SET. THEORY

Set :

It is a collection of well defined distinct objects.

Ex : Set of natural numbers $N = \{1, 2, 3 \cdots\}$
The States in India

Set — roster notation
— set builder notation

Roster Notation: -

All the elements of the set are listed, if possible, separated by commas and enclosed with in braces.

Ex : 1) The Set V of all vowels in the english alphabet

$V = \{a, e, i, o, u\}$

Builder Notation : -

We define the elements of the set by Specifying a property that they have in common.

Ex: -

1) The Set $V = \{x \mid x$ is a vowel in the English Alphabet$\}$
is the same as $A = \{a, e, i, o, u\}$

2) the set $B = \{x \mid x$ is an even +ve integer not exceeding $10\}$
is the same as $B = \{2, 4, 6, 8, 10\}$.

Set operations :

Definition: 1

The union of two sets A and B, denoted by $A \cup B$, is the set of elements that belongs to A or to B or to both.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Ex:- If $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and $C = \{3, 4, 5\}$

then $A \cup B = \{1, 2, 3, 4\}$

$B \cup C = \{2, 3, 4, 5\}$

$A \cup C = \{1, 2, 3, 4, 5\}$

Definition: 2

The intersection of two sets A and B denoted by $A \cap B$. is the set of elements that belongs to both A and B.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Ex: If $A = \{1, 2, 3\}$ $B = \{2, 3, 4\}$ and $C = \{3, 4, 5\}$

then $A \cap B = \{2, 3\}$; $B \cap C = \{3, 4\}$ and $A \cap C = \{3\}$

Definition: 3

If $A \cap B$ is the empty set, viz, If A and B do not have any element in common then the sets A and B are said to be disjoint.

Ex: If $A = \{1, 3, 5\}$ and $B = \{2, 4, 6, 8\}$ then $A \cap B = \phi$ and hence A and B are disjoint.

Difinition : 4

If U is the Universal set and A is any set, then the set of elements which belong to U but which do not belong to A is called the Complement of A and is denoted by A' (or) A^C (or) $\bar{A}$.

$$A' = \{ x \mid x \in U \text{ and } x \notin A \}$$

For Example: If $U = \{1,2,3,4,5\}$ and $A = \{1,3,5\}$ then $\bar{A} = \{2,4\}$.

Difinition : 5

If A and B are any two sets, then the set of elements that belongs to A but do not belong to B is called the difference of A and B or relative complement of B with respect to A and is denoted by $A-B$ or $A \setminus B$.

viz., $$A - B = \{ x \mid x \in A \text{ and } x \notin B \}.$$

For example: if $A = \{1,2,3\}$ and $B = \{1,3,5,7\}$

then $A - B = \{2\}$ and $B - A = \{5,7\}$.

Cartesian Product :-

If A and B are sets, the set of all ordered pairs whose first component belongs to A and second components belongs to B is called the cartesian product of A and B is denoted by $A \times B$.

$$A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$$

Ex: if $A = \{a, b, c\}$ and $B = \{1, 2\}$.. then

$$A \times B = \{ (a,1), (a,2), (b,1), (b,2), (c,1), (c,2) \}$$

$$B \times A = \{ (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) \}$$

$$A \times B \neq B \times A$$

The Algebraic laws of set theory

Set Identities

Name of the law	Identity
1. Identity laws	$A \cup \phi = A$, $A \cap \phi = A$
2. Domination laws	$A \cup U = U$, $A \cap \phi = \phi$
3 Idempotent laws	$A \cup A = A$, $A \cap A = A$
4. Inverse (or) Complement	$A \cup \bar{A} = U$, $A \cap \bar{A} = \phi$
5. Double Complement law	$\bar{\bar{A}} = A$
6. commutative laws	$A \cup B = B \cup A$, $A \cap B = B \cap A$

Associative

$$(A\cup B)\cup C = A\cup(B\cup C)$$
$$(A\cap B)\cap C = A\cap(B\cap C)$$

8. Distributive

$$A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$$
$$A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$$

9. Absorption laws

$$A\cup(A\cap B) = A$$
$$A\cap(A\cup B) = A$$

10. De morgan's law

$$\overline{A\cup B} = \bar{A}\cap\bar{B}$$
$$\overline{A\cap B} = \bar{A}\cup\bar{B}$$

The Duality Principle:-

Note: The dual of any statement is obtained by replacing $\cup$ by $\cap$, $\cap$ by $\cup$, ϕ by U, U by ϕ.

Problems:-

1) Let us use the set builder notation to estabilish this identity. $A\cap B = B\cap A$

Proof:-

$$A\cap B = \{x \mid x\in A\cap B\}$$
$$A\cap B = \{x \mid x\in A \text{ and } x\in B\}$$
$$= \{x \mid x\in B \text{ and } x\in A\}$$
$$= \{x \mid x\in B\cap A\}$$
$$A\cap B = B\cap A$$

Hence proved.

