

**Module - 4 Analytic Functions**

Definition of Analytic Function – Cauchy Riemann equations – Properties of analytic functions – Determination of analytic function using Milne Thomson's method – Conformal mappings: Magnification, Rotation, Inversion, Reflection – Bilinear Transformation – Cauchy's integral theorem (without proof) – Cauchy's integral theorem applications – Application of Bilinear transformation and Cauchy's Integral in Engineering.

**Analytic function (or) Holomorphic function (or) Regular function.**

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

**Entire (or) an Integral function.**

A function which is analytic everywhere in the finite plane except at  $z = \infty$  is called an entire function.

Example:  $e^z$ ,  $\sin z$ ,  $\cosh z$ .

**Necessary conditions for  $f(z)$  to be analytic.**

The necessary conditions for a complex function  $f(z) = u(x,y) + i v(x,y)$  to be analytic in a region R are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad (\text{i.e.) C - R equations.}$$

**Sufficient conditions for  $f(z)$  to be analytic.**

If the partial derivatives  $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}$  exist and continuous in D and satisfies the conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \quad \text{Then the function } f(z) \text{ is analytic in a domain D.}$$

**Harmonic function.**

Any function which possess continuous second order partial derivatives and which satisfies Laplace equation is called a harmonic function. (i.e) If  $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ , then f is harmonic then

**Show that the function  $u = 2x - x^3 + 3xy^2$  is harmonic.**

**Solution:** Given  $u = 2x - x^3 + 3xy^2$

$$u_x = 2 - 3x^2 + 3y^2 \quad u_y = 6xy$$

$$u_{xx} = -6x \quad u_{yy} = 6x$$

$$u_{xx} + u_{yy} = -6x + 6x = 0$$

Hence u is harmonic

**Show that the function  $u = \frac{1}{2} \log(x^2 + y^2)$  is harmonic and determine its conjugate. Also find  $f(z)$ .**

$$\text{Given } u = \frac{1}{2} \log(x^2 + y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} (2x) = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} (2y) = \frac{y}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2)(1) - 2y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2 - x^2 + x^2 - y^2}{(x^2 + y^2)^2} = 0$$

Hence  $u$  is harmonic function

To find conjugate of  $u$

$$\phi_1(x, y) = \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}$$

$$\phi_1(z, o) = \frac{1}{z}$$

$$\phi_2(x, y) = \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

$$\phi_2(z, o) = 0$$

By Milne Thomson Methods

$$f'(z) = \phi_1(z, o) - i\phi_2(z, o)$$

$$\int f'(z) dz = \int \frac{1}{z} dz + 0$$

$$= \log z + c$$

$$f(z) = \log re^{i\theta}$$

$$f(z) = u + iv = \log r + i\theta$$

$$u = \log r, \quad v = \theta$$

$$u = \log \sqrt{x^2 + y^2} \quad \left[ \because r^2 = x^2 + y^2, \quad \theta = \tan^{-1} \left( \frac{y}{x} \right) \right]$$

$$v = \tan^{-1} \left( \frac{y}{x} \right) \therefore \text{Conjugate of } u \text{ is } \tan^{-1} \left( \frac{y}{x} \right).$$

### Conformal transformation.

A mapping or transformation which preserves angles in magnitude and in direction between every pair of curves through a point is said to be conformal transformation.

**Isogonal transformation.**

A transformation under which angles between every pair of curves through a point are preserved in magnitude but altered in sense is said to be isogonal at that point.

**Bilinear transformation (or) Mobius transformation (or) linear fractional transformation.**

The transformation  $w = \frac{az+b}{cz+d}$ ,  $ad - bc \neq 0$  where  $a, b, c, d$  are complex numbers is called a bilinear transformation. This is also called as Mobius or linear fractional transformation.

**Cross Ratio.**

The cross ratio of four points  $z_1, z_2, z_3, z_4$  is given by  $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$ .

**Show that  $f(z) = |z|^2$  is differentiable at  $z = 0$  but not analytic at  $z = 0$ .**

**Solution:** Let  $z = x + iy$  and  $\bar{z} = x - iy$

$$|z|^2 = z\bar{z} = x^2 + y^2$$

$$f(z) = |z|^2 = (x^2 + y^2) + i0$$

$$u = x^2 + y^2, \quad v = 0$$

$$u_x = 2x, \quad v_x = 0$$

$$u_y = 2y, \quad v_y = 0$$

So the C-R equations  $u_x = v_y$  and  $u_y = -v_x$  are not satisfied everywhere except at  $z = 0$ .

So  $f(z)$  may be differentiable only at  $z = 0$ . Now  $u_x = 2x$ ,  $v_y = 0$  and  $u_y = 2y$ ,  $v_x = 0$  are continuous everywhere and in particular at  $(0, 0)$ . So  $f(z)$  is differentiable at  $z = 0$  only and not analytic.

**Obtain the invariant points of the transformation  $w = \frac{z-1}{z+1}$**

**Solution:** Given:  $w = \frac{z-1}{z+1}$

The invariant points are obtained by replacing  $w$  by  $z$ .

$$\text{i.e., } z = \frac{z-1}{z+1} \Rightarrow z^2 + 1 = 0 \therefore z = \pm i$$

**Can  $v = \tan^{-1}\left(\frac{y}{x}\right)$  be the imaginary part of an analytic function? If so construct an analytic function  $f(z) = u + iv$ , taking  $v$  as the imaginary part and hence find  $u$ .**

**Solution:**

$$\text{Let } v = \tan^{-1}\left(\frac{y}{x}\right)$$

$$v_x = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(\frac{-y}{x^2}\right) = \frac{-y}{x^2 + y^2}; \quad v_{xx} = - \left( \frac{(x^2 + y^2) \cdot 0 - y(2x)}{(x^2 + y^2)^2} \right) = \frac{2xy}{(x^2 + y^2)^2}$$

$$v_y = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}; \quad v_{yy} = - \left( \frac{(x^2 + y^2) \cdot 0 - x(2y)}{(x^2 + y^2)^2} \right) = \frac{-2xy}{(x^2 + y^2)^2}$$

$v_{xx} + v_{yy} = 0 \Rightarrow v$  is harmonic and hence  $v$  can be the imaginary part of an analytic function.

