Joint ICTP-IAEA Workshop on Monte Carlo Radiation Transport and Associated Data Needs for Medical Applications

28 October – 8 November 2024

ICTP, Trieste, Italy

Lecture 12

Statistical uncertainties in EGSnrc

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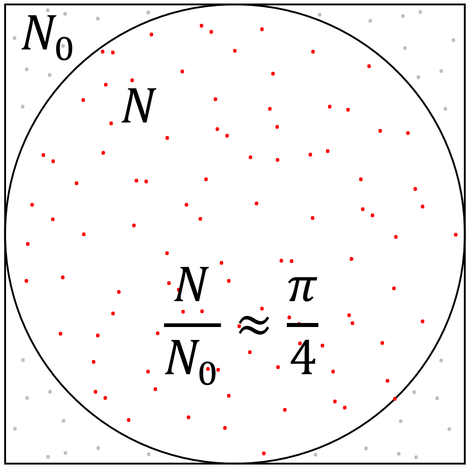
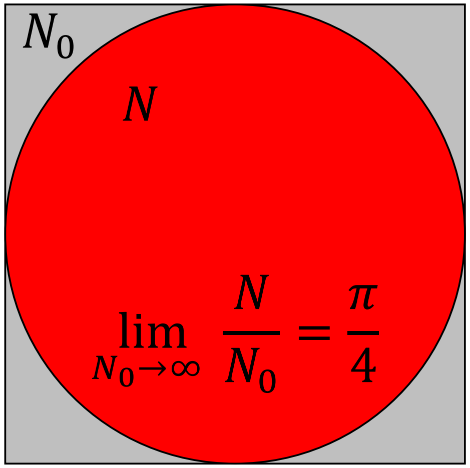
The Monte Carlo method uses random numbers to estimate deterministic quantities very precisely....

Yet, that's no contradiction.

Monte Carlo is a statistical sampling method

- The **Monte Carlo** method uses **statistical sampling** to **estimate expectation values**.
  - This way, the technique estimates the expected behaviour of the system for an **infinitely large number of source particles**.
  - Here's the famous example for estimating the number π:

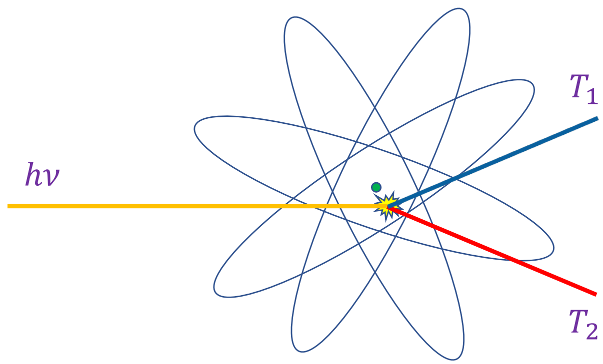




Monte Carlo is a statistical sampling method

- The **Monte Carlo** method uses **statistical sampling** to **estimate expectation values**.
  - In the context of radiation transport, Monte Carlo simulations reproduce the **stochastic nature** of particles with respect to

    1. Interactions with individual atoms/molecules;



Interaction site's differential cross section

$$\frac{\mathrm{d}^2\,\sigma_\mathrm{a}}{\mathrm{d}\Omega_1\mathrm{d}\Omega_2} = f(h\nu, \Omega_1, \Omega_2)$$

