UNIT-3

TWO-DIMENSIONAL RANDOM VARIABLES

Defn:- let S be the sample space. let $X=X(S)$ & $Y=Y(S)$ be two functions each assigning a real numbers to each outcome $s \in S$ then (X,Y) is a two dimensional random variables.

Two types of two Dimensional R.V are.

(i) Discrete R.V
ii) Continuous R.V

T.D Discrete Random Variable:-

If the possible values of (X,Y) are finite or countably infinite, then (X,Y) is called a two dimensional discrete random variable

TD continuous Random Variable:-

If (X,Y) can assumes all values in a specified Region R in xy-Plane then (X,Y) is called a Two dimensional continuous random Variable.

Joint Probability function (or) Joint Probability mass fn:-

If (X,Y) be a TDDRV such that $P(X=x_i, y=y_j) = P(x_i, y_j) = P_{ij}$ is called the Joint probability fn. (or) joint probability mass fn.

If (i) $P_{ij} \geq 0 \quad \forall i \,\&\, j$

(ii) $\sum_j \sum_i P_{ij} = 1$

Joint Probability density fn:-

If (X,Y) be a TD continuous Random Variable then $f(x,y)$ is called the joint probability density fn. of (X,Y).

If (i) $f(x,y) \geq 0 \quad \forall (x,y) \in R$, R is the region.

(ii) $\iint_R f(x,y)\,dx\,dy = 1$. (or) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$ (or) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$

Joint cumulative distribution fn.:-

In discrete case:-

$$F(x,y) = P[X \leq x, Y \leq y] = \sum_{j=1}^{m}\sum_{i=1}^{n} P(x_i, y_j)$$

In continuous case:-

$$F(x,y) = P[X \leq x, Y \leq y] = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dx\,dy.$$

Marginal Probability function (or) Marginal distribution:-

(i) In discrete case:-

* $P[X = x_i] = \sum_{j=1}^{m} P(x_i, y_j) = P_{i*}$ → marginal probability fn. of X.

* $P[Y = y_j] = \sum_{i=1}^{n} P(x_i, y_j) = P_{*j}$ → marginal probability fn. of Y.

(ii) In continuous case:-

* Marginal prob. density fn. of X is

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

* Marginal prob. density fn. of y is

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$$

Conditional distribution:-

In discrete case:-

* Conditional probability fn. X given $Y = y_j$ is

$$P[X = x_i / Y = y_j] = \frac{P[X = x_i, Y = y_j]}{P[Y = y_j]}$$

* Conditional probability fn. y given $X=x_i$ is

$$P[Y=y_j/X=x_i] = \frac{P[X=x_i, Y=y_j]}{P[X=x_i]}$$

In continuous case :-

* The conditional probability fn. of x given y is

$$f(x/y) = \frac{f(x,y)}{f(y)} \quad ; \quad f(y) \geq 0$$

* The conditional probability fn. of y given x is

$$f(y/x) = \frac{f(x,y)}{f(x)} \quad ; \quad f(x) > 0$$

Independent Condition :-

In discrete case :-

* $P[X=x_i, Y=y_j] = p(x) \cdot p(y)$ (or) $P_{ij} = P_i * P_j$

In continuous case :-

* $f(x,y) = f(x) \cdot f(y)$

Note :-

1) $P[X=x_i, Y=y_j] = P[X=x_i \cap Y=y_j]$

2) $P[a_1 \leq x \leq b_1 \cap a_2 \leq y \leq b_2] = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x,y)\, dx\, dy$

3) $f(x,y)$ or $f_{xy}(x,y)$ are both represents Joint probability fn. .

4) $f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$.

5) $F(\infty,\infty) = 1$; $F(-\infty, y) = F(x, -\infty) = 0$; $0 \leq F(x,y) \leq 1$.

Problems based on discrete Random Variable :-

1. The joint probability mass fn. of (X, Y) is given by $P(x, y) = K(2x + 3y)$; $x = 0, 1, 2$; $y = 1, 2, 3$. Find all the marginal and conditional probability distributions. Also find the probability distribation fn. of $(X+Y)$.

Soln :- given $P(x, y) = K(2x + 3y)$ for $x = 0, 1, 2$ & $y = 1, 2, 3$.

