4) Given a random variable Ω with density $f(\omega)$ and another random variable ϕ uniformly distributed in $(-\pi, \pi)$ and independent of Ω and $x(t) = a\cos(\Omega t+\phi)$ prove that $x(t)$ is WSS process.

Soln:- given $x(t) = a\cos(\Omega t+\phi)$ x is uniformly distr. $f(x)$ is pdf

ϕ is uniform distributed

$$\left(f(x) = \begin{cases} \frac{1}{b-a} & a<x<b \\ 0 & \text{otherwise} \end{cases}\right)$$

$$f(\phi) = \begin{cases} \frac{1}{2\pi} & -\pi < \phi < \pi \\ 0 & \text{otherwise.} \end{cases}$$

(i) mean $E[x(t)] = E[a\cos(\Omega t+\phi)]$

$= a\,E[\cos\Omega t\cos\phi - \sin\Omega t \cdot \sin\phi]$

$= a[E(\cos\Omega t)\,E(\cos\phi) - E(\sin\Omega t)\,E(\sin\phi)]$ since Ω & ϕ are indep

$= a\left[E[\cos\Omega t]\int_{-\pi}^{\pi}\cos\phi\,\frac{1}{2\pi}\,d\phi - E[\sin\Omega t]\int_{-\pi}^{\pi}\sin\phi\,\frac{1}{2\pi}\,d\phi\right]$

$= \frac{a}{2\pi}\left[E[\cos\Omega t]\,[\sin\phi]_{-\pi}^{\pi} - E[\sin\Omega t]\,[-\cos\phi]_{-\pi}^{\pi}\right]$

$= \frac{a}{2\pi}\left[E[\cos\Omega t]\,(\sin\pi - \sin(-\pi)) + E[\sin\Omega t]\,\{\cos\pi - \cos(-\pi)\}\right]$

$= \frac{a}{2\pi}\left[E(\cos\Omega t)\,[0-0] + E[\sin\Omega t]\,[-1+(1)]\right]$

$E[x(t)] = 0$ which is a constant.

(ii) $R(t_1, t_2) = E[x(t_1)\cdot x(t_2)]$

$= E[a\cos(\Omega t_1+\phi)\cdot a\cos(\Omega t_2+\phi)]$

$= a^2\,E\left[\frac{1}{2}\left[\cos(\Omega(t_1+t_2)+2\phi) + \cos(\Omega(t_1-t_2)+0)\right]\right]$

$= \frac{a^2}{2}\left[\int_{-\pi}^{\pi}(\cos\Omega(t_1+t_2)+2\phi)\frac{1}{2\pi}\,d\phi\right] + \int_{-\pi}^{\pi}\cos\Omega(t_1-t_2)\,\frac{1}{2\pi}\,d\phi$

$= \frac{a^2}{4\pi}\left[\int_{-\pi}^{\pi}\cos(\Omega(t_1+t_2)+2\phi)\,d\phi + \int_{-\pi}^{\pi}\cos\Omega(t_1-t_2)\,d\phi\right]$

$$= \frac{a^2}{4\pi}\left[\left[\frac{\sin(-\Omega(t_1+t_2)+2\phi)}{2}\right]_{-\pi}^{\pi} + \cos\Omega(t_1-t_2)[\phi]_{-\pi}^{\pi}\right]$$

$$= \frac{a^2}{4\pi}\left[\frac{1}{2}(\sin(-\Omega(t_1+t_2)+2\pi) - \sin(-\Omega(t_1+t_2)-2\pi)\right.$$
$$\left. + \cos\Omega(t_1-t_2)[\pi+\pi]\right]$$

$$= \frac{a^2}{4\pi}\left[\frac{1}{2}(\sin\Omega(t_1+t_2) - \sin\Omega(t_1+t_2)) + 2\pi\cos\Omega(t_1-t_2)\right]$$

$$= \frac{a^2}{\cancel{4\pi}_{2}}\left[0 + \cancel{2\pi}\cos\Omega(t_1-t_2)\right]$$

$$= \frac{a^2}{2}\cos\Omega(t_1-t_2)$$

$R(t_1,t_2)$ is a fn. of t_1-t_2

$\therefore$ $x(t)$ is a WSS.

