VECTOR CALCULUS

EVERY CHILD MUST BE ENCOURAGED TO GET AS MUCH EDUCATION AS HE HAS THE ABILITY TO TAKE. WE WANT THIS NOT ONLY FOR HIS SAKE—BUT FOR THE NATION'S SAKE. NOTHING MATTERS MORE TO THE FUTURE OF OUR COUNTRY: NOT MILITARY PREPARATIONS—FOR ARMED MIGHT IS WORTHLESS IF WE LACK THE BRAIN POWER TO BUILD A WORLD OF PEACE; NOT OUR PRODUCTIVE ECONOMY—FOR WE CANNOT SUSTAIN GROWTH WITHOUT TRAINED MANPOWER; NOT OUR DEMOCRATIC SYSTEM OF GOVERNMENT—FOR FREEDOM IS FRAGILE IF CITIZENS ARE IGNORANT.

—LYNDON B. JOHNSON

2.1 INTRODUCTION

Chapter 1 has focused mainly on vector addition, subtraction, and multiplication in Cartesian coordinates, and Chapter 2 extended all these to other coordinate systems. This chapter deals with vector calculus—integration and differentiation of vectors.

The concepts introduced in this chapter provide a convenient language for expressing certain fundamental ideas in electromagnetics or mathematics in general. A student may feel uneasy about these concepts at first—not seeing what they are "good for." Such a student is advised to concentrate simply on learning the mathematical techniques and to wait for their applications in subsequent chapters.

2.2 DIFFERENTIAL LENGTH, AREA, AND VOLUME

Differential elements in length, area, and volume are useful in vector calculus. They are defined in the Cartesian, cylindrical, and spherical coordinate systems.

A. CARTESIAN COORDINATE SYSTEMS

From Figure 2.1, we notice that the differential displacement at $d\mathbf{l}$ is the vector from point $S(x, y, z)$ to point $B(x + dx, y + dy, z + dz)$.

1. Differential displacement is given by

$$\boxed{d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z} \tag{2.1}$$

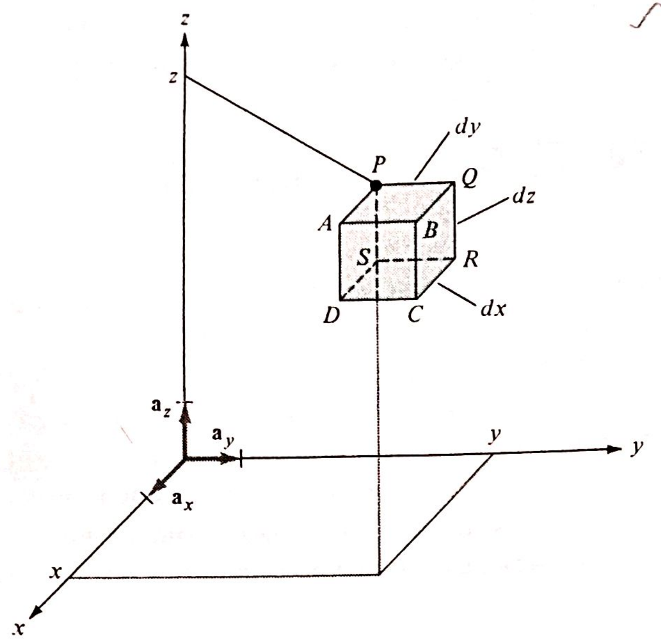

Figure 2.1 Differential elements in the right-handed Cartesian coordinate system.

2. Differential normal surface area is given by

$$\boxed{\begin{aligned} d\mathbf{S} = {} & dy\,dz\,\mathbf{a}_x \\ & dx\,dz\,\mathbf{a}_y \\ & dx\,dy\,\mathbf{a}_z \end{aligned}} \tag{2.2}$$

and illustrated in Figure 2.2.

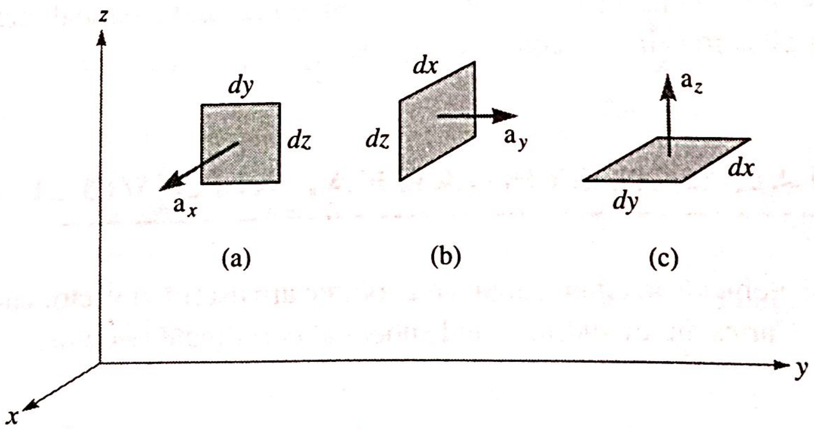

Figure 2.2 Differential normal surface areas in Cartesian coordinates: **(a)** $d\mathbf{S} = dy\,dz\,\mathbf{a}_x$, **(b)** $d\mathbf{S} = dx\,dz\,\mathbf{a}_y$, **(c)** $d\mathbf{S} = dx\,dy\,\mathbf{a}_z$.

3. Differential volume is given by

$$\boxed{dv = dx\,dy\,dz} \tag{2.3}$$

These differential elements are very important as they will be referred to throughout the book. The student is encouraged not to memorize them, but to learn how to derive them from Figures 2.1 and 2.2. Notice from eqs. (2.1) to (2.3) that $d\mathbf{l}$ and $d\mathbf{S}$ are vectors, whereas dv is a scalar. Observe from Figure 2.1 that if we move from point P to Q (or Q to P), for example, $d\mathbf{l} = dy\, \mathbf{a}_y$ because we are moving in the y-direction, and if we move from Q to S (or S to Q), $d\mathbf{l} = dy\, \mathbf{a}_y + dz\, \mathbf{a}_z$ because we have to move dy along y, dz along z, and $dx = 0$ (no movement along x). Similarly, to move from D to Q (or Q to D) would mean that $d\mathbf{l} = dx\, \mathbf{a}_x + dy\, \mathbf{a}_y + dz\, \mathbf{a}_z$.

The way $d\mathbf{S}$ is defined is important. The differential surface (or area) element $d\mathbf{S}$ may generally be defined as

$$d\mathbf{S} = dS\, \mathbf{a}_n \tag{2.4}$$

where dS is the area of the surface element and $\mathbf{a}_n$ is a unit vector normal to the surface dS (and directed away from the volume if dS is part of the surface describing a volume). If we consider surface $ABCD$ in Figure 2.1, for example, $d\mathbf{S} = dy\, dz\, \mathbf{a}_x$, whereas for surface $PQRS$, $d\mathbf{S} = -dy\, dz\, \mathbf{a}_x$ because $\mathbf{a}_n = -\mathbf{a}_x$ is normal to $PQRS$.

What we have to remember at all times about differential elements is $d\mathbf{l}$ and how to get $d\mathbf{S}$ and dv from it. When $d\mathbf{l}$ is remembered, $d\mathbf{S}$ and dv can easily be found. For example, $d\mathbf{S}$ along $\mathbf{a}_x$ can be obtained from $d\mathbf{l}$ in eq. (2.1) by multiplying the components of $d\mathbf{l}$ along $\mathbf{a}_y$ and $\mathbf{a}_z$; that is, $dy\, dz\, \mathbf{a}_x$. Similarly, $d\mathbf{S}$ along $\mathbf{a}_z$ is the product of the components of $d\mathbf{l}$ along $\mathbf{a}_x$ and $\mathbf{a}_y$; that is $dx\, dy\, \mathbf{a}_z$. Also, dv can be obtained from $d\mathbf{l}$ as the product of the three components of $d\mathbf{l}$, that is, $dx\, dy\, dz$. The idea developed here for Cartesian coordinates will now be extended to other coordinate systems.

B. CYLINDRICAL COORDINATE SYSTEMS

From Figure 2.3, the differential elements in cylindrical coordinates can be found as follows:

1. Differential displacement is given by

$$\boxed{d\mathbf{l} = d\rho\, \mathbf{a}_\rho + \rho\, d\phi\, \mathbf{a}_\phi + dz\, \mathbf{a}_z} \tag{2.5}$$

2. Differential normal surface area is given by

$$\boxed{\begin{aligned} d\mathbf{S} &= \rho\, d\phi\, dz\, \mathbf{a}_\rho \\ &\quad d\rho\, dz\, \mathbf{a}_\phi \\ &\quad \rho\, d\rho\, d\phi\, \mathbf{a}_z \end{aligned}} \tag{2.6}$$

and illustrated in Figure 2.4.

