Module – 3 Laplace Transforms

Laplace Transforms of standard functions – Transforms properties – Transforms of Derivatives and Integrals – Initial value theorems (without proof) and verification for some problems – Final value theorems (without proof) and verification for some problems – Inverse Laplace transforms using partial fractions – Inverse Laplace transforms using second shifting theorem – LT using Convolution theorem – problems only – ILT using Convolution theorem – problems only – LT of periodic functions – problems only – Solve linear second order ordinary differential equations with constant coefficients only – Solution of Integral equation and integral equation involving convolution type – Application of Laplace Transform in Engineering.

Introduction

Laplace Transformation named after a Great French mathematician **PIERRE SIMON DE LAPLACE** (1749-1827) who used such transformations in his researches related to "Theory of Probability". The powerful practical Laplace transformation techniques were developed over a century later by the English electrical Engineer **OLIVER HEAVISIDE** (1850-1925) and were often called "Heaviside - Calculus".

Transformation

A "Transformation" is an operation which converts a mathematical expression to a different equivalent form.

Laplace Transform

Let $f(t)$ be a given function which is defined for all positive values of t.

If $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$ exists, then F(s) is called ***Laplace transform*** of $f(t)$.

Exponential Order

A function $f(t)$ is said to be of **exponential order** if $\lim_{t\to\infty} e^{-st} f(t) = 0$.

Sufficient conditions for the existence of Laplace transforms

The Laplace transform of $f(t)$ exists if

- **i.** $f(t)$ is continuous or piecewise continuous in $[a, b]$ where $a > 0$.
- **ii.** $f(t)$ is of exponential order.

Example

L [tan t] does not exist since tan t is not piecewise continuous. i.e., tan t has infinite number of infinite discontinuities at $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$

Note:

(i) Not all $f(t)$ are Laplace transformable.

(ii) The above two conditions are not *necessary*.

Laplace transform for some basic functions

S. No.	$f(t)$	$L\{f(t)\}$
1	e^{at}	$\frac{1}{s-a}$, $s-a>0$
2	e^{-at}	$\frac{1}{s+a}$, $s+a>0$
3	$\sin at$	$\frac{a}{s^2+a^2}$, $s>0$
4	$\cos at$	$\frac{s}{s^2+a^2}$, $s>0$
5	$\sinh at$	$\frac{a}{s^2-a^2}$, $s>\lvert a\rvert$
6	$\cosh at$	$\frac{s}{s^2-a^2}$, $s>\lvert a\rvert$
7	1	$\frac{1}{s}$
8	K	$\frac{K}{s}$
9	t	$\frac{1}{s^2}$
10	t^n	$\frac{n!}{s^{n+1}}$, $n = 0, 1, 2, \ldots$.
11	t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$, n is not an integer.
12	$t\,e^{at}$	$\frac{1}{(s-a)^2}$
13	Periodic function with period 'p'	$\frac{1}{1-e^{-sp}}\int_0^p e^{-st} f(t)\,dt$

Properties of Laplace transform:

S. No.	Property	Laplace Transform
1	Linear Property	$L\left(af(t)\pm bg(t)\right)=aL\left(f(t)\right)\pm bL\left(g(t)\right)$
2	First shifting theorem	$L\left(e^{-at}f(t)\right)=F(s+a)$ $L\left(e^{at}f(t)\right)=F(s-a)$
3	Second shifting theorem	If $L(f(t))=F(s)$ and $g(t)=\begin{cases} f(t-a), t>a \\ 0, t<a \end{cases}$, then $L(g(t))=e^{-as}F(s)$.
4	Change of scale property	$L\left(f(at)\right)=\frac{1}{a}F\left(\frac{s}{a}\right),\ a>0$
5	Multiplication by *t*	$L\left(t^n f(t)\right)=(-1)^n\frac{d^n}{ds^n}F(s)$
6	Division by *t*	$L\left(\frac{f(t)}{t}\right)=\int_s^\infty F(s)\,ds,$ provided $\lim\limits_{t\to 0}\frac{f(t)}{t}$ exists
7	Transforms of integrals	$L\left(\int_0^t f(t)dt\right)=\frac{L\left[f(t)\right]}{s}$
8	**Initial Value theorem:** If $L\left(f(t)\right)=F(s)$ then $\lim\limits_{t\to 0}f(t)=\lim\limits_{s\to\infty}sF(s)$	
9	**Final value theorem:** If $L\left(f(t)\right)=F(s)$ then $\lim\limits_{t\to\infty}f(t)=\lim\limits_{s\to 0}sF(s)$	
10	**Convolution of two functions:** The convolution of two functions *f*(*t*) and *g*(*t*) is defined as $\int_0^t f(u)g(t-u)du=f(t)*g(t)$	
11	**Convolution theorem:** The Laplace transform of convolution of two functions is equal to the product of their Laplace transforms. (i.e) $L\left[f(t)*g(t)\right]=L\left[f(t)\right]L\left[g(t)\right]$.	

Problems based on Laplace Transforms

1. **Find** $L\left(2e^{-3t}+3t^2-4\sin 2t+2\cos 3t\right)$**.**

Solution:

$$L\left(2e^{-3t}+3t^2-4\sin 2t+2\cos 3t\right)=\frac{2}{s+3}+3\left(\frac{2}{s^3}\right)-4\left(\frac{2}{s^2+4}\right)+2\left(\frac{s}{s^2+9}\right)$$

2. **Find** $\boldsymbol{L[e^{3t+5}]}$.

Solution:

$$\mathbf{L[e^{3t}.e^5] = e^5L[e^{3t}] = e^5\left(\frac{1}{s-3}\right)}$$

3. **Find the Laplace transform of** $\boldsymbol{f(t)=cos^2(3t)}$**.**

Solution: $L[\cos^2 3t] \quad = L\left[\frac{1+\cos 6t}{2}\right]=\frac{L(1)+L(\cos 6t)}{2} \because \cos^2 t=\frac{1+\cos 2t}{2}$

$$= \frac{1}{2s}+\frac{s}{2(s^2+36)} \because L(1)=\frac{1}{s}, L(\cos at)=\frac{s}{s^2+a^2}$$

$$\therefore \quad L[\cos^2 3t] = \frac{s^2+18}{s(s^2+36)}$$

4. **Find the Laplace transform of** $\boldsymbol{sin^3(2t)}$**.**

Solution: $L\left[sin^3(2t)\right]=\frac{1}{4}L[3\sin 2t-\sin 6t] = \frac{3}{4}L[\sin 2t]-\frac{1}{4}L[\sin 6t]$

$$\left(\because \sin^3 t=\frac{1}{4}[3\sin t-\sin 3t]\right)$$

$$= \frac{3}{4}\left(\frac{2}{s^2+4}\right)-\frac{1}{4}\left(\frac{6}{s^2+36}\right) = \frac{6}{4}\left(\frac{1}{s^2+4}-\frac{1}{s^2+36}\right).$$

5. **Find** $L\left[\sin 8t\cos 4t+\cos^3 4t+5\right]$.

Solution:

$$L\left[\sin 8t\cos 4t+\cos^3 4t+5\right]=L\left[\sin 8t\cos 4t\right]+L\left[\cos^3 4t\right]+L[5]$$

$$L\left[\sin 8t+\cos 4t\right]=L\left[\frac{\sin 12t+\sin 4t}{2}\right] \left[\because \sin A\cos B=\frac{\sin(A+B)+\sin(A-B)}{2}\right]$$

$$=\frac{1}{2}\{L[\sin 12t]+L(\sin 4t)\}$$

$$=\frac{1}{2}\left\{\frac{12}{s^2+144}+\frac{4}{s^2+16}\right\}$$

$$L\left[\cos^3 4t\right]=L\left[\frac{\cos 12t+3\cos 4t}{4}\right]\left[\because \cos^3\theta=\frac{\cos 3\theta+3\cos\theta}{4}\right]$$

$$=\frac{1}{4}\{L(\cos 12t)+3L(\cos 4t)\}$$

$$=\frac{1}{4}\left[\frac{s}{s^2+144}+\frac{3s}{s^2+16}\right]$$

$$L[5]=5L[1]=5\left[\frac{1}{s}\right]=\frac{5}{s}.$$

$$L\left[\sin 8t\cos 4t+\cos^3 4t+5\right]=\frac{1}{2}\left\{\frac{12}{s^2+144}+\frac{4}{s^2+16}\right\}+\frac{1}{4}\left\{\frac{s}{s^2+144}+\frac{3s}{s^2+16}\right\}+\frac{5}{s}.$$

6. **Find the Laplace transform of unit step function**

Solution: The Unit step function is $u_a(t)=\begin{cases}0, & t<a\\ 1, & t>a,\ a\ge 0\end{cases}$

The Laplace transform $L[f(t)]=\int_0^\infty e^{-st}f(t)\,dt=\int_a^\infty e^{-st}(1)\,dt=\left[\frac{e^{-st}}{-s}\right]_a^\infty=-\frac{1}{s}\left[e^{-\infty}-e^{-as}\right]=\frac{e^{-as}}{s}.$

7. **Find** $L\left[t^{3/2}\right]$.

Solution:

We know that $L\left[t^n\right]=\dfrac{\Gamma(n+1)}{s^{n+1}}$

$$L\left[t^{3/2}\right]=\frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{3}{2}+1}}=\frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{s^{\frac{5}{2}}}\ \because\Gamma(n+1)=n\Gamma(n)$$

$$=\frac{\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)}{s^{5/2}}=\frac{\frac{3}{2}.\frac{1}{2}\ \Gamma\left(\frac{1}{2}\right)}{S^{5/2}}$$

$$=\frac{3\sqrt{\pi}}{4s^{5/2}}\quad\left[\therefore\Gamma(1/2)=\sqrt{\pi}\right]$$

Problems based on First Shifting Property

8. **Find the Laplace transform of e^{-t} sin 2t.**

Solution:

$L[e^{-t}\sin 2t] = L[e^{-at} f(t)] = F(s + a) = F(s + 1)$

$F(s) = L[f(t)] = L(\sin 2t) = \frac{2}{s^2+4}$

$F(s + 1) = \frac{2}{(s+1)^2+4} = \frac{2}{s^2+2s+5}$

9. **Find** $L\left[e^{-at}\cos bt\right]$.

