CIA-03 (Maths)

P.A. $f(x) = \begin{cases} a^2 - x^2 & \text{if } |x| < a \\ 0 & \text{if } |x| > a \end{cases}$

$$\int_0^\infty \frac{\sin t - \cos t}{t^3} \cdot dt$$

Soln

$$\begin{cases} a^2 - x^2 & \text{if } |x| < a \\ 0 & \text{if } |x| > a. \end{cases}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2) \cdot (\cos sx + i \sin sx) \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2) \cos sx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} \cancel{\sin sx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2) \cos sx \cdot dx$$

$$= \frac{2}{\sqrt{2\pi}} \left[(a^2 - x^2) \cdot \left(\frac{\sin sx}{s} \right) - (-2x) \left(\frac{-\cos sx}{s^2} \right) + (-2) \left(-\frac{\sin sx}{s^3} \right) \right]_0^a$$

$$= \frac{2}{\sqrt{2\pi}} \left[\left(0 - \frac{2a \cos sa}{s^2} + \frac{2 \sin sx}{s^3} \right) \right]$$

$$= \frac{4}{\sqrt{2\pi}} \left[\frac{\sin sa - as \cos sa}{s^3} \right]$$

$$F(s) = 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin sa - sa \cos sa}{s^3} \right]$$

Apply inverse fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot e^{-isx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \left(2\sqrt{\frac{2}{\pi}}\right) \int_{-\infty}^{\infty} \left(\frac{\sin sa - sa\cos sa}{s}\right) e^{-isx} \, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin sa - sa\cos sa}{s}\right) \cos sx \cdot dx + \frac{2i}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin sa - sa\cos sx}{s^2}\right) \cdot \sin sx \cdot ds$$

$$f(x) = \frac{4}{\pi} \int_{0}^{\infty} \left(\frac{\sin sa - sa\cos sa}{s^3}\right) \cdot \cos sx \cdot ds$$

take $a = 1$ and $x = 0$

$$\left.\begin{matrix} f(x) \\ \text{at } x=0 \end{matrix}\right\} = \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3} \, ds$$

$$\left.\begin{matrix} f(x) \\ \text{at } x=0 \end{matrix}\right\} = f(0)$$

$$= a^2 - 0$$

$$= 1$$

$$\frac{\pi}{4} = \int_{0}^{\infty} \left(\frac{\sin t - t\cos t}{t^3}\right) dt$$ [s is a dummy variable]

ii) using parseval identity:

$$\int_{-\infty}^{\infty} |f(x)|^2 \cdot dx = \int_{-\infty}^{\infty} |F(s)|^2 \cdot ds$$

$$= \int_{-a}^{a} |f(x)|^2 \, dx$$

$$= \int_{-a}^{a} (a^2 - x^2) \cdot dx$$

$$= 2\int_{0}^{a} (a^2 - x^2) \cdot dx$$

$$= 2\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= 2\left[a^3 - \frac{a^3}{3}\right]$$

$$= 2\left[\frac{3a^3 - a^3}{3}\right]$$

$$= \frac{4a^3}{3}$$

$$= 2\int_0^1 (1-x^2)^2\, dx$$

$$= 2\int_0^1 \left[1 - 2x^2 + x^4\right] dx$$

$$= 2\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_0^1$$

$$= 2\left[1 - \frac{2}{3} + \frac{1}{5}\right]$$

$$= 2\left[\frac{8}{15}\right]$$

$$\boxed{LHS = \frac{16}{15}} \quad \text{—①}$$

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \left|2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)\right|^2$$

$$= \frac{8}{\pi}\left(\frac{\sin s - s\cos s}{s^3}\right)^2$$

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \frac{8}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$= \frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds \quad \text{—②}$$

① = 2

$$\frac{16}{15} = \frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$\frac{\pi}{15} = \int_0^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right) dt$$

17 b $\int_0^\infty \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$

By using parseval identity

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F_s[f(x)] + F_s[g(x)] \cdot dx$$

$$\int_0^\infty \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$$ $\rightarrow$ let us assume that $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$

$$\therefore LHS \rightarrow \int_0^\infty f(x) \cdot g(x) \cdot dx = \int_0^\infty e^{-ax} \cdot e^{-bx} \cdot dx$$

$$= \int_0^\infty e^{-(a+b)x}\, dx$$

$$= \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= \frac{1}{a+b}$$

$$\therefore \int_0^\infty f(x) \cdot g(x) \cdot dx = \frac{1}{a+b} \quad - (1)$$

RHS

$$\int_0^\infty F_s[f(x)] + F_s[g(x)] \cdot dx$$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \cdot dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{x}{x^2+a^2}\right]$$ [$\because$ s is a dummy variable

$- (2)$

$$F_s[g(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \sin sx \, dx$$

$$F_s[g(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx} \sin sx \, dx$$

$$F_s[g(x)] = \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2 + b^2} \right]$$

$$F_s[g(x)] = \sqrt{\frac{2}{\pi}} \left[\frac{x}{x^2 + b^2} \right] \quad [\because s \text{ is a dummy variable}]$$

$\leftarrow$ ③

$$\int_0^\infty F_s[g(x)] \cdot F_s[f(x)] = \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x}{(x^2+a^2)} \cdot \frac{x}{(x^2+b^2)} \, dx$$

$$\int_0^\infty F_s[g(x)] \cdot F_s[f(x)] = \frac{2}{\pi} \int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)} \, dx \quad (\text{using } 2 \text{ and } 3)$$

$\leftarrow$ ④

Equate ① and ④

$$\frac{1}{a+b} = \frac{2}{\pi} \int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}$$

$$\frac{\pi}{2(a+b)} = \int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}$$

