Module – 2 Vector Calculus

Review of vectors in 2, 3 dimensions – Gradient, divergence, curl – Solenoidal, Irrotational fields – Vector identities (without proof) – Directional derivatives – Line integrals, Surface integrals, Volume integrals – Green's theorem (without proof) – Gauss divergence theorem (without proof), Verification, Applications to Cubes, parallelopiped only – Stoke's theorem (without proof) – Verification, Applications to Cubes, parallelopiped only – Applications of Line and Volume integrals in Engineering.

Basic Formulae

1. $\nabla = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$

2. $\nabla\varphi = grad\ \varphi = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}$

3. Directional derivative = $\nabla\varphi \bullet \frac{\vec{a}}{|\vec{a}|}$

4. Normal derivative = $|\nabla\varphi|$

5. Unit normal vector $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$

6. Angle between the surfaces $\cos\theta = \frac{\nabla\varphi_1 \bullet \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}$

7. Let $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

Differentiate partially w.r.t. *x*

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Differentiate partially w.r.t. *y* $\frac{\partial r}{\partial y} = \frac{y}{r}$

Differentiate partially w.r.t. z $\frac{\partial r}{\partial z} = \frac{z}{r}$

1. **Find $\nabla\phi$ if $\phi = \log\left(x^2+y^2+z^2\right)$.**

Solution:

$$\nabla\phi = \vec{i}\ \frac{\partial\phi}{\partial x} + \vec{j}\ \frac{\partial\phi}{\partial y} + \vec{k}\ \frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}\left(\log(x^2+y^2+z^2)\right) + \vec{j}\frac{\partial}{\partial y}\log(x^2+y^2+z^2) + \vec{k}\frac{\partial}{\partial z}\log(x^2+y^2+z^2)$$

$$= \vec{i}\frac{2x}{(x^2+y^2+z^2)} + \vec{j}\frac{2y}{(x^2+y^2+z^2)} + \vec{k}\frac{2z}{(x^2+y^2+z^2)}$$

$$= \frac{2}{x^2+y^2+z^2}\left(x\vec{i}+y\vec{j}+z\vec{k}\right) = \frac{2\vec{r}}{r^2} \qquad \because (\vec{r} = x\vec{i}+y\vec{j}+z\vec{k}\ \&\ r^2 = x^2+y^2+z^2)$$

2. **Find $\nabla\phi$ if $\phi = x\,y\,z$ at the point (1, 2, 3).**

Solution:

$$\nabla\varphi = grad\ \varphi = \vec{i}\ \frac{\partial\varphi}{\partial x} + \vec{j}\ \frac{\partial\varphi}{\partial y} + \vec{k}\ \frac{\partial\varphi}{\partial z}$$

$$\nabla\varphi = \vec{i}\ y\,z + \vec{j}\ \ x\,z + \vec{k}\ x\,y$$

$$\nabla\varphi \text{ at } (1,2,3) = 6\vec{i} + 3\vec{j} + 2\vec{k}$$

3. **Find ∇r .**

Solution:

$$\nabla r = \vec{i}\ \frac{\partial r}{\partial x} + \vec{j}\ \frac{\partial r}{\partial y} + \vec{k}\ \frac{\partial r}{\partial z}$$

$$\nabla r = \vec{i}\ \frac{x}{r} + \vec{j}\ \frac{y}{r} + \vec{k}\ \frac{z}{r} = \frac{\vec{r}}{r}$$

4. **Find the unit normal vector to the surface $x^2 + xy + z^2 = 4$ at the point (1, –1, 2).**

Solution:

Let $\varphi = x^2 + xy + z^2 - 4$

$$\nabla\varphi = grad\ \varphi = \vec{i}\ \frac{\partial\varphi}{\partial x} + \vec{j}\ \frac{\partial\varphi}{\partial y} + \vec{k}\ \frac{\partial\varphi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2x+y, \qquad \frac{\partial\phi}{\partial y} = x, \qquad \frac{\partial\phi}{\partial z} = 2z$$

$$[\nabla\phi]_{(1,-1,2)} = \left[(2x+y)\vec{i} + x\vec{j} + 2z\vec{k}\right]_{(1,-1,2)} = \vec{i} + \vec{j} + 4\vec{k}$$

The unit normal vector is

$$\therefore \quad \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\vec{i}+\vec{j}+4\vec{k}}{\sqrt{1^2+1^2+4^2}} = \frac{\vec{i}+\vec{j}+4\vec{k}}{\sqrt{18}}.$$

5. **Find the unit normal vector to the surface $x^2 + y^2 + z^2 = 1$ at the point (1, 1, 1).**

Ans $\hat{n} = \dfrac{i+j+k}{\sqrt{3}}$

6. **Find the directional derivative of $\phi = 3x^2 + 2y - 3z$ at (1, 1, 1) in the direction $2\vec{i} + 2\vec{j} - \vec{k}$.**

Solution: The gradient of ϕ is $\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z}$

$$\frac{\partial\phi}{\partial x} = 6x, \qquad \frac{\partial\phi}{\partial y} = 2, \qquad \frac{\partial\phi}{\partial z} = -3$$

$$\nabla\phi = 6x\,i + 2\,j - 3k$$

Directional derivative of ϕ is

$$\vec{a} = 2\vec{i} + 2\vec{j} - \vec{k}$$

$$|\vec{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3$$

$$\nabla\phi.\frac{\vec{a}}{|\vec{a}|} = \left[(6x\vec{i} + 2\vec{j} - 3\vec{k})\cdot\left(\frac{2\vec{i}+2\vec{j}-\vec{k}}{3}\right)\right]_{(1,1,1)} = \frac{19}{3}$$

7. **Find the directional derivative of $\phi = 2xy + z^2$ at (1, – 1 , 3) in the direction $i + 2j + 2k$.**

Ans $\dfrac{14}{3}$

8. **Find the directional derivative of $\phi = x^2 + y^2 + 4xyz$ at (1, – 2 , 2) in the direction $2i - 2j + k$.**

Ans $-\dfrac{44}{3}$

9. **Find the directional derivative of $\phi = x^2 - y^2 + 2z^2$ at P (1, 2 , 3) in the direction of line PQ where Q is (5, 0, 4).**

Solution:

$$\nabla\varphi = grad\,\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$\nabla\varphi = grad\ \varphi = \vec{i}\ 2x + \vec{j}\,(-2y) + \vec{k}\ 4z$

$\nabla\varphi$ at $(1,2,3) = 2\vec{i} - 4\vec{j} + 12\vec{k}$

$\vec{a} = OQ - OP = (5\vec{i} + 0\vec{j} + 4\vec{k}) - (\vec{i} + 2\vec{j} + 3\vec{k}) = 4\vec{i} - 2\vec{j} + \vec{k}$

Directional derivative = $\nabla\varphi \bullet \dfrac{\vec{a}}{|\vec{a}|}$

$$= (2\vec{i} - 4\vec{j} + 12\vec{k}) \bullet \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

10. **In what direction from (3, 1, –2) is the directional derivative of $\phi = x^2y^2z^4$ a maximum? Find the magnitude of this maximum.**

Solution: Given $\phi = x^2y^2z^4$

$\dfrac{\partial\phi}{\partial x} = 2xy^2z^4,\qquad \dfrac{\partial\phi}{\partial y} = 2x^2yz^4,\qquad \dfrac{\partial\phi}{\partial z} = 4x^2y^2z^3$

$\nabla\phi = (2xy^2z^4)\vec{i} + (2x^2yz^4)\vec{j} + (4x^2y^2z^3)\vec{k}$

$[\nabla\phi]_{(3,\,1,\,-2)} = 96\vec{i} + 288\vec{j} - 288\vec{k} = 96(\vec{i} + 3\vec{j} - 3\vec{k})$

$\therefore$ The maximum directional derivative occurs in the direction of $\nabla\phi = 96(\vec{i} + 3\vec{j} - 3\vec{k})$

The magnitude of this maximum directional derivative is

$|\nabla\phi| = 96\sqrt{1^2+3^2+(-3)^2} = 96\sqrt{1+9+9} = 96\sqrt{19}.$

11. **In what direction from (1, 1, –2) is the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ a maximum? Find the magnitude of this maximum.**

Ans Directional derivative is maximum in the direction of $2\vec{i} - 4\vec{j} - 16\vec{k}$

Maximum directional derivative = $\sqrt{276}$

12. **Find the angle between the surfaces $x \log z = y^2 - 1$ and $x^2y = 2 - z$ at the point (1, 1, 1).**

Solution: Let $\phi_1 = y^2 - x\log z - 1$

$\dfrac{\partial\phi}{\partial x} = -\log z\,,\qquad \dfrac{\partial\phi}{\partial y} = 2y,\qquad \dfrac{\partial\phi}{\partial z} = -\dfrac{x}{z}$

$\nabla\phi_1 = -\log z\ \vec{i} + 2y\vec{j} - \dfrac{x}{z}\vec{k}$, $(\nabla\phi_1)_{(1,1,1)} = 2\vec{j} - \vec{k}$ and $|\nabla\phi_1| = \sqrt{2^2+(-1)^2} = \sqrt{5}$

Let $\phi_2 = x^2y - 2 + z$

$$\frac{\partial\phi}{\partial x} = 2xy\ , \qquad \frac{\partial\phi}{\partial y} = x^2, \qquad \frac{\partial\phi}{\partial z} = 1$$

