Properties of Laplace Transform

Property	$x(t)$	$X(s)$
	$x_1(t)$	$X_1(s)$
	$x_2(t)$	$X_2(s)$
1). Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$
2) Time shifting	$x(t-t_0)$	$e^{-st_0}X(s)$
3) Frequency Shifting	$x(t)e^{j\Omega_0 t}$	$X(s-\Omega_0)$
4) Time differentiation	$\frac{dx(t)}{dt}$	$sX(s)-x(0^-)$
	$\frac{d^2x(t)}{dt^2}$	$s^2X(s)-sx(0^-)-\frac{dx(0^-)}{dt}$
	$\frac{d^nx(t)}{dt^n}$	$s^nX(s)-s^{n-1}x(0^-)\ldots-\frac{d^{n-1}}{dt^{n-1}}x(0^-)$
5) Time Integration	$\int_0^t x(\tau)d\tau$	$\frac{X(s)}{s}$
	$\int_{-\infty}^t x(\tau)d\tau$	$\frac{X(s)}{s}+\frac{1}{s}\int_{-\infty}^0 x(\tau)d\tau$
6) Frequency Differentiation	$-tx(t)$	$\frac{dX(s)}{ds}$
7) Frequency Integration	$\frac{x(t)}{t}$	$\int_s^\infty X(s)ds$
8) Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$

Property	$x(t)$	$X(s)$
9) Time convolution	$x_1(t) * x_2(t)$	$X_1(s)\, X_2(s)$
10) Frequency convolution	$x_1(t)\, x_2(t)$	$X_1(s) * X_2(s)$
11) Initial value theorem	$x(0-)$	$\lim_{s \to \infty} s X(s)$
12) Final value theorem	$x(\infty)$	$\lim_{s \to 0} s X(s)$

① Find the Laplace transform of the signal $x(t) = 2e^{-2t}u(t) + 4e^{-4t}u(t)$

$$X(s) = L[x(t)] = L\left[2e^{-2t}u(t) + 4e^{-4t}u(t)\right]$$

Using linearity property

$$X(s) = 2L[e^{-2t}u(t)] + 4L[e^{-4t}u(t)]$$

$$= \frac{2}{s+2} + \frac{4}{s+4} \quad \text{Roc: } Re(s) > -2.$$

② Given $x(t) = e^{-t}u(t)$, find the inverse Laplace transform of $e^{-3s} X(2s)$.

we know that $L[e^{-t}u(t)] = \frac{1}{s+1}$

Using time scaling property we have

$$L[x(at)] = \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

If $X(s) = \frac{1}{s+1}$

$$X(2s) = \frac{1}{\frac{s}{2}+1} = \frac{2}{s+2} \Rightarrow 2X(2s) = \frac{4}{s+2}.$$

$L^{-1}[2X(2s)] = x/2 \quad (\therefore a = 1/2)$

$L^{-1}[X(2s)] = \frac{1}{2} x(t/2)$

Using time shifting property

$L[x(t-t_0)] = e^{-st_0} X(s)$

$L^{-1}[e^{-st_0} X(s)] = x(t-t_0)$

$L^{-1}[e^{-3s} X(2s)] = \frac{1}{2} x\left(\frac{t-3}{2}\right) = e^{-\left(\frac{t-3}{2}\right)} u\left(\frac{t-3}{2}\right)$

(3) Find the L.T of $x(t) = u(t-2)$

Time shifting property

$L[x(t-t_0)] = e^{-st_0} X(s)$

$L[u(t)] = 1/s$

$L[u(t-2)] = \frac{e^{-2s}}{s}$

(4) Find the Laplace transform of the signals

(i) $x(t) = e^{-at} \sin \Omega_0 t \, u(t)$

Frequency shifting property:

$L[e^{-at} x(t)] = X(s+a)$

$L[\sin \Omega_0 t \, u(t)] = \frac{\Omega_0}{s^2 + \Omega_0^2}$

Using frequency shifting ppty.

