Image Denoising: homework 3

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Exercice 8. 1 :

we assume in this exercice that we have P and the noise are independent and the noise is white satisfying:

$$\mathbb{P}(\check{P} \mid P) = \frac{1}{(2\pi\sigma^2)^{k^2/2}} e^{-\frac{\|\check{P} - P\|^2}{2\sigma^2}}$$

We have also : $\check{P} = P + N$.

With N is a patch of a white noise.

This noise has a mean equal to zero and uncorrelated elements with equal constant variance. Thus, we have:

$$\begin{cases} E(N) = 0 \\ Var(N) = \sigma^2 I \end{cases}$$

Also, $\overline{P}$ and C_P are the characteristics of P.

$$\Rightarrow \quad C_{\check{P}} = Cov(\check{P}) = Cov(P + N)$$

$$= Cov(P) + Cov(N) \text{ (independent)}$$

$$= C_P + \sigma^2 I$$

$$E\check{P} = E(P + N) = E(P) + \underbrace{E(N)}_{=0}$$

$$= \overline{P}$$

$$\Rightarrow \quad C_{\check{P}} = C_P + \sigma^2 I \text{ and } E\check{P} = \overline{P}.$$

Exercice 8.3 :

The goal of this exercice is to compare the two denoising equations :
$$\begin{cases} \hat{P}_1 = \bar{P} + (C_{\breve{p}} - \sigma^2 I)\, C_{\breve{p}}^{-1} (\breve{P} - \bar{P}) \\ \hat{P}_2 = \bar{P}^1 + C_{p}^1 (C_p^1 + \sigma^2 I)^{-1} (\breve{P} - \bar{P}^1) \end{cases}$$
with their corresponding wiener filtering equations on an orthonormal basis.

1 → $\{G_i\}_{i=1}^{k}$ the set of eigenvectors of $C_{\breve{p}}$ that correspond to the eigenvalue λ_i.

* $C_{\breve{p}}$ is invertible ⇒ G_i is the i^{th} eigenvector that correspond to the eigenvalue $1/\lambda_i$ of $C_{\breve{p}}^{-1}$.

* $\hat{P}_1 = \bar{P} + (C_{\breve{p}} - \sigma^2 I) C_{\breve{p}}^{-1} (\breve{P} - \bar{P})$

$$\left(\{G_i\} \text{ orthonormal basis}\right) = \sum_{i=1}^{k} \langle \bar{P} + (C_{\breve{p}} - \sigma^2 I)\, C_{\breve{p}}^{-1} (\breve{P} - \bar{P}),\, G_i \rangle G_i$$

$$= \sum_{i=1}^{k} \left[\langle \bar{P}, G_i \rangle G_i + \langle (C_{\breve{p}} - \sigma^2 I)\, C_{\breve{p}}^{-1} (\breve{P} - \bar{P}), G_i \rangle G_i \right]$$

$$= \sum_{i=1}^{k} \left[\langle \bar{P}, G_i \rangle G_i + \langle \breve{P} - \bar{P}, \left[(C_{\breve{p}} - \sigma^2 I)\, C_{\breve{p}}^{-1}\right]^T G_i \rangle G_i \right]$$

$$= \sum_{i=1}^{k} \left[\langle \bar{P}, G_i \rangle G_i + \langle \breve{P} - \bar{P}, (C_{\breve{p}}^{-1})^T (C_{\breve{p}} - \sigma^2 I)^T G_i \rangle G_i \right]$$

$$(C_{\breve{p}} \text{ is symmetric}) = \sum_{i=1}^{k} \left[\langle \bar{P}, G_i \rangle G_i + \langle \breve{P} - \bar{P}, C_{\breve{p}}^{-1} (C_{\breve{p}} - \sigma^2 I) G_i \rangle G_i \right]$$

$$= \sum_{i=1}^{k} \langle \bar{P}, G_i \rangle G_i + \langle \breve{P} - \bar{P}, \frac{\lambda_i - \sigma^2}{\lambda_i} G_i \rangle G_i$$