$\nabla\phi_2 = (2xy)\vec{i} + x^2\vec{j} + (1)\vec{k}$, $(\nabla\phi_2)_{(1,1,1)} = 2\vec{i} + \vec{j} + \vec{k}$ and $|\nabla\phi_2| = \sqrt{2^2+1^2+1^2} = \sqrt{6}$

$$cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{\left(2\vec{j}-\vec{k}\right).\left(2\vec{i}+\vec{j}+\vec{k}\right)}{\left(\sqrt{5}\right)\left(\sqrt{6}\right)} = \frac{0+2-1}{\sqrt{30}} \Rightarrow \theta = cos^{-1}\left(\frac{1}{\sqrt{30}}\right).$$

13. **Find the angle between the surfaces $z = x^2 + y^2 - 3$ and $x^2 + y^2 + z^2 = 9$ at the point (2, – 1 , 2).**

Ans $\cos\theta = \dfrac{8}{3\sqrt{21}}$

14. **Find the angle between the normals to the surface $x^2 = yz$ at the points (1, 1, 1) and (2, 4, 1).**

Solution:

Given $\varphi = x^2 - yz$

$$\nabla\varphi = 2x\vec{i} - z\vec{j} - y\vec{k}$$

$\nabla\varphi_1 /(1,1,1) = 2\vec{i} - \vec{j} - \vec{k}$ $\qquad$ $\nabla\varphi_2 /(2,4,1) = 4\vec{i} - \vec{j} - 4\vec{k}$

$|\nabla\varphi_1| = \sqrt{4+1+1} = \sqrt{6}$ $\qquad$ $|\nabla\varphi_2| == \sqrt{16+1+16} = \sqrt{33}$

$$\cos\theta = \frac{\nabla\varphi_1 \circ \nabla\varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|} = \frac{(2\vec{i}-\vec{j}-\vec{k})\circ(4\vec{i}-\vec{j}-4\vec{k})}{\sqrt{6}\sqrt{33}} = \frac{13}{\sqrt{6}\sqrt{33}}$$

15. **Find ‘a’ and ‘b’ so that the surfaces $\mathbf{ax^3 - by^2z = (a+3)x^2}$ and $\mathbf{4x^2y - z^3 = 11}$ cut orthogonally at (2, –1, –3).**

Solution:

$$Let\ \phi_1 = ax^3 - by^2z - (a+3)x^2$$

$$\frac{\partial\phi}{\partial x} = 3ax^2 - (a+3)2x, \qquad \frac{\partial\phi}{\partial y} = -2byz, \qquad \frac{\partial\phi}{\partial z} = -by^2$$

$$\therefore \nabla\phi_1 = [3ax^2 - (a+3)2x]\vec{i} - 2byz\vec{j} - by^2\vec{k}$$

$At\ (2,-1,-3)\ \nabla\phi_1 = (8a-12)\vec{i} - 6b\vec{j} - b\vec{k}$

$Let\ \phi_2 = 4x^2y - z^3 - 11$

$\frac{\partial\phi}{\partial x} = 8xy, \quad \frac{\partial\phi}{\partial y} = -4x^2, \quad \frac{\partial\phi}{\partial z} = -3z^2$

$\therefore \nabla\phi_2 = 8xy\vec{i} - 4x^2\vec{j} - 3z^2\vec{k}$

$At\ (2,-1,-3)\ \nabla\phi_2 = 16\vec{i} - 16\vec{j} - 27\vec{k}$

Since the surfaces cut orthogonally at $(2,-1,-3)$,

$\nabla\phi_1.\nabla\phi_2 = 0$

$\Rightarrow -16(8a-12) - 16(6b) + 27b = 0$

$\Rightarrow -128a + 192 - 69b = 0$

$\Rightarrow 128a + 69b = 192 \qquad \rightarrow (1)$

Since the point s $(2,-1,-3)$ *lies on the surface* $\phi_1(x,y,z) = 0$, *we have*

$8a + 3b - 4a = 12$

$\Rightarrow 4a + 3b = 12 \qquad \rightarrow (2)$

Solving (1) & (2) *we get* $a = -2.333 \quad b = 7.111$

16. **Find a and b such that the surfaces ax^2 – byz = (a+2)x and 4x^{2}y+z^3=4 cut orthogonally at (1, –1, 2).**

Solution:

Let $\phi_1 = ax^2 - byz - (a+2)x$

$\nabla\varphi_1 = \vec{i}[2ax - (a-2)] + \vec{j}(-bz) + \vec{k}(-by)$

$\nabla\varphi_1$ **at (1, –1, 2)** $= \vec{i}[a-2] - 4b\vec{j} + b\vec{k}$

$|\nabla\varphi_1| = \sqrt{(a-2)^2 + 17b^2}$

$\phi_2 = 4x^2y + 2^3 - 4$

$\nabla\phi_2 = 8xy\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}$

$\nabla\phi_2{}_{(1,-1,2)} = -8\vec{i} + 4\vec{j} + 12\vec{k}$

$|\nabla\varphi_2| = \sqrt{64+16+144} = \sqrt{224}$

$$\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$= \frac{-8(a-2)-16b+12b}{\sqrt{(a-2)^2+17b^2}\sqrt{224}}$$

Given $\theta = 90^0$, $\cos 90^0 = 0$

$$\therefore 0 = \frac{-8a+16-16b+12b}{\sqrt{(a-2)^2+17b^2}\sqrt{224}}$$

$$= -8a+16-16b+12b = 0$$

$$= 2a+b-4 = 0 \qquad \ldots (1)$$

Since the point (1,-1,2) lies on the surface $\phi_1(x,y,z) = 0$,

a – 2b – (a+2) = 0

b = -1

$\therefore$ (1) $\Rightarrow$ 2a + (-1)-4 = 0 a = $\frac{5}{2}$

17. **Find** $\nabla(r^n)$

Solution:

$$\nabla\left(r^n\right) = \vec{i}\ \frac{\partial\left(r^n\right)}{\partial x} + \vec{j}\ \frac{\partial\left(r^n\right)}{\partial y} + \vec{k}\ \frac{\partial\left(r^n\right)}{\partial z}$$

$$= \vec{i}\ \ nr^{n-1}\frac{x}{r} + \vec{j}\ \ nr^{n-1}\frac{y}{r} + \vec{k}\ \ nr^{n-1}\frac{z}{r}$$

$$= \vec{i}\, nr^{n-2}x + \vec{j}\, nr^{n-2}y + \vec{k}\, nr^{n-2}z$$

$$= nr^{n-2}(x\vec{i} + y\vec{j} + z\vec{k})\ (\because \vec{r} = x\vec{i} + y\vec{j} + z\vec{k})$$

$$\therefore \nabla\left(r^n\right) = nr^{n-2}\vec{r}.$$

DIVERGENCE, CURL, SOLENOIDAL, IRROTATIONAL

Let $\vec{F} = F_1\,\vec{i} + F_2\,\vec{j} + F_3\,\vec{k}$

1. $div\,\vec{F} = \nabla \bullet \vec{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$

2. $Curl\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

3. Solenoidal $\nabla \bullet \vec{F} = 0$

4. Irrotation al $\nabla \times \vec{F} = \vec{0}$

18. Find $curl\vec{F}$ if $\vec{F} = xy\vec{\imath} + yz\vec{\jmath} + zx\vec{k}$.

Solution:

Given $\vec{F} = xy\vec{\imath} + yz\vec{\jmath} + zx\vec{k}$

$$curl\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \vec{\imath}\,(0 - y) - \vec{\jmath}(z - 0) + \vec{k}\,(0 - x)$$

$$= -y\,\vec{\imath} - z\,\vec{\jmath} - x\,\vec{k}$$

19. **Find ‘a’, such that $\vec{F} = (3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal.**

Solution: We know that $\vec{F}$ is Solenoidal if $div\,\vec{F} = 0$ *or* $\nabla.\vec{F} = 0$

$$\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right).[(3x-2y+z)\,\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}\,] = 0$$

$$\frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + ay - z) + \frac{\partial}{\partial z}(x - y + 2z) = 0$$

$\Rightarrow$ 3 + a + 2 =0

$\Rightarrow$ 5 + a = 0 $\therefore$ a = –5.

20. **Find the constant a, b, c so that** $\vec{F}=(x+2y+az)\vec{i}+(bx-3y-z)\vec{j}+(4x+cy+2z)\vec{k}$ **is irrotational.**

Solution:

Given $\vec{F}$ is irrotational **i.e.,** $\nabla\times\vec{F}=\vec{0}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}=\vec{0}$$

$$\vec{i}\left(\frac{\partial}{\partial y}(4x+cy+2z)-\frac{\partial}{\partial z}(bx-3y-z)\right)-\vec{j}\left(\frac{\partial}{\partial x}(4x+cy+2z)-\frac{\partial}{\partial z}(x+2y+az)\right)$$
$$+\vec{k}\left(\frac{\partial}{\partial x}(bx-3y-z)-\frac{\partial}{\partial y}(x+2y+az)\right)=\vec{0}$$

$$=i.e.,\quad \vec{i}(c+1)-\vec{j}(4-a)+\vec{k}(b-2)=0$$
$$=\therefore c+1=0, 4-a=0, and\ b-2=0$$
$$\Rightarrow a=4, b=2, c=-1$$

21. **Find the constant a, b, c so that** $\vec{F}=(axy+bz^3)\vec{i}+(3x^2-cz)\vec{j}+(3xz^2-y)\vec{k}$ **is irrotational.**

Ans ***a*** **= 6,** ***b*** **= 1,** ***c*** **= 1**

22. **Prove that** $r^n\,\vec{r}$ **is an irrotational vector for any value of 'n' but is solenoidal only if n = – 3 .**