$L[e^{-at} \sin \Omega_0 t \, u(t)] = \frac{\Omega_0}{(s+a)^2 + \Omega_0^2}$

(ii) $x(t) = [4e^{-2t}\cos 5t - 3e^{-2t}\sin 5t]u(t)$

$L[4e^{-2t}\cos 5t - 3e^{-2t}\sin 5t]u(t)$

$\Rightarrow L[4e^{-2t}\cos 5t\, u(t)] - L[3e^{-2t}\sin 5t\, u(t)]$

$L[\cos \Omega_0 t\, u(t)] = \dfrac{s}{s^2+\Omega_0^2} \Rightarrow L[\cos 5t\, u(t)] = \dfrac{s}{s^2+25}$

$L[\sin \Omega_0 t\, u(t)] = \dfrac{\Omega_0}{s^2+\Omega_0^2} \Rightarrow L[\sin 5t\, u(t)] = \dfrac{5}{s^2+25}$

- Using freq. shifting property: $L[e^{-at}x(t)] = X(s+a)$

$e^{-2t}\cos 5t = \dfrac{(s+5)}{(s+5)^2+25}$

$e^{-2t}\sin 5t = \dfrac{5}{(s+5)^2+25}$

$\therefore \boxed{L[x(t)] = \dfrac{4(s+5)}{(s+5)^2+25} - \dfrac{15}{(s+2)^2+25}}$

(5) Find Laplace Transform of:

$\dfrac{d^2y(t)}{dt^2} + 3\dfrac{dy(t)}{dt} + 2y(t) = \dfrac{d}{dt}x(t)$

Time defferentiation property

$\dfrac{d^n x(t)}{dt^n} = s^n X(s) - s^{n-1}x(0^-) \ldots - \dfrac{d^{n-1}x(0^-)}{dt}$

$$\frac{d^2y(t)}{dt^2} = s^2Y(s) - sy(0^-) - \frac{dy}{dt}(0^-)$$

$$\frac{dy(t)}{dt} = sY(s) - y(0^-)$$

$\therefore$ Applying L.T to both sides

$$\left[s^2Y(s) - sy(0^-) - \frac{dy}{dt}(0^-)\right] + 3\left[sY(s) - y(0^-)\right] + 2Y(s) = sX(s) - x(0^-).$$

(b) Find the inverse Laplace Transform of:

$$X(s) = \frac{1}{s(s+2)}$$

Solution: Let $X_1(s) = \frac{1}{s+2}$.

$$X(s) = \frac{X_1(s)}{s}$$

$$x_1(t) = L^{-1}[X_1(s)] = L^{-1}\left[\frac{1}{s+2}\right] = e^{-2t}u(t)$$

Using time integral property

$$L\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{X(s)}{s}$$

$$\therefore L^{-1}\left[\frac{X(s)}{s}\right] = \int_{-\infty}^{t} x(\tau)\,d\tau$$

$$L^{-1}\left[\frac{X_1(s)}{s}\right] = L^{-1}\left[\frac{1}{s(s+2)}\right] = \int_{-\infty}^{t} x_1(\tau)\,d\tau = \int_{-\infty}^{t} e^{-2\tau}u(\tau)\,d\tau$$

$$\Rightarrow \int_0^t e^{-2\tau}\, d\tau = -\frac{1}{2} e^{-2\tau}\Big|_0^t = \frac{1-e^{-2t}}{2}$$

(7) Find the Laplace Transform of

$x(t) = -te^{-2t}u(t)$

Frequency differentiation property:

$$L[e^{-at}u(t)] = \frac{1}{s+a}.$$

$$L[e^{-2t}u(t)] = \frac{1}{s+2}$$

Using differentiation in s-domain property

$$L[-t\,x(t)] = \frac{dX(s)}{ds}$$

$$L[-te^{-2t}u(t)] = \frac{d}{ds}\left[\frac{1}{s+2}\right]$$

$$\boxed{L[-te^{-2t}u(t)] = \frac{-1}{(s+2)^2}}$$

(8) Given the transform pair

$$L[x(t)] = \frac{2s}{s^2-2}.$$

determine the L.T of $x(2t)$

If $L[x(t)] = X(s)$

then $L[x(at)] = \frac{1}{|a|} X(s/a)$ ← Time scaling property

Given $L[x(t)] = \frac{2s}{s^2-2}$

Using time scaling property, we can write

$$L[x(2t)] = \frac{1}{2} \frac{2(s/2)}{(s/2)^2-2} = \frac{1}{2} \frac{s}{\frac{s^2}{4}-2}$$

$$L[x(2t)] = \frac{2s}{s^2-8}$$

9) Given $x_1(t) = e^{-2t}u(t)$ and $x_2(t) = e^{-3t}u(t)$. Determine $Y(s)$ where $y(t) = x_1(t-2) * x_2(-t+3)$

