Answer ALL Questions.

PART C – (2 x 12 = 24 marks)

| | | |
|---|---|---|
| 17.a | Verify Green's theorem for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region defined by the lines $x = 0, y = 0$ *and* $x + y = 1$. | CO2 K2 |
| | OR | |
| 17.b | Verify Gauss divergence theorem for $\vec{F} = x^2 i + y^2 j + z^2 k$ taken over the rectangular parallopiped formed by $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$ | CO2 K2 |
| | | |
| 18. a i) | Prove that $\vec{F} = (y^2 \cos x + z^3)i + (2y \sin x - 4)j + 3xz^2 k$ is irrotational. Find the scalar potential. | CO2 K2 |
| a.ii) | Find Laplace transform of $f(t) = \begin{cases} a\ sinwt, 0 < t < \pi/w, \\ 0, \quad \frac{\pi}{w} < t < \frac{2\pi}{w} \end{cases}$ | CO3 K2 |
| | OR | |
| 18. b.i) | Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ displaces a particle in xy plane from (0,0) to (1,1) along the parabola $y^2 = x$ | CO2 K2 |
| 18. b.ii) | Verify initial value theorem for $f(t) = 1 + e^{-t}(\sin t + \cos t)$ | CO3 K2 |

Problem (1)

Verify green's theorem for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)\,dy$

where $C$ is the boundary of the region defined by the lines $x=0$; $y=0$; $x+y=1$.

**Sol:-**

By Green's theorem, $\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d$

R.H.S

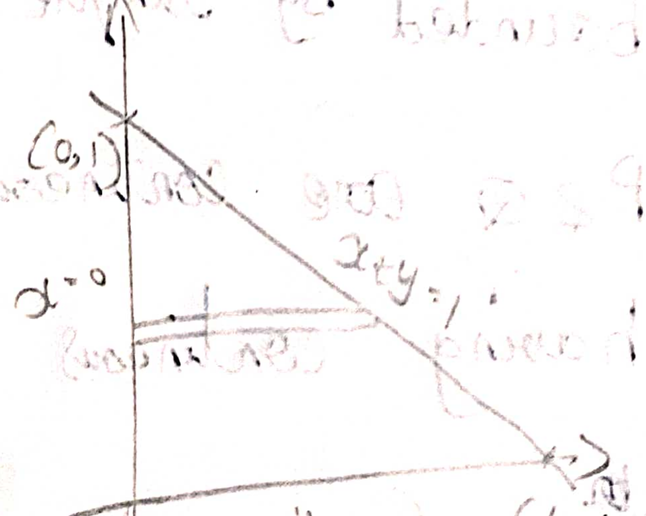
given, the region bounded by the curvs

$x=0$; $y=0$; $x+y=1$

In the region

$x$ varies from 0 to $1-y$

$y$ varies from 0 to 1



Since, $P = 3x^2 - 8y^2$ and $Q = 4y - 6xy$

$$\frac{\partial P}{\partial y} = -16y \qquad \frac{\partial Q}{\partial x} = -6y$$

$$\iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \int_0^1 \int_0^{1-y} -6y + 16y \, dx \, dy$$

$$= \int_0^1 \int_0^{1-y} 10y \, dx \, dy \Rightarrow \int_0^1 [10yx]_0^{1-y} \, dy$$

$$= \int_0^1 10y(1-y) \, dy \Rightarrow 10 \int_0^1 y - y^2 \, dy$$

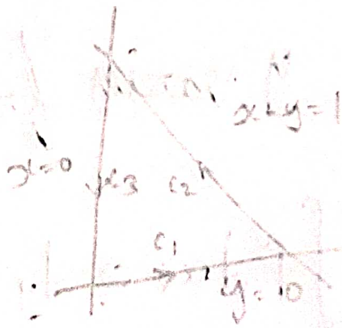
$$\Rightarrow 10 \left[ \frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 \Rightarrow 10 \left[ \frac{1}{2} - \frac{1}{3} \right] \Rightarrow 10 \left[ \frac{1}{6} \right]$$

$$\Rightarrow \frac{5}{3}$$

$$\iint_R \left[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx \, dy = \frac{5}{3} \quad \text{—— (1)}$$

L.H.S

$$\therefore \oint_C P\,dx + Q\,dy = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy \quad \text{—— (2)}$$



For $C_1$

Here $y = 0$

$dy = 0$

$\therefore \; Pdx + Qdy = (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

