

UNIT-3

①

TWO-DIMENSIONAL RANDOM VARIABLES

Defn:- let S be the sample space. let $X = X(S)$ & $Y = Y(S)$ be two functions each assigning a real numbers to each outcome $s \in S$ then (X, Y) is a two dimensional random Variables.

Two types of two dimensional R.V are .

- i) Discrete R.V
- ii) Continuous R.V

TD Discrete Random Variable:-

If the possible values of (X, Y) are finite or countably infinite, then (X, Y) is called a two dimensional discrete random variable.

TD continuous Random Variable:-

If (X, Y) can assumes all values in a specified Region R in xy -Plane then (X, Y) is called a Two dimensional continuous random Variable.

Joint Probability function (or) Joint Probability mass f_{ij} :-

If (X, Y) be a TDDR.V such that $P(X=x_i, Y=y_j) = P(x_i, y_j) = p_{ij}$ is called the joint Probability f_{ij} (or) joint probability mass f_{ij} .

If (i) $p_{ij} \geq 0 \quad \forall i, j$

(ii) $\sum_j \sum_i p_{ij} = 1$

Joint Probability density f_{ij} :-

If (X, Y) be a TD continuous Random Variable then $f(x, y)$ is called the joint probability density f_{ij} of (X, Y) .

If i) $f(x, y) \geq 0 \quad \forall (x, y) \in R$, R is the region.

ii) $\iint_R f(x, y) dx dy = 1$ or $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$ (or) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx = 1$

Joint cumulative distribution fn. :-

In discrete case :-

$$F(x, y) = P[X \leq x, Y \leq y] = \sum_{j=1}^m \sum_{i=1}^n P(x_i, y_j)$$

In continuous case :-

$$F(x, y) = P[X \leq x, Y \leq y] = \int_{-\infty}^x \int_{-\infty}^y f(x, y) dx dy$$

Marginal Probability function (or) Marginal distribution :-

i) In discrete case :-

* $P[X = x_i] = \sum_{j=1}^m P(x_i, y_j) = P_{i*} \rightarrow$ marginal probability fn. of x .

* $P[Y = y_j] = \sum_{i=1}^n P(x_i, y_j) = P_{*j} \rightarrow$ marginal probability fn. of y .

ii) In continuous case :-

* Marginal ^{prob.} density fn. of x is

$$f_x(x) = f(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

* Marginal ^{prob.} density fn. of y is

$$f_y(y) = f(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

Conditional distribution :-

In discrete case :-

* Conditional probability fn. of x given $y = y_j$ is

$$P[X = x_i / Y = y_j] = \frac{P[X = x_i, Y = y_j]}{P[Y = y_j]}$$

* Conditional probability f_{h_i} of x given $x = x_i$ is

$$P[Y = y_j / X = x_i] = \frac{P[X = x_i, Y = y_j]}{P[X = x_i]}$$

In continuous case:-

* The conditional probability f_{h_i} of x given y is

$$f(x/y) = \frac{f(x,y)}{f(y)} ; f(y) > 0$$

* The conditional probability f_{h_i} of y given x is

$$f(y/x) = \frac{f(x,y)}{f(x)} ; f(x) > 0$$

Independent condition:-

In discrete case:-

$$* P[X = x_i, Y = y_j] = P(x) \cdot P(y) \text{ (or) } P_{ij} = P_i \cdot P_j$$

In continuous case:-

$$* f(x,y) = f(x) \cdot f(y)$$

Note:-

$$1) P[X = x_i, Y = y_j] = P[X = x_i \cap Y = y_j]$$

$$2) P[a_1 \leq x \leq b_1 \cap a_2 \leq y \leq b_2] = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x,y) dx dy$$

3) $f(x,y)$ or $f_{xy}(x,y)$ are both represents Joint probability f_{h_i} .

$$4) f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y).$$

$$5) F(\infty, \infty) = 1 ; F(-\infty, y) = F(x, -\infty) = 0 ; 0 \leq F(x,y) \leq 1.$$

Problems based on discrete Random Variable :-

1. The joint probability mass fn. of (X, Y) is given by $P(X, Y) = K(2X + 3Y)$; $X = 0, 1, 2$; $Y = 1, 2, 3$. Find all the marginal and conditional probability distributions. Also find the probability distribution fn. of $(X+Y)$.

Soln:- Given $P(X, Y) = K(2X + 3Y)$ for $X = 0, 1, 2$ & $Y = 1, 2, 3$.

