Type IV : RHS $e^{ax+by}\phi(x,y)$

$$PI = \frac{1}{f(D,D')} e^{ax+by}\phi(x,y) = e^{ax+by}\frac{1}{f(D+a, D'+b)}\phi(x,y)$$

Example:- Solve $(D^2 - 3DD' + 2D'^2)z = (2+4x)e^{x+2y}$

Solution:- To find CF

Put $D = m$ & $D' = 1$ and equate it to zero

$$m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0$$

$$\Rightarrow m = 1, m = 2$$

$\therefore$ the CF is $f_1(y+x) + f_2(y+2x)$

To find PI :-

$$PI = \frac{1}{f(D,D')}(2+4x)e^{x+2y}$$

$$= \frac{1}{D^2 - 3DD' + 2D'^2}(2+4x)e^{x+2y}$$

$$= e^{x+2y}\cdot\frac{1}{(D+1)^2 - 3(D+1)(D'+2) + 2(D'+2)^2}(2+4x)$$

$$= e^{x+2y}\cdot\frac{1}{D^2+2D+1-3DD'-6D-3D'-6+2D'^2+8D'+8}(2+4x)$$

$$= e^{x+2y}\cdot\frac{1}{D^2+2D'^2-4D+5D'-3DD'+3}(2+4x)$$

$$= e^{x+2y}\cdot\frac{1}{3\left[1+\left(\frac{D^2+2D'^2-4D+5D'-3DD'}{3}\right)\right]}(2+4x)$$

$$= e^{x+2y}\cdot\frac{1}{3}\left[1+\left(\frac{D^2}{3}+\frac{2D'^2}{3} - \frac{4}{3} + \frac{5D'}{3} - \frac{3DD'}{3}\right)\right]^{-1}(2+4x)$$

$$= e^{x+2y} \cdot \frac{1}{3}\left[1 + \frac{4D}{3}\right](2+4x)$$

$$= \frac{e^{x+2y}}{3} \cdot \left(2+4x+\frac{16}{3}\right)$$

$$= \frac{e^{x+2y}}{3}\left[4x + \frac{22}{3}\right]$$

—x—

<u>Homework:</u> Solve $(D^2 - D'^2)z = e^{x-y}\sin(x+2y)$

<u>Solution:-</u> <u>The CF</u>

Put $D = m$ & $D' = 1$

$m^2 - 1 = 0 \Rightarrow m^2 = 1 \Rightarrow m = \pm 1$

The CF is $f_1(y+x) + f_2(y-x)$

<u>To find PI</u>

$$PI = \frac{1}{D^2 - D'^2} e^{x-y}\sin(x+2y)$$

$$= e^{x-y}\frac{1}{(D+1)^2 - (D'+2)^2}\sin(x+2y)$$

$$= e^{x-y}\frac{1}{D^2+2D+1-D'^2-4D'-4}\sin(x+2y)$$

$$= e^{x-y} \cdot \frac{1}{-1+2D+1+4-4D'-4}\sin(x+2y) \quad \Bigg| \begin{array}{l} D^2 = -1^2 = -1 \\ D'^2 = -2^2 = -4 \end{array}$$

$$= e^{x-y}\frac{1}{2D-4D'}\sin(x+2y)$$

$$= e^{x-y} \cdot \frac{(2D+4D')}{(2D-4D')(2D+4D')}\sin(x+2y)$$

$$= e^{x-y} \cdot \frac{(2D+4D')}{4D^2 - 16D'^2}\sin(x+2y)$$

$$= e^{x-y} \cdot \frac{(2\cos(x+2y) + 4\cos(x+2y)\cdot 2)}{4(-1) - 16(-4)}$$

$$= e^{x-y}\,\frac{10\cos(x+2y)}{60} = \frac{e^{x-y}}{6}\cos(x+2y)$$

$\therefore$ The solution is $z = CF + PI$

$$z = f_1(y+x) + f_2(y-x) + \frac{e^{x-y}}{6}\cos(x+2y)$$

Type V :: General rule

Example 1 :- Solve $(D^2 + DD' - 6D'^2)z = y\cos x$

Solution :- To find CF

Put $D = m$ & $D' = 1$ and equate it to zero

$m^2 + m - 6 = 0 \Rightarrow (m+3)(m-2) = 0 \Rightarrow m = -3,\ m = +2$

$\therefore$ The CF is $f_1(y-3x) + f_2(y+2x)$

To find PI

$$PI = \frac{1}{D^2 + DD' - 6D'^2}\, y\cos x$$

$$= \frac{1}{(D+3D')(D-2D')}\, y\cos x = \frac{1}{(D+3D')}\int (a-2x)\cos x\, dx \qquad \boxed{y = a-2x}$$

$$= \frac{1}{(D+3D')}\left[(a-2x)\sin x - (-2)(-\cos x)\right]$$

$$= \frac{1}{(D+3D')}(y\sin x - 2\cos x) = \int((a+3x)\sin x - 2\cos x)\,dx \qquad \boxed{y = a+3x}$$

$$= (a+3x)(-\cos x) - (+3)(-\sin x) - 2\sin x$$

$$= -y\cos x - 3\sin x - 2\sin x$$

$$= -y\cos x + \sin x.$$

$\therefore$ The solution is $z = CF + PI$

$$z = f_1(y+3x) + f_2(y+2x) + \sin x - y\cos x.$$

Example 2:- Solve $(D^2 + 2DD' + D'^2)z = 2\cos y - x\sin y$

Solution:- To find CF

Put $D = m$ & $D' = 1$ and equate it to zero

$m^2 + 2m + 1 = 0 \Rightarrow (m+1)(m+1) = 0 \Rightarrow m = -1, -1$

$\therefore$ The CF is $f_1(y-x) + x f_2(y-x)$

To find PI :

$$PI. = \frac{1}{D^2 + 2DD' + D'^2}(2\cos y - x\sin y)$$

$$= \frac{1}{(D+D')(D+D')}(2\cos y - x\sin y)$$

$$= \frac{1}{D+D'}\int 2\cos(a+x) - x\sin(a+x)$$

$\boxed{y = a+x}$

$$= \frac{1}{D+D'}\left\{2\sin(a+x) - \left[x(-\cos(a+x)) - 1(-\sin(a+x))\right]\right\}$$

$$= \frac{1}{D+D'}\left[2\sin y + x\cos y - \sin y\right] = \frac{1}{D+D'}\left[\sin y + x\cos y\right]$$

$$= \int \left[\sin(a+x) + x\cos(a+x)\right].dx \qquad \boxed{y = a+x}$$

$$= \left[-\cos(a+x) + x(\sin a+x) - (1)(-\cos(a+x))\right]$$

$$= -\cos y + x\sin y + \cos y = x\sin y$$

$\therefore$ The solution is $z = CF + PI$

i.e., $z = f_1(y-x) + x f_2(y-x) + x\sin y$.