Solve $\frac{y^2 z}{x} p + xzq = y^2$

Solution:- This is of the form $Pp + Qq = R$

where $P = \frac{y^2 z}{x}$; $Q = xz$; $R = y^2$

The subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2} \Rightarrow \frac{x\,dx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}$$

Taking the first two ratios

$$\frac{x\,dx}{y^2 \cancel{z}} = \frac{dy}{x\cancel{z}}$$

$$x^2 dx = y^2 dy$$

Integrating, we get

$$\frac{x^3}{3} = \frac{y^3}{3} + c'$$

$$\Rightarrow \boxed{x^3 - y^3 = c.}$$

Taking the first & the last ratios

$$\frac{x\,dx}{\cancel{y^2} z} = \frac{dz}{\cancel{y^2}}$$

$$x\,dx = z\,dz$$

Integrating, we get

$$\frac{x^2}{2} = \frac{z^2}{2} + d$$

$$\boxed{x^2 - z^2 = d.}$$

$\therefore$ The solution is $\phi(x^3 - y^3, x^2 - z^2) = 0.$

Example:-

Solve $p \tan x + q \tan y = \tan z$

Solution: This is of the form $Pp + Qq = R$

$P = \tan x$; $Q = \tan y$; $R = \tan z$

The subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

i.e., $\dfrac{dx}{\tan x} = \dfrac{dy}{\tan y} = \dfrac{dz}{\tan z}$

Taking the first two ratios

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

Integrating, we get

$$\int \frac{dx}{\tan x} = \int \frac{dy}{\tan y}$$

$$\int \cot x \, dx = \int \cot y \, dy$$

$$\log \sin x = \log \sin y + \log c$$

$\Rightarrow$ $\log \sin x - \log \sin y = \log c$

$$\cancel{\log}\left(\frac{\sin x}{\sin y}\right) = \cancel{\log} c$$

$\Rightarrow$ $\boxed{\dfrac{\sin x}{\sin y} = c.}$

Taking the second & the third ratios

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

Integrating, we get

$$\int \frac{dy}{\tan y} = \int \frac{dz}{\tan z}$$

$$\int \cot y \, dy = \int \cot z \, dz$$

$$\log \sin y = \log \sin z + \log d$$

$\Rightarrow$ $\cancel{\log}\left(\dfrac{\sin y}{\sin z}\right) = \cancel{\log} d$

$$\boxed{\frac{\sin y}{\sin z} = d}$$

$\therefore$ The solution is

$$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$$

H.W ① Solve: $(y^2+z^2)p - xyq + xz = 0$

(Hint: Compare the second & third ratios; use the multipliers x, y, z)

② Solve $x^2(y-z)p + y^2(z-x) = z^2(x-y)$

(Hint: Use the multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$; $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$)

Homogeneous Linear PDE with Constant Coefficients

An equation of the form

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + k_n \frac{\partial^n z}{\partial y^n} = F(x,y) \quad \text{——— (1)}$$

where $k_1, k_2 \ldots k_n$ are constants is called the homogeneous linear PDE with constant coefficients.

The complete solution of (1) contains two parts, namely the complementary function and the particular integral

To find CF :

(1) can be written as

$$(D^n + k_1 D^{n-1} D' + \cdots + k_n D'^n) z = F(x,y)$$

where $D = \frac{\partial}{\partial x}$; $D' = \frac{\partial}{\partial y}$.

Consider the equation $(D^2 + k_1 DD' + k_2 D'^2) z = 0$

Replacing D by m and D' by 1 we get

$$m^2 + k_1 m + k_2 = 0 \quad \text{——— (2)}$$

Solving (2) we get

Case (i): If the roots are distinct say $m_1 \neq m_2$

then the CF is

$$z = f(y + m_1 x) + g(y + m_2 x)$$

Case (ii): If the roots are equal (same) say $m_1 = m_2$

then the CF is

$$z = f(y + m_1 x) + x\,\phi(y + m_1 x)$$

—x—

Solve $4\frac{\partial^2 z}{\partial x^2} + 12\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial y^2} = 0$

Solution: The given eqn can be written as

$$(4D^2 + 12DD' + 9D'^2)z = 0 \quad \text{where } D = \frac{\partial}{\partial x}, \ D' = \frac{\partial}{\partial y}$$

To find CF

Put $D = m$ & $D' = 1$, then

$$4m^2 + 12m + 9 = 0$$

$$m = \frac{-12 \pm \sqrt{144 - 4.4.9}}{2.4} = \frac{-12 \pm \sqrt{144 - 144}}{8}$$

$$= -3/2, \ -3/2$$

Here the roots are same.