By Milne's method,  $f(z) = \int \{v_y(z, 0) + i v_x(z, 0)\} dz + c$

$$v_x = \frac{-y}{x^2 + y^2}; \quad v_x(z, 0) = 0;$$

$$v_y = \frac{x}{x^2 + y^2}; \quad v_y(z, 0) = \frac{1}{z}$$

$$f(z) = \int \frac{dz}{z} + c = \log z + c = \log r + i\theta + c_1 + ic_2 \quad (\because z = re^{i\theta})$$

$$= \underbrace{\left(\frac{1}{2} \log(x^2 + y^2) + c_1\right)}_u + i \underbrace{\tan^{-1}\left(\frac{y}{x}\right)}_v \quad \left(\because r = \sqrt{x^2 + y^2} \text{ \& } \theta = \tan^{-1}\left(\frac{y}{x}\right)\right)$$

$(c_2 = 0)$

$$\therefore u = \frac{1}{2} \log(x^2 + y^2) + c_1$$

**Prove that  $u = x^2 - y^2$  &  $v = \frac{-y}{x^2 + y^2}$  are harmonic functions but not harmonic conjugate.**

**Solution:**

$$u = x^2 - y^2$$

$$u_x = 2x$$

$$u_y = -2y$$

$$u_{xx} = 2$$

$$u_{yy} = -2$$

$$\therefore u_{xx} + u_{yy} = 0$$

$$v = \frac{-y}{x^2 + y^2}$$

$$v_x = \frac{2xy}{(x^2 + y^2)^2}$$

$$v_y = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$v_{xx} = \frac{2y(y^2 - 3x^2)}{(x^2 + y^2)^3}$$

$$v_{yy} = \frac{2y(3x^2 - y^2)}{(x^2 + y^2)^3}$$

$$v_{xx} + v_{yy} = 0$$

Hence  $u$  and  $v$  are harmonic.

But  $u_x \neq v_y$  &  $v_x \neq -u_y$

C-R equations are not satisfied. Hence  $u+iv$  are not an analytic function. So they are not harmonic conjugate.

**Prove that  $w = \frac{z}{z+a}$  where  $a \neq 0$  is analytic whereas  $w = \frac{\bar{z}}{\bar{z}+a}$  is not analytic.**

Solution:

$$w = \frac{z}{z+a} = \frac{x+iy}{x+iy+a} = \frac{x+iy}{(x+a)+iy} = \frac{x+iy}{(x+a)+iy} \left( \frac{(x+a)-iy}{(x+a)-iy} \right)$$

$$= \frac{(x+iy)((x+a)-iy)}{(x+a)^2 + y^2} = \frac{x(x+a) + y^2}{(x+a)^2 + y^2} + i \frac{(x+a)y - xy}{(x+a)^2 + y^2}$$

$$w = \underbrace{\frac{x(x+a) + y^2}{(x+a)^2 + y^2}}_u + i \underbrace{\frac{ay}{(x+a)^2 + y^2}}_v$$

$$u = \frac{x(x+a) + y^2}{(x+a)^2 + y^2};$$

$$u_x = \frac{((x+a)^2 + y^2)(2x+a) - (x(x+a) + y^2)(2(x+a))}{((x+a)^2 + y^2)^2}$$

$$= \frac{2x(x+a) + 2xy^2 - 2x^2(x+a) - 2xy^2 - 2ax(x+a) - 2ay^2}{((x+a)^2 + y^2)^2}$$

$$= \frac{(x+a)(2x^2 + 2ax + ax + a^2 - 2x^2 - 2ax) - ay^2}{((x+a)^2 + y^2)^2}$$

$$u_x = \frac{a((x+a)^2 - y^2)}{((x+a)^2 + y^2)^2} \dots (1)$$

$$u_y = \frac{((x+a)^2 + y^2)(2y) - (x(x+a) + y^2)(2y)}{((x+a)^2 + y^2)^2}$$

$$= \frac{2y((x+a)^2 + y^2 - (x(x+a) + y^2))}{((x+a)^2 + y^2)^2}$$

$$= \frac{2y(x^2 + ax + a^2 + y^2 - x^2 - ax - y^2)}{((x+a)^2 + y^2)^2}$$

$$u_y = \frac{2ay(x+a)}{((x+a)^2 + y^2)^2} \dots (2)$$

$$v = \frac{ay}{(x+a)^2 + y^2};$$

$$v_x = \frac{((x+a)^2 + y^2)(0) - (ay)(2(x+a))}{((x+a)^2 + y^2)^2}$$

$$v_x = \frac{-2ay(x+a)}{((x+a)^2 + y^2)^2} \dots (3)$$

$$v_y = \frac{((x+a)^2 + y^2)(a) - (ay)(2y)}{((x+a)^2 + y^2)^2}$$

$$= \frac{a((x+a)^2 + y^2 - 2y^2)}{((x+a)^2 + y^2)^2}$$

$$v_y = \frac{a((x+a)^2 - y^2)}{((x+a)^2 + y^2)^2} \dots (4)$$

From (1) and (4),  $u_x = v_y$

From (2) and (3),  $u_y = -v_x$

Also  $u_x, u_y, v_x, v_y$  are continuous functions in  $x$  and  $y$ .

