Divergence Theorem :-

* point form (or) differential form

$$\nabla \cdot \vec{D} = \rho_v$$

$\vec{D} \rightarrow$ Electric flux density.

* Integral form

$$\oint_S \bar{D} \cdot \bar{ds} = \int_v \rho_v \cdot dv.$$

$$\boxed{\Rightarrow \oint_S \bar{D} \cdot \bar{ds} = \int_V (\nabla \cdot \vec{D}) \cdot dv}$$

Divergence Theorem.

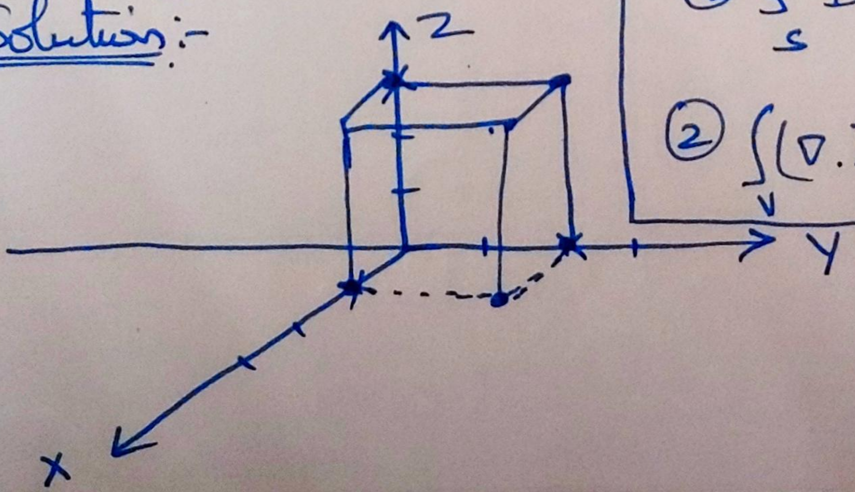
$c/m^2 \cdot m^2 = c$ $\qquad$ $c/m^2 \cdot 1/m \cdot m^3 = c$.

1. Given $\bar{D} = 2xy\,\hat{a}_x + x^2\,\hat{a}_y \; c/m^2$.

Find the total charge lying within the region $0 \leq x \leq 1$; $0 \leq y \leq 2$; $0 \leq z \leq 3$ by two different methods.

Solution:-

$$\boxed{\begin{array}{l} ① \oint_S \bar{D}\,\bar{ds} \\ ② \int_V (\nabla \cdot \vec{D}) \cdot dv \end{array}}$$

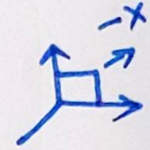


Given:-

$\vec{D} = 2xy\,\hat{a}_x + x^2\,\hat{a}_y \ \text{C/m}^2$.

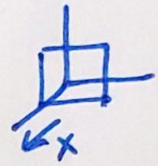
$0 \le x \le 1$

$x = 0$: $d\vec{S}_1 = dy.dz\,(-\hat{a}_x)$



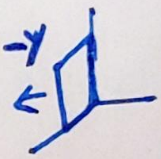
{-ve direction}

$x = 1$: $d\vec{S}_2 = dy.dz\,(\hat{a}_x)$.

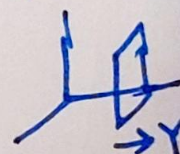


$0 \le y \le 2$.

$y = 0$: $d\bar{S}_3 = dx\,dz\,(-\hat{a}_y)$

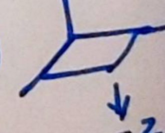


$y = 2$: $d\bar{S}_4 = dx\,dz\,(\hat{a}_y)$.

