Image Denoising - Part 2 - Homework 3

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Exercise 2.1:

The gradient descent is an iterative minimization method with the following update equation:

$$u^{t+1} = u^t - \eta \ \ E_\theta^{FoE}(u^t, v)$$

The gradient descent of E_θ^{FoE} is calculated as follow:

$$E_\theta^{FoE}(u_t, v) = \frac{1}{2\sigma^2}||u_t - v||^2 + \sum_{x\in\Omega}\sum_{i=1}^{N}\phi_i(k_i * u_t(x)) + constants$$

$$\nabla E_\theta^{FoE}(u_t, v) = \frac{1}{\sigma^2}(u_t - v) + \sum_{x\in\Omega}\sum_{i=1}^{N}\nabla(k_i * u_t(x))\phi_i^{'}(k_i * u_t(x))$$

We have: $k_i * u_t(x) = \sum_{j=1}^{d} u_t(x-j)k_i(j)$ with $k_i \in \mathbb{R}^d$.

$$\Longrightarrow \nabla_{u_t}(k_i * u_t(x)) = \begin{pmatrix} 0 \\ . \\ . \\ 0 \\ k_i(d) \\ . \\ . \\ k_i(1) \\ 0 \\ . \\ . \\ 0 \end{pmatrix} \text{ this vector has the same size as the vector } u_t.$$

$$\Longrightarrow \nabla E_\theta^{FoE}(u_t, v) = \frac{1}{\sigma^2}(u_t - v) + \sum_{i=1}^{N}\sum_{x\in\Omega}\begin{pmatrix} k_i(d) \\ . \\ . \\ k_i(1) \end{pmatrix}\phi_i^{'}(k_i * u_t)(x)$$

$$= \frac{1}{\sigma^2}(u_t - v) + \sum_{i=1}^{N}\bar{k_i} * \phi_i^{'}(k_i * u_t)$$

Where:

- $\bar{k_i}$ is a π-rotation of the convolution kernel k_i.
- $\phi^{'}(u)$, for an image u, denotes the image resulting from applying $\phi^{'}$ to every pixel u

  i.e. $\phi^{'}(u)(x) = \phi^{'}(u(x))$

Conclusion:

$$u^{t+1} = u^t - \eta \ \ (\frac{1}{\sigma^2}(u^t - v) + \sum_{i=1}^{N} \bar{k_i} * \phi_i^{'}(k_i * u^t))$$

Exercise 2.2:

We assume that the data is distributed according to p(u). The limit of the log-likelihood when $m \to \infty$:

$$L_\infty(\theta) = \mathbb{E}\{log\ (p_\theta(u))\} = \int log\ (p_\theta(u))p(u)du = \lim_{m\to\infty} \ \frac{1}{m}\sum_{j=1}^{m} log\ p_\theta(u_j))$$

$$\underset{\theta}{\text{Max}}\ L_\infty(\theta) \Longleftrightarrow \underset{\theta}{\text{Max}}\ \int log\ (p_\theta(u))p(u)du$$

$$\Longleftrightarrow \underset{\theta}{\text{Min}}\ -\int log\ (p_\theta(u))p(u)du$$

$$\Longleftrightarrow \underset{\theta}{\text{Min}}\ \{-\int log\ (p_\theta(u))p(u)du + \int log\ (p(u))p(u)du\}$$

the second term is independent from θ

$$\Longleftrightarrow \underset{\theta}{\text{Min}}\ \int log\ (\frac{p(u)}{p_\theta(u)})p(u)du$$

$$\Longleftrightarrow \underset{\theta}{\text{Min}}\ KL(p(u)||p_\theta(u))$$