Solution:

$$\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$$
$$r=|r|=\sqrt{x^2+y^2+z^2}$$
$$\frac{\partial r}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2+z^2}}=\frac{x}{\sqrt{x^2+y^2+z^2}}=\frac{x}{r}$$

Similarly

$$\frac{\partial r}{\partial y}=\frac{2y}{2\sqrt{x^2+y^2+z^2}}=\frac{y}{\sqrt{x^2+y^2+z^2}}=\frac{y}{r}$$
$$\frac{\partial r}{\partial z}=\frac{2z}{2\sqrt{x^2+y^2+z^2}}=\frac{z}{\sqrt{x^2+y^2+z^2}}=\frac{z}{r}$$

$$r^n=\left(x^2+y^2+z^2\right)^{n/2}$$
$$r^n r=r^n\left(x\vec{i}+y\vec{j}+z\vec{k}\right)$$

$$\nabla\times\left(r^n r\right)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ r^n x & r^n y & r^n z\end{vmatrix}$$

$$=\vec{i}\left(\frac{\partial}{\partial y}(r^n z)-\frac{\partial}{\partial z}\left(r^n y\right)\right)-\vec{j}\left(\frac{\partial}{\partial x}\left(r^n z\right)-\frac{\partial}{\partial z}\left(r^n x\right)\right)+\vec{k}\left(\frac{\partial}{\partial x}\left(r^n y\right)-\frac{\partial}{\partial y}\left(r^n x\right)\right)$$

$$=\vec{i}\left(znr^{n-1}\frac{\partial r}{\partial y}-ynr^{n-1}\frac{\partial r}{\partial z}\right)-\vec{j}\left(znr^{n-1}\frac{\partial r}{\partial x}-xnr^{n-1}\frac{\partial r}{\partial z}\right)+\vec{k}\left(ynr^{n-1}\frac{\partial r}{\partial x}-xnr^{n-1}\frac{\partial r}{\partial y}\right)$$

$$=\vec{i}\left(znr^{n-1}\frac{y}{r}-ynr^{n-1}\frac{z}{r}\right)-\vec{j}\left(znr^{n-1}\frac{x}{r}-xnr^{n-1}\frac{z}{r}\right)+\vec{k}\left(ynr^{n-1}\frac{x}{r}-xnr^{n-1}\frac{y}{r}\right)$$

$$=0\vec{i}+0\vec{j}+0\vec{k}=0$$

$\therefore$ $\boldsymbol{r^n r}$ is irrotational for all values of n.

$$div\left(r^n\,\vec{r}\right)=\nabla\bullet\left(r^n\,\vec{r}\right)=\left(\vec{i}\,\frac{\partial}{\partial x}+\vec{j}\,\frac{\partial}{\partial y}+\vec{k}\,\frac{\partial}{\partial z}\right)\left(r^n\,(x\vec{i}+y\vec{j}+z\vec{k})\right)$$

$$=\frac{\partial}{\partial x}\left(r^n\,x\right)+\frac{\partial}{\partial y}\left(r^n\,y\right)+\frac{\partial}{\partial z}\left(r^n\,z\right)$$

$$=r^n+xnr^{n-1}\frac{\partial r}{\partial x}+r^n+ynr^{n-1}\frac{\partial r}{\partial y}+r^n+znr^{n-1}\frac{\partial r}{\partial z}$$

$$=3r^n+nr^{n-2}(x^2+y^2+z^2)=3r^n+nr^{n-2}(r^2)=3r^n+nr^n=(3+n)r^n$$

If n = -3 then $\nabla\bullet\left(r^n\,\vec{r}\right)=0$.

$\therefore$ $\boldsymbol{r^n r}$ is solenoidal only if n = – 3.

23. **If $\vec{F}=x^3\vec{i}+y^3\vec{j}+z^3\vec{k}$, then find div curl $\vec{F}$.**

Solution: $div\ curl\,\overrightarrow{F}=\nabla.\,(\nabla\times\overrightarrow{F})$

$$\nabla\times\overrightarrow{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^3 & y^3 & z^3\end{vmatrix}$$

$$=\vec{i}(0-0)-\vec{j}(0-0)+\vec{k}(0-0)=\vec{0}$$

$$\nabla\times\vec{F}=\vec{0}$$

$$\therefore \nabla.(\nabla\times\vec{F})=0$$

24. **If $\nabla\phi=(y^2-2xyz^3)\vec{i}+(3+2xy-x^2z^3)\vec{j}+(6z^3-3x^2yz^2)\vec{k}$ find ϕ.**

Solution:

$$\nabla\varphi=\vec{i}\,\frac{\partial\varphi}{\partial x}+\vec{j}\,\frac{\partial\varphi}{\partial y}+\vec{k}\,\frac{\partial\varphi}{\partial z} \qquad \textbf{(1)}$$

$$\nabla\varphi=\left(y^2-2\,x\,y\,z^3\right)\vec{i}+\left(3+2\,x\,y-x^2\,z^3\right)\vec{j}+\left(6\,z^3-3\,x^2\,y\,z^2\right)\vec{k} \qquad \textbf{(2)}$$

Comparing (1) and (2)

$$\frac{\partial\varphi}{\partial x}=y^2-2\,x\,y\,z^3 \qquad \textbf{(3)}$$

$$\frac{\partial\varphi}{\partial y}=3+2\,x\,y-x^2\,z^3 \qquad \textbf{(4)}$$

$$\frac{\partial\varphi}{\partial z}=6\,z^3-3\,x^2\,y\,z^2 \qquad \textbf{(5)}$$

Integrating (3) w.r.t. *x* (keeping *y* and *z* as constant)

$$\varphi=y^2\,x-x^2\,y\,z^3+f_1\,(y,z)$$

Integrating (4) w.r.t. *y* (keeping *x* and *z* as constant)

$$\varphi=3\,y+x\,y^2-x^2\;y\,z^3+f_2\,(x,z)$$

Integrating (5) w.r.t. *z* (keeping *x* and *y* as constant)

$$\varphi=\frac{3}{2}z^4-x^2\,y\,z^3+f_3\,(x,y)$$

Hence $\varphi=y^2\,x-x^2\,y\,z^3+3\,y+\frac{3}{2}z^4+c$ where c is a constant, $c=\;f_1\,(y,z)+\;f_2\,(x,z)+\;f_3\,(x,y)$

25. **If $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$ find $\phi(x,y,z)$ given that $\phi(1,-2,2) = 4$.**

Solution:

$$\nabla\phi = \vec{i}\ \frac{\partial\phi}{\partial x} + \vec{j}\ \frac{\partial\phi}{\partial y} + \vec{k}\ \frac{\partial\phi}{\partial z} \quad \rightarrow (1)$$

Given $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k} \quad \rightarrow (2)$

$\therefore$ *comparing* (1) & (2)

$$\frac{\partial\phi}{\partial x} = 2xyz^3 \quad \rightarrow (3)$$

$$\frac{\partial\phi}{\partial y} = x^2z^3 \quad \rightarrow (4)$$

$$\frac{\partial\varphi}{\partial z} = 3x^2\ yz^2 \quad \rightarrow \mathbf{(5)}$$

Integrating (3) w.r.t. *x* (keeping *y* and *z* as constant)

$\varphi = x^2\ yz^3 + f_1(y,z)$

Integrating (4) w.r.t. *y* (keeping *x* and *z* as constant)

$\varphi = x^2\ yz^3 + f_2(x,z)$

Integrating (5) w.r.t. *z* (keeping *x* and *y* as constant)

$\varphi = x^2\ yz^3 + f_3(x,y)$

Hence $\varphi = x^2\ yz^3 + c$ where c is a constant, $c = f_1(y,z) + f_2(x,z) + f_3(x,y)$

Given $\varphi(1,-2,2) = 4$

$\varphi(1,-2,2) = -16 + c = 4$

$c = 20$

Hence $\varphi = x^2\ yz^3 + 20$

26. **Show that the vector $\overline{F}=(y^2\cos x+z^3)\vec{i}+(2y\sin x-4)\vec{j}+3xz^2\vec{k}$ is irrotational and find the scalar potential function.**

Solution:

$$curl\bar{F}=\nabla\times\vec{F}=\vec{0}$$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y^2\cos x+z^3 & 2y\sin x-4 & 3xz^2\end{vmatrix}$$

$$=\vec{i}\left(\frac{\partial}{\partial y}\left(3xz^2\right)-\frac{\partial}{\partial z}\left(2y\sin x-4\right)\right)-\vec{j}\left(\frac{\partial}{\partial x}\left(3xz^2\right)-\frac{\partial}{\partial z}\left(y^2\cos x+z^3\right)\right)$$

$$+\vec{k}\left(\frac{\partial}{\partial y}\left(y^2\cos x+z^3\right)-\frac{\partial}{\partial x}\left(2y\sin x-4\right)\right)$$

$$=\vec{i}\left(0-0\right)-\vec{j}\left(3z^2-3z^2\right)+\vec{k}\left(2y\cos x-2y\cos x\right)=\vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

