1. Consider a stable LTI system that is characterized by the differential equation,

$$\frac{d^2y(t)}{dt^2} + 4\frac{d}{dt}y(t) + 3y(t) = \frac{d}{dt}x(t) + 2x(t).$$

(a) Determine the frequency response of the system.
(b) Determine the Impulse response of the system.
(c) If $x(t) = e^{-t}.u(t)$. Compute the output $y(t)$.

Solution:-

(a) $$\frac{d^2y(t)}{dt^2} + 4\frac{d}{dt}y(t) + 3y(t) = \frac{d}{dt}x(t) + 2x(t).$$

Taking Fourier Transform on both sides, we get.

$$F\left(\frac{d^2y(t)}{dt^2}\right) + 4F\left(\frac{d}{dt}y(t)\right) + 3.F\left[y(t)\right]$$

$$= F\left[d/dt\, x(t)\right] + 2.F\left[x(t)\right]$$

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$$(j\omega)^2 \cdot Y(j\omega) + 4.(j\omega) \cdot Y(j\omega) + 3 \cdot Y(j\omega)$$
$$= j\omega \cdot X[j\omega] + 2 \cdot X[j\omega].$$

$$Y[j\omega]\left[(j\omega)^2 + 4(j\omega) + 3\right] = X[j\omega](j\omega+2).$$

$$\frac{Y[j\omega]}{X[j\omega]} = \frac{j\omega+2}{(j\omega)^2 + 4(j\omega) + 3} \Rightarrow H[j\omega]$$

$$= \frac{j\omega+2}{(j\omega+1)(j\omega+3)}$$

(b)

$$\Rightarrow \frac{A}{j\omega+1} + \frac{B}{j\omega+3} = \frac{j\omega+2}{(j\omega+1)(j\omega+3)}.$$

$$A(j\omega+3) + B(j\omega+1) = (j\omega+2)$$

Let $j\omega = -3$

$$0 + B(-3+1) = (-3+2)$$

$$\boxed{B = 1/2}.$$

Let $j\omega = -1$.

$$A(-1+3) + 0 = -1+2.$$

$$\boxed{A = 1/2}.$$

$$\therefore H[j\omega] = \frac{1}{2(j\omega+1)} + \frac{1}{2(j\omega+3)}.$$

Taking Inverse F.T.

$$h(t) = \frac{1}{2} e^{-t} u(t) + \frac{1}{2} e^{-3t} u(t).$$

$$\therefore h(t) = \frac{1}{2}\left[e^{-t} + e^{-3t}\right] u(t)$$

(c). $x(t) = e^{-t} u(t).$

$$h(t) = \frac{1}{2} e^{-t} u(t) + \frac{1}{2} e^{-3t} u(t).$$

$$y(t) = x(t) * h(t).$$

$$F[y(t)] = F[x(t) * h(t)].$$

$$Y(j\omega) = X(j\omega) \cdot H(j\omega)$$

w.r.t $X(j\omega) = \dfrac{1}{1+j\omega}.$

& $H(j\omega) = \dfrac{1}{2(j\omega+1)} + \dfrac{1}{2(j\omega+3)}.$

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$$\therefore Y(j\omega) = \frac{1}{(1+j\omega)}\left[\frac{1}{2(j\omega+1)} + \frac{1}{2(j\omega+3)}\right].$$

$$= \frac{1}{2(1+j\omega)^2} + \frac{1}{2(1+j\omega)(3+j\omega)}.$$

$$= \frac{(3+j\omega) + (1+j\omega).}{2(1+j\omega)^2(3+j\omega)} = \frac{4+2j\omega}{2(1+j\omega)^2(3+j\omega)}$$

$$Y(j\omega) = \frac{2+j\omega}{(1+j\omega)^2(3+j\omega)}.$$

$$Y(j\omega) = \frac{A}{1+j\omega} + \frac{B}{(1+j\omega)^2} + \frac{C}{3+j\omega} = \frac{j\omega+2}{(1+j\omega)^2(3+j\omega)}$$

$$\Rightarrow A(1+j\omega)(3+j\omega) + B(3+j\omega) + C(1+j\omega)^2 = (j\omega+2)$$

Let $j\omega = -3$

$$\Rightarrow 0+0+C[1-3]^2 = (-3+2)$$

$$= \boxed{C = -1/4}$$

Let $j\omega = -1.$

$$0 + B(3-1) + 0 = -1+2.$$

$$\boxed{B = 1/2}$$

Equating coefficients of $(j\omega)^2$.

$$A + C = 0.$$

$$A - 1/4 = 0 \quad \Rightarrow \boxed{A = 1/4}.$$

$$Y(j\omega) = \frac{1/4}{(1+j\omega)} + \frac{1/2}{(1+j\omega)^2} + \frac{(-1/4)}{(3+j\omega)}.$$

$$\boxed{\therefore y(t) = \left[\tfrac{1}{4} e^{-t} + \tfrac{1}{2} t e^{-t} - \tfrac{1}{4} e^{-3t}\right] u(t)}$$

$\boxed{2}$ For the cascade system shown, if

$$h_1(n) = 2\left(-\tfrac{1}{2}\right)^n u(n) - \left(-\tfrac{1}{2}\right)^{n-1} u(n-1) \ \&$$

$$H_2(e^{j\omega}) = \frac{2}{\left(1 - e^{-j\omega} + \tfrac{1}{4} e^{-j2\omega}\right)}$$

Find the overall impulse response, $h(n)$ of the system,

$$x(n) \longrightarrow \boxed{h_1(n)} \longrightarrow \boxed{h_2(n)} \longrightarrow y(n)$$

