Date ____

Page ____

Laplace Transition of Periodic function

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}$$

where T is period of the function $f(t)$

function is periodic if $f(x+T) = f(t)$

$T \rightarrow$ least +ve integer

Eg. $f(x) = \sin x$

$f(x+\pi) = \sin(x+\pi) = -\sin x$

$f(x+2\pi) = \sin x$.

$f(x+4\pi) = \ldots$

Q Find the Laplace Transform of the function

$$f(t) = \begin{cases} \sin \omega t & 0 \le t < \frac{\pi}{\omega} \\ 0 & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$$

Ans $f(t)$ is periodic function with period $T = \frac{2\pi}{\omega}$

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}$$

$$= \frac{\int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\,dt}{1 - e^{-s\frac{2\pi}{\omega}}}$$

$$= \frac{\int_0^{\pi/\omega} e^{-st} \cdot \sin \omega t \cdot dt + \cancel{\int_{\pi/\omega}^{2\pi/\omega} e^{-st} \cdot 0 \cdot dt}}{1 - e^{-s\frac{2\pi}{\omega}}}$$

$$\left[\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)\right]$$

$$= \frac{\left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\frac{\pi}{\omega}}}{1-e^{-\frac{2s\pi}{\omega}}}$$

$$= \frac{\left[\frac{e^{-\frac{s\pi}{\omega}}}{s^2+\omega^2}(0-\omega(-1))\right] - \left[\frac{1}{s^2+\omega^2}(0-\omega(1))\right]}{1-e^{-\frac{2\pi s}{\omega}}}$$

$$\left[\begin{array}{l} e^a \cdot e^b = e^{a+b} \\ (e^a)^2 = e^a \cdot e^a = e^{2a} \end{array}\right]$$

$$= \frac{\frac{\omega e^{-\frac{s\pi}{\omega}} + \omega}{s^2+\omega^2}}{1-\left(e^{-\frac{\pi s}{\omega}}\right)^2}$$

$$= \frac{\omega\left(\cancel{1+e^{-\frac{2\pi}{\omega}}}\right)}{\left(\cancel{1+e^{-\frac{\pi s}{\omega}}}\right)\left(1-e^{-\frac{\pi s}{\omega}}\right)\left(s^2+\omega^2\right)}$$

$$= \frac{\omega}{1-e^{-\frac{\pi s}{\omega}}} \cdot \frac{1}{s^2+\omega^2}$$

Q Find the L.T. of the periodic function

$$f(t) = \begin{cases} t & 0 \le t \le a \\ 2a-t & a \le t \le 2a \end{cases}$$

and $f(t+2a) = f(t)$

Ans $f(t)$ is periodic periodic $T = 2a$

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

$$= \frac{\int_0^{2a} e^{-st} f(t)\,dt}{1-e^{-2as}}$$

Page

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\cdot t\,dt + \int_a^{2a} e^{-st}(2a-t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left(t\cdot\frac{e^{-st}}{-s}\right)-\left(1\cdot\frac{e^{-st}}{s^2}\right)+0\right]_0^a$$

$$+\left[\left((2a-t)\frac{e^{-st}}{-s}\right)-(-1)\cdot\frac{e^{-st}}{s^2}\right]_a^{2a}$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[\left(\frac{-a\cdot e^{-sa}}{s}-\frac{e^{-sa}}{s^2}\right)-\left(0-\frac{1}{s^2}\right)\right]\right.$$

$$\left.+\left[\left(0+\frac{e^{-sa}}{s^2}\right)-\left(-a\cdot\frac{e^{-st}}{s}+\frac{e^{-sa}}{s^2}\right)\right]\right\}$$

$$= \frac{1}{1-e^{-2as}}\left(\cancel{\frac{-ae^{-sa}}{s}}-\frac{e^{-sa}}{s^2}+\frac{1}{s^2}+\frac{e^{-2as}}{s^2}+\cancel{\frac{ae^{-sa}}{s}}-\frac{e^{-sa}}{s^2}\right)$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-2e^{-sa}+1+e^{-2as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\cdot\frac{(1-e^{-as})^2}{s^2}$$