Solution:

$$L\left[e^{-at}\cos bt\right] = \left[L(\cos bt)\right]_{s\to s+a}$$

$$= \left[\frac{s}{s^2+b^2}\right]_{s\to s+a}$$

$$= \left[\frac{s+a}{(s+a)^2+b^2}\right]$$

Problems based on Multiplication by *t*

10. **Find the Laplace transform of $e^{-2t}t^{1/2}$.**

Solution: $L\left[e^{-2t}t^{1/2}\right] = L\left[t^{1/2}\right]_{s\to s+2}$

$$\because \textit{If } L[f(t)] = F(s), \text{then } L\left[e^{-at} f(t)\right] = F(s)\big|_{s\to s+a}$$

$$= \left[\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{3/2}}\right]_{s\to s+2} \quad = \left[\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}}\right]_{s\to s+2}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{(s+2)^{3/2}} \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \Gamma n+1 = n\Gamma n\right)$$

11. **Obtain the Laplace transform of** $\sin 2t - 2t\cos 2t$.

Solution: $L[\sin 2t - 2t\cos 2t] \;=\; L[\sin 2t] - 2L[t\cos 2t] = L[\sin 2t] - 2\left(-\frac{d}{ds}L[\cos 2t]\right)$

$$= \frac{2}{s^2+4} + 2\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = \frac{2}{s^2+4} + 2\left(\frac{\left(s^2+4\right)(1) - s(2s)}{\left(s^2+4\right)^2}\right)$$

$$= \frac{2\left(s^2+4\right) + 2\left(4-s^2\right)}{\left(s^2+4\right)^2}$$

$$\therefore L[\sin 2t - 2t\cos 2t] = \frac{16}{\left(s^2+4\right)^2}$$

12. **Find** $L\left(t e^t\right)$**.**

Solution

$$L\left(t\, f(t)\right) = -\frac{d}{ds} L(f(t))$$

$$L\left(t e^t\right) = -\frac{d}{ds} L(e^t)$$

$$= -\frac{d}{ds} L\left(\frac{1}{s-1}\right) = \frac{1}{(s-1)^2}$$

13. **Find** $L\left(t \sin 2t\right)$**.**

Solution

$$L\left(t\, f(t)\right) = -\frac{d}{ds} L(f(t))$$

$$L\left(t \sin 2t\right) = -\frac{d}{ds} L(\sin 2t)$$

$$= -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{\left(s^2+4\right)^2}$$

14. **Find the Laplace transform of $f(t) = \boldsymbol{t^2 \cos t}$.**

Solution

$$L\left[t^2 \cos t\right] = \left[\frac{d^2}{ds^2} L[\cos t]\right] = \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{d}{ds}\left(\frac{\left(s^2+1\right).1 - 1.2s.s}{\left(s^2+1\right)^2}\right) = \frac{d}{ds}\left(\frac{1-s^2}{\left(s^2+1\right)^2}\right)$$

$$= \frac{\left(s^2+1\right)^2(-2s) - \left(1-s^2\right)2\left(s^2+1\right)2s}{\left(s^2+1\right)^3} = \frac{-2s\left(3-s^2\right)}{\left(s^2+1\right)^3}$$

15. **Find the Laplace Transform of $\boldsymbol{f(t) = e^{-t} t\ cost}$.**

Solution

$$L[e^{-t} tcost] = -\frac{d}{ds} L[\cos t]_{s \to s+1} = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]_{s \to s+1}$$

$$= -\left[\frac{(s^2+1)(1) - s(2s)}{(s^2+1)^2}\right]_{s \to s+1}$$

$$= \left[\frac{s^2-1}{(s^2+1)^2}\right]_{s \to s+1}$$

$$= \frac{(s+1)^2-1}{((s+1)^2+1)^2} = \frac{s^2+2s}{(s^2+2s+2)^2}$$

$$= \frac{s(s+2)}{(s^2+2s+2)^2}$$

16. **Find the Laplace transform of $\boldsymbol{f(t) = te^{-3t} cos2t}$.**

Solution

$$L[f(t)] = L[te^{-3t} cos2t] = -\frac{d}{ds} L[\cos 2t]_{s \to s+3} = -\frac{d}{ds}\left[\frac{s}{s^2+4}\right]_{s \to s+3}$$

$$= -\left[\frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}\right]_{s \to s+3} = \left[\frac{s^2-4}{(s^2+4)^2}\right]_{s \to s+3}$$

$$= \frac{(s+3)^2-4}{((s+3)^2+4)^2}$$

$$= \frac{s^2+6s+5}{(s^2+6s+13)^2}$$

17. **Find $L\left[t^2 e^{-t}\cos t\right]$.**

Solution:

$$L\left[t^2 e^{-t}\cos t\right] = L\left[t^2\cos t\right]_{s\to s+1}$$

$$=\left[(-1)^2\frac{d^2}{ds^2}L[\cos t]\right]_{s\to s+1} = \left[\frac{d^2}{ds^2}\left[\frac{s}{s^2+1}\right]\right]_{s\to s+1}$$

$$=\left[\frac{d}{ds}\frac{\left(s^2+1\right)1-s.2s}{\left(s^2+1\right)^2}\right]_{s\to s+1}$$

$$=\left[\frac{d}{ds}\frac{1-s^2}{\left(s^2+1\right)^2}\right]_{s\to s+1} = \left[\frac{2s^3-6s}{\left(s^2+1\right)^3}\right]_{s\to s+1} = \frac{2(s+1)^3-6(s+1)}{\left((s+1)^2+1\right)^3}$$

18. **Find $L\left[t^2 e^{t}\sin t\right]$**

Solution:

$$L\left[t^2 e^{t}\sin t\right] = (-1)^2\frac{d^2}{ds^2}L\left[e^t\sin t\right] \quad \ldots (1)$$

Now $L\left[e^t\sin t\right] = \left[L[\sin t]\right]_{s\to(s-1)} = \dfrac{1}{(s-1)^2+1} \quad \ldots \; (2)$

Substituting (2) in (1) we get

$$L\left[t^2 e^{t}\sin t\right] = \frac{d}{ds}\left[\frac{0-2(s-1)}{\left((s-1)^2+1\right)^2}\right] = \frac{d}{ds}\left[\frac{-2(s-1)}{\left(s^2-2s+2\right)^2}\right]$$

$$=\frac{\left(s^2-2s+2\right)^2(-2)+2(s-1)2\left(s^2-2s+2\right)(2s-2)}{\left(s^2-2s+2\right)^4}$$

$$=\frac{2\left(s^2-2s+2\right)\left[-\left(s^2-2s+2\right)+4(s-1)^2\right]}{\left(s^2-2s+2\right)^4}$$

$$= \frac{2\left(s^2-2s+2\right)\left[-s^2+2s-2+4s^2+4-8s\right]}{\left(s^2-2s+2\right)^4}$$

$$\therefore F(s) = \frac{2\left(s^2-2s+2\right)\left[3s^2-6s+2\right]}{\left(s^2-2s+2\right)^4} = \frac{2\left(3s^2-6s+2\right)}{\left(s^2-2s+2\right)^3}$$

Problems based on Division by *t*

19. **Find $\boldsymbol{L\left[\frac{sint}{t}\right]}$.**

Solution

$$L\left[\frac{sint}{t}\right]=L\left[\frac{f(t)}{t}\right]=\int_s^\infty F(s)ds$$

$$\text{F(s)} = \text{L } [sint]=\frac{1}{s^2+1^2}$$

$$\int_s^\infty F(s)ds=\int_s^\infty \frac{1}{s^2+1}ds=[tan^{-1}(s)]_s^\infty$$

$$=[tan^{-1}\infty - tan^{-1}s]=\left[\frac{\pi}{2}-tan^{-1}s\right]=cot^{-1}s$$

20. **Find the Laplace transform of $\boldsymbol{\frac{e^{-t}\sin t}{t}}$.**

Solution:

$$L\left(\frac{e^{-t}\sin t}{t}\right) = \int_s^\infty L\left(e^{-t}\sin t\right)ds$$

$$= \int_s^\infty L\left(\sin t\right)_{s+1} ds = \int_s^\infty \left(\frac{1}{s^2+1}\right)_{s+1} ds = \int_s^\infty \frac{1}{(s+1)^2+1} ds$$

$$= \left[\tan^{-1}(s+1)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$$

21. **Find $\boldsymbol{L\left[\frac{\sin^2 t}{t}\right]}$.**

Solution:

$$L\left[\frac{\sin^2 t}{t}\right] = L\left[\frac{1-\cos 2t}{2t}\right] = \frac{1}{2}L\left[\frac{1-\cos 2t}{t}\right] = \frac{1}{2}\int_s^\infty L\left[1-\cos 2t\right]\ ds$$

$$= \frac{1}{2}\int_s^\infty \left\{L[1] - L[\cos 2t]\right\} ds \; = \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log\left(s^2+4\right)\right]_s^\infty \; = \frac{1}{2}\left[\log\frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{1}{\sqrt{1+\frac{4}{s^2}}}\right]_s^\infty \; = \frac{1}{2}\left[\log 1 - \log\frac{1}{\sqrt{1+\frac{4}{s^2}}}\right] = \frac{1}{2}\left[0 - \log\frac{s}{\sqrt{s^2+4}}\right]$$

$$F(s) \; = \frac{1}{2}\log\left(\frac{s}{\sqrt{s^2+4}}\right)^{-1} \; = \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$$

22. **Find the Laplace Transform of $f(t) = \frac{1-cost}{t}$.**

Solution

$$L[1-cost] = \frac{1}{s} - \frac{s}{s^2+1}$$

$$L\left[\frac{1-cost}{t}\right] = \int_s^\infty L[1-cost]ds = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= -\frac{1}{2}[\log(s^2+1) - logs^2]_s^\infty$$

$$= -\frac{1}{2}\left[log\frac{s^2+1}{s^2}\right]_s^\infty = -\frac{1}{2}\left[log\left(1+\frac{1}{s^2}\right)\right]_s^\infty$$

$$= -\frac{1}{2}\log 1 + \frac{1}{2}log\left[1+\frac{1}{s^2}\right] \; = \frac{1}{2}log\left(\frac{s^2+1}{s^2}\right)$$