Hence  $w = \frac{z}{z+a}$  is analytic.

$$\text{Now } w = \frac{\bar{z}}{\bar{z}+a} = \frac{x-iy}{x-iy+a} = \frac{x-iy}{(x+a)-iy} = \frac{x-iy}{(x+a)-iy} \left( \frac{(x+a)+iy}{(x+a)+iy} \right)$$

$$= \frac{(x-iy)((x+a)+iy)}{(x+a)^2 + y^2} = \frac{x(x+a) + y^2}{(x+a)^2 + y^2} + i \frac{-(x+a)y + xy}{(x+a)^2 + y^2}$$

$$w = \underbrace{\frac{x(x+a) + y^2}{(x+a)^2 + y^2}}_u + i \underbrace{\frac{-ay}{(x+a)^2 + y^2}}_v$$

$$u = \frac{x(x+a) + y^2}{(x+a)^2 + y^2};$$

$$u_x = \frac{a((x+a)^2 - y^2)}{((x+a)^2 + y^2)^2} \dots (5)$$

$$u_y = \frac{2ay(x+a)}{((x+a)^2 + y^2)^2} \dots (6)$$

$$v = \frac{-ay}{(x+a)^2 + y^2};$$

$$v_x = \frac{2ay(x+a)}{((x+a)^2 + y^2)^2} \dots (7)$$

$$v_y = \frac{-a((x+a)^2 - y^2)}{((x+a)^2 + y^2)^2} \dots (8)$$

From (5) and (8),  $u_x \neq v_y$

From (6) and (7),  $u_y \neq -v_x$

Hence  $w = \frac{\bar{z}}{\bar{z} + a}$  is not analytic.

### Properties of Analytic function

#### Property : 1

**The function  $f(z) = u + iv$  is analytic, show that  $u = \text{constant}$  and  $v = \text{constant}$  are orthogonal**

**Proof:**

If  $f(z) = u + iv$  is an analytic function of  $z$ , then it satisfies C-R equations

$$u_x = v_y, u_y = -v_x$$

$$\text{Given } u(x, y) = C_1 \dots \dots \dots (1)$$

$$v(x, y) = C_2 \dots \dots \dots (2)$$

By total differentiation

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

Differentiate equation (1) & (2) we get  $du = 0$ ,  $dv = 0$

$$\therefore \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\partial u / \partial x}{\partial u / \partial y} = m_1 (\text{say})$$

$$\frac{dy}{dx} = \frac{-\partial v / \partial x}{\partial v / \partial y} = m_2 (\text{say})$$

$$\therefore m_1 m_2 = -\frac{\partial u / \partial x}{\partial u / \partial y} \times \frac{-\partial v / \partial x}{\partial v / \partial y} \quad (\because u_x = v_y \quad u_y = -v_x)$$

$$\therefore m_1 m_2 = -1$$

The curves  $u(x, y) = C_1$  and  $v(x, y) = C_2$  cut orthogonally.

**Property : 2****Prove that an analytic function with constant modulus is constant.****Proof:**Let  $f(z) = u + iv$  be analytic

By C.R equations satisfied

i.e.,  $u_x = v_y, u_y = -v_x$ 

$$\therefore f(z) = u + iv$$

$$\Rightarrow |f(z)| = \sqrt{u^2 + v^2} = C \Rightarrow |f(z)|^2 = u^2 + v^2 = C^2$$

$$u^2 + v^2 = C^2 \dots\dots\dots(1)$$

Diff (1) with respect to  $x$ 

$$2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} = 0$$

$$uu_x + vv_x = 0 \dots\dots\dots(2)$$

Diff (1) with respect to  $y$ 

$$2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} = 0$$

$$-uv_x + vu_x = 0 \dots\dots\dots(3)$$

$$(2) \times u + (3) \times v \Rightarrow (u^2 + v^2)u_x = 0$$

$$\Rightarrow u_x = 0$$

$$(2) \times v - (3) \times u \Rightarrow (u^2 + v^2)v_x = 0$$

$$\Rightarrow v_x = 0$$

$$\text{W.K.T } f'(z) = u_x + iv_x = 0$$

$$f'(z) = 0 \quad \text{Integrate w.r.to } z$$

$$f(z) = C$$

**Property : 3****8. Prove that the real and imaginary parts of an analytic function are harmonic function.****Proof:**Let  $f(z) = u + iv$  be an analytic function of  $z$ . Then by C- R equations we have,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \dots\dots\dots(1) \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \dots\dots\dots(2)$$

Differentiating (1) partially with respect to  $x$ , we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \dots\dots\dots(3)$$

Differentiating (2) partially with respect to  $y$ , we get



$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} \dots\dots\dots (4)$$

Adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

$\therefore u$  satisfies the Laplace equation.

Similarly

Differentiating (1) partially with respect to  $y$ , we get

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} \dots\dots\dots (5)$$

Differentiating (2) partially with respect to  $x$ , we get

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \dots\dots\dots (6)$$

Adding (5) and (6), we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0$$

$\therefore v$  satisfies the Laplace equation.

Hence the real and imaginary parts of an analytic function are harmonic function.

#### Property : 4

**9. The real part of an analytic function  $f(z)$  is constant, prove that  $f(z)$  is a constant function.**

**Proof:**

Let  $f(z) = u + iv$

Given  $u = \text{constant}$ .  $\Rightarrow u_x = 0$  and  $u_y = 0$

by C-R equations,  $u_x = 0 \Rightarrow v_y = 0$  and  $u_y = 0 \Rightarrow v_x = 0$

$$f'(z) = u_x + iv_x = 0 + i0 = 0$$

Integrating,  $f(z) = c$  (where  $c$  is a constant)

**10. If  $f(z)$  is an analytic function, prove that  $\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right\} |f(z)|^2 = 4|f'(z)|^2$**

**Proof:**

Let  $f(z) = u + iv$  be analytic.

