(b) Use the method of map-entered variables to find an expression for $F$ from the following map. Treat $C$ and $C'$ as if they were independent variables. Is the result a correct representation of $F$? Is it minimum?

| $B \backslash A$ | 0 | 1 |
|---|---|---|
| 0 | | $C$ |
| 1 | $C'$ | **1** |

(c) Work Problem 6.6.

**9.** In this unit you have learned a "turn-the-crank" type procedure for finding minimum sum-of-products forms for switching functions. In addition to learning how to "turn the crank" and grind out minimum solutions, you should have learned several very important concepts in this unit. In particular, make sure you know:

(a) What a prime implicant is
(b) What an essential prime implicant is
(c) Why the minimum sum-of-products form is a sum of prime implicants
(d) How don't-cares are handled when using the Quine-McCluskey method and the prime implicant chart

**10.** Reread the objectives of the unit. If you are satisfied that you can meet the objectives, take the readiness test.

# Quine-McCluskey Method

The Karnaugh map method described in Unit 5 is an effective way to simplify switching functions which have a small number of variables. When the number of variables is large or if several functions must be simplified, the use of a digital computer is desirable. The Quine-McCluskey method presented in this unit provides a systematic simplification procedure which can be readily programmed for a digital computer.

The Quine-McCluskey method reduces the minterm expansion (standard sum-of-products form) of a function to obtain a minimum sum of products. The procedure consists of two main steps:

1. Eliminate as many literals as possible from each term by systematically applying the theorem $XY + XY' = X$. The resulting terms are called prime implicants.
2. Use a prime implicant chart to select a minimum set of prime implicants which, when ORed together, are equal to the function being simplified and which contain a minimum number of literals.

# 6.1 Determination of Prime Implicants

In order to apply the Quine-McCluskey method to determine a minimum sum-of-products expression for a function, the function must be given as a sum of minterms. (If the function is not in minterm form, the minterm expansion can be found by using one of the techniques given in Section 5.3.) In the first part of the Quine-McCluskey method, all of the prime implicants of a function are systematically formed by combining minterms. The minterms are represented in binary notation and combined using

$$XY + XY' = X \tag{6-1}$$

where $X$ represents a product of literals and $Y$ is a single variable. Two minterms will combine if they differ in exactly one variable.

In order to find all of the prime implicants, all possible pairs of minterms should be compared and combined whenever possible. To reduce the required number of comparisons, the binary minterms are sorted into groups according to the number of 1's in each term. Thus,

$$f(a, b, c, d) = \Sigma\, m(0, 1, 2, 5, 6, 7, 8, 9, 10, 14) \tag{6-2}$$

is represented by the following list of minterms:

| | | |
|---|---|---|
| group 0 | 0 | 0000 |
| group 1 | 1 | 0001 |
| | 2 | 0010 |
| | 8 | 1000 |
| group 2 | 5 | 0101 |
| | 6 | 0110 |
| | 9 | 1001 |
| | 10 | 1010 |
| group 3 | 7 | 0111 |
| | 14 | 1110 |

In this list, the term in group 0 has zero 1's, the terms in group 1 have one 1, those in group 2 have two 1's, and those in group 3 have three 1's.

Two terms can be combined if they differ in exactly one variable. Comparison of terms in nonadjacent groups is unnecessary because such terms will always differ in at least two variables and cannot be combined using $XY + XY' = X$. Similarly, the comparison of terms within a group is unnecessary because two terms with the same number of 1's must differ in at least two variables. Thus, only terms in adjacent groups must be compared.

First, we will compare the term in group 0 with all of the terms in group 1. Terms 0000 and 0001 can be combined to eliminate the fourth variable, which yields 000–. Similarly, 0 and 2 combine to form 00–0 ($a'b'd'$), and 0 and 8 combine to form –000 ($b'c'd'$). The resulting terms are listed in Column II of Table 6-1.

Whenever two terms combine, the corresponding decimal numbers differ by a power of 2 (1, 2, 4, 8, etc.). This is true because when the binary representations differ in exactly one column and if we subtract these binary representations, we get a 1 only in the column in which the difference exists. A binary number with a 1 in exactly one column is a power of 2.