$$= \sum_{i=1}^{k} 1 - \frac{\lambda_i - \sigma^2}{\lambda_i} \langle \bar{P}, G_i \rangle G_i + \sum_{i=1}^{k} \frac{\lambda_i - \sigma^2}{\lambda_i} \langle \breve{P}, G_i \rangle G_i$$

$$\hat{P}_1 = \sum_{i=1}^{k} \frac{\sigma^2}{\lambda_i} \langle \bar{P}, G_i \rangle G_i + \sum_{i=1}^{k} \frac{\lambda_i - \sigma^2}{\lambda_i} \langle \breve{P}, G_i \rangle G_i$$

$$\Rightarrow \hat{P_1} = \sum_{i=1}^{k} \frac{\sigma'^2}{d_i} \langle \bar{P}, G_i \rangle G_i + \sum_{i=1}^{k} a(i) \langle \check{P}, G_i \rangle G_i$$

with $a(i) = \dfrac{d_i - \sigma'^2}{d_i}$

Here, we found that the second part of the formula can be written as:

$\sum a(i) \langle \check{P}, G_i \rangle G_i$ which correspond to wiener filtering.

However, the only difference between the two expressions is the added term : $\sum_{i=1}^{k} \frac{\sigma'^2}{d_i} \langle \bar{P}, G_i \rangle G_i$. This term depends on the vector $\bar{P}$ which is the average of the patches.

2- $\hat{P_2} = \bar{P}^1 + C_P^1 (C_P^1 + \sigma'^2 I)^{-1} (\check{P} - \bar{P}^1)$

* $\{J_i\}_{i=1}^{n}$ is the set of the eigenvectors of C_P^1 that correspond to the eigen values λ_i^1

* C_P^1 is invertible and symmetric.
  $\Rightarrow$ J_i is the i^{th} eigenvector that correspond to the eigenvalue $\frac{1}{\lambda_i^1}$ of the matrix $(C_P^1)^{-1}$.

* $\hat{P_2} = \bar{P}^1 + C_P^1 (C_P^1 + \sigma'^2 I)^{-1} (\check{P} - \bar{P}^1)$

($\{J_i\}$ orthonormal basis) $= \sum_{i=1}^{k} \langle \bar{P}^1 + C_P^1 (C_P^1 + \sigma'^2 I)^{-1} (\check{P} - \bar{P}^1), J_i \rangle J_i$

$= \sum_{i=1}^{k} \langle \bar{P}^1, J_i \rangle J_i + \langle C_P^1 (C_P^1 + \sigma'^2 I)^{-1} (\check{P} - \bar{P}^1), J_i \rangle J_i$

(C_P^1 is symmetric) $= \sum_{i=1}^{k} \langle \bar{P}^1, J_i \rangle J_i + \langle \check{P} - \bar{P}^1, \left(C_P^1 (C_P^1 + \sigma'^2 I)^{-1}\right)^T J_i \rangle J_i$

$= \sum_{i=1}^{k} \langle \bar{P}^1, J_i \rangle J_i + \langle \check{P} - \bar{P}^1, (C_P^1 + \sigma'^2 I)^{-1} C_P^1 J_i \rangle J_i$

$= \sum_{i=1}^{k} \langle \bar{P}^1, J_i \rangle J_i + \langle \check{P} - \bar{P}^1, \frac{\lambda_i}{d_i + \sigma'^2} J_i \rangle J_i$

$= \sum_{i=1}^{k} \left(1 - \frac{d_i}{d_i + \sigma'^2}\right) \langle \bar{P}^1, J_i \rangle J_i + \sum_{i=1}^{k} \langle \check{P}_i, \frac{d_i}{d_i + \sigma'^2} J_i \rangle J_i$

$$\hat{P}_2 = \sum_{i=1}^{k} \frac{\sigma^2}{d_i + \sigma^2} \langle \bar{P}^1, g_i \rangle g_i + \sum_{i=1}^{k} \frac{d_i}{d_i + \sigma^2} \langle \tilde{P}, g_i \rangle g_i$$

$$\Rightarrow \hat{P}_2 = \sum_{i=1}^{k} \frac{\sigma^2}{d_i + \sigma^2} \langle \bar{P}^1, g_i \rangle g_i + \sum_{i=1}^{k} \alpha(i) \langle \tilde{P}, g_i \rangle g_i$$

The second part of the formula can be written as:

$\sum \alpha(i) \langle \tilde{P}, g_i \rangle g_i$ with $\alpha(i) = \dfrac{d_i}{d_i + \sigma^2}$

which correspond to wiener filtering.

