Homework 1

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Exercice n 1: We have $p_n = P(X=n) = \frac{\lambda^n e^{-\lambda}}{n!}$

* The expectation: $E_x = \sum_{n\in\mathbb{N}} n p_n = \sum_{n\geq 1} \frac{n \lambda^n e^{-\lambda}}{n!}$

$$= \lambda e^{-\lambda} \sum_{n\geq 1} \frac{\lambda^{n-1}}{(n-1)!} = \lambda e^{-\lambda} \sum_{n\geq 0} \frac{\lambda^n}{n!}$$

$$\boxed{E_n = \lambda e^{-\lambda} e^{\lambda} = \lambda}$$

* The standard deviation:

$$\sigma^2(x) = \sum_{n\geq 1} n^2 p_n - (E_x)^2$$

$$= \sum_{n\geq 1} \frac{n^2 \lambda^n e^{-\lambda}}{n!} - \lambda^2$$

$$= \lambda e^{-\lambda} \sum_{n\geq 1} n \frac{\lambda^{n-1}}{(n-1)!} - \lambda^2$$

$$= \lambda e^{-\lambda} \left(\sum_{n\geq 0} (n+1) \frac{\lambda^n}{n!} \right) - \lambda^2$$

$$= \lambda e^{-\lambda} \left[\sum_{n\geq 0} \frac{n \lambda^n}{n!} + \sum_{n\geq 0} \frac{\lambda^n}{n!} \right] - \lambda^2$$

$$= \lambda e^{-\lambda} \left[\lambda \sum_{n\geq 1} \frac{\lambda^{n-1}}{(n-1)!} + \sum_{n\geq 0} \frac{\lambda^n}{n!} \right] - \lambda^2$$

$$= \lambda e^{-\lambda} \left[\lambda \sum_{n\geq 0} \frac{\lambda^n}{n!} + \sum_{n\geq 0} \frac{\lambda^n}{n!} \right] - \lambda^2$$

$$= \lambda e^{-\lambda} \left[\lambda e^{\lambda} + e^{\lambda} \right] - \lambda^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

$\Rightarrow$ Standard deviation: $\boxed{\sigma(x) = \sqrt{\lambda}}$

Exercice n 2:

We have * $(x_i)_{i=1,\dots,n}$ are independent $\sim P(\lambda_i)$

* $\lambda = \sum_{i=1}^{n} \lambda_i$ and $y = \sum_{i=1}^{n} y_i$

The aim of the exercice is to show that y is a poisson variable with parameter λ. To do this, let's take first $y = x_1 + x_2$ with $x_1 \sim P(\lambda_1)$

$x_2 \sim P(\lambda_2)$

Now, we have:

$$P(y=k) = \sum_{i=0}^{k} P(x_1 = i, x_2 = k-i) = P(x_1 + x_2 = k)$$

$$\overset{x_1 \text{ and } x_2 \text{ independent}}{=} \sum_{i=0}^{k} P(x_1 = i)\, P(x_2 = k-i)$$

$$= \sum_{i=0}^{k} \frac{\lambda_1^{i} e^{-\lambda_1}}{i!} \times \frac{\lambda_2^{k-i} e^{-\lambda_2}}{(k-i)!}$$

$$= \frac{e^{-\lambda_1} e^{-\lambda_2}}{k!} \sum_{i=0}^{k} \underbrace{\frac{k!}{i!(k-i)!}}_{= C_k^i} \lambda_1^{i} \lambda_2^{k-i}$$

$$= \frac{e^{-\lambda_1} e^{-\lambda_2}}{k!} (\lambda_1 + \lambda_2)^k$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} (\lambda_1 + \lambda_2)^k$$

Thus, $y \sim P(\lambda_1 + \lambda_2)$

We succeded to prove the result for two random variables.
Therefore, this result may be generalized recursively for $y = \sum_{i=1}^{n} x_i$ which show that $y \sim P\left(\sum_{i=1}^{n} \lambda_i\right)$.