3. Differential volume is given by

$$\boxed{dv = \rho\, d\rho\, d\phi\, dz} \tag{2.7}$$

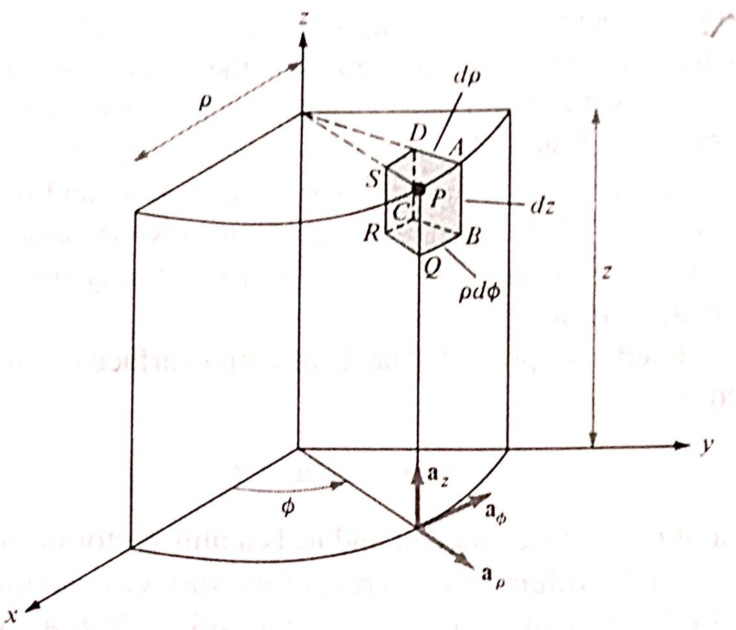

Figure 2.3 Differential elements in cylindrical coordinates.

As mentioned in the preceding section on Cartesian coordinates, we need only remember $d\mathbf{l}$; $d\mathbf{S}$ and dv can easily be obtained from $d\mathbf{l}$. For example, $d\mathbf{S}$ along $\mathbf{a}_z$ is the product of the components of $d\mathbf{l}$ along $\mathbf{a}_\rho$ and $\mathbf{a}_\phi$, that is, $d\rho\ \rho\ d\phi\ \mathbf{a}_z$. Also dv is the product of the three components of $d\mathbf{l}$, that is, $d\rho\ \rho\ d\phi\ dz$.

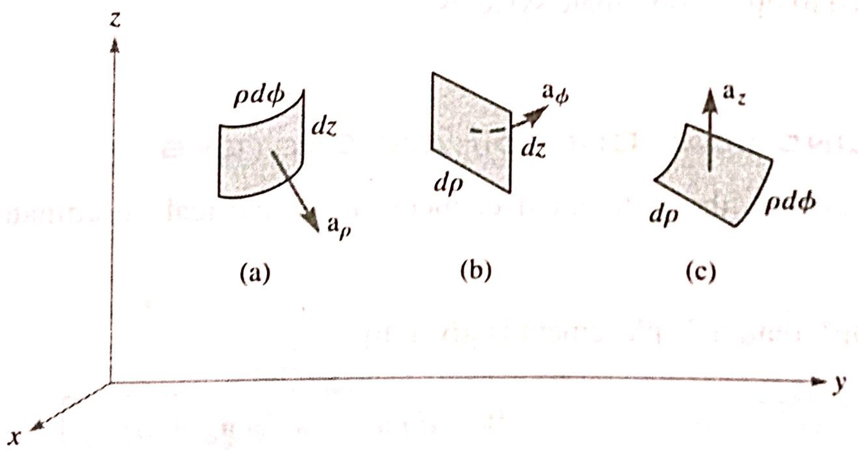

Figure 2.4 Differential normal surface areas in cylindrical coordinates: **(a)** $d\mathbf{S} = \rho\ d\phi\ dz\ \mathbf{a}_\rho$, **(b)** $d\mathbf{S} = d\rho\ dz\ \mathbf{a}_\phi$, **(c)** $d\mathbf{S} = \rho\ d\rho\ d\phi\ \mathbf{a}_z$.

C. SPHERICAL COORDINATE SYSTEMS

From Figure 2.5, the differential elements in spherical coordinates can be found as follows.

1. The differential displacement is

$$\boxed{d\mathbf{l} = dr\ \mathbf{a}_r + r\ d\theta\ \mathbf{a}_\theta + r \sin\theta\ d\phi\ \mathbf{a}_\phi} \qquad (2.8)$$

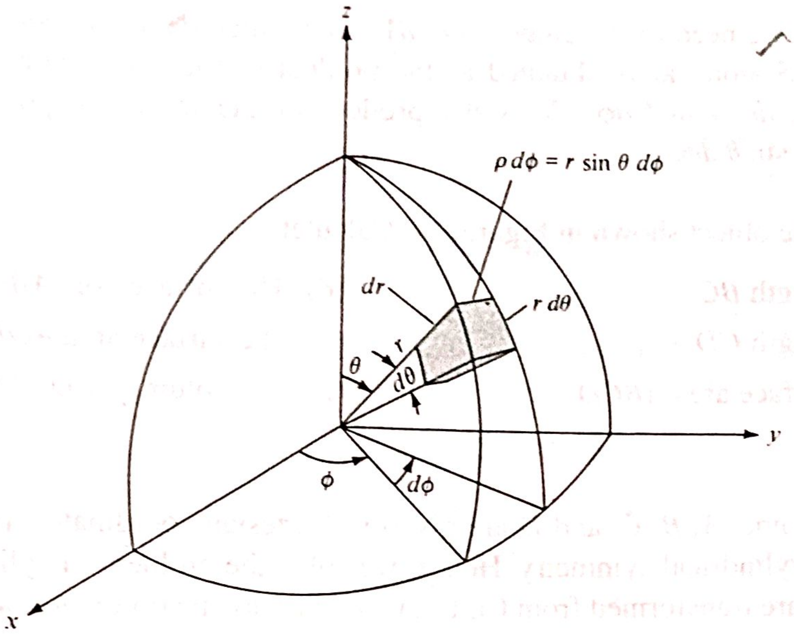

Figure 2.5 Differential elements in the spherical coordinate system.

2. The differential normal surface area is

$$\boxed{\begin{aligned} d\mathbf{S} = {} & r^2 \sin\theta\, d\theta\, d\phi\, \mathbf{a}_r \\ & r \sin\theta\, dr\, d\phi\, \mathbf{a}_\theta \\ & r\, dr\, d\theta\, \mathbf{a}_\phi \end{aligned}} \tag{2.9}$$

and illustrated in Figure 2.6.

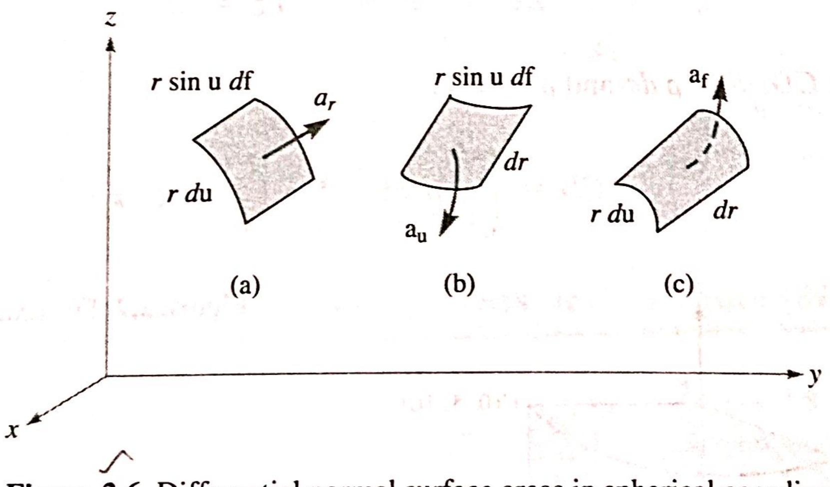

Figure 2.6 Differential normal surface areas in spherical coordinates: **(a)** $d\mathbf{S} = r^2 \sin\theta\, d\theta\, d\phi\, \mathbf{a}_r$, **(b)** $d\mathbf{S} = r \sin\theta\, dr\, d\phi\, \mathbf{a}_\phi$, **(c)** $d\mathbf{S} = r\, dr\, d\theta\, \mathbf{a}_\phi$.

3. The differential volume is

$$\boxed{dv = r^2 \sin\theta\, dr\, d\theta\, d\phi} \tag{2.10}$$

2.3 LINE, SURFACE, AND VOLUME INTEGRALS

The familiar concept of integration will now be extended to cases in which the integrand involves a vector. By "line" we mean the path along a curve in space. We shall use terms such as *line, curve,* and *contour* interchangeably.

The **line integral** $\int_L \mathbf{A} \cdot d\mathbf{l}$ is the integral of the tangential component of **A** along curve L.

Given a vector field **A** and a curve L, we define the integral

$$\int_L \mathbf{A} \cdot d\mathbf{l} = \int_a^b |\mathbf{A}| \cos \theta \, dl \tag{2.11}$$

as the *line integral* of **A** around L (see Figure 2.9). If the path of integration is a closed curve such as *abca* in Figure 2.9, eq. (2.11) becomes a closed contour integral

$$\oint_L \mathbf{A} \cdot d\mathbf{l} \tag{2.12}$$

which is called the *circulation* of **A** around L.

