Ordinary Differential Equations

* For any given linear differential equation

let $\frac{d}{dx} = D \qquad \frac{d^2}{dx^2} = D^2$

* Convert the given differential equation into the form $\phi(D).y = F(x)$

where $\phi(D)$ is a function in D

For example if DE is

$$\frac{d^2y}{dx} + 3\frac{dy}{dx} - 3\sin x + y = 0$$

$$\underbrace{(D^2 + 3D + 1)}_{\phi(D)} y = \underbrace{3\sin x}_{F(x)}$$

* The solution of any differential equation consists of two parts

$$y = CF + PI$$

CF → Complementary function

PI → Particular Integral

NOTE :- If $F(x) = 0$ in any DE then its solution just comprises of CF and no PI is involved in its solution

* To find the Complementary Function

→ Form the Auxilary equation by putting $D=m$ in $\phi(D)=0$

→ Solve and get the values of m, i.e roots of $\phi(D)=0$

• If all roots are unequal and real

$$C.F = Ae^{m_1 x} + Be^{m_2 x} + \cdots\cdots$$

• If two roots are equal $(m_1 = m_2 = m)$

$$C.F = (A + Bx)e^{mx} + Ce^{m_3 x} + \cdots\cdots$$

• If three roots are equal $(m_1 = m_2 = m_3 = m)$
{Analogous to above case}

$$C.F = (A + Bx + Cx^2)e^{mx} + De^{m_4 x} +$$

• If roots are imaginary $(m = \alpha \pm i\beta)$

$$C.F = e^{\alpha x}\{\cos\beta x + \sin\beta x\}$$

To find the Particular Integral

- The particular integral is given by

$$PI = \frac{1}{\phi(D)} \cdot F(x)$$

Type 1. If $F(x) = e^{ax}$ then

$$PI = \frac{1}{\phi(D)} F(x) = \frac{1}{\phi(D)} e^{ax} = \boxed{\frac{1}{\phi(a)} e^{ax}}$$

(provided if $\phi(a) \neq 0$)

If $\phi(a) = 0$ then

$$PI = \frac{1}{\phi(D)} e^{ax} = x \cdot \frac{1}{\phi'(D)} e^{ax} = x \cdot \frac{1}{\phi'(a)} e^{ax}$$

$$\boxed{\phi'(D) = \frac{d(\phi(D))}{dx}}$$

(provided $\phi'(a) \neq 0$

* If $\phi'(a) = 0$ then $PI = \frac{x^2}{\phi''(a)} e^{ax}$ and so on

until $\frac{x^n}{\phi^n(a)} e^{ax}$; $\phi^n(a) \neq 0$

** If there is any constant before e^{ax} $\{be^{ax}\}$ then b comes as it is in the PI

TYPE 2 :: $F(x) = \sin ax \,/\, \cos ax$

then $P.I = \frac{1}{\phi(D)} F(x) = \frac{1}{\phi(D)} \sin ax / \cos ax$

$= x \cdot \frac{1}{\phi'(-a^2)} \sin ax / \cos ax \quad , \quad \phi'(-a^2) \neq 0$

(i.e) on $\phi(D)$ replace D^2 by $-a^2$ provided $\phi(D) \neq 0$

If $\phi(D) = 0$ when $D^2 = -a^2$ then

Similarly if $\phi'(-a^2) = 0$ ~~then~~

$P.I = x^2 \cdot \frac{1}{\phi''(-a^2)} \sin ax / \cos ax$

- On doing the above step we get an expression in D, now convert that D into D^2 by rationalisation.

{ for clarity refer the attached example problems }

If $F(x)$ contains $\sin^2 x / \cos^2 x$ then convert it into simple trigonometric ratios by.

$$\boxed{\begin{aligned} \cos^2 x &= \frac{1 + \cos 2x}{2} \\ \sin^2 x &= \frac{1 - \cos 2x}{2} \end{aligned}}$$

Example 3.7. *Solve:* $(D^2 + 3D + 2)y = \sin x$

Solution: Given $(D^2 + 3D + 2)y = \sin x$
(i.e.) $\phi(D)y = F(x)$
The auxiliary equation is $m^2 + 3m + 2 = 0$

