18MAB102T- Advanced Calculus and Complex Analysis SET-A

ANSWER KEY-CYCLE TEST-II - C_1-slot

PART-A (4 × 5 marks = 20 marks)

1) $\nabla\phi = (2xy+4z^2)\vec{i} + x^2\vec{j} + 8xz\vec{k}$ $\therefore (\nabla\phi)_{(1,-2,-1)} = \vec{j} - 8\vec{k}$ (2 marks)

$\therefore$ Directional derivative $= \dfrac{\nabla\phi \cdot \vec{a}}{|\vec{a}|} = \dfrac{(\vec{j}-8\vec{k})\cdot(2\vec{i}-\vec{j}-2\vec{k})}{3} = \dfrac{15}{3} = 5$ (3 marks)

2) TST $\vec{F}$ is conservative. ie, TST $\vec{F}$ is irrotational.

$$\therefore \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = \vec{0}$$ $\therefore \vec{F}$ is conservative. (5 marks)

3) $L\left[\dfrac{1+2t}{\sqrt{t}}\right] = L\left[t^{-\frac{1}{2}}\right] + 2L\left[t^{\frac{1}{2}}\right]$

$= \left[\dfrac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}}\right] + 2\left[\dfrac{\Gamma(\frac{1}{2}+1)}{s^{\frac{1}{2}+1}}\right] = \left[\dfrac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}}\right] + 2\left[\dfrac{\Gamma(\frac{3}{2})}{s^{\frac{3}{2}}}\right] = \sqrt{\dfrac{\pi}{s}}\left[1+\dfrac{1}{s}\right]$ (2 marks)

(3 marks)

4) $L\left[e^{-4t}(\sin 2t \cos t)\right] = \left\{L\left[\sin 2t \cos t\right]\right\}_{s\to(s+4)}$ (by First shifting property) (1 marks)

$= \frac{1}{2}\left\{L[\sin 3t] + L[\sin t]\right\}_{s\to(s+4)} = \frac{1}{2}\left\{\frac{3}{s^2+9} + \frac{1}{s^2+1}\right\}_{s\to(s+4)}$ (3 marks)

$= \frac{1}{2}\left\{\frac{3}{(s+4)^2+9} + \frac{1}{(s+4)^2+1}\right\}$ (1 marks)

PART-B (3×10 marks = 30 marks)

5) $$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2\cos x+z^3) & (2y\sin x-4) & (3xz^2+2) \end{vmatrix} = \vec{i}(0-0) - \vec{j}(3z^2-3z^2) + \vec{k}(2y\cos x - 2y\cos x)$$

$= \vec{0}$ $\therefore \vec{F}$ is Irrotational.

$\therefore \vec{F}$ is conservative. (4 marks)

$\therefore \vec{F} = \nabla\phi$

Comparing coefficients, we get,

$\dfrac{\partial\phi}{\partial x} = y^2\cos x + z^3 \Rightarrow \phi = y^2\sin x + xz^3 + f(y,z)$

$\dfrac{\partial\phi}{\partial y} = 2y\sin x - 4 \Rightarrow \phi = y^2\sin x - 4y + f(x,z)$

$\dfrac{\partial\phi}{\partial z} = 3xz^2 + 2 \Rightarrow \phi = xz^3 + 2z + f(x,y)$

$\boxed{\therefore \phi = y^2\sin x + xz^3 - 4y + 2z + c}$

(6 marks)

6) To verify Gauss Divergence Theorem.

ie, To verify:- $\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV$ (1 mark)

$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = 2(x+y+z)$. (1 mark)

$\therefore RHS = \int_0^c\int_0^b\int_0^a 2(x+y+z)\,dx\,dy\,dz$ (1 mark)

$\therefore RHS = 2\int_0^c\int_0^b \left(\frac{a^2}{2} + ay + az\right) dy\,dz = 2\int_0^c \left(\frac{a^2 b}{2} + \frac{ab^2}{2} + abz\right) dz = abc(a+\ldots$ (2 m...

