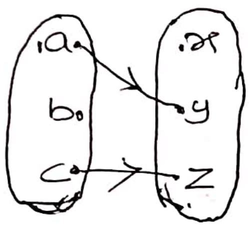
Functions

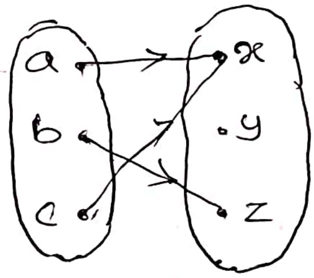
Definitions :-

A relation f from a set X to another set Y is called **function** if for every $x \in X$ there is an unique $y \in Y$ such that $(x,y) \in f$. It is repre--sented as $f : X \to Y$.

Let $f : X \to Y$ be any function if $y = f(x)$, then x is called an pre image and y is called the image of x.

Ex:- State whether or not each of the diagram given below define a function of $A = \{a,b,c\}$ into $B = \{x,y,z\}$.





it is not function. assigned to 'b'.

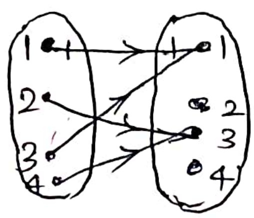
It is function.

Here we can ~~representation~~ a function $f(a) = x$; $f(b) = z$ & $f(c) = x$.

Note :- X is domain of f denoted $D(f)$ and Y is called co-domain or range of f and it is denoted by $R(f)$. Domain of f is $D(f) = X$ and $R(f) \subseteq Y$

Problems :-

1) The following fig. defines the function f which maps the set $\{1,2,3,4\}$ into itself. Find the range of f.



Here $f(1) = 1$; $f(2) = 3$; $f(3) = 1$
$f(4) = 3$

Range of $f = R(f) = \{1,3\}$

2) If the fn. f is defined by $f(x) = x^2+1$ on the set $\{-2, -1, 0, 1, 2\}$ then find the range of f.

Soln:

$f(x) = x^2+1$

$f(-2) = (-2)^2+1 = 5$, $f(-1) = (-1)^2+1 = 2$, $f(0) = 0+1 = 1$

$f(1) = 1^2+1 = 2$; $f(2) = 2^2+1 = 5$

The range of f is $R(f) = \{1, 2, 5\}$

Types of function:-

One-to-one :- (Injective)

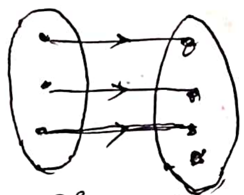
A function $f: X \to Y$ is called one-to-one (1-1) (or) injective, if the distinct elements of X are mapped into distinct elements of Y.

(ie) ~~Iff~~ f is 1-1 iff $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$

(or)

$f(x_1) = f(x_2)$ whenever $x_1 = x_2$

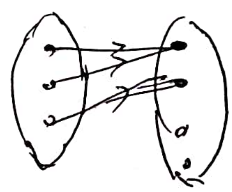
Ex:-



x y

One-to-one

(∵ It is one-to-one because each every element of ~~x~~ is mapped distinct elements of y)

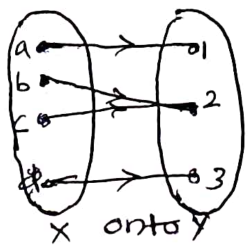


not one-to-one

onto :- (surjective)

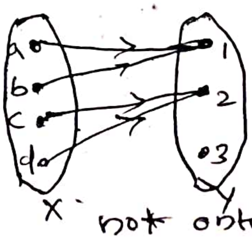
A function $f: X \to Y$ is called onto or surjective ~~if and if only~~ if, for every $y \in Y$, there exists atleast one element $x \in X$, such that $f(x) = y$.

(ie) if $f(x) = y$ then f is onto.



(since all y is having fn.)