Then  $u_x = v_y$  and  $u_y = -v_x$  (1)

Also  $u_{xx} + u_{yy} = 0$  and  $v_{xx} + v_{yy} = 0$  (2)

Now  $|f(z)|^2 = u^2 + v^2$  and  $f'(z) = u_x + iv_x$

$$\therefore \frac{\partial}{\partial x} |f(z)|^2 = 2u \cdot u_x + 2v \cdot v_x$$

$$\text{and } \frac{\partial^2}{\partial x^2} |f(z)|^2 = 2[u_x^2 + u \cdot u_{xx} + v_x^2 + v \cdot v_{xx}] \quad (3)$$

$$\text{Similarly } \frac{\partial^2}{\partial y^2} |f(z)|^2 = 2[u_y^2 + u \cdot u_{yy} + v_y^2 + v \cdot v_{yy}] \quad (4)$$

Adding (3) and (4)

$$\begin{aligned} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 &= 2[u_x^2 + u_y^2 + u(u_{xx} + u_{yy}) + v_x^2 + v_y^2 + v(v_{xx} + v_{yy})] \\ &= 2[u_x^2 + v_x^2 + u(0) + v_x^2 + u_x^2 + v(0)] \\ &= 4[u_x^2 + v_x^2] \end{aligned}$$

**11. Find the map of the circle (i)  $|z| = 3$  under the transformation  $w = 2z$**

**(ii)  $|z| = 1$  by the transformation  $w = z + 2 + 4i$**

**Solution (i) :** Given  $w = 2z$ ,  $|z| = 3$

$$|w| = 2|z|$$

$$|w| = 2(3) = 6$$

Hence the image of the circle  $|z| = 3$  in the  $z$ -plane maps to the circle  $|w| = 6$  in the  $w$ -plane.

**Solution (ii) :**

$$\text{Given: } w = z + 2 + 4i$$

$$u + iv = x + iy + 2 + 4i = (x + 2) + i(y + 4)$$

$$u = x + 2, \quad v = y + 4$$

$$\Rightarrow x = u - 2, \quad y = v - 4$$

$$\Rightarrow |z| = 1$$

$$x^2 + y^2 = 1 \quad \text{Hence } (u - 2)^2 + (v - 4)^2 = 1.$$

$\therefore$  The circle in the  $z$ -plane is mapped into the circle in the  $w$ -plane with centre (2, 4) and radius 1.

**Find the image of the infinite strip  $\frac{1}{4} < y < \frac{1}{2}$  under the transformation  $w = \frac{1}{z}$**

**Solution:**

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$z = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2} \Rightarrow x = \frac{u}{u^2+v^2} \quad \text{---(1)} \quad y = -\frac{v}{u^2+v^2} \quad \text{---(2)}$$

Given strip is  $\frac{1}{4} < y < \frac{1}{2}$  when  $y = \frac{1}{4}$

$$\frac{1}{4} = -\frac{v}{u^2+v^2} \quad (\text{by 2})$$

$$u^2 + (v+2)^2 = 4 \dots\dots\dots(3)$$

which is a circle whose centre is at  $(0, -2)$  in the  $w$ -plane and radius 2.

$$\text{When } y = \frac{1}{2}$$

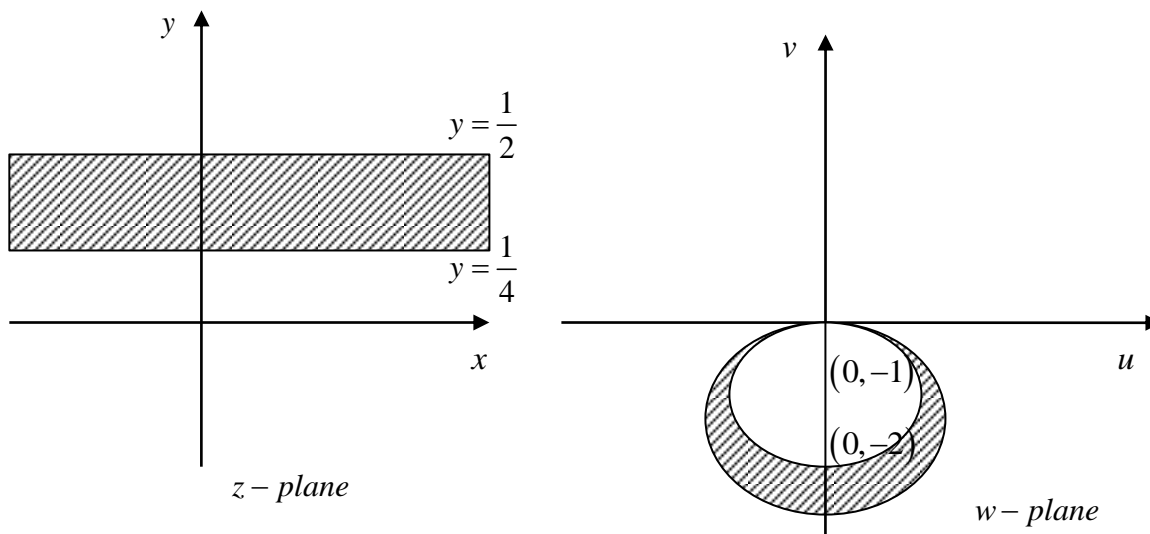
$$\frac{1}{2} = \frac{-v}{u^2+v^2} \quad (\text{by 2})$$

$$u^2 + v^2 + 2v = 0$$

$$u^2 + (v+1)^2 = 1 \dots\dots\dots(4)$$

which is a circle whose centre is at  $(0, -1)$  and radius is 1 in the  $w$ -plane.

