Unit IV

Analysis of Discrete Time (DT) Signals & Systems

1: Discrete Fourier Transform (DFT)

- DFT is a powerful computation tool that allows us to evaluate the Fourier transform $X(e^{j\omega})$ on a digital computer or specially designed digital hardware.
- Unlike DTFT which is defined for sequences with infinite or finite length, the DFT is defined only for the sequences with finite length.

The expressions for DFT & IDFT are

DFT:
$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}$$

and IDFT:
$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi kn}{N}}$$

$X(K) = DFT[x(n)]$

$x(n) = IDFT[X(K)]$

$x(n) \Longleftrightarrow X(K)$

Twiddle factor: $W_N = e^{-j\frac{2\pi}{N}}$

Magnitude of twiddle factor

$$\left|e^{-\frac{j2\pi}{N}}\right| = 1$$

and the phase of twiddle factor is

$$e^{-\frac{j2\pi}{N}} = -\frac{2\pi}{N}.$$

Twiddle factor is a vector on unit circle.

Consider the term W_N^{kn}.

$kn = r$.

$\therefore W_N^r$ for $N=8$, $r = 0, 1, 2, 3$, is given below

$$W_8^0 = W_8^8 = W_8^{16} = 1.$$

$$W_8^1 = W_8^9 = W_8^{17} = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}.$$

$$W_N = e^{-\frac{j2\pi}{N}}$$

$$W_N^{kn} = e^{-\frac{j2\pi kn}{N}}.$$

$$W_8^0 = e^{-\frac{j2\pi(0)}{8}} = e^0 = 1 \Rightarrow W_8^0 = 1$$

$$W_8^1 = e^{-\frac{j2\pi(1)}{8}} = e^{-j\pi/4} = \cos\pi/4 - j\sin\pi/4 = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}.$$

$$W_8^1 = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}.$$

$$W_8^2 = e^{-j\pi/2} = -j$$

$$W_8^4 = e^{-j\pi} = -1$$

Examples.

1. Find 4-point DFT of the following sequences.

(i) $x(n) = \{1, -2, 3, 4\}$

Solution:

$N = 4$, Sequence length $L = 4$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nk}{N}}, \quad k = 0, 1, 2, 3$$

$$X(0) = \sum_{n=0}^{3} x(n) e^{0} = \sum_{n=0}^{3} x(n) = x(0) + x(1) + x(2) + x(3)$$

$$\Rightarrow 1 - 2 + 3 + 4 = 6.$$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-\frac{j n\pi(1)\times 2}{4}} \Rightarrow \sum_{n=0}^{3} x(n) e^{-\frac{j\pi n}{2}}$$

$$\Rightarrow x(0) + x(1) e^{-j\pi/2} + x(2) e^{-j\pi} + x(3) e^{-j\frac{3\pi}{2}}$$

$$\Rightarrow 1 - 2\left(\cos\pi/2 - j\sin\pi/2\right) + 3(\cos\pi - j\sin\pi)$$

$$+ 4\left[\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right] = 1 - 2(-j) + 3(-1) + 4j$$

$$X(1) \Rightarrow -2 + j6$$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi n}.$$

$$= x(0) + x(1) e^{-j\pi} + x(2) e^{-j2\pi} + x(3) e^{-j3\pi}$$

$$= 1 - 2(-1) + 3(1) + 4(-1) = 2$$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j\frac{3\pi n}{2}}$$

$$= x(0) + x(1) e^{-j\frac{3\pi}{2}} + x(2) e^{j3\pi} + x(3) e^{-j\frac{9\pi}{2}}$$

$$= 1 - 2(j) + 3(-1) + 4(-j)$$

$$= -2 - j6$$

$$X(K) = \{6, -2+j6, 2, -2-j6\}.$$

② Find the IDFT of the following:

$$X(K) = \{1, 1-j2, -1, 1+j2\}$$

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi k n}{N}} \quad n = 0, 1, \ldots N-1$$