Surface	Eqn	$\vec{n}$	$\vec{F}$	$\vec{F}\cdot\vec{n}$	ds
S_1:	$z=0$	$-\vec{k}$	$x^2\vec{i} + y^2\vec{j}$	0	—
S_2:	$z=c$	$\vec{k}$	$x^2\vec{i} + y^2\vec{j} + c^2\vec{k}$	c^2	$dx\,dy$
S_3:	$y=0$	$-\vec{j}$	$x^2\vec{i} + z^2\vec{k}$	0	—
S_4:	$y=b$	$\vec{j}$	$x^2\vec{i} + b^2\vec{j} + z^2\vec{k}$	b^2	$dx\,dz$
S_5:	$x=0$	$-\vec{i}$	$y^2\vec{j} + z^2\vec{k}$	0	—
S_6:	$x=a$	$\vec{i}$	$a^2\vec{i} + y^2\vec{j} + z^2\vec{k}$	a^2	$dy\,dz$

(3 marks)

$\therefore LHS = \iint_S \vec{F}\cdot\vec{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$

$= \int_0^b\int_0^a c^2\,dx\,dy + \int_0^c\int_0^a b^2\,dx\,dz + \int_0^c\int_0^b a^2\,dy\,dz$

$= c^2ab + b^2ac + a^2bc$

$= abc(a+b+c)$ (2 marks)

$\therefore LHS = RHS$

Hence Gauss Div. Thm. Verified.

7) **IVT:** $\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,F(s)$ (1 mark)

Given $f(t) = 1 + e^{-t}(\sin t + \cos t)$

$\therefore L[f(t)] = \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{(s+1)}{(s+1)^2+1}$

$LHS = \lim_{t\to 0}\left[1 + e^{-t}(\sin t + \cos t)\right] = 2$ (1 mark)

$RHS = \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{(s+1)}{(s+1)^2+1}\right] = \lim_{s\to\infty}\left\{1 + \frac{s}{s^2\left[\left(1+\frac{1}{s}\right)^2 + \frac{1}{s^2}\right]} + \frac{s^2\left(1+\frac{1}{s}\right)}{s^2\left[\left(1+\frac{1}{s}\right)^2 + \frac{1}{s^2}\right]}\right\}$ (2 marks)

$= 1 + 0 + 1 = 2$

$\therefore \boxed{LHS = RHS = 2}$ (2 marks)

Hence Initial Value theorem verified.

FVT: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$ (1 mark)

$\lim_{t\to\infty}\left[1 + e^{-t}(\sin t + \cos t)\right] = 1$ (1 mark)

$RHS = \lim_{s\to 0} s\left[\frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{(s+1)}{(s+1)^2+1}\right] = 1$

$\therefore LHS = RHS = 1$ Hence Final Value Thm. verified. (2 marks)

8) $L[f(t)] = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} f(t)\,dt = \frac{1}{1-e^{-2\pi s}}\left\{\int_0^{\pi} e^{-st}(t)\,dt + \int_{\pi}^{2\pi} e^{-st}(2\pi - t)\,dt\right\}$ (1 mark)

$= \frac{1}{1-e^{-2\pi s}}\left\{\left[(t)\left[\frac{e^{-st}}{-s}\right] - (1)\left[\frac{e^{-st}}{s^2}\right]\right]_0^{\pi} + \left[(2\pi - t)\left[\frac{e^{-st}}{-s}\right] - (-1)\left[\frac{e^{-st}}{s^2}\right]\right]_{\pi}^{2\pi}\right\}$ (1 mark)

$= \frac{1}{1-e^{-2\pi s}}\left\{\frac{1 - 2e^{-\pi s} + e^{-2\pi s}}{s^2}\right\} = \frac{1}{s^2}\left[\frac{(1-e^{-\pi s})^2}{(1-e^{-\pi s})(1+e^{-\pi s})}\right]$ (4 marks) (2 marks)

$= \frac{1}{s^2}\left[\frac{e^{\pi s/2} - e^{-\pi s/2}}{e^{\pi s/2} + e^{-\pi s/2}}\right] = \frac{1}{s^2}\tanh\left(\frac{\pi s}{2}\right)$ (2 marks)

18MAB102T – Advanced Calculus and Complex Analysis SET-B

ANSWER KEY — CYCLE TEST-II – C_1-slot.