Hence the infinite strip  $\frac{1}{4} < y < \frac{1}{2}$  is transformed into the region between circles  $u^2 + (v+1)^2 = 1$  and  $u^2 + (v+2)^2 = 4$  in the  $w$ -plane.



**Find the image of  $|z - 2i| = 2$  under the transformation  $w = \frac{1}{z}$**

**Solution:**

$$\text{Given } w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$\text{Now } w = u + iv$$

$$z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$\text{i.e., } x + iy = \frac{u-iv}{u^2+v^2} \therefore x = \frac{u}{u^2+v^2} \dots\dots\dots(1) \quad y = \frac{-v}{u^2+v^2} \dots\dots\dots(2)$$

Given  $|z - 2i| = 2$

$$|x + iy - 2i| = 2 \Rightarrow |x + i(y - 2)| = 2$$

$$x^2 + (y - 2)^2 = 4 \Rightarrow x^2 + y^2 - 4y = 0 \dots\dots\dots (3)$$

Sub (1) and (2) in (3)

$$\left(\frac{u}{u^2 + v^2}\right)^2 + \left(\frac{-v}{u^2 + v^2}\right)^2 - 4\left[\frac{-v}{u^2 + v^2}\right] = 0$$

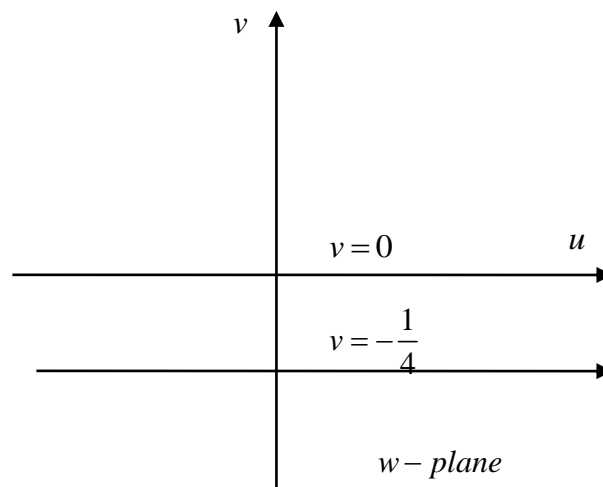
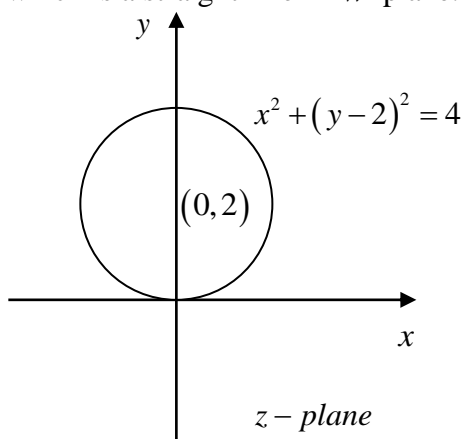
$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + \left[\frac{4v}{u^2 + v^2}\right] = 0$$

$$\frac{(u^2 + v^2) + 4v(u^2 + v^2)}{(u^2 + v^2)^2} = 0$$

$$\frac{(1 + 4v)(u^2 + v^2)}{(u^2 + v^2)^2} = 0$$

$$1 + 4v = 0 \Rightarrow v = -\frac{1}{4} \quad (\because u^2 + v^2 \neq 0)$$

which is a straight line in  $w$ -plane.



**Show that the transformation  $w = \frac{1}{z}$  transforms in general, circles and straight lines into circles and straight lines.**

**Solution:**

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$\Rightarrow x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

$$\Rightarrow x = \frac{u}{u^2 + v^2} \text{ and } y = \frac{-v}{u^2 + v^2}$$

Consider the equation  $a(x^2 + y^2) + bx + cy + d = 0$  -----(1)

This equation represents a circle if  $a \neq 0$  and a straight line if  $a = 0$

Under the transformation  $w = \frac{1}{z}$  equation (1) becomes

$$d(u^2 + v^2) + bu - cv + a = 0 \text{ -----(2)}$$

This equation represents a circle if  $d \neq 0$  and a straight line if  $d = 0$

Value of a & d	Equation (1) and (2)	Conclusion
$a \neq 0, d \neq 0$	Equation (1) and (2) represents a circle, not passing through the origin, in the z-plane and w-plane	The transformation maps a circle not passing through the origin in z-plane into a circle not passing through the origin in w-plane
$a \neq 0, d = 0$	Equation (1) represents a circle passing through the origin in the z-plane and equation (2) represents a straight line not passing through the origin in w-plane	The transformation maps a circle passing through the origin in z-plane into a straight line not passing through the origin in w-plane
$a = 0, d \neq 0$	Equation (1) represents a straight line not passing through the origin in the z-plane and equation (2) represents a circle passing through the origin in w-plane	The transformation maps a straight line not passing through the origin in the z-plane into a circle passing through the origin in w-plane
$a = 0, d = 0$	Equation (1) and (2) represents a straight line passing through the origin in the z-plane and w-plane	The transformation maps represents a straight line passing through the origin in z-plane into a straight line passing through the origin in w-plane

Thus the transformation  $w = \frac{1}{z}$  maps the totality of circles and straight lines as circles or straight lines.