$\therefore \; Pdx + Qdy = 3x^2\,dx$

$x$ varies from 0 to 1

$$\therefore \int_{C_1} Pdx + Qdy = \int_0^1 3x^2\,dx \Rightarrow \left[3 \cdot \frac{x^3}{3}\right]_0^1$$

$$\int_{C_1} Pdx + Qdy = 1 \quad \text{③}$$

For $C_3$

$x = 0$

$dx = 0$

$\therefore \; Pdx + Qdy = 4y\,dy$

$y$ varies from 1 to 0

$$\int_{C_2} Pdx + Q\,dy = \int_1^0 4y\,dy \Rightarrow \left[2y^2\right]_1^0 \Rightarrow -2$$

④

For $C_2$

$x+y=1$

$y=1-x$

$dy = 0 - dx$

$x$ varies from 1 to 0

$$\int_{C_2} P\,dx + Q\,dy = \int_1^0 (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

$$= \int_1^0 [3x^2 - 8(1-x)^2]\,dx + [4(1-x) - 6x(1-x)](-dx)$$

$$= \int_1^0 [3x^2 - 8(1-x)^2 - 4(1-x) + 6(x-x^2)]\,dx$$

$$= \left[\frac{\cancel{3}x^3}{\cancel{3}} - \frac{8(1-x)^3}{-3} - 4\left[x - \frac{x^2}{2}\right] + 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]\right]_1^0$$

$$= 0 + \frac{8}{3} - 4[0] + 6[0] - 1 - 0 + 4\left(1-\frac{1}{2}\right) - 6\left(\frac{1}{2}-\frac{1}{3}\right)$$

$$= \frac{8}{3} - 1 + 2 - \cancel{6}\left(\frac{1}{\cancel{6}}\right) \Rightarrow \frac{8}{3} \quad \text{--- (5)}$$

By (2)

$$\oint_C P\,dx + Q\,dy = 1 + \frac{8}{3} - 2 \Rightarrow \frac{5}{3} \quad \text{--- (6)}$$

From (1) and (6)

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Green's theorem is verified.

From (1) and (2),

$$\iiint_V \nabla \circ \vec{F} dv = \left( \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S6} \right) \vec{F} \circ \hat{n} \, dS$$

Hence Gauss Divergence theorem is verified.

**42.** Verify Gauss divergence theorem for $\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$ taken over the cube bounded by the planes $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1$.

**Solution:**

$$\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$$

$$\nabla \circ \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2x + 2y + 2z = 2(x + y + z)$$

$$RHS = \iiint_V \nabla \circ \vec{F} dv = 2\int_0^1\int_0^1\int_0^1 (x + y + z) dx dy dz = 2\int_0^1\int_0^1 \left[\frac{x^2}{2} + xy + xz\right]_0^1 dy dz = 2\int_0^1\int_0^1 \left[\frac{1}{2} + y + z\right] dy dz$$

$$= 2\int_0^1 \left[\frac{y}{2} + \frac{y^2}{2} + yz\right]_0^1 dz = 2\int_0^1 \left[\frac{1}{2} + \frac{1}{2} + z\right]_0^1 dz = 2\int_0^1 [1 + z]_0^1 dz = 2\left[z + \frac{z^2}{2}\right]_0^1$$

$$= 2\left(1 + \frac{1}{2}\right) = 2\left(\frac{3}{2}\right) = 3 \text{.........................(1)}$$

| Surface | $\hat{n}$ | $\vec{F} \circ \hat{n}$ | Equation | $\vec{F} \circ \hat{n}$ on S | $dS$ | $\iint_s \vec{F} \circ \hat{n} dS$ |
|---|---|---|---|---|---|---|
| $S_1$ | $\vec{i}$ | $x^2$ | $x = 1$ | 1 | $dydz$ | $\int_0^1\int_0^1 dydz$ |
| $S_2$ | $-\vec{i}$ | $-x^2$ | $x = 0$ | 0 | $dydz$ | 0 |
| $S_3$ | $\vec{j}$ | $y^2$ | $y = 1$ | 1 | $dxdz$ | $\int_0^1\int_0^1 dxdz$ |
| $S_4$ | $-\vec{j}$ | $-y^2$ | $y = 0$ | 0 | $dxdz$ | 0 |
| $S_5$ | $\vec{k}$ | $z^2$ | $z = 1$ | 1 | $dxdy$ | $\int_0^1\int_0^1 dxdy$ |
| $S_6$ | $-\vec{k}$ | $-z^2$ | $z = 0$ | 0 | $dxdy$ | 0 |





$$LHS = \iint_S \vec{F} \circ \hat{n} dS = \left( \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S6} \right) \vec{F} \circ \hat{n} \, dS$$

$$= \int_0^1\int_0^1 dydz + \int_0^1\int_0^1 (0) dydz + \int_0^1\int_0^1 dxdz + \int_0^1\int_0^1 (0) dxdz + \int_0^1\int_0^1 dxdy + \int_0^1\int_0^1 (0) dxdy$$

$$= 1 + 1 + 1 = 3 \text{..............................................................................(2)}$$

From (1) and (2),

$$\iiint_V \nabla \circ \vec{F} dv = \left( \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S6} \right) \vec{F} \circ \hat{n} \, dS$$

Hence Gauss Divergence theorem is verified.