For $N = 4$

$$x(n) = \frac{1}{4} \sum_{K=0}^{3} X(K) e^{j\frac{\pi k n}{2}} \quad n = 0, 1, 2, 3$$

$$x(0) = \frac{1}{4} \sum_{K=0}^{3} X(K) = \frac{1}{4}\left[1 + (1-j2) - 1 + (1+j2)\right] = 0.5$$

$$x(0) = 0.5$$

$$x(1) = \frac{1}{4} \sum_{K=0}^{3} X(K) e^{j\frac{\pi K}{2}}$$

$$\Rightarrow \frac{1}{4}\left[X(0) + X(1) e^{j\frac{\pi}{2}} + X(2) e^{j\pi} + X(3) e^{j\frac{3\pi}{2}}\right]$$

$$\Rightarrow \frac{1}{4}\left[1 + (1-j2)(j) - 1(-1) + (1+j2)(-j)\right]$$

$$= \frac{1}{4}\left[1 + \cancel{j} + 2 + 1 - \cancel{j} + 2\right] = \frac{3}{2} = 1.5$$

$$x(2) = \frac{1}{4} \sum_{K=0}^{3} X(k) e^{j\pi K}$$

$$= \frac{1}{4}\left[X(0) + X(1) e^{j\pi} + X(2) e^{j2\pi} + X(3) e^{j3\pi}\right]$$

$$= \frac{1}{4}\left[1 + (1-j2)(-1) - 1(1) + (1+j2)(-1)\right]$$

$$= \frac{1}{4}\left[1 - 1 + \cancel{j2} - 1 - 1 - \cancel{j2}\right] = -0.5$$

$$x(3) = \frac{1}{4} \sum_{K=0}^{3} X(K) e^{j\frac{3\pi K}{2}}$$

$$= \frac{1}{4}\left[X(0) + X(1) e^{j\frac{3\pi}{2}} + X(2) e^{j3\pi} + X(3) e^{j\frac{9\pi}{2}}\right]$$

$$\Rightarrow \frac{1}{4}\left[1 + (1-j2)(-j) + (-1)(-1) + (1+j2)(j)\right]$$

$$\Rightarrow \frac{1}{4}\left[1 - j - 2 + 1 + j - 2\right] = -0.5$$

$$x(n) = \{0.5, 1.5, -0.5, -0.5\}.$$

(3) Find the 8 point DFT of the sequence

$x(n) = \{1, 1, 1, 1, 1, 1, 0, 0\}$.

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi K n}{N}} \qquad K = 0, 1, \ldots N-1$$

For $N = 8$

$$X(K) = \sum_{n=0}^{7} x(n) e^{-j\frac{\pi n K}{4}} \qquad K = 0, 1, \ldots 7$$

$X(0) = x(0) + x(1) + x(2) + x(3) + x(4) + x(5) + x(6) + x(7)$

$= 1+1+1+1+1+1+0+0 = 6 \Rightarrow \boxed{X(0) = 6}$

$X(1) = \sum_{n=0}^{7} x(n) e^{-j\frac{\pi n}{4}}$

$= x(0) + x(1) e^{-j\pi/4} + x(2) e^{-j\pi/2} + x(3) e^{-j\frac{3\pi}{4}}$

$+ x(4) e^{-j\pi} + x(5) e^{-j\frac{5\pi}{4}} + x(6) e^{-j\frac{3\pi}{2}} + x(7) e^{-j\frac{7\pi}{4}}$

$\Rightarrow 1 + 0.707 - j0.707 - j - 0.707 - j0.707 - 1 - 0.707$

$+ j0.707 \Rightarrow -0.707 - j1.707$

$\boxed{X(1) = -0.707 - j1.707}$.