X \ Y	1	2	3	$P[X=x]$
0	3K	6K	9K	18K
1	5K	8K	11K	24K
2	7K	10K	13K	30K
$P[Y=y]$	15K	24K	33K	72K $\to \Sigma P(x_i, y_j)$

$P(0,1) = K(0+3) = 3K$
$P(0,2) = K(0+3(2)) = 6K$
$P(0,3) = K(0+3(3)) = 9K$
$P(1,1) = K(2+3(1)) = 5K$
$P(1,2) = K(2+3(2)) = 8K$
$P(1,3) = K(2+3(3)) = 11K$
$P(2,1) = K(2(2)+3(1)) = 7K$
$P(2,2) = K(4+3(2)) = 10K$
$P(2,3) = K(4+3(3)) = 13K$

WKT $P(x, y)$ is the Joint Probability mass fn. $\Sigma\Sigma P(x_i, y_j) = 1$

$$\therefore \quad 72K = 1 \Rightarrow \boxed{K = \frac{1}{72}}$$

$\therefore$ The table becomes,

X \ Y	1	2	3	$P[X=x]$	
0	$\frac{3}{72}$	$\frac{6}{72}$	$\frac{9}{72}$	$\frac{18}{72}$	$P[X=0]$
1	$\frac{5}{72}$	$\frac{8}{72}$	$\frac{11}{72}$	$\frac{24}{72}$	$P[X=1]$
2	$\frac{7}{72}$	$\frac{10}{72}$	$\frac{13}{72}$	$\frac{30}{72}$	$P[X=2]$
$P[Y=y]$	$\frac{15}{72}$	$\frac{24}{72}$	$\frac{33}{72}$	1	
	$P[Y=1]$	$P[Y=2]$	$P[Y=3]$		

* Marginal probability distribution fn. of x :-

$P[x=0] = \frac{18}{72}$

$P[x=1] = \frac{24}{72}$

$P[x=2] = \frac{30}{72}$.

* Marginal probability distribution fn. of y :-

$P[y=1] = \frac{15}{72}$; $P[y=2] = \frac{24}{72}$; $P[y=3] = \frac{33}{72}$.

*(i) Conditional distribution of x given $y=y$:-

$$P[x=x / y=y] = \frac{P[x=x, y=y]}{P[y=y]}$$ $\left(\because P(A|B) = \frac{P(A\cap B)}{P(B)}\right)$

When $y=1$ $\Rightarrow$

$$P[x=0 / y=1] = \frac{P[x=0, y=1]}{P[y=1]} = \frac{3/72}{15/72} = \frac{3}{15}$$

$$P[x=1 / y=1] = \frac{P[x=1, y=1]}{P[y=1]} = \frac{5/72}{15/72} = \frac{5}{15}$$

$$P[x=2 / y=1] = \frac{P[x=2, y=1]}{P[y=1]} = \frac{7/72}{15/72} = \frac{7}{15}$$

When $y=2$

$$P[x=0 / y=2] = \frac{P[x=0, y=2]}{P[y=2]} = \frac{6/72}{24/72} = \frac{6}{24}$$

$$P[x=1 / y=2] = \frac{P[x=1, y=2]}{P[y=2]} = \frac{8/72}{24/72} = \frac{8}{24}$$

$$P[x=2 / y=2] = \frac{P[x=2, y=2]}{P[y=2]} = \frac{10/72}{24/72} = \frac{10}{24}.$$

When $y=3$:-

$$P[x=0 / y=3] = \frac{P[x=0, y=3]}{P[y=3]} = \frac{9/72}{33/72} = \frac{9}{33}$$

$$P[x=1 / y=3] = \frac{P[x=1, y=3]}{P[y=3]} = \frac{11/72}{33/72} = \frac{11}{33}$$

$$P[x=2 / y=3] = \frac{P[x=2, y=3]}{P[y=3]} = \frac{13/72}{33/72} = \frac{13}{33}$$

* Conditional distribution of y given x=x :-

$$P[Y=y/X=x] = \frac{P[X=x, Y=y]}{P[X=x]}$$

When x=0 :

$$P[Y=1/X=0] = \frac{P[X=0, Y=1]}{P[X=0]} = \frac{3/72}{18/72} = \frac{3}{18}$$

$$P[Y=2/X=0] = \frac{P[X=0, Y=2]}{P[X=0]} = \frac{6/72}{18/72} = 6/18$$

$$P[Y=3/X=0] = \frac{P[X=0, Y=3]}{P[X=0]} = \frac{9/72}{18/72} = 9/18$$

When x=1 :

$$P[Y=1/X=1] = \frac{P[X=1, Y=1]}{P[X=1]} = \frac{5/72}{24/72} = 5/24$$

$$P[Y=2/X=1] = \frac{P[X=1, Y=2]}{P[X=1]} = \frac{8/72}{24/72} = 8/24$$

$$P[Y=3/X=1] = \frac{P[X=1, Y=3]}{P[X=1]} = \frac{11/72}{24/72} = 11/24$$

When x=2 :-

$$P[Y=1/X=2] = \frac{P[X=2, Y=1]}{P[X=2]} = \frac{7/72}{30/72} = 7/30$$

$$P[Y=2/X=2] = \frac{P[X=2, Y=2]}{P[X=2]} = \frac{\frac{10}{72}}{30/72} = \frac{10}{30}$$

$$P[Y=3/X=2] = \frac{P[X=2, Y=3]}{P[X=2]} = \frac{13/72}{30/72} = \frac{13}{30}$$

