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Tutorial 2

(1) $\int_0^1\int_0^2\int_0^3 xyz\,dx\,dy\,dz = \int_0^1\int_0^2 \frac{9}{2}yz\,dy\,dz$

$= \frac{9}{2}\cdot\frac{4}{2}\int_0^1 z\,dz = \frac{9}{2}\cdot\frac{4}{2}\cdot\frac{1}{2} = \frac{9}{2}.$

(2) $\int_0^1\int_{y^2}^1\int_0^{1-x} x\,dz\,dy\,dx = \int_0^1\int_{y^2}^1 \left[xz\right]_0^{1-x} dy\,dx$

$\int_0^1\int_{y^2}^1 x(1-x)\,dx\,dy$

$= \int_0^1 \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{y^2}^1 dy = \int_0^1 \frac{1}{2}-\frac{1}{3}-\frac{y^4}{2}+\frac{y^6}{3}\,dy.$

$\int_0^1 \frac{1}{6}-\frac{y^4}{2}+\frac{y^6}{3}\,dy.$

$\left[\frac{y}{6}-\frac{y^5}{10}+\frac{y^7}{21}\right]_0^1 = \frac{1}{6}-\frac{1}{10}+\frac{1}{21} = \frac{36}{315} = \frac{4}{35}.$

(3) $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1^2-x^2-y^2-z^2}} = \int_0^1\int_0^{\sqrt{1^2-x^2}} \left[\sin^{-1}\frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$

$= \int_0^1\int_0^{\sqrt{1-x^2}} \pi/_2\,dy\,dx$

$= \int_0^1 \pi/_2 \sqrt{1^2-x^2}$

$= \pi/_2 \int_0^1 \frac{x}{2}\sqrt{1^2-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\Big]$

$= \frac{1}{2}\cdot\frac{\pi^2}{2^2} = \pi^2/8$ //.

(2)

(96)

(4) $\int_0^1\int_0^{1-z}\int_0^{1-y-z} xyz\, dx\, dy\, dz.$

$\int_0^1\int_0^{1-z} yz\left(\frac{x^2}{2}\right)_0^{1-y-z} dy\, dz. = \int_0^1\int_0^{1-z} yz\,\frac{(1-y-z)^2}{2}\, dy.dz.$

$\int_0^1\int_0^{1-z} \frac{yz}{2}\left[(1-z)^2+y^2-2(1-z)(y)\right] dy\,dz$

$\int_0^1\int_0^{1-z} \frac{(1-z^2)yz}{2} + \frac{y^3 z}{2} - \frac{2(1-z)(y^2)z}{2}\, dy\,dz$

$\int_0^1 \frac{(1-z)^2 z}{2}\left[\frac{y^2}{2}\right]_0^{1-z} + z/2\left[\frac{y^4}{4}\right]_0^{1-z} - \frac{2z(1-z)}{2}\left[y^3/3\right]_0^{1-z} dz.$

$\int_0^1 \frac{(1-z)^4 z}{4} + \frac{(1-z)^4}{8}.z - \frac{2z(1-z)^4}{6} dz.$

$\int_0^1 \frac{(1-z)^4 z\,dz}{24.}$ $\quad 1-z=t \quad -dz=dt.$

$\int_0^1 \frac{t^4(1-t)\,dt}{24} = \frac{1}{24}\int_0^1 t^4 - t^5\, dt. = \frac{-1}{24}\left[\left(\frac{t^5}{5}\right)_1^0 - \left(\frac{t^6}{6}\right)_1^0\right]$

$-\frac{1}{24}\left[\left[0-\frac{1}{5}\right]-\left[0-\frac{1}{6}\right]\right] = \frac{-1}{24}\left[-\frac{1}{5}+\frac{1}{6}\right]$

$= \frac{-1}{24}\times\frac{-1}{30} = \frac{1}{720}$ //.