$\therefore$ C.F is

$$z = f_1\left(y - \frac{3}{2}x\right) + x f_2\left(y - \frac{3}{2}x\right)$$

Solve: $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$

Solution: The given equation can be written as

$$(D^2 + 3DD' + 2D'^2)z = 0 \quad \text{--- (1)} \qquad D = \frac{\partial}{\partial x}, \ D' = \frac{\partial}{\partial y}$$

To find CF

Put $D = m$ & $D' = 1$ in (1)

$$m^2 + 3m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$\Rightarrow m = -2, \ m = -1$$

The roots are distinct.

$\therefore$ CF is $z = f_1(y - 2x) + f_2(y - x)$

~~Solve~~ Particular Integral :-

Type I :- RHS $= e^{ax+by}$

In this case put $D = a$; $D' = b$.

Now $P.I = \frac{1}{f(D, D')} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by}$

provided $f(a,b) \neq 0$

Solve :- $r - 4s + 4t = e^{2x+y}$

Solution :- The given equation can be written as

$$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$$

i.e., $(D^2 - 4DD' + 4D'^2) z = e^{2x+y}$ ——— (1)

To find CF

Put $D = m$ & $D' = 1$ in (1)

(1) $\Rightarrow$ $m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$

$m = 2, 2$

$\therefore$ The CF is $z = f_1(y+2x) + x f_2(y+2x)$

To find PI

$$P.I = \frac{1}{f(D, D')} e^{2x+y}$$

Here $a = 2, b = 1$

Put $D = a = 2$

$D' = b = 1$

$$= \frac{1}{D^2 - 4DD' + 4D'^2} e^{2x+y}$$

$$= \frac{1}{2^2 - 4(2)(1) + 4(1)^2} e^{2x+y}$$

$$\frac{1}{4-8+4} e^{2x+y} = \frac{1}{0} e^{2x+y}$$

$$= \frac{x}{2D-4D'} e^{2x+y} = \frac{x^2}{+2} e^{2x+y}$$

$$= \frac{x^2 e^{2x+y}}{2}.$$

$\therefore$ The solution is $Z = CF + PI$

i.e., $Z = f_1(y+2x) + x f_2(y+2x) + \frac{x^2}{2} e^{2x+y}$

<u>Solve:</u> $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 5\frac{\partial^3 z}{\partial x \partial y^2} - 2\frac{\partial^3 z}{\partial y^3} = e^{2x+y}$

<u>Solution:</u> The given equation can be written as

$$(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)Z = e^{2x+y} \quad \text{(1)}$$

<u>To find CF</u>

Put $D = m$ & $D' = 1$ in (1) and equate it to zero

$$m^3 - 4m^2 + 5m - 2 = 0$$

<u>Put $m = 1$;</u> $1 - 4 + 5 - 2 = 0$

$\therefore$ $\boxed{m = 1}$ is a root.

1	1	-4	5	-2
		1	-3	2
	1	-3	2	0

$$m^2 - 3m + 2 = 0$$

$$(m-2)(m-1) = 0$$

$\Rightarrow$ $\boxed{m = 2,\ m = 1}$

$\therefore$ The roots are $m = 1, 1, 2$.

To find CF. $\cos x \cos 2y$

$\therefore$ The CF is $z = f_1(y+x) + x f_2(y+x) + f_3(y+2x)$

To find PI :

$$PI = \frac{1}{f(D,D')} e^{2x+y}$$

$$= \frac{1}{D^3 - 4D^2D' + 5DD'^2 - 2D'^3} e^{2x+y}$$

Here $a=2, b=1$

Put $\therefore D = a = 2$

$D' = b = 1$

$$= \frac{1}{8 - 16 + 10 - 2} e^{2x+y}$$

$$= \frac{1}{0} e^{2x+y}$$

$$= \frac{x}{3D^2 - 8DD' + 5D'^2} e^{2x+y}$$

$$= \frac{x}{3(4) - 8(2)(1) + 5(1)} e^{2x+y} = \frac{x}{1} e^{2x+y}$$

$\therefore$ $PI = x e^{2x+y}$

$\therefore$ The solution is

$z = CF + PI$

$= f_1(y+x) + x f_2(y+x) + f_3(y+2x) + x e^{2x+y}$.

