Shortest distance is $\sqrt{6}$, longest distance is $3\sqrt{6}$

18 (i) The A.E is $m^2+3m+2 = 0$

ie, $(m+2)(m+1) = 0$ —(1)

$\Rightarrow m_1 = -2, m_2 = -1$

$\therefore$ The CF $= Ae^{-2x}+Be^{-x}$ —(1)

$$P.I = \frac{1}{D^2+3D+2}(x^2+\sin x) \quad \text{—(1)}$$

$$= \underbrace{\frac{1}{D^2+3D+2}x^2}_{PI_1} + \underbrace{\frac{1}{D^2+3D+2}\sin x}_{PI_2}$$

$$PI_1 = \frac{1}{2\left(1+\frac{D^2+3D}{2}\right)}x^2 = \frac{1}{2}\left[1+\left(\frac{D^2+3D}{2}\right)\right]^{-1}x^2 \quad \text{—(1)}$$

$$= \frac{1}{2}\left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2\right]x^2$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}-\frac{3D}{2}+\frac{9D^2}{4}\right]x^2 \quad \text{—(1)}$$

$$= \frac{1}{2}\left[x^2-\frac{2}{2}-\frac{3(2x)}{2}+\frac{9(2)}{4}\right] \quad \text{—(1)}$$

$$= \frac{1}{2}\left[x^2-3x+\frac{7}{2}\right] \quad \text{—(1)}$$

$$PI_2 = \frac{1}{-1+3D+2}\sin x \quad \text{—(1)}$$

$$= \frac{(3D-1)}{(3D+1)(3D-1)}\sin x \quad \text{—(1)}$$

$$= \frac{(3D-1)}{9D^2-1}\sin x$$

$$= \frac{3\cos x-\sin x}{-10} \quad \text{—(1)}$$

$\therefore$ The solution is $y = CF+PI$ —(1)

ie, $y = Ae^{-2x}+Be^{-x}+\frac{1}{2}\left[x^2-3x+\frac{7}{2}\right]$

$-\frac{1}{10}(3\cos x-\sin x)$ —(1)

The given differential equation
Can be written as

$$(D^2+1)\,y = \operatorname{Cosec} x \quad \text{—— (1)}$$

The A.E is $m^2+1=0$ —— (1)

$\Rightarrow m = \pm i$ —— (1)

$\therefore$ CF is $A\cos x + B\sin x$

$\Rightarrow A f_1 + B f_2$ —— (1)

$PI = Pf_1 + Qf_2$ where —— (1)

$$P = -\int \frac{f_2 R}{W}\,dx \;\&\; Q = \int \frac{f_1 R}{W}\,dx \quad \text{—— (2)}$$

$$W = f_1 f_2' - f_2 f_1' \quad \text{—— (1)}$$

$$= \cos^2 x + \sin^2 x = 1$$

$$P = -\int \frac{\sin x \,.\, \operatorname{Cosec} x}{1}\,dx$$

$$= -\int dx = -x \quad \text{—— (1)}$$

$$Q = \int \frac{\cos x \,.\, \operatorname{Cosec} x}{1}\,dx$$

$$= \int \frac{\cos x}{\sin x}\,dx = \log \sin x \quad \text{—— (1)}$$

$\therefore$ The solution is $y = CF + PI$ —— (1)

i.e, $y = A\cos x + B\sin x - x\cos x + \log \sin x \,.\, \sin x.$ —— (1)