SRM Institute of Science and Technology

Tiruchirappalli Campus

Cycle Test - II - Nov 2022 - Set A

Calculus and Linear Algebra

Answer Key

Part - A

1. $\boxed{a}$ 2. $\boxed{d}$ 3. $\boxed{a}$ 4. $\boxed{d}$ 5. $\boxed{c}$

6. $\boxed{a}$ 7. $\boxed{d}$ 8. $\boxed{d}$ 9. $\boxed{a}$ 10. $\boxed{b}$

Part - B

11. Given $z = xy^2 + x^2y$; $x = at^2$, $y = 2at$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \text{ ——— } (1)$$

$$= (y^2 + 2xy)\, 2at + (2xy + x^2)\, 2a \text{ — } (1)$$

$$= (4a^2t^2 + 4a^2t^3)\, 2at + (4a^2t^3 + a^2t^4)\, 2a$$

$$= 8a^3t^3(1+t) + 2a^3t^3(4+t)$$

$$= 2a^3t^3(4 + 4t + 4 + t)$$

$$= 2a^3t^3(5t + 8) \text{ ——— } (2)$$

12. Given $u = \sin^{-1}\left(\frac{x^2+y^2}{x-y}\right)$.

$$\Rightarrow \sin u = \frac{x^2+y^2}{x-y} \longrightarrow (1)$$

$$\therefore f = \sin u = \frac{x^2+y^2}{x-y}$$

$$f(x,y) = \frac{x^2+y^2}{x-y} \Rightarrow f(tx,ty) = \frac{t^2x^2+t^2y^2}{tx-ty}$$

$$= \frac{t^2(x^2+y^2)}{t(x-y)} = t\, f(x,y)$$

$\therefore f$ is a homogeneous function of degree 1. — (1)

P.T.

$\therefore$ By Euler's theorem

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=1.f \quad \text{---} (1)$$

$$\Rightarrow \quad x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)=\sin u$$

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\sin u$$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{\sin u}{\cos u}=\tan u. \quad \text{---} (1)$$

13. ~~If~~ The functions are functionally dependent if

$$J\left(\frac{u,v,w}{x,y,z}\right)=0 \quad \text{---} (1)$$

Now
$$J\left(\frac{u,v,w}{x,y,z}\right)=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix} \quad \text{---} (1)$$

$$=\begin{vmatrix} y+z & x+z & x+y\\ 2x & 2y & 2z\\ 1 & 1 & 1\end{vmatrix} \quad \text{---} (1)$$

$$=(y+z)(2y-2z)-(x+z)(2x-2z)+(x+y)(2x-2y)$$

$$=\cancel{2y^2}-\cancel{2yz}+\cancel{2yz}-\cancel{2z^2}-\cancel{2x^2}+\cancel{2xz}-\cancel{2xz}+\cancel{2z^2}+\cancel{2x^2}-\cancel{2xy}+\cancel{2xy}-\cancel{2y^2}$$

$$=0 \quad \text{---} (1)$$

Hence the functions are functionally dependent

14. The given differential equation can be written as

$$(D^2+6D+8)y=3e^{-4x} \quad \text{where } D=\frac{d}{dx}$$

A.E is $m^2+6m+8=0$ ——— (1)

$$(m+4)(m+2)=0$$

$\Rightarrow m_1=-4,\ m_2=-2 \quad \therefore CF=Ae^{-4x}+Be^{-2x}$ — (1)

$P.I = \frac{1}{D^2+6D+8} 3e^{-4x}$

$= 3 \cdot \frac{1}{16-24+8} e^{-4x} = 3 \cdot \frac{1}{0} e^{-4x}$

$= 3 \frac{x}{2D+6} e^{-4x} = 3 \cdot \frac{x e^{-4x}}{-8+6} = -\frac{3x}{2} e^{-4x}$ ——— (1)

$\therefore$ The solution is $y = CF + PI$

$= Ae^{-4x} + Be^{-2x} - \frac{3x}{2} e^{-4x}$ ——— (1)

——x——

15. Given $(D^2+4)y = \sin 2x$

A.E is $m^2+4=0 \Rightarrow m^2 = -4 \Rightarrow m = \pm\sqrt{-4}$ ——— (1)

$= \pm 2i$

$\therefore CF = A\cos 2x + B\sin 2x$ ——— (1)