We can conclude that $\boxed{y \sim P(\lambda) \text{ with } \lambda = \sum_{i=1}^{n} \lambda_i}$

Exercice n 3:

" The denoising procedure with the standard variance stabilizing transformation (VST) follows three steps:
1. apply VST to approximate homoscedasticity
2. Denoise the transformed data
3. Apply an inverse VST "

First, we have a noisy image $\hat{u} = u + g(u)\, n$

with $\begin{cases} n \sim N(0,1) \\ u \text{ is a noiseless image.} \end{cases}$

1. Applying VST:

Let's take f: a soft function.

we have $f(\hat{u}) \simeq f(u) + f'(u)\, g(u)\, n$

Using Taylor approximation

We have also : $g(u)^2 = Var(\check{u}) = u$
(assuming that we have a linear noise)
Thus, $g(u) = \sqrt{u}$.
Now, we want to take a real and smooth function f such that $f(\check{u})$ gets a uniform standard deviation independant of u.
The easy case is : $f(u) = a\sqrt{u}$; with a is a constant
$\Rightarrow f(\check{u}) \simeq f(u) + \frac{a}{2\cancel{\sqrt{u}}} \cancel{\sqrt{u}}\, n \simeq f(u) + \underbrace{\frac{a}{2} n}_{\text{the noise term}}$

(2) The denoising part consists in eliminating the noise part from the previous result and thus, we get :

$$\boxed{f(\check{u}) \simeq f(u)}$$

(3) Applying the inverse of VST ; which is not just an algebric inverse of the VST and must be optimed to avoid biais. $u^* = f^{-1}(f(\check{u})) \simeq f^{-1}(f(u)) \simeq u$.

Following these three steps, we found that we have denoised the noisy image using the standard Variance Stabilizing transformation.

Exercice n 5 :

The operator Dinf minimizing the mean square error (MSE)

$$D\,inf = \underset{D}{\arg\min}\; E\{\| U - D\check{U}\|^2\}$$

with $D\check{U} = \sum_{i=1}^{M} a(i) \langle \check{U}, G_i\rangle G_i$

" We want to show that : $\begin{cases} a(i) = \dfrac{|\langle U, G_i\rangle|^2}{|\langle U, G_i\rangle|^2 + \sigma^2} \\ E(\|U - D\check{U}\|^2) = \sum_{i=1}^{M} \dfrac{|\langle U, G_i\rangle|^2 \sigma^2}{|\langle U, G_i\rangle|^2 + \sigma^2} \end{cases}$"

We have : $D\,inf = \underset{D}{\arg\min}\; E\{\| U - D\check{U}\|^2\}$

$$E\{\|U - D\check{U}\|^2\} = E\{\| U - \sum_{i=1}^{M} a(i) \langle \check{U}, G_i\rangle G_i\|^2\}$$

$$= E\{\| \underbrace{\sum_{i=1}^{M} \langle U, G_i\rangle G_i}_{\{G_i\}\text{ orthonormal base}} - \sum_{i=1}^{M} a(i) \langle \check{U}, G_i\rangle G_i\|^2\}$$

$$= E\left\{ \left\| \sum_{i=1}^{M} \langle U, G_i\rangle G_i - a(i)\left[\langle U, G_i\rangle + \langle N, G_i\rangle\right] G_i \right\|^2 \right\}$$

$$= E\left\{ \left\| \sum_{i=1}^{M} \langle U - a(i)U - a(i)N, G_i\rangle G_i \right\|^2 \right\}$$

$\{G_i\}$ orthogonal basis

$$= E\left\{ \sum_{i=1}^{M} \langle U - a(i)U - a(i)N, G_i\rangle^2 \right\}$$

$$= E\left\{ \sum_{i=1}^{M} (1-a(i))^2 |\langle U, G_i\rangle|^2 + a(i)^2 |\langle N, G_i\rangle|^2 - 2\, a(i)(1-a(i)) \langle U, G_i\rangle\langle N, G_i\rangle \right\}$$