**Find the image of the circle  $|z - 1| = 1$  under the transformation  $w = z^2$**

**Solution:**

In polar form  $z = r e^{i\theta}$ ,  $w = R e^{i\phi}$

Given

$$|z-1|=1$$

$$|re^{i\theta}-1|=1$$

$$|r \cos \theta + i r \sin \theta - 1| = 1$$

$$|(r \cos \theta - 1) + i r \sin \theta| = 1$$

$$(r \cos \theta - 1)^2 + (r \sin \theta)^2 = 1^2$$

$$r^2 - 2r \cos \theta = 0$$

$$r = 2 \cos \theta \text{ --- (1)}$$

Now, we have

$$w = z^2$$

$$R e^{i\phi} = (r e^{i\theta})^2$$

$$R e^{i\phi} = r^2 e^{i2\theta}$$

$$R = r^2, \quad \phi = 2\theta$$

$$(1) \Rightarrow$$

$$r^2 = (2 \cos \theta)^2$$

$$= 4 \cos^2 \theta$$

$$= 4 \left[ \frac{1 + \cos 2\theta}{2} \right]$$

$$r^2 = 2(1 + \cos 2\theta)$$

$$R = 2(1 + \cos \phi)$$

**Find the bilinear transformation of the points  $-1, 0, 1$  in  $z$ - plane onto the points  $0, i, 3i$  in  $w$ - plane.**

**Solution:**

$$\text{Given } z_1 = -1, w_1 = 0$$

$$z_2 = 0, w_2 = i$$

$$z_3 = 1, w_3 = 3i$$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(i-3i)}{(w-3i)(i-0)} = \frac{(z-(-1))(0-1)}{(z-1)(0-(-1))}$$

$$\frac{w(-2i)}{(w-3i)(i)} = \frac{(z+1)(-1)}{(z-1)(1)}$$

$$\frac{2w}{w-3i} = \frac{z+1}{z-1}$$

$$2wz - 2w = wz + w - 3iz - 3i$$

$$w(2z - 2 - z - 1) = -3i(z + 1)$$

$$w(z - 3) = -3i(z + 1)$$

$$\therefore w = -3i \frac{(z+1)}{(z-3)}$$

**Find the bilinear transformation which maps the points  $z = \infty, i, 0$  into  $w = 0, i, \infty$  respectively.**

**Solution:**

Given  $z_1 = \infty, w_1 = 0$

$z_2 = i, w_2 = i$

$z_3 = 0, w_3 = \infty$

Cross-ratio

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$\frac{(w - w_1)w_3 \left( \frac{w_2}{w_3} - 1 \right)}{w_3 \left( \frac{w}{w_3} - 1 \right) (w_2 - w_1)} = \frac{z_1 \left( \frac{z}{z_1} - 1 \right) (z_2 - z_3)}{(z - z_3) z_1 \left( \frac{z_2}{z_1} - 1 \right)}$$

$$\frac{(w - w_1) \left( \frac{w_2}{w_3} - 1 \right)}{\left( \frac{w}{w_3} - 1 \right) (w_2 - w_1)} = \frac{\left( \frac{z}{z_1} - 1 \right) (z_2 - z_3)}{(z - z_3) \left( \frac{z_2}{z_1} - 1 \right)}$$

$$\frac{(w - 0)(0 - 1)}{(0 - 1)(i - 0)} = \frac{(0 - 1)(i - 0)}{(z - 0)(0 - 1)}$$

$$\frac{w}{i} = \frac{i}{z}, \quad w = \frac{i^2}{z}, \quad \therefore w = -\frac{1}{z}$$

**Find the bilinear transformation which maps the points  $z = 1, i, -1$  into the points**

**$w = 0, 1, \infty$ .**

**Solution:**

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

Here,  $w_3 = \infty$

$$\frac{(w-w_1)w_3\left(\frac{w_2}{w_3}-1\right)}{(w_1-w_2)w_3\left(1-\frac{w}{w_3}\right)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{\infty}-1\right)}{(w_1-w_2)\left(1-\frac{w}{\infty}\right)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-w_1)(-1)}{(w_1-w_2)(1)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-w_1)}{(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)}{(1-0)} = \frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

$$w = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$w = \frac{(z-1)(i+1)}{(z+1)(i-1)} \cdot \frac{(i+1)}{(i+1)} = i \frac{(z-1)}{(z+1)}$$

**Find the bilinear transformation which maps the points  $0, 1, \infty$  in  $z$ -plane into itself in  $w$ -plane.**

**Solution:**

Given  $z_1=0, w_1=0, z_2=1, w_2=1, z_3=\infty, w_3=\infty$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)w_3\left(\frac{w_2}{w_3}-1\right)}{w_3\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)z_3\left(\frac{z_2}{z_3}-1\right)}{z_3\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)\left(\frac{z_2}{z_3}-1\right)}{\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w-0)(0-1)}{(0-1)(1-0)} = \frac{(z-0)(0-1)}{(0-1)(1-0)}$$

$$w = z$$



**Find the bilinear transformation which maps the points  $z=1, i, -1$  into the points  $w= i, 0, -i$ . Hence find the image of  $|z| < 1$**

**Solution:**

**We know that**

$$\begin{aligned} \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} &= \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \\ \Rightarrow \frac{(w-i)(0+i)}{(i-0)(-i-w)} &= \frac{(z-1)(i+1)}{(1-i)(-1-z)} \\ \Rightarrow \frac{(w-i)(i)}{-(i)(w+i)} &= \frac{(z-1)(1+i)}{-(1-i)(z+1)} \\ \Rightarrow \frac{(w-i)}{(w+i)} &= \frac{(z-1)(1+i)}{(z+1)(1-i)} \times \frac{(1+i)}{(1+i)} \\ \Rightarrow \frac{(w-i)}{(w+i)} &= \frac{(z-1)}{(z+1)} \times \frac{(1-1+2i)}{(1+1)} \\ \Rightarrow \frac{(w-i)}{(w+i)} &= \frac{(z-1)}{(z+1)} \times \frac{(2i)}{(2)} \\ \Rightarrow \frac{(w-i)}{(w+i)} &= \frac{i(z-1)}{(z+1)} \dots\dots\dots(1) \\ \Rightarrow \frac{(w-i)}{(w+i)} &= \frac{iz-i}{z+1} \end{aligned}$$