X onto Y

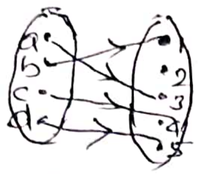


(∵ since 3 is not ~~mapped~~ mapping on X)

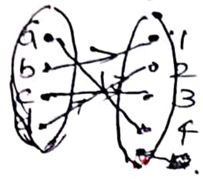
X not onto

One-to-one onto :- (Invertible)

A function $f: X \to Y$ is called one-to-one onto or) (bijective) if f is both one-to-one and onto.
Obviously, if X and Y are finite such that $f: X \to Y$ is bijective then X and Y have the same number of the elements.



(Here the given fn. one-to-one but not onto)

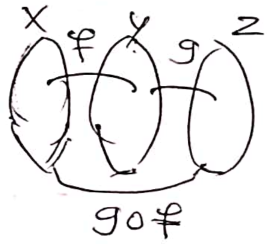


(Here the given fn. is 1-1 and onto)

Composition of functions :-

If $f: A \to B$ and $g: B \to C$ then the composition $g \circ f: A \to C$

$$g \circ f(x) = g(f(x)) \quad \forall x \in A$$



Inverse of a function :-

If $f: A \to B$ and $g: B \to A$ then the function g is called the inverse of the function f if

$g \circ f = I_A$ and $f \circ g = I_B$.

$y = f(x)$ then $x = g(y)$

Thus the fn. $g: B \to A$ is called the inverse of $f: A \to B$ of $x = g(y)$ whenever $y = f(x)$.

Problems :-

1) let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 4x - 1$, $g(x) = \cos x$. Find $f \circ g$ & $g \circ f$.

Soln:

To find $g \circ f : R \to R$

$g \circ f(x) = g(f(x)) = g(4x-1)$; $f \circ g(x) = f(g(x))$

$g \circ f = \cos(4x-1)$; $= f(\cos x)$

$f \circ g = 4\cos x - 1$

$\therefore g \circ f \neq f \circ g$.

2) If $f(x) = x^2$; $g(x) = 3x$; $f: R \to R$, $g: R \to R$

Find $f \circ g$ and $g \circ f$.

Soln:

$f \circ g(x) = f(g(x)) = f(3x) = (3x)^2 = 9x^2$

$g \circ f(x) = g(f(x)) = g(x^2) = 3x^2$

$\therefore f \circ g \neq g \circ f$

3) Show that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x-1$ is a bijection.

Soln: $\therefore$ For a fn. to be a bijection, it must be 1-1 and onto.

$f: \mathbb{R} \to \mathbb{R}$

* Set of all real numbers

(i) To prove 1-1,

let $f(x_1) = f(x_2)$

$3x_1 - 1 = 3x_2 - 1$

$3x_1 = 3x_2$

$\therefore x_1 = x_2$

f is 1-1

(ii) To prove onto :-

Let $y \in R$ $\exists$ $x = \frac{y+1}{3} \in R$.

$\ni f(x) = y$

f is onto

($f(x) = y = 3x-1$
$3x = y+1$
$\therefore x = \frac{y+1}{3}$)
$y = 3x-1$)

$\therefore f$ is bijection.

Hw

4) Let $f(x) = x^3$ and $g(x) = 4x+3$ be fns on R, then $f \circ g$ is

4) If $A=\{x\in R \mid x\neq 2\}$, $B=\{x\in R \mid x\neq 1\}$ Define $f: A\to B$ by (3)

$f(x)=\frac{x}{x-2}$ Prove that f is 1-1 and onto. Also find f^{-1}.

Soln:- (i) 1-1 :

$f(x)=f(y)$

$\frac{x}{x-2}=\frac{y}{y-2}$

$\cancel{xy}-2x=\cancel{xy}-2y$

$\cancel{2}x=\cancel{2}y \Rightarrow x=y$

$\therefore f$ is 1-1

$f(x)=y$

$ff^{-1}(x)=f^{-1}(y)$

$x=f^{-1}(y)$

ii) onto :-

$f(x)=\frac{x}{x-2}$

$y=\frac{x}{x-2}$

y is defined $x\neq 2$ $\forall y\neq 1\in B$ $\exists x\neq 2\in A$

$\exists f(x)=y$

$\therefore f$ is onto.

iii) f^{-1} :-

$x=f^{-1}(y)$

$x=\frac{2y}{y-1}$ $y\neq 1$

$\therefore f^{-1}(x)=\frac{2x}{x-1}$ $x\neq 1$

$\therefore y=\frac{x}{x-2}$

$(x-2)y=x$

$x=xy-2y$

$2y=xy-x$

$2y=x(y-1)$

$x=\frac{2y}{y-1}$

5) If $A=\{x\in R \mid x\neq 1/2\}$ and $f: A\to R$ defined by $f(x)=\frac{4x}{2x-1}$,

(i) Find range of f

(ii) Show that f is invertible.

