Unit 1

(1) Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and hence find A^{-1}.

<u>Soln:</u>

Let $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

The char. eqn is, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

where S_1 = Sum of leading diagonal elts

$= 1+2+1 = \underline{4}$.

S_2 = Sum of the minors of the leading diagonal elt's.

$= \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix}$

$= 2-6 + 1-7 + 2-12 = \underline{-20}$

$S_3 = |A| = \begin{vmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{vmatrix}$

$= 1(2-6) - 3(4-3) + 7(8-2)$

$= -4 - 3 + 42$

$\underline{|A| = 35}$

$\therefore$ The char. eqn is, $\boxed{\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0}$.

<u>Verification</u>

To verify CHT, we have to prove that

$A^3 - 4A^2 - 20A - 35I = 0$.

$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+12+7 & 3+6+14 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+4+1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = A^2 \times A$$

$$= \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

$$\therefore A^3 - 4A^2 - 20A - 35I = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - \begin{bmatrix} 80 & 92 & 92 \\ 60 & 88 & 148 \\ 40 & 36 & 56 \end{bmatrix}$$

$$- \begin{bmatrix} 20 & 60 & 140 \\ 80 & 40 & 60 \\ 20 & 40 & 20 \end{bmatrix} - \begin{bmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence C-H- Theorem is Verified.

To find A^{-1}

Consider, $A^3 - 4A^2 - 20A - 35I = 0$

Multiply by A^{-1}.

$\Rightarrow A^2 - 4A - 20I - 35A^{-1} = 0$

$\Rightarrow -35A^{-1} = -A^2 + 4A + 20I$

$35A^{-1} = A^2 - 4A - 20I$

$A^{-1} = \frac{1}{35}\left[A^2 - 4A - 20I\right]$

$$= \frac{1}{35}\left\{\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - \begin{bmatrix} 4 & 12 & 28 \\ 16 & 8 & 12 \\ 4 & 8 & 12 \end{bmatrix} - \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}\right\}$$

$$A^{-1} = \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -18 \end{bmatrix}$$

(2) Deduce the quadratic form $6x_1^2+3x_2^2+3x_3^2-4x_1x_2-2x_2x_3+4x_3x_1$ to a canonical form and hence find rank, index and signature.

Soln:

Step 1 To find the matrix of the Q.F.

$$A = \begin{bmatrix} \text{Coeff of } x_1^2 & \tfrac{1}{2}\text{ Co-eff of } x_1x_2 & \tfrac{1}{2}\text{ Co-eff of } x_1x_3 \\ \tfrac{1}{2}\text{ Coeff of } x_2x_1 & \text{Co-eff of } x_2^2 & \tfrac{1}{2}\text{ Co-eff of } x_2x_3 \\ \tfrac{1}{2}\text{ Coeff of } x_3x_1 & \tfrac{1}{2}\text{ Coeff of } x_3x_2 & \text{Co-eff of } x_3^2 \end{bmatrix}$$

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Step 2: To find char. eqn.. $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$.

Here, $s_1 = 12$; $s_2 = 36$; $s_3 = 32$.

$\Rightarrow \boxed{\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0}$, is the char. eqn.

Step 3 To find Eigen values

$$\begin{array}{r|rrrr} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

$\Rightarrow \lambda = 2$ is an root.

$\Rightarrow \lambda^2 - 10\lambda + 16 = 0$

$\lambda^2 - 8\lambda - 2\lambda + 16 = 0$

$(\lambda-8)(\lambda-2) = 0$

$\lambda = 2, 8$ are another roots.

$\therefore$ Eigen Values are $\lambda = 2, 3, 8$.

step 4: To find Eigen vectors:

let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the Eigenvectors, then the eqn.

$$(A - \lambda I) X = 0.$$

$$\Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} (6-\lambda)x_1 - 2x_2 + 2x_3 &= 0 \\ -2x_1 + (3-\lambda)x_2 - x_3 &= 0 \\ 2x_1 - x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\} \text{ --- (A)}$$

Case(i) when $\lambda = 8$,

$$(A) \Rightarrow \begin{aligned} -2x_1 - 2x_2 + 2x_3 &= 0. \\ -2x_1 - 5x_2 - x_3 &= 0 \\ 2x_1 - x_2 - 5x_3 &= 0. \end{aligned}$$

Taking 1st two eqn, & cross multiply rule, we get.