2. Transport in dense media.

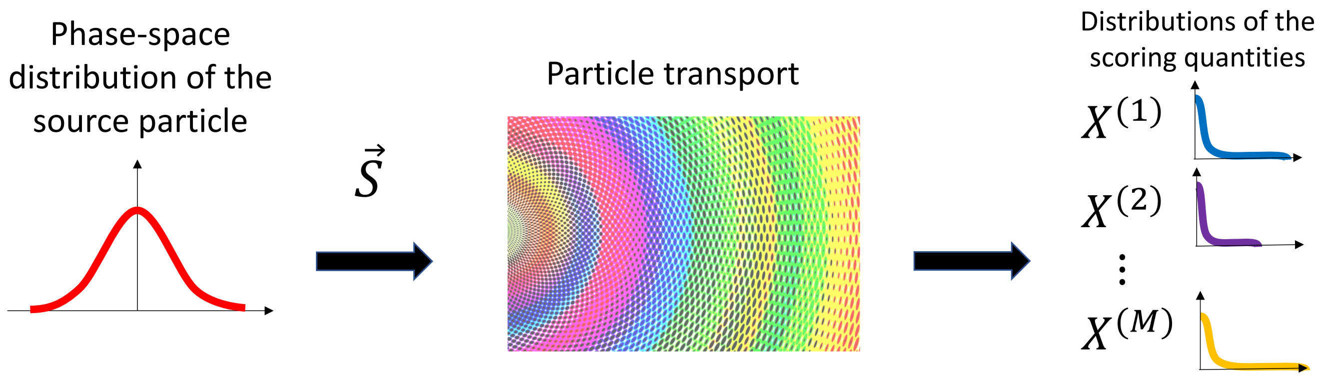
Macroscopic cross section

$$\Sigma = \rho N_\mathrm{A}\frac{\sigma_\mathrm{a}}{A}$$

Probability for the particle to reach interaction site is $P = e^{-\Sigma s}$ $\leftrightarrow$ Probability distribution of the step s is $p(s) = \Sigma e^{-\Sigma s}$

Definition of history in the context of Monte Carlo

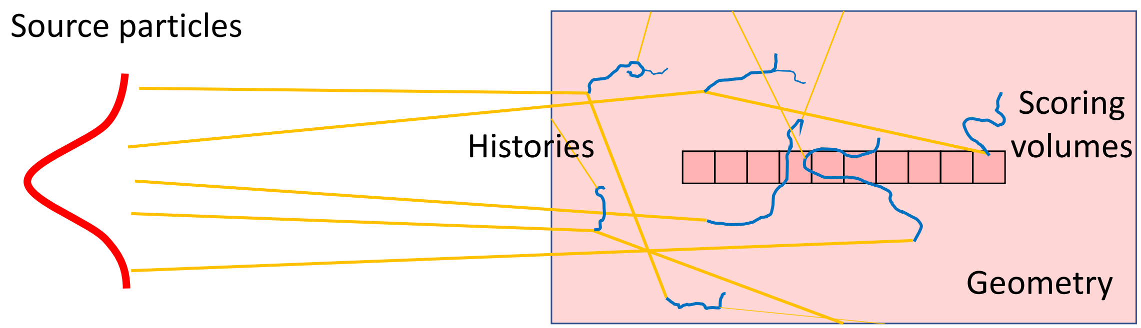
- A **history** is a **statistical sample** of an independent **source particle** being **transported** in the **geometry** of interest.
  - Mathematically speaking, a history consists of the following steps:
    1. **One independent sample** of the **phase-space probability distribution** of the **source particle**. Let's just call this *a single sample the source particle*;
    2. **Transport** of the **source particle** through the **geometry**, including subsequent secondary particles[1];
    3. Contribution to **scoring quantities**.



[1] This also includes any VRT that can be applied to achieve a history.