(iii) Find domain (f^{-1}), range (f^{-1}) and f^{-1}

Proof :- $f(x) = \frac{4x}{2x-1} \Rightarrow f(x) = y \Rightarrow x = \frac{y}{2y-4}$ $(y \neq 2)$

(i) Range of $f = \{y \in R \mid y \neq 2\}$ x is defined for $y \neq 2$

(ii) To prove that f is invertiable.

(ie) f is 1-1 and onto

(i) f is 1-1 :-

$f(x_1) = f(x_2)$

$\frac{4x_1}{2x_1-1} = \frac{4x_2}{2x_2-1}$

$x_1 = x_2$

$\therefore f$ is 1-1

f is onto :-

$\forall\ y \neq 2 \in R, \exists\ x \neq \frac{1}{2} \ni$

$\Rightarrow f(x) = y$

$\Rightarrow f$ is onto.

$\therefore f$ is invertiable.

(iii) Domain of f^{-1}:

$f : A \to R,\ f^{-1} : R \to A$

$dom(f^{-1}) = Range(f)$

$= \{y \in R / y \neq 2\}$

Range of f^{-1} :-

$= dom(f)$

$= \{x \in R / x \neq \frac{1}{2}\}$

$\therefore f^{-1}$:

$x = f^{-1}(y)$

$x = \frac{y}{2y-4}$; $y \neq 2$

$(\because f(x) = y = \frac{4x}{2x-1}$

$f^{-1}(x) = \frac{x}{2x-4}$; $x \neq 2$

HW: If $S = \{1,2,3,4,5\}$ if the fn.s $f, g, h : S \to S$ are given by $f = \{(1,2), (2,1), (3,4), (4,5), (5,3)\}$

$g = \{(1,3), (2,5), (3,1), (4,2), (5,4)\}$

$h = \{(1,2), (2,2), (3,4), (4,3), (5,1)\}$

1) verify whether $f \circ g = g \circ f$

2) Explain why f & g inverse but h does not.

3) find f^{-1} & g^{-1} 4) S.T $(f \circ g)^{-1} = g^{-1} \circ f^{-1} \neq f^{-1} \circ g^{-1}$

6. If $f: Z \to N \cup \{0\}$ is defined by $f(x) = \begin{cases} 2x-1 & \text{if } x > 0 \\ -2x & \text{if } x \le 0 \end{cases}$

(i) prove that is 1-1 and onto (ii) Determine f^{-1}.

Soln: -

(i) To prove f is 1-1 and onto

let $x_1, x_2 \in Z$ and $f(x_1) = f(x_2)$. Then either $f(x_1)$ and $f(x_2)$ both odd or even.

If they are both odd then

$$2x_1 - 1 = 2x_2 - 1$$

$$x_1 = x_2$$

If they are both even then

$$-2x_1 = -2x_2$$

$$x_1 = x_2$$

$\therefore f(x_1) = f(x_2)$ whenever $x_1 = x_2$

$f(x)$ is one to one.

To prove f is onto: -

let $y \in N$ If y is odd its pre image is $\frac{y+1}{2}$

$\left(\because f(x) = y = 2x-1\right.$
$2x = y+1$
$\left.\boxed{x = \frac{y+1}{2}}\right)$

If y is even its pre image $\frac{-y}{2}$

$(\because y = -2x$
$\Rightarrow x = -y/2$

for any $y \in N$ the pre image is $\frac{y+1}{2} \in Z$ (or) $\frac{-y}{2} \in Z$

$\therefore f(x)$ is onto.