$$= \frac{1}{(1+e^{-as})(1-e^{-as})}\cdot\frac{(1-e^{-as})^2}{s^2}$$

$$= \frac{1-e^{-as}}{1+e^{-as}}\cdot\frac{1}{s^2}$$

$$= \frac{1}{s^2}\left[\frac{1-e^{-\frac{as}{2}}\cdot e^{-\frac{as}{2}}}{1+e^{-\frac{as}{2}}\cdot e^{-\frac{as}{2}}}\right]$$

Date

Page

$$\frac{1}{s^2}\left[\frac{e^{\frac{as}{2}} - e^{-\frac{as}{2}}}{e^{as/2} + e^{-as/2}}\right]$$

$$= \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$$

Q $f(t) = \begin{cases} 1, & \text{when } 0 < t \le a/2 \\ -1, & \text{when } a/2 < t < a \end{cases}$

Ans $\frac{1}{s} \tanh \frac{sa}{4}$.

~~Ans $T = a$~~

Q $f(t) = \begin{cases} E, & 0 < t < a/2 \\ -E, & a/2 < t < a \end{cases}$

Ans $\frac{E}{s} \tanh \frac{as}{4}$

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Initial And Final Value Theorem

Statement of IVT

Let $L[f(t)] = F(s)$, $f'(t)$ exists,

Then

$$\boxed{\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)}$$

Statement of FVT

$$\boxed{\lim_{t \to} f(t) = \lim_{s \to 0} sF(s)}$$

Q. Find IVT ~~of~~ $f(t) = a \cdot e^{-bt}$

IVT

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

LHS $\lim_{t \to 0^+} f(t)$

Let

$f(t) = ae^{-bt}$

$$\lim_{t \to 0} f(t) = \lim_{t \to 0} \left[ae^{-bt}\right] = ae^{0} = a.$$

RHS $\lim_{s \to \infty} sF(s)$

$$F(s) = L[f(t)]$$

$$= L[ae^{-bt}]$$

$$= aL[e^{-bt}]$$

$$= a \cdot \frac{1}{s+b}$$

$$sF(s) = s \cdot \frac{a}{s+b}.$$

$$\lim_{t \to \infty} sF(s) = \lim_{s \to \infty} \left[a \left[\frac{s}{s+b} \right) \right]$$

$$= \lim_{s \to \infty} \left[a \left[\frac{s}{s(1+b/s)} \right] \right]$$

$$= \lim_{s \to \infty} \left[a \cdot \frac{1}{1+b/s} \right]$$

$$= a$$

LHS = RHS = a

Q FVT $f:$ ae^{-bt}

FVT

LHS let $f(t) = ae^{-bt}$

lim

Q $f(t) = e^{-t} \cdot \sin t$

IVT

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

LHS

Let $f(t) = e^{-t} \cdot \sin t$

$$\lim_{t \to 0} f(t) = \lim_{t \to 0} \left[e^{-t} \cdot \sin t \right]$$

$$= e^{0} \cdot \sin 0$$

$$= 1 \cdot 0 = 0$$

RHS

$F(s) = L[f(t)]$

$F(s) = L\left[e^{-t} \cdot \sin t\right]$

$$= \left[\frac{1}{s^2+1}\right]_{s \to s+1}$$

$$F(s) = \frac{1}{(s+1)^2+1}$$

$$sF(s) = \frac{s}{(s+1)^2+1}$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{s}{(s+1)^2+1}\right]$$

$$= \lim_{s\to\infty}\left[\frac{s}{s^2(1+1/s)^2+1}\right]$$

$$= \lim_{s\to\infty}\left[\frac{\cancel{s}}{s^{\cancel{2}}\left((1+1/s)^2+1/s^2\right)}\right]$$

$$= \lim_{s\to\infty}\frac{1}{s(1+1/s)^2+1/s^2}$$

$$= 0$$

LHS = RHS