Applying componendo and dividendo rule, we get

$$\begin{aligned} \Rightarrow \frac{w-i+w+i}{w-i-w-i} &= \frac{iz-i+z+1}{iz-i-z-1} \Rightarrow \frac{2w}{-2i} = \frac{iz-i+z+1}{iz-i-z-1} \\ \Rightarrow \boxed{w = -i \left[ \frac{(1+i)z+1-i}{(i-1)z-1-i} \right]} &= \frac{(1-i)z-1-i}{(i-1)z-1-i} \end{aligned}$$

**To find the image of  $|z| < 1$**

From (1),

$$\frac{(z-1)}{(z+1)} = -i \frac{(w-i)}{(w+i)} = \frac{-iw-1}{w+i}$$

Applying Componendo and dividendo rule, we get

$$\frac{z-1+z+1}{z-1-z-1} = \frac{-iw-1+w+i}{-iw-1-w-i}$$

$$\Rightarrow \frac{2z}{-2} = \frac{(1-i)w+i-1}{(-1-i)w-1-i}$$

$$\Rightarrow \boxed{z = \frac{(1-i)w+i-1}{(1+i)w+1+i}}$$

$$\text{Now } |z| < 1 \Rightarrow \left| \frac{(1-i)w+i-1}{(1+i)w+1+i} \right| < 1$$

$$\Rightarrow |(1-i)w+i-1| < |(1+i)w+1+i|$$

$$\Rightarrow |(1-i)(u+iv)+i-1| < |(1+i)(u+iv)+1+i|$$

$$\Rightarrow |u+iv-iu+v+i-1| < |u+iv+iu-v+1+i|$$

$$\Rightarrow |u+v-1+i(1-u+v)| < |u-v+1+i(1+u+v)|$$

$$\Rightarrow \sqrt{(u+v-1)^2 + (1-u+v)^2} < \sqrt{(u-v+1)^2 + (1+u+v)^2}$$

$$\Rightarrow u^2 + v^2 + 1 - 2u - 2v + 2uv + 1 + u^2 + v^2 - 2u + 2v - 2uv < \\ u^2 + v^2 + 1 + 2u - 2v - 2uv + 1 + u^2 + v^2 + 2u + 2v + 2uv$$

$$\Rightarrow -2u - 2u < 2u + 2u$$

$$\Rightarrow -4u < 4u$$

$$\Rightarrow -8u < 0$$

$$\Rightarrow \boxed{u > 0}$$

$\therefore$  the image of  $|z| < 1$  in  $z$ -plane is right half of  $w$ -plane  $u > 0$ .

**21. Prove that  $w = \frac{z}{1-z}$  maps the upper half of the  $z$ -plane into the upper half of the  $w$ -plane.**

**What is the image of the circle  $|z| = 1$  under this transformation?**

**Solution:**

$$w = \frac{z}{1-z} \Rightarrow w(1-z) = z$$

$$w - wz = z$$

$$w = (1+w)z$$

$$z = \frac{w}{1+w} \dots\dots\dots(1)$$

$$\text{put } z = x+iy, \quad w = u+iv$$

$$\begin{aligned}
 x+iy &= \frac{u+iv}{1+u+iv} = \frac{(u+iv)}{(1+u)+iv} \frac{(1+u)-iv}{(1+u)-iv} \\
 &= \frac{u(1+u)-iuv+iv(1+u)+v^2}{(1+u)^2+v^2} \\
 &= \frac{(u+u^2+v^2)+iv}{(1+u)^2+v^2}
 \end{aligned}$$

Equating real and imaginary parts

$$x = \frac{(u+u^2+v^2)}{(1+u)^2+v^2}, \quad y = \frac{v}{(1+u)^2+v^2}$$

$$y=0 \Rightarrow \frac{v}{(1+u)^2+v^2} = 0$$

$$y>0 \Rightarrow \frac{v}{(1+u)^2+v^2} > 0$$

$$\Rightarrow v > 0$$

Thus the upper half of the  $z$ - plane is mapped onto the upper half of the  $w$ - plane.

Image of  $|z|=1$ :

by (1)

$$\begin{aligned}
 |z|=1 &\Rightarrow \left| \frac{w}{1+w} \right| = 1 \\
 &\Rightarrow \frac{|w|}{|1+w|} = 1 \\
 &\Rightarrow |w| = |1+w| \\
 &\Rightarrow |u+iv| = |1+u+iv| \\
 &\Rightarrow \sqrt{u^2+v^2} = \sqrt{(1+u)^2+v^2} \\
 &\Rightarrow u^2+v^2 = (1+u)^2+v^2 \\
 &\Rightarrow u^2 = 1+u^2+2u \\
 &\Rightarrow 2u+1=0 \\
 &\Rightarrow u = -\frac{1}{2}
 \end{aligned}$$

**22. Determine the analytic function whose real part is  $\frac{\sin 2x}{\cosh 2y - \cos 2x}$**

**Solution:**

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$\phi_1(x, y) = \frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2 \cos 2x) - \sin 2x(2 \sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\begin{aligned}\phi_1(z, 0) &= \frac{(1 - \cos 2z)(2 \cos 2z) - 2 \sin^2 2z}{(1 - \cos 2z)^2} \\ &= \frac{(1 - \cos 2z)(2 \cos 2z) - 2(1 - \cos^2 2z)}{(1 - \cos 2z)^2} \\ &= \frac{(1 - \cos 2z)(2 \cos 2z) - 2(1 - \cos 2z)(1 + \cos 2z)}{(1 - \cos 2z)^2} \\ &= \frac{-2}{1 - \cos 2z} = -\frac{1}{\sin^2 z} = -\operatorname{cosec}^2 z\end{aligned}$$

$$\phi_2(x, y) = \frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x(2 \sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(z, 0) = 0$$