Given a vector field **A**, continuous in a region containing the smooth surface S, we define the *surface integral* or the *flux* of **A** through S (see Figure 2.10) as

$$\Psi = \int_S |\mathbf{A}| \cos \theta \, dS = \int_S \mathbf{A} \cdot \mathbf{a}_n \, dS$$

or simply

$$\boxed{\Psi = \int_S \mathbf{A} \cdot d\mathbf{S}} \tag{2.13}$$

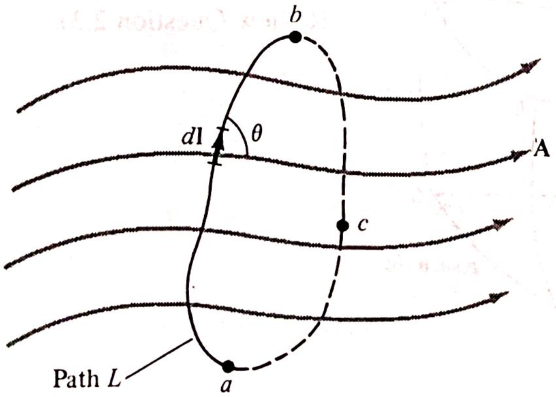

Figure 2.9 Path of integration of vector field **A**.

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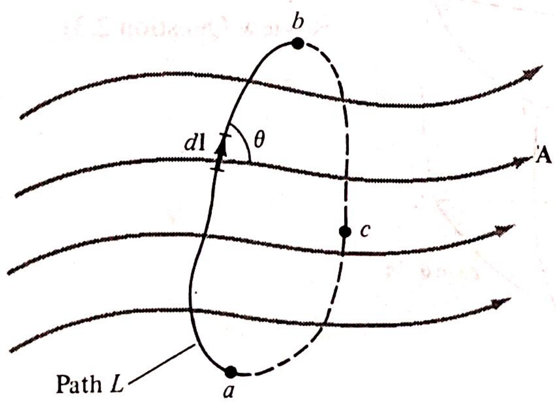

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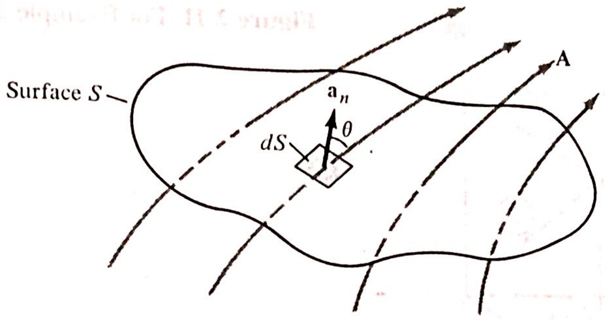


Figure 2.10 The flux of a vector field **A** through surface S.

where, at any point on S, $\mathbf{a}_n$ is the unit normal to S. For a closed surface (defining a volume), eq. (2.13) becomes

$$\Psi = \oint_S \mathbf{A} \cdot d\mathbf{S} \tag{2.14}$$

which is referred to as the *net outward flux of* **A** from S. Notice that a closed path defines an open surface whereas a closed surface defines a volume (see Figures 2.12 and 2.17).

We define the integral

$$\int_v \rho_v \, dv \tag{2.15}$$

as the *volume integral* of the scalar ρ_v over the volume v. The physical meaning of a line, surface, or volume integral depends on the nature of the physical quantity represented by **A** or ρ_v. Note that $d\mathbf{l}$, $d\mathbf{S}$, and dv are all as defined in Section 2.2.

*. **DEL OPERATOR :–** The del operator, written ∇, is the vector differential operator. In Cartesian coordinates,

$$\boxed{\nabla = \frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z} \longrightarrow (1)$$

This vector differential operator, otherwise known as the gradient operator, is not a vector in itself, but when operates on a scalar function, for example, a vector ensues.

The operator is useful in defining

(i). The gradient of a scalar 'V', written as ∇V.

(ii). The divergence of a vector 'A', written as $\nabla \cdot \vec{A}$.

(iii). The curl of a vector 'A', written as $\nabla \times \vec{A}$

(iv). The Laplacian of a scalar, written as $\nabla^2 V$

To obtain ∇ in terms of $\rho, \phi,$ and z, we know that

$$\rho = \sqrt{x^2+y^2} \quad \text{and} \quad \tan\phi = \frac{y}{x}.$$

$$\vec{a}_x = \cos\phi\,\vec{a}_\rho - \sin\phi\,\vec{a}_\phi$$

$$\vec{a}_y = \sin\phi\,\vec{a}_\rho + \cos\phi\,\vec{a}_\phi$$

From Chain rule differentiation formulas we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial x} \longrightarrow (2)$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial y} \longrightarrow (3)$$

$$\frac{\partial \rho}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\left(x^2+y^2\right)^{1/2}\right) = \frac{1}{2}\cdot\left(x^2+y^2\right)^{\frac{1}{2}-1}\cdot 2x$$

$$= \left(x^2+y^2\right)^{-1/2}\cdot x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{\rho} = \frac{\rho\cos\phi}{\rho} = \cos\phi.$$

$$\boxed{\therefore \quad \frac{\partial \rho}{\partial x} = \cos\phi} \longrightarrow (4)$$

$$\frac{\partial \rho}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{x^2+y^2}\right) = \frac{\partial}{\partial y}\left(x^2+y^2\right)^{1/2} = \frac{1}{2}\cdot\left(x^2+y^2\right)^{\frac{1}{2}-1}\cdot 2y$$

$$= \frac{y}{\sqrt{x^2+y^2}} = \frac{\rho\sin\phi}{\rho} = \sin\phi$$

$$\boxed{\therefore \quad \frac{\partial \rho}{\partial y} = \sin\phi} \longrightarrow (5)$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}(y/x)\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot y\cdot\frac{\partial}{\partial x}\left(\frac{1}{x}\right)$$

$$= \frac{y\cdot x^2}{x^2+y^2}\cdot(-1)\,x^{-2} = \cancel{\frac{-y}{\sqrt{x}}} \; \frac{-y}{x^2+y^2} = \frac{-\rho\sin\phi}{\rho^2}$$

$$\boxed{\therefore \quad \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{\rho}} \longrightarrow (6)$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left(\tan^{-1}(y/x)\right) = \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} = \frac{x^2}{x^2+y^2}\cdot\frac{1}{x}$$

$$= \frac{x}{x^2+y^2} = \frac{\rho\cos\phi}{\rho^2} = \frac{\cos\phi}{\rho}$$

$$\boxed{\therefore \quad \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{\rho}} \longrightarrow (7)$$

~~Substituting (4)-(7) in (2) + (3) we get~~ we know

$$\Rightarrow \nabla = \frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z$$

substituting (2) + (3) in above equation yields

$$\Rightarrow \nabla = \left(\frac{\partial}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial x}\right)\left(\cos\phi\,\vec{a}_\rho - \sin\phi\,\vec{a}_\phi\right) +$$

$$\left(\frac{\partial}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial y}\right)\left(\sin\phi\,\vec{a}_\rho + \cos\phi\,\vec{a}_\phi\right) + \frac{\partial}{\partial z}\vec{a}_z.$$

$$\Rightarrow \nabla = \left(\frac{\partial}{\partial \rho}\cos\phi + \frac{\partial}{\partial \phi}\frac{-\sin\phi}{\rho}\right)\left(\cos\phi\,\vec{a}_\rho - \sin\phi\,\vec{a}_\phi\right) +$$

$$\left(\frac{\partial}{\partial \rho}\sin\phi + \frac{\partial}{\partial \phi}\frac{\cos\phi}{\rho}\right)\left(\sin\phi\,\vec{a}_\rho + \cos\phi\,\vec{a}_\phi\right) + \frac{\partial}{\partial z}\vec{a}_z.$$

$$\Rightarrow \nabla = \left(\sin^2\phi + \cos^2\phi\right)\frac{\partial}{\partial \rho}\vec{a}_\rho + \frac{1}{\rho}\left(\sin^2\phi + \cos^2\phi\right)\frac{\partial}{\partial \phi}\vec{a}_\phi + \frac{\partial}{\partial z}\vec{a}_z$$

$$\boxed{\therefore \nabla = \frac{\partial}{\partial \rho}\vec{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\vec{a}_\phi + \frac{\partial}{\partial z}\vec{a}_z} \longrightarrow (8)$$

CYLINDRICAL COORDINATE

similarly, to obtain ∇ interms of r, θ, and ϕ, we use

$$r = \sqrt{x^2+y^2+z^2}, \quad \tan\theta = \frac{\sqrt{x^2+y^2}}{z}, \quad \tan\phi = \frac{y}{x}.$$

$$\vec{a}_x = \sin\theta\cos\phi\,\vec{a}_r + \cos\theta\cos\phi\,\vec{a}_\theta - \sin\phi\,\vec{a}_\phi$$

$$\vec{a}_y = \sin\theta\sin\phi\,\vec{a}_r + \cos\theta\sin\phi\,\vec{a}_\theta + \cos\phi\,\vec{a}_\phi$$

$$\vec{a}_z = \cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta.$$

From chain rule differentiation formulas, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial x} \longrightarrow (9)$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial y}. \longrightarrow (10)$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial z}. \longrightarrow (11)$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2} = \frac{1}{2}(x^2+y^2+z^2)^{\frac{1}{2}-1}\cdot 2x = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x}{r} = \frac{r\sin\theta\cos\phi}{r} = \sin\theta\cos\phi.$$

$$\boxed{\therefore \frac{\partial r}{\partial x} = \sin\theta\cos\phi} \longrightarrow (12)$$