Cross Correlation function and its properties :-

Cross correlation :-

Defn :- let $\{x(t)\}$ and $\{y(t)\}$ be two random processes then the cross correlation between them is defined as

$$R_{xy}(t, t+\tau) = E[x(t) \cdot y(t+\tau)] = R_{xy}(\tau)$$

Properties :-

1) $R_{xy}(\tau) = R_{yx}(-\tau)$

2) $|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) \cdot R_{yy}(0)}$

3) $|R_{xy}(\tau)| \leq \frac{1}{2}[R_{xx}(0) + R_{yy}(0)]$

4) ~~If the~~ processes $\{x(t)\}$ and $\{y(t)\}$ are orthogonal then $R_{xy}(\tau) = 0$.

5) If the processes $\{x(t)\}$ and $\{y(t)\}$ are independent then $R_{xy}(\tau) = \mu_x \cdot \mu_y$ (or) $R_{xy}(\tau) = E(x) \cdot E(y)$.

1) If $R_{xx}(\tau)$ is the auto correlation fn of random process $\{x(t)\}$, prove a) $R_{xx}(\tau)$ is an even fn of τ
b) $R_{xx}(\tau) \leq R(0)$.

Proof:-

a) $R_{xx}(\tau)$ is even function of τ (ie) To prove, $R_{xx}(\tau) = R_{xx}(-\tau)$

$$R_{xx}(\tau) = E[x(t) \cdot x(t+\tau)]$$

$$R_{xx}(-\tau) = E[x(t) \cdot x(t-\tau)]$$

put $t - \tau = p$

$t = p + \tau$

$$= E[x(p+\tau) \cdot x(p)]$$

$$= E[x(p) \cdot x(p+\tau)]$$

$$R_{xx}(-\tau) = R_{xx}(\tau)$$

(ie) $R_{xx}(\tau) = R_{xx}(-\tau)$

(or)

$R(\tau) = R(-\tau)$ Hence Proved

~~To~~ Prove

b) $|R(\tau)| \leq R(0)$

Proof:- The cauchy-schwarz inequality is

$$[E[xy]]^2 \leq E(x^2) \cdot E(y^2)$$

put $x = x(t)$ & $y = x(t+\tau)$

$$[E[x(t) \cdot x(t+\tau)]]^2 \leq E[x^2(t)] \cdot E[x^2(t+\tau)]$$

$$|R_{xx}(\tau)|^2 \leq [E[x^2(t)]]^2$$

(since $E[x(t)]$ & $Var[x(t)]$ are constant for stationary process)

taking square root on both sides,

$$|R_{xx}(\tau)| \leq E(x^2(t))$$

$$|R_{xx}(\tau)| \leq R_{xx}(0)$$

$(R_{xx}(0) = E[x^2(t)]$

Hence proved, (or) $|R(\tau)| \leq R(0)$

2) If $\{x(t)\}$ is a wss process with mean μ and auto covariance fn.

$$C_{xx}(\tau) = \begin{cases} \sigma_x^2\left(1-\frac{|\tau|}{\tau_0}\right) & \text{for } 0 \le |\tau| \le \tau_0 \\ 0 & \text{for } |\tau| \ge \tau_0 \end{cases}$$

Find the variance of the time average of $\{x(t)\}$ over $(0,T)$. Also examine that the process $\{x(t)\}$ is mean ergodic.