The only difference between the two equations is the added terms: $\sum_{i=1}^{k} \dfrac{\sigma^2}{d_i + \sigma^2} \langle \bar{P}^1, g_i \rangle g_i$.

This term depends on the vector $\bar{P}^1$ which correspond to the average of the patches.

Exercice 8.4:

* we have bayes formula:

$$P(\tilde{P} \mid P) = \frac{P(P \mid \tilde{P}) \cdot P(\tilde{P})}{P(P)}$$

* Fubini Tonelli theorem:

we have $P \longmapsto \int \left| P(\tilde{P})\, P(P \mid \tilde{P})\, \|P - \tilde{P}\|^2 \, dP \right| < +\infty$

$\tilde{P} \longmapsto \int \left| P(\tilde{P})\, P(P \mid \tilde{P})\, \|P - \tilde{P}\|^2 \, d\tilde{P} \right| < +\infty$

Thus, $\displaystyle\iint P(\tilde{P})\, P(P \mid \tilde{P})\, \|P - \tilde{P}\|^2 \, dP \, d\tilde{P}$

$$= \iint P(\tilde{P})\, P(P \mid \tilde{P})\, \|P - \tilde{P}\|^2 \, d\tilde{P} \, dP$$

Now, we have:

$$MSE = \int P(P) \int P(\tilde{P} \mid P)\, \|P - \tilde{P}\|^2 \, d\tilde{P} \, dP$$

$$MSE \overset{\text{bayes}}{=} \int P(\cancel{P}) \int \frac{P(P|\hat{P})\ P(\check{P})}{P(\cancel{P})} \|P - \check{P}\|^2 d\check{P}\ dP$$

$$= \iint P(P|\check{P})\ P(\check{P}) \quad \|P - \check{P}\|^2\ d\check{P}\ dP$$

$$\overset{\text{Fubini}}{=} \int P(\check{P}) \int P(P|\check{P}) \quad \|P - \check{P}\|^2\ dP\ d\check{P}$$

$$\Rightarrow \quad MSE = \int P(\check{P}) \int P(P|\check{P}) \quad \|P - \check{P}\|^2\ dP\ d\check{P}$$

Exercice 8.5:

We have $MMSE(\check{P}) = E(\|\hat{P} - \check{P}\|^2 \mid \check{P})$

$$= \int \mathbb{P}(P|\check{P})\,(P - \hat{P})^2\, dP$$

$$\frac{\partial MMSE(\check{P})}{\partial \hat{P}} = -2 \int \mathbb{P}(P|\check{P})\,(P - \hat{P})\, dP = 0$$

$$\Rightarrow \int \frac{\mathbb{P}(\check{P}|P) \cdot \mathbb{P}(P)}{\mathbb{P}(\check{P})} P\ dP = \int \frac{\mathbb{P}(\check{P}|P)\,\mathbb{P}(P)}{\mathbb{P}(\check{P})} \hat{P}\, dP$$

$$= \frac{1}{\mathbb{P}(\check{P})} \cdot \hat{P} \underbrace{\int \mathbb{P}(\check{P}|P)\,\mathbb{P}(P)\,dP}_{\mathbb{P}(\check{P})}$$

$$= \hat{P} \cdot \frac{\mathbb{P}\cancel{(\check{P})}}{\mathbb{P}\cancel{(\check{P})}} = \hat{P}$$

$$\Rightarrow \quad \hat{P} = \int \frac{\mathbb{P}(\check{P}|P)\ \mathbb{P}(P)}{P(\check{P})} P\, dP.$$

This is the optimal estimator for the bayesian (MMSE).

$\Rightarrow$ We can say that this formula permits to prove that the MMSE is the one that minimizes MSE.