23. **Find $L\left[\frac{cos\,at - cos\,bt}{t}\right]$.**

Solution

$$L\left[\frac{cosat - cosbt}{t}\right] = \int_s^\infty L[cosat - cosbt]ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \left[\frac{1}{2}log(s^2+a^2) - \frac{1}{2}log(s^2+b^2)\right]_s^\infty$$

$$=\frac{1}{2}\left[log\,\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty=\frac{1}{2}\left[log\,\frac{s^2\left(1+\frac{a^2}{s^2}\right)}{s^2\left(1+\frac{b^2}{s^2}\right)}\right]_s^\infty$$

$$=\frac{1}{2}\left[log1-log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right)\right]=\frac{1}{2}\ log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

24. **Using Laplace transform, evaluate $\int_0^\infty t\,e^{-2t}\sin t\,dt$**

Solution: $\int_0^\infty e^{-2t} f(t)dt=\left[\int_0^\infty e^{-st} f(t)dt\right]_{s=2}$ $=$ $\left[L[t\sin t]\right]_{s=2}=\left[-\frac{d}{ds}L\left[\sin t\right]\right]_{s=2}$

$$=-\frac{d}{ds}\left(\frac{1}{s^2+1}\right)=-\left(\frac{-2s}{\left(s^2+1\right)^2}\right)=\frac{4}{25}$$

Problems based on Convolution Theorem

25. **Evaluate $\int_0^t \sin u\cos(t-u)du$ using Laplace Transform.**

Solution: Let $L\left[\int_0^t \sin u\cos(t-u)du\right]=L\left[\sin t * \cos t\right]$

= L[sin t] L[cos t] (by Convolution theorem)

$$=\frac{1}{\left(s^2+1\right)}\frac{s}{\left(s^2+1\right)}=\frac{s}{\left(s^2+1\right)^2}$$

$$\int_0^t \sin u\cos(t-u)du=L^{-1}\left[\frac{s}{\left(s^2+1\right)^2}\right]=\frac{1}{2}L^{-1}\left[\frac{2s}{\left(s^2+1\right)^2}\right]=\frac{t}{2}\sin t\ \left(\because L^{-1}\left[\frac{2s}{\left(s^2+a^2\right)^2}\right]=t\sin at\right)$$

26. **Find the Laplace transform of $\int_0^t t\,e^{-t}\sin t\,dt$**

Solution:

$$L\left[\sin t\right]=\frac{1}{s^2+1}$$

$$L[t\sin t] = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\left(\frac{(s^2+1)0-2s}{(s^2+1)^2}\right) = \frac{2s}{(s^2+1)^2}$$

$$\therefore\ L\left[te^{-t}\ \sin t\right] = \left.\frac{2s}{(s^2+1)^2}\right|_{s\to s+1} = \frac{2(s+1)}{\left((s+1)^2+1\right)^2} = \frac{2(s+1)}{(s^2+2s+2)^2}$$

$$L\left[\int_0^t t\,e^{-t}\sin t\,dt\right] = \frac{1}{s}L\left[te^{-t}\ \sin t\right]$$

$$\therefore \qquad = \frac{1}{s}\frac{2(s+1)}{(s^2+2s+2)^2}$$

27. **Find the Laplace transform of $e^{-t}\int_0^t t\cos t\,dt$.**

$$L\left[e^{-t}\int_0^t t\cos t\,dt\right] = \left[L\left(\int_0^t t\cos t\,dt\right)\right]_{s\to s+1} = \left[\frac{1}{s}L(t\cos t)\right]_{s\to(s+1)}$$

$$= \left[\frac{1}{s}\left(-\frac{d}{ds}L(\cos t)\right)\right]_{s\to(s+1)} = \left[-\frac{1}{s}\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]_{s\to(s+1)}$$

$$= \left[-\frac{1}{s}\left(\frac{s^2+1-2s^2}{(s^2+1)^2}\right)\right]_{s\to(s+1)} = \left[-\frac{1}{s}\left(\frac{1-s^2}{(s^2+1)^2}\right)\right]_{s\to(s+1)}$$

$$\therefore F(s) = \left[\frac{s^2-1}{s(s^2+1)^2}\right]_{s\to(s+1)} = \left[\frac{(s+1)^2-1}{(s+1)\left[(s+1)^2+1\right]^2}\right] = \frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$$

28. **Find the Laplace transform of** $e^{-4t}\int_0^t t\ \sin 3t dt$.

Solution:

$$L[\sin 3t] = \frac{3}{s^2+9}$$

$$L\left[t\sin 3t\right] = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = -\left(\frac{\left(s^2+9\right)0 - 3(2s)}{\left(s^2+9\right)^2}\right) = \frac{6s}{\left(s^2+9\right)^2}$$

$$L\left(\int_0^t t\sin 3t dt\right) = \frac{L(t\sin 3t)}{s} = \frac{6}{(s^2+9)^2}$$

$$L\left(e^{-4t}\int_0^t t\sin 3t dt\right) = L\left(\int_0^t t\sin 3t dt\right)\Bigg|_{s\to s+4} = \frac{6}{\left(\left(s+4\right)^2+9\right)^2} = \frac{6}{\left(s^2+8s+16+9\right)^2}$$

$$\therefore\ L\left(e^{-4t}\int_0^t t\sin 3t dt\right) = \frac{6}{\left(s^2+8s+25\right)^2}$$

Problems based on Initial and Final Value Theorems

29. **Verify initial value theorem for the function *f* (*t*) = 2 – cos t.**

Solution

Initial value theorem states that $\lim\limits_{t\to 0} f\left(t\right) = \lim\limits_{s\to\infty} sF\left(s\right)$

L. H. S. = $\lim\limits_{t\to 0} f(t) = 2 - \cos 0 = 1$

R. H. S. = $\lim\limits_{s\to\infty} s\,L(f(t)) = \lim\limits_{s\to\infty} s\,L(2-\cos t)$

$$= \lim_{s\to\infty} s\left(2 - \frac{s^2}{s^2+1}\right) == \lim_{s\to\infty} s\left(2 - \frac{1}{1+\frac{1}{s^2}}\right) = 2 - 1 = 1$$

L.H.S=R.H.S

Initial value theorem verified.

30. **Verify initial and final value theorems for the function f(t) = 1 + e^{-t} (sin *t* + cos *t*).**

Solution:

Initial value theorem states that $\lim\limits_{t\to 0} f\left(t\right) = \lim\limits_{s\to\infty} sF\left(s\right)$

$$L\left[f\left(t\right)\right] = F\left(s\right)$$

$$= \frac{1}{s} + L\left[\sin t + \cos t\right]_{s\to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} = \frac{1}{s} + \frac{s+2}{(s+1)^2+1}$$

$$\text{L.H.S.} = \lim_{t\to 0} f(t) = 1+1 = 2$$

$$\text{R.H.S} = \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{s+2}{(s+1)^2+1}\right] = \lim_{s\to\infty}\left[1 + \frac{s(s+2)}{(s+1)^2+1}\right]$$

$$= \lim_{s\to\infty}\left[1 + \frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left[1+\frac{2}{s}+\frac{2}{s^2}\right]}\right] = \lim_{s\to\infty}\left[1 + \frac{1+\frac{2}{s}}{1+\frac{2}{s}+\frac{2}{s^2}}\right] = 1+1 = 2$$

L.H.S=R.H.S

Initial value theorem verified.

Final value theorem states that $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\text{L.H.S.} = \lim_{t\to\infty}\left[1 + e^{-t}(\sin t + \cos t)\right] = 1+0 = 1$$

$$\text{R.H.S} = \lim_{s\to 0}\left[1 + \frac{s(s+2)}{(s+1)^2+1}\right] = 1+0 = 1$$

L.H.S.=R.H.S Hence final value theorem verified.

Problems based on Periodic Functions

Periodic function

A function $f(t)$ is said to be **periodic function** if $f(t+p) = f(t)$ for all t. The least value of $p > 0$ is called the **period** of $f(t)$. For example, $\sin t$ and $\cos t$ are periodic functions with period 2π. $\tan t$ is a periodic function with period π.

31. **Find the Laplace transform of the square wave function defined by** $f(t) = \begin{cases} E, & 0 < t < \frac{a}{2} \\ -E, & \frac{a}{2} < t < a \end{cases}$ **&** $f(t+a) = f(t)$.

Solution:

$$L[f(t)] = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} e^{-st}\ f(t)\,dt + \int_{a/2}^{a} e^{-st}\ f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} Ee^{-st}\ dt + \int_{a/2}^{a} e^{-st}(-E)\,dt\right] = \frac{E}{1-e^{-as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{a/2} - \left(\frac{e^{-st}}{-s}\right)_{a/2}^{a}\right]$$

$$= \frac{E}{s\left(1-e^{-as}\right)}\left[-\left(e^{-\frac{as}{2}}-1\right)+\left(e^{-as}-e^{-\frac{as}{2}}\right)\right]$$

$$= \frac{E}{s\left(1-e^{-as}\right)}\left[-e^{-\frac{as}{2}}+1+e^{-as}-e^{-\frac{as}{2}}\right]$$

$$= \frac{E}{s\left(1-e^{-\frac{as}{2}}\right)\left(1+e^{-\frac{as}{2}}\right)}\left(1-e^{-\frac{as}{2}}\right)^2 = \frac{E}{s}\left(\frac{1-e^{-\frac{as}{2}}}{1+e^{-\frac{as}{2}}}\right)$$

$$\therefore F(s) = \frac{E}{s}\left[\frac{e^{sa/4}-e^{-sa/4}}{e^{sa/4}+e^{-sa/4}}\right] = \frac{E}{s}\tanh\left(\frac{sa}{4}\right)$$

32. **Find the Laplace transform of the rectangular wave given by** $f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$.