Definition of history in the context of Monte Carlo

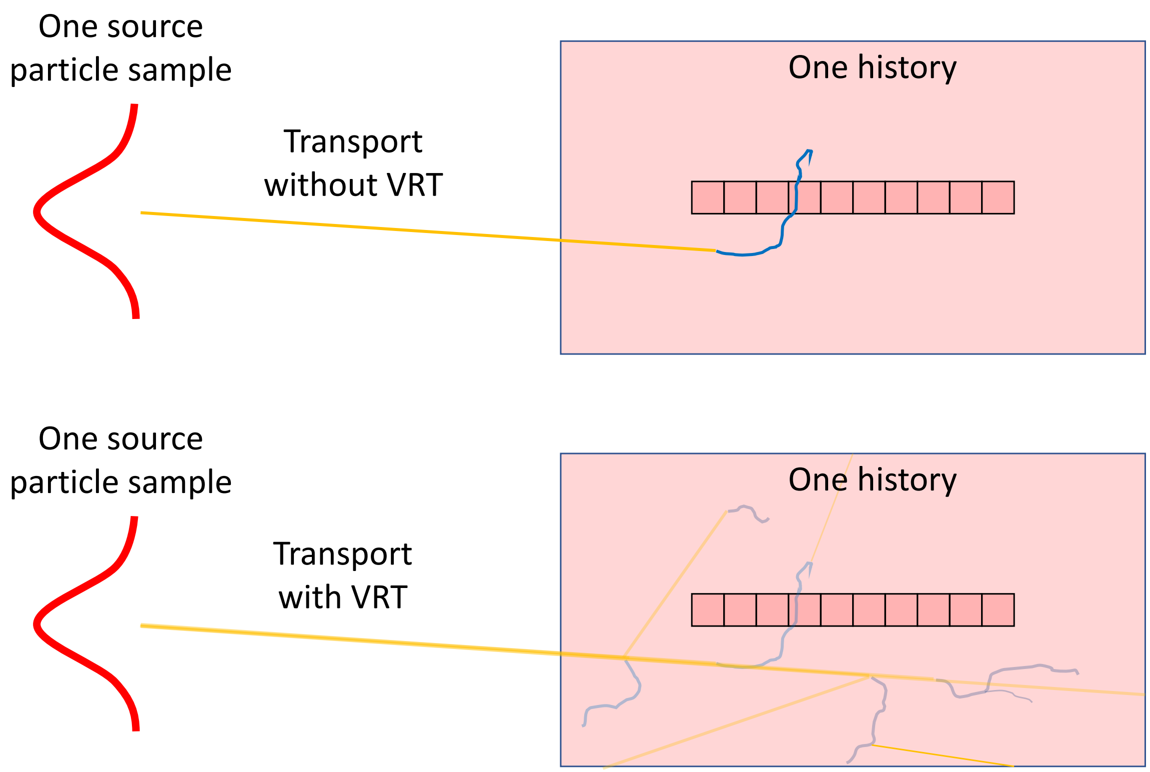
- A **history** is a **statistical sample** of an independent **source particle** being **transported** in the **geometry** of interest.
  - Because source particles are statistically independent, **histories must be statistically independent**.



In this figure, each sampled source particle has an associated a history.

Definition of history in the context of Monte Carlo

- A **history** is a **statistical sample** of an independent **source particle** being **transported** in the **geometry** of interest.
  - In the case *Variance Reduction Techniques* are used, one history **includes all subsequent particles** which weight is defined accordingly.



Random variables, samples, and scoring quantities

- During the simulation of radiation transport, scoring quantities - also known as tallies - have the following properties:
  - A **scoring quantity** is an **estimator** of an **expectation value**.
  - To each **scoring quantity** is associated a **random variable**[2], let's note it X. We define the expectation value of X to be μ_X, and its variance to be σ_X^2.
  - After N histories, the **sample associated** with the **scoring quantity** is an ensemble of independent random variables

$$\{X_i\}_{i=1}^{N} = \{X_1, X_2, \cdots, X_N\}$$

    Each value X_i is a single sample of the random variable X obtained from history i. Every element X_i statistically behaves like X, meaning that each has the same expectation value (or mean) and the same variance.

[2] The random variable has a probability distribution defined by the source and the geometry. In the context of radiation transport, probability distributions cannot be written in a closed analytical form due to the complexity of combining particle interactions with matter successively in heterogeneous geometries.

Random variables, samples, and scoring quantities

- The use of pseudo-random number generators is at the core of statistical sampling in Monte Carlo
  - Two algorithms are available: `ranmar` and `ranlux`. The default in EGSnrc is `ranlux`.
- You can control the sequence by choosing your random seeds
  - Example with `dosrznrc`

```
INITIAL RANDOM NO. SEEDS= 7, 15
```

  - Example with `egs_chamber`

```
:start rng definition:
   type = ranmar # generator type
  initial seeds = 7 15 # initial seeds
:stop rng definition:
```

- To remember
  - The **same input file** with **identical random seeds** will yield **identical results**.
  - The **same input file** with **different random seeds** will yield **statistically independent results**.

Random variables, samples, and scoring quantities

- Once the simulation is finished, your sample $\{X_i\}_{i=1}^N$ is used to characterize your **scoring quantity** with the following **two statistical estimators**
  1. The **mean** of X, defined as[3]

$$\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$$

  2. The **standard deviation of the mean** of X, defined as[4]

$$s_{\overline{X}} = \sqrt{\frac{1}{N-1}\left[\frac{1}{N}\sum_{i=1}^{N} X_i^2 - \overline{X}^2\right]}$$

[3]The mean is the estimator of the expectation value of X.
[4]The standard deviation of the mean is the estimator of the square root of the variance of the mean of X.