$\Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, -2 \Rightarrow m_1 = -1, m_2 = -2$
$\Rightarrow C.F. = C_1e^{-x} + C_2e^{-2x}$

$$P.I. = \frac{1}{\phi(D)}F(x) = \frac{1}{D^2+3D+2}\sin x = \frac{1}{-1+3D+2}\sin x$$

$$= \frac{1}{3D+1}\sin x$$

$$P.I. = \frac{(3D-1)}{(3D-1)(3D+1)}\sin x = \frac{(3D-1)}{(9D^2-1)}\sin x$$

$$= \frac{1}{(-9-1)}(3D\sin x - \sin x)$$

$$= -\frac{1}{10}(3\cos x - \sin x)$$

The complete solution: $y = CF + PI$

$$y = C_1e^{-x} + C_2e^{-2x} - \frac{1}{10}(3\cos x - \sin x)$$

Example 3.8. *Solve:* $(D^2 + 6D + 8)y = e^{-2x} + \cos^2 x$

Solution: Given $(D^2 + 6D + 8)y = e^{-2x} + \cos^2 x$
(i.e.) $\phi(D)y = F(x)$
The auxiliary equation is $m^2 + 6m + 8 = 0$
$\Rightarrow (m+2)(m+4) = 0 \Rightarrow m = -2, -4 \Rightarrow m_1 = -2, m_2 = -4$
$\Rightarrow C.F. = C_1e^{-2x} + C_2e^{-4x}$

$$P.I. = \frac{1}{\phi(D)}F(x) = \frac{1}{D^2+6D+8}(e^{-2x} + \cos^2 x)$$

$$= \frac{1}{D^2+6D+8}e^{-2x} + \frac{1}{D^2+6D+8}\cos^2 x$$

$$= \frac{1}{4-12+8}e^{-2x} + \frac{1}{D^2+6D+8}\left(\frac{1+\cos 2x}{2}\right)$$

$$= x.\frac{1}{2D+6}e^{-2x} + \frac{1}{2}\frac{1}{D^2+6D+8}e^{0x} + \frac{1}{2}\frac{1}{D^2+6D+8}\cos 2x$$

$$= x.\frac{1}{-4+6}e^{-2x} + \frac{1}{2}\left(\frac{1}{0+0+8}\right)e^{0x} + \frac{1}{2}\left(\frac{1}{-4+6D+8}\right)\cos 2x$$

$$= \frac{x}{2}e^{-2x} + \frac{1}{16} + \frac{1}{2}\frac{1}{6D+4}\cos 2x$$

$$= \frac{x}{2}e^{-2x} + \frac{1}{16} + \frac{1}{2}\frac{(6D-4)}{(6D+4)(6D-4)}\cos 2x$$

$$= \frac{x}{2}e^{-2x} + \frac{1}{16} + \frac{1}{2}\frac{(6D-4)}{(36D^2-16)}\cos 2x$$

$$= \frac{x}{2}e^{-2x} + \frac{1}{16} + \frac{1}{2}\frac{(6D-4)}{[36(-4)-16]}\cos 2x$$

$$= \frac{x}{2}e^{-2x} + \frac{1}{16} + \frac{1}{2}\frac{1}{-160}[6D\cos 2x - 4\cos 2x]$$

$$= \frac{x}{2}e^{-2x} + \frac{1}{16} - \frac{1}{320}[-12\sin 2x - 4\cos 2x]$$

$$= \frac{x}{2}e^{-2x} + \frac{1}{16} + \frac{1}{80}[3\sin 2x + \cos 2x]$$

The complete solution: $y = CF + PI$

TYPE 3 : If $F(x) = x^n$ $\{n = +\text{ve Integer}\}$

$$PI = \frac{1}{\phi(D)} F(x) = \frac{1}{\phi(D)} x^n = \frac{1}{[1 \pm f(D)]} x^n$$

$$= [1 \pm f(D)]^{-1} x^n$$

* Express $\phi(D)$ as $1 \pm f(D)$ bring it to the Nr and expand $[1 \pm f(D)]^{-1}$ as binomial series. Operate x^n on each term of this expression)

* All Binomial expansions given below.