To find Scalar potential ϕ we assume $\vec{F}=\nabla\phi$

$$\vec{F}=\nabla\phi=(y^2\cos x+z^3)\vec{i}+(2y\sin x-4)\vec{j}+3xz^2\vec{k}$$

$$\left(\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}\right)=(y^2\cos x+z^3)\vec{i}+(2y\sin x-4)\vec{j}+3xz^2\vec{k}$$

comparing coefficient of $\vec{i},\vec{j}\ \&\ \vec{k}$

$$\frac{\partial\phi}{\partial x}=y^2\cos x+z^3 \qquad \rightarrow(1)$$

$$\frac{\partial\phi}{\partial y}=2y\sin x-4 \qquad \rightarrow(2)$$

$$\frac{\partial\phi}{\partial z}=3xz^2 \qquad \rightarrow(3)$$

Integrating (1) w.r.t. *x* (keeping *y* and *z* as constant)

$$\varphi=y^2(\sin x)+xz^3+f_1(y,z)$$

Integrating (2) w.r.t. *y* (keeping *x* and *z* as constant)

$$\varphi=y^2\sin x-4y+f_2(x,z)$$

Integrating (3) w.r.t. *z* (keeping *x* and *y* as constant)

$$\varphi=xz^3+f_3(x,y)$$

Hence $\varphi = y^2 \sin x + x z^3 - 4y + c$ where c is a constant, $c = f_1(y,z) + f_2(x,z) + f_3(x,y)$

27. **Show that the vector $\vec{F} = \left(6xy + z^3\right)\vec{i} + \left(3x^2 - z\right)\vec{j} + \left(3xz^2 - y\right)\vec{k}$ is irrotational and find the scalar potential function.**

Solution:

Given $\vec{F} = \left(6xy + z^3\right)\vec{i} + \left(3x^2 - z\right)\vec{j} + \left(3xz^2 - y\right)\vec{k}$

$curl\vec{F} = \nabla \times \vec{F} = \vec{0}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix} = \vec{i}(-1+1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x - 6x) = \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

To find scalar potential ϕ we assume $\vec{F} = \nabla\phi$

$$\vec{F} = \nabla\phi = \left(6xy + z^3\right)\vec{i} + \left(3x^2 - z\right)\vec{j} + \left(3xz^2 - y\right)\vec{k}$$

$$\left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\ \frac{\partial\phi}{\partial y} + \vec{k}\ \frac{\partial\phi}{\partial z}\right) = \left(6xy + z^3\right)\vec{i} + \left(3x^2 - z\right)\vec{j} + \left(3xz^2 - y\right)\vec{k}$$

comparing coefficient of $\vec{i}, \vec{j}$ & $\vec{k}$

$$\frac{\partial\phi}{\partial x} = \left(6xy + z^3\right) \qquad \rightarrow (1)$$

$$\frac{\partial\phi}{\partial y} = \left(3x^2 - z\right) \qquad \rightarrow (2)$$

$$\frac{\partial\phi}{\partial z} = \left(3xz^2 - y\right) \qquad \rightarrow (3)$$

Integrating (1) w.r.t. *x* (keeping *y* and *z* as constant)

$\varphi = 3x^2\, y + x z^3 + f_1(y,z)$

Integrating (2) w.r.t. *y* (keeping *x* and *z* as constant)

$\varphi = 3x^2\, y - y z + f_2(x,z)$

Integrating (3) w.r.t. *z* (keeping *x* and *y* as constant)

$\varphi = x z^3 - y z + f_3(x,y)$

Hence $\varphi = 3x^2\, y + x z^3 - y z + c$ where c is a constant, $c = f_1(y,z) + f_2(x,z) + f_3(x,y)$

28. If $\vec{A}$ and $\vec{B}$ are irrotational, then prove that $\vec{A}\times\vec{B}$ is solenoidal.

Solution:

$\vec{A}$ and $\vec{B}$ are irrotational.

$\therefore \nabla\times\vec{A}=\vec{0}$ and $\nabla\times\vec{B}=\vec{0}$

Now $\nabla\bullet\left(\vec{A}\times\vec{B}\right)=\left(\nabla\times\vec{A}\right)\bullet\vec{B}-\left(\nabla\times\vec{B}\right)\bullet\vec{A}=0-0=0$

$\therefore \vec{A}\times\vec{B}$ is solenoidal.

29. Prove that (i) $div\,\vec{r}=3$ (ii) $curl\,\vec{r}=0$.

Solution:

$\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$

$$div\,\vec{r}=\nabla\bullet\vec{r}=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\bullet\left(x\vec{i}+y\vec{j}+z\vec{k}\right)$$

$$=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)=1+1+1=3$$

$$Curl\,\vec{r}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x & y & z\end{vmatrix}=0\vec{i}+0\vec{j}+0\vec{k}$$

30. If $\phi=x^2-y^2$, then prove that $\nabla^2\varphi=0$.

Solution:

$\nabla^2\varphi=\nabla\bullet\nabla\varphi$

$$=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\bullet\left(\vec{i}\frac{\partial\varphi}{\partial x}+\vec{j}\frac{\partial\varphi}{\partial y}+\vec{k}\frac{\partial\varphi}{\partial z}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\right)$$

$$=\frac{\partial}{\partial x}\left(2x\right)+\frac{\partial}{\partial y}\left(-2y\right)+\frac{\partial}{\partial z}\left(0\right)=2-2=0$$

31. Prove that $curl\,(grad\;\varphi) = 0$.

Solution:

$$Curl\,(grad\;\varphi) = \nabla\times\nabla\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial \varphi}{\partial x} & \dfrac{\partial \varphi}{\partial y} & \dfrac{\partial \varphi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2\varphi}{\partial y\,\partial z} - \frac{\partial^2\varphi}{\partial z\,\partial y}\right) - \vec{j}\left(\frac{\partial^2\varphi}{\partial x\,\partial z} - \frac{\partial^2\varphi}{\partial z\,\partial x}\right) + \vec{k}\left(\frac{\partial^2\varphi}{\partial x\,\partial y} - \frac{\partial^2\varphi}{\partial y\,\partial x}\right)$$

$= 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k}$ (Since mixed partial derivatives are equal.)

32. State Green's Theorem.

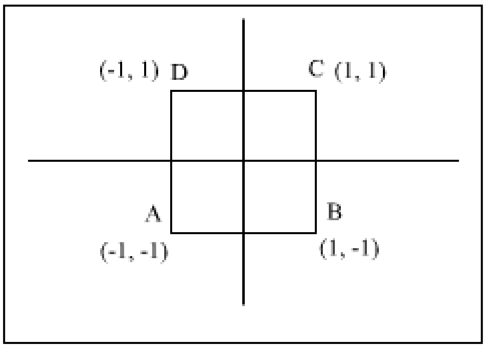
Statement: If $P(x, y)$ and $Q(x, y)$ are continuous functions of x, y with continuous partial derivatives $\dfrac{\partial P}{\partial y}$ and $\dfrac{\partial Q}{\partial x}$ in a region R of the xy plane bounded by a simple closed curve C, then $\displaystyle\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$ where C is the curve traversed in the counter clockwise direction.

33. Verify Green's theorem for $\displaystyle\int_c \left[x^2(1+y)dx + (x^3+y^3)dy\right]$ where C is the boundary of the region defined by the lines x = ± 1 and y = ± 1.

Solution:

By Green's theorem

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



Given $\displaystyle\int_c x^2(1+y)dx + (y^3+x^3)dy$

$P = x^2(1+y)$ $\qquad Q = y^3 + x^3$

$\dfrac{\partial P}{\partial y} = x^2$ $\qquad \dfrac{\partial Q}{\partial x} = 3x^2$

Consider

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

$$= \int_{-1}^{1} \int_{-1}^{1} (3x^2 - x^2) dydx$$

$$= \int_{-1}^{1} \int_{-1}^{1} (2x^2) dydx$$

$$= \int_{-1}^{1} 2 \left[\frac{x^3}{3} \right]_{-1}^{1} dy$$

$$= \int_{-1}^{1} \frac{2}{3} \left[1^3 - (-1)^3 \right] dy = \int_{-1}^{1} \frac{2}{3} 2 \, dy = \int_{-1}^{1} \left[\frac{4}{3} \right] dy$$

$$= \left[\frac{4}{3} \right] [y]_{-1}^{1} = \left[\frac{4}{3} \right] [1 - (-1)] = \frac{8}{3} \qquad \rightarrow (1)$$

Consider

$$\int_c Pdx + Qdy = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

Along AB, $y = -1$, $dy = 0$ *and* x *varies from* -1 *to* 1

$$\therefore \int_{AB} Pdx + Qdy = \int_{-1}^{1} x^2 (1 + y) dx = \int_{-1}^{1} x^2 (1 - 1) dx = 0$$

Along BC, $x = 1$, $dx = 0$ *and* y *varies from* -1 *to* 1

$$\therefore \int_{BC} Pdx + Qdy = \int_{-1}^{1} (x^3 + y^3) dy = \int_{-1}^{1} (1 + y^3) dy$$

$$= \left[y + \frac{y^4}{4} \right]_{-1}^{1} = \left[1 + \frac{1}{4} \right] - \left[-1 + \frac{1}{4} \right]$$

$$= 1 + \frac{1}{4} + 1 - \frac{1}{4} = 2$$

Along CD, $y = 1$, $dy = 0$ *and* x *varies from* 1 *to* -1

$$\therefore \int_{CD} Pdx + Qdy = \int_{-1}^{1} x^2 (1 + y) dx = \int_{-1}^{1} 2x^2 dx = \left[\frac{2x^3}{3} \right]_{1}^{-1} = \frac{2}{3} \left[(-1)^3 - (1)^3 \right] = \frac{2}{3} [-1 - 1] = -\frac{4}{3}$$

Along DA, $x = -1$, $dx = 0$ *and* y *varies from* 1 *to* -1

$$\therefore \int_{DA} Pdx + Qdy = \int_{1}^{-1} (x^3 + y^3) dy = \int_{1}^{-1} (-1 + y^3) dy$$

$$= \left[\frac{y^4}{4} - y \right]_{1}^{-1} = \frac{1}{4} + 1 - \frac{1}{4} + 1 = 2$$

$$\int_C Pdx + Qdy = 0 + 2 - \frac{4}{3} + 2 = 4 - \frac{4}{3} = \frac{8}{3} \qquad \to (2)$$

$\therefore (1) = (2)$

Hence the theorem is verified.