Random variables, samples, and scoring quantities

- It is crucial to distinguish the standard deviation of the mean $s_{\overline{X}}$ from the standard deviation s_X, the latter being is an estimator of the square root of the variance σ_X. That quantity is nonzero and is independent of the number of histories.
  - Indeed, we can use the definition of the standard deviation of X to show that $s_{\overline{X}} = \frac{s_X}{\sqrt{N}}$. The expectation value of $s_{\overline{X}}$ is therefore $\frac{\sigma_X}{\sqrt{N}}$. That is

    $$s_{\overline{X}} \propto \frac{1}{\sqrt{N}}$$

    This is a very important result that will **help you to determine how many histories** you need for your simulation.
  - Since σ_X is non zero, for an infinite number of history, the standard deviation of the mean is

    $$s_{\overline{X}} = \lim_{N\to\infty} \frac{s_X}{\sqrt{N}} = 0$$

    In other words, an ideal simulation with a an infinite number of history would converge perfectly to the value we're aim for[5].

[5] That's only true for statistical uncertainties. In practice, non-statistical uncertainties limit the accuracy of the simulation.

Monte Carlo results

- At this point, we realize that **Monte Carlo simulation results always come in pairs**.
  - We have the **estimation value**, and its **statistical uncertainty**.

$$\text{simulation result} = \overline{X} \pm s_{\overline{X}}$$

or

$$\text{simulation result} = \overline{X} \pm u_{\overline{X}} \quad \text{(in \%)}$$

When using $s_{\overline{X}}$, we report the **absolute statistical uncertainty** on the results.
When using $u_{\overline{X}}$, we report the **relative statistical uncertainty** relative to the result, defined as

$$u_{\overline{X}} = \frac{s_{\overline{X}}}{\overline{X}} \quad \text{(unitless)}$$

or equivalently

$$u_{\overline{X}} = \frac{s_{\overline{X}}}{\overline{X}} \cdot 100\% \quad \text{(in \%)}$$

Correlated data

- When more than one scoring quantity is defined, they are often statistically correlated.
- The covariance between two random variables X and Y is a measure of how much the two random variables go in the same direction (positive) or opposite direction (negative or anti-) with respect to their means, from one single paired sample (X, Y) to another.
- The estimator of the covariance of two simulations results $\overline{X}$ and $\overline{Y}$ is given by

$$s_{\overline{X},\overline{Y}} = \frac{1}{N-1}\left[\frac{1}{N}\sum_{i=1}^{N} X_i Y_i - \overline{X}\,\overline{Y}\right]$$

- In EGSnrc, we use the *Correlated Sampling* technique to compute the uncertainty of the ratio $\frac{X}{Y}$ as follows:

$$s_{\frac{\overline{X}}{\overline{Y}}} = \frac{\overline{X}}{\overline{Y}}\sqrt{\left(\frac{s_{\overline{X}}}{\overline{X}}\right)^2 + \left(\frac{s_{\overline{Y}}}{\overline{Y}}\right)^2 - \frac{2 s_{\overline{X},\overline{Y}}}{\overline{X}\,\overline{Y}}}$$

History-by-history estimation

- In EGSnrc, the statistics are implemented on a history-by-history basis[6]. This means that to compute the estimators, one does not need to store all samples of scoring quantities. Indeed, let's observe the previous estimators considering the example of two scoring quantities X and Y:

For $\overline{X}$: $\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$ and $s_{\overline{X}} = \sqrt{\frac{1}{N-1}\left[\frac{1}{N}\sum_{i=1}^{N} X_i^2 - \overline{X}^2\right]}$

For $\overline{Y}$: $\overline{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$ and $s_{\overline{Y}} = \sqrt{\frac{1}{N-1}\left[\frac{1}{N}\sum_{i=1}^{N} Y_i^2 - \overline{Y}^2\right]}$

For $\overline{X}$ and $\overline{Y}$: $s_{\overline{X},\overline{Y}} = \frac{1}{N-1}\left[\frac{1}{N}\sum_{i=1}^{N} X_iY_i - \overline{X}\,\overline{Y}\right]$

We realize here that only those quantity need to be kept during the simulation

$$\sum_{i=1}^{N} X_i \ , \ \sum_{i=1}^{N} X_i^2 \ , \ \sum_{i=1}^{N} Y_i \ , \ \sum_{i=1}^{N} Y_i^2 \ , \ \sum_{i=1}^{N} X_iY_i$$

[6] Walters, B. R. B., Kawrakow, I., & Rogers, D. W. O. (2002). History by history statistical estimators in the BEAM code system. Medical physics, 29(12), 2745-2752.