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 +$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 +$$

(iv) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

Example 3.13. Solve: $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = x^2 + 3x - 1$

Solution: Given $(D^2 - 5D + 6)y = x^2 + 3x - 1$
(i.e.) $\phi(D)y = F(x)$
The auxiliary equation is $m^2 - 5m + 6 = 0$
$\Rightarrow (m-2)(m-3) = 0 \Rightarrow m = 2, 3$
$\Rightarrow m_1 = 2, m_2 = 3$ and $m_1 \neq m_2$
$\Rightarrow C.F. = C_1e^{2x} + C_2e^{3x}$

Now P.I.$= \dfrac{1}{\phi(D)}F(x) = \dfrac{1}{(D^2 - 5D + 6)}(x^2 + 3x - 1)$

$$= \frac{1}{6}\left[1 + \left(\frac{D^2 - 5D}{6}\right)\right](x^2 + 3x - 1)$$

$$= \frac{1}{6}\left[1 + \left(\frac{D^2 - 5D}{6}\right)\right]^{-1}(x^2 + 3x - 1)$$

$$= \frac{1}{6}\left[1 - \left(\frac{D^2 - 5D}{6}\right) + \left(\frac{D^2 - 5D}{6}\right)^2 - \dots\right](x^2 + 3x - 1)$$

$$= \frac{1}{6}\left[1 - \frac{D^2}{6} + \frac{5D}{6} + \frac{D^4}{36} - \frac{10D^3}{36} + \frac{25D^2}{36}\right](x^2 + 3x - 1)$$ [Omit-

ting higher powers as F(x) is differentiable 2 times]

$$=\frac{1}{6}\left[1-\frac{D^2}{6}+\frac{5D}{6}+\cancel{\frac{25D^2}{36}}\right](x^2+3x-1)$$

$$+\frac{25}{36}\left[D^2(x^2+3x-1)\right]$$

$$=\frac{1}{6}\left[(x^2+3x-1)-\frac{1}{6}D^2(x^2+3x-1)+\frac{5}{6}D(x^2+3x-1)\right]$$

$$=\frac{1}{6}\left[x^2+3x-1-\frac{1}{6}(2)+\frac{5}{6}(2x+3)+\frac{25}{36}(2)\right]$$

$$=\frac{1}{6}\left[x^2+3x-1+\frac{1}{3}+\frac{5}{6}(2x)+\frac{5}{6}(3)+\frac{25}{36}(2)\right]$$

$$=\frac{1}{6}\left[x^2+\left(3+\frac{5}{3}\right)x+\left(\frac{1}{3}-1+\frac{5}{2}+\frac{25}{18}\right)\right]$$

$$=\frac{1}{6}\left[x^2+\frac{14}{3}x+\frac{58}{18}\right]=\frac{1}{6}\left[x^2+\frac{14}{3}x+\frac{26}{9}\right]$$

The complete solution $y=CF+PI$

$$y=C_1e^{2x}+C_2e^{3x}+\frac{1}{6}\left[x^2+\frac{14}{3}x+\frac{26}{9}\right]$$

Solve: $(D^2-5D+6)y=x^2+4e^{3x}$

TYPE 4 : $F(x) = e^{ax} f(x)$

$f(x) = x^n \mid \sin ax \mid \cos ax$, then

$$P.I = \frac{1}{\phi(D)} F(x) = \frac{1}{\phi(D)} e^{ax} f(x)$$

$$= e^{ax} \frac{1}{\phi(D+a)} f(x)$$

replace D by D+a

* In most cases the expression of D we get in $\phi(D)$ would relate to previous forms especially binomial expansions

Example 3.19. *Solve:* $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x} + e^{3x}\sin x$

Solution: Given $(D^2 + 4D + 4)y = e^{-2x} + e^{3x}\sin x$
(i.e.) $\phi(D)y = F(x)$
The auxiliary equation is $m^2 + 4m + 4 = 0$
$\Rightarrow (m+2)^2 = 0 \Rightarrow m = -2, -2$
Roots are real and equal.
$\Rightarrow C.F. = (C_1 + C_2x)e^{-2x}$

$$\text{Now } P.I. = \frac{1}{\phi(D)}F(x) = \frac{1}{(D+2)^2}(e^{-2x} + e^{3x}\sin x)$$