$\therefore f(x)$ is invertiable.

ii) To find f^{-1}: -

$$y = f(x) = \begin{cases} 2x-1 & \text{if } x > 0 \\ \frac{-x}{2} & \text{if } x < 0 \end{cases}$$

$$f^{-1}(y) = x = \begin{cases} \frac{y+1}{2} & \text{if } x = 1, 3, 5, \ldots \\ \frac{-y}{2} & \text{if } x = 0, 2, 4 \ldots \end{cases}$$

$$\therefore f^{-1}(x) = \begin{cases} \frac{x+1}{2} & \text{if } x = 1, 3, 5 \ldots \\ \frac{-x}{2} & \text{if } x = 0, 2, 4 \ldots \end{cases}$$

7) If $A = \{1,2,3,4,5\}$, $B = \{1,2,3,8,9\}$ and the functions $f : A \to B$ and $g : A \to A$ are defined by $f = \{(1,8), (3,9), (4,3), (2,1), (5,2)\}$ and $g = \{(1,2), (3,1), (2,2), (4,3), (5,2)\}$ find $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$ if they exist.

Soln :-

(i) $f \circ g = f(g(x))$. Here $g : A \to A$ and $f : A \to B$

and $f \circ g : A \to B$

$f \circ g(1) = f(g(1)) = f(2) = 1$

$f \circ g(2) = f(g(2)) = f(2) = 1$

$f \circ g(3) = f(g(3)) = f(1) = 8$

$f \circ g(4) = f(g(4)) = f(3) = 9$

$f \circ g(5) = f(g(5)) = f(2) = 1$

$g = \{(1,2), (3,1), (2,2), (4,3), (5,2)\}$

$f = \{(1,8), (3,9), (4,3), (2,1), (5,2)\}$

(or)

$f \circ g = \{(1,1), (3,8), (2,1), (4,9), (5,1)\}$

$\therefore f \circ g = \{(1,1), (2,1), (3,8), (4,9), (5,1)\}$

ii) To find $g \circ f$:-

Domain of $g = \{1,2,3,4,5\}$

range of $f = \{1,8,9,3,2\}$

range of $f \not\subseteq$ Domain of g

$\therefore$ $g \cdot f$ is not defined.

iii) To find $f \circ f$:-

Domain of $f = \{1,2,3,4,5\}$

range of $f = \{8,9,3,1,2\}$

range of $f \not\subseteq$ Domain of f

$f \cdot f$ is not defined.

iv) To find $g \cdot g$:-

Domain of $g = \{1,2,3,4,5\}$, range of $g = \{1,2,3,4,5\}$

range of $g \subseteq$ Domain of g

$g = \{(1,2), (3,1), (2,2), (4,3), (5,2)\}$

$g = \{(1,2), (3,1), (2,2), (4,3), (5,2)\}$

$g \circ g(1) = g(g(1)) = g(2) = 2$

$g \circ g(2) = g(g(2)) = g(2) = 2$

$g \cdot g(3) = g(g(3)) = g(1) = 2$

$g \cdot g(4) = g(g(4)) = g(3) = 1$

$g \cdot g(5) = g(g(5)) = g(2) = 2$

$g \cdot g = \{(1,2), (2,2), (3,2), (4,1), (5,2)\}$

Problems :

1) If $f, g, h : R \to R$ defined by $f(x) = x^3 - 4x$; $g(x) = \frac{1}{x^2+1}$; $h(x) = x^4$, verify if .

$\{(f \circ g) \circ h\}(x) = \{f \circ (g \circ h)\}(x)$ after estimating values.