Non-statistical uncertainties

- In EGSnrc, the **uncertainties reported are statistical** (or type A) in nature. However, there are **non-statistical uncertainties** (or type B), which are sometimes referred to as *systematic uncertainties*. Type B uncertainties contribute to the results due to how EGSnrc is implemented and on what data is being used.

  1. Self-consistency of transport algorithms

  - The strategy to asses this contribution is to perform simulations for which we know the result theoretically. The main example is the application of Fano's theorem to determine what energy will be scored by a homogeneous source in a medium of homogeneous properties, but with variations in mass density. This is know as the **Fano test**.
  - Typical Fano test results with an air cavity in cobalt-60[7] or in the presence of a magnetic field[8] show that **EGSnrc is consistent within its own cross section at a level of 0.1%**.
  - I encourage you to perform the Fano test to quantify the self-consistency of your own simulations[9].

[7] Kawrakow, I. (2000). Accurate condensed history Monte Carlo simulation of electron transport. II. Application to ion chamber response simulations. Medical physics, 27(3), 499-513.

[8] Malkov, V. N., & Rogers, D. W. O. (2016). Charged particle transport in magnetic fields in EGSnrc. Medical physics, 43(7), 4447-4458.

[9] Bouchard, H., de Pooter, J., Bielajew, A., & Duane, S. (2015). Reference dosimetry in the presence of magnetic fields: conditions to validate Monte Carlo simulations. Physics in Medicine & Biology, 60(17), 6639.

Non-statistical uncertainties

- In EGSnrc, the **uncertainties reported are statistical** (or type A) in nature. However, there are **non-statistical uncertainties** (or type B), which are sometimes referred to as *systematic uncertainties*. Type B uncertainties contribute to the results due to how EGSnrc is implemented and on what data is being used.

2. Uncertainty from interaction data

- This contribution to uncertainties mainly arises from the mean excitation energies (I-value) via stopping powers and cross section data.
- A study by Wulff *et al.*[10] showed that uncertainties of up to 0.40% can be observed in ion chambers quality correction factors in the context of radiotherapy beams.
- Another study by Muir & Rogers[11] showed that this uncertainty can go up to 0.63% in the same context.

[10] Wulff, J., Heverhagen, J. T., Zink, K., & Kawrakow, I. (2010). Investigation of systematic uncertainties in Monte Carlo-calculated beam quality correction factors. Physics in Medicine & Biology, 55(16), 4481.

[11] Muir, B. R., & Rogers, D. W. O. (2010). Monte Carlo calculations of, the beam quality conversion factor. Medical physics, 37(11), 5939-5950.

Confidence intervals from statistical uncertainties

- In the situation where statistical uncertainties dominate non-statistical uncertainties, we can predict the behaviour of the errors using the *Central limit theorem*, which essentially tells us that

  The probability distribution of $\overline{X}$ *for number of histories* $N \to \infty$ *is a Gaussian distribution, independently of the probability distribution of* X*.*

  - Since the probability for the expectation value of X to be within a certain confidence interval is given by

    $$\alpha = P\left(\mu_X \in \left[\overline{X} - a\, s_{\overline{X}}, \overline{X} + a\, s_{\overline{X}}\right]\right)$$

    with a positive.