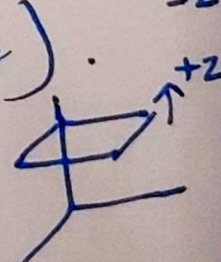


$0 \le z \le 3$

$z = 0$: $d\bar{S}_5 = dx\,dy\,(-\hat{a}_z)$



$z = 3$: $d\bar{S}_6 = dx\,dy\,(\hat{a}_z)$.



$$\bar{D}\cdot\bar{ds}_1 = \left[2xy\,\hat{a}_x + x^2\hat{a}_y\right] dy\,dz\,(-\hat{a}_x)$$

$$= -2xy\,dy\,dz.$$

$$\bar{D}\cdot\bar{ds}_2 = \left[2xy\,\hat{a}_x + x^2\hat{a}_y\right] dy\,dz\,(\hat{a}_x)$$

$$= 2xy\,dy\,dz$$

$$\bar{D}\cdot\bar{ds}_3 = \left[2xy\,\hat{a}_x + x^2\hat{a}_y\right] dx\,dz\,(-\hat{a}_y)$$

$$= -x^2\,dx\,dz.$$

$$\vec{D}\cdot\bar{ds}_4 = \left[2xy\,\hat{a}_x + x^2\hat{a}_y\right] dx\,dz\,(\hat{a}_y).$$

$$= x^2\,dx\,dz.$$

$$\vec{D}\cdot\bar{ds}_5 = \left[2xy\,\hat{a}_x + x^2\hat{a}_y\right] dx\,dy\,(-\hat{a}_z)$$

$$= 0.$$

$$\vec{D}\cdot\bar{ds}_6 = \left[2xy\,\hat{a}_x + x^2\hat{a}_y\right] dx\,dy\,(\hat{a}_z)$$

$$= 0.$$

$$\Rightarrow Q = \oint \vec{D}\cdot\vec{ds} \Rightarrow -\int_{y=0}^{2}\int_{z=0}^{3} 2xy\,dy\,dz\Big|_{x=0}$$

$$+\int_{y=0}^{2}\int_{z=0}^{3} 2xy\,dy\,dz\Big|_{x=1} + \int_{x=0}^{1}\int_{z=0}^{3} (-x^2)\,dx\,dz\Big|_{y=0}$$

$$+ \int_{x=0}^{1}\int_{y=0}^{3} x^2\,dx\,dz\Big|_{y=2}.$$

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$$\Rightarrow Q = -\int_{y=0}^{2}\int_{z=0}^{3} 2xy\,dy\,dz\Big|_{x=0} + \int_{y=0}^{2}\int_{z=0}^{3} 2xy\,dy\,dz\Big|_{x=1}$$

$$-\int_{x=0}^{1}\int_{z=0}^{3} x^2\,dx\,dz\Big|_{y=0} + \int_{x=0}^{1}\int_{z=0}^{3} x^2\,dx\,dz\Big|_{y=2}.$$

$$\Rightarrow -2(0)\left[\frac{y^2}{2}\right]_0^2\left[z\right]_0^3 + 2(1)\left[\frac{y^2}{2}\right]_0^2\left[z\right]_0^3$$

$$-\left[\frac{x^3}{3}\right]_0^1\left[z\right]_0^3 + \left[\frac{x^3}{3}\right]_0^1\left[z\right]_0^3.$$

$$\Rightarrow 0 + 2\left[\frac{2^2}{2} - 0\right]\left[3-0\right] - \left(\frac{1^3}{3} - 0\right)\left[3-0\right]$$

$$+\left(\frac{1^3}{3} - 0\right)\left[3-0\right]$$

$$\Rightarrow 0 + 12 - 1 + 1 = 12\,C$$

$$\boxed{Q = 12\,C}$$

$$Q = \int_v (\bar{\nabla} \cdot \bar{D})\, dv.$$

w.k.t. $\vec{D} = 2xy\,\hat{a}_x + x^2\,\hat{a}_y$ C/m^2.

$$\vec{\nabla} \cdot \vec{D} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial z}(0).$$

$$= 2y + 0 + 0 = 2y.$$

$$Q = \int_v 2y \cdot dx \cdot dy \cdot dz.$$

$$= \int_{x=0}^{1} \int_{y=0}^{2} \int_{z=0}^{3} 2y \cdot dx \cdot dy \cdot dz.$$

$$= 2\left\{\left[\frac{y^2}{2}\right]_0^2 \cdot \left[x\right]_0^1 \cdot \left[z\right]_0^3\right\}$$

$$= 2\left\{\left(\frac{4}{2}\right)(1)(3)\right\} \Rightarrow 12\,C$$

$$\boxed{Q = 12\,C}$$