Solution:

Given $x_1(t) = e^{-2t}u(t)$ & $x_2(t) = e^{-3t}u(t)$

$X_1(s) = L[e^{-2t}u(t)] = \frac{1}{s+2}$ and $X_2(s) = \frac{1}{s+3}$

Let $x(t) = x_1(t-2)$ & $h(t) = x_2(-t+3)$ then

$$y(t) = x(t) * h(t)$$

Using time convolution property

$$Y(s) = X(s)H(s)$$

$$X(s) = L[x_1(t-2)]$$

$$L[x_1(t-2)] = e^{-2s}X_1(s) \quad \text{time shifting property}$$

$$= e^{-2s}\left(\frac{1}{s+2}\right)$$

$H(s) = L[x_2(-t+3)]$

The signal $x_2(-t+3)$ can be obtained by time reversal of $x(t)$ & then shifting it by 3 units.

Using time reversal property

$$L[x_2(-t)] = X_2(-s) = \frac{1}{-s+3}$$

Using time shifting property

$$H(s) = L[x_2(-t+3)] = e^{-3s} X_2(-s)$$

$$= \frac{e^{-3s}}{3-s}$$

$$Y(s) = X(s)H(s) = \left(\frac{e^{-2s}}{s+2}\right)\left(\frac{e^{-3s}}{3-s}\right)$$

(i) Determine the initial value of

$$X(s) = \frac{2s+3}{s(s^2+5s+6)}$$

$$x(0^+) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \cancel{s}\left[\frac{2s+3}{\cancel{s}(s^2+5s+6)}\right]$$

$$\lim_{s\to\infty} \frac{2s+3}{s^2+5s+6} \qquad \boxed{s = 1/x.}$$

$$\Rightarrow \lim_{x\to 0} \frac{2/x + 3}{\frac{1}{x^2}+\frac{5}{x}+6} \Rightarrow \lim_{x\to 0} \frac{2x+3x^2}{6x^2+5x+1} = \frac{2(0)+3(0)}{6(0)+5(0)+1}$$

$$\lim_{x\to 0} \Rightarrow 0.$$

SRM

Find the final value of $X(s) = \frac{s-1}{s(s+1)}$

$$x(\infty) = \lim_{s\to 0} s\,X(s) = \lim_{s\to 0} \frac{\cancel{s}(s-1)}{\cancel{s}(s+1)}$$

$$\lim_{s\to 0} \frac{s-1}{s+1} = -1$$

(iii) Find the initial & final values of :

(a) $\frac{s+5}{s^2+3s+2}$

Initial value

$$x(0^+) = \lim_{s\to\infty} s\,X(s) = \lim_{s\to\infty} \frac{s(s+5)}{s^2+3s+2} = \lim_{s\to\infty} \frac{s^2+5}{s^2+3s+2}$$

$$s = 1/x \quad \therefore\ x(0^+) = \lim_{x\to 0} \frac{1+5x}{2x^2+3x+1}$$

$$\Rightarrow 1$$

Final value

$$x(\infty) = \lim_{s\to 0} s\,X(s) = \frac{s(s+5)}{s^2+3s+2} = 0.$$

(b) $\frac{s^2+5s+7}{s^2+3s+2}$

Initial value

$$x(0^+) = \lim_{s\to\infty} s\,X(s) = \frac{s(s^2+5s+7)}{s^2+3s+2} = \infty$$

$\therefore$ Initial value does not exist

Final value:

$$x(\infty) = \lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{s(s^2+5s+17)}{s^2+3s+2} = 0.$$