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \hline \begin{matrix} -2 & 2 \\ -5 & -1 \end{matrix} & \begin{matrix} 2 & -2 \\ -1 & -2 \end{matrix} & \begin{matrix} -2 & -2 \\ -2 & -5 \end{matrix} \end{array}$$

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4} = 1k.$$

$$\Rightarrow \frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6} = k \text{ (say)}.$$

$$k = 1/6, \quad \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$\therefore$ The Eigen Vector $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

Case (ii) $\lambda = 2$ in (A), we get

$$4x_1 - 2x_2 + 2x_3 = 0$$
$$-2x_1 + x_2 - x_3 = 0$$
$$2x_1 - x_2 + x_3 = 0.$$

The above eqn's are same.

Consider any one,

$$2 \cdot x_1 - x_2 + x_3 = 0.$$

put $x_3 = 0$ & $x_1 = 1$.

$$2(1) - x_2 = 0$$
$$-x_2 = -2$$
$$x_2 = 2$$

$\therefore$ The Eigen vector $X_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

Case (iii) when $\lambda = 2$, (is repeated)

Since we are going to diagonalize the matrix through the orthogonal transformation, we have to find X_3 which is orthogonal to X_1 and X_2.

$\Rightarrow$ X_3 is orthogonal to X_2

$\Rightarrow$ $x_1 + 2x_2 + 0x_3 = 0.$

$\Rightarrow$ $2x_2 = -x_1$

(or) $\boxed{x_1 = -2x_2}$

X_3 is orthogonal to X_1 too,

$2x_1 - x_2 + x_3 = 0.$

$x_3 = x_2 - 2x_1$

$= x_2 + 4x_2$

$\boxed{x_3 = 5x_2}$

Put $x_2 = 1$, we get, $x_1 = -2$

$x_3 = 5.$

$\therefore$ $X_3 = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$, is the 3rd Eigen Vector.

Step 5 Normalized Eigen Vectors:

Eigen Vectors:	$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$	$X_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$	$X_3 = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$
Normalised form	$\begin{pmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$	$\begin{pmatrix} 1/\sqrt{3} \\ 2/\sqrt{3} \\ 0 \end{pmatrix}$	$\begin{pmatrix} -2/\sqrt{30} \\ 1/\sqrt{30} \\ 5/\sqrt{30} \end{pmatrix}$

Step 6 To find Modal matrix.

The normalised model matrix is,

$$N = \begin{bmatrix} 2/\sqrt{6} & 1/\sqrt{5} & -2/\sqrt{30} \\ -1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 1/\sqrt{6} & 0 & 5/\sqrt{30} \end{bmatrix} \text{ and } N^T = \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ -2/\sqrt{30} & \frac{1}{\sqrt{30}} & 5/\sqrt{30} \end{bmatrix}$$

Step 7 To find N^TAN.

$$AN = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{6} & 1/\sqrt{5} & -2/\sqrt{30} \\ -1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 1/\sqrt{6} & 0 & 5/\sqrt{30} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{16}{\sqrt{6}} & -2/\sqrt{5} & \frac{-4}{\sqrt{30}} \\ \frac{-8}{\sqrt{6}} & \frac{4}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ \frac{8}{\sqrt{6}} & 0 & \frac{10}{\sqrt{30}} \end{bmatrix}$$

$$N^TAN = \begin{bmatrix} \frac{2}{\sqrt{6}} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & \frac{2}{\sqrt{5}} & 0 \\ -2/\sqrt{30} & \frac{1}{\sqrt{30}} & 5/\sqrt{30} \end{bmatrix} \begin{bmatrix} \frac{16}{\sqrt{6}} & -2/\sqrt{5} & \frac{-4}{\sqrt{30}} \\ \frac{-8}{\sqrt{6}} & 4/\sqrt{5} & 2/\sqrt{30} \\ 8/\sqrt{6} & 0 & \frac{10}{\sqrt{30}} \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$= D$ which is a Diagonal matrix.

Step 8 Canonical form.

$$Y^T(N^TAN)Y = (y_1, y_2\ y_3)\begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= 8y_1^2 + 2y_2^2 + 2y_3^3.$$

$\Rightarrow$ positive definite.

Unit 2

① A rectangular box, open at the top, is to have to volume of 32cc, find the dimensions of the box, that requires the least material for its construction.

Soln: let x, y, z be the dimensions of the box.

The surface area of the box $= xy + 2yz + 2zx$

Since the box is opened at the top.