**TABLE 6-1**
Determination of Prime Implicants



| | Column I | Column II | Column III |
|---|---|---|---|
| group 0 | 0 0000 ✓ | 0, 1 000– ✓ | 0, 1, 8, 9 –00– |
| group 1 | 1 0001 ✓ | 0, 2 00–0 ✓ | 0, 2, 8, 10 –0–0 |
| | 2 0010 ✓ | 0, 8 –000 ✓ | ~~0, 8, 1, 9 –00–~~ |
| | 8 1000 ✓ | 1, 5 0–01 | ~~0, 8, 2, 10 –0–0~~ |
| group 2 | 5 0101 ✓ | 1, 9 –001 ✓ | 2, 6, 10, 14 -- 10 |
| | 6 0110 ✓ | 2, 6 0–10 ✓ | ~~2, 10, 6, 14 -- 10~~ |
| | 9 1001 ✓ | 2, 10 –010 ✓ | |
| | 10 1010 ✓ | 8, 9 100– ✓ | |
| group 3 | 7 0111 ✓ | 8, 10 10–0 ✓ | |
| | 14 1110 ✓ | 5, 7 01–1 | |
| | | 6, 7 011– | |
| | | 6, 14 –110 ✓ | |
| | | 10, 14 1–10 ✓ | |

Because the comparison of group 0 with groups 2 and 3 is unnecessary, we proceed to compare terms in groups 1 and 2. Comparing term 1 with all terms in group 2, we find that it combines with 5 and 9 but not with 6 or 10. Similarly, term 2 combines only with 6 and 10, and term 8 only with 9 and 10. The resulting terms are listed in Column II. Each time a term is combined with another term, it is checked off. A term may be used more than once because $X + X = X$. Even though two terms have already been combined with other terms, they still must be compared and combined if possible. This is necessary because the resultant term may be needed to form the minimum sum solution. At this stage, we may generate redundant terms, but these redundant terms will be eliminated later. We finish with Column I by comparing terms in groups 2 and 3. New terms are formed by combining terms 5 and 7, 6 and 7, 6 and 14, and 10 and 14.

Note that the terms in Column II have been divided into groups, according to the number of 1's in each term. Again, we apply $XY + XY' = X$ to combine pairs of terms in Column II. In order to combine two terms, the terms must have the same variables, and the terms must differ in exactly one of these variables. Thus, it is necessary only to compare terms which have dashes (missing variables) in corresponding places and which differ by exactly one in the number of 1's.

Terms in the first group in Column II need only be compared with terms in the second group which have dashes in the same places. Term 000– (0, 1) combines only with term 100– (8, 9) to yield –00–. This is algebraically equivalent to $a'b'c + ab'c' = b'c'$. The resulting term is listed in Column III along with the designation 0, 1, 8, 9 to indicate that it was formed by combining minterms 0, 1, 8, and 9. Term (0, 2) combines only with (8, 10), and term (0, 8) combines with both (1, 9) and (2, 10). Again, the terms which have been combined are checked off. Comparing terms from the second and third groups in Column II, we find that (2, 6) combines with (10, 14), and (2, 10) combines with (6, 14).

Note that there are three pairs of duplicate terms in Column III. These duplicate terms were formed in each case by combining the same set of four minterms in a different order. After deleting the duplicate terms, we compare terms from the two groups in Column III. Because no further combination is possible, the process terminates. In general, we would keep comparing terms and forming new groups of terms and new columns until no more terms could be combined.

The terms which have not been checked off because they cannot be combined with other terms are called prime implicants. Because every minterm has been included in at least one of the prime implicants, the function is equal to the sum of its prime implicants. In this example we have

$$f = \underset{(1,5)}{a'c'd} + \underset{(5,7)}{a'bd} + \underset{(6,7)}{a'bc} + \underset{(0,1,8,9)}{b'c'} + \underset{(0,2,8,10)}{b'd'} + \underset{(2,6,10,14)}{cd'} \tag{6-3}$$

In this expression, each term has a minimum number of literals, but the number of terms is not minimum. Using the consensus theorem to eliminate redundant terms yields

$$f = a'bd + b'c' + cd' \tag{6-4}$$

which is the minimum sum-of-products expression for $f$. Section 6.2 discusses a better method of eliminating redundant prime implicants using a prime implicant chart.

Next, we will define implicant and prime implicant and relate these terms to the Quine-McCluskey method.

**Definition** Given a function $F$ of $n$ variables, a product term $P$ is an *implicant* of $F$ iff for every combination of values of the $n$ variables for which $P = 1$, $F$ is also equal to 1.

In other words, if for some combination of values of the variables, $P = 1$ and $F = 0$, then $P$ is *not* an implicant of $F$. For example, consider the function

$$F(a, b, c) = a'b'c' + ab'c' + ab'c + abc = b'c' + ac \tag{6-5}$$

If $a'b'c' = 1$, then $F = 1$; if $ac = 1$, then $F = 1$; etc. Hence, the terms $a'b'c'$, $ac$, etc., are implicants of $F$. In this example, $bc$ is *not* an implicant of $F$ because when $a = 0$ and $b = c = 1$, $bc = 1$ and $F = 0$. In general, if $F$ is written in sum-of-products form, every product term is an implicant. Every minterm of $F$ is also an implicant of $F$, and so is any term formed by combining two or more minterms. For example, in Table 6-1, all of the terms listed in any of the columns are implicants of the function given in Equation (6-2).