* We have $\begin{cases} E\left(|\langle N, G_i\rangle|^2\right) = \sigma^2 \\ E\left(\langle N, G_i\rangle\right) = 0 \end{cases}$

$$E\left(\left\| U - D\hat{U} \right\|^2\right) = \sum_{i=1}^{M} \Big[(1-a(i))^2 E\left(|\langle U, G_i\rangle|^2\right) + a(i)^2 E\big(\underbrace{|\langle N, G_i\rangle|^2}_{\sigma^2}\big) - 2a(i)(1-a(i))\, E\big(\underbrace{\langle U, G_i\rangle\langle N, G_i\rangle}_{=0}\big) \Big]$$

$$= \sum_{i=1}^{M} \left[(1-a(i))^2 |\langle U, G_i\rangle|^2 + a^2(i)\sigma^2 \right]$$

Minimizing the previous expression with respect to D

$\iff$ minimizing for each component of the function because D is considered as the sum of independent elements.

$\iff \forall\, i \in [\![1, M]\!]$

$$\frac{\partial E\left(\| U - D\hat{U} \|^2\right)}{\partial a(i)} = \frac{\partial\left(\sum_{i=1}^{M} (1-a(i))^2 |\langle U, G_i\rangle|^2 + a(i)^2\sigma^2 \right)}{\partial a(i)} = 0$$

$$\iff \cancel{2}a(i)\,\sigma^2 + \cancel{2}(a(i) - 1)\, |\langle U, G_i\rangle|^2 = 0$$

$$\iff a(i)\left[\sigma^2 + |\langle U, G_i\rangle|^2\right] = |\langle U, G_i\rangle|^2$$

$$\iff \boxed{a(i) = \frac{|\langle U, G_i\rangle|^2}{\sigma^2 + |\langle U, G_i\rangle|^2}}$$

The MSE:

$$E\left(\|U - D_{inf}\tilde{U}\|^2\right)$$

$$= \sum_{i=1}^{M} (1-a(i))^2 |\langle U, G_i\rangle|^2 + a(i)\,\sigma^2$$

$$= \sum_{i=1}^{M} \left[\left(1 - \frac{|\langle U, G_i\rangle|^2}{\sigma^2 + |\langle U, G_i\rangle|^2}\right)^2 |\langle U, G_i\rangle|^2 + \frac{\sigma^2 \left(|\langle U, G_i\rangle|^2\right)^2}{\left(\sigma^2 + |\langle U, G_i\rangle|^2\right)^2} \right]$$

$$= \sum_{i=1}^{M} \left(\frac{\sigma^2}{|\langle U, G_i\rangle|^2 + \sigma^2}\right)^2 |\langle U, G_i\rangle|^2 + \sigma^2 \left(\frac{|\langle U, G_i\rangle|^2}{\sigma^2 + |\langle U, G_i\rangle|^2}\right)^2$$

$$= \sum_{i=1}^{M} \frac{\sigma^4 |\langle U, G_i\rangle|^2 + \sigma^2 |\langle U, G_i\rangle|^4}{\left(|\langle U, G_i\rangle|^2 + \sigma^2\right)^2}$$

$$= \sum_{i=1}^{M} \frac{\cancel{\left(\sigma^2 + |\langle U, G_i\rangle|^2\right)} \left(\sigma^2 |\langle U, G_i\rangle|^2\right)}{\left(|\langle U, G_i\rangle|^2 + \sigma^2\right)^{\cancel{2}}}$$

$$= \sum_{i=1}^{M} \frac{\sigma^2 |\langle U, G_i\rangle|^2}{|\langle U, G_i\rangle|^2 + \sigma^2}$$

$$\boxed{E\left(\|U - D_{inf}\tilde{U}\|^2\right) = \sum_{i=1}^{M} \frac{\sigma^2 |\langle U, G_i\rangle|^2}{|\langle U, G_i\rangle|^2 + \sigma^2}}$$

Exercice n6:

" If we restricts $a(i) \in \{0, 1\}$

$\Rightarrow$ the projection operator that minimizes the MSE is obtained with $a(i) = \begin{cases} 1 & |\langle U, G_i\rangle| \geq c\sigma^2 \\ 0 & \text{otherwise} \end{cases}$

for some well chosen $c > 1$ the corresponding MSE satisfies $E\left(\|U - D_{inf}\tilde{U}\|^2\right) \leq \sum_i \min\left(|\langle U, G_i\rangle|^2, c\sigma^2\right)$"

and this inequality becomes an equality for $c = 1$.

