Image Denoising : Homework 4

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Exercice n 3.1 : optimal convex combination of a blind-spot network and the noise image.

We have images are contaminated with noise that :
(i) unbiased $E_v(v \mid u) = u$ for all u.
(ii) spatially independent.

1 - Show that the MSE of $F^\lambda(v)$ can be decomposed as:

$$E_{v,u}\left(\|F^\lambda(v) - u\|^2\right) = \lambda^2 E_{v,u}(\|F(v) - u\|^2) + (1-\lambda)^2 E_{v,u}(\|u-v\|^2) + 2\lambda(1-\lambda) E_{v,u}\left(\langle F(v) - u, v-u\rangle\right)$$

We have $F^\lambda(v) = \lambda F(v) + (1-\lambda) v$

The MSE of $F^\lambda(v)$:

$$E_{v,u}\left(\|F^\lambda(v) - u\|^2\right) = E_{v,u}\left(\|\lambda F(v) + (1-\lambda)v - u\|^2\right)$$

$$= E_{u,v}\left(\|\lambda(F(v) - u) + (1-\lambda)(v-u)\|^2\right)$$

$$= E_{u,v}\left(\lambda^2 \|F(v) - u\|^2 + (1-\lambda)^2 \|v-u\|^2 + 2\lambda(1-\lambda)\langle F(v)-u, v-u\rangle\right)$$

$$\boxed{E_{u,v}(\|F^\lambda(v)-u\|^2) = \lambda^2 E_{u,v}\left(\|F(v) - u\|^2\right) + (1-\lambda)^2 E_{u,v}\left(\|v-u\|^2\right) + 2\lambda(1-\lambda) E_{u,v}\left(\langle F(v) - u, v-u\rangle\right)}$$

2 - Show that $E_{v,u}\left(\langle F(v) - u, v-u\rangle\right) = 0$

$$E_{v,u}\left(\langle F(v) - u, v-u\rangle\right) = E_u E_v\left(\langle F(v) - u, u-v\rangle \mid u\right)$$

(we have split the scalar product over sets $J \in \mathcal{J}$, because $\mathcal{J}$ a partition)

$$= E_u E_v\left\{\sum_{J\in\mathcal{J}} \langle F(v)_J - u_J, u_J - v_J\rangle \mid u\right\} \quad (3.26)$$

or $E_v\left(\langle F(v)_J - u_J, u_J - v_J\rangle \mid u\right)$

$$= E_{v_J, v_{J^c}}\left(\langle F(v)_J - u_J, u_J - v_J\rangle \mid u\right)$$

$$= E_{v_{J^c}}\left(E_{v_J}\left\{\langle F(v)_J - u_J, u_J - v_J\rangle \mid v_{J^c}, u\right\} \mid u\right)$$

$$= E_{v_{J^c}}\left(\langle F(v)_J - u_J, E_{v_J}\{u_J - v_J \mid v_{J^c}, u\}\rangle \mid u\right)$$

$$= E_{v_{J^c}} \left\{ \langle F(v)_J - u_J , u_J - E\{ v_J \mid u \} \rangle \mid u \right) = 0$$

This result because the noise in J is conditionally independent from the noise in J^c given u and unbiased.

We found that : $E_v \left(\langle F(v)_J - u_J , u_J - v_J \rangle \mid u \right) = 0$

Now, $E_{v,u} \left(\langle F(v) - u , v - u \rangle \right)$

$$= E_u \Big(\sum_{J \in \mathcal{J}} \underbrace{E_v \left(\langle F(v)_J - u_J , u_J - v_J \rangle \mid u \right)}_{=0} \Big) = 0$$

3_ Deduce that the λ^* minimizes the MSE is given by :

$$\lambda^* = \frac{E_u \{ \mathbb{V}\{ v \mid u \} \}}{E_{v,u} \{ \| F(v) - u \|^2 \} + E_u \{ \mathbb{V} \{ v \mid u \} \}}$$

$$\begin{aligned} MSE &= \lambda^2 E_{v,u} \{ \| F(v) - u \|^2 \} + (1-\lambda)^2 E_{v,u} \{ \| v - u \|^2 \} \\ &\quad + 2\lambda(1-\lambda) \underbrace{E_{u,v} \{ \langle F(v) - u , v - u \rangle \}}_{=0} \\ &= \lambda^2 E_{v,u} \{ \| F(v) - u \|^2 \} + (1-\lambda)^2 E_{v,u} \{ \| v - u \|^2 \} \end{aligned}$$