Volume $= 32 \Rightarrow xyz = 32 \Rightarrow xyz - 32 = 0$.

$\therefore F(x,y,z) = (xy + 2yz + 2zx) + \lambda(xyz - 32) = 0$

LO

$$\frac{\partial F}{\partial x} = y + 2z + \lambda(yz)$$

$$\frac{\partial F}{\partial y} = x + 2z + \lambda(zx)$$

$$\frac{\partial F}{\partial z} = 2y + 2z + \lambda(xy)$$

$$\frac{\partial F}{\partial \lambda} = xyz - 32.$$

$$\frac{\partial F}{\partial x} = 0$$

$$y + 2z + \lambda(yz) = 0$$

$$\lambda = \frac{-y-2z}{yz}.$$

$$\frac{\partial F}{\partial y} = 0$$

$$\Rightarrow \lambda = \frac{-x-2z}{xz}$$

$$\frac{\partial F}{\partial z} = 0$$

$$\lambda = \frac{-2y-2x}{xy}.$$

$$\Rightarrow \frac{-y-2z}{yz} = \frac{-x-2z}{xz} = \frac{-2y-2x}{xy}.$$

Comparing, $\frac{-y-2z}{yz} = \frac{-x-2z}{xz} \Rightarrow x = y.$

Comparing $\frac{-x-2z}{xz} = \frac{-2y-2x}{xy} \Rightarrow y = 2z.$

Thus $x = y = 2z.$

$$\frac{\partial F}{\partial \lambda} = 0 \quad \Rightarrow xyz = 32$$

$$x(x)\left(\frac{x}{2}\right) = 32$$

$$x^3 = 64$$

$$\boxed{x = 4}$$

Then $y = 4$, & $z = 2$.

$\therefore$ The dimension of the box is $(4, 4, 2)$

② Find the volume of the greatest rectangular parallelopiped than can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Soln:

Let $2x, 2y, 2z$ be the dimension of the rectangular parallelopiped. So we have to maximize $8xyz$ subject to $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

$$\therefore \quad F(x,y,z) = 8xyz + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$

$$\frac{\partial F}{\partial x} = 8yz + \frac{2\lambda x}{a^2} \; ; \; \frac{\partial F}{\partial y} = 8xz + \frac{2\lambda y}{b^2} \; ; \; \frac{\partial F}{\partial z} = 8xy + \frac{2\lambda z}{c^2}$$

$\frac{\partial F}{\partial x} = 0.$	$\frac{\partial F}{\partial y} = 0$	$\frac{\partial F}{\partial z} = 0$
$\Rightarrow \lambda = \frac{a^2 yz}{x}$	$\lambda = \frac{b^2 xz}{y}$	$\lambda = \frac{c^2 xy}{z}$

$\Rightarrow$ Consider, $\frac{a^2 y\cancel{z}}{x} = \frac{b^2 x\cancel{z}}{y}.$

$\Rightarrow \frac{y^2}{b^2} = \frac{x^2}{a^2}.$

Consider,

$\frac{b^2 xz}{y} = \frac{c^2 xy}{z}$

$\Rightarrow y^2/b^2 = z^2/c^2.$

Thus $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}.$

$$\frac{\partial F}{\partial \lambda} = \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right).$$

$\frac{\partial F}{\partial \lambda} = 0$

$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$\Rightarrow \frac{3x^2}{a^2} = 1 \quad \Rightarrow \quad x = \frac{a}{\sqrt{3}}$

|||ly, $y = \frac{b}{\sqrt{3}}$

$c = \frac{c}{\sqrt{3}}.$

maximum Volume $= 8xyz$

$= \frac{8abc}{3\sqrt{3}}$

③ Expand $e^x \cos y$ in power of x and y as far as the terms of the 3^{rd} degree.

Soln:

Given : $f(x,y) = e^x \cos y$.

$$f(x,y) = f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b)$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] + \cdots \quad \text{①}$$

here $a=0,\ b=0$.