Solution:

Given $f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$

This function is periodic in the interval $(0, 2b)$ with period 2*b*.

$$L\left[f(t)\right] = \frac{1}{1-e^{-2bs}}\int_0^{2b} e^{-st}\ f(t)\,dt$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^{b} e^{-st}\ f(t)\,dt + \int_b^{2b} e^{-st}\ f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^{b} e^{-st}\ dt + \int_b^{2b} e^{-st}(-1)\,dt\right] = \frac{1}{1-e^{-2bs}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{b} - \left(\frac{e^{-st}}{-s}\right)_b^{2b}\right]$$

$$= \frac{1}{s\left(1-e^{-2bs}\right)}\left[-\left(e^{-bs}-1\right)+\left(e^{-2bs}-e^{-bs}\right)\right]$$

$$= \frac{1}{s\left(1-e^{-2bs}\right)}\left[-e^{-bs}+1+\left(e^{-bs}\right)^2-e^{-bs}\right]$$

$$= \frac{1}{s\left(1-e^{-bs}\right)\left(1+e^{-bs}\right)}\left(1-e^{-bs}\right)^2 = \frac{1}{s}\left(\frac{1-e^{-bs}}{1+e^{-bs}}\right)$$

$$\therefore F(s) = \frac{1}{s}\left[\frac{e^{sb/2}-e^{-sb/2}}{e^{sb/2}+e^{-sb/2}}\right] = \frac{1}{s}\tanh\left(\frac{sb}{2}\right)$$

33. **Find the Laplace transform of f(t) =** $\begin{cases} t, & 0 \le t \le a \\ 2a-t, & a \le t \le 2a \end{cases}$ **and** $f(t+2a) = f(t)$ **for all** t.

Solution:

$$L\left[f(t)\right] = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^{a} e^{-st} f(t)\,dt + \int_a^{2a} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^{a} e^{-st}\, t\,dt + \int_a^{2a} e^{-st}(2a-t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[t\left(\frac{e^{-st}}{-s}\right)-(1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right)-(-1)\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[-t\left(\frac{e^{-st}}{s}\right)-\left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \left[-(2a-t)\left(\frac{e^{-st}}{s}\right)+\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[\left(-a\,\frac{e^{-as}}{s}-\frac{e^{-as}}{s^2}\right)-\left(-\frac{1}{s^2}\right)\right]+\left[\frac{e^{-2as}}{s^2}-\left(-\frac{ae^{-as}}{s}+\frac{e^{-as}}{s^2}\right)\right]\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-as}}{s}-\frac{e^{-as}}{s^2}+\frac{1}{s^2}+\frac{e^{-2as}}{s^2}+\frac{ae^{-as}}{s}-\frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1+e^{-2as}-2e^{-as}}{s^2}\right] = \frac{\left(1-e^{-sa}\right)^2}{s^2\left(1-e^{-as}\right)\left(1+e^{-as}\right)}$$

$$\therefore\ F(s) = \frac{1-e^{-sa}}{s^2\left(1+e^{-as}\right)} = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

34. **Find the Laplace transform of the rectangular wave given by** $f(t) = \begin{cases} \sin \omega t, & 0 < t < \dfrac{\pi}{\omega} \\ 0, & \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega} \end{cases}$.

Solution:

This function is periodic function with period $\frac{2\pi}{\omega}$ in the interval $\left(0, \frac{2\pi}{\omega}\right)$.

$$L\left[f(t)\right] = \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\int_0^{\frac{2\pi}{\omega}} e^{-st}\ f(t)\,dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\int_0^{\frac{\pi}{\omega}} e^{-st}\sin \omega t\,dt + 0\right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{-st}}{s^2+\omega^2}\left[-s\sin \omega t - \omega\cos \omega t\right]\right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{\frac{-s\pi}{\omega}}\omega + \omega}{s^2+\omega^2}\right]$$

$$= \frac{\omega\left(e^{\frac{-s\pi}{\omega}}+1\right)}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(1+e^{\frac{-\pi s}{\omega}}\right)\left(s^2+\omega^2\right)} = \frac{\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(s^2+\omega^2\right)}$$

INVERSE LAPLACE TRANSFORMS

Inverse Laplace transform for some basic functions:

S. No.	F(s)	$f(t)=L^{-1}(F(s))$
1	$\frac{1}{s-a},\ s-a>0$	e^{at}
2	$\frac{1}{s+a},\ s+a>0$	e^{-at}
3	$\frac{a}{s^2+a^2},\ s>0$	$\sin at$
4	$\frac{s}{s^2+a^2},\ s>0$	$\cos at$
5	$\frac{a}{s^2-a^2},\ s>\|a\|$	$\sinh at$
6	$\frac{s}{s^2-a^2},\ s>\|a\|$	$\cosh at$
7	$\frac{1}{s}$	1
8	$\frac{1}{s^2}$	t
9	$\frac{n!}{s^{n+1}}$	t^n
10	$\frac{s-a}{(s-a)^2+b^2}$	$e^{at}\cos bt$
11	$\frac{1}{(s-a)^2+b^2}$	$e^{at}\frac{\sin bt}{b}$
12	$\frac{s-a}{(s-a)^2-b^2}$	$e^{at}\cosh bt$
13	$\frac{1}{(s-a)^2-b^2}$	$e^{at}\frac{\sinh bt}{b}$
14	$\frac{1}{(s-a)^2}$	$t\,e^{at}$
15	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$
16	$\frac{s}{(s^2+a^2)^2}$	$\frac{t\sin at}{2a}$

Properties of Inverse Laplace transforms:

S. No.	Property	Laplace Transform
1	Linear Property	$L^{-1}\left[a\,F(s)+b\,G(s)\right]=a\,L^{-1}[F(s)]+b\,L^{-1}[G(s)]$
2	First shifting theorem	$L^{-1}\left[F(s-a)\right]=e^{at}\,f(t)$ $L^{-1}\left[F(s+a)\right]=e^{-at}\,f(t)$
3	Second shifting theorem	$L^{-1}[e^{-as}F(s)]=\begin{cases} f(t-a), t>a \\ 0, t<a \end{cases}$
4	Change of scale property	$L^{-1}\left[F(as)\right]=\frac{1}{a}\,f\left(\frac{t}{a}\right)$
5	Multiplication by s	$L^{-1}\left[s\,F(s)\right]=f'(t)$
6	Division by s	$L^{-1}\left[\frac{F(s)}{s}\right]=\int_0^t f(t)dt$
7	Inverse Laplace Transforms of integrals	$L^{-1}\left[\int_s^\infty F(s)\,ds\right]=\frac{f(t)}{t}$
8	Inverse Laplace Transforms of derivatives	$L^{-1}\left[F(s)\right]=-\frac{1}{t}L^{-1}[F'(s)]$
9	**Convolution theorem for Inverse Laplace Transforms:** $L^{-1}\left[F(s)\bullet G(s)\right]=f(t)*g(t)$	

35. **Find** $L^{-1}\left(\frac{1}{s-3}+\frac{1}{s}+\frac{s}{s^2-9}\right)$.

Solution:

$$L^{-1}\left(\frac{1}{s-3}+\frac{1}{s}+\frac{s}{s^2-9}\right)=e^{3t}+1+\cosh 3t$$

36. **Find** $L^{-1}\left(\frac{s}{(s+2)^2}\right)$**.**

Solution:

$$L^{-1}\left(\frac{s}{(s+2)^2}\right)=L^{-1}\left(\frac{s+2-2}{(s+2)^2}\right)=L^{-1}\left(\frac{1}{(s+2)}\right)-2L^{-1}\left(\frac{1}{(s+2)^2}\right)=e^{-2t}-2te^{-2t}$$

37. **Find** $L^{-1}\left(\frac{1}{s^2+2s+5}\right)$**.**

Solution:

$$L^{-1}\left(\frac{1}{s^2+2s+5}\right)=L^{-1}\left(\frac{1}{(s+1)^2+4}\right)=\frac{e^{-t}\sin 2t}{2}$$

38. **Find** $\boldsymbol{L^{-1}\left[\frac{1}{s^2+6s+13}\right]}$**.**

Solution:

$$L^{-1}\left[\frac{1}{s^2+6s+13}\right]=L^{-1}\left[\frac{1}{(s+3)^2+4}\right]=L^{-1}\left[\frac{1}{(s+3)^2+2^2}\right]$$

$$=\frac{1}{2}L^{-1}\left[\frac{2}{(s+3)^2+2^2}\right]=\frac{1}{2}\mathrm{e}^{-3\mathrm{t}}\sin 2t.$$

39. **Find** $\boldsymbol{L^{-1}\left(\frac{s}{s^2+4s+5}\right)}$**.**

Solution:

$$L^{-1}\left(\frac{s}{s^2+4s+5}\right)=L^{-1}\left(\frac{(s+2)-2}{(s+2)^2+1}\right)=e^{-2t}L^{-1}\left(\frac{s-2}{s^2+1}\right)$$

$$=e^{-2t}\left[L^{-1}\left(\frac{s}{s^2+1}\right)-2L^{-1}\left(\frac{1}{s^2+1}\right)\right]$$

$$=e^{-2t}\left[\cos t-2\sin t\right]$$

40. **Find** $\boldsymbol{L^{-1}\left[\frac{s+2}{s^2+2s+2}\right]}$**.**

Solution: $L^{-1}\left[\frac{s+2}{s^2+2s+2}\right]=L^{-1}\left[\frac{(s+1)+1}{(s+1)^2+1}\right]\ \because\ L^{-1}[F(s+a)]=e^{-at}L^{-1}[F(s)]$

$$= L^{-1}\left[\frac{(s+1)}{(s+1)^2+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= e^{-t}\left(L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]\right)$$

$$\therefore L^{-1}\left[\frac{s+2}{s^2+2s+2}\right] = e^{-t}(\cos t + \sin t)$$

Problems based on Multiplication by s

41. **Find the inverse Laplace transform of $\dfrac{s}{(s+2)^2}$.**

Solution:

$$L^{-1}\left(\frac{s}{(s+2)^2}\right) = L^{-1}\left(s.\frac{1}{(s+2)^2}\right)$$

$$= \frac{d}{dt}L^{-1}\left(\frac{1}{(s+2)^2}\right) = \frac{d}{dt}e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$= \frac{d}{dt}\left(e^{-2t}t\right) = e^{-2t} + t\left(-2e^{-2t}\right) = e^{-2t}(1-2t)$$