Soln :-

$$f \circ g(x) = f(g(x)) = f\left(\frac{1}{x^2+1}\right) = \left(\frac{1}{x^2+1}\right)^3 - 4\left(\frac{1}{x^2+1}\right)$$

$$\{(f \circ g) \circ h\}(x) = (f \circ g) h(x) = (f \circ g)(x^4) = \left(\frac{1}{x^8+1}\right)^3 - 4\left(\frac{1}{x^8+1}\right) \to ①$$

$$(g \circ h)(x) = g(h(x)) = g(x^4) = \frac{1}{x^8+1}$$

$$f \circ (g \circ h(x)) = f\left(\frac{1}{x^8+1}\right) = \left(\frac{1}{x^8+1}\right)^3 - 4\left(\frac{1}{x^8+1}\right) \to ②$$

from ① & ②

$$\therefore \{(f \circ g) \circ h\}(x) = \{f \circ (g \circ h)\}(x)$$

Hence proved.

2) If $S = \{1, 2, 3, 4, 5\}$ and if the functions f, g and h are from $S \to S$ and are given by

$$f = \{(1,2), (2,1), (3,4), (4,5), (5,3)\}$$

$$g = \{(1,3), (2,5), (3,1), (4,2), (5,4)\}$$

$$h = \{(1,2), (2,2), (3,4), (4,3), (5,1)\}$$

a) verify $g \circ f = f \circ g$

b) Explain why f and g have inverses but h does not.

c) Find f^{-1} & g^{-1}

d) Show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1} \neq f^{-1} \circ g^{-1}$

Soln :-

(a) $f \circ g(x) = f(g(x))$

$$f = \{(1,2), (2,1), (3,4), (4,5), (5,3)\}$$

$$g = \{(1,3), (2,5), (3,1), (4,2), (5,4)\}$$

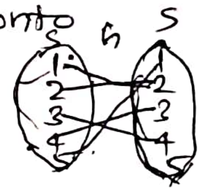
$$g \circ f(x) = g(f(x))$$

$$g \circ f = \{(1,5), (2,3), (3,2), (4,4), (5,1)\}$$

$$f \circ g = \{(1,4), (2,3), (3,2), (4,1), (5,5)\}$$

$\therefore$ $g \circ f \neq f \circ g$

(b) Both f and g are 1-1 and onto. They are invertible. But h is not 1-1 and not onto.



so h has no inverse.

c) $$f^{-1} = \{(2,1), (1,2), (4,3), (5,4), (3,5)\}$$

$$g^{-1} = \{(3,1), (5,2), (1,3), (2,4), (4,5)\}$$

d) To prove $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

$$(f \circ g)^{-1} = \{(4,1), (3,2), (2,3), (1,4), (5,5)\} \rightarrow ①$$

$$g^{-1} \circ f^{-1} = \{(2,3), (1,4), (4,1), (5,5), (3,2)\} \rightarrow ②$$

from ① & ② $\Rightarrow (f \circ g)^{-1} = g^{-1} \circ f^{-1}$

$$f^{-1} \circ g^{-1} = \{(3,2), (5,1), (1,5), (2,3), (4,4)\} \rightarrow ③$$

from ① & ② & ③

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1} \neq f^{-1} \circ g^{-1}$$

Hence Proved.

Property based on compositions of functions:

Property :1

Composition of functions is associative,

viz. $f: A \to B$, $g: B \to C$ and $h: C \to D$ are functions

then $h \circ (g \circ f) = (h \circ g) \circ f$

Proof :-

Since $f: A \to B$ and $g: B \to C$ then we've $g \circ f = A \to C$

Since $g \circ f = A \to C$ and $h: C \to D$ then $h \circ (g \circ f) = A \to D$ $\to (1)$

$h: C \to D$ and $g: B \to C$ then we have $h \circ g: B \to D$

Since $h \circ g: B \to D$ and $f: A \to B$ then $(h \circ g) \circ f = A \to D$ $\to (2)$

from (1) & (2) $h \circ (g \circ f) = (h \circ g) \circ f$

The Composition of functions is associative.

Property :2

When $f: A \to B$ and $g: B \to C$ are functions then $g \circ f: A \to C$ is an injection, surjection or bijection according as f and g are injection, surjection or bijection

Proof:-

(i) Let $a_1, a_2 \in A$. Then to prove $g \circ f$ is 1-1.