8. a $y_{n+2} - 5y_{n+1} + 6y_n = 1 \quad y_0 = 0 ; y_1 = 1$

Take Z transform

$$[z^2 F(z) - z^2 y(0) - z y(1)] - 5[z F(z) - z y(0)] + 6 F(z) = \frac{z}{z-1}$$

$$z^2 F(z) - 0 - z - 5z F(z) + 0 + 6 \cdot F(z) = \frac{z}{z-1}$$

$$F(z)[z^2 - 5z + 6] - z = \frac{z}{(z-1)}$$

$$F(z)[(z-2)(z-3)] = \frac{z}{z-1} + z$$

$$F(z)[(z-2)(z-3)] = \cancel{\frac{z+1(z-1)}{z-1}} \quad \frac{z + z(z-1)}{(z-1)}$$

$$F(z)[(z-2)(z-3)] = \cancel{\frac{z+z-1}{z-1}} \quad \frac{z + z^2 - z}{(z-1)}$$

$$\cancel{F(z) = \frac{2z-1}{(z-1)(z-2)(z-3)}}$$

$$F(z)[(z-2)(z-3)] = \frac{z^2}{(z-1)}$$

$$F(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$$

$$\frac{F(z)}{z} = \frac{z}{(z-1)(z-2)(z-3)}$$

$$\frac{z}{(z-1)(z-2)(z-3)} = \frac{A}{(z-1)} + \frac{B}{(z-2)} + \frac{C}{(z-3)}$$

$$z = A(z-2)(z-3) + B(z-1)(z-3) + C(z-1)(z-2)$$

$z = 1$

$1 = A(1-2)(1-3)$

$1 = A(-1)(-2)$

$\boxed{A = 2}$

$z = 2$

$2 = B(2-1)(2-3)$

$2 = B(1)(-1)$

$\boxed{B = -2}$

& $Z=3$

$3 = C(3-1)(3-2)$

$3 = C(2)(1)$

$\boxed{C = 3/2}$

$$\frac{F(Z)}{Z} = \frac{2}{(Z-1)} + \frac{(-2)}{(Z-2)} + \frac{3/2}{(Z-3)}$$

$$F(Z) = \frac{2Z}{(Z-1)} + \frac{(-2)Z}{(Z-2)} + \frac{3/2\,Z}{Z-3}$$

$$Z^{-1}[F(Z)] = Z^{-1}\left[\frac{2Z}{(Z-1)}\right] + Z^{-1}\left[\frac{-2Z}{Z-2}\right] + Z^{-1}\left[\frac{3/2\,Z}{Z-3}\right]$$

$$Z^{-1}[F(Z)] = 2(1)^n + (-2)(2)^n + \frac{3}{2}(3)^n$$

$$\boxed{Z^{-1}[F(Z)] = 2(1)^n - 2(2)^n + \frac{3}{2}(3)^n}$$

18 b. i) $Z^{-1}\left[\frac{Z^3}{(Z-1)^2(Z-2)}\right]$

$$F(Z) = \frac{Z^3}{(Z-1)^2(Z-2)}$$

$$\frac{F(Z)}{Z} = \frac{Z^2}{(Z-1)^2(Z-2)}$$

$$\frac{Z^2}{(Z-1)^2(Z-2)} = \frac{A}{(Z-1)} + \frac{B}{(Z-1)^2} + \frac{C}{(Z-2)}$$

$$Z^2 = A(Z-1)(Z-2) + B(Z-2) + C(Z-1)^2$$

$\boxed{Z=1}$

$1 = B(1-2)$

$1 = B(-1)$

$\boxed{B = -1}$

$Z=2$

$(2)^2 = C(2-1)^2$

$4 = C(1)^2$

$\boxed{C=4}$

Put equate z^2

$$\frac{na^{n-1}}{n(1)^{n-1}}$$

$1 = A + C$

$1 = A + 4$

$\boxed{A = -3}$

$$\frac{F(z)}{z} = \frac{(-3)}{(z-1)} + \frac{(-1)}{(z-1)^2} + \frac{4}{(z-2)}$$

$$F(z) = z\left[\frac{-3}{z-1}\right] + \left[\frac{-z}{(z-1)^2}\right] + \frac{4z}{z-2}$$

$$z^{-1}[F(z)] = z^{-1}\left[\frac{-3z}{z-1}\right] + z^{-1}\left[\frac{-z}{(z-1)^2}\right] + z^{-1}\left[\frac{4z}{z-2}\right]$$

$$z^{-1}[F(z)] = -3(1)^n + (-1)(n1^{n-1}) + 4(2)^n$$

$$\boxed{z^{-1}[F(z)] = -3(1)^n - (n) + 4(2)^n}$$

ii) $z\left[\frac{1}{n+1}\right]$

$$z[x(n)] = \sum_{n=0}^{\infty} x(n) z^{n-1}$$

$$z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot z^{-n}$$

$$z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \frac{1}{(n+1) z^n}$$

$$= 1 + \frac{1}{2z} + \frac{1}{3z^2} + \cdots$$

$$= z\left[\frac{1}{z} + \frac{1}{2}\left[\frac{1}{z^2}\right] + \frac{1}{3}\left[\frac{1}{z^3}\right] + \cdots\right]$$

$$= z\left[-\log\left(1 - \frac{1}{z}\right)^{-1}\right]$$

$$= \cancel{z\,du}\; z\left[-\log\left(1-\frac{1}{z}\right)^{-1}\right]$$

$$= z\left[\log\left(\frac{z-1}{z}\right)^{-1}\right]$$

$$= z\log\left(\frac{z}{z-1}\right).$$

$$\therefore\; Z\left[\frac{1}{n+1}\right] = z\log\left[\frac{z}{z-1}\right]$$