(ii) $\overline{A\cap B} = \bar{A}\cup\bar{B}$

Proof:-

$$\overline{A\cap B} = \{x \mid x\notin A\cap B\}$$

$x\in A\cap B$

$\Rightarrow x\in A$ and $x\in B$

$$= \{x \mid x\notin A \text{ or } x\notin B\}$$
$$= \{x \mid x\in\bar{A} \text{ or } x\in\bar{B}\} = \{x\in\bar{A}\cup\bar{B}\}$$
$$\overline{A\cap B} = \bar{A}\cup\bar{B}$$

3. Prove that $(A-C) \cap (C-B) = \phi$ analytically where A, B, C sets. ~~Verify graphically~~.

Proof:-

$$(A-C)\cap(C-B) = \{x \mid x\in A \text{ and } x\notin C \text{ and } x\in C \text{ and } x\notin B\}$$

$$= \{x \mid x\in A \text{ and } (x\in C \text{ and } x\in \bar{C}) \text{ and } x\in \bar{B}\}$$

$$= \{x \mid x\in A \text{ and } x\in \phi \text{ and } x\in \bar{B}\}$$

$$= \{x \mid x\in A\cap\phi \text{ and } x\in \bar{B}\}$$

$$= \{x \mid x\in \phi \text{ and } x\in \bar{B}\}$$

$$= \{x \mid x\in \phi\cap \bar{B}\}$$

$$(A-C)\cap(C-B) = \{x \mid x\in\phi\} = \phi$$

4. Prove that $A-(B\cap C) = (A-B)\cup(A-C)$ analytically where A, B and C are set's.

Proof:-

$$A-(B\cap C) = \{x \mid x\in A-(B\cap C)\}$$

$$= \{x \mid x\in A \text{ and } x\notin (B\cap C)\}$$

$$= \{x \mid x\in A \text{ and } (x\notin B \text{ or } x\notin C)\}$$

$$= \{x \mid (x\in A \text{ and } x\notin B) \text{ or } (x\in A \text{ and } x\notin C)\}$$

$$= \{x \mid x\in (A-B), \text{ or } x\in (A-C)\}$$

$$= \{x \mid x\in (A-B) \text{ or } x\in (A-C)\}$$

$$= \{x \mid x\in (A-B)\cup(A-C)\}$$

$$A-(B\cap C) = (A-B)\cup(A-C).$$ Hence Proved.

If A, B and C are sets, Prove, both analytically and graphically, that $A \cap (B-C) = (A\cap B) - (A \cap C)$

Proof:-

L.H.S

$$A \cap (B-C) = \{x \mid x \in A \text{ and } x \in (B-C)\}$$

$$= \{x \mid x \in A \text{ and } (x \in B \text{ and } x \notin C)\}$$

$$= \{x \mid x \in A \cap B \cap \bar{C}\}$$

$$A \cap (B-C) = A \cap B \cap \bar{C} \longrightarrow (1)$$

R.H.S

$$(A\cap B) - (A \cap C) = \{x \mid x \in (A \cap B) \text{ and } x \in \overline{(A \cap C)}\}$$

$$= \{x \mid x \in (A\cap B) \text{ and } (x \in \bar{A} \cup \bar{C})\} \quad (\because \text{by Demorgan's law})$$

$$= \{x \mid x \in (A \cap B) \text{ and } (x \in \bar{A} \text{ or } x \in \bar{C})\}$$

(by distributive law)

$$= \{x \mid (x \in (A\cap B) \text{ and } x \in \bar{A}) \text{ or } (x \in A\cap B \text{ and } x \in \bar{C})\}$$

$$= \{x \mid x \in (A \cap B \cap \bar{A}) \text{ or } x \in (A \cap B \cap \bar{C})\}$$

$$= \{x \mid x \in (A \cap \bar{A} \cap B) \text{ or } x \in (A \cap B \cap \bar{C})\}$$

$$= \{x \mid x \in (\phi \cap B) \text{ or } x \in (A \cap B \cap \bar{C})\}$$

$$= \{x \mid x \in \phi \text{ or } x \in (A \cap B \cap \bar{C})\}$$

$$= \{x \mid x \in A \cap B \cap \bar{C}\}$$

$$= A \cap B \cap \bar{C} \longrightarrow (2)$$

From (1) & (2) $\Rightarrow$ Hence Proved.

6. If A, B and C sets prove that $\overline{A \cup B \cap C} = (\bar{C}$ ...
Using set identities.

Proof:-

$\overline{A \cup B \cap C} = \bar{A} \cap \overline{(B \cap C)}$ by De Morgan's law.

$= \bar{A} \cap (\bar{B} \cup \bar{C})$ by Demorgan's law.

$= (\bar{B} \cup \bar{C}) \cap \bar{A}$ by commutative law.

$= (\bar{C} \cup \bar{B}) \cap \bar{A}$ by commutative law.

= R.H.S

Hence Proved.