$X \backslash Y$	1	2	3	$P[X=x]$
0	3K	6K	9K	18K
1	5K	8K	11K	24K
2	7K	10K	13K	30K
$P[Y=y]$	15K	24K	33K	72K $\rightarrow \sum P(x_i, y_j)$

$$\begin{aligned}
 P(0,1) &= K(0+3) = 3K \\
 P(0,2) &= K(0+3 \cdot 2) = 6K \\
 P(0,3) &= K(0+3 \cdot 3) = 9K \\
 P(1,1) &= K(2+3 \cdot 1) = 5K \\
 P(1,2) &= K(2+3 \cdot 2) = 8K \\
 P(1,3) &= K(2+3 \cdot 3) = 11K \\
 P(2,1) &= K(4+3 \cdot 1) = 7K \\
 P(2,2) &= K(4+3 \cdot 2) = 10K \\
 P(2,3) &= K(4+3 \cdot 3) = 13K
 \end{aligned}$$

WKT

$P(X, Y)$ is the joint probability mass fn. $\sum \sum P(x_i, y_j) = 1$

$$\therefore 72K = 1 \Rightarrow \boxed{K = \frac{1}{72}}$$

\therefore The table becomes,

$X \backslash Y$	1	2	3	$P[X=x]$
0	$\frac{3}{72}$	$\frac{6}{72}$	$\frac{9}{72}$	$\frac{18}{72}$
1	$\frac{5}{72}$	$\frac{8}{72}$	$\frac{11}{72}$	$\frac{24}{72}$
2	$\frac{7}{72}$	$\frac{10}{72}$	$\frac{13}{72}$	$\frac{30}{72}$
$P[Y=y]$	$\frac{15}{72}$	$\frac{24}{72}$	$\frac{33}{72}$	1

$$P[X=0]$$

$$P[X=1]$$

$$P[X=2]$$

$$P[Y=1] \quad P[Y=2] \quad P[Y=3]$$

* Marginal probability distribution fn. of x :-

$$P[X=0] = \frac{18}{72}$$

$$P[X=1] = \frac{24}{72}$$

$$P[X=2] = \frac{30}{72}$$

* Marginal probability distribution fn. of y :-

$$P[Y=1] = \frac{15}{72} ; P[Y=2] = \frac{24}{72} ; P[Y=3] = \frac{33}{72}$$

* Conditional distribution of x given y :-

$$P[X=x / Y=y] = \frac{P[X=x, Y=y]}{P[Y=y]}$$

$$(\because P(A|B) = \frac{P(A \cap B)}{P(B)})$$

When Y=1 \Rightarrow

$$P[X=0 / Y=1] = \frac{P[X=0, Y=1]}{P[Y=1]} = \frac{\frac{3}{72}}{\frac{15}{72}} = \frac{3}{15}$$

$$P[X=1 / Y=1] = \frac{P[X=1, Y=1]}{P[Y=1]} = \frac{\frac{5}{72}}{\frac{15}{72}} = \frac{5}{15}$$

$$P[X=2 / Y=1] = \frac{P[X=2, Y=1]}{P[Y=1]} = \frac{\frac{7}{72}}{\frac{15}{72}} = \frac{7}{15}$$

When Y=2

$$P[X=0 / Y=2] = \frac{P[X=0, Y=2]}{P[Y=2]} = \frac{\frac{6}{72}}{\frac{24}{72}} = \frac{6}{24}$$

$$P[X=1 / Y=2] = \frac{P[X=1, Y=2]}{P[Y=2]} = \frac{\frac{8}{72}}{\frac{24}{72}} = \frac{8}{24}$$

$$P[X=2 / Y=2] = \frac{P[X=2, Y=2]}{P[Y=2]} = \frac{\frac{10}{72}}{\frac{24}{72}} = \frac{10}{24}$$

When Y=3 :-

$$P[X=0 / Y=3] = \frac{P[X=0, Y=3]}{P[Y=3]} = \frac{\frac{9}{72}}{\frac{33}{72}} = \frac{9}{33}$$

$$P[X=1 / Y=3] = \frac{P[X=1, Y=3]}{P[Y=3]} = \frac{\frac{11}{72}}{\frac{33}{72}} = \frac{11}{33}$$

$$P[X=2 / Y=3] = \frac{P[X=2, Y=3]}{P[Y=3]} = \frac{\frac{13}{72}}{\frac{33}{72}} = \frac{13}{33}$$

* Conditional distribution of y given $x=x$:-

$$P[Y=y/X=x] = \frac{P[X=x, Y=y]}{P[X=x]}$$

When $x=0$:

$$P[Y=1/X=0] = \frac{P[X=0, Y=1]}{P[X=0]} = \frac{3/72}{18/72} = \frac{3}{18}$$

$$P[Y=2/X=0] = \frac{P[X=0, Y=2]}{P[X=0]} = \frac{6/72}{18/72} = \frac{6}{18}$$

$$P[Y=3/X=0] = \frac{P[X=0, Y=3]}{P[X=0]} = \frac{9/72}{18/72} = \frac{9}{18}$$

When $x=1$:

$$P[Y=1/X=1] = \frac{P[X=1, Y=1]}{P[X=1]} = \frac{5/72}{24/72} = \frac{5}{24}$$

$$P[Y=2/X=1] = \frac{P[X=1, Y=2]}{P[X=1]} = \frac{8/72}{24/72} = \frac{8}{24}$$

$$P[Y=3/X=1] = \frac{P[X=1, Y=3]}{P[X=1]} = \frac{11/72}{24/72} = \frac{11}{24}$$