Solution :-

$$H_1(e^{j\omega}) = \left.\frac{2-z^{-1}}{1+\frac{1}{2}z^{-1}}\right|_{z=e^{j\omega}}$$

$$= \frac{2-e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}$$

$$H(e^{j\omega}) = H_1(e^{j\omega}) \cdot H_2(e^{j\omega})$$

$$= \left(\frac{2-e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}\right)\left(\frac{2}{1-e^{-j\omega}+\frac{1}{4}e^{-j2\omega}}\right)$$

$$= 2\times 2\left(\frac{1-\frac{1}{2}e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}\right)\left(\frac{1}{(1-\frac{1}{2}e^{-j\omega})^2}\right)$$

$$= 4 \cdot \frac{1}{(1+\frac{1}{2}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})}$$

$$= 4\left(\frac{\frac{1}{2}}{1+\frac{1}{2}e^{-j\omega}} + \frac{\frac{1}{2}}{1-\frac{1}{2}e^{-j\omega}}\right)$$ By partial fraction method.

$$\boxed{h(n) = 2\left[\left(-\tfrac{1}{2}\right)^n + \left(\tfrac{1}{2}\right)^n\right]u(n)}$$

(3). For a system excited by $x(t) = e^{-3t}u(t)$, the impulse response is.,

$$h(t) = e^{-2t}u(t) + e^{2t}u(-t)$$. Find the output for the system.

<u>Solution</u> :-

Input $x(t) = e^{-3t}u(t)$.

$$X(s) = \frac{1}{s+3}.$$

Impulse response.

$$h(t) = e^{-2t}u(t) + e^{2t}u(-t)$$

$e^{2t}u(t) \Rightarrow \frac{1}{s-2}$

$$H(s) = \frac{1}{s+2} + \frac{1}{-s+2}.$$

$$H(j\omega) = \frac{1}{j\omega+2} + \frac{1}{-j\omega+2} = \frac{4}{(j\omega+2)(-j\omega+2)}$$

The output $y(t)$ is Convolution of $x(t)$ & $h(t)$.

$$Y(j\omega) = X(j\omega) \cdot H(j\omega).$$

$$= \left(\frac{1}{j\omega+3}\right)\left[\frac{4}{(j\omega+2)(-j\omega+2)}\right]$$

$$\therefore Y(j\omega) = \frac{4}{(j\omega+3)(j\omega+2)(-j\omega+2)}$$

Taking Partial fractions.

$$Y(j\omega) = \frac{A}{j\omega+3} + \frac{B}{j\omega+2} + \frac{c}{-j\omega+2}$$

$$A = (j\omega+3)\left[Y(j\omega)\right]\Big|_{j\omega=-3}$$

$$= \frac{4}{(j\omega+2)(-j\omega+2)}\Big|_{j\omega=-3} \Rightarrow \frac{-4}{5}.$$

$$B = (j\omega+2)\left[Y(j\omega)\right]\Big|_{j\omega=-2}.$$

$$= \frac{4}{(j\omega+3)(-j\omega+2)}\Big|_{j\omega=-2} = 1.$$

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$$C = (-j\omega+2)\, Y(j\omega)\Big|_{j\omega=2} = \frac{4}{(j\omega+3)(j\omega+2)}\Bigg|_{j\omega=2}$$

$$\Rightarrow 1/5.$$

$$\therefore Y[j\omega] = \frac{-4}{5}\frac{1}{(j\omega+3)} + \frac{1}{(j\omega+2)} + \frac{1}{5}\frac{1}{(-j\omega+2)}$$

Taking Inverse Fourier Transform.

$$y(t) = IFT[Y(j\omega)] =$$

$$\boxed{y(t) = \frac{-4}{5}e^{-3t}u(t) + e^{-2t}u(t) + \frac{1}{5}e^{2t}u(-t)}$$

(4) Consider a stable LTI system characterized by the differential equation

$$\frac{d}{dt} y(t) + 2\,y(t) = x(t).$$

Find its impulse response.

<u>Solution</u> :-

Taking Fourier Transform,

$$j\omega \left[Y(j\omega)\right] + 2\left[Y(j\omega)\right] = X(j\omega).$$

$-y(0)$

neglecting initial value conditions.

$$Y(j\omega)\left[j\omega+2\right] = X(j\omega)$$

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{1}{j\omega+2}$$

$H(j\omega)$

$$\boxed{H(j\omega) = \frac{1}{j\omega+2}.}$$

The Impulse response of the system,

$$h(t) = \mathrm{IFT}\left[H(j\omega)\right]$$

$$\boxed{h(t) = e^{-2t}u(t)}$$

5. Consider a Causal LTI System with frequency response $H(j\omega) = \frac{1}{4+j\omega}$. For a particular input $x(t)$, the system is observed to produce the output,

$$y(t) = e^{-2t}u(t) - e^{-4t}u(t).$$

Find the input $x(t)$.

<u>Solution</u>:-

$$Y(j\omega) = F[y(t)]$$

$$= \frac{1}{j\omega+2} - \frac{1}{j\omega+4}$$

$$= \frac{2}{(j\omega+2)(j\omega+4)}.$$

Input $X(\omega) = \frac{Y(\omega)}{H(\omega)} \Rightarrow \frac{2/(j\omega+2)(j\omega+4)}{1/(4+j\omega)}$

$$= \frac{2}{j\omega+2}.$$

Taking IFT we get $x(t)$

Input

$\Rightarrow x(t) = F^{-1}\left[X(\omega)\right]$

$= F^{-1}\left[\frac{2}{j\omega+2}\right] = 2e^{-2t}u(t)$.

$$\boxed{x(t) = 2e^{-2t}u(t)}$$