$$= x^2\frac{1}{2}e^{-2x} + e^{3x}\frac{1}{(D+5)^2}\sin x$$

$$= \frac{x^2}{2}e^{-2x} + e^{3x}\frac{1}{D^2 + 10D + 25}\sin x$$

$$= \frac{x^2}{2}e^{-2x} + e^{3x}\frac{1}{-1+10D+25}\sin x$$

$$= \frac{x^2}{2}e^{-2x} + e^{3x}\frac{1}{24+10D}\sin x$$

$$= \frac{x^2}{2}e^{-2x} + e^{3x}\frac{1}{2(12+5D)}\sin x$$

$$= \frac{1}{(D+2)^2}e^{-2x} + \frac{1}{(D+2)^2}e^{3x}\sin x$$

$$= x.\frac{1}{2(D+2)}e^{-2x} + e^{3x}\frac{1}{(D+3+2)^2}\sin x$$

$$= \frac{x^2}{2}e^{-2x} + \frac{e^{3x}}{2}\frac{(12-5D)}{(12-5D)(12+5D)}\sin x$$

$$= \frac{x^2}{2}e^{-2x} + \frac{e^{3x}}{2}\frac{(12-5D)}{(144-25D^2)}\sin x$$

$$= \frac{x^2}{2}e^{-2x} + \frac{e^{3x}}{2}\frac{1}{(144-25D^2)}(12\sin x - 5D\sin x)$$

$$= \frac{x^2}{2}e^{-2x} + \frac{e^{3x}}{338}(12\sin x - 5\cos x)$$

The complete solution is $y = CF + PI$

$$y = (C_1 + C_2x)e^{-2x} + \frac{x^2}{2}e^{-2x} + \frac{e^{3x}}{338}(12\sin x - 5\cos x)$$

$d^3y \quad d^2y \quad dy = x^2e^{-2x}$

TYPE 5 : If $F(x) = x^n \sin ax \,/\, x^n \cos ax$

$$PI = \frac{1}{\phi(D)} F(x) = \frac{1}{\phi(D)} x^n \sin ax \,/\, x^n \cos ax$$

It becomes $\frac{1}{\phi(D)} x^n \cos ax + i \frac{1}{\phi(D)} x^n \sin ax$

$$= \frac{1}{\phi(D)} x^n (\cos ax + i \sin ax) = \frac{1}{\phi(D)} x^n e^{iax}$$

$$= e^{iax} \frac{1}{\phi(D+a)} x^n$$

** Best way to understand, see an example

Scanned by CamScanner

Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x\sin x$

$\Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$
$\Rightarrow C.F. = (c_1 + c_2 x)e^x$

$$\text{Now } P.I = \frac{1}{\phi(D)}F(x) = \frac{1}{(D^2 - 2D + 1)}x\sin x$$

$$= \text{Imaginary part of } \frac{1}{(D^2 - 2D + 1)}x(\cos x + i\sin x)$$

$$= \text{I.P of } \frac{1}{(D^2 - 2D + 1)}xe^{ix} \quad (\Rightarrow e^{ix} = \cos x + i\sin x)$$

$$= \text{I.P of } \left\{e^{ix}\frac{1}{[(D+i)^2 - 2(D+i) + 1]}x\right\}$$

$$= \text{I.P of } \left\{e^{ix}\frac{1}{[(D^2 - 2(1-i)D - 2i]}x\right\}$$

$$= \text{I.P of } \left\{e^{ix}\frac{1}{-2i\left[1 - \left(\frac{D^2 - 2(1-i)D}{2i}\right)\right]}x\right\}$$

$$= \text{I.P of } \left\{e^{ix}\frac{1}{-2i}\left[1 - \left(\frac{D^2 - 2(1-i)D}{2i}\right)\right]^{-1}x\right\}$$

$$= \text{I.P of } \left\{e^{ix}\frac{i}{2}\left[1 + \left(\frac{D^2 - 2(1-i)D}{2i}\right) + \cdots\right]x\right\}$$

$$= \text{I.P of } \left\{e^{ix}\frac{i}{2}[1 + (1+i)D]x\right\}$$

$$= \text{I.P of } \left\{ e^{ix} \frac{i}{2}[x + (1+i)] \right\}$$

$$= \text{I.P of } \left\{ \frac{i}{2}(\cos x + i \sin x)(x + i + 1) \right\}$$

$$= \text{I.P } \left\{ \frac{i}{2}(x \cos x + ix \sin x + i \cos x - \sin x + \cos x + i \sin x) \right\}$$

$$= \text{I.P } \left\{ \frac{1}{2}(ix \cos x - x \sin x - \cos x - i \sin x + i \cos x - \sin x) \right\}$$