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2+z^2)^{1/2} = \frac{1}{2}(x^2+y^2+z^2)^{\frac{1}{2}-1}2y = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{y}{r} = \frac{r\sin\theta\sin\phi}{r} = \sin\theta\sin\phi.$$

$$\boxed{\therefore \frac{\partial r}{\partial y} = \sin\theta\sin\phi} \longrightarrow (13)$$

$$\frac{\partial r}{\partial z} = \frac{z}{r} = \frac{r\cos\theta}{r} = \cos\theta.$$

$$\boxed{\therefore \frac{\partial r}{\partial z} = \cos\theta} \longrightarrow (14).$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)\right) = \frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\cdot\frac{\partial}{\partial x}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$= \frac{z^2}{x^2+y^2+z^2}\cdot\frac{1}{z}\cdot\frac{1}{2}\cdot(x^2+y^2)^{\frac{1}{2}-1}\cdot 2x = \frac{z}{x^2+y^2+z^2}\times\frac{x}{\sqrt{x^2+y^2}}$$

$$\Rightarrow \frac{\partial\theta}{\partial x} = \frac{r\cos\theta}{r^2} \times \frac{r\sin\theta\cos\phi}{r\sin\theta} = \frac{1}{r}\cos\theta\cos\phi$$ $(\because \rho = \sqrt{x^2+y^2}$ & $\rho = r\sin\theta)$

$$\therefore \boxed{\frac{\partial\theta}{\partial x} = \frac{1}{r}\cos\theta\cos\phi} \longrightarrow (15)$$

$$\frac{\partial\theta}{\partial y} = \frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)\right) = \frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2} \times \frac{1}{z} \times \frac{1}{2} \times \frac{1}{\sqrt{x^2+y^2}} \times 2y$$

$$= \frac{z}{x^2+y^2+z^2} \times \frac{y}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r^2} \times \frac{r\sin\theta\sin\phi}{r\sin\theta} = \frac{1}{r}\cos\theta\sin\phi$$

$$\therefore \boxed{\frac{\partial\theta}{\partial y} = \frac{1}{r}\cos\theta\sin\phi} \longrightarrow (16)$$

$$\frac{\partial\theta}{\partial z} = \frac{\partial}{\partial z}\left(\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)\right) = \frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2} \cdot \frac{\partial}{\partial z}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$= \frac{z^2}{x^2+y^2+z^2} \times \sqrt{x^2+y^2} \times (-1) \times z^{-2} = \frac{-\rho}{r^2} = \frac{-r\sin\theta}{r^2}$$

$$\therefore \boxed{\frac{\partial\theta}{\partial z} = -\frac{1}{r}\sin\theta} \longrightarrow (17)$$

From (6) & (7), we know

$$\frac{\partial\phi}{\partial x} = \frac{-\sin\phi}{\rho} = \frac{-\sin\phi}{r\sin\theta} \longrightarrow (18)$$

$$\& \ \frac{\partial\phi}{\partial y} = \frac{\cos\phi}{\rho} = \frac{\cos\phi}{r\sin\theta} \longrightarrow (19)$$

we know $\nabla = \frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z$

substituting (9) - (11) and unit vectors in the above equation

$$\Rightarrow \nabla = \left(\frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial x} + \frac{\partial}{\partial\phi}\frac{\partial\phi}{\partial x}\right)\left(\sin\theta\cos\phi\,\vec{a}_r + \cos\theta\cos\phi\,\vec{a}_\theta - \sin\phi\,\vec{a}_\phi\right)$$

$$+ \left(\frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial y} + \frac{\partial}{\partial\phi}\frac{\partial\phi}{\partial y}\right)\left(\sin\theta\sin\phi\,\vec{a}_r + \cos\theta\sin\phi\,\vec{a}_\theta + \cos\phi\,\vec{a}_\phi\right)$$

$$+\left(\frac{\partial}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial z}\right)\left(\cos\theta\,\vec{a}_r-\sin\theta\,\vec{a}_\theta\right).$$

$$\Rightarrow \nabla=\left(\frac{\partial}{\partial r}\sin\theta\cos\phi+\frac{\partial}{\partial\theta}\frac{1}{r}\cos\theta\cos\phi+\frac{\partial}{\partial\phi}\frac{-\sin\phi}{r\sin\theta}\right)\left(\sin\theta\cos\phi\,\vec{a}_r\right.$$

$$\left.+\cos\theta\cos\phi\,\vec{a}_\theta-\sin\phi\,\vec{a}_\phi\right)+\left(\frac{\partial}{\partial r}\sin\theta\sin\phi+\frac{\partial}{\partial\theta}\frac{1}{r}\cos\theta\sin\phi+\frac{\partial}{\partial\phi}\frac{\cos\phi}{r\sin\theta}\right)$$

$$\left(\sin\theta\sin\phi\,\vec{a}_r+\cos\theta\sin\phi\,\vec{a}_\theta+\cos\phi\,\vec{a}_\phi\right)+\left(\frac{\partial}{\partial r}\cos\theta+\frac{\partial}{\partial\theta}\frac{-\sin\theta}{r}\right)$$

$$\left(\cos\theta\,\vec{a}_r-\sin\theta\,\vec{a}_\theta\right)$$

$$\Rightarrow \nabla=\sin^2\theta\cos^2\phi\frac{\partial}{\partial r}\vec{a}_r+\sin\theta\cos\theta\cos^2\phi\frac{\partial}{\partial r}\vec{a}_\theta-\sin\theta\sin\phi\cos\phi\frac{\partial}{\partial r}\vec{a}_\phi$$

$$+\frac{1}{r}\sin\theta\cos\theta\cos^2\phi\frac{\partial}{\partial\theta}\vec{a}_r+\frac{1}{r}\cos^2\theta\cos^2\phi\frac{\partial}{\partial\theta}\vec{a}_\theta-\frac{1}{r}\cos\theta\sin\phi\cos\phi\frac{\partial}{\partial\theta}\vec{a}_\phi$$

$$-\frac{1}{r}\sin\phi\cos\phi\frac{\partial}{\partial\phi}\vec{a}_r-\frac{1}{r}\frac{\cos\theta\sin\phi\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\vec{a}_\theta+\frac{1}{r}\frac{\sin^2\phi}{\sin\theta}\frac{\partial}{\partial\phi}\vec{a}_\phi$$

$$+\sin^2\theta\sin^2\phi\frac{\partial}{\partial r}\vec{a}_r+\sin\theta\cos\theta\sin^2\phi\frac{\partial}{\partial r}\vec{a}_\theta+\sin\theta\sin\phi\cos\phi\frac{\partial}{\partial r}\vec{a}_\phi$$

$$+\frac{1}{r}\sin\theta\cos\theta\sin^2\phi\frac{\partial}{\partial\theta}\vec{a}_r+\frac{1}{r}\cos^2\theta\sin^2\phi\frac{\partial}{\partial\theta}\vec{a}_\theta+\frac{1}{r}\cos\theta\sin\phi\cos\phi\frac{\partial}{\partial\theta}\vec{a}_\phi$$

$$+\frac{1}{r}\sin\phi\cos\phi\frac{\partial}{\partial\phi}\vec{a}_r+\frac{1}{r}\frac{\cos\theta\sin\phi\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\vec{a}_\theta+\frac{1}{r}\frac{\cos^2\phi}{\sin\theta}\frac{\partial}{\partial\phi}\vec{a}_\phi$$

$$+\cos^2\theta\frac{\partial}{\partial r}\vec{a}_r-\sin\theta\cos\theta\frac{\partial}{\partial r}\vec{a}_\theta-\frac{1}{r}\sin\theta\cos\theta\frac{\partial}{\partial\theta}\vec{a}_r$$

$$+\frac{1}{r}\sin^2\theta\frac{\partial}{\partial\theta}\vec{a}_\theta$$

$$\Rightarrow \nabla=\sin^2\theta\cos^2\phi\frac{\partial}{\partial r}\vec{a}_r+\sin^2\theta\sin^2\phi\frac{\partial}{\partial r}\vec{a}_r+\cos^2\theta\frac{\partial}{\partial r}\vec{a}_r$$

$$+\frac{1}{r}\cos^2\theta\cos^2\phi\frac{\partial}{\partial\theta}\vec{a}_\theta+\frac{1}{r}\cos^2\theta\sin^2\phi\frac{\partial}{\partial\theta}\vec{a}_\theta+\frac{1}{r}\sin^2\theta\frac{\partial}{\partial\theta}\vec{a}_\theta$$

$$+\frac{1}{r}\frac{\sin^2\phi}{\sin\theta}\frac{\partial}{\partial\phi}\vec{a}_\phi+\frac{1}{r}\frac{\cos^2\phi}{\sin\theta}\frac{\partial}{\partial\phi}\vec{a}_\phi$$

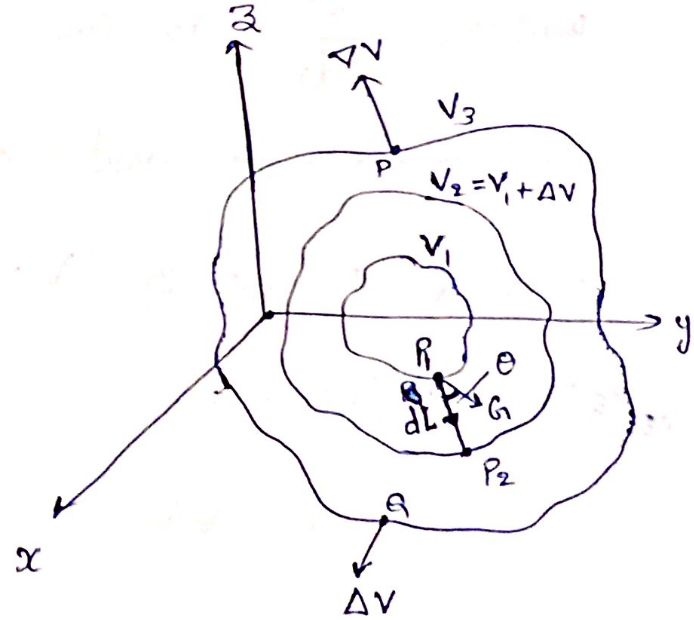
$$\Rightarrow \quad \nabla = \frac{\partial}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\vec{a}_\phi$$

$$\therefore \boxed{\nabla = \frac{\partial}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\vec{a}_\phi} \rightarrow (20) \quad \text{SPHERICAL COORDINATE}$$

* GRADIENT OF A SCALAR:-

The gradient of a scalar field 'V' is a vector that represents both the magnitude and the direction of the maximum space rate of increase of 'V'.