**Definition** A *prime implicant* of a function $F$ is a product term implicant which is no longer an implicant if any literal is deleted from it.

In Equation (6-5), the implicant $a'b'c'$ *is not* a *prime* implicant because $a'$ can be eliminated, and the resulting term ($b'c'$) is still an implicant of $F$. The implicants $b'c'$ and $ac$ are *prime implicants* because *if* we delete a literal from either term, the term will no longer be an implicant of $F$. Each prime implicant of a function has a minimum number of literals in the sense that no more literals can be eliminated from it by combining it with other terms.

The Quine-McCluskey method, as previously illustrated, finds all of the product term implicants of a function. The implicants which are nonprime are checked off in the process of combining terms so that the remaining terms are prime implicants.

A minimum sum-of-products expression for a function consists of a sum of some (but not necessarily all) of the prime implicants of that function. In other words, a sum-of-products expression which contains a term which is not a prime implicant cannot be minimum. This is true because the nonprime term does not contain a minimum number of literals—it can be combined with additional minterms to form a prime implicant which has fewer literals than the nonprime term. Any nonprime term in a sum-of-products expression can thus be replaced with a prime implicant, which reduces the number of literals and simplifies the expression.

## 6.2 The Prime Implicant Chart

Given all the prime implicants of a function, the prime implicant chart can be used to select a minimum set of prime implicants. The minterms of the function are listed across the top of the chart, and the prime implicants are listed down the side.

A prime implicant is equal to a sum of minterms, and the prime implicant is said to cover these minterms. If a prime implicant covers a given minterm, an X is placed at the intersection of the corresponding row and column. Table 6-2 shows the prime implicant chart derived from Table 6-1. All of the prime implicants (terms which have not been checked off in Table 6-1) are listed on the left.

In the first row, X's are placed in columns 0, 1, 8, and 9, because prime implicant $b'c'$ was formed from the sum of minterms 0, 1, 8, and 9. Similarly, X's are placed in columns 0, 2, 8, and 10 opposite the prime implicant $b'd'$ and so forth.

**TABLE 6-2**
Prime Implicant Chart



| | | 0 | 1 | 2 | 5 | 6 | 7 | 8 | 9 | 10 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (0, 1, 8, 9) | $b'c'$ | X | X | | | | | X | ⊗ | | |
| (0, 2, 8, 10) | $b'd'$ | X | | X | | | | X | | X | |
| (2, 6, 10, 14) | $cd'$ | | | X | | X | | | | X | ⊗ |
| (1, 5) | $a'c'd$ | | X | | X | | | | | | |
| (5, 7) | $a'bd$ | | | | X | | X | | | | |
| (6, 7) | $a'bc$ | | | | | X | X | | | | |

If a minterm is covered by only one prime implicant, then that prime implicant is called an *essential* prime implicant and must be included in the minimum sum of products. Essential prime implicants are easy to find using the prime implicant chart. If a given column contains only one X, then the corresponding row is an essential prime implicant. In Table 6-2, columns 9 and 14 each contain one X, so prime implicants $b'c'$ and $cd'$ are essential.

Each time a prime implicant is selected for inclusion in the minimum sum, the corresponding row should be crossed out. After doing this, the columns which correspond to all minterms covered by that prime implicant should also be crossed out. Table 6-3 shows the resulting chart when the essential prime implicants and the corresponding rows and columns of Table 6-2 are crossed out. A minimum set of prime implicants must now be chosen to cover the remaining columns. In this example, $a'bd$ covers the remaining two columns, so it is chosen. The resulting minimum sum of products is

$$f = b'c' + cd' + a'bd$$

which is the same as Equation (6-4). Note that even though the term $a'bd$ is included in the minimum sum of products, $a'bd$ is *not* an *essential* prime implicant. It is the sum of minterms $m_5$ and $m_7$; $m_5$ is also covered by $a'c'd$, and $m_7$ is also covered by $a'bc$.

**TABLE 6-3**



| | | 0 | 1 | 2 | 5 | 6 | 7 | 8 | 9 | 10 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (0, 1, 8, 9) | $b'c'$ | X | X | | | | | X | X | | |
| (0, 2, 8, 10) | $b'd'$ | X | | X | | | | X | | X | |
| (2, 6, 10, 14) | $cd'$ | | | X | | X | | | | X | X |
| (1, 5) | $a'c'd$ | | X | | X | | | | | | |
| (5, 7) | $a'bd$ | | | | X | | X | | | | |
| (6, 7) | $a'bc$ | | | | | X | X | | | | |

When selecting prime implicants for a minimum sum, the essential prime implicants are chosen first because all essential prime implicants must be included in every minimum sum. After the essential prime implicants have been chosen, the minterms which they cover can be eliminated from the prime implicant chart by crossing out the corresponding columns. If the essential prime implicants do not cover all of the minterms, then additional nonessential prime implicants are needed. In simple cases, the nonessential prime implicants needed to form the minimum solution may be selected by trial and error. For larger prime implicant charts, additional procedures for chart reduction can be employed.[1] (Also, see Problem 6.21.) Some functions have two or more minimum sum-of-products expressions, each having the same number of terms and literals. The next example shows such a function.