7. If A, B and C are sets, Prove that

~~...~~ $-C$ $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof:- $A \times (B \cap C) = \{(x, y) \mid x \in A \text{ and } y \in (B \cap C)\}$

$= \{(x, y) \mid x \in A \text{ and } (y \in B \text{ and } y \in C)\}$

$= \{(x, y) \mid (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C)\}$

$= \{(x, y) \mid ((x, y) \in A \times B) \text{ and } ((x, y) \in A \times C)\}$

$= \{(x, y) \mid (x, y) \in (A \times B) \cap (A \times C)\}$

$= \{(x, y) \mid (x, y) \in ((A \times B) \cap (A \times C))\}$

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

Hence Proved.

Let $A=\{1,2,3\}$, $B=\{2,3,4\}$, $C=\{5,6,7\}$ and $D=\{6,7,8\}$ then find $(A\cap B)\times(C\cap D)$.

Soln:

$A\cap B=\{2,3\}$, $C\cap D=\{6,7\}$

$(A\cap B)\times(C\cap D) =$

$= \{(2,6),(2,7),(3,6),(3,7)\}$

9) Simplify the following set using set identities
$(A\cap B)\cup(B\cap((C\cap D)\cup(C\cap \bar{D})))$.

Soln:-

$= (A\cap B)\cup(B\cap((C\cap D)\cup(C\cap\bar{D})))$ — by distributive law

$= (A\cap B)\cup(B\cap(C\cap(D\cup\bar{D})))$ — by inverse law (or complement law)

$= (A\cap B)\cup(B\cap(C\cap U))$ — by identity law

$= (A\cap B)\cup(B\cap C)$ — by commutative law

$= (B\cap A)\cup(B\cap C)$ — by distributive law

$= B\cap(A\cup C)$

10) Write the dual of the following statements:

(i) $A=(\bar{B}\cap A)\cup(A\cap B)$

(ii) $(A\cap B)\cup(\bar{A}\cap B)\cup(A\cap\bar{B})\cup(\bar{A}\cap\bar{B})=U$

Soln:-

$\because$ The dual of (i) is

$A=(\bar{B}\cup A)\cap(A\cup B)$

$\because$ The dual of (ii) is

$(A\cup B)\cap(\bar{A}\cup B)\cap(A\cup\bar{B})\cap(\bar{A}\cup\bar{B})=\phi$

Partition of a set:-

If S is a non-empty set, a collection of disjoint non empty subsets of S whose union is S is called a partition of S. (In other words, the collection of subsets A_i is a partition of S if and only if

(i) $A_i \neq \phi$ for each i

(ii) $A_i \cap A_j = \phi$, for $i \neq j$ and

(iii) $\underset{i}{\cup} A_i = S$ where $\underset{i}{\cup} A_i$ represents the union of the subsets A_i for all i.

Note:-

The subsets in a partion are also called blocks of the partition.

Ex:- if $S = \{1, 2, 3, 4, 5, 6\}$

(i) $\{\{1,2,3\}, \{4,5\}, \{6\}\}$ is a partition

(ii) $[\{1,2\}, \{3,4\}, \{5,6\}]$ is a partition.

(iii) $[\{1,3\}, \{3,5\}, \{2,4,6\}]$ is not a partition.

since $\{1,3\}$ & $\{3,5\}$ are not disjoint.

(iv) $[\{1,3,5\} \{2,4\}]$ is not a partition. Since the Union of subsets is not S, as the element 6 missing.

$\Rightarrow S = \{1, 2, 3, 4\}$

(i) $[\{1\}, \{2\}, \{3\}, \{4\}]$ is a partition.

(ii) $[\{1,2\}, \{3,4\}]$.

(iii) $[\{1\}, \{2,3,4\}]$ is a partition.

(iv) $[\{1\}, \{2,3\}, \{4\}]$

$[\{1,2,3\} \{4\}]$ $[\{1,4\}, \{2,3\}]$

Symmetric difference :-

Defn:

Let A and B are any two sets. Then symmetric difference or Boolean sum of A and B is the set A+B defined by

$$A \Delta B = (A-B) \cup (B-A) \text{ (or) } (A \cup B) - (A \cap B) = A \oplus B$$
$$= (A \cap \bar{B}) \cup (B \cap \bar{A})$$

Ex:- 1) Let $A = \{1,2,3,4,5,6\}$ and $B = \{2,4,6,8\}$

$A+B = \{1,3,5,8\}$

$A \cup B = \{1,2,3,4,5,6,8\}$

$A \cap B = \{2,4,6\}$

$A-B = \{1,3,5\}$

$B-A = \{8\}$

$A+B = \{1,3,5,8\}$

2) If $A = \{a,b,c,d\}$ find $P(A) = ?$ and $|A| = 4$
$|P(A)| = 8$?

3) write all proper subsets of $A = \{a,b,c\}$.

Ans: The proper subsets are

$= \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}\}$

H.W

$A = \{1,2,3\}$ (i) Find all subsets of A
2) Find all proper subsets of A?