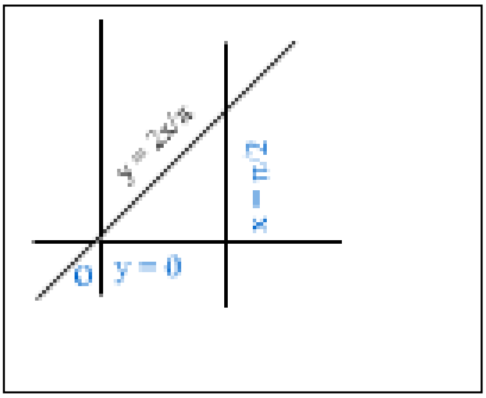
34. **Using Green's theorem , evaluate $\int_C (y - \sin x)dx + \cos x\, dy$ where C is the triangle bounded by the lines y = 0, $x = \frac{\pi}{2}$ and y $= \left(\frac{2}{\pi}\right)$ x**

Solution:

Green's theorem states that

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Given $\int_C (y - \sin x)dx + \cos x\, dy$



$P = y - \sin x \qquad\qquad Q = \cos x$

$\frac{\partial P}{\partial y} = 1 \qquad\qquad \frac{\partial Q}{\partial x} = -\sin x$

$$\int_C (y - \sin x)dx + \cos x\, dy = \iint_R (-\sin x - 1)\, dx\, dy$$

$$\iint_R (-\sin x - 1)\, dx\, dy = \int_0^1 \int_{\frac{\pi y}{2}}^{\frac{\pi}{2}} (-\sin x - 1)\, dx\, dy$$

$$= \int_0^1 [\cos x - x]_{\frac{\pi y}{2}}^{\frac{\pi}{2}} dy$$

$$= \int_0^1 \left[\left(\cos\frac{\pi}{2} - \frac{\pi}{2}\right) - \left(\cos\frac{\pi y}{2} - \frac{\pi y}{2}\right)\right] dy$$

$$= \left[-\frac{\pi y}{2} - \frac{\sin\frac{\pi y}{2}}{\frac{\pi}{2}} + \frac{\pi y^2}{4}\right]_0^1 = \left[-\frac{\pi}{2} - \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} + \frac{\pi}{4}\right] - [0] = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

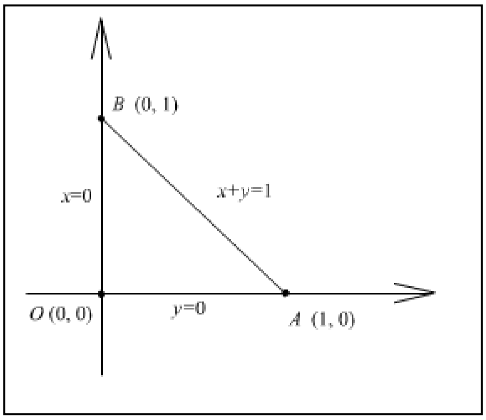
35. **Verify Green's theorem for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region defined by the lines x = 0, y = 0 and x + y =1.**

Solution:

Green's theorem states that

Given $\int_C \left(3x^2 - 8y^2\right)dx + \left(4y - 6xy\right)dy$

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$



$$P = 3x^2 - 8y^2$$

$$\frac{\partial P}{\partial y} = -16y$$

$$Q = 4y - 6xy$$

$$\frac{\partial Q}{\partial x} = -6y$$

Evaluation of RHS:

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \iint_R \left(-6y + 16y\right)dxdy$$

$$= \int_0^1 \int_0^{1-y} 10y\, dx\, dy = \int_0^1 10y\left[x\right]_0^{1-y} dy$$

$$= \int_0^1 10y(1-y)dy$$

$$= 10\int_0^1 \left(y - y^2\right)dy$$

$$= 10\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= 10\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{10}{6}$$

$$= \frac{5}{3}$$

Evaluation of LHS:

$$\int_C (Pdx+Qdy) = \int_{OA} (Pdx+Qdy) + \int_{AB} (Pdx+Qdy) + \int_{BO} (Pdx+Qdy)$$

Along OA: $y=0 \Rightarrow dy=0$

$$\int_{OA} Pdx+Qdy = \int_{OA} (3x^2)\,dx$$

$$=\left[\frac{3x^3}{3}\right]_0^1 = 1-0=1$$

Along AB:

$$x+y=1 \Rightarrow y=1-x$$
$$\Rightarrow dy=-dx$$

$$\int_{AB} Pdx+Qdy = \int_{AB} (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

$$= \int_{AB} [3x^2-8(1-x)^2]dx + [4(1-x)-6x(1-x)](-dx)$$

$$= \int_1^0 (-11x^2+26x-12)\,dx$$

$$=\left[\frac{-11x^3}{3}+\frac{26x^2}{2}-12x\right]_1^0 = (0)-(\frac{-11}{3}+\frac{26}{2}-12) = \frac{11}{3}-1=\frac{8}{3}$$

Along BO: $x=0 \Rightarrow dx=0$

$$\int_{BO} Pdx+Qdy = \int_{BO} 4y\,dy$$

$$=\left[\frac{4y^2}{2}\right]_1^0 = 2[0-(1)]$$

$$=-2$$

$$\therefore \int_C Pdx+Qdy = 1+\frac{8}{3}-2=\frac{5}{3}$$

Hence Green's theorem is verified.

36. Prove that the area bounded by a simple closed curve C is given by $\frac{1}{2}\int_C (xdy - ydx)$. **Hence find area of the ellipse x = a cos θ, y = b sin θ.**

Solution: W.K.T. Green's theorem is

$$\int_C (udx + vdy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy \qquad \ldots 1$$

Here $v = \frac{x}{2}$ $\qquad u = -\frac{y}{2}$

$\frac{\partial v}{\partial x} = \frac{1}{2}$ $\qquad \frac{\partial u}{\partial y} = -\frac{1}{2}$

$$(1) \Rightarrow \int_C \left(\frac{x}{2} dy - \frac{y}{2} dx \right) = \iint_R \left(\frac{1}{2} + \frac{1}{2} \right) dxdy$$

$$\frac{1}{2}\int_C (xdy - ydx) = \iint_R dxdy$$

$$\frac{1}{2}\int_C xdy - ydx = \text{Area of the ellipse} \qquad \ldots 2$$

Given x = a cos θ, $\qquad$ y = b sin θ

dx = -a sin θ dθ, $\qquad$ dy = b cos θ dθ

θ varies from 0 to 2π.

$$(2) \Rightarrow \text{Area of the ellipse} = \frac{1}{2}\int_C xdy - ydx$$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos\theta)(-b\cos\theta d\theta) - (b\sin\theta)(-a\sin\theta d\theta)$$

$$= \frac{1}{2}\int_0^{2\pi} [ab\cos\theta\cos\theta + ab\sin\theta\sin\theta] d\theta$$

$$= \frac{ab}{2}\int_0^{2\pi}(\cos^2\theta+\sin^2\theta)d\theta = \frac{ab}{2}\int_0^{2\pi}d\theta = \frac{ab}{2}[\theta]_{\theta=0}^{\theta=2\pi}$$

Area of the ellipse $= \frac{ab}{2}[2\pi] = \pi ab$

37. State Stoke's theorem (Relation between Line and Surface Integrals).

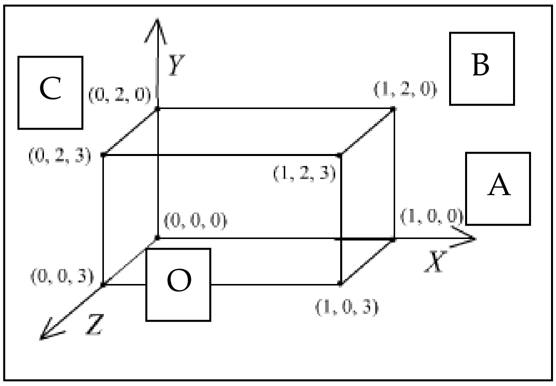
Statement: If S is an open surface bounded by a simple closed curve C and if a vector function $\vec{F}$ is continuous and has continuous first order partial derivatives in S and on C, then

$\iint_S curl\vec{F}\cdot\hat{n}\,ds = \int_C \vec{F}\cdot d\vec{r}$ where $\hat{n}$ is the outward unit normal vector at any point of S.

38. Verify Stoke's theorem for the vector $\vec{F} = xy\vec{i} - 2yz\,\vec{j} - xz\vec{k}$, where S is the open surface of the rectangular parallelopiped formed by the planes x = 0, y = 0, x = 1, y = 2 and z = 3 above the XOY plane.