$P.I = \frac{1}{D^2+4} \sin 2x = \frac{1}{-4+4} \sin 2x$

$= \frac{x}{2D} \sin 2x = \frac{x}{2} \int \sin 2x \, dx = \frac{x}{2}\left(-\frac{\cos 2x}{2}\right)$

$= -\frac{x}{4}\cos 2x$ ——— (1)

$\therefore$ The solution is $y = CF + PI$

$= A\cos 2x + B\sin 2x - \frac{x}{4}\cos 2x$ ——— (1)

16. Let $x = e^z \Rightarrow z = \log x$ ——— (1). Then $xD = D'$, $x^2D^2 = D'(D'-1)$; $D' = \frac{d}{dz}$

$\therefore$ The given equation can be written as

$(D'(D'-1) + D')y = z$

$(D'^2 - \cancel{D'} + \cancel{D'})y = z$ ——— (1)

A.E is $m^2 = 0 \Rightarrow m_1 = 0, m_2 = 0$

$\therefore$ CF is $(Az+B)e^{0z} = Az+B$ ——— (1)

$P.I = \frac{1}{D'^2} z = \frac{1}{D'} \int z \, dz = \frac{1}{D'}\left(\frac{z^2}{2}\right) = \int \frac{z^2}{2} dz = \frac{z^3}{6}$

$\therefore$ The solution is $y = CF + PI = (Az+B) + \frac{z^3}{6} \Rightarrow (A\log x + B) + \frac{(\log x)^3}{6}$ ——— (1)

17 (i) Let $f(x,y) = e^x \cos y \Rightarrow f(0,0) = 1$ —— (1)

$f_x(x,y) = e^x \cos y \Rightarrow f_x(0,0) = 1$ —— (1)

$f_{xx}(x,y) = e^x \cos y \Rightarrow f_{xx}(0,0) = 1$ —— (1)

$f_{xxx}(x,y) = e^x \cos y \Rightarrow f_{xxx}(0,0) = 1$ —— (1)

$f_y(x,y) = -e^x \sin y \Rightarrow f_y(0,0) = 0$ —— (1)

$f_{yy}(x,y) = -e^x \cos y \Rightarrow f_{yy}(0,0) = -1$ —— (1)

$f_{yyy}(x,y) = e^x \sin y \Rightarrow f_{yyy}(0,0) = 0$ —— (1)

$f_{xy}(x,y) = -e^x \sin y \Rightarrow f_{xy}(0,0) = 0$ —— (1)

$f_{xxy}(x,y) = -e^x \sin y \Rightarrow f_{xxy}(0,0) = 0$ —— (1)

$f_{xyy}(0,0) = -e^x \cos y \Rightarrow f_{xyy}(0,0) = -1$ —— (1)

The Taylor series of $f(x,y)$ about the point (a,b) is

$$f(x,y) = f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b) + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$ —— (1)

$$+ \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] + \cdots$$

Here $(a,b) = (0,0)$

$$\therefore e^x \cos y = 1 + x(1) + y(0) + \frac{1}{2!}\left[x^2(1) + 2xy(0) + y^2(-1)\right]$$

$$+ \frac{1}{3!}\left[x^3(1) + 3x^2y(0) + 3xy^2(-1) + y^3(0)\right]$$

$$= 1 + x + \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^3}{6} - \frac{xy^2}{2}$$ —— (1)

(ii) Let x, y, z be the dimensions of the box. then —— (1)

the volume is $xyz = 32$ (given)

the surface area is $xy + 2yz + 2zx$ (as it is open at the top) —— (1)

$\therefore$ The problem is Min $f = xy + 2yz + 2zx$ —— (1)

subject to the condition $\phi = xyz - 32 = 0$

Let $F = f + \lambda\phi$ where λ is the Lagrange's multiplier —— (1)

$= (xy + 2yz + 2zx) + \lambda(xyz - 32)$

For stationary values

$F_x = 0 \Rightarrow (y+2z) + \lambda(yz) = 0$ —— ① ——(1)

$F_y = 0 \Rightarrow (x+2z) + \lambda(xz) = 0$ —— ② ——(1)

$F_z = 0 \Rightarrow (2y+2x) + \lambda(xy) = 0$ —— ③ ——(1)

$F_\lambda = 0 \Rightarrow xyz - 32 = 0$ —— ④ ——(1)

From ①, ② & ③ we get

$$-\lambda = \frac{y+2z}{yz} = \frac{x+2z}{xz} = \frac{2x+2y}{xy} \quad \text{——(1)}$$

$$\Rightarrow \frac{x(y+2z)}{xyz} = \frac{y(x+2z)}{xyz} = \frac{z(2x+2y)}{xyz}$$

$$\Rightarrow x(y+2z) = y(x+2z) = z(2x+2y)$$

Taking first two ratios, we get

$xy + 2xz = xy + 2yz$

$\Rightarrow \boxed{x = y}$ ——(1)

Taking second & third ratios, we get

$y(x+2z) = z(2x+2y)$

$xy + 2yz = 2xz + 2yz$

$\boxed{y = 2z}$ ——(1)

$\therefore x = y = 2z$

Sub in ④ we get.