42. **Find $L^{-1}\left(\dfrac{s}{(s+2)^3}\right)$.**

Solution: $L^{-1}\left(\dfrac{s}{(s+2)^3}\right) = L^{-1}\left(\dfrac{s+2-2}{(s+2)^3}\right)$

$$= L^{-1}\left(\frac{1}{(s+2)^2}\right) - 2\ L^{-1}\left(\frac{1}{(s+2)^3}\right)$$

$$= e^{-2t}L^{-1}\left(\frac{1}{s^2}\right) - e^{-2t}L^{-1}\left(\frac{2}{s^3}\right)$$

$$= e^{-2t}\left(t-t^2\right).$$

43. **Find $L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right]$.**

Solution: Let F(s) = $\tan^{-1}\left(\frac{1}{s}\right)$

$$F'(s) = \frac{1}{1+(1/s)^2}\left(\frac{-1}{s^2}\right) = \frac{-1}{s^2+1}$$

By property $L^{-1}\left[F'(s)\right] = -L^{-1}\left[\frac{1}{s^2+1}\right] = -\sin t$

$$\therefore\ L^{-1}(F'(s)) = -\sin t\,;\ L^{-1}(F(s)) = \frac{-1}{t}L^{-1}\left[F'(s)\right]$$

$$\therefore L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right] = \frac{\sin t}{t}$$

44. **Find $L^{-1}\left[\cot^{-1}(s+1)\right]$.**

Solution: Let $L^{-1}\left[\cot^{-1}(s+1)\right] = f(t)$

$$\therefore\ L\left[f(t)\right] = \cot^{-1}(s+1)$$

$$L\left[tf(t)\right] = -\frac{d}{ds}\left[\cot^{-1}(s+1)\right] = \frac{1}{(s+1)^2+1}$$

$$tf(t) = L^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right] = e^{-t}\sin t$$

$$\therefore\ f(t) = \frac{e^{-t}\sin t}{t}$$

45. **Find the inverse Laplace transform of $\log\left(\frac{1+s}{s^2}\right)$.**

Solution:

Let $L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right] = f(t)$

$$\therefore\ L\left[f(t)\right] = \log\left(\frac{1+s}{s^2}\right)$$

$$L\left[t\,f(t)\right]=\frac{-d}{ds}\left[\log\left(\frac{1+s}{s^2}\right)\right]=\frac{-d}{ds}\left[\log(1+s)-\log\left(s^2\right)\right]=-\frac{1}{1+s}+\frac{1}{s^2}2s$$

$$L\left[t\,f(t)\right]=\frac{2}{s}-\frac{1}{s+1}$$

$$t\,f(t)=L^{-1}\left[\frac{2}{s}-\frac{1}{s+1}\right]=2L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{1}{s+1}\right]=2(1)-e^{-t}$$

$$\therefore\ f(t)=\frac{2-e^{-t}}{t}$$

$$\therefore\ L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right]=\frac{2-e^{-t}}{t}$$

Problems based on Partial Fractions

46. **Find** $L^{-1}\left(\dfrac{s-5}{s^2-3s+2}\right)$**.**

Solution:

$$L^{-1}\left(\frac{s-5}{s^2-3s+2}\right)=L^{-1}\left(\frac{A}{s-1}+\frac{B}{s-2}\right)=L^{-1}\left(\frac{4}{s-1}\right)+L^{-1}\left(\frac{-3}{s-2}\right)=4e^{t}-3e^{2t}$$

47. **Find** $\boldsymbol{L^{-1}\left[\dfrac{5s^2-15s-11}{(s+1)(s-2)^3}\right]}$**.**

Solution:

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3}=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}+\frac{D}{(s-2)^3}$$

$$5s^2-15s-11=A(s-2)^3+B(s+1)(s-2)^2+C(s+1)(s-2)+D(s+1)$$

Put $s=-1 \Rightarrow \boxed{A=-\frac{1}{3}}$

Equating the coefficients of $s^3 \Rightarrow \boxed{B=\frac{1}{3}}$

Put $s=2 \Rightarrow \boxed{D=-7}$

Put $s=0 \Rightarrow \boxed{C=4}$

$$\therefore \frac{5s^2-15s-11}{(s+1)(s-2)^3}=\frac{-1/3}{s+1}+\frac{1/3}{s-2}+\frac{4}{(s-2)^2}-\frac{7}{(s-2)^3}$$

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]=-\frac{1}{3}L^{-1}\left[\frac{1}{s+1}\right]+\frac{1}{3}L^{-1}\left[\frac{1}{s-2}\right]+4L^{-1}\left[\frac{1}{(s-2)^2}\right]-7L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$=-\frac{1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]-7e^{2t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$=-\frac{1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}t-\frac{7}{2}e^{2t}L^{-1}\left[\frac{2}{s^3}\right]$$

$$\therefore f(t)=-\frac{1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}t-\frac{7}{2}e^{2t}t^2$$

Problems based on Convolution Theorem

48. **Using Convolution theorem, find $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right]$.**

Solution:

$$L^{-1}[F(s)G(s)]=L^{-1}[F(s)]*L^{-1}[G(s)]$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*L^{-1}\left[\frac{1}{s^2+a^2}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*\frac{1}{a}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$=\cos at*\frac{1}{a}\sin at=\frac{1}{a}[\cos at*\sin at]$$

$$=\frac{1}{a}\int_0^t \cos au\ \sin a(t-u)\,du=\frac{1}{a}\int_0^t \sin(at-au)\cos au\,du$$

$$=\frac{1}{a}\int_0^t \frac{\sin(at-au+au)+\sin(at-au-au)}{2}\,du$$

$$=\frac{1}{2a}\int_0^t [\sin at+\sin a(t-2u)]\,du$$

$$= \frac{1}{2a}\left[\sin at\ u + \left(\frac{-\cos a(t-2u)}{-2a}\right)\right]_0^t$$

$$= \frac{1}{2a}\left[u\ \sin at + \left(\frac{\cos a(t-2u)}{2a}\right)\right]_0^t$$

$$= \frac{1}{2a}\left[t\ \sin at + \left(\frac{\cos at}{2a}\right) - \left(0 + \frac{\cos at}{2a}\right)\right]$$

$$f(t) = \frac{1}{2a}\left[t\sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a}\right] = \frac{1}{2a} t\sin at$$

49. **Find the inverse Laplace transform of $\frac{s}{(s^2+a^2)(s^2+b^2)}$ using convolution theorem.**

Solution:

$$L^{-1}[F(s)G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

$$\therefore\ L^{-1}\left[\frac{s}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{1}{s^2+b^2}\right]$$

$$= \frac{1}{b}\cos at * \sin bt$$

$$= \frac{1}{b}\int_0^t \cos au\ \ \sin b(t-u)\ du$$

$$= \frac{1}{2b}\int_0^t \left[\sin(au+bt-bu) - \sin(au-bt+bu)\right]\ du$$

$$= \frac{1}{2b}\int_0^t \left[\sin((a-b)u+bt) - \sin((a+b)u-bt)\right]\ du$$

$$= \frac{1}{2b}\left[\frac{-\cos(bt+(a-b)u)}{a-b} + \frac{\cos((a+b)u-bt)}{a+b}\right]_0^t$$

$$= \frac{1}{2b}\left[\left(\frac{-\cos(bt+at-bt)}{a-b} + \frac{\cos(at+bt-bt)}{a+b}\right) - \left(\frac{-\cos bt}{a-b} + \frac{\cos bt}{a+b}\right)\right]$$

$$= \frac{1}{2b}\left[\left(\frac{-\cos(at)}{a-b}+\frac{\cos(at)}{a+b}\right)-\left(\frac{-\cos bt}{a-b}+\frac{\cos bt}{a+b}\right)\right]$$

$$= \frac{1}{2b}\left(\frac{-2b\cos at}{a^2-b^2}+\frac{2b\cos bt}{a^2-\mathrm{b}^2}\right)$$

$$f(t)=\frac{\cos bt-\cos at}{a^2-b^2}$$

50. **Find the inverse Laplace transform of $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ by using convolution theorem.**

Solution:

$$L^{-1}\left[F(s)G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*L^{-1}\left[\frac{s}{s^2+b^2}\right]=\cos at*\cos bt$$

$$=\int_0^t \cos au \ \ \cos b(t-u)\ du$$

$$=\frac{1}{2}\int_0^t \left[\cos(au+bt-bu)+\cos(au-bt+bu)\right]\ du$$

$$=\frac{1}{2}\int_0^t \left[\cos((a-b)u+bt)+\cos((a+b)u-bt)\right]\ du$$

$$=\frac{1}{2}\left[\frac{\sin(bt+(a-b)u)}{a-b}+\frac{\sin((a+b)u-bt)}{a+b}\right]_0^t$$

$$=\frac{1}{2}\left[\left(\frac{\sin(bt+at-bt)}{a-b}+\frac{\sin(at+bt-bt)}{a+b}\right)-\left(\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right)\right]$$

$$=\frac{1}{2}\left[\left(\frac{\sin(at)}{a-b}+\frac{\sin(at)}{a+b}\right)-\left(\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right)\right]$$

$$=\frac{1}{2}\left(\frac{2a\sin(at)}{a^2-b^2}-\frac{2b\sin(bt)}{a^2-\mathrm{b}^2}\right)$$

$$f(t)=\frac{a\sin(at)-b\sin(bt)}{a^2-b^2}$$

51. **Find the inverse Laplace transform of** $\dfrac{s}{(s^2+1)(s^2+4)}$**.**

Solution:

$$L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right]=L^{-1}\left[\frac{s}{s^2+1}\;\frac{1}{s^2+4}\right]=L^{-1}\left[\frac{s}{s^2+1}\right]*\frac{1}{2}L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$=\frac{1}{2}\cos t*\sin 2t$$

$$=\frac{1}{2}\int_0^t \cos u\;\sin 2(t-u)\,du$$

$$=\frac{1}{4}\int_0^t\left[\sin(u+2t-2u)-\sin(u-2t+2u)\right]du \quad [2\cos A\sin B=\sin(A+B)-\sin(A-B)]$$

$$=\frac{1}{4}\int_0^t\left[\sin(2t-u)-\sin(u-2t)\right]du$$

$$=\frac{1}{4}\left[\frac{-\cos(2t-u)}{-1}+\frac{\cos(u-2t)}{1}\right]_0^t$$

$$=\frac{1}{4}\left[\cos t-\cos 2t+\cos t-\cos 2t\right]$$

$$=\frac{1}{4}\left[2\cos t-2\cos 2t\right]$$

$$\therefore\; f(t)=\frac{1}{2}\left[\cos t-\cos 2t\right]$$

52. **Using Convolution theorem, find the inverse Laplace transform of** $\dfrac{2}{(s+1)(s^2+4)}$**.**