When $x=2$:-

$$P[Y=1/X=2] = \frac{P[X=2, Y=1]}{P[X=2]} = \frac{7/72}{30/72} = \frac{7}{30}$$

$$P[Y=2/X=2] = \frac{P[X=2, Y=2]}{P[X=2]} = \frac{10/72}{30/72} = \frac{10}{30}$$

$$P[Y=3/X=2] = \frac{P[X=2, Y=3]}{P[X=2]} = \frac{13/72}{30/72} = \frac{13}{30}$$

Probability distribution of $x+y$:-

$x+y$	Probability	
1	$P(0,1) = \frac{3}{72}$	$\frac{3}{72}$
2	$P(0,2) + P(1,1) = \frac{6}{72} + \frac{5}{72} = \frac{11}{72}$	$\frac{11}{72}$
3	$P(0,3) + P(1,2) + P(2,1) = \frac{9}{72} + \frac{8}{72} + \frac{7}{72} = \frac{24}{72}$	$\frac{24}{72}$
4	$P(1,3) + P(2,2) = \frac{11}{72} + \frac{10}{72} = \frac{21}{72}$	$\frac{21}{72}$
5	$P(2,3) = \frac{13}{72}$	$\frac{13}{72}$

2. For the bivariate probability distribution of (X, Y) given below. Find $P[X \leq 1]$, $P[Y \leq 3]$, $P[X \leq 1, Y \leq 3]$, $P[X \leq 1 | Y \leq 3]$, $P[Y \leq 3 | X \leq 1]$ and $P[X+Y \leq 4]$.

$X \backslash Y$	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Soln: -

$X \backslash Y$	1	2	3	4	5	6	$P[X=x]$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{10}{16}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$\frac{8}{64}$
$P[Y=y]$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	$\frac{16}{64}$	1

$P[Y=1]$ $P[Y=2]$ $P[Y=3]$ $P[Y=4]$ $P[Y=5]$ $P[Y=6]$

(i) To find $P[X \leq 1]$:-

$$P[X \leq 1] = P[X=0] + P[X=1]$$

$$P[X \leq 1] = \frac{8}{32} + \frac{10}{16} = \underline{\underline{\frac{28}{32}}}$$

(ii) To find $P[Y \leq 3]$:-

$$P[Y \leq 3] = P[Y=1] + P[Y=2] + P[Y=3]$$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{5+6+11}{64} = \underline{\underline{\frac{23}{64}}}$$

iii) To find $P[X \leq 1, Y \leq 3]$:-

$$P[X \leq 1, Y \leq 3] = P[0,1] + P[0,2] + P[0,3] + P[1,1] + P[1,2] + P[1,3]$$

$$= 0 + 0 + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8}$$

$$= \frac{1+2+2+4}{32} = \frac{9}{32}$$

iv) $P[X \leq 1 / Y \leq 3] = \frac{P[X \leq 1, Y \leq 3]}{P[Y \leq 3]}$ $(P(A/B) = \frac{P(A \cap B)}{P(B)})$

$$= \frac{\frac{9}{32}}{23/64} = \frac{9}{32} \times \frac{64}{23} = \frac{18}{23} //$$

v) $P[Y \leq 3 / X \leq 1] = \frac{P[Y \leq 3, X \leq 1]}{P[X \leq 1]} = \frac{P[X \leq 1, Y \leq 3]}{P[X \leq 1]}$

$$= \frac{9/32}{28/32} = \frac{9}{32} \times \frac{32}{28} = \frac{9}{28} //$$

vi) $P[X+Y \leq 4]$:-

$$P[X+Y \leq 4] = P[0,1] + P[0,2] + P[0,3] + P[0,4] + P[1,1] + P[1,2] + P[1,3] + P[2,1] + P[2,2]$$

$$= 0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{3} + \frac{1}{32}$$

$$= \frac{1+2+2+2+4+1+1}{32} = \frac{13}{32}$$

HW 10 1) The joint distribution of X_1 and X_2 is given by $f(x_1, x_2) = \frac{x_1 + x_2}{21}$; $x_1 = 1, 2$ and 3 ; $x_2 = 1$ and 2 . Find the marginal distribution of X_1 and X_2 .

M.d of $X_1 \Rightarrow P[X_1=1] = 5/21, P[X_1=2] = 7/21 ; P[X_1=3] = 9/21$

M.d of $X_2 \Rightarrow P[X_2=1] = 9/21 ; P[X_2=2] = 12/21$.