PART-A (4×5 marks = 20 marks)

1) $\nabla\phi_1 = 2x\vec{i} + z\vec{j} + y\vec{k}$; $(\nabla\phi_1)_{(1,1,1)} = 2\vec{i}+\vec{j}+\vec{k}$ $\therefore |\nabla\phi_1| = \sqrt{6}$ ($1\frac{1}{2}$ marks)

$\nabla\phi_2 = \vec{i} + 2\vec{j} - \vec{k}$ $\therefore |\nabla\phi_2| = \sqrt{6}$ ———— ($1\frac{1}{2}$ marks)

$\cos\theta = \dfrac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \dfrac{2+2-1}{\sqrt{6}\sqrt{6}} = \dfrac{1}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$. (2 marks)

2) Workdone $= \int_C \vec{F}\cdot d\vec{r} = \int_C (x^2-y^2+x)\,dx + (2x+y)\,dy$ (2 marks)

C: $y = x^2$, $dy = 2x\,dx$

$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^1 (x^2 - x^4 + x)\,dx + (2x+x^2)(2x\,dx) = \int_0^1 (-x^4 + \cancel{5}x^2 + \cancel{y}x)\,dx = \cancel{\frac{59}{30}}\ \frac{37}{15}$

$+2x^3$ (3 marks)

3) $L[\cosh at \cos at] = L\left[\left(\dfrac{e^{at}+e^{-at}}{2}\right)\cos at\right] = \dfrac{1}{2}\left\{L[e^{at}\cos at] + L[e^{-at}\cos at]\right\}$ (2 marks)

$= \dfrac{1}{2}\left\{\dfrac{(s-a)}{(s-a)^2+a^2} + \dfrac{(s+a)}{(s+a)^2+a^2}\right\}$ (3 marks)

4) $L[e^t \sin^2 t] = L\left[e^t\left[\dfrac{1-\cos 2t}{2}\right]\right] = \dfrac{1}{2}\left\{L[e^t] - L[e^t\cos 2t]\right\}$ (2 marks)

$= \dfrac{1}{2}\left\{\left(\dfrac{1}{s-1}\right) - \dfrac{(s-1)}{(s-1)^2+4}\right\}$ (3 marks)

PART-B (3×10 marks = 30 marks)

5) $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x+yz) & (4y+xz) & -(6z-xy) \end{vmatrix} = \vec{i}(x-x) - \vec{j}(y-y) + \vec{k}(z-z)$

$= \vec{0}$ $\therefore \vec{F}$ is irrotational.

Hence $\vec{F}$ is conservative. (4 marks)

$\therefore \vec{F} = \nabla\phi = (2x+yz)\vec{i} + (4y+xz)\vec{j} - (6z-xy)\vec{k}$

$\dfrac{\partial\phi}{\partial x} = 2x+yz \Rightarrow \phi = \int(2x+yz)\,dx = x^2 + xyz + f(y,z)$

$\dfrac{\partial\phi}{\partial y} = 4y+xz \Rightarrow \phi = 2y^2 + xyz + f(x,z)$

$\dfrac{\partial\phi}{\partial z} = -(6z-xy) \Rightarrow \phi = -(3z^2 - xyz) + f(x,y)$ (6 marks)

$\boxed{\therefore \phi = x^2 + 2y^2 - 3z^2 + xyz + C}$

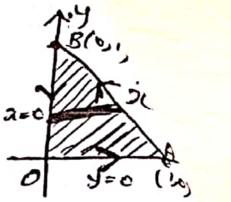
6) To verify Green's Thm.

i.e., To verify $\oint_C M\,dx + N\,dy = \iint_R \left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)dx\,dy$ (1 mark)

$M = 3x^2 - 8y^2$; $N = 4y - 6xy$

$\dfrac{\partial M}{\partial y} = -16y$; $\dfrac{\partial N}{\partial x} = -6y$ (1 mark)