$$= \text{I.P } \left\{ \frac{1}{2}(-x \sin x - \cos x - \sin x) \right.$$

$$\left. + \frac{i}{2}(x \cos x - \sin x + \cos x) \right\}$$

$$= \frac{1}{2}(x \cos x - \sin x + \cos x)$$

The complete solution is $y = CF + PI$

$$y = (c_1 + c_2 x)e^x + \frac{1}{2}(x \cos x - \sin x + \cos x)$$

• $\frac{d^2y}{dx^2} + 4y = x\sin x$

The auxiliary equation is $m^2 + 4 = 0 \Rightarrow m^2 = -4$

$\Rightarrow m = \pm i2 = 0 \pm i2 \Rightarrow \alpha \pm i\beta = 0 \pm i2 \Rightarrow \alpha = 0, \beta = 2$

Roots are imaginary.

$\Rightarrow CF = C_1 \cos 2x + C_2 \sin 2x$

Now $P.I. = \frac{1}{\phi(D)} F(x) = \frac{1}{\phi(D)} x f(x)$

$= x . \frac{1}{\phi(D)} f(x) - \frac{\phi'(D)}{[\phi(D)]^2} f(x)$

$= x \frac{1}{D^2 + 4} \sin x - \frac{2D}{(D^2 + 4)^2} \sin x$

$= x \frac{1}{-1 + 4} \sin x - \frac{2}{(D^2 + 4)^2} \cos x$

$= \frac{x}{3} \sin x - \frac{2}{(-1 + 4)^2} \cos x$

$= \frac{x}{3} \sin x - \frac{2}{9} \cos x$

The complete solution is $y = CF + PI$

$y = C_1 \cos 2x + C_2 \sin 2x + \frac{x}{3} \sin x - \frac{2}{9} \cos x$ 19

Example 3.23. *Solve:* $\frac{d^2y}{dx^2} - y = xe^x \sin x$

Solve $\frac{d^2y}{dx^2} - y = xe^x \sin x$

Solution: Given $(D^2-1)y = xe^x \sin x$
(i.e.) $\phi(D)y = F(x)$
The auxiliary equation is $m^2 - 1 = 0 \Rightarrow m = \pm 1$
$\Rightarrow m_1 = 1, m_2 = -1$ and $m_1 \neq 1, m_2$
$\Rightarrow CF = C_1e^{-x} + C_2e^x$

$$P.I. = \frac{1}{\phi(D)}F(x) = \frac{1}{D^2-1}xe^x \sin x$$

First type 4 then type 5

$$= e^x \frac{1}{(D+1)^2-1} x\sin x = e^x \frac{1}{(D^2+2D)} x \sin x$$

$$P.I. = e^x \left[x\frac{1}{(D^2+2D)}\sin x - \frac{(2D+2)}{(D^2+2D)^2}\sin x\right]$$

$$= e^x \left[x\frac{1}{(-1+2D)}\sin x - \frac{(2D+2)}{(-1+2D)^2}\sin x\right]$$

$$= e^x \left[x\frac{2D+1}{(4D^2-1)}\sin x - \frac{2(D+1)}{4D^2-4D+1}\sin x\right]$$

$$= e^x \left[x\frac{2D+1}{(-4-1)}\sin x - \frac{2(D+1)}{-4-4D+1}\sin x\right]$$

$$= e^x \left[\frac{x}{-5}(2\cos x + \sin x) - \frac{2(D+1)}{(-3-4D)}\sin x\right]$$

$$= e^x \left[-\frac{x}{5}(2\cos x + \sin x) - \frac{2(D+1)(-3+4D)}{(9-16D^2)}\sin x\right]$$

$$= e^x \left[-\frac{x}{5}(2\cos x + \sin x) - \frac{2(4D^2+D-3)}{(9+16)}\sin x\right]$$

$$= e^{x}\left[-\frac{x}{5}(2\cos x + \sin x) - \frac{2}{25}(-4\sin x + \cos x - 3\sin x)\right]$$

$$= e^{x}\left[-\frac{x}{5}(2\cos x + \sin x) - \frac{2}{25}(\cos x - 7\sin x)\right]$$

The complete solution is $y = CF + PI$ $y = C_1e^{-x} + C_2e^{x} - e^{x}\left[\frac{x}{5}(2\cos x + \sin x) + \frac{2}{25}(\cos x - 7\sin x)\right]$

EXERCISE