Solve $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = e^{2x-y}$

Solution:- The given equation can be written as

$$(D^2 - 3DD' + 2D'^2)z = e^{2x-y} \quad \text{--- (1)}$$

To find CF :

Put $D = m$ & $D' = 1$ in (1) & equate it to zero

$m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0$

$\Rightarrow m = 2, \ m = 1$

$\therefore$ The CF is $z = f_1(y+2x) + f_2(y+x)$

To find PI

$$PI = \frac{1}{f(D,D')} e^{2x-y} = \frac{1}{D^2 - 3DD' + 2D'^2} e^{2x-y}$$

Here $a = 2, b = -1$

Put $D = a = 2$

$D' = b = -1$

$$= \frac{1}{4 - 3(2)(-1) + 2(-1)^2} e^{2x-y}$$

$$= \frac{1}{4+6+2} e^{2x-y}$$

$$= \frac{1}{12} e^{2x-y}$$

$\therefore$ The solution is $z = CF + PI$

$$z = f_1(y+2x) + f_2(y+x) + \frac{1}{12} e^{2x-y}$$

Type II :- RHS = $\sin(ax+by)$ or $\cos(ax+by)$

$$PI = \frac{1}{f(D,D')} \sin(ax+by)$$

Replace D^2 by $-a^2$, DD' by $-ab$ and D'^2 by $-b^2$

— x —

Example :- Solve $r+s-6t = \cos(2x+y)$

Solution :- The given pde can be written as

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = \cos(2x+y)$$

This can be written as

To find CF :- $(D^2 + DD' - 6D'^2)z = \cos(2x+y)$ — ①

Put $D = m$ & $D' = 1$ in ① and equate it to zero.

ie, $m^2 + m - 6 = 0$

$(m+3)(m-2) = 0$

$\Rightarrow m = -3$ & $m = 2$

$\therefore$ The CF is $f_1(y-3x) + f_2(y+2x)$

To find PI :-

$$PI = \frac{1}{f(D,D')} \cos(2x+y)$$

$$= \frac{1}{D^2 + DD' - 6D'^2} \cos(2x+y)$$

Here $a=2, b=1$
$D^2 = -a^2 = -4$
$DD' = -(2)(1) = -2$
$D'^2 = -1^2 = -1$

$$= \frac{1}{-4-2+6(-1)} \cos(2x+y)$$

$$= \frac{1}{0} \cos(2x+y)$$

$$= \frac{x}{2D+D'} \cos(2x+y)$$

$$= x \cdot \frac{(2D-D')}{(2D+D')(2D-D')} \cos(2x+y)$$

$$= x \cdot \frac{(2D - D')}{4D^2 - D'^2} \cos(2x+y)$$

$$= \frac{x \cdot \{2[-\sin(2x+y)\,2] + \sin(2x+y)\}}{4(-4) - (-1)}$$

$$= \frac{x \cdot [-3\sin(2x+y)]}{-15} = \frac{x\sin(2x+y)}{5}.$$

$\therefore$ The solution is $z = CF + PI$

i.e., $z = f_1(y-3x) + f_2(y+2x) + \dfrac{x\sin(2x+y)}{5}$.

Example:

Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = \cos(x+y)$

Solution: To find CF

Put $D = m$ & $D' = 1$ and equate it to zero

$(m^3 + m^2 - m - 1) = 0$

Put $m = 1$; $1 + 1 - 1 - 1 = 0$

$\therefore m = 1$ is a root.

$$\begin{array}{r|rrrr} 1 & 1 & 1 & -1 & -1 \\ & & 1 & 2 & 1 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array}$$

$m^2 + 2m + 1 = 0$

$(m+1)(m+1) = 0 \Rightarrow m = -1, -1$

$\therefore$ The CF is $f_1(y+x) + f_2(y-x) + x f_3(y-x)$

To find PI: Here $a = 1$, $b = 1$

$$P.I. = \frac{1}{f(D,D')}\cos(x+y) = \frac{1}{D^3 + D^2D' - DD'^2 - D'^3}\cos(x+y)$$

$D^2 = -a^2 = -1$

$DD' = -ab = -1$

$D'^2 = -b^2 = -1$

$$= \frac{1}{-D - D' + D + D'}\cos(x+y) = \frac{1}{0}\cos(x+y)$$

$$= \frac{x}{3D^2 + 2DD' - D'^2}\cos(x+y) = \frac{x}{3(-1) + 2(-1) - (-1)}\cos(x+y)$$

$$= \frac{x}{-4}\cos(x+y)$$

$\therefore$ The solution is $z = CF + PI \Rightarrow z = f_1(y+x) + f_2(y-x) + x f_3(y-x) - \dfrac{x}{4}\cos(x+y)$