By Milne's Thomson method,

$$\begin{aligned}f(z) &= \int \phi_1(z, 0) dz - i \int \phi_2(z, 0) dz \\ &= \int -\operatorname{cosec}^2 z dz - i0 \\ &= \cot z + c\end{aligned}$$

**23. If  $f(z) = u + iv$  is an analytic function and  $u - v = e^x(\cos y - \sin y)$  find  $f(z)$  in terms of  $z$** **Solution:**

$$f(z) = u + iv \quad (1)$$

$$if(z) = iu - v \quad (2)$$

$$\therefore (1+i)f(z) = (u-v) + i(u+v)$$

$$F(z) = U + iV, \quad \text{where } F(z) = (1+i)f(z), \quad U = u-v, \quad V = u+v$$

$$\therefore U = u-v = e^x(\cos y - \sin y)$$

$$\phi_1(x, y) = \frac{\partial U}{\partial x} = e^x[\cos y - \sin y]$$

$$\phi_1(z, 0) = e^z$$

$$\phi_2(x, y) = \frac{\partial U}{\partial y} = e^x[-\sin y - \cos y]$$

$$\phi_2(z, 0) = e^z(-1) = -e^z$$

By Milne's Thomson Method

$$F(z) = \int \phi_1(z, 0) dz - i \int \phi_2(z, 0) dz$$

$$= \int e^z dz - i \int -e^z dz = e^z + i e^z$$

$$= (1+i)e^z$$

$$(1+i)f(z) = (1+i)e^z + C_1$$

$$f(z) = e^z + C$$

**24. Find the regular function whose imaginary part is  $e^{-x}(x \cos y + y \sin y)$**

**Solution:**

$$v = e^{-x}(x \cos y + y \sin y)$$

$$\phi_2(x, y) = \frac{\partial v}{\partial x} = e^{-x}[\cos y] + (x \cos y + y \sin y)(-e^{-x})$$

$$\phi_2(z, 0) = e^{-z} + (z)(-e^{-z}) = e^{-z} - ze^{-z} = e^{-z}(1-z)$$

$$\phi_1(x, y) = \frac{\partial u}{\partial y} = e^{-x}[-x \sin y + y \cos y + \sin y(1)]$$

$$\phi_1(z, 0) = e^{-z}[0+0+0] = 0$$

By Milne's Thomson Method

$$f(z) = \int \phi_1(z, 0) dz + i \int \phi_2(z, 0) dz$$

$$= \int 0 dz + i \int (1-z)e^{-z} dz$$

$$= i \left[ (1-z) \left[ \frac{e^{-z}}{-1} \right] - (-1) \left[ \frac{e^{-z}}{(-1)^2} \right] \right] + C$$

$$= i \left[ -(1-z)e^{-z} + e^{-z} \right] + C$$

$$= i \left[ -e^{-z} + ze^{-z} + e^{-z} \right] + C = i \left[ ze^{-z} \right] + C$$

**25. Determine the analytic function  $w = u + iv$  if  $u = e^{2x}(x \cos 2y - y \sin 2y)$ .**

**Solution:**

$$\text{Given } u = e^{2x}[x \cos 2y - y \sin 2y]$$

$$\phi_1(x, y) = \frac{\partial u}{\partial x} = e^{2x} \cos 2y + [x \cos 2y - y \sin 2y] 2e^{2x}$$

$$\therefore \phi_1(z, 0) = e^{2z} + 2ze^{2z} \dots \dots \dots (1)$$

$$\phi_2(x, y) = \frac{\partial u}{\partial y} = e^{2x}[-x2 \sin 2y - 2y \cos 2y - \sin 2y]$$

$$\therefore \phi_2(z, 0) = 0 \dots \dots \dots (2)$$

By Milne Thomson method

$$F'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

From (1) & (2)

$$\begin{aligned}\int F'(z) dz &= \int (e^{2z} + 2ze^{2z}) dz \\ &= \frac{e^{2z}}{2} + 2 \left[ z \frac{e^{2z}}{2} - (1) \frac{e^{2z}}{4} \right] \\ &= \frac{e^{2z}}{2} + ze^{2z} - \frac{e^{2z}}{2}, \quad \therefore F(z) = ze^{2z}\end{aligned}$$

**26. Construct the analytic function  $f(z) = u + iv$  given that  $2u + 3v = e^x (\cos y - \sin y)$ .**

**Solution:**

$$2u + 3v = e^x [\cos y - \sin y]$$

$$f(z) = u + iv \dots \dots \dots (1)$$

$$3if(z) = 3iu - 3v \dots \dots \dots (2)$$

$$(1) \times 2 \Rightarrow 2f(z) = 2u + i2v \dots \dots \dots (3)$$

$$(3) - (2) \Rightarrow (2 - 3i)f(z) = (2u + 3v) + i(2v - 3u) \dots \dots \dots (4)$$

$$F(z) = U + iV$$

$$\therefore 2u + 3v = U = e^x [\cos y - \sin y]$$

$$\phi_1(x, y) = \frac{\partial U}{\partial x} = e^x \cos y - e^x \sin y$$

$$\phi_1(z, 0) = e^z$$

$$\phi_2(x, y) = \frac{\partial U}{\partial y} = -e^x \sin y - e^x \cos y$$

$$\phi_2(z, 0) = -e^z$$

By Milne Thomson method

$$F'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$\int F'(z) dz = \int e^z dz - i \int -e^z dz$$

$$F(z) = (1 + i)e^z + C \dots \dots \dots (5)$$

From (4) & (5)

$$(1+i)e^z + C = (2-3i)f(z)$$

$$f(z) = \frac{1+i}{2-3i}e^z + \frac{C}{2-3i}$$

$$f(z) = \frac{-1+5i}{13}e^z + \frac{C}{2-3i}$$