We have the projector $a(i) = \begin{cases} 1 & \text{if } |\langle U, G_i\rangle|^2 \geq c\sigma^2 \\ 0 & \text{otherwise} \end{cases}$

* if $c > 1$

$\rightarrow |\langle U, G_i\rangle| \geq c\sigma^2 \Rightarrow a(i) = 1.$

$$\Rightarrow MSE = \sum_{i=1}^{n} \frac{\sigma^2 \, |\langle U, G_i\rangle|^2}{\sigma^2 + |\langle U, G_i\rangle|^2}$$

$$= \sum_{i=1}^{n} \sigma^2 a(i) = \sum_{i=1}^{n} \sigma^2 \leq \sum_{i=1}^{n} c\sigma^2$$

$$\Rightarrow MSE \leq \sum_i \min\{c\sigma^2, |\langle U, G_i\rangle|^2\}$$

because $|\langle U, G_i\rangle|^2 \geq c\sigma^2$

$\Rightarrow \min\{c\sigma^2, |\langle U, G_i\rangle|^2\} = c\sigma^2$

$\rightarrow |\langle U, G_i\rangle|^2 < c\sigma^2 \Rightarrow a(i) = 0$

$$MSE = \sum_{i=1}^{n} (1 - \underbrace{a(i)}_{=0})^2 |\langle U, G_i\rangle|^2 + \underbrace{a(i)^2}_{=0} \sigma^2$$

$$= \sum_{i=1}^{m} |\langle U, G_i\rangle|^2$$

$$= \sum_{i=1}^{n} \min\{|\langle U, G_i\rangle|^2, c\sigma^2\}$$

because $|\langle U, G_i\rangle|^2 < c\sigma^2$

$\Rightarrow \min\{|\langle U, G_i\rangle|^2, c\sigma^2\} = |\langle U, G_i\rangle|^2$

$$\Rightarrow MSE \leq \min\{|\langle U, G_i\rangle|^2, c\sigma^2\} \text{ for } c > 1$$

* if $c = 1$

$\rightarrow |\langle U, G_i\rangle|^2 \geq \sigma^2 \Rightarrow a(i) = 1$

$$MSE = \sum_{i=1}^{n} (1 - \underbrace{a(i)}_{=1})^2 |\langle U, G_i\rangle|^2 + \underbrace{a(i)^2}_{=1} \sigma^2$$

$$= \sum_{i=1}^{n} \sigma^2 = \sum_{i=1}^{n} \min\{\sigma^2, |\langle U, G_i\rangle|^2\}$$

$\rightarrow |\langle U, G_i\rangle|^2 < \sigma^2 \Rightarrow a(i) = 0$

$$MSE = \sum_{i=1}^{n} (1 - \underbrace{a(i)}_{0})^2 |\langle U, G_i\rangle|^2 + \underbrace{a(i)^2}_{0} \sigma^2$$

$$= \sum_{i=1}^{n} |\langle U, G_i\rangle|^2 = \sum_{i=1}^{M} \min\{\sigma^2, |\langle U, G_i\rangle|^2\}$$

$\Rightarrow \quad MSE = \sum_i \min\{\sigma_i^2, |\langle U, G_i\rangle|^2\}$

To conclude: given the projector $a(i)$; $MSE \leq \sum_i \min\{c\sigma_i^2, |\langle U, G_i\rangle|^2\}$ with equality if $c = 1$.

Exercice n°7: "DCT and IDCT are isometries in $\mathbb{R}^N$ and inverse of each other."