A mathematical expression for the gradient can be obtained by evaluating the difference in field 'dV' between points P_1 and P_2, where V_1, V_2, and V_3 are contours on which 'V' is constant.



From calculus,

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$\Rightarrow \quad dV = \left(\frac{\partial V}{\partial x}\vec{a}_x + \frac{\partial V}{\partial y}\vec{a}_y + \frac{\partial V}{\partial z}\vec{a}_z\right)\cdot\left(dx\,\vec{a}_x + dy\,\vec{a}_y + dz\,\vec{a}_z\right)$$

For convenience, let $\vec{G} = \frac{\partial V}{\partial x}\vec{a}_x + \frac{\partial V}{\partial y}\vec{a}_y + \frac{\partial V}{\partial z}\vec{a}_z$

Then $\quad dV = \vec{G}\cdot\vec{dL} = G\cos\theta\, dL$

or $\frac{dV}{dL} = G \cos\theta \longrightarrow (i)$

where 'dL' is the differential displacement from P_1 to P_2 and 'θ' is the angle between G and dL.

From eq. (i), we notice that $\frac{dV}{dL}$ is a maximum when $\theta = 0$, when dL is in the direction of G. Hence

$$\left.\frac{dV}{dL}\right|_{max} = \frac{dV}{dn} = G$$

where $\frac{dV}{dn}$ is the normal derivative. Thus 'G' has its magnitude and direction as those of the maximum rate of change of 'V'. By definition, G is the gradient of 'V'. Therefore

$$\text{grad } V = \nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z \quad \text{(CARTESIAN)}$$

$$\nabla V = \frac{\partial V}{\partial \rho} a_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi} a_\phi + \frac{\partial V}{\partial z} a_z \quad \text{(CYLINDRICAL)}$$

$$\nabla V = \frac{\partial V}{\partial r} a_r + \frac{1}{r}\frac{\partial V}{\partial \theta} a_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} a_\phi \quad \text{(SPHERICAL)}$$

The following computation formulas on gradient, which are easily proved, should be noted:

(i) $\nabla(U+V) = \nabla U + \nabla V$

(ii). $\nabla(VU) = V\nabla U + U\nabla V$

(iii) $\nabla \left[\frac{V}{U}\right] = \frac{U \nabla V - V \nabla U}{U^2}$

(iv). $\nabla V^n = n V^{n-1} \nabla V$

where U & V are scalars and 'n' is an integer.

Fundamental properties of the gradient of a scalar field V:-

(a). The magnitude of ∇V equals the maximum rate of change in 'V' per unit distance.

(b). ∇V points in the direction of the maximum rate of change in 'V'.

(c). ∇V at any point is perpendicular to the constant 'V' surface that passes through that point. (point P & Q).

(d). The projection (or component) of ∇V in the direction of a unit vector '$\vec{a}$' is $\nabla V \cdot \vec{a}$ and is called the directional derivative of 'V' along '$\vec{a}$'.

(e). If $\vec{A} = \nabla V$, 'V' is said to be the scalar potential of $\vec{A}$.

*. DIVERGENCE OF A VECTOR :-

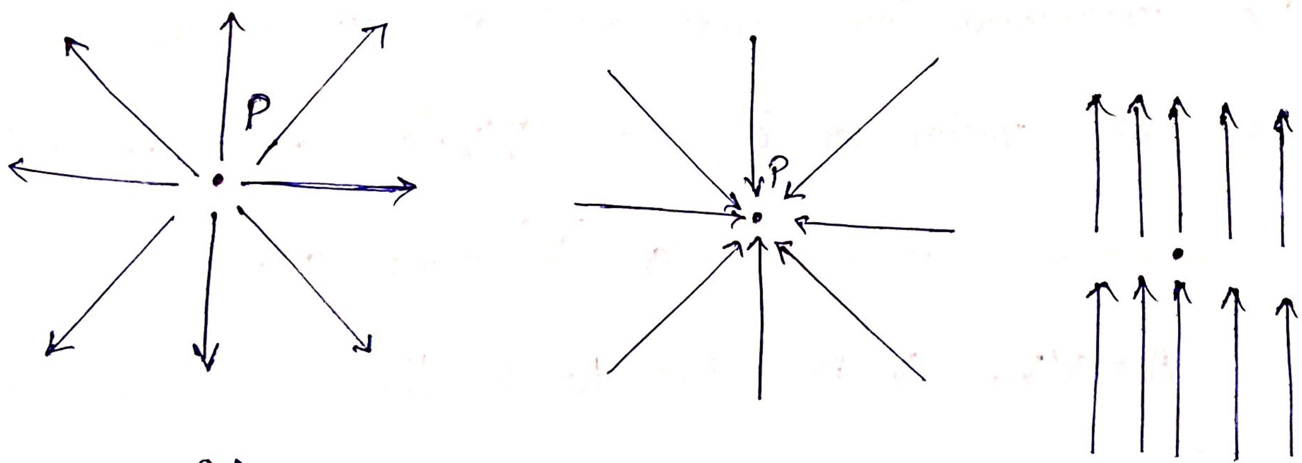
The divergence of 'A' at a given point 'P' is the outward flux per unit volume as the volume shrinks about 'P.'

Hence,

$$\text{div}\,\vec{A} = \nabla\cdot\vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A}\cdot d\vec{s}}{\Delta v}$$

where Δv is the volume enclosed by the closed surface 'S' in which 'P' is located.

* Physically, we may regard the divergence of the vector field 'A' at a given point as a measure of how much the field diverges or emanates from that point.



(a) (b) (c)

Fig: Illustration of the divergence of a vector field at P. (a) Positive divergence (b) negative divergence (c) zero divergence.

The divergence of 'A' in a Cartesian system is given by

$$\boxed{\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}} \longrightarrow (1)$$

In cylindrical system

$$\boxed{\nabla\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}} \longrightarrow (2)$$

In Spherical coordinates

$$\boxed{\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}} \longrightarrow (3)$$

Properties of the divergence of a vector field :-

(i). It produces a scalar field (because scalar product is involved).

(ii). $\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$

(iii). $\nabla\cdot(V\vec{A}) = V\,\nabla\cdot\vec{A} + \vec{A}\cdot\nabla V$

*. DIVERGENCE THEOREM (or) GAUSS - OSTROGRADSKY THEOREM.

The divergence theorem states that the total outward flux of a vector field $\vec{A}$' through the closed surface 'S' is the same as the volume integral of the divergence of $\vec{A}$'.

To prove the divergence theorem, subdivide volume 'v' into a large number of small cells. If the k^{th} cell has volume Δv_k and is bounded by surface S_k

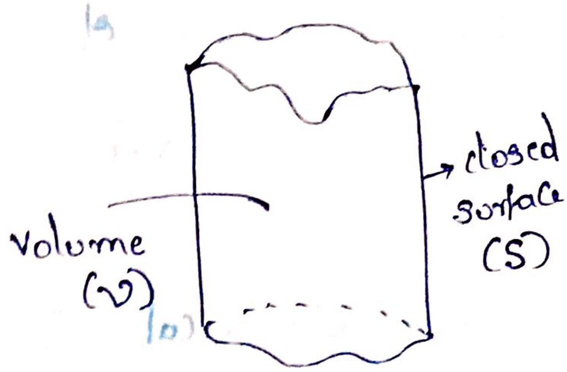


Fig:- Volume 'v' enclosed by surface 'S'.

$$\boxed{\oint_S \vec{A}\cdot\vec{ds} = \sum_k \oint_{S_k} \vec{A}\cdot\vec{ds} = \sum_k \frac{\oint_{S_k} \vec{A}\cdot\vec{ds}}{\Delta v_k}\,\Delta v_k} \longrightarrow (1)$$

Since the outward flux to one cell is inward to some neighboring cells, there is cancellation on every interior surface, so the sum of the integrals over the 'S_k's is the same as the surface integral over the surface 'S'.

Taking the limit of the right hand side of eq.(1) gives

$$\oint_S \vec{A}\cdot\vec{ds} = \int_v \nabla\cdot\vec{A}\, dv$$

$$\therefore \boxed{\oint_S \vec{A}\cdot\vec{ds} = \int_v \nabla\cdot\vec{A}\, dv} \longrightarrow (2)$$

which is the divergence theorem.