$(g \circ f)(a_1) = (g \circ f)(a_2)$

$g(f(a_1)) = g(f(a_2))$

$f(a_1) = f(a_2)$ (g is injective)

$a_1 = a_2$ ($\because f$ is injective)

$\therefore$ $g \circ f$ is 1-1

ii) To Prove $g \circ f$ is surjective (onto) :-

Let $c \in C$, since g is onto, there is an element $b \in B$ such that $c = g(b)$.
since f is onto, there is an element $a \in A$ such that $b = f(a)$.

Now $(g \circ f)(a) = g(f(a)) = g(b) = c$

$\Rightarrow$ $g \circ f : A \to C$ is onto.

from (i) & (ii) $\Rightarrow$ $g \circ f : A \to C$ is bijective if f and g are bijective.

Property based on Inverse function :-

Property :- 3

The inverse of a function f, if exists, is unique.

Proof :- Let g and h be inverse of f then by the defn. $g \circ f = I_A$, $f \circ g = I_B$ and also $h \circ f = I_A$

$$h = h \circ I_B = h \circ (f \circ g) = (h \circ f) \circ g = I_A \circ g = g$$

so $h = g$

Property : 4.

If $f : A \to B$, $g : B \to C$ are invertible (inverse exists) functions, then $g \circ f : A \to C$ is also invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

(or)

The inverse of the composition of two functions is equal to the composition of the inverse of the functions in the reverse order.

Proof :-

Given f and g are 1-1 and onto.

so $g \circ f$ is also bijective.

$\Rightarrow g \circ f$ is invertiable.

Since $f : A \to B$ and $g : B \to C$, we have

$f^{-1} : B \to A$ and $g^{-1} : C \to B$.

For any $a \in A$, let $b = f(a)$ and $c = g(b)$.

$\Rightarrow \quad f^{-1}(b) = a \; ; \; g^{-1}(c) = b$

$(g \circ f)(a) = g(f(a)) = g(b) = c$

$\Rightarrow \quad a = (g \circ f)^{-1}(c) \longrightarrow (1)$

and $\quad (f^{-1} \circ g^{-1})(c) = f^{-1}(g^{-1}(c)) = f^{-1}(b) = a \longrightarrow (2)$

from (1) & (2) $\quad (g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Property :- 5

The necessary and sufficient condition for the function $f : A \to B$ to be invertible (for f^{-1} to exist) is that f is one to one and onto.

Proof :- case(i): If f is invertible, then to prove f is 1-1 and onto.

(i) To prove 1-1 :-

Pf:- let $f : A \to B$ is invertible.

Then there exists a unique function $g : B \to A$ Such that $g \circ f = I_A$ & $f \circ g = I_B$

let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$

where $f(a_1), f(a_2) \in B$ [$\because f : A \to B$ is function)

since $g : B \to A$ is a function

$g\{f(a_1)\} = g\{f(a_2)\}$

$(g \circ f)(a_1) = (g \circ f)(a_2)$

$I_A \, a_1 = I_A \, a_2 \Rightarrow a_1 = a_2$

$\therefore$ Hence f is one to one.

(ii) To prove f is onto :-

Since g is a function $g(b) \in A$ for $b \in B$ $(\because g: B \to A)$

now $b = I_B(b) = f \circ g(b) = f(g(b))$

$\therefore$ for every $b \in B$, there exist an element $g(b) \in A$ such that $f(g(b)) = b$.

$\therefore$ So f is onto.

case (ii) : If f is 1-1 and onto, then to prove f is invertible.

Let $f : A \to B$ is bijective.

Since f is onto, for each $b \in B$ there exist an element $a \in A$ such that $f(a) = b$.

Hence, we can define a function $g: B \to A$ by $g(b) = a$ where $f(a) = b \longrightarrow (1)$

If possible, let $g(b) = a_1$ & $g(b) = a_2$ where $a_1 \neq a_2$

This means that $f(a_1) = f(a_2) = b$ where $a_1 \neq a_2$

which is not possible ($\because$ since f is one to one)

Thus $g: B \to A$ is a unique function.

Hence from (1)

$$g \circ f = I_A, \; f \circ g = I_B$$

$\therefore$ So f is invertible.