Solution:

$$L^{-1}\left[\frac{2}{(s+1)(s^2+4)}\right]=L^{-1}\left[\frac{1}{s+1}\;\frac{2}{s^2+4}\right]=L^{-1}\left[\frac{1}{s+1}\right]*L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$=e^{-t}*\sin 2t$$

$$=\int_0^t e^{-u}\;\sin 2(t-u)\,du$$

$$= \int_0^t e^{-u} \sin(2t - 2u)\, du$$

$$= \int_0^t e^{-u} \left[\sin 2t \cos 2u - \cos 2t \sin 2u\right] du$$

$$= \int_0^t e^{-u} \sin 2t \cos 2u \, du - \int_0^t e^{-u} \cos 2t \sin 2u \, du$$

$$= \sin 2t \int_0^t e^{-u} \cos 2u \, du - \cos 2t \int_0^t e^{-u} \sin 2u \, du$$

$$= \sin 2t \left[\frac{e^{-u}}{1+4}\left(-\cos 2u + 2\sin 2u\right)\right]_0^t - \cos 2t \left[\frac{e^{-u}}{1+4}\left(-\sin 2u - 2\cos 2u\right)\right]_0^t$$

$$= \sin 2t \left[\left(\frac{e^{-t}}{5}\left(-\cos 2t + 2\sin 2t\right)\right) - \left(\frac{1}{5}(-1)\right)\right] - \cos 2t \left[\frac{e^{-t}}{5}\left(-\sin 2t - 2\cos 2t\right) - \left(\frac{1}{5}(-2)\right)\right]$$

$$= \sin 2t \left[\frac{e^{-t}}{5}\left(-\cos 2t + 2\sin 2t\right) + \frac{1}{5}\right] - \cos 2t \left[\frac{e^{-t}}{5}\left(-\sin 2t - 2\cos 2t + \frac{2}{5}\right)\right]$$

$$= \frac{e^{-t}}{5}\left[-\sin 2t \cos 2t + 2\sin^2 2t + \sin 2t \cos 2t + 2\cos^2 2t\right] + \frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t$$

$$= \frac{e^{-t}}{5}\left[2(1)\right] + \frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t$$

$$f(t) = \frac{1}{5}\left[2e^{-t} + \sin 2t - 2\cos 2t\right]$$

53. **Find the inverse Laplace transform of $\dfrac{s^2}{\left(s^2+1\right)\left(s^2+4\right)}$.**

Solution:

$$L^{-1}\left[F(s)G(s)\right] = L^{-1}\left[F(s)\right] * L^{-1}\left[G(s)\right]$$

$$\therefore\ L^{-1}\left[\frac{s^2}{\left(s^2+1^2\right)\left(s^2+2^2\right)}\right] = L^{-1}\left[\frac{s}{s^2+1^2}\right] * L^{-1}\left[\frac{s}{s^2+2^2}\right]$$

$$= \cos t * \cos 2t$$

$$= \int_0^t \cos u \; \cos 2(t-u)\; du$$

$$= \frac{1}{2}\int_0^t \left[\cos(u+2t-2u)+\cos(u-2t+2u)\right]\; du$$

$$= \frac{1}{2}\int_0^t \left[\cos(-u+2t)+\cos(3u-2t)\right]\; du$$

$$= \frac{1}{2}\left[\frac{\sin(2t-u)}{-1}+\frac{\sin(3u-2t)}{3}\right]_0^t$$

$$= \frac{1}{2}\left[\left(\frac{\sin t}{-1}+\frac{\sin t}{3}\right)-\left(\frac{\sin 2t}{-1}-\frac{\sin 2t}{3}\right)\right]$$

$$= \frac{1}{2}\left(\frac{2\sin t}{-3}-\frac{4\sin 2t}{-3}\right)$$

$$f(t) = \frac{\sin t - 2\sin 2t}{-3}$$

54. Find $L^{-1}\left(\dfrac{e^{-2s}}{\left(s^2+s+1\right)^2}\right)$.

Solution:

$$L^{-1}\left(\frac{e^{-2s}}{\left(s^2+s+1\right)^2}\right) = L^{-1}\left(\frac{e^{-s}}{s^2+s+1}\;\frac{e^{-s}}{s^2+s+1}\right)$$

$$= L^{-1}\left(\frac{1}{s^2+s+1}\right)_{t\to t-1} * L^{-1}\left(\frac{1}{s^2+s+1}\right)_{t\to t-1}$$

$$= L^{-1}\left(\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right)_{t\to t-1} * L^{-1}\left(\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right)_{t\to t-1}$$

$$= e^{-t/2} L^{-1}\left(\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)_{t\to t-1} * e^{-t/2} L^{-1}\left(\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)_{t\to t-1}$$

$$= \left[e^{-t/2}\frac{\sin\left(\frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}} * e^{-t/2}\frac{\sin\left(\frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}}\right]_{t\to t-1}$$

$$= \frac{2}{\sqrt{3}} e^{-(t-1)/2}\sin\left(\frac{\sqrt{3}}{2}(t-1)\right) * \frac{2}{\sqrt{3}} e^{-(t-1)/2}\sin\left(\frac{\sqrt{3}}{2}(t-1)\right)$$

$$= \frac{4}{3}\left[e^{-(t-1)/2}\sin\left(\frac{\sqrt{3}}{2}(t-1)\right) * e^{-(t-1)/2}\sin\left(\frac{\sqrt{3}}{2}(t-1)\right)\right]$$

$$= \frac{4}{3}\int_0^t e^{-\frac{u-1}{2}} e^{-\frac{t-u-1}{2}} \sin\left(\frac{\sqrt{3}}{2}u-\frac{\sqrt{3}}{2}\right)\ \sin\left(\frac{\sqrt{3}}{2}t-\frac{\sqrt{3}}{2}u-\frac{\sqrt{3}}{2}\right)du$$

$$= \frac{4}{3}\int_0^t e^{-\left(\frac{t}{2}-1\right)} \frac{1}{2}\ \cos\left(\frac{\sqrt{3}}{2}u-\frac{\sqrt{3}}{2}t\right)-\cos\left(\frac{\sqrt{3}}{2}t-\sqrt{3}\right)du$$

$$= \frac{2}{3} e^{-\left(\frac{t-2}{2}\right)}\left[\frac{\sin\left(\frac{\sqrt{3}}{2}u-\frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}} - \cos\left(\frac{\sqrt{3}}{2}t-\sqrt{3}\right)u\right]_0^t$$

$$= e^{-\left(\frac{t-2}{2}\right)}\left[\frac{4}{3\sqrt{3}}\sin\frac{\sqrt{3}}{2}t-\frac{2}{3}t\cos\left(\frac{\sqrt{3}}{2}t-\sqrt{3}\right)\right]$$

Problems based on solving differential equations

55. **Solve using Laplace transform $\frac{dy}{dt}+y=e^{-t}$ given that y(0) = 0.**

Solution: Taking L.T. on both sides, we get $L\left[y'(t)\right]+L\left[y(t)\right]=L\left[e^{-t}\right]$

$$sL\left[y(t)\right]-y(0)+L\left[y(t)\right]=L\left[e^{-t}\right]$$

$$sL\left[y(t)\right]-0+L\left[y(t)\right]=\frac{1}{s+1}$$

$$(s+1)\ L[y(t)]=\frac{1}{s+1}$$

$$L[y(t)]=\frac{1}{(s+1)^2}$$

$$\therefore y(t) = L^{-1}\left(\frac{1}{(s+1)^2}\right)=e^{-t}L\left(\frac{1}{s^2}\right)= e^{-t}\,t \qquad \left(\because\ L\left[e^{-at}f(t)\right]=F(s+a)\right)$$

56. **Using Laplace transform to solve the differential equation** $\boldsymbol{y'' + y' = t^2 + 2t}$**, given** $\boldsymbol{y = 4,\ y' = -2}$ **when** $\boldsymbol{t = 0}$

Solution:

Given $y'' + y' = t^2 + 2t$

$$L\left[y''+y'\right]=L\left[t^2+2t\right]$$

$$\left[s^2L\left[y(t)\right]-sy(0)-y'(0)\right]+\left[sL\left[y(t)\right]-y(0)\right]=\frac{2}{s^3}+\frac{2}{s^2}$$

$$L\left[y(t)\right]\left(s^2+s\right)=\frac{2}{s^3}+\frac{2}{s^2}+4s-2+4$$

$$L\left[y(t)\right]s(s+1)=\frac{2}{s^3}+\frac{2}{s^2}+4s+2$$

$$L\left[y(t)\right]=\frac{2+2s+4s^4+2s^3}{s^4(s+1)}$$

$$L\left[y(t)\right]=\frac{2}{s}+\frac{2}{s^4}+\frac{2}{s+1}$$

$$y(t)=L^{-1}\left[\frac{2}{s}+\frac{2}{s^4}+\frac{2}{s+1}\right]$$

$$=2+2\frac{t^3}{6}+2e^{-t}$$

$$y(t)=2+\frac{t^3}{3}+2e^{-t}$$

57. **Solve $\left(D^2+3D+2\right)y=e^{-3t}$, given $y(0)=1$, and $y'(0)=-1$ using Laplace Transforms.**

Solution:

Given $y''+3y'+2y=e^{-3t}$

Taking Laplace transforms on both sides.