Solution:

By Stoke's theorem



$$\oint_C \vec{F}\bullet d\vec{r} = \iint_S \nabla\times\vec{F}\bullet\hat{n}\,dS$$

$$\oint_C \vec{F}\bullet d\vec{r} = x\,y\,dx - 2\,y\,z\,dy - x\,z\,dz$$

Evaluation of $L.H.S$:

$$\int_C \vec{F}.\vec{dr} = \int_{OA}\vec{F}.\vec{dr} + \int_{AB}\vec{F}.\vec{dr} + \int_{BD}\vec{F}.\vec{dr} + \int_{DO}\vec{F}.\vec{dr}$$

Along OA : $y=0,\ z=0,\ dy=0,\ dz=0$

$$\int_{OA}\vec{F}.\vec{dr} = 0$$

Along AB : $x=1,\ z=0,\ dx=0,\ dz=0$

$$\int_{AB}\vec{F}.\vec{dr} = \int_{AB} 0 = 0$$

Along BC : $y=2,\ z=0,\ dy=0, dz=0$

$$\int_{BC}\vec{F}.\vec{dr} = \int_{BC}(2x)dx = \int_1^0 2x\ dx = \left[\frac{2x^2}{2}\right]_1^0 = 0-1 = -1$$

Along CO: $x=0,\ z=0, dx=0, dz=0$

$$\int_{CO} \vec{F}.\vec{dr} = \int_{CO} 0 = 0$$

$$\therefore \int_C \vec{F}.\vec{dr} = 0+0-1+0=-1$$

Evaluation of RHS:

$$\iint_S \nabla\times\vec{F}.n\ ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5}$$

Given, $\vec{F} = xy\vec{i} - 2yz\vec{j} - xz\vec{k}$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2yz & -xz \end{vmatrix} = \vec{i}\left[0-(-2y)\right] - \vec{j}\left[-z-0\right] + \vec{k}\left[0-x\right]$$

$$= 2y\vec{i} + (z)\vec{j} - x\vec{k}$$

Over S_1: x = 0 , $n = -\vec{i}$

$$\iint_{S_1} \left(\nabla\times\vec{F}\right).n\ ds = \int_0^3\int_0^2 \left[2y\vec{i}\right].(-\vec{i})dydz$$

$$= \int_0^3\int_0^2 -2ydy\,dz$$

$$= \int_0^3\int_0^2 -2y\,dy\,dz = \int_0^3 \left[\frac{-2y^2}{2}\right]_0^2 dz$$

$$= -4(z)_0^3 = -12$$

Over S_2: x =1 , $n = \vec{i}$

$$\iint_{S_2} \left(\nabla\times\vec{F}\right).n\ ds = \int_0^3\int_0^2 \left[2y\vec{i}\right].\left(\vec{i}\right) dydz$$

$$= \int_0^3\int_0^2 2y\,dy\,dz = \int_0^3 \left[\frac{2y^2}{2}\right]_0^2 dz = 12$$

Over S_3: y =0 , $n = -\vec{j}$

$$\iint_{S_3}\left(\nabla\times\vec{F}\right).n\ ds = \int_0^3\int_0^1\left[z\vec{j}\right].\left(-\vec{j}\ \right)dxdz = -\int_0^3\int_0^1 (z)\ dx\,dz$$

$$= -\int_0^3 (xz)_0^1 = -\int_0^3 (z)\,dz = -\left(\frac{z^2}{2}\right)_0^3 = -\frac{9}{2}$$

Over S_4: y = 1 , $n = \vec{j}$

$$\iint_{S_4}\left(\nabla\times\vec{F}\right).n\ ds = \int_0^3\int_0^1 z\,\vec{j}.\vec{j}\,dxdz$$

$$= \int_0^3\int_0^1 (z)\ dx\,dz = \int_0^3 (xz)_0^1\ dz$$

$$= \left(\frac{z^2}{2}\right)_0^3 = \frac{9}{2}$$

Over S_5 : z = 1 , $n = \vec{k}$

$$\iint_{S_5}\left(\nabla\times\vec{F}\right).n\ ds = \int_0^2\int_0^1\left(-x\vec{k}\right).\vec{k}dxdy$$

$$= \int_0^2\int_0^1 (-x)dxdy = \int_0^2\left(-\frac{x^2}{2}\right)_0^1 dy$$

$$= \int_0^2\left(\frac{-1}{2}\right)dy = \left(\frac{-1}{2}\right)(y)_0^2 = -1$$

$$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} = -12 + 12 - \frac{9}{2} + \frac{9}{2} - 1 = -1$$

$\therefore$ L.HS = R.HS.

Hence Stoke's theorem is verified.

39. Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\,\vec{j}$ in the rectangular region bounded by the lines x = 0, x = a, y = 0 and y = b.

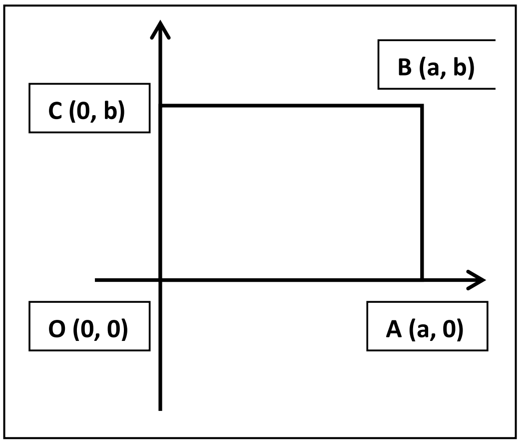
Solution:

Given $\vec{F} = \left(x^2 + y^2\right)\vec{i} - 2xy\vec{j}$

By Stoke's theorem $\int_C \vec{F}.d\vec{r} = \iint_S curl\,\vec{F}.\vec{n}\,ds$

Evaluation of LHS:

$$\int_C \vec{F}.dr = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$



Along OA : $y = 0 \Rightarrow dy = 0,$ x varies from 0 to a

$$\therefore \int_{OA} \vec{F}.dr = \int_0^a \left(x^2\right)dx$$

$$= \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$$

Along AB: $x = a \Rightarrow dx = 0,$ y varies from 0 to b

$$\int_{AB} \vec{F}.dr = \int_0^b -2ay\ dy$$

$$= -2a\left(\frac{y^2}{2}\right)_0^b = -ab^2$$

Along BC: $y = b,$ $dy = 0$, x varies from a to 0

$$\int_{BC} \vec{F}.d\vec{r} = \int_a^0 \left(x^2 + b^2\right)dx$$

$$= \left(\frac{x^3}{3} + b^2x\right)_a^0$$

$$= -\frac{a^3}{3} - ab^2$$

Along CO: $x = 0,$ $dx = 0,$ y varies from b to 0

$$\int_{CO} \vec{F}.d\vec{r} = \int_b^0 \left(0 + y^2\right)0 + 0 = 0$$

$$\therefore \int_C \vec{F}.d\vec{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 + 0 = -2ab^2$$

Evaluation of RHS:

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[-2y-2y] = -4y\vec{k}$$

As the region is in the xy plane we can take $\vec{n} = \vec{k}$ and $ds = dxdy$

$$\iint_S curl\,\vec{F}.\vec{n}\,ds = \iint -4y\vec{k}.\vec{k}\,dx\,dy$$

$$= -4\int_0^b\int_0^a y\;dx\,dy$$

$$= -4\left(\frac{y^2}{2}\right)_0^b (x)_0^a$$

$$= -2ab^2$$

$$\therefore \int_C \vec{F}.d\vec{r} = \iint_S curl\vec{F}.\vec{n}ds$$

Hence Stoke's theorem is verified.

40. State Gauss Divergence Theorem (Relation between Surface and Volume Integrals).

Statement: If V is the volume bounded by a closed surface S and if a vector function $\vec{F}$ is having continuous first order partial derivatives on S , then $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V div\vec{F}\;dV$.

where $\hat{n}$ is the outward unit normal vector to the surface.

41. Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$ taken over the cube bounded by the planes $x=0,\ x=1, y=0,\ y=1, z=0, z=1.$

Solution:

$$\vec{F} = 4xz\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$$

$$\nabla\circ\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla\circ\vec{F} = 4z - 2y + y = 4z - y$$

$$RHS = \iiint_V \nabla \circ \vec{F} dv = \int_0^1\int_0^1\int_0^1 (4z-y)dxdydz = \int_0^1\int_0^1 [4zx - yx]_0^1 dydz = \int_0^1\int_0^1 [4z-y]dydz$$

$$= \int_0^1 \left[4zy - \frac{y^2}{2}\right]_0^1 dz = \int_0^1 \left[4z - \frac{1}{2}\right] dz = \left[4\frac{z^2}{2} - \frac{z}{2}\right]_0^1 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2} \text{..........................(1)}$$

Surface	$\hat{n}$	$\vec{F} \circ \hat{n}$	Equation	$\vec{F} \circ \hat{n}$ on S	dS	$\iint_s \vec{F} \circ \hat{n}\, dS$
S_1	$\vec{i}$	$4xz$	$x=1$	$4z$	$dydz$	$\int_0^1\int_0^1 4zdydz$
S_2	$-\vec{i}$	$-4xz$	$x=0$	0	$dydz$	0
S_3	$\vec{j}$	$-y^2$	$y=1$	-1	$dxdz$	$\int_0^1\int_0^1 (-1)dxdz$
S_4	$-\vec{j}$	y^2	$y=0$	0	$dxdz$	0
S_5	$\vec{k}$	yz	$z=1$	y	$dxdy$	$\int_0^1\int_0^1 ydxdy$
S_6	$-\vec{k}$	$-yz$	$z=0$	0	$dxdy$	0

$$LHS = \iint_S \vec{F} \circ \hat{n}\, dS = \left(\iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S6}\right) \vec{F} \circ \hat{n}\, dS$$

$$= \int_0^1\int_0^1 4zdydz + \int_0^1\int_0^1 (0)dydz + \int_0^1\int_0^1 (-1)dxdz + \int_0^1\int_0^1 (0)dxdz + \int_0^1\int_0^1 ydxdy + \int_0^1\int_0^1 (0)dxdy$$

$$= 4\int_0^1\int_0^1 zdydz + 0 - \int_0^1\int_0^1 dxdz + 0 + \int_0^1\int_0^1 ydxdy + 0$$

$$= 4\int_0^1 z(y)_0^1 dz - \int_0^1 (x)_0^1 dz + \int_0^1 y(x)_0^1 dy = 4\int_0^1 zdz - \int_0^1 dz + \int_0^1 ydy = \left(4\frac{z^2}{2}\right)_0^1 - (z)_0^1 + \left(\frac{y^2}{2}\right)_0^1$$

$$= \frac{4}{2} - 1 + \frac{1}{2} = 2 - 1 + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2} \text{...(2)}$$

From (1) and (2),

$$\iiint_V \nabla \circ \vec{F}dv = \left(\iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S6} \right) \vec{F} \circ \hat{n}\, dS$$

Hence Gauss Divergence theorem is verified.