5. **Show that the vector $\overline{F}=(y^2\cos x+z^3)\vec{i}+(2y\sin x-4)\vec{j}+3xz^2\vec{k}$ is irrotational and find the scalar potential function.**

**Solution:**

$$curl\vec{F}=\nabla\times\overline{F}=\vec{0}$$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y^2\cos x+z^3 & 2y\sin x-4 & 3xz^2\end{vmatrix}$$

$$=\vec{i}\left(\frac{\partial}{\partial y}\left(3xz^2\right)-\frac{\partial}{\partial z}(2y\sin x-4)\right)-\vec{j}\left(\frac{\partial}{\partial x}\left(3xz^2\right)-\frac{\partial}{\partial z}\left(y^2\cos x+z^3\right)\right)$$

$$+\vec{k}\left(\frac{\partial}{\partial y}\left(y^2\cos x+z^3\right)-\frac{\partial}{\partial x}(2y\sin x-4)\right)$$

$$=\vec{i}\,(0-0)-\vec{j}\left(3z^2-3z^2\right)+\vec{k}\,(2y\cos x-2y\cos x)=\vec{0}$$

$\therefore$ $\overline{F}$ is irrotational.

To find Scalar potential $\phi$ we assume $\overline{F}=\nabla\phi$

$$\overline{F}=\nabla\phi=(y^2\cos x+z^3)\vec{i}+(2y\sin x-4)\vec{j}+3xz^2\vec{k}$$

$$\left(\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\ \frac{\partial\phi}{\partial y}+\vec{k}\ \frac{\partial\phi}{\partial z}\right)=(y^2\cos x+z^3)\vec{i}+(2y\sin x-4)\vec{j}+3xz^2\vec{k}$$

*comparing coefficient of* $\vec{i},\vec{j}\,\&\,\vec{k}$

$$\frac{\partial\phi}{\partial x}=y^2\cos x+z^3 \qquad \rightarrow(1)$$

$$\frac{\partial\phi}{\partial y}=2y\sin x-4 \qquad \rightarrow(2)$$

$$\frac{\partial\phi}{\partial z}=3xz^2 \qquad \rightarrow(3)$$

**Integrating (1) w.r.t. $x$ (keeping $y$ and $z$ as constant)**

$$\varphi=y^2\,(\sin x)+x\,z^3+f_1\,(y,z)$$

**Integrating (2) w.r.t. $y$ (keeping $x$ and $z$ as constant)**

$$\varphi=y^2\sin x-4\,y+f_2\,(x,z)$$

**Integrating (3) w.r.t. $z$ (keeping $x$ and $y$ as constant)**

$$\varphi=x\,z^3+f_3\,(x,y)$$





Hence $\varphi=y^2\sin x+x\,z^3-4\,y+c$ where $c$ is a constant, $c=f_1\,(y,z)+f_2\,(x,z)+f_3\,(x,y)$

Find the L.T of half sign wave rectifier function $f(t) = \begin{cases} a \sin\omega t, & 0 \leq t \leq \pi/\omega \\ 0, & \pi/\omega \leq t \leq 2\pi/\omega \end{cases}$

Given period $P = 2\pi/\omega$

$$L(f(x)) = \frac{1}{1-e^{-PS}} \int_0^P e^{-St} f(t)\, dt$$

$$\frac{1}{1-e^{-\frac{2\pi}{\omega}s}} \int_0^{2\pi/\omega} e^{-st} f(t)\, dt$$

$$\frac{1}{1-e^{-\frac{2\pi}{\omega}s}} \left[ \int_0^{\pi/\omega} e^{-st} (a \sin \omega t)\, dt \int_{\pi/\omega}^{2\pi/\omega} e^{-st} (0)\, dt \right.$$

$$\frac{a}{1-e^{-\frac{2\pi}{\omega}s}} \left[ \int_0^{\pi/\omega} e^{-st} \sin\omega t \; dt \right]$$

Formula

$$\because \int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b\cos bx\right]$$