Soln:- we know that

Case(i):-

$$Var[\bar{x}_T] = \frac{1}{T}\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right) c(\tau)\, d\tau$$

$$= \frac{2}{T}\sigma_x^2\int_0^T\left(1-\frac{|\tau|}{T}\right)\left(1-\tau/\tau_0\right) d\tau \quad (0<T<\tau_0)$$

$$= \frac{2\sigma_x^2}{T}\int_0^T\left(1-\frac{\tau}{\tau_0}-\frac{\tau}{T}+\frac{\tau^2}{T\tau_0}\right) d\tau$$

$$= \frac{2\sigma_x^2}{T}\left[\tau - \frac{\tau^2}{2\tau_0} - \frac{\tau^2}{2T} + \frac{\tau^3}{3T\tau_0}\right]_0^T$$

$$= \frac{2\sigma_x^2}{T}\left[T - \frac{T^2}{2\tau_0} - \frac{T^{\cancel{2}}}{2\cancel{T}} + \frac{T^{\cancel{3}2}}{3\cancel{T}\tau_0} - 0\right]$$

$$= \frac{2\sigma_x^2}{T}\left[T - \frac{T}{2} - \frac{T^2}{2\tau_0} + \frac{T^2}{3\tau_0}\right)$$

$$= \frac{2\sigma_x^2}{T}\left[\frac{2T-T}{2} - \frac{1}{\tau_0}\left(\frac{3T^2-2T^2}{6}\right)\right]$$

$$= \frac{2\sigma_x^2}{T}\left[\frac{T}{2} - \frac{T^2}{6\tau_0}\right] = \frac{\cancel{2}\sigma_x^2}{\cancel{T}} \times \frac{\cancel{T}}{\cancel{2}}\left(1 - \frac{T}{3\tau_0}\right)$$

$$= \sigma_x^2\left(1 - \frac{T}{3\tau_0}\right).$$

<u>Case (iii) ($T > \tau_0$)</u>

$$\text{var}[\bar{x}_T] = \frac{2\sigma_x^2}{T}\int_0^{\tau_0}\left(1 - \frac{\tau}{T}\right)\left(1 - \frac{\tau}{\tau_0}\right)d\tau$$

$$= \frac{2\sigma_x^2}{T}\int_0^{\tau_0}\left[1 - \frac{\tau}{\tau_0} - \frac{\tau}{T} + \frac{\tau^2}{T\tau_0}\right]d\tau$$

$$= \frac{2\sigma_x^2}{T}\left[\tau - \frac{\tau^2}{2\tau_0} - \frac{\tau^2}{2T} + \frac{\tau^3}{3T\tau_0}\right]_0^{\tau_0}$$

$$= \frac{2\sigma_x^2}{T}\left[\tau_0 - \frac{\tau_0^{\cancel{2}}}{2\cancel{\tau_0}} - \frac{\tau_0^2}{2T} + \frac{\tau_0^{\cancel{3}2}}{3T\cancel{\tau_0}} - 0\right]$$

$$= \frac{2\sigma_x^2}{T}\left[\tau_0 - \frac{\tau_0}{2} - \frac{\tau_0^2}{2T} + \frac{\tau_0^2}{3T}\right]$$

$$= \frac{2\sigma_x^2}{T}\left[\frac{\tau_0}{2} - \frac{\tau_0^2}{T}\left(\frac{1}{2} + \frac{1}{3}\right)\right]$$

$$= \frac{2\sigma_x^2}{T}\left[\frac{\tau_0}{2} - \frac{\tau_0^2}{6T}\right] = \frac{2\sigma_x^2}{T} \times \frac{1}{2}\left[\tau_0 - \frac{\tau_0^2}{3T}\right]$$

$$= \frac{\sigma_x^2}{T}\left[\tau_0 - \frac{\tau_0^2}{3}\right]$$

When T is sufficiently large

$$\lim_{T\to\infty} \text{var}\,\bar{x}_T = \lim_{T\to\infty} \frac{\sigma_x^2}{T}\left(\tau_0 - \frac{\tau_0^2}{3}\right) = 0$$

$\therefore$ $\bar{x}_T$ is mean ergodic.