42. Verify Gauss divergence theorem for $\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$ taken over the cube bounded by the planes $x = 0,\ x = 1, y = 0,\ y = 1, z = 0, z = 1.$

Solution:

$$\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$$

$$\nabla \circ \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2x + 2y + 2z = 2(x + y + z)$$

$$RHS = \iiint_V \nabla \circ \vec{F}dv = 2\int_0^1\int_0^1\int_0^1 (x + y + z)dxdydz = 2\int_0^1\int_0^1 \left[\frac{x^2}{2} + xy + xz \right]_0^1 dydz = 2\int_0^1\int_0^1 \left[\frac{1}{2} + y + z \right] dydz$$

$$= 2\int_0^1 \left[\frac{y}{2} + \frac{y^2}{2} + yz \right]_0^1 dz = 2\int_0^1 \left[\frac{1}{2} + \frac{1}{2} + z \right]_0^1 dz = 2\int_0^1 [1 + z]_0^1 dz = 2\left[z + \frac{z^2}{2} \right]_0^1$$

$$= 2\left(1 + \frac{1}{2}\right) = 2\left(\frac{3}{2}\right) = 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Surface	$\hat{n}$	$\vec{F} \circ \hat{n}$	Equation	$\vec{F} \circ \hat{n}$ on S	dS	$\iint_s \vec{F} \circ \hat{n}\, dS$
S_1	$\vec{i}$	x^2	$x = 1$	1	$dydz$	$\int_0^1\int_0^1 dydz$
S_2	$-\vec{i}$	$-x^2$	$x = 0$	0	$dydz$	0
S_3	$\vec{j}$	y^2	$y = 1$	1	$dxdz$	$\int_0^1\int_0^1 dxdz$
S_4	$-\vec{j}$	$-y^2$	$y = 0$	0	$dxdz$	0
S_5	$\vec{k}$	z^2	$z = 1$	1	$dxdy$	$\int_0^1\int_0^1 dxdy$
S_6	$-\vec{k}$	$-z^2$	$z = 0$	0	$dxdy$	0

$$LHS = \iint_S \vec{F} \circ \hat{n}\, dS = \left(\iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S6} \right) \vec{F} \circ \hat{n}\, dS$$

$$= \int_0^1\int_0^1 dydz + \int_0^1\int_0^1 (0) dydz + \int_0^1\int_0^1 dxdz + \int_0^1\int_0^1 (0) dxdz + \int_0^1\int_0^1 dxdy + \int_0^1\int_0^1 (0) dxdy$$

$$= 1+1+1 = 3 ....(2)$$

From (1) and (2),

$$\iiint_V \nabla \circ \vec{F} dv = \left(\iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S6} \right) \vec{F} \circ \hat{n}\, dS$$

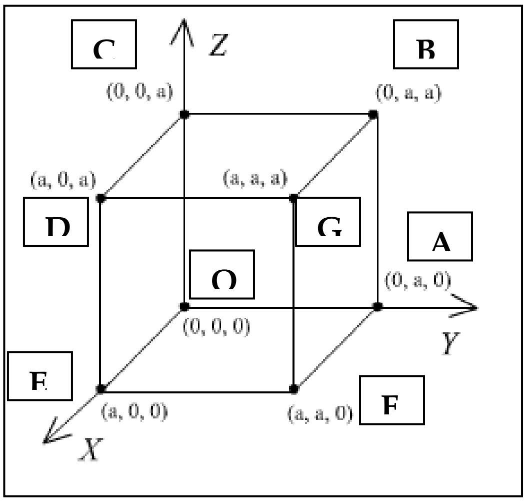
Hence Gauss Divergence theorem is verified.

43. **Verify Gauss divergence theorem for $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ taken over the cube bounded by x = 0, x = a, y = 0, y = a, z = 0 and z = a.**

Solution:

By Gauss Divergence theorem $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V div\vec{F}\ dV$

S_1	OABC
S_2	DEFG
S_3	OCDE
S_4	ABGF
S_5	OEFA
S_6	CDGB



Evaluation of LHS:

$$\iint_S \vec{F} \cdot nds = \iint_{S_1} \vec{F} \cdot \hat{n}\, ds + \iint_{S_2} \vec{F} \cdot nds + ... + \iint_{S_6} \vec{F} \cdot nds$$

Over S_1: x = 0, $\hat{n} = -\vec{i}$

$$\iint_{S_1} \vec{F} \cdot \hat{n}\, ds = \int_0^a\int_0^a (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}).(-\vec{i})\, dy\, dz = \int_0^a\int_0^a -x^3\, dy\, dz$$

$$= 0$$

Over S2: x = a, $\hat{n} = \vec{i}$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}).(\vec{i})\,dy\,dz = \int_0^a\int_0^a x^3\,dy\,dz$$

$$= \int_0^a\int_0^a a^3\,dy\,dz = a^3\int_0^a [y]_0^a dz = a^3\int_0^a a\,dz$$

$$= a^4\,[z]_0^a = a^4(a) = a^5$$

Over S3: y = 0, $\hat{n} = -\vec{j}$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}).(-\vec{j})\,dx\,dz = \int_0^a\int_0^a -y^3\,dx\,dz = 0$$

Over S4: y = a, $\hat{n} = \vec{j}$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}).(\vec{j})\,dx\,dz = \int_0^a\int_0^a y^3\,dx\,dz$$

$$= \int_0^a\int_0^a a^3\,dx\,dz = a^3\int_0^a [x]_0^a\,dz = a^3\int_0^a [a-0]\,dz = a^4\,[z]_0^a = a^4\,(a) = a^5$$

Over S5: z = 0, $\hat{n} = -\vec{k}$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}).(\overrightarrow{-k})\,dx\,dy = \int_0^a\int_0^a -z^3\,dx\,dy = 0$$

Over S6: z = a, $\hat{n} = \vec{k}$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}).(\vec{k})\,dx\,dy = \int_0^a\int_0^a z^3\,dx\,dy$$

$$= a^3\int_0^a [x]_0^a dy = a^3\int_0^a a\;dy = a^4\,[y]_0^a = a^4(a) = a^5$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = 0 + a^5 + 0 + a^5 + 0 + a^5 = 3a^5$$

Evaluation of RHS:

$$\nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\ \frac{\partial}{\partial y} + \vec{k}\ \frac{\partial}{\partial z}\right)\left(x^3\vec{i} + y^3\vec{j}\ \ + z^3\vec{k}\ \right)$$

$$\nabla.\vec{F} = 3x^2 + 3y^2 + 3z^2$$

$$\iiint_V \nabla.\vec{F}\, dV = \int_0^a\int_0^a\int_0^a 3x^2 + 3y^2 + 3z^2\, dx\, dy\, dz$$

$$= 3\int_0^a\int_0^a\int_0^a x^2 + y^2 + z^2\, dx\, dy\, dz$$

$$= 3\int_0^a\int_0^a \left[\frac{x^3}{3} + (y^2 + z^2)x\right]_0^a dy\, dz$$

$$= 3\int_0^a\int_0^a \left[\frac{a^3}{3} + (y^2 + z^2)\mathrm{a}\right] dy\, dz$$

$$= 3\int_0^a \left[\frac{a^3}{3}y + a\frac{y^3}{3} + az^2 y\right]_0^a dz$$

$$= 3\int_0^a \frac{a^4}{3} + \frac{a^4}{3} + a^2z^2\ dz$$

$$= 3\left[\frac{a^4}{3}z + \frac{a^4}{3}z + a^2\frac{z^3}{3}\right]_0^a$$

$$= 3\left[\frac{a^5}{3} + \frac{a^5}{3} + \frac{a^5}{3}\right]$$

$$= \frac{9a^5}{3} = 3a^5$$

Hence Gauss Divergence theorem is verified.

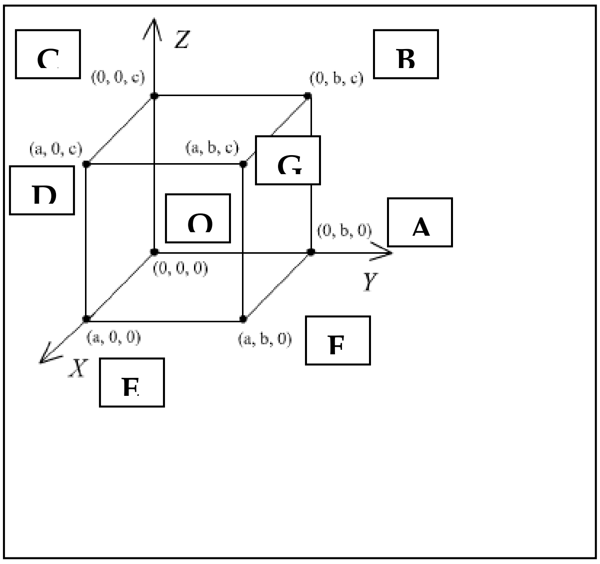
44. Verify Gauss divergence theorem for $\vec{\mathbf{F}} = \left(\mathbf{x^2 - yz}\right)\vec{\mathbf{i}} + \left(\mathbf{y^2 - zx}\right)\vec{\mathbf{j}} + \left(\mathbf{z^2 - xy}\right)\vec{\mathbf{k}}$ taken over the rectangular parallelopiped bounded by the planes x = 0, x = a, y = 0, y = b, z = 0, and z = c.