$f(x,y) = e^x \cos y$. $\qquad f(0,0) = 1$

$f_x(x,y) = e^x \cos y$ $\qquad f_x(0,0) = 1$

$f_{xx}(x,y) = e^x \cos y$ $\qquad f_{xx}(0,0) = 1$

$f_{xxx}(x,y) = e^x \cos y$ $\qquad f_{xxx}(0,0) = 1$

$f_y(x,y) = -e^x \sin y$ $\qquad f_y(0,0) = 0$

$f_{yy}(x,y) = -e^x \cos y$ $\qquad f_{yy}(0,0) = -1$

$f_{yyy}(x,y) = e^x \sin y$ $\qquad f_{yyy}(0,0) = 0$

$f_{xy}(x,y) = -x e^x \sin y$ $\qquad f_{xy}(0,0) = 0$

$f_{xxy}(x,y) = -x^2 e^x \sin y$ $\qquad f_{xxy}(0,0) = 0$

$f_{xyy}(x,y) = -x e^x \cos y$. $\qquad f_{xyy}(0,0) = 0$.

$$\text{①} \Rightarrow e^x \cos y = 1 + x(1) + y(0) + \frac{1}{2}\left[x^2(1) + 0 + y^2(-1)\right] + \frac{1}{6}\left[x^3(1) + 3(0) + 3(0) + 0\right] + \cdots$$

$$e^x \cos y = 1 + x + \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^3}{6} + \cdots$$

Unit 3 Solve.

① $(D^2-4D+3)y = \sin 3x$

Soln:

The A.E is,

$$m^2-4m+3=0$$

$$(m-3)(m-1)=0$$

$$m=1,3$$

$\therefore$ The CF is $= Ae^{x} + Be^{3x}$. — ①.

$$PI = \frac{1}{D^2-4D+3}\sin 3x,$$

Replace D^2 by -9,

$$= \frac{1}{-9-4D+3}\sin 3x$$

$$= \frac{1}{-6-4D}\sin 3x.$$

$$= \frac{-4D+6}{16D^2-36}\sin 3x,$$ Replace $D^2=-9$

$$= \frac{(-4D+6)\sin 3x}{-144-36} = \frac{-4(\cos 3x)\cdot 3+6\sin 3x}{-180}$$

$$= \frac{-12\cos 3x+6\sin 3x}{-180} = \frac{12\cos 3x-6\sin 3x}{180}$$

$$PI = \frac{2\cos 3x-\sin 3x}{30}$$

$\therefore$ General soln is;

$$y = Ae^{x}+Be^{3x} + \frac{2\cos 3x-\sin 3x}{30}$$ //.

Unit 4.

(1) Find the circle of curvature of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$.

Soln:

Given: $\sqrt{x}+\sqrt{y}=\sqrt{a}$.

diff w.r.t 'x', we get,

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}=0.$$

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx}=-\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx}=\frac{-\sqrt{y}}{\sqrt{x}}.$$

$$\Rightarrow \boxed{\frac{dy}{dx}_{\left(\frac{a}{4},\frac{a}{4}\right)}=-1}$$

$$\boxed{y_1=-1}$$

$$\frac{d^2y}{dx^2}=\frac{-\left[\sqrt{x}\cdot\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}-\sqrt{y}\cdot\frac{1}{2\sqrt{x}}\right]}{x}$$

$$\frac{d^2y}{dx^2}_{\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{-\left[\sqrt{\frac{a}{4}}\cdot\frac{1}{2\sqrt{\frac{a}{4}}}(-1)-\sqrt{\frac{a}{4}}\cdot\frac{1}{2\sqrt{\frac{a}{4}}}\right]}{\frac{a}{4}}$$

$$=\frac{-\left(-\frac{1}{2}-\frac{1}{2}\right)}{\frac{a}{4}}$$

$$=\frac{1}{\frac{a}{4}}=\frac{4}{a}.$$

$$\Rightarrow \boxed{y_2=\frac{4}{a}}$$

$$\bar{x}=x-\frac{y_1(1+y_1^2)}{y_2}$$

$$=\frac{a}{4}-\frac{(-1)(1+(-1)^2)}{\frac{4}{a}}$$

$$=\frac{a}{4}+\frac{a}{4}(1+1)=\frac{a}{4}+\frac{a}{2}=\frac{3a}{4}.$$

$$\therefore \boxed{\bar{x}=\frac{3a}{4}}$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}.$$

$$= \frac{a}{4} + \frac{(1+1)}{(4/a)}$$

$$= \frac{a}{4} + \frac{a}{4}(2)$$

$$= \frac{a}{4} + \frac{a}{2}$$

$$\boxed{\bar{y} = \frac{3a}{4}}$$

$$\therefore \rho = \frac{[1+(y_1)^2]^{3/2}}{y_2}. = \frac{[1+(-1)^2]^{3/2}}{(4/a)}$$

$$= \frac{a}{4}.(2)^{3/2} = \frac{a}{\cancel{4}}.\cancel{2}\sqrt{2}$$

$$\boxed{\rho = \frac{a\sqrt{2}}{2}}$$

$\therefore$ The circle of curvature at $(a/4, a/4)$ is,

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$\left(x-\frac{3a}{4}\right)^2 + \left(y-\frac{3a}{4}\right)^2 = \frac{\cancel{2}a^2}{\cancel{4}2} = \frac{a^2}{2}$$

$$\Rightarrow \left(x-\frac{3a}{4}\right)^2 + \left(y-\frac{3a}{4}\right)^2 = \frac{a^2}{2}$$