$$L\left(y''+3y'+2y\right)=L\left(e^{-3t}\right)$$

$$L\left[y''(t)\right]+3L\left[y'(t)\right]+2L\left[y(t)\right]=\frac{1}{s+3}$$

$$\left[s^2L\left[y(t)\right]-sy(0)-y'(0)\right]+3\left[sL\left[y(t)\right]-y(0)\right]+2L\left[y(t)\right]=\frac{1}{s+3}$$

$$\left[s^2L\left[y(t)\right]-s(1)-(-1)\right]+3\left[sL\left[y(t)\right]-1\right]+2L\left[y(t)\right]=\frac{1}{s+3}$$

$$L\left[y(t)\right]\left[s^2+3s+2\right]=\frac{1}{s+3}+s+2$$

$$L\left[y(t)\right]=\frac{s^2+5s+7}{(s+3)\left(s^2+3s+2\right)},\ y(t)=L^{-1}\left[\frac{s^2+5s+7}{(s+1)(s+2)(s+3)}\right]$$

$$y(t)=L^{-1}\left[\frac{3/2}{s+1}-\frac{1}{s+2}+\frac{1/2}{s+3}\right]$$

$$y(t)=\frac{3}{2}L^{-1}\left[\frac{1}{s+1}\right]-L^{-1}\left[\frac{1}{s+2}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s+3}\right]$$

$$y(t)=\frac{3}{2}e^{-t}-e^{-2t}+\frac{1}{2}e^{-3t}$$

58. **Solve $y''+2y'-3y=\sin t$, given $y(0)=0,\ y'(0)=0$.**

Solution:

Given $y''+2y'-3y=\sin t$

$$L\left[y''(t)+2y'(t)-3y(t)\right]=L[\sin t]$$

$$L\left[y''(t)\right]+2L\left[y'(t)\right]-3L\left[y(t)\right]=L[\sin t]$$

$$\left[s^2L\left[y(t)\right]-sy(0)-y^{'}(0)\right]+2\left[sL\left[y(t)\right]-y(0)\right]-3L\left[y(t)\right]=\frac{1}{s^2+1}$$

$$\left[s^2L\left[y(t)\right]-s(0)-0\right]+2\left[sL\left[y(t)\right]-(0)\right]-3L\left[y(t)\right]=\frac{1}{s^2+1}$$

$$s^2L\left[y(t)\right]+2sL\left[y(t)\right]-3L\left[y(t)\right]=\frac{1}{s^2+1}$$

$$L\left[y(t)\right]\left(s^2+2s-3\right)=\frac{1}{s^2+1}$$

$$L\left[y(t)\right]=\frac{1}{\left(s^2+1\right)\left(s^2+2s-3\right)}$$

$$y(t)=L^{-1}\left[\frac{1}{\left(s^2+1\right)\left(s^2+2s-3\right)}\right]==L^{-1}\left[\frac{1}{(s-1)(s+3)\left(s^2+1\right)}\right]$$

Now

$$\frac{1}{(s-1)(s+3)\left(s^2+1\right)}=\frac{A}{s-1}+\frac{B}{s+3}+\frac{Cs+D}{\left(s^2+1\right)}$$

$$1=A(s+3)\left(s^2+1\right)+B(s-1)\left(s^2+1\right)+(Cs+D)(s-1)(s+3)$$

Put $s=1\Rightarrow \boxed{A=\frac{1}{8}}$

Put $s=-3\Rightarrow \boxed{B=\frac{-1}{40}}$

Equating coeff. of $s^3\Rightarrow \boxed{C=\frac{-1}{10}}$

Equating the constant terms$\Rightarrow \boxed{D=\frac{-1}{5}}$

$$\therefore\ \frac{1}{(s-1)(s+3)\left(s^2+1\right)}=\frac{1/8}{s-1}+\frac{-1/40}{s+3}+\frac{(-1/10)s-1/5}{\left(s^2+1\right)}$$

$$L^{-1}\left[\frac{1}{(s-1)(s+3)\left(s^2+1\right)}\right]=L^{-1}\left[\frac{1/8}{s-1}+\frac{-1/40}{s+3}+\frac{(-1/10)s-1/5}{\left(s^2+1\right)}\right]$$

$$=\frac{1}{8}L^{-1}\left[\frac{1}{s-1}\right]-\frac{1}{40}L^{-1}\left[\frac{1}{s+3}\right]-\frac{1}{10}L^{-1}\left[\frac{s+2}{s^2+1}\right]$$

$$=\frac{1}{8}e^{t}-\frac{1}{40}e^{-3t}-\frac{1}{10}\left[L^{-1}\left[\frac{s}{s^2+1}\right]+L^{-1}\left[\frac{2}{s^2+1}\right]\right]$$

$$=\frac{1}{8}e^{t}-\frac{1}{40}e^{-3t}-\frac{1}{10}\left[\cos t+2\sin t\right]$$

59. **Solve the equation y″+ 9y = cos2t with y(0) =1, y $\left(\frac{\pi}{2}\right)$ = –1.**

Solution:

Given $\left(D^2+9\right)y=\cos 2t$

Taking Laplace transforms on both sides

$$L\left[y''(t)\right]+9L\left[y(t)\right]=L\left[\cos 2t\right]$$

$$s^2L\left[y(t)\right]-sy(0)-y'(0)+9L\left[y(t)\right]=\frac{s}{s^2+4}$$

Using the initial conditions

$y(0)=1,$ and taking $y'(0)=k$

We have

$$s^2L\left[y(t)\right]-(s)(1)-k+9L\left[y(t)\right]=\frac{s}{s^2+4}$$

$$\Rightarrow L\left[y(t)\right]=\frac{s}{\left(s^2+4\right)\left(s^2+9\right)}+\frac{s+k}{s^2+9}$$

$$=\frac{s}{5\left(s^2+4\right)}-\frac{s}{5\left(s^2+9\right)}+\frac{s}{s^2+9}+\frac{k}{s^2+9}$$

$$\therefore\ y(t)=\frac{1}{5}L^{-1}\left[\frac{s}{s^2+4}\right]-\frac{1}{5}L^{-1}\left[\frac{s}{s^2+9}\right]+L^{-1}\left[\frac{s}{s^2+9}\right]+k\,L^{-1}\left[\frac{s}{s^2+9}\right]$$

$$=\frac{1}{5}\cos 2t-\frac{1}{5}\cos 3t+\cos 3t+\frac{k}{3}\sin 3t$$

Put $t=\frac{\pi}{2}$ we get $y\left(\frac{\pi}{2}\right)=\frac{1}{5}(-1)-\frac{1}{5}(0)+0+\frac{k}{3}(-1)=-\frac{1}{5}-\frac{k}{3}$

But given $y\left(\frac{\pi}{2}\right)=-1$

$\therefore\ -1=-\frac{1}{5}-\frac{k}{3}$

$\Rightarrow k=\frac{12}{5}$

$\therefore\ y(t)=\frac{1}{5}\cos 2t-\frac{1}{5}\cos 3t+\cos 3t+\frac{4}{5}\sin 3t$

$y(t)=\frac{4}{5}[\cos 3t+\sin 3t]+\frac{1}{5}\cos 2t$

60. **Solve $x''+2x'+5x=e^{-t}\sin t$, where $x(0)=0,\ x'(0)=1$ using Laplace Transforms.**

Solution:

Given $x''+2x'+5x=e^{-t}\sin t$

Taking Laplace transforms on both side

$L\left[x''+2x'+5x\right]=L\left[e^{-t}\sin t\right]$

$L\left[x''(t)\right]+2L\left[x'(t)\right]+5L\left[x(t)\right]=\frac{1}{s^2+2s+2}$

$\left[s^2L\left[x(t)\right]-sx(0)-x'(0)\right]+2\left[sL\left[x(t)\right]-x(0)\right]+5L\left[x(t)\right]=\frac{1}{s^2+2s+2}$

$\left[s^2L\left[x(t)\right]-s(0)-1\right]+2\left[sL\left[x(t)\right]-(0)\right]+5L\left[x(t)\right]=\frac{1}{s^2+2s+2}$

$L\left[x(t)\right]\left[s^2+2s+5\right]=\frac{1}{s^2+2s+2}+1$

$L\left[x(t)\right]\left[s^2+2s+5\right]=\frac{s^2+2s+3}{s^2+2s+2}$

$L\left[x(t)\right]=\frac{s^2+2s+3}{\left(s^2+2s+2\right)\left(s^2+2s+5\right)}=\frac{(s+1)^2+2}{\left((s+1)^2+1\right)\left((s+1)^2+4\right)}$

$$x(t) = L^{-1}\left[\frac{(s+1)^2+2}{\left((s+1)^2+1\right)\left((s+1)^2+4\right)}\right]$$

$$x(t) = e^{-t}L^{-1}\left[\frac{s^2+2}{\left(s^2+1\right)\left(s^2+4\right)}\right]$$

$$x(t) = e^{-t}L^{-1}\left[\frac{1/3}{s^2+1}+\frac{2/3}{s^2+4}\right]$$

$$= e^{-t}\left[\frac{1}{3}\sin t+\frac{1}{3}\sin 2t\right]$$

$$= \frac{e^{-t}}{3}\left[\sin t+\sin 2t\right]$$

61. **Using Laplace transform to solve the differential equation**

$$\boldsymbol{y'' - 3y' + 2y = 4t + e^{3t}}, \textbf{ where } \boldsymbol{y(0) = 1,\ y'(0) = -1}$$

Solution:

Given $y'' - 3y' + 2y = 4t + 3e^{t}$

$$L\left[y'' - 3y' + 2y\right] = L\left[4t + 3e^{t}\right]$$

$$L\left[y''(t)\right] - 3L\left[y'(t)\right] + 2L\left[y(t)\right] = 4L[t] + 3L\left[e^{3t}\right]$$

$$\left[s^2L\left[y(t)\right] - sy(0) - y'(0)\right] - 3\left[sL\left[y(t)\right] - y(0)\right] + 2L\left[y(t)\right] = \frac{4}{s^2} + \frac{3}{s-3}$$

$$\left[s^2L\left[y(t)\right] - s(1) - (-1)\right] - 3\left[sL\left[y(t)\right] - 1\right] + 2L\left[y(t)\right] = \frac{4}{s^2} + \frac{3}{s-3}$$

$$\left[s^2L\left[y(t)\right] - s + 1\right] - 3\left[sL\left[y(t)\right] - 1\right] + 2L\left[y(t)\right] = \frac{4}{s^2} + \frac{3}{s-3}$$

$$L\left[y(t)\right]\left(s^2-3s+2\right)=s-4+\frac{4}{s^2}+\frac{3}{s-3}$$

$$L\left[y(t)\right]\left(s^2-3s+2\right)=\frac{(s-4)s^2(s-3)+4(s-4)+3s^2}{s^2(s-3)}$$

$$L\left[y(t)\right]=\frac{s^4-7s^3+13s^2+4s-12}{\left(s^2-3s+2\right)s^2(s-3)}$$

$$L\left[y(t)\right]=\frac{s^4-7s^3+13s^2+4s-12}{(s-2)(s-1)s^2(s-3)}$$

$$y(t)=L^{-1}\left[\frac{s^4-7s^3+13s^2+4s-12}{(s-2)(s-1)s^2(s-3)}\right]$$

$$=L^{-1}\left[\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-1}+\frac{D}{s-2}+\frac{E}{s-3}\right]$$

$$=L^{-1}\left[\frac{3}{s}+\frac{2}{s^2}+\frac{-1/2}{s-1}+\frac{-2}{s-2}+\frac{1/2}{s-3}\right]$$

$$y(t)=3+2t-\frac{1}{2}e^t-2e^{2t}+\frac{1}{2}e^{3t}$$

62. **Solve $y''-3y'+2y=e^{2t}$, $y(0)=-3$, $y'(0)=5$.**

Solution:

Given $y''-3y'+2y=e^{2t}$

$$L\left[y''-3y'+2y\right]=L\left[e^{2t}\right]$$

$$L\left[y''\right]-3L\left[y'\right]+2L[y]=L\left[e^{2t}\right]$$

$$\left[s^2L\left[y(t)\right]-sy(0)-y'(0)\right]-3\left[sL\left[y(t)\right]-y(0)\right]+2L\left[y(t)\right]=\frac{1}{s-2}$$

$$\left[s^2L\left[y(t)\right]-s(-3)-5\right]-3\left[sL\left[y(t)\right]-(-3)\right]+2L\left[y(t)\right]=\frac{1}{s-2}$$

$$s^2L\left[y(t)\right]+3s-5-3sL\left[y(t)\right]-9+2L\left[y(t)\right]=\frac{1}{s-2}$$

$$L\left[y(t)\right]\left[s^2-3s+2\right]+3s-14=\frac{1}{s-2}$$

$$\therefore \quad L\left[y(t)\right]\left[s^2-3s+2\right]=\frac{1}{s-2}-3s+14$$

$$\therefore \quad L\left[y(t)\right]=\frac{-3s^2+20s-27}{(s-2)\left(s^2-3s+2\right)}$$

$$y(t)=L^{-1}\left[\frac{-3s^2+20s-27}{(s-2)\left(s^2-3s+2\right)}\right]$$

$$y(t)=L^{-1}\left[\frac{-3s^2+20s-27}{(s-1)(s-2)^2}\right]$$

$$\frac{-3s^2+20s-27}{(s-1)(s-2)^2}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}$$

$$-3s^2+20s-27=A(s-2)^2+B(s-1)(s-2)+C(s-1)$$

Put $s=1 \Rightarrow \boxed{A=-10}$

Put $s=2 \Rightarrow \boxed{C=1}$

Equating the coeff. of $s^2 \Rightarrow \boxed{B=7}$

$$\therefore \quad \frac{-3s^2+20s-27}{(s-1)(s-2)^2}=\frac{-10}{s-1}+\frac{7}{s-2}+\frac{1}{(s-2)^2}$$

$$L^{-1}\left[\frac{-3s^2+20s-27}{(s-1)(s-2)^2}\right]=L^{-1}\left[\frac{-10}{s-1}\right]+L^{-1}\left[\frac{7}{s-2}\right]+L^{-1}\left[\frac{1}{(s-2)^2}\right]$$

$$=-10e^t+7e^{2t}+e^{2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$=-10e^t+7e^{2t}+te^{2t}$$

63. **Use Laplace Transform to solve** $(D^2-3D+2)y=e^{3t}$ **with** $y(0)=1$ **and** $y'(0)=0$**.**

Solution:

$$y''-3y'+2y=e^{3t} \text{ -----------(1)}$$

$$L(y'')-3L(y')+2L(y)=L(e^{3t})$$

$$\left(s^2L(y)-sy(0)-y'(0)\right)-3(sL(y)-y(0))+2L(y)=\frac{1}{s-3}$$

$$\left(s^2 - 3s + 2\right)L(y) - s + 3 = \frac{1}{s-3}$$

$$(s-1)(s-2)L(y) = \frac{1}{s-3} + s - 3$$

$$L(y) = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)}$$

$$y(t) = L^{-1}\left[\frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)}\right]$$

Consider $\frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$

$$s^2 - 6s + 10 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

Put $s = 1,\ A = \frac{5}{2}$, put $s = 2$, $B = -2$ and for $s = 3,\ C = \frac{1}{2}$

$$y(t) = L^{-1}\left[\frac{5/2}{s-1} + \frac{-2}{s-2} + \frac{1/2}{s-3}\right]$$

$$y(t) = \frac{5}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}$$

64. **Using Lapalce Transform, Solve** $\frac{d^2y}{dt^2} + 4y = \sin 2t$ **, given** $y(0) = 3$ & $y'(0) = 4$ **.**

Solution:

$$y'' + 4y = \sin 2t$$

$$L(y'') + 4L(y) = L(\sin 2t)$$

$$\left(s^2 L(y) - sy(0) - y'(0)\right) + 4L(y) = \frac{2}{s^2 + 4}$$

$$\left(s^2 + 4\right)L(y) - 3s - 4 = \frac{2}{s^2 + 4}$$

$$\left(s^2 + 4\right)L(y) = \frac{2}{s^2 + 4} + 3s + 4$$

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$$L(y) = \frac{3s^3 + 4s^2 + 12s + 18}{(s^2 + 4)^2}$$

Consider $\frac{3s^3 + 4s^2 + 12s + 18}{(s^2+4)^2} = \frac{(As + B)}{s^2 + 4} + \frac{(Cs + D)}{(s^2 + 4)^2}$

$$3s^3 + 4s^2 + 12s + 18 = (As + B)\left(s^2 + 4\right) + (Cs + D)$$

Comparing the co.eff of s^3, $A = 3$

Comparing the co.eff of s^2, $B = 4$

Comparing the co.eff of s, $C = 0$

Comparing the constant term $D = 2$

$$y(t) = L^{-1}\left[\frac{(3s+4)}{s^2+4} + \frac{(0.s+2)}{(s^2+4)^2}\right]$$

$$= \quad 3L^{-1}\left(\frac{s}{s^2+4}\right) + 2L^{-1}\left(\frac{2}{s^2+4}\right) + L^{-1}\left(\frac{2}{(s^2+4)^2}\right)$$

$$= 3\cos 2t + 2\sin 2t + \frac{t^3 e^{-2t}}{6}$$

65. **Solve $\frac{dx}{dt} - 2x + 3y = 0$; $\frac{dy}{dt} - y + 2x = 0$ with $x(0) = 8$, $y(0) = 3$.**

The given differential equation canbe written as

$$x'(t) - 2x + 3y = 0 \quad y'(t) - y + 2x = 0$$

Taking Laplace transforms weget,

$$L\left[x'(t) - 2x + 3y\right] = L[0]$$

$$sL\left[x(t)\right] - x(0) - 2L\left[x(t)\right] + 3L\left[y(t)\right] = 0$$

$$sL\left[x(t)\right] - 8 - 2L\left[x(t)\right] + 3L\left[y(t)\right] = 0$$

$$L\left[x(t)\right](s-2) + 3L\left[y(t)\right] = 8 \qquad (1)$$

And $\quad L\left[y'(t) - y + 2x\right] = L[0]$

$$sL\left[y(t)\right] - y(0) - L\left[y(t)\right] + 2L\left[x(t)\right] = 0$$

$$sL\left[y(t)\right] - 3 - L\left[y(t)\right] + 2L\left[x(t)\right] = 0$$

$$2L\left[x(t)\right] + (s-1)L\left[y(t)\right] = 3 \qquad (2)$$

Solving (1) and (2) we get,

$$L\left[x(t)\right] = \frac{8s-17}{(s+1)(s-4)} = \frac{5}{s+1} + \frac{3}{s-4},$$

$$\therefore\ x(t) = L^{-1}\left[\frac{5}{s+1} + \frac{3}{s-4}\right],$$

$$\boxed{x(t) = 5e^{-t} + 3e^{4t}}$$

and $$L\left[y(t)\right]=\frac{3s-22}{(s+1)(s-4)}=\frac{5}{s+1}-\frac{2}{s-4}$$

$$y(t)=L^{-1}\left[\frac{5}{s+1}-\frac{2}{s-4}\right]=5e^{-t}-2e^{4t}$$

66. **Determine y which satisfies the equation** $\boldsymbol{\frac{dy}{dt}+2y+\int_0^t y\,dt=2\cos t,\ \ y(0)=1}$

Solution:

Given $y'(t)+2y(t)+\int_0^t y(t)\,dt=2\cos t,\ \ y(0)=1$

$$L\left[y'(t)\right]+2L\left[y(t)\right]+L\left[\int_0^t y(t)\,dt\right]=L\left[2\cos t\right]$$

$$sL\left[y(t)\right]-y(0)+2L\left[y(t)\right]+\frac{1}{s}L\left[y(t)\right]=\frac{2s}{s^2+1}$$

$$sL\left[y(t)\right]-1+2L\left[y(t)\right]+\frac{1}{s}L\left[y(t)\right]=\frac{2s}{s^2+1}$$

$$L\left[y(t)\right]=\frac{s}{s^2+1}$$

$$y(t)=L^{-1}\left[\frac{s}{s^2+1}\right]=\cos t$$

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