NOTE:- The theorem applies to any volume 'v' bounded by the closed surface 'S' ~~such that~~ provided that A and $\nabla\cdot A$ are continuous in the region.

*. CURL OF A VECTOR :-

The curl of $\vec{A}$ is an axial (or rotational) vector whose magnitude is the maximum circulation of $\vec{A}$ per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the circulation maximum.

$$\text{curl}\,\vec{A} = \nabla \times \vec{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \vec{A} \cdot \vec{dL}}{\Delta S} \right)_{max} \vec{a}_n$$

where the area ΔS is bounded by the curve 'L' and $\vec{a}_n$ is the unit vector normal to the surface ΔS and is determined by using the right-hand rule.

In Cartesian coordinates :-

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

or $$\nabla \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \vec{a}_x - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \vec{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \vec{a}_z .$$

In Cylindrical Coordinates :-

$$\nabla \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \vec{a}_\rho & \rho \vec{a}_\phi & \vec{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

or $$\nabla \times \vec{A} = \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \vec{a}_\rho - \left(\frac{\partial A_z}{\partial \rho} - \frac{\partial A_\rho}{\partial z} \right) \vec{a}_\phi$$

$$+ \frac{1}{\rho} \left(\frac{\partial (\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi} \right) \vec{a}_z$$

In Spherical Coordinates :-

$$\nabla \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \vec{a}_r & r \vec{a}_\theta & r\sin\theta \, \vec{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r\sin\theta \, A_\phi \end{vmatrix}$$

or $$\nabla \times A = \frac{1}{r\sin\theta} \left(\frac{\partial (A_\phi \sin\theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi} \right) \vec{a}_r$$

$$- \frac{1}{r} \left(\frac{\partial (r A_\phi)}{\partial r} - \frac{1}{\sin\theta} \frac{\partial A_r}{\partial \phi} \right) \vec{a}_\theta$$

$$+ \frac{1}{r} \left(\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right) \vec{a}_\phi$$

Properties of the curl:-

(i). The curl of a vector field is another vector field.

(ii). $\nabla \times (\vec{A}+\vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$

(iii). $\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) + (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B}$

(iv). $\nabla \times (V\vec{A}) = V\nabla\times\vec{A} + \nabla V \times \vec{A}$

(v). The divergence of the curl of a vector field vanishes;

that is $\nabla\cdot(\nabla\times\vec{A}) = 0$

(vi). The curl of the gradient of a scalar field vanishes;

that is $\nabla \times \nabla V = 0$.

The physical significance of the curl of a vector field is, the curl provides the maximum value of the circulation of the field per unit area and indicates the direction along which this maximum value occurs.

The curl of a vector field 'A' at a point 'P' may be regarded as a measure of the circulation or how much the field curls around 'P'.

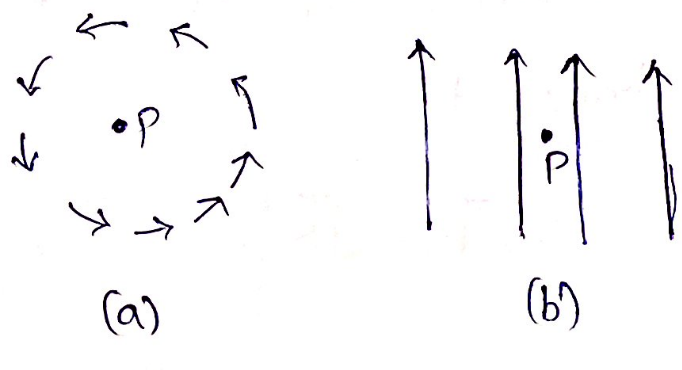


Fig:- Illustration of a curl: (a) curl at 'P' points out of the page (b). curl at 'P' is zero.

* STOKES'S THEOREM :-

Stokes's theorem states that the circulation of a vector field $\vec{A}$ around a (closed) path 'L' is equal to the surface integral of the curl of $\vec{A}$ over the open surface 'S' bounded by 'L', provided $\vec{A}$ and $\nabla \times \vec{A}$ are continuous on 'S'.

The surface S is subdivided into a large number of cells. If the k^{th} cell has surface area ΔS_K and is bounded by path L_K.

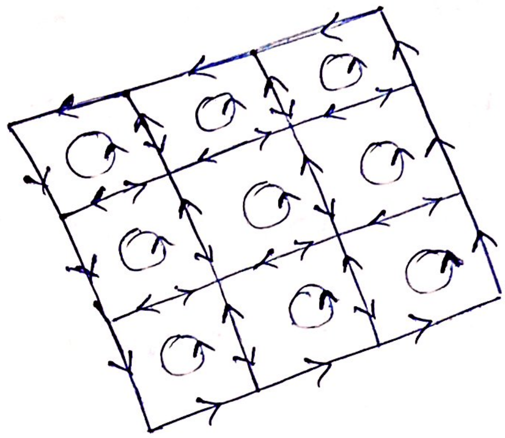


Fig :- Illustration of Stokes's theorem.

$$\oint_L \vec{A}\cdot d\vec{L} = \sum_K \oint_{L_K} \vec{A}\cdot d\vec{L} = \sum_K \frac{\oint_{L_K} \vec{A}\cdot d\vec{L}}{\Delta S_K} \Delta S_K \longrightarrow (1)$$

As shown in Fig, there is cancellation on every interior path, so the sum of the line integrals around the L_K's is the same as the line integral around the bounding curve L. Therefore, taking the limit of the right-hand side of eq. (1) leads to

$$\boxed{\oint_L \vec{A}\cdot d\vec{L} = \int_S (\nabla \times \vec{A})\cdot d\vec{S}}$$

which is Stokes's theorem.

*. LAPLACIAN OF A SCALAR:-

The Laplacian of a scalar field 'V', written as $\nabla^2 V$, is the divergence of the gradient of 'V'.

Thus in Cartesian coordinates,

$$\text{Laplacian } V = \nabla \cdot \nabla V = \nabla^2 V$$

$$= \left[\frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z\right] \cdot \left[\frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z\right]$$

$$\boxed{\therefore \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}$$

Note:- The Laplacian of a scalar field is another scalar field.

In Cylindrical Coordinates

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}}$$

In Spherical Coordinates

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}}$$

A Scalar field 'V' is said to be harmonic in a given region if its Laplacian vanishes in that region. In otherwords

$$\nabla^2 V = 0.$$

Laplacian of a Vector :-

$\times \quad \nabla^2 A = \nabla \cdot \nabla \vec{A} \longrightarrow$ No sense

$\downarrow$

gradient is not applied on a vector.

The Laplacian of a vector can be written as

$$\checkmark \quad \boxed{\nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla \times \nabla \times \vec{A}}$$

VECTOR CALCULUS

EVERY CHILD MUST BE ENCOURAGED TO GET AS MUCH EDUCATION AS HE HAS THE ABILITY TO TAKE. WE WANT THIS NOT ONLY FOR HIS SAKE—BUT FOR THE NATION'S SAKE. NOTHING MATTERS MORE TO THE FUTURE OF OUR COUNTRY: NOT MILITARY PREPARATIONS—FOR ARMED MIGHT IS WORTHLESS IF WE LACK THE BRAIN POWER TO BUILD A WORLD OF PEACE; NOT OUR PRODUCTIVE ECONOMY—FOR WE CANNOT SUSTAIN GROWTH WITHOUT TRAINED MANPOWER; NOT OUR DEMOCRATIC SYSTEM OF GOVERNMENT—FOR FREEDOM IS FRAGILE IF CITIZENS ARE IGNORANT.

—LYNDON B. JOHNSON

2.1 INTRODUCTION

Chapter 1 has focused mainly on vector addition, subtraction, and multiplication in Cartesian coordinates, and Chapter 2 extended all these to other coordinate systems. This chapter deals with vector calculus—integration and differentiation of vectors.

The concepts introduced in this chapter provide a convenient language for expressing certain fundamental ideas in electromagnetics or mathematics in general. A student may feel uneasy about these concepts at first—not seeing what they are "good for." Such a student is advised to concentrate simply on learning the mathematical techniques and to wait for their applications in subsequent chapters.

2.2 DIFFERENTIAL LENGTH, AREA, AND VOLUME

Differential elements in length, area, and volume are useful in vector calculus. They are defined in the Cartesian, cylindrical, and spherical coordinate systems.

A. CARTESIAN COORDINATE SYSTEMS

From Figure 2.1, we notice that the differential displacement at $d\mathbf{l}$ is the vector from point $S(x, y, z)$ to point $B(x + dx, y + dy, z + dz)$.

1. Differential displacement is given by

$$\boxed{d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z} \tag{2.1}$$

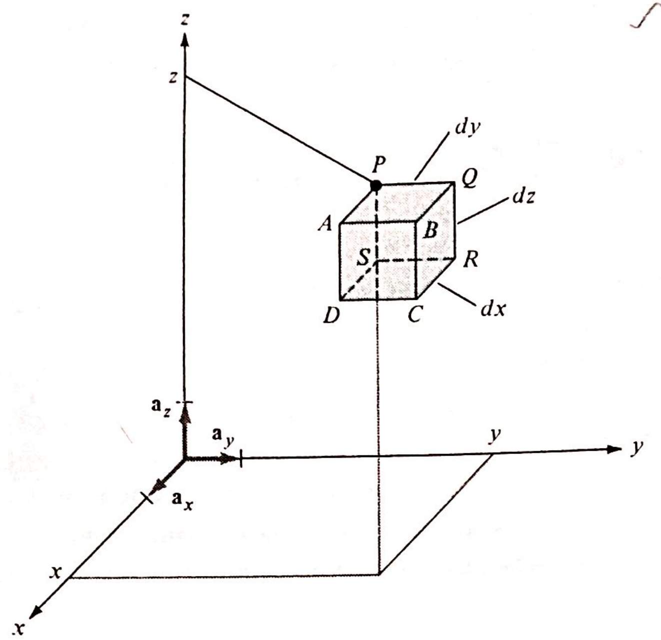


Figure 2.1 Differential elements in the right-handed Cartesian coordinate system.