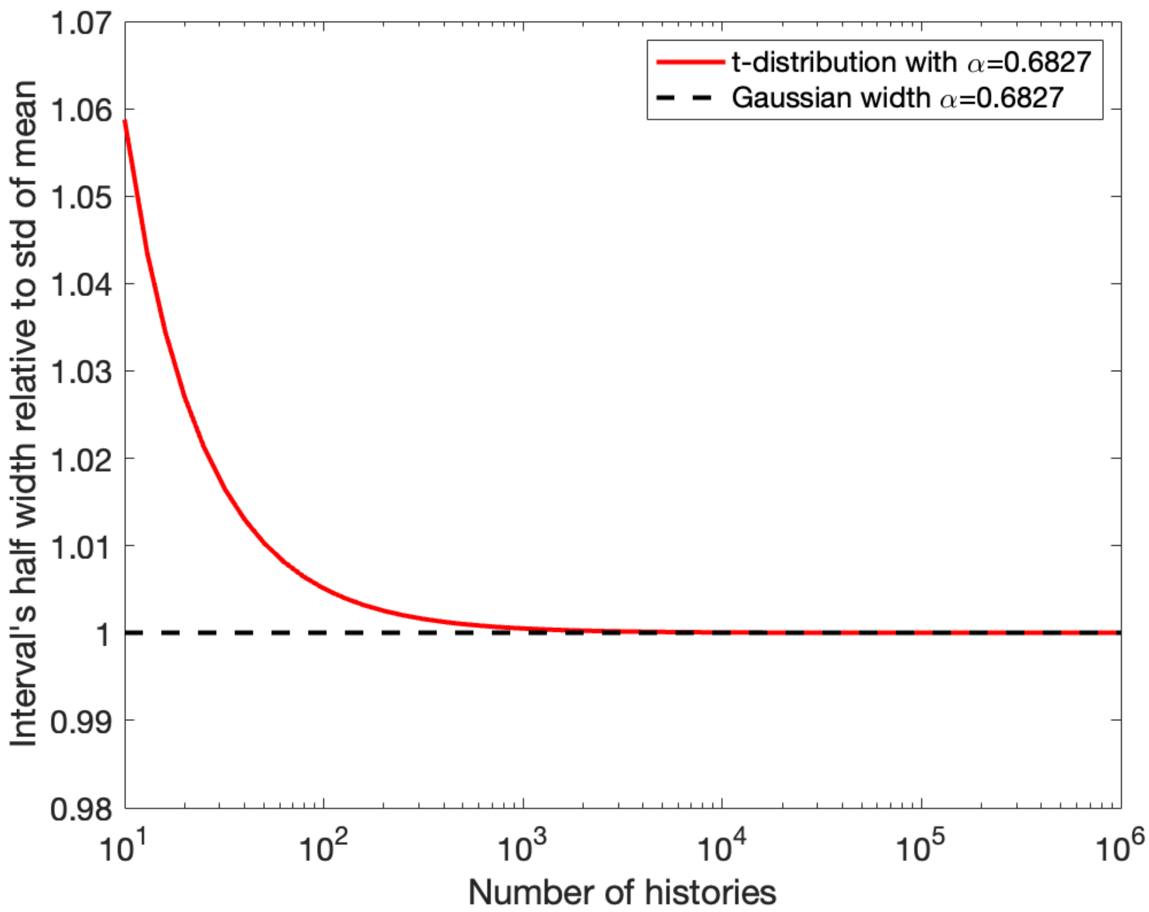
  - Then, the consequence of the CLM is that

    $$\alpha = P\left(t \in [-a, a]\right)$$

    with t following a *t*-distribution with $N - 1$ degrees of freedom. This can be shown in a few lines.

Confidence intervals from statistical uncertainties

- When N is large, e.g., $N > 10^3$, the t-distribution behaves like the standard normal distribution.



Confidence intervals from statistical uncertainties

- When N is large, e.g., $N > 10^3$, the t-distribution behaves like standard normal distribution.
- For such large N, the value a is found such that

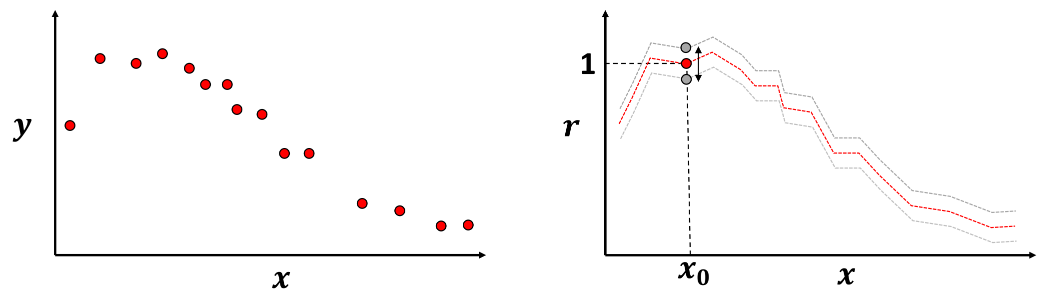
$$\alpha = \int_{-a}^{a} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \mathrm{d}z$$

$$\Rightarrow \quad \alpha = 1 - 2\left(1 - \int_{-\infty}^{a} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \mathrm{d}z\right)$$

The expression in parentheses is typical what is found in tables, since it is the cumulative distribution function (cdf) of a standard normal distribution evaluated at a.