Example 3 :- Solve $(D^2 - DD')z = \cos x \cos 2y$

Solution :- **To find CF :** Put $D = m$ & $D' = 1$ and equate it to 0

$$m^2 - m = 0 \Rightarrow m(m-1) = 0 \Rightarrow m = 0, m = 1$$

$\therefore$ the CF is $f_1(y + 0x) + f_2(y + x)$

To find PI :-

$$PI = \frac{1}{f(D, D')} \cos x \cos 2y$$

$$= \frac{1}{D^2 - DD'} \left[\frac{\cos(x+2y) + \cos(x-2y)}{2}\right]$$

$$= \frac{1}{2}\left[\underbrace{\frac{1}{D^2 - DD'}\cos(x+2y)}_{PI_1} + \underbrace{\frac{1}{D^2 - DD'}\cos(x-2y)}_{PI_2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{-1-(-2)}\cos(x+2y) + \frac{1}{-1-(2)}\cos(x-2y)\right]$$

$$= \frac{1}{2}\left[\cos(x+2y) - \frac{1}{3}\cos(x-2y)\right]$$

PI_1

$a = 1 ; b = 2$

$D^2 = -a^2 = -1$

$DD' = -ab = -2$

$D'^2 = -b^2 = -4$

$PI_2 : a = 1, b = -2$

$D^2 = -1$

$DD' = +2$

$D'^2 = -4$

$\therefore$ The solution is $z = CF + PI$

$$z = f_1(y) + f_2(y+x) + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

HW :- (i) Solve $(D^2 - DD')z = \sin x \cos 2y$

(ii) Solve $(D^2 - 4D'^2)z = \sin(2x + y)$

(iii) Solve $(D^3 - 3D^2D' - 4DD'^2 + 12D'^3)z = \sin(2x + y)$

(iv) Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = 3\sin(x + y)$

Type III :: RHS $= \dfrac{1}{f(D,D')} x^r y^s$

$$= \frac{1}{1+\phi(D,D')} x^r y^s = [1+\phi(D,D')]^{-1} x^r y^s$$

$[1+\phi(D,D')]^{-1}$ is to be expanded in powers of D & D'

Example:- Solve $(D^3 - 7DD'^2 - 6D'^3)z = x^2 y$

Solution:- To find CF Put $D = m$ & $D' = 1$ & equate it to 0.

$$m^3 - 7m - 6 = 0$$

Put $m = -1$; $-1 + 7 - 6 = 0 \Rightarrow m = -1$ is a root

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$m^2 - m - 6 = 0$

$(m-3)(m+2) = 0$

$\Rightarrow m = 3,\ m = -2$

$\therefore$ the CF is $f_1(y-x) + f_2(y+3x) + f_3(y-2x)$

To find PI:

$$PI = \frac{1}{f(D,D')} x^2 y = \frac{1}{D^3 - 7DD'^2 - 6D'^3} x^2 y$$

$$= \frac{1}{D^3\left[1 - \left(\frac{7DD'^2 + 6D'^3}{D^3}\right)\right]} x^2 y$$

$$= \frac{1}{D^3}\left[1 + \left(\frac{7D'^2}{D^2} + \frac{6D'^3}{D^3}\right)\right]^{-1} x^2 y$$

$(1+x)^{-1} = 1 - x + x^2 - x^3 \cdots\cdots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 \cdots\cdots$

$$= \frac{1}{D^3}\left[1 - \left(\frac{7D'^2}{D^2} + \frac{6D'^3}{D^3}\right) + \cdots\right] x^2 y$$

$$= \frac{1}{D^3}\left[x^2 y\right] = \frac{x^5 y}{60}$$

$\frac{x^3}{3} \rightarrow \frac{x^4}{4} \rightarrow \frac{x^5}{5}$

$= \frac{x^5}{60}$

$\therefore$ The solution is $z = CF + PI$

$$z = f_1(y-x) + f_2(y+3x) + f_3(y-2x) + \frac{x^5 y}{60}$$

Example:- Solve $(D^2+3DD'+2D'^2)z = x+y$

Solution:- To find CF Put $D=m$ & $D'=1$ and equate it to zero

$$m^2+3m+2=0$$

$$\Rightarrow (m+2)(m+1)=0$$

$$\Rightarrow m=-2,\ m=-1$$

$\therefore$ the CF is $f_1(y-2x)+f_2(y-x)$

To find PI:

$$PI = \frac{1}{f(D,D')}(x+y)$$

$$= \frac{1}{D^2+3DD'+2D'^2}(x+y)$$

$$= \frac{1}{D^2\left[1+\left(\frac{3DD'+2D'^2}{D^2}\right)\right]}(x+y) = \frac{1}{D^2}\left[1+\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)\right]^{-1}(x+y)$$

$$= \frac{1}{D^2}\left[1-\frac{3D'}{D}\right](x+y)$$

$$= \frac{1}{D^2}\left[(x+y)-\frac{3}{D}(1)\right]$$

$$= \frac{1}{D^2}(x+y)-\frac{3}{D^3}(1) = \frac{x^3}{6}+\frac{yx^2}{2}-\frac{\cancel{3}x^3}{\cancel{6}_2}$$

$\therefore$ The solution is

$$Z = CF + PI$$

$$Z = f_1(y-2x)+f_2(y-x)-\frac{x^3}{3}+\frac{yx^2}{2}$$