(2) Find the eqn of the circle of curvature of the parabola $y^2 = 12x$ at the pt $(3, 6)$.

Soln:

Given: $y^2 = 12x$

$2y \frac{dy}{dx} = 12$

$\frac{dy}{dx} = \frac{12 \cdot 6}{2y} = 6/y$.

$\left.\frac{dy}{dx}\right|_{(3,6)} = \frac{6}{6} = 1$.

$\Rightarrow \boxed{y_1 = 1}$

$\frac{d^2y}{dx^2} = -\frac{6}{y^2} \cdot \frac{dy}{dx}$

$\left.\frac{d^2y}{dx^2}\right|_{(3,6)} = -\frac{\cancel{6}}{36_6}\left(\left.\frac{dy}{dx}\right|_{(3,6)}\right)$

$= -\frac{1}{6}(1)$.

$\boxed{y_2 = -1/6}$

$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{(-1/6)} = (-6)(2)^{3/2} = (-6)2\sqrt{2} = -12\sqrt{2}$

$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$

$= 3 - \frac{(1)(1+(1)^2)}{(-1/6)} = 3 + 6(2)$. $\Rightarrow \boxed{15 = \bar{x}}$

$\bar{y} = y + \left(\frac{1+y_1^2}{y_2}\right) = 6 + \left(\frac{1+1}{-1/6}\right) = +6 - 6(2) \Rightarrow \boxed{-6. = \bar{y}}$

$\therefore$ Center of curvature $(\bar{x}, \bar{y}) = (15, -6)$.

Circle of curvature is $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

$\Rightarrow (x-15)^2 + (y+6)^2 = 288$

$\boxed{\because (-12\sqrt{2})^2 = 288}$

(3) Find the ~~evolute~~ evolute of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Soln:

The parametric eqn's of the ellipse are

$x = a\cos\theta$ | $y = b\sin\theta$.

$\frac{dx}{d\theta} = -a\sin\theta$ | $\frac{dy}{d\theta} = b\cos\theta$.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-b\cos\theta}{a\sin\theta} = -\frac{b}{a}\cot\theta.$$

$$\boxed{y_1 = -\frac{b}{a}\cot\theta}$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(-\frac{b}{a}\cot\theta\right)\cdot\frac{d\theta}{dx}.$$

$$= -\frac{b}{a}(-\operatorname{cosec}^2\theta)\cdot\frac{1}{-a\sin\theta}$$

$$= -\frac{b}{a^2}\operatorname{cosec}^2\theta\cdot\operatorname{cosec}\theta$$

$$y_2 = \frac{d^2y}{dx^2} = -\frac{b}{a^2}\operatorname{cosec}^3\theta.$$

$$\boxed{y_2 = -\frac{b}{a^2}\operatorname{cosec}^3\theta}$$

$$\bar{X} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a\cos\theta - \frac{-\frac{b}{a}\cot\theta\left(1+\frac{b^2}{a^2}\cot^2\theta\right)}{-\frac{b}{a^2}\operatorname{cosec}^3\theta}$$

$$\cancel{= a\cos\theta - \left[\frac{\frac{ab}{a}\cot\theta\frac{(a^2+b^2\cot^2\theta)}{a^2}}{\frac{b}{a}\operatorname{cosec}^3\theta}\right]}$$

$$= a\cos\theta - a\cot\theta \cdot \sin^3\theta \left(1 + \frac{b^2}{a^2}\cot^2\theta\right)$$

$$= a\cos\theta - a\frac{\cos\theta}{\sin\theta} \cdot \sin^{3}\theta \left(1 + \frac{b^2}{a^2} \cdot \frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$= a\cos\theta - a\cos\theta\sin^2\theta \left(1 + \frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$= a\cos\theta - a\cos\theta \cdot \sin^2\theta - \frac{b^2}{a}\cos^3\theta \cdot$$