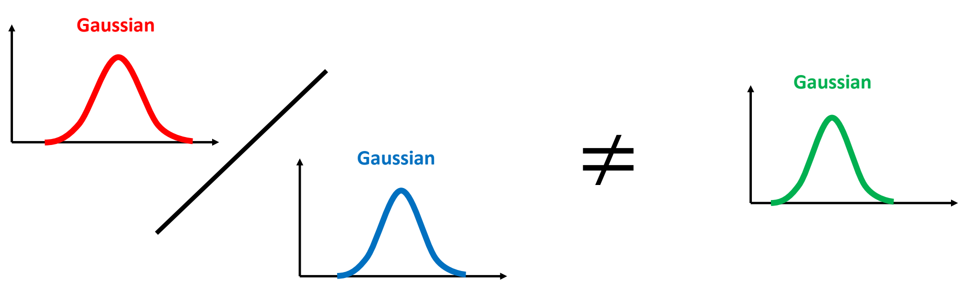
Normalization of simulation results

- Normalization of Monte Carlo results can be tricky, since the normalization value
  1. **1.** Is a random variable itself, so its statistical error is propagated on all results of the simulation.



  2. **2.** Is statistically correlated to other simulation results, and often the is no option (to my knowledge) in EGSnrc to account for it.

  3. **3.** Degrades de probability distribution of the results, i.e., the ratio distribution is not Gaussian! Hence, statistical inference requires additional considerations.



Normalization of simulation results

- Here are a few helpful tips for healthy data normalization
  1. When normalizing to a single value, e.g. a percent depth-dose distribution, at the very least one must **account for** the following **error propagation rule**

$$s_{\frac{\overline{X}}{\overline{Y}}} \approx \frac{\overline{X}}{\overline{Y}}\sqrt{\left(\frac{s_{\overline{X}}}{\overline{X}}\right)^2 + \left(\frac{s_{\overline{Y}}}{\overline{Y}}\right)^2}$$

     Except at the normalization point where the uncertainty is zero by definition.

  2. Ideally, one would choose a **normalization value** that is **unambiguous** and that has a **low uncertainty compared to the other results**. That is
     (a) When normalizing to a point, choose one that is clearly defined, e.g. $z = 10$ cm for a percent depth-dose distribution.
     (b) If you must choose an ambiguous point, e.g., d_{max} of a precent depth-dose curve, make sure the uncertainty is very low (e.g., 0.1%).
     (c) When many scoring quantities are involved, you could normalize over the mean of all results. In typical simulations, this has little to no impact on the relative uncertainties because the uncertainty of the mean of all results is very small. Thus, the normalized data retains its Gaussian distribution characteristics.

Take home message: what do I need to remember?

- A history is a statistically independent sample of a scoring quantity;
- Monte Carlo results always come in pairs and are calculated from those estimators

$$\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i \quad \text{and} \quad s_{\overline{X}} = \sqrt{\frac{1}{N-1}\left[\frac{1}{N}\sum_{i=1}^{N} X_i^2 - \overline{X}^2\right]}$$

- The uncertainty is always proportional to the inverse of the square root of the number of histories;

$$s_{\overline{X}} \propto \frac{1}{\sqrt{N}}$$

- Statistical correlations between results should always be accounted for in the user code to get accurate uncertainty estimations (typically for relative quantities);

$$s_{\frac{\overline{X}}{\overline{Y}}} = \frac{\overline{X}}{\overline{Y}}\sqrt{\left(\frac{s_{\overline{X}}}{\overline{X}}\right)^2 + \left(\frac{s_{\overline{Y}}}{\overline{Y}}\right)^2 - \frac{2s_{\overline{X},\overline{Y}}}{\overline{X}\,\overline{Y}}}$$

- Normalization values should have a low uncertainty. The rule for error uncertainty propagation should be applied. Statistical inference can be affected by the normalization technique used (e.g., confidence intervals, hypothesis testing, etc.).