(5) $\iiint_R x-y+z\, dx.dy.dz$ where R is given by $1\le x\le 2,\ 2\le y\le 3,\ 1\le z\le 3.$

$\int_1^3\int_2^3\int_1^2 (x-y+z)\,dx.dy.dz.$ $\int_1^3\int_2^3\left[\frac{x^2}{2}-yx+zx\right]_1^2 dy\,dz$

$=\int_1^3\int_2^3\left[\frac{3}{2}-y+z\right]dy\,dz$ $=\int\left(\frac{3}{2}y-\frac{y^2}{2}+zy\right)_2^3 dz$

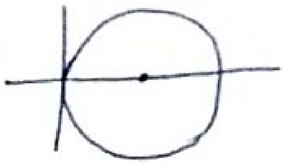
$\int_1^3\left(\frac{3}{2}-\frac{5}{2}+z\right) dz$

$$\left(\frac{3}{2}z - \frac{5}{2}z + \frac{z^2}{2}\right)_1^3$$

$$3 - \frac{\cancel{10}^{5}}{\cancel{2}} + 4 = 2$$

6) i) Find the area of $r^2 = a^2\cos 2\theta$ by double integration.

$$4 \cdot \int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$



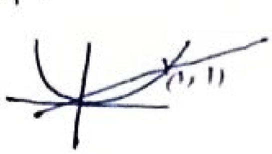
$$4 \cdot \int_0^{\pi/4} \left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$4 \cdot \int_0^{\pi/4} \frac{a^2 \cos 2\theta}{2}\, d\theta$$

$$4 \cdot \frac{a^2}{4}\left[\sin 2\theta\right]_0^{\pi/4}$$

$$= 4 \cdot \frac{a^2}{4}[1] = a^2$$

ii) $y = x$ & $y = x^2$ in first quad.

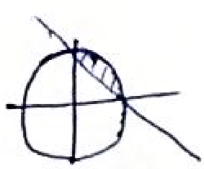


$$\int_0^1 \int_{x^2}^{x} dx\,dy$$

$$= \int_0^1 (x - x^2)\,dx$$

$$\frac{x^2}{2} - \frac{x^3}{3} = \frac{1}{6}$$

7) area bounded $y = 2 - x$ & $x^2 + y^2 = 4$.



$y + x = 2.$

$$\int_0^2 \int_{2-y}^{\sqrt{4-y^2}} dx\,dy. \qquad \int_0^2 \sqrt{4-y^2} - 2 + y\,dy$$

$$\left(\frac{y}{2}\sqrt{4-y^2} + \frac{4}{2}\sin^{-1}\frac{y}{2} - 2y + \frac{y^2}{2}\right)_0^2$$

$$= 0 + 2\,\pi/2 - 4 + 2 = \pi - 2$$

8 Find the volume of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Volume = 8 × volume in first octant

$$= 8 \times \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$= 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\; dy\cdot dx$$

put $r^2 = \left(1-\frac{x^2}{a^2}\right)b^2$

$$= \frac{8c}{b}\int_0^a\int_0^r \sqrt{r^2-y^2}\;dy\,dx$$

$$= \frac{8c}{b}\int_0^a \left[\frac{r^2}{2}\sin^{-1}\frac{y}{r} + \frac{y}{2}\sqrt{r^2-y^2}\right]_0^r$$

$$= \frac{2c\pi}{b}\int_0^a r^2\,dx = \frac{2c\pi}{b}\int_0^a \left(1-\frac{x^2}{a^2}\right)b^2\,dx$$

$$= 2cb\pi\left(x-\frac{x^3}{3a^2}\right)_0^a = \frac{4\pi}{3}abc.$$

(9) Find the area lying inside the circle $r = a\sin\theta$ & outside the cardioid. $r = a(1-\cos\theta)$

Eliminating r we get $\sin\theta+\cos\theta = 1 \Rightarrow \sin^2\theta+\cos^2\theta+\sin 2\theta = 1$

$\therefore \sin 2\theta = 0$

$\theta = \pi/2$ or 0.

$$\int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta = \int_0^{\pi/2} \frac{a^2\sin^2\theta - a^2(1-\cos\theta)^2}{2}\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}(2\cos\theta - 2\cos^2\theta)\,d\theta = a^2(1-\pi/4)$$

$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$

$$= \frac{a^2}{2}\int_0^{\pi/2}(2\sin\theta - 2\cos^2\theta)\,d\theta$$

$$= a^2(1-\pi/4)$$

$$= \frac{a^2}{2}[\pi/2 - 2].$$

10 Find the area $\iint r^3\,dr\,d\theta$ over the bounded between circles $r=2\cos\theta$ $r=4\cos\theta$.

$$\int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r^3\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left(\frac{r^4}{4}\right)_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{(256-16)}{4}\cos^4\theta\, d\theta$$

$$2\cdot\int_0^{\pi/2}\left[60\cdot\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]\right]$$

$$= \frac{\cancel{2}\times 60\times\pi\times 3}{8\times \cancel{2}} = \frac{15\times 3\pi}{2}$$

$$= 45\pi/2.$$