Consider the linear mapping

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^4$$

$$\Phi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix} \tag{1}$$

$$x_i = \mu_{ij} b_1 + \mu_{ij} b_2 + \mu_{ij} b_3$$

- Find the transformation matrix A_Φ

By definition this is the matrix mapping the basis vectors in $\mathbb{R}^3$ to $\mathbb{R}^4$

$$\Phi(b_i) = \lambda_{i,1} c_1 + \lambda_{i,2} c_2 + \lambda_{i,3} c_3 + \lambda_{i,4} c_4 \tag{2}$$

note that $\dim(\Phi(b_i)) = 4$

we know a canonical basis for $\mathbb{R}^3$

$$b_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, b_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

so we know $\mu_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mu_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mu_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

so $b_i = x_i$

$$\Phi\left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}\right) = \begin{bmatrix} 3b_1 + 2b_2 + b_3 \\ b_1 + b_2 + b_3 \\ b_1 - 3b_2 \\ 2b_1 + 3b_2 + b_3 \end{bmatrix} \tag{4}$$

through inspection we find the transformation matrix

$$A_\Phi = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix} \tag{5}$$

- Determine $\mathrm{rk}(A_\Phi)$

we can use the rank nullity theorem to solve this

$$\dim(\mathrm{im}(\Phi)) = \dim(\mathbb{R}^4) - \dim(\ker(\Phi)) \tag{6}$$

to find the kernel we can see apply gaussian elimination and check in the row eschelon form if a basis vector can be writtern as linear combination of others

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}, -2R_1 + R_3 \tag{7}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 0 & 1 & -1 \end{bmatrix}, -3R_1 + R_0 \tag{8}$$

$$\begin{bmatrix} 0 & -1 & -2 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 0 & 1 & -1 \end{bmatrix}, -1R_1 + R_2 \tag{9}$$

$$\begin{bmatrix} 0 & -1 & -2 \\ 1 & 1 & 1 \\ 0 & -4 & -1 \\ 0 & 1 & -1 \end{bmatrix}, R_1 \text{ switch } R_0 \tag{10}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -4 & -1 \\ 0 & 1 & -1 \end{bmatrix}, R_1 + R_3 \tag{11}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -4 & -1 \\ 0 & 0 & -3 \end{bmatrix}, R_1 + R_3 \tag{12}$$

thus the $\dim(\ker(\Phi)) = 1$ the $\mathbf{0}_{\mathbb{R}^3}$ is the only element present, we conclude

$$\dim(\operatorname{im}(\Phi)) = 3 \tag{13}$$

and that indeed true as we have 3 basis vectors.

- Compute the kernel and image of Φ. What are $\dim(\ker(\Phi))$ and $\dim(\operatorname{im}(\Phi))$?

$$\ker(\Phi) = \{\mathbf{0}\} \tag{14}$$

$$\operatorname{im}(\Phi) = \left\{ \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ -1 \\ -4 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ -2 \\ -1 \\ -3 \end{bmatrix} \right\} \tag{15}$$

the dimensions are 0 and 3 respectively