$a = -s$
$b = \omega$

$$= \frac{a}{1-e^{-\frac{2\pi}{\omega}s}}\left[\frac{e^{-st}}{s^2+\omega^2}\left(-s \sin \omega t - \omega \cos \omega t\right)\right]_0^{\pi/\omega}$$

$$= \frac{a}{1-e^{-\frac{2\pi}{\omega}s}}\left[\frac{e^{-\frac{s\pi}{\omega}}}{s^2+\omega^2}\left(0+\omega\right) - \frac{1}{s^2+\omega^2}\left(-\omega\right)\right]$$

$$= \frac{a}{1-e^{\frac{2\pi}{\omega}s}}\left[\frac{\omega\left[e^{-s\frac{\pi}{\omega}}+1\right]}{s^2+\omega^2}\right]$$

$$= \frac{a\omega}{\left(1-e^{-s\pi/\omega}\right)\cancel{\left(1+e^{-\frac{s\pi}{\omega}}\right)}} \cdot \frac{\cancel{e^{-\frac{s\pi}{\omega}}+1}}{s^2+\omega^2}$$

$$L\{f(t)\} = \frac{a\omega}{\left(s^2+\omega^2\right)\left(1-e^{-s\pi/\omega}\right)}$$

48. **Find the work done when a force $\vec{F}=(x^2-y^2+x)\vec{i}-(2xy+y)\vec{j}$ moves a particle in the XY - plane from (0, 0) to (1,1) along the parabola $y^2=x$.**

**Solution:**

Given $\overline{F}=\left(x^2-y^2+x\right)\overline{i}-\left(2xy+y\right)\overline{j}$

$d\vec{r}=dx\vec{i}+dy\vec{j}+dz\vec{k}$

$\overline{F}.\overline{dr}=\left(x^2-y^2+x\right)dx-\left(2xy+y\right)dy$.





Given $y^2=x$

$2ydy=dx$

$$\therefore \overline{F}.d\overline{r}=\left(x^2-x+x\right)dx-\left(2y^3+y\right)dy$$

$$=x^2dx-\left(2y^3+y\right)dy$$

$$\int_C \overline{F}.\overline{dr}=\int_0^1 x^2dx-\int_0^1\left(2y^3+y\right)dy$$

$$=\left[\frac{x^3}{3}\right]_0^1-\left[\frac{2y^4}{4}+\frac{y^2}{2}\right]_0^1$$

$$=\left(\frac{1}{3}-0\right)-\left[\left(\frac{2}{4}+\frac{1}{2}\right)-(0+0)\right]=\frac{-2}{3}$$

$\therefore$ Work done $=\dfrac{2}{3}$

Verify Initial value theorem and final value theorem $f(t) = 1 + e^{-t}(\sin t + \cos t)$

<u>I. V. T.</u>

$$\lim_{t \to 0} f(t) = \lim_{t \to 0}\left[1 + e^{-t}(\cos t + \sin t)\right]$$

$$= 1 + e^{-0}(\cos(0) + \sin(0))$$

$$= 1 + 1(1 + 0)$$

$$= 2$$

RHS) W.R.t $F(s) = L(f(t))$

$$F(s) = L(1 + e^{-t}(\sin t + \cos t))$$

$$= L(1) + L(e^{-t}\sin t) + L(e^{-t}\cos t)$$

$$= L(1) + L(\sin t)_{s \to s+1} + L(\cos t)_{s \to s+1}$$

$$= \frac{1}{s} + \left[\frac{1}{s^2+1}\right]_{s \to s+1} + \left[\frac{s}{s^2+1}\right]_{s \to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$= \frac{1}{s} + \frac{s+1+1}{(s+1)^2+1}$$

$$= \frac{1}{s} + \frac{s+2}{s^2+2s+1+1}$$

$$F(s) = \frac{1}{s} + \frac{s+2}{s^2+2s+2}$$

$$sF(s) = \frac{s}{s} + \frac{s(s+2)}{s^2+2s+2}$$

$$sF(s) = 1 + \frac{s^2+2s}{s^2+2s+2}$$

$$\lim_{s \to \infty} sF(s) = 1 + \lim_{s \to \infty} \frac{s^2+2s}{s^2+2s+2}$$

$$= 1 + \lim_{s \to \infty} \frac{\cancel{s^2}(1 + 2/s)}{\cancel{s^2}(1 + \frac{2}{s} + \frac{2}{s^2})} = 1 + \frac{1+0}{1+0+0} = 2$$

1 = 2 Verified

E.V.T $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

L.H.S :-

$$\lim_{t \to \infty} f(t) = \lim_{t \to \infty} \left(1 + e^{-t}(\cos t + \sin t)\right)$$

$$= 1 + \lim_{t \to \infty} e^{-t}(\cos t + \sin t)$$

$$= 1 + 0 \Rightarrow 1 \quad \text{——} \; (3)$$

R.H.S

$$sF(s) = 1 + \frac{s^2 + 2s}{s^2 + 2s + 2}$$

$$\lim_{s \to 0} sF(s) = \lim_{s \to 0} \left(1 + \frac{s^2 + 2s}{s^2 + 2s + 2}\right)$$

$$= 1 + 0$$

from (3) & (4) $= 1$ —— (4)

from (3) & (4) ~~E.V.T~~

Verified