$X(2) = \sum_{n=0}^{7} x(n) e^{-jn\pi/2}$

$\Rightarrow x(0) + x(1) e^{-j\pi/2} + x(2) e^{-j\pi} + x(3) e^{-j\frac{3\pi}{2}} + x(4) e^{-j2\pi}$

$+ x(5) e^{-j5\pi/2} + x(6) e^{-j3\pi} + x(7) e^{-j7\pi/2}$

$\Rightarrow 1 - j - 1 + j + 1 - j$

$\boxed{X(2) = 1 - j}$.

$X(3) = \sum_{n=0}^{7} x(n) e^{-j\frac{3n\pi}{4}}$

$= x(0) + x(1) e^{-j\frac{3\pi}{4}} + x(2) e^{-j\frac{3\pi}{2}} + x(3) e^{-j\frac{9\pi}{4}}$

$+ x(4) e^{-j3\pi} + x(5) e^{-j\frac{15\pi}{4}} + x(6) e^{-j\frac{9\pi}{2}}$

$+ x(7) e^{-j\frac{21\pi}{4}}$

$\Rightarrow 1 - 0.707 - j0.707 + j + 0.707 - j0.707 - 1 + 0.707$

$+ j0.707$

SRM

$X(3) = 0.707 + j0.293$

$$X(4) = \sum_{n=0}^{7} x(n) e^{-j\pi n}.$$

$$= x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$
$$+ x(4)e^{-j4\pi} + x(5)e^{-j5\pi} + x(6)e^{-j6\pi} + x(7)e^{-j7\pi}$$

$$\Rightarrow 1 - 1 + 1 - 1 + 1 - 1 = 0.$$

$$\boxed{X(4) = 0}$$

$$X(5) = \sum_{n=0}^{7} x(n) e^{-j\frac{5\pi n}{4}}.$$

$$= x(0) + x(1)e^{-j\frac{5\pi}{4}} + x(2)e^{-j\frac{5\pi}{2}} + x(3)e^{-j\frac{15\pi}{4}}$$
$$+ x(4)e^{-j5\pi} + x(5)e^{-j\frac{25\pi}{4}} + x(6)e^{-j\frac{15\pi}{2}}$$
$$+ x(7)e^{-j\frac{35\pi}{4}}$$

$$\Rightarrow 1 - 0.707 + j0.707 - j + 0.707 + j0.707 - 1 +$$
$$0.707 - j0.707$$

$$\boxed{X(5) = 0.707 - j0.293.}$$

$$X(6) = \sum_{n=0}^{7} x(n) e^{-j\frac{3\pi n}{2}}$$

$$= x(0) + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi} + x(3)e^{-j\frac{9\pi}{2}}$$
$$+ x(4)e^{-j6\pi} + x(5)e^{-j\frac{15\pi}{2}} + x(6)e^{-j9\pi}$$
$$+ x(7)e^{-j21\pi/2} = 1 + j - 1 - j + 1 + j$$

$$\boxed{X(6) = 1 + j}$$

$$X(7) = \sum_{n=0}^{7} x(n) e^{-j\frac{7\pi n}{4}}$$

$$= x(0) + x(1) e^{-j7\pi/4} + x(2) e^{-j\frac{7\pi}{2}} + x(3) e^{-j21\pi/4}$$
$$+ x(4) e^{-j7\pi} + x(5) e^{-j35\pi/4} + x(6) e^{-j21\pi/2}$$
$$+ x(7) e^{-j\frac{49\pi}{4}}$$

$$\Rightarrow 1 + 0.707 + j0.707 + j - 0.707 + j\,0.707 - 1 - 0.707 - j0.707$$

$$\boxed{X(7) = -0.707 + j1.707}$$

$$X(K) = \{6, -0.707 - j1.707, 1-j, 0.707 + j0.293, 0,$$
$$0.707 - j0.293, 1+j, -0.707 + j1.707\}$$