$$= a\cos\theta(1 - \sin^2\theta) - \frac{b^2}{a}\cos^3\theta.$$

$$= a\cos\theta(\cos^2\theta) - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos^3\theta - \frac{b^2}{a}\cos^3\theta$$

$$\boxed{\bar{x} = \frac{a^2 - b^2}{a}(\cos^3\theta)} \quad \text{—— (1)}$$

$$\bar{y} = y + \left(\frac{1 + y_1^2}{y_2}\right)$$

$$= b\sin\theta + \frac{1 + \frac{b^2}{a^2}\cot^2\theta}{-\frac{b}{a^2}\operatorname{cosec}^3\theta.}$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta\left(1 + \frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - \frac{b}{b}\sin\theta\cos^2\theta.$$

$$= b\sin\theta - \frac{a^2\sin^3\theta}{b} - b\sin\theta\cos^2\theta$$

taking $b\sin\theta$ out side from this two terms.

$$= b\sin\theta(1 - \cos^2\theta) - \frac{a^2}{b}\sin^3\theta$$

$$= b\sin\theta(\sin^2\theta) - \frac{a^2}{b}\sin^3\theta$$

$$= \frac{b^2 - a^2}{b}(\sin^3\theta) \Rightarrow \boxed{-\left(\frac{a^2 - b^2}{b}\right)\sin^3\theta = \bar{y}}$$

(2)

To eliminate 'θ' between (1) & (2), we get

(1) $\Rightarrow$ $a\bar{x} = (a^2-b^2)\cos^3\theta$.

$(a\bar{x})^{2/3} = (a^2-b^2)^{2/3}\cos^2\theta$. —— (3)

(2) $\Rightarrow$ $b\bar{y} = \left[-(a^2-b^2)\right]\sin^3\theta$

$(b\bar{y})^{2/3} = (a^2-b^2)^{2/3}\sin^2\theta$ —— (4)

(3) + (4) $\Rightarrow$ $(a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2-b^2)^{2/3}(1)$

$\Rightarrow (ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$ is the

eqn of the evolute of the given ellipse.

(4) Find the envelope of the family of st. line given by $x\cos\alpha + y\sin\alpha = a\sec\alpha$, where '$\alpha$' is the parameter.

Soln:

Given: $x\cos\alpha + y\sin\alpha = a\sec\alpha$.

$\div$ by $\cos\alpha$, we get,

$$x + y\tan\alpha = a\frac{\sec\alpha}{\cos\alpha}$$

$$= a\sec^2\alpha$$

$$= a(1+\tan^2\alpha)$$

$$= a + a\tan^2\alpha.$$

$$\Rightarrow x + y\tan\alpha - a - a\tan^2\alpha = 0.$$

$$\Rightarrow -a\tan^2\alpha + y\tan\alpha + x - a = 0$$

$$\Rightarrow a\tan^2\alpha - y\tan\alpha + (a-x) = 0$$

which is a quadratic eqn in $\tan\alpha$.

$A = a$; $B = -y$; $C = a - x$.

$$\Rightarrow B^2 - 4AC = 0.$$

$\boxed{y^2 - 4a(a-x) = 0}$, which is the envelope of the given st. line.

(5) Find the envelope of the family of st-lines.

$y = mx + \sqrt{a^2m^2 + b^2}$, where '$m$' is the parameter.

Soln:

$$y = mx + \sqrt{a^2m^2 + b^2}.$$

$$y - mx = \sqrt{a^2m^2 + b^2}.$$

Squaring on both sides,

$(y-mx)^2 = a^2m^2 + b^2$

$y^2 + m^2x^2 - 2ymx - a^2m^2 - b^2 = 0$

$m^2(x^2-a^2) - 2myx + (y^2-b^2) = 0$, is a quadratic eqn in 'm'.

$\Rightarrow A = x^2 - a^2$; $B = -2xy$; $C = y^2 - b^2$.

The envelope ie, $B^2 - 4AC = 0$.

$4x^2y^2 - 4(x^2-a^2)(y^2-b^2) = 0$

$\cancel{x^2y^2} - \cancel{x^2y^2} + x^2b^2 + a^2y^2 - a^2b^2 = 0.$

$x^2b^2 + y^2a^2 = a^2b^2$

$\div a^2b^2$,

$\Rightarrow \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$, is the envelope of the given eqn.