Solution:

By Gauss Divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V div\vec{F}\;dV$$

S1	OABC
S2	DEFG
S3	OCDE
S4	ABGF
S5	OEFA
S6	CDGB



Evaluation of LHS:

$$\iint_S \vec{F}\cdot n ds = \iint_{S_1} \vec{F}\cdot \hat{n}\, ds + \iint_{S_2} \vec{F}\cdot nds + \dots + \iint_{S_6} \vec{F}\cdot nds$$

Over S_1: x = 0, $\hat{n} = -\vec{i}$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^b (0-yz)(-1)dy\,dz = \int_0^c\int_0^b (yz)\,dy\,dz = \int_0^c\left[z\left(\frac{y^2}{2}\right)_0^b\right]dz = \frac{b^2}{2}\left(\frac{z^2}{2}\right)_0^c = \frac{b^2c^2}{4}$$

Over S_2: x = a, $\hat{n} = \vec{i}$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^b (-yz+a^2)\,dy\,dz = \int_0^b\left[-y\left(\frac{z^2}{2}\right)_0^c + a^2\,[z]_0^c\right]dy$$

$$= -\frac{c^2}{2}\left(\frac{y^2}{2}\right)_0^b + ca^2\,[y]_0^b = a^2bc - \frac{b^2c^2}{4}$$

Over S_3: y = 0, $\hat{n} = -\vec{j}$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^a (xz)\,dx\,dz = \int_0^c\left(\frac{x^2}{2}z\right)_0^a dz = \frac{a^2}{2}\left(\frac{c^2}{2}\right) = \frac{a^2c^2}{4}$$

Over S_4: y = b, $\hat{n} = \vec{j}$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^a (-xz+b^2)\,dx\,dz = \int_0^c \left[-z\left(\frac{a^2}{2}\right) + b^2 a \right] dz = ab^2c - \frac{a^2c^2}{4}$$

Over S_5: z = 0, $\hat{n} = -\vec{k}$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a (xy)\,dx\,dy = \int_0^b \left[y\left(\frac{x^2}{2}\right)_0^a \right] dy = \frac{a^2b^2}{4}$$

Over S_6: z = c, $\hat{n} = \vec{k}$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a (-xy+c^2)\,dx\,dy = \int_0^b \left[-y\left(\frac{a^2}{2}\right) + c^2 a \right] dy = abc^2 - \frac{a^2b^2}{4}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \frac{b^2c^2}{4} + a^2bc - \frac{b^2c^2}{4} + \frac{a^2c^2}{4} + a\ b^2c - \frac{a^2c^2}{4} + \frac{a^2b^2}{4} + a\ bc^2 - \frac{a^2b^2}{4}$$

$$= a^2bc + ab^2c + abc^2 = abc(a+b+c)$$

Evaluation of RHS:

$$\nabla.\vec{F} = 2(x+y+z)$$

$$\iiint_V \nabla.\vec{F}\,dV = \int_0^c\int_0^b\int_0^a 2(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^c\int_0^b \left[\frac{x^2}{2} + xy + xz \right]_0^a dy\,dz$$

$$= 2\int_0^c\int_0^b \left[\frac{a^2}{2} + ay + az \right] dy\,dz$$

$$= 2\int_0^c \left[\frac{a^2}{2}y + a\frac{y^2}{2} + ayz \right]_0^b dz$$

$$= 2\left[\frac{a^2bz}{2} + \frac{ab^2z}{2} + \frac{abz^2}{2} \right]_0^c$$

$$= 2\left[\frac{a^2bc}{2}+\frac{ab^2c}{2}+\frac{abc^2}{2}\right]= a^2bc+ab^2c+abc^2 = abc(a+b+c)$$

Hence Gauss divergence theorem is verified.

45. **Evaluate** $\iint_S \vec{F}.\hat{n}ds$ **where** $\overline{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ **and S is the surface bounding the region $x^2+y^2 = 4$, z = 0 and z = 3.**

Solution:

Gauss divergence theorem is

$$\iint_S \vec{F}.\hat{n}ds = \iiint_V \nabla.\vec{F}dv$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4x)+\frac{\partial}{\partial y}\left(-2y^2\right)+\frac{\partial}{\partial z}\left(z^2\right)\right]dv$$

$$= \iiint_V [4-4y+2z]dv \quad = \int_{-2}^{2}\int_{\sqrt{4-x_2}}^{\sqrt{4-x^2}}\int_0^3 (4-4y+2z)dzdydx$$

$$= \int_{-2}^{2}\int_{\sqrt{4-x_2}}^{\sqrt{4-x^2}}\int_0^3\left[4x-4yz+\frac{2z^2}{2}\right]_0^3 dydx \quad = \int_{-2}^{2}\int_{\sqrt{4-x_2}}^{\sqrt{4-x^2}}[(12-12y+9)-0]dydx$$

$$= \int_{-2}^{2}\int_{\sqrt{4-x_2}}^{\sqrt{4-x^2}}[21-12y]dx = \int_{-2}^{2}\left[21y-12\frac{y^2}{2}\right]_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2}42\sqrt{4-x^2}\,dx = (42)(2)\int_0^2\sqrt{4-x^2}\,dx = 84\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$= 84\pi$

46. **If** $F = ax\vec{i} + by\vec{j} + cz\vec{k}$, **a, b, c are constants, show that** $\iint_S \vec{F}\bullet\hat{n}\,ds = \frac{4\pi}{3}(a+b+c)$ **where S is the surface of a unit sphere.**

Solution:

W.K.T. Gauss's divergence theorem

$$\iint_S \vec{F} \bullet \hat{n} ds = \iiint_V \nabla \bullet \vec{F} dV \;\; = \iiint_V \left(\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz) \right) dV$$

$$= \iiint_V (a+b+c) dV \; = (a+b+c)V = (a+b+c)\frac{4}{3}\pi(1)^3$$

$$\iint_S \vec{F} \bullet \hat{n} ds = \frac{4}{3}\pi(a+b+c)$$

47. **Evaluate** $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$ **and C is the straight line from A (0, 0, 0) to B (2, 1, 3).**

Solution:

Given $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\overrightarrow{F}.d\vec{r} = 3x^2 dx + (2xz - y)dy + zdz$

The equation of AB is $\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$ (say) $\left(\because \; \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \right)$

$$\begin{aligned} &\Rightarrow x = 2t \Rightarrow dx = 2dt \\ &\quad y = t \Rightarrow dy = dt \\ &\quad z = 3t \Rightarrow dz = 3dt \end{aligned} \quad , \qquad \int_C \overrightarrow{F}. \; \overrightarrow{dr} = \int_0^1 3x^2 dx + (2xz - y)dy + zdz$$

$$= \int_0^1 \left(36t^2 + 8t\right) dt = \left[36\frac{t^3}{3} + 8\left(\frac{t^2}{2}\right) \right]_0^1 = 16$$

48. **Find the work done when a force** $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ **moves a particle in the XY - plane from (0, 0) to (1,1) along the parabola** $y^2 = x$.

Solution:

Given $\overrightarrow{F} = \left(x^2 - y^2 + x\right)\vec{i} - \left(2xy + y\right)\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\overrightarrow{F}. \overrightarrow{dr} = \left(x^2 - y^2 + x\right)dx - \left(2xy + y\right)dy$.

Given $y^2 = x$

$2ydy = dx$

$$\therefore \vec{F}.d\vec{r} = \left(x^2 - x + x\right)dx - \left(2y^3 + y\right)dy$$
$$= x^2dx - \left(2y^3 + y\right)dy$$

$$\int_C \overrightarrow{F}.\overrightarrow{dr} = \int_0^1 x^2dx - \int_0^1 \left(2y^3 + y\right)dy$$
$$= \left[\frac{x^3}{3}\right]_0^1 - \left[\frac{2y^4}{4} + \frac{y^2}{2}\right]_0^1$$
$$= \left(\frac{1}{3} - 0\right) - \left[\left(\frac{2}{4} + \frac{1}{2}\right) - \left(0 + 0\right)\right] = \frac{-2}{3}$$

$\therefore$ Work done $= \dfrac{2}{3}$

49. **If** $F = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$ **, evaluate** $\int_C \vec{F} \bullet d\vec{r}$ **from (0,0,0) to (1,1,1) along the curve x = t, y = t^2, z = t^3.**

Solution:

The end points are (0,0,0) and (1,1,1).

These points correspond to t = 0 and t = 1.

$\therefore$ dx = dt, dy = 2t dt, dz = $3t^2$dt

$$\int_C \vec{F} \bullet d\vec{r} = \int_C \left(3x^2 + 6y\right)dx - 14yzdy + 20xz^2dz$$

$$= \int_0^1 (3t^2 + 6t^2)dt - 14t^5(2tdt) + 20t^7(3t^2)dt = \int_0^1 (9t^2 - 28t^6 + 60t^9)dt = 5$$

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