2. Differential normal surface area is given by

$$\boxed{\begin{aligned} d\mathbf{S} = dy\,dz\,\mathbf{a}_x \\ dx\,dz\,\mathbf{a}_y \\ dx\,dy\,\mathbf{a}_z \end{aligned}} \tag{2.2}$$

and illustrated in Figure 2.2.

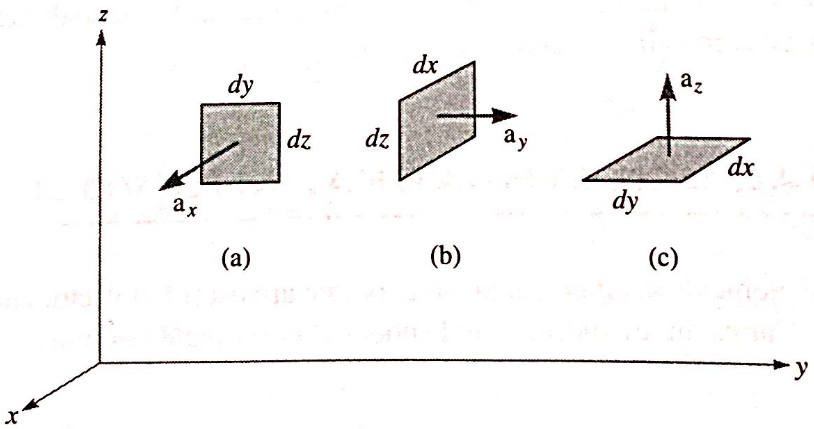


Figure 2.2 Differential normal surface areas in Cartesian coordinates: (a) $d\mathbf{S} = dy\,dz\,\mathbf{a}_x$, (b) $d\mathbf{S} = dx\,dz\,\mathbf{a}_y$, (c) $d\mathbf{S} = dx\,dy\,\mathbf{a}_z$.

3. Differential volume is given by

$$\boxed{dv = dx\,dy\,dz} \tag{2.3}$$

These differential elements are very important as they will be referred to throughout the book. The student is encouraged not to memorize them, but to learn how to derive them from Figures 2.1 and 2.2. Notice from eqs. (2.1) to (2.3) that $d\mathbf{l}$ and $d\mathbf{S}$ are vectors, whereas dv is a scalar. Observe from Figure 2.1 that if we move from point P to Q (or Q to P), for example, $d\mathbf{l} = dy\, \mathbf{a}_y$ because we are moving in the y-direction, and if we move from Q to S (or S to Q), $d\mathbf{l} = dy\, \mathbf{a}_y + dz\, \mathbf{a}_z$ because we have to move dy along y, dz along z, and $dx = 0$ (no movement along x). Similarly, to move from D to Q (or Q to D) would mean that $d\mathbf{l} = dx\, \mathbf{a}_x + dy\, \mathbf{a}_y + dz\, \mathbf{a}_z$.

The way $d\mathbf{S}$ is defined is important. The differential surface (or area) element $d\mathbf{S}$ may generally be defined as

$$d\mathbf{S} = dS\, \mathbf{a}_n \tag{2.4}$$

where dS is the area of the surface element and $\mathbf{a}_n$ is a unit vector normal to the surface dS (and directed away from the volume if dS is part of the surface describing a volume). If we consider surface $ABCD$ in Figure 2.1, for example, $d\mathbf{S} = dy\, dz\, \mathbf{a}_x$, whereas for surface $PQRS$, $d\mathbf{S} = -dy\, dz\, \mathbf{a}_x$ because $\mathbf{a}_n = -\mathbf{a}_x$ is normal to $PQRS$.

What we have to remember at all times about differential elements is $d\mathbf{l}$ and how to get $d\mathbf{S}$ and dv from it. When $d\mathbf{l}$ is remembered, $d\mathbf{S}$ and dv can easily be found. For example, $d\mathbf{S}$ along $\mathbf{a}_x$ can be obtained from $d\mathbf{l}$ in eq. (2.1) by multiplying the components of $d\mathbf{l}$ along $\mathbf{a}_y$ and $\mathbf{a}_z$; that is, $dy\, dz\, \mathbf{a}_x$. Similarly, $d\mathbf{S}$ along $\mathbf{a}_z$ is the product of the components of $d\mathbf{l}$ along $\mathbf{a}_x$ and $\mathbf{a}_y$; that is $dx\, dy\, \mathbf{a}_z$. Also, dv can be obtained from $d\mathbf{l}$ as the product of the three components of $d\mathbf{l}$, that is, $dx\, dy\, dz$. The idea developed here for Cartesian coordinates will now be extended to other coordinate systems.

B. CYLINDRICAL COORDINATE SYSTEMS

From Figure 2.3, the differential elements in cylindrical coordinates can be found as follows:

1. Differential displacement is given by

$$\boxed{d\mathbf{l} = d\rho\, \mathbf{a}_\rho + \rho\, d\phi\, \mathbf{a}_\phi + dz\, \mathbf{a}_z} \tag{2.5}$$

2. Differential normal surface area is given by

$$\boxed{\begin{aligned} d\mathbf{S} &= \rho\, d\phi\, dz\, \mathbf{a}_\rho \\ &\quad d\rho\, dz\, \mathbf{a}_\phi \\ &\quad \rho\, d\rho\, d\phi\, \mathbf{a}_z \end{aligned}} \tag{2.6}$$

and illustrated in Figure 2.4.

3. Differential volume is given by

$$\boxed{dv = \rho\, d\rho\, d\phi\, dz} \tag{2.7}$$

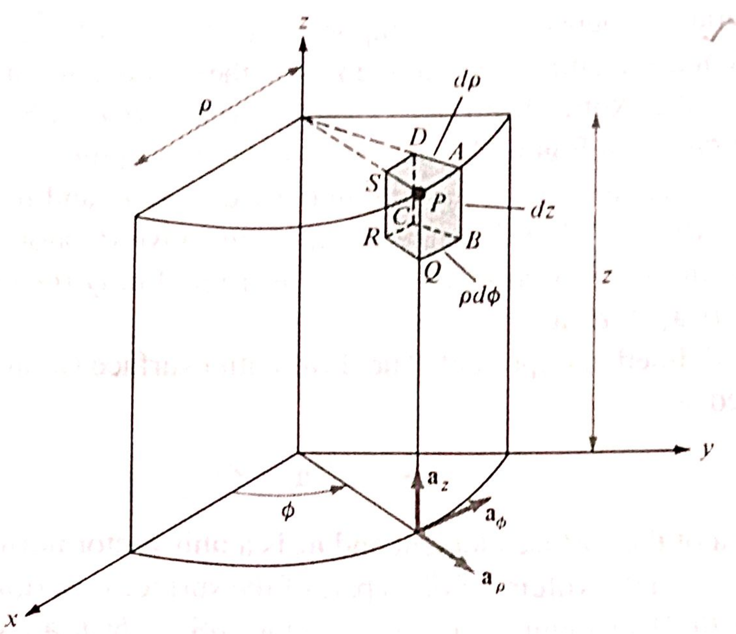


Figure 2.3 Differential elements in cylindrical coordinates.

As mentioned in the preceding section on Cartesian coordinates, we need only remember $d\mathbf{l}$; $d\mathbf{S}$ and dv can easily be obtained from $d\mathbf{l}$. For example, $d\mathbf{S}$ along $\mathbf{a}_z$ is the product of the components of $d\mathbf{l}$ along $\mathbf{a}_\rho$ and $\mathbf{a}_\phi$, that is, $d\rho\ \rho\ d\phi\ \mathbf{a}_z$. Also dv is the product of the three components of $d\mathbf{l}$, that is, $d\rho\ \rho\ d\phi\ dz$.

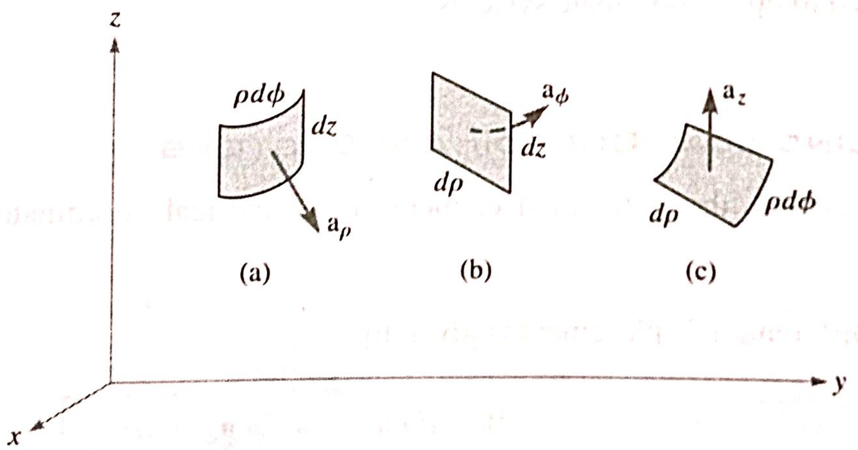


Figure 2.4 Differential normal surface areas in cylindrical coordinates: **(a)** $d\mathbf{S} = \rho\ d\phi\ dz\ \mathbf{a}_\rho$, **(b)** $d\mathbf{S} = d\rho\ dz\ \mathbf{a}_\phi$, **(c)** $d\mathbf{S} = \rho\ d\rho\ d\phi\ \mathbf{a}_z$.