Probability distribution of X+Y :-

X+Y	Probability	
1	$P(0,1) = 3/72$	$3/72$
2	$P(0,2) + P(1,1) = \frac{6}{72} + \frac{5}{72} = \frac{11}{72}$	$11/72$
3	$P(0,3) + P(1,2) + P(2,1) = \frac{9}{72} + \frac{8}{72} + \frac{7}{72} = \frac{24}{72}$	$24/72$
4	$P(1,3) + P(2,2) = \frac{11}{72} + \frac{10}{72} = \frac{21}{72}$	$21/72$
5	$P(2,3) = \frac{13}{72}$	$13/72$

2. For the bivariate probability distribution of (X, Y) given below. find $P[X \leq 1]$, $P[Y \leq 3]$, $P[X \leq 1, Y \leq 3]$, $P[X \leq 1 / Y \leq 3]$, $P[Y \leq 3 / X \leq 1]$ and $P[X+Y \leq 4]$.

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Soln: -

X \ Y	1	2	3	4	5	6	$P[X=x]$	
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{8}{32}$	$P[X=0]$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{10}{16}$	$P[X=1]$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$\frac{8}{64}$	$P[X=2]$
$P[Y=y]$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	$\frac{16}{64}$	1	
	$P[Y=1]$	$P[Y=2]$	$P[Y=3]$	$P[Y=4]$	$P[Y=5]$	$P[Y=6]$		

(i) <u>To find $P[X \leq 1]$</u> :-

$$P[X \leq 1] = P[X=0] + P[X=1]$$

$$P[X \leq 1] = \frac{8}{32} + \frac{10}{16} = \frac{28}{32}$$

ii) <u>To find $P[Y \leq 3]$</u> :-

$$P[Y \leq 3] = P[Y=1] + P[Y=2] + P[Y=3]$$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{6+6+11}{64} = \frac{23}{64}$$

iii) <u>To find $P[x \leq 1, y \leq 3]$:-</u>

$$P[x \leq 1, y \leq 3] = P[0,1] + P[0,2] + P[0,3] + P[1,1] + P[1,2] + P[1,3]$$

$$= 0 + 0 + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8}$$

$$= \frac{1+2+2+4}{32} = \frac{9}{32}$$

iv) $$P[x \leq 1 / y \leq 3] = \frac{P[x \leq 1, y \leq 3]}{P[y \leq 3]} \qquad \left(P(A/B) = \frac{P(A \cap B)}{P(B)}\right)$$

$$= \frac{\frac{9}{32}}{23/64} = \frac{9}{\cancel{32}} \times \frac{\overset{2}{\cancel{64}}}{23} = \frac{18}{23}$$

v) $$P[y \leq 3 / x \leq 1] = \frac{P[y \leq 3, x \leq 1]}{P[x \leq 1]} = \frac{P[x \leq 1, y \leq 3]}{P[x \leq 1]}$$

$$= \frac{9/32}{28/32} = \frac{9}{\cancel{32}} \times \frac{\cancel{32}}{28} = \frac{9}{28}$$

vi) <u>$P[x + y \leq 4]$:-</u>

$$P[x + y \leq 4] = P[0,1] + P[0,2] + P[0,3] + P[0,4] + P[1,1] + P[1,2] + P[1,3] + P[2,1] + P[2,$$

$$= 0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{32} + \frac{1}{32}$$

$$= \frac{1+2+2+2+4+1+1}{32} = \frac{13}{32}$$

HW 1) The joint distribution of x_1 and x_2 is given by

$f(x_1, x_2) = \frac{x_1 + x_2}{21}$; $x_1 = 1, 2$ and 3 ; $x_2 = 1$ and 2.

Find the marginal distribution of x_1 and x_2.

M.d of $x_1 \Rightarrow P[x_1 = 1] = 5/21$, $P[x_1 = 2] = 7/21$; $P[x_1 = 3] = 9/21$

M.D of $x_2 \Rightarrow P[x_2 = 1] = 9/21$; $P[x_2 = 12/21]$.