* DCT : for $0 \leq k \leq N-1$; for $0 \leq j \leq N-1$

$$Y_k = 2\alpha_k \sum_{j=0}^{N-1} X_j \cos\left(\pi\left(j+\tfrac{1}{2}\right)\tfrac{k}{N}\right)$$

with $\alpha_k = \begin{cases} \sqrt{\frac{1}{4N}} & k = 0 \\ \sqrt{\frac{1}{2N}} & k = 1, \dots, N-1 \end{cases}$

DCT: $X \longmapsto AX$ $\begin{cases} \text{with } A_{ij} = 2\alpha_i \cos\left(\pi\left(j+\frac{1}{2}\right)\frac{i}{N}\right) \\ \text{with } \alpha_i = \begin{cases} \sqrt{\frac{1}{4N}} & i=0 \\ \sqrt{\frac{1}{2N}} & 1 \leq i \leq N-1 \end{cases} \end{cases}$

The DCT is a linear transformation in function of X.
Now, if A is an orthogonal matrix, this transformation will act as an isometry.

In other words: DCT is an isometry $\Leftrightarrow A^T A = Id$.

$$\Rightarrow (A^TA)_{ij} = \sum_{k=0}^{N-1} (A^T)_{ik} A_{kj}$$

$$= \sum_{k=0}^{N-1} A_{ki} A_{kj}$$

$$= \sum_{k=0}^{N-1} 2\alpha_k \cos\left(\pi\left(i+\tfrac{1}{2}\right)\tfrac{k}{N}\right) \times 2\alpha_k \cos\left(\pi\left(j+\tfrac{1}{2}\right)\tfrac{k}{N}\right)$$

$$= 4\sum_{k=0}^{N-1} \alpha_k^2 \cos\left(\pi\left(i+\tfrac{1}{2}\right)\tfrac{k}{N}\right) \cos\left(\pi\left(j+\tfrac{1}{2}\right)\tfrac{k}{N}\right)$$

$$= 4\sum_{k=0}^{N-1} \frac{\alpha_k^2}{2} \left[\cos\left(\tfrac{\pi k}{N}(i+j+1)\right) + \cos\left(\tfrac{\pi k}{N}(i-j)\right)\right]$$

$$= 2\sum_{k=0}^{N-1}\alpha_k^2\left[\cos\left(\frac{\pi k}{N}(i+j+1)\right)+\cos\left(\frac{\pi k}{N}(i-j)\right)\right]$$

$$= 2\sum_{k=0}^{N-1}\alpha_k^2\cos\left(\frac{\pi k}{N}(i+j+1)\right)+2\sum_{k=0}^{N-1}\alpha_k^2\cos\left(\frac{\pi k}{N}(i-j)\right)$$

$$= 2\sum_{k=0}^{N-1}\alpha_k^2\,\mathrm{Re}\left(e^{i\pi\frac{k}{N}(i+j+1)}\right)+2\sum_{k=0}\alpha_k^2\,\mathrm{Re}\left(e^{i\frac{\pi k}{N}(i-j)}\right)$$

$$= 2\,\mathrm{Re}\left(\sum_{k=0}^{N-1}\alpha_k^2\left[e^{i\frac{\pi k}{N}(i+j+1)}+e^{i\pi\frac{k}{N}(i-j)}\right]\right)$$

$$= 2\,\mathrm{Re}\left(2\alpha_0^2+\sum_{k=1}^{N-1}\alpha_k^2\left[e^{i\frac{\pi k}{N}(i+j+1)}+e^{i\frac{\pi k}{N}(i-j)}\right]\right)$$

$$= 2\,\mathrm{Re}\left(\frac{1}{4N}\times 2+\frac{1}{2N}\sum_{k=1}^{N-1}e^{i\frac{\pi k}{N}(i+j+1)}+e^{i\frac{\pi k}{N}(i-j)}\right)$$

$$= \mathrm{Re}\left(\frac{1}{N}+\frac{1}{N}\sum_{k=1}^{N-1}e^{i\pi\frac{k}{N}(i+j+1)}+e^{i\pi\frac{k}{N}(i-j)}\right)$$

for $i\neq j$

$$(A^TA)_{ij}=\frac{1}{N}\mathrm{Re}\left(1+\frac{e^{\frac{i\pi}{N}(i+j+1)}-e^{i\pi(i+j+1)}}{1-e^{i\frac{\pi}{N}(i+j+1)}}+\frac{e^{\frac{i\pi(i-j)}{N}}-e^{i\pi(i-j)}}{1-e^{i\pi\frac{(i-j)}{N}}}\right)$$

we have here two cases:

(1) if $i+j+1$ is pair $\Rightarrow$ $i+j$ is impair

$\Rightarrow$ i or j is impair

$\Rightarrow$ $i-j$ is impair

(2) if $i+j+1$ is impair $\Rightarrow$ $i+j$ is pair

$\Rightarrow \begin{cases} i \text{ pair}, j \text{ pair} \\ i \text{ impair}, j \text{ impair}\end{cases}$

$\Rightarrow$ $i-j$ is pair

$*$: ~~$i+j+1$ is impair~~ $\begin{cases} e^{i\pi(i+j+1)} = -1 \\ e^{i\pi(i-j)} = +1 \end{cases}$

$$\Rightarrow (A^TA)_{ij}=\frac{1}{N}\mathrm{Re}\left(\cancel{1}+\frac{e^{\frac{i\pi(i+j+1)}{N}}+1}{1-e^{\frac{i\pi}{N}(i+j+1)}}-\cancel{1}\right)$$

$$=\frac{1}{N}\mathrm{Re}\left(\frac{\cancel{e^{i\frac{(i+j+1)}{2N}\pi}}\left(e^{\frac{i\pi(i+j+1)}{2N}}+e^{\frac{-i\pi(i+j+1)}{2N}}\right)}{\cancel{e^{i\frac{(i+j+1)\pi}{2N}}}\left(e^{\frac{i\pi(i+j+1)}{2N}}-e^{\frac{-i\pi(i+j+1)}{2N}}\right)}\right.$$

$$= \frac{1}{N} \, Re \left(\frac{\overbrace{2\cos\left(\frac{\pi}{2N}(i+j+1)\right)}^{\text{imaginair term}}}{2i \sin\left(\frac{\pi}{2N}(i+j+1)\right)} \right) = 0$$

$\times$ if $i+j+1$ is pair : $\begin{cases} e^{i(i+j+1)} = 1 \\ e^{i(i-j)} = -1 \end{cases}$

$$(A^TA)_{ij} = \frac{1}{N} Re\left(\cancel{1} - \cancel{1} + \frac{e^{i\frac{\pi}{N}(i-j)} + 1}{1 - e^{i\frac{\pi}{N}(i-j)}} \right)$$

$$= \frac{1}{N} \quad Re \left(\frac{e^{\frac{i\pi}{2N}(i-j)}}{e^{\frac{i\pi}{2N}(i-j)}} \; \frac{e^{\frac{i\pi(i-j)}{2N}} + e^{-\frac{i\pi(i-j)}{2N}}}{e^{\frac{i\pi(i-j)}{2N}} - e^{-\frac{i\pi(i-j)}{2N}}} \right)$$

$$= \frac{1}{N} \quad Re \left(\frac{2\cos\left(\frac{\pi}{2N}(i-j)\right)}{2i \sin\left(\frac{\pi}{2N}(i-j)\right)} \right) = 0$$

$\Rightarrow \forall \; i \neq j \quad \Rightarrow \quad (A^TA)_{ij} = 0$

Now, for $i = j$:

$$(A^TA)_{ij} = 2 \; Re \left(\sum_{k=0}^{N-1} \alpha_k^2 \left[e^{i\pi\frac{k}{N}(2j+1)} + 1 \right] \right)$$

$$= 2 \; Re \left(\frac{1}{4N} \times 2 + \frac{1}{2N} \left[\sum_{k=1}^{N-1} e^{\frac{i\pi k}{N}(2j+1)} + N-1 \right] \right)$$