C. SPHERICAL COORDINATE SYSTEMS

From Figure 2.5, the differential elements in spherical coordinates can be found as follows.

1. The differential displacement is

$$\boxed{d\mathbf{l} = dr\ \mathbf{a}_r + r\ d\theta\ \mathbf{a}_\theta + r \sin\theta\ d\phi\ \mathbf{a}_\phi} \tag{2.8}$$

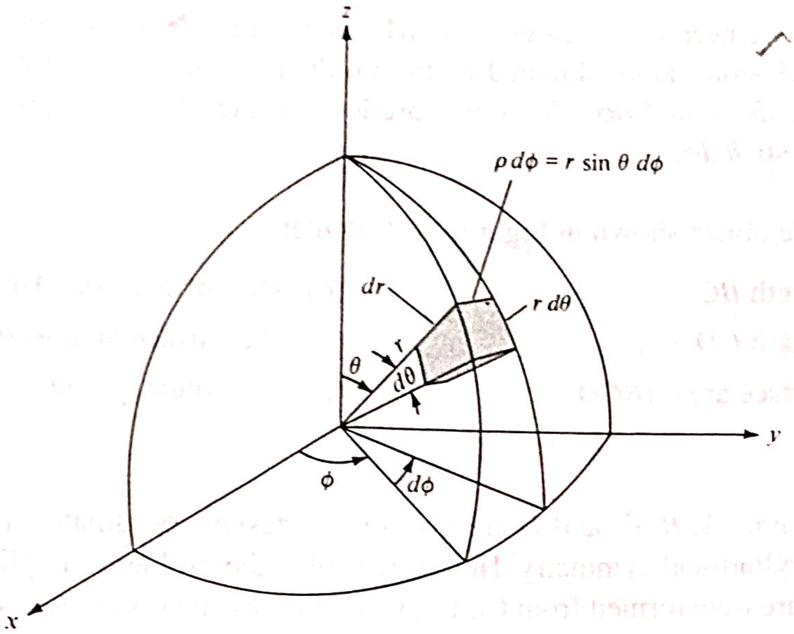


Figure 2.5 Differential elements in the spherical coordinate system.

2. The differential normal surface area is

$$\boxed{\begin{aligned} d\mathbf{S} &= r^2 \sin\theta\, d\theta\, d\phi\, \mathbf{a}_r \\ &\quad r \sin\theta\, dr\, d\phi\, \mathbf{a}_\theta \\ &\quad r\, dr\, d\theta\, \mathbf{a}_\phi \end{aligned}} \tag{2.9}$$

and illustrated in Figure 2.6.

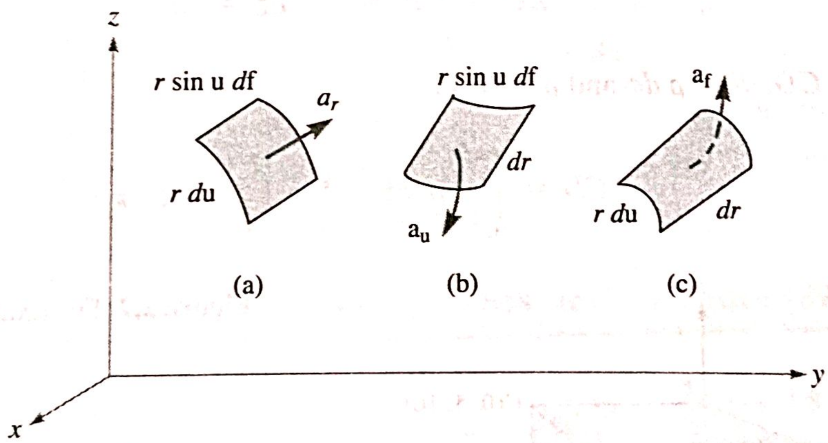


Figure 2.6 Differential normal surface areas in spherical coordinates: **(a)** $d\mathbf{S} = r^2 \sin\theta\, d\theta\, d\phi\, \mathbf{a}_r$, **(b)** $d\mathbf{S} = r \sin\theta\, dr\, d\phi\, \mathbf{a}_\phi$, **(c)** $d\mathbf{S} = r\, dr\, d\theta\, \mathbf{a}_\phi$.

3. The differential volume is

$$\boxed{dv = r^2 \sin\theta\, dr\, d\theta\, d\phi} \tag{2.10}$$

2.3 LINE, SURFACE, AND VOLUME INTEGRALS

The familiar concept of integration will now be extended to cases in which the integrand involves a vector. By "line" we mean the path along a curve in space. We shall use terms such as *line, curve,* and *contour* interchangeably.

> The **line integral** $\int_L \mathbf{A} \cdot d\mathbf{l}$ is the integral of the tangential component of **A** along curve L.

Given a vector field **A** and a curve L, we define the integral

$$\int_L \mathbf{A} \cdot d\mathbf{l} = \int_a^b |\mathbf{A}| \cos\theta \, dl \tag{2.11}$$

as the *line integral* of **A** around L (see Figure 2.9). If the path of integration is a closed curve such as *abca* in Figure 2.9, eq. (2.11) becomes a closed contour integral

$$\oint_L \mathbf{A} \cdot d\mathbf{l} \tag{2.12}$$

which is called the *circulation* of **A** around L.

Given a vector field **A**, continuous in a region containing the smooth surface S, we define the *surface integral* or the *flux* of **A** through S (see Figure 2.10) as

$$\Psi = \int_S |\mathbf{A}| \cos\theta \, dS = \int_S \mathbf{A} \cdot \mathbf{a}_n \, dS$$

or simply

$$\boxed{\Psi = \int_S \mathbf{A} \cdot d\mathbf{S}} \tag{2.13}$$

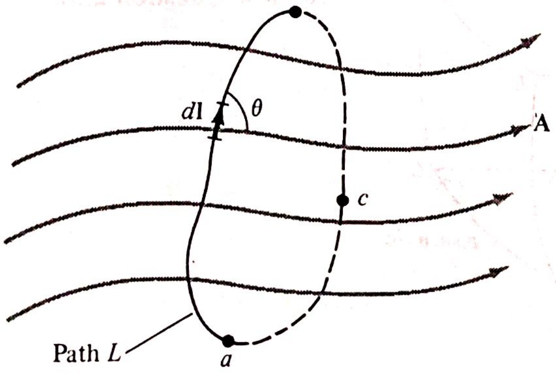


Figure 2.9 Path of integration of vector field **A**.

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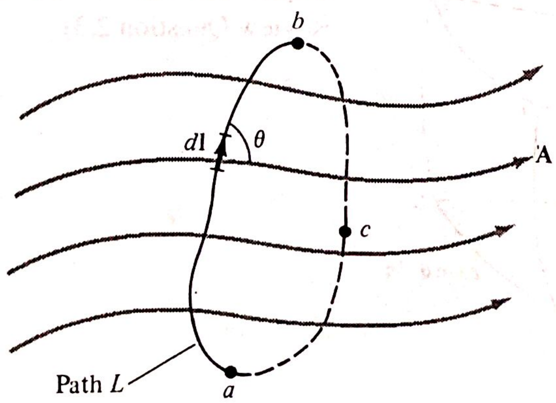

Figure 2.9 Path of integration of vector field **A**.

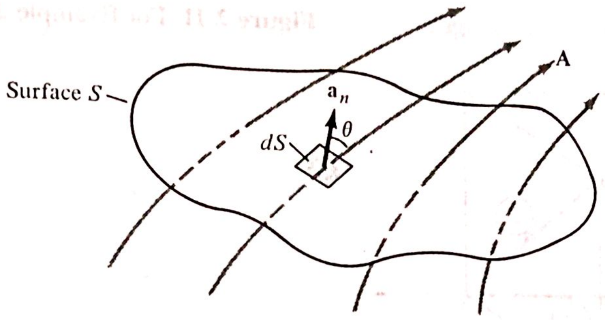

Figure 2.10 The flux of a vector field **A** through surface S.

where, at any point on S, $\mathbf{a}_n$ is the unit normal to S. For a closed surface (defining a volume), eq. (2.13) becomes

$$\Psi = \oint_S \mathbf{A} \cdot d\mathbf{S} \tag{2.14}$$

which is referred to as the *net outward flux of* **A** from S. Notice that a closed path defines an open surface whereas a closed surface defines a volume (see Figures 2.12 and 2.17).

We define the integral

$$\int_v \rho_v \, dv \tag{2.15}$$

as the *volume integral* of the scalar ρ_v over the volume v. The physical meaning of a line, surface, or volume integral depends on the nature of the physical quantity represented by **A** or ρ_v. Note that $d\mathbf{l}$, $d\mathbf{S}$, and dv are all as defined in Section 2.2.