Let E be vector space. Let f and g be two automorphisms on E such that $f \circ g = \mathrm{id}_E$. Show that $\ker(f) = \ker(g \circ f), \mathrm{im}(g) = \mathrm{im}(g \circ f)$ and that $\ker(f) \cap \mathrm{im}(g) = \{O_E\}$

remember that automorphisms are isomorphisms in the same structure, in this case the vector space E.

this implies $f \circ f^{-1} = \mathrm{id}_E$ and $g \circ g^{-1} = \mathrm{id}_E$ and we also know $f \circ g = \mathrm{id}_E$

suppose $\ker(f) = X$ where X is the set of vectors in E that map to 0_E after applying f. now consider $f^{-1} \circ f = \mathrm{Id}_E$ we know that $\ker(\mathrm{Id}_E) = \{0_E\}$ by definition of identity, this implies $\ker(f^{-1} \circ f) = \{O_E\}$.

Now consider $f \circ f^{-1}$, this is valid due to being an automorphism and as such a isomorphism, then

$$\ker(f^{-1}) = Y \tag{1}$$

and $\ker(f^{-1} \circ f) = \ker(\mathrm{Id}_E) = \{0_E\}$

now suppose $