(4) Find the IDFT of the sequence

$$X(K) = \{5, 0, 1-j, 0, 1, 0, 1+j, 0\}.$$

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi Kn}{N}} \quad n = 0, 1, \ldots N-1$$

For $N = 8$

$$x(n) = \frac{1}{8} \sum_{K=0}^{N-1} X(K) e^{j\frac{\pi K n}{4}} \; ; \; n = 0, 1, \ldots 7.$$

$$x(0) = \frac{1}{8}\left[\sum_{K=0}^{7} X(K)\right] = \frac{1}{8}[5 + 0 + 1 - j + 0 + 1 + 0 + 1 + j + 0] = 1$$

$$x(1) = \frac{1}{8}\left[\sum_{K=0}^{7} X(K) e^{\frac{j\pi K}{4}}\right] = \frac{1}{8}[5 + (1-j)(j) + 1(-1) + (1+j)(-j)]$$
$$= \frac{1}{8}[6] = 0.75$$

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$$x(2)=\frac{1}{8}\left[\sum_{K=0}^{7} X(K)e^{j\frac{\pi K}{2}}\right]=\frac{1}{8}[5+(1-j)(-1)+1(1)+(1+j)(-1)]$$

$$=\frac{1}{8}[4]=0.5$$

$$x(3)=\frac{1}{8}\left[\sum_{K=0}^{7} X(K)e^{j\frac{3\pi K}{4}}\right]=\frac{1}{8}[5+(1-j)(-j)+1(-1)+(1+j)(j)]$$

$$=\frac{1}{8}[2]=0.25$$

$$x(4)=\frac{1}{8}\left[\sum_{K=0}^{7} X(K)e^{j\pi K}\right]=\frac{1}{8}[5+(1-j)(1)+1(1)+(1+j)(1)]$$

$$=1$$

$$x(5)=\frac{1}{8}\left[\sum_{K=0}^{7} X(K)e^{j\frac{5\pi K}{4}}\right]=\frac{1}{8}[5+(1-j)(j)+1(-1)+(1+j)(-j)]$$

$$=\frac{1}{8}[6]=0.75$$

$$x(6)=\frac{1}{8}\left[\sum_{K=0}^{7} X(K)e^{j\frac{3\pi K}{2}}\right]=\frac{1}{8}[5+(1-j)(-1)+1(1)+(1+j)(-1)]$$

$$=\frac{1}{8}[4]=0.5$$

$$x(7)=\frac{1}{8}\left[\sum_{K=0}^{7} X(K)e^{j\frac{7\pi K}{4}}\right]=\frac{1}{8}[5+(1-j)(-j)+1(-1)+(1+j)(j)]$$

$$=\frac{1}{8}[2]=0.25$$

$$x(n)=\{1,0.75,0.5,0.25,1,0.75,0.5,0.25\}.$$

Properties of DFT

Property	Time Domain	Frequency Domain
Periodicity	$x(n) = x(n+N)$	$X(K) = X(K+N)$
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(K) + a_2 X_2(K)$
Time reversal	$x(N-n)$	$X(N-K)$
Circular Time shift	$x((n-l))_N$	$X(K)\, e^{-j\frac{2\pi k l}{N}}$
Circular freq. Shift	$x(n)\, e^{j\frac{2\pi l n}{N}}$	$X((K-l))_N$
Circular Convolution	$x_1(n) \, N \, x_2(n)$	$X_1(K)\, X_2(K)$
Circular correlation	$x_1(n) \, N \, y^*(-n)$	$X(K)\, Y^*(K)$
Multiplication of 2 sequences	$x_1(n)\, x_2(n)$	$\frac{1}{N}\left[X_1(K) \, N \, X_2(K)\right]$
Complex Conjugate	$x^*(n)$	$X^*(N-K)$
Parsevals Theorem	$\sum_{n=0}^{N-1} x(n)\, y^*(n)$	$\frac{1}{N}\sum_{K=0}^{N-1} X(K)\, Y^*(K)$