**Example**

A prime implicant chart which has two or more X's in every column is called a *cyclic* prime implicant chart. The following function has such a chart:

$$F = \Sigma\, m(0, 1, 2, 5, 6, 7) \tag{6-6}$$

Derivation of prime implicants:

| | | | | |
|---|---|---|---|---|
| 0 | 000 | ✓ | 0, 1 | 00– |
| 1 | 001 | ✓ | 0, 2 | 0–0 |
| 2 | 010 | ✓ | 1, 5 | –01 |
| 5 | 101 | ✓ | 2, 6 | –10 |
| 6 | 110 | ✓ | 5, 7 | 1–1 |
| 7 | 111 | ✓ | 6, 7 | 11– |

Table 6-4 shows the resulting prime implicant chart. All columns have two X's, so we will proceed by trial and error. Both (0, 1) and (0, 2) cover column 0, so we will try (0, 1). After crossing out row (0, 1) and columns 0 and 1, we examine column 2, which is covered by (0, 2) and (2, 6). The best choice is (2, 6) because it covers two of the remaining columns while (0, 2) covers only one of the remaining columns. After crossing out row (2, 6) and columns 2 and 6, we see that (5, 7) covers the remaining columns and completes the solution. Therefore, one solution is $F = a'b' + bc' + ac$.

**TABLE 6-4**

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| | | | 0 | 1 | 2 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| ① → | (0, 1) | $a'b'$ | X | X | | | | |
| | (0, 2) | $a'c'$ | X | | X | | | |
| | (1, 5) | $b'c$ | | X | | X | | |
| ② → | (2, 6) | $bc'$ | | | X | | X | |
| ③ → | (5, 7) | $ac$ | | | | X | | X |
| | (6, 7) | $ab$ | | | | | X | X |

[1]For a discussion of such procedures, see E. J. McCluskey, *Logic Design Principles* (Prentice-Hall, 1986).

However, we are not guaranteed that this solution is minimum. We must go back and solve the problem over again starting with the other prime implicant that covers column 0. The resulting table (Table 6-5) is

**TABLE 6-5**



| | | | 0 | 1 | 2 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $P_1$ | (0, 1) | $a'b'$ | × | × | | | | |
| $P_2$ | (0, 2) | $a'c'$ | × | | × | | | |
| $P_3$ | (1, 5) | $b'c$ | | × | | × | | |
| $P_4$ | (2 6) | $bc'$ | | | × | | × | |
| $P_5$ | (5, 7) | $ac$ | | | | × | | × |
| $P_6$ | (6, 7) | $ab$ | | | | | × | × |

Finish the solution and show that $F = a'c' + b'c + ab$. Because this has the same number of terms and same number of literals as the expression for $F$ derived in Table 6-4, there are two minimum sum-of-products solutions to this problem. Compare these two minimum solutions for Equation (6-6) with the solutions obtained in Figure 5-9 using Karnaugh maps. Note that each minterm on the map can be covered by two different loops. Similarly, each column of the prime implicant chart (Table 6-4) has two X's, indicating that each minterm can be covered by two different prime implicants.

# 6.3 Petrick's Method

Petrick's method is a technique for determining all minimum sum-of-products solutions from a prime implicant chart. The example shown in Tables 6-4 and 6-5 has two minimum solutions. As the number of variables increases, the number of prime implicants and the complexity of the prime implicant chart may increase significantly. In such cases, a large amount of trial and error may be required to find the minimum solution(s). Petrick's method is a more systematic way of finding all minimum solutions from a prime implicant chart than the method used previously. Before applying Petrick's method, all essential prime implicants and the minterms they cover should be removed from the chart.

We will illustrate Petrick's method using Table 6-5. First, we will label the rows of the table $P_1$, $P_2$, $P_3$, etc. We will form a logic function, $P$, which is true when all of the minterms in the chart have been covered. Let $P_1$ be a logic variable which is true when the prime implicant in row $P_1$ is included in the solution, $P_2$ be a logic variable which is true when the prime implicant in row $P_2$ is included in the solution, etc. Because column 0 has X's in rows $P_1$ and $P_2$, we must choose row $P_1$ or $P_2$ in order to cover minterm 0. Therefore, the expression $(P_1 + P_2)$ must be true. In order to cover minterm 1, we must choose row $P_1$ or $P_3$; therefore, $(P_1 + P_3)$ must be true. In order