Let's derive the bellow expression with respect to λ :

$$\frac{\partial E_{v,u} \left(\| F^\lambda(v) - u \|^2 \right)}{\partial \lambda} = 2\lambda E_{v,u} \{ \| F(v) - u \|^2 \} - 2(1-\lambda) E_{v,u} \{ \| v - u \|^2 \} = 0$$

$$\Rightarrow \lambda \left(E_{v,u} \{ \| F(v) - u \|^2 \} + E_{v,u} \{ \| v - u \|^2 \} \right) = E_{v,u} \{ \| v - u \|^2 \}$$

or we have : $E_{v,u} \left(\| v - u \|^2 \right) = E_u \left(\mathbb{V} \{ v \mid u \} \right)$ (3.18)

$$\Rightarrow \boxed{\lambda^* = \frac{E_u \left(\mathbb{V} \{ v \mid u \} \right)}{E_{v,u} \{ \| F(v) - u \|^2 \} + E_u \left(\mathbb{V} \{ v \mid u \} \right)}}$$

4- Suppose now that the noise has a variance $\boxed{\mathbb{V}\{v|u\} = d\sigma^2}$ for all $u \in \mathbb{R}^d$.
Use proposition 3.4 and express λ^* in terms of σ^2 and the self-supervised risk $R_{N2S}(F)$.

(3.2) * $$E_v\left(\|F(v) - u\|^2\right) = E_{v,u}\left(\|F(v) - u\|^2\right) + E_u\left(\mathbb{V}\{v|u\}\right)$$

and * $$R_{N2S}(F) = E_v\left(\|F(v) - u\|^2\right)$$

we have : $$\lambda^* = \frac{E_u\left(\mathbb{V}\{v|u\}\right)}{E_{v,u}\left(\|F(v) - u\|^2\right) + E_u\left(\mathbb{V}\{v|u\}\right)}$$

$$= \frac{d\sigma^2}{E_v\left(\|F(v) - u\|^2\right)} = \frac{d\sigma^2}{R_{N2S}(F)}$$

$$\Rightarrow \boxed{\lambda^* = \frac{d\sigma^2}{R_{N2S}(F)}}$$

Exercice 3.2 : Bias - Variance decomposition:
Given an estimator $\hat{u}(v)$ of u and v is a noisy version of u.
Show that, for a given u, the MSE can be expressed as follow:

$$E_v\left(\|\hat{u}(v) - u\|^2 \mid u\right) = \|E_v\{\hat{u}(v)|u\} - u\|^2 + E_v\left(\|\hat{u}(v) - E_v\{\hat{u}(v)|u\}\|^2 \mid u\right)$$

we have :

$$MSE = E_v\left(\|\hat{u}(v) - u\|^2 \mid u\right)$$

$$= E_v\left(\|\hat{u}(v) - u + E_v\{\hat{u}(v)|u\} - E_v\{\hat{u}(v)|u\}\|^2 \mid u\right)$$

$$= E_v\Big(\|E_v\{\hat{u}(v)|u\} - u\|^2 + \|\hat{u}(v) - E_v\{\hat{u}(v)|u\}\|^2 + 2\left\langle E_v\{\hat{u}(v)|u\} - u, \hat{u}(v) - E_v\{\hat{u}(v)|u\}\right\rangle \Big| u\Big)$$

$$= E_V\left(\underbrace{\| E_V(\hat{u}|u) - u \|^2}_{\text{(independent of } v)} \middle| u\right) + E_V\left(\| \hat{u}(v) - E_V(\hat{u}(v)|u) \|^2 \middle| u\right)$$

$$+ 2 E_V\left(\langle E_V(\hat{u}(v)|u) - u \,,\, \hat{u}(v) - E_V(\hat{u}(v)|u) \rangle \middle| u\right)$$

* $E_V(\| E_V(\hat{u}(v)|u) - u \|^2 \,|u) = \| E_V(\hat{u}(v)|u) - u \|^2$

* The expectation is a linear function and the scalar product is bilinear :

$$E_V(\langle E_V(\hat{u}(v)|u) - u \,,\, \hat{u}(v) - E_V(\hat{u}(v)|u) \rangle \,|\, u)$$

$$= \langle E_V(\hat{u}(v)|u) - u \,,\, E_V(\hat{u}(v) - E_V(\hat{u}(v)|u)\,|u)\,|u) \rangle$$

$$= \langle E_V(\hat{u}(v)|u) - u \,,\, \underbrace{E_V(\hat{u}(v)|u) - E_V(\hat{u}(v)|u)}_{=0} \rangle$$

$$= 0$$

Thus,

$$\boxed{E_V(\| \hat{u}(v) - u \|^2 \,|u) = \| E_V(\hat{u}(v)\,|u) - u \|^2 + E_V\{ \| \hat{u}(v) - E_V(\hat{u}(v)|u) \|^2 \,|u \}}$$