$4z^3 = 32 \Rightarrow z^3 = 8 \Rightarrow z = 2$

$\therefore x = 4 \,;\, y = 4$ ——(1)

$\therefore$ The dimensions of the box are $x = 4$, $y = 4$ & $z = 2$

18 (i) Put $x = e^z \Rightarrow z = \log x$, where $D' = \frac{d}{dz}$ ——(1)

$\therefore xD = D'$ & $x^2D^2 = DD'$ ——(1)

$\therefore$ The equation becomes

$[D'(D'-1) + 4D' + 2]\, y = \sin z + e^z$ ——(1)

i.e., $[D'^2 - D' + 4D' + 2]\, y = \sin z + e^z$

$[D'^2 + 3D' + 2]\, y = \sin z + e^z$ ——(1)

The A.E is $m^2 + 3m + 2 = 0$ ——(1)

$(m+2)(m+1) = 0 \Rightarrow m_1 = -2,\ m_2 = -1$

$\therefore$ The CF $= Ae^{-2z} + Be^{-z}$ ——(1)

$$P.I = \frac{1}{D'^2+3D'+2}(\sin z + e^z) \quad \text{—— (1)}$$

$$= \frac{1}{D'^2+3D'+2}\sin z + \frac{1}{D'^2+3D'+2}e^z$$

$$= \frac{1}{-1+3D'+2}\sin z + \frac{1}{1+3+2}e^z$$

$$= \frac{(3D'-1)}{(3D'+1)(3D'-1)}\sin z + \frac{1}{6}e^z \quad \text{—— (3)}$$

$$= \frac{(3D'-1)}{9D'^2-1}\sin z + \frac{1}{6}e^z$$

$$= \frac{3\cos z - \sin z}{-10} + \frac{1}{6}e^z$$

$\therefore$ The solution is $y = CF + PI$ —— (1)

i.e., $y = Ae^{-2z} + Be^{-z} - \frac{(3\cos z - \sin z)}{10} + \frac{1}{6}e^z$. —— (1)

$$= \frac{A}{x^2} + \frac{B}{x} - \frac{1}{10}\left(3\cos(\log x) - \sin(\log x)\right] + \frac{1}{6}x$$

(ii) The given equation can be written as

$$(D^2+1)y = \tan x \quad \text{—— (1)}$$

The A.E is $m^2+1 = 0$ —— (1)

$m = \pm i$

$\therefore$ CF is $A\cos x + B\sin x$

$= Af_1 + Bf_2$ —— (1)

$PI = Pf_1 + Qf_2$ —— (1)

where $P = -\int \frac{f_2 R}{W}dx$ —— (1)

& $Q = \int \frac{f_1 R}{W}dx$ —— (1)

$W = f_1 f_2' - f_2 f_1'$ —— (1)

$= \cos^2 x + \sin^2 x$ —— (1)

$= 1.$

$\therefore P = -\int \frac{\sin x \cdot \tan x}{1}dx$

$= -\int \frac{\sin^2 x}{\cos x}dx$

$-\int \left(\frac{1-\cos^2 x}{\cos x}\right)dx = -\int \frac{1}{\cos x}dx + \int \cos x\, dx$

$= \int \sec x\, dx - \int \cos x\, dx$

$= -\log(\sec x + \tan x) + \sin x$ —— (1)

$Q = \int \frac{\cos x \cdot \tan x}{1}dx = \int \sin x\, dx$

$= -\cos x$ —— (1)

$\therefore$ The solution is $y = CF + PI$ —— (1)

i.e., $y = A\cos x + B\sin x + \left[-\log(\sec x + \tan x) + \sin x\right]\cos x$

$- \cos x \sin x$ —— (1)

$= A\cos x + B\sin x - \log(\sec x + \tan x)\cos x$