$\therefore$ RHS $= \iint_R (-6y+16y)\,dx\,dy = \int_0^1 \int_0^{1-y} 10y\,dx\,dy = 10\int_0^1 y(1-y)\,dy = \frac{5}{3}$ (2 marks)



Line	Eqn/:	$\vec{F}\cdot d\vec{r} = (3x^2-8y^2)dx + (4y-6xy)dy$
OA	$y=0$, $dy=0$, $0\le x\le 1$	$3x^2\,dx$
AB	$x+y=1$, $y=1-x$, $dy=-dx$, $1\le x\le 0$	$(-11x^2+26x-12)\,dx$
BO	$x=0$, $dx=0$, $1\le y\le 0$	$4y\,dy$

(3 marks)

$\therefore$ LHS $= \oint_C M\,dx + N\,dy = \int_{OA} + \int_{AB} + \int_{BO}$

$$= \int_0^1 3x^2\,dx + \int_1^0 (-11x^2+26x-12)\,dx + \int_1^0 4y\,dy$$

$$= 1 + \frac{8}{3} - 2 = \frac{5}{3}$$ (2 marks)

$\therefore$ LHS = RHS $= \frac{5}{3}$

Hence Green's theorem verified.

7) Given: $f(t) = (t+2)^2 e^{-t} = e^{-t}(t^2+4t+4)$ $\therefore L[f(t)] = \frac{2}{(s+1)^3} + \frac{4}{(s+1)^2} + \frac{4}{(s+1)}$ (2 marks)

IVT: $\underset{t\to 0}{Lt}\, f(t) = \underset{s\to\infty}{Lt}\, sF(s)$ (1 mark)

LHS $= \underset{t\to 0}{Lt}\,(t+2)^2 e^{-t} = 4$

RHS $= \underset{s\to\infty}{Lt}\, s\left[\frac{2}{(s+1)^3} + \frac{4}{(s+1)^2} + \frac{4}{(s+1)}\right]$

$$= \underset{s\to\infty}{Lt}\left[\frac{2s}{s^3(1+\frac{1}{s})^3} + \frac{4s}{s^2(1+\frac{1}{s})^2} + \frac{4s}{s(1+\frac{1}{s})}\right]$$

$= 4$

$\therefore$ LHS = RHS = 4

Hence Initial value theorem verified.

FVT: $\underset{t\to\infty}{Lt}\, f(t) = \underset{s\to 0}{Lt}\, sF(s)$ (1 mark) $\underset{t\to\infty}{Lt}\, e^{-t}(t+2)^2 = 0$ (1 mark)

$\underset{s\to 0}{Lt}\, s\left[\frac{2}{(s+1)^3} + \frac{4}{(s+1)^2} + \frac{4}{(s+1)}\right] = 0$ (2 marks) $\therefore$ LHS = RHS = 0

Hence Final value thm verified.

8) $L[f(t)] = \frac{1}{1-e^{-4s}}\int_0^4 e^{-st} f(t)\,dt$ (1 mark) $= \frac{1}{1-e^{-4s}}\left\{\int_0^2 e^{-st}(t)\,dt + \int_2^4 e^{-st}(4-t)\,dt\right\}$ (1 mark)

$$= \frac{1}{1-e^{-4s}}\left\{\left[(t)\left[\frac{e^{-st}}{-s}\right] - (1)\left[\frac{e^{-st}}{s^2}\right]\right]_0^2 + \left[(4-t)\left[\frac{e^{-st}}{-s}\right] - (-1)\left[\frac{e^{-st}}{s^2}\right]\right]_2^4\right\}$$ (4 marks)

$$= \frac{1}{1-e^{-4s}}\left\{\frac{1-2e^{-2s}+e^{-4s}}{s^2}\right\} = \frac{1}{s^2}\left[\frac{(1-e^{-2s})^2}{(1-e^{-2s})(1+e^{-2s})}\right] = \frac{(1-e^{-2s})}{s^2(1+e^{-2s})}$$

$$= \frac{1}{s^2}\left(\frac{e^{s}-e^{-s}}{e^{s}+e^{-s}}\right) = \frac{1}{s^2}\tanh(s).$$ (2 marks) (2 marks)