$$= 2 \; Re \left(\cancel{\frac{1}{2N}} - \cancel{\frac{1}{2N}} + \frac{1}{2} + \frac{1}{2N} \sum_{k=1}^{N-1} e^{\frac{i\pi k}{N}(2j+1)} \right)$$

$$= Re \left(1 + \frac{1}{N} \sum_{k=1}^{N-1} e^{\frac{i\pi k}{N}(2j+1)} \right)$$

$$= 1 + \frac{1}{N} Re \left(\frac{e^{\frac{i\pi}{N}(2j+1)} \left(1 - e^{i\pi/N (N-1)(2j+1)}\right)}{1 - e^{\frac{i\pi}{N}(2j+1)}} \right)$$

$$= 1 + \frac{1}{N} Re \left(\frac{e^{\frac{i\pi}{N}(2j+1)} - e^{i\pi(2j+1)}}{1 - e^{\frac{i\pi}{N}(2j+1)}} \right)$$

$$= 1 + \frac{1}{N} \mathrm{Re}\left(\frac{e^{i\pi\frac{(2j'+1)}{N}} + 1}{1 - e^{i\pi\frac{(2j'+1)}{N}}} \right)$$

$$= 1 + \frac{1}{N} \mathrm{Re}\left(\frac{\cancel{e^{i\pi\frac{(2j'+1)}{2N}}}}{\cancel{e^{i\pi\frac{(2j'+1)}{2N}}}} \; \frac{e^{i\pi\frac{(2j'+1)}{2N}} + e^{-i\pi\frac{(2j'+1)}{2N}}}{e^{i\pi\frac{(2j'+1)}{2N}} - e^{-i\pi\frac{(2j'+1)}{2N}}} \right)$$

$$= 1 + \frac{1}{N} \mathrm{Re}\left(\frac{2\cos\left(\frac{\pi(2j+1)}{2N}\right)}{2i\sin\left(\frac{\pi(2j+1)}{2N}\right)} \right)$$

$$= 1 + \frac{1}{N} \times 0 = 1$$

Conclusion $A^T A = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

$$= I.$$

Thus A is an orthogonal matrix since $A^T A = I$ and therefore $f = DCT$ is an is isometry.

* Now, let's move to the IDCT:

for $k = 0, \dots N-1$

$j = 0, \dots, N-1$

$$IDCT = \left\{ X_j = \beta_0 Y_0 + \sum_{k=1}^{N-1} \beta_k 2 Y_k \cos\left(\pi\left(j+\tfrac{1}{2}\right)\frac{k}{N}\right)\right.$$

with $\beta_k = \begin{cases} \sqrt{\frac{1}{N}} & k = 0 \\ \sqrt{\frac{1}{2N}} & k \neq 0 \end{cases}$

$\Rightarrow$ IDCT : $Y \longmapsto BY$

$\Bigg\{$ with $B_{ij} = 2\beta_j \cos\left(\pi\left(i+\tfrac{1}{2}\right)\frac{j}{N}\right)$

with $\beta_j = \begin{cases} \sqrt{\frac{1}{4N}} & j = 0 \\ \sqrt{\frac{1}{2N}} & j \neq 0 \end{cases}$

we have $A_{ij} = d\, \alpha_i \cos\left(\pi \left(j+\frac{1}{2}\right) \frac{i}{N}\right)$

and $B_{ij} = d\, \beta_j \cos\left(\pi \left(i+\frac{1}{2}\right) \frac{j}{N}\right)$

we have $\beta_j = \alpha_i \Rightarrow \left.\begin{array}{l} A_{ij} = B_{ji} \\ B_{ij} = A_{ji} \end{array}\right\} \Rightarrow B = A^T$

$\Rightarrow$ IDCT $= Y \longmapsto A^T Y$ is an isometry

because $(A^T)(A^T)^T = A^T A = Id$.

In addition, the two matrices are transposed of each other then, the IDCT is an inverse transform of the DCT.

$\Rightarrow$ DCT and the IDCT are isometries on $\mathbb{R}^N$ and inverse of each other (DCT: $X \longmapsto AX$; IDCT : $Y \mapsto BY$ with $B = A^T$)

Exercice n 8 :

Let's have the problem : $\begin{cases} \text{Min} \sum_k \alpha_k^2 E\left(\left(P_k - E(P_k)\right)^2\right) \\ \text{st} \quad \sum_k \alpha_k = 1 \\ \quad\quad \alpha_k \geq 0 \quad \forall k \end{cases}$

we want demontrate that this problem implies the existence of some $\lambda \in \mathbb{R}$ such that : $2\alpha_k \sigma_k^2 = \lambda \quad \forall k$

* we suppose $\begin{cases} f(\alpha) = \sum_k \alpha_k^2 \sigma_k^2 & \text{which is a convex function of } \alpha. \\ g(\alpha) = 1 - \sum_k \alpha_k & \text{which is an affin function} \end{cases}$

$C = \{ \alpha \in \mathbb{R}^n ; \sum_k \alpha_k = 1 \}$ is a convex and compact set

$\Rightarrow$ f is a convex and lower semi continuous on C which is a convex and a compact set

$\Rightarrow$ the optimization problem is convex and feasible

$\Rightarrow \exists\, \hat{\alpha} \in C \;/\; f(\hat{\alpha}) = \inf f(\alpha)$.

Now, we can write the Lagrangian :

$$\mathcal{L}(\alpha, \lambda) = \sum \alpha_k^2 \sigma_k^2 + \lambda\left(1 - \sum \alpha_k\right)$$

Using KKT conditions: $(\hat{\alpha}, \hat{\lambda})$ is a saddle point

$$\iff \nabla \mathcal{L}(\hat{\alpha}, \hat{\lambda}) = 0$$

$$\iff \forall k \quad \frac{\partial \mathcal{L}}{\partial \alpha_k} = 0$$

$$\iff 2 \alpha_k \sigma'^2_k - \lambda = 0 \quad \forall k$$

$$\iff \boxed{\hat{\lambda} = 2 \alpha_k \sigma'^2_k \quad \forall k}$$

$\Rightarrow$ The constrained optimization problem implies the existence of $\lambda \in \mathbb{R}$ such that $2 \alpha_k \sigma'^2_k = \lambda \ \forall k$.

Exercise n 9:

a_k : coeffient of thresholding.

Under the hard thresholding: $a_k \in \{0,1\}$ and N_{P_k} is the number of non null values.

Parseval theorem: for a patch X_k: $Var(X_k) = E\left(\left(X_k - E(X_k)\right)^2\right)$

$$= \sum_k \sigma'^2 a_k^2$$

* we have

$$\begin{cases} \alpha_k = \dfrac{\sigma'^{-2}_k}{\sum_k \sigma'^{-2}_k} \\ \\ \sigma'^2_k = \sum_k \sigma'^2 a_k^2 = N_{P_k} \sigma'^2 \end{cases}$$

$$\Rightarrow \alpha_k = \frac{\cancel{\sigma'^{-2}} N_{P_k}^{-2}}{\sum \cancel{\sigma'^{-2}} N_{P_k}^{-2}} = \frac{N_{P_k}^{-2}}{\sum_k N_{P_k}^{-2}} \quad (4.22)$$

* Now, the coeffients are given by winer coeffient:

for a patch X_k; we have $a_k = e_{P_k}$.

$$\Rightarrow \sigma'^2_k = \sigma'^2 \sum_{k=1}^{N} e_{P_k}^2 = \sigma'^2 \| e_{P_k} \|^2$$

$$\Rightarrow \alpha_k = \frac{\sigma'^{-2}_k}{\sum_{k=1}^{N} \sigma'^{-2}_k} = \frac{\cancel{\sigma'^{-2}} \| e_{P_k} \|^{-2}}{\cancel{\sigma'^{-2}} \sum_{k=1}^{N} \| e_{P_k} \|^{-2}}$$

$$\boxed{\alpha_k = \frac{\| e_{P_k} \|^{-2}}{\sum_{k=1}^{N} \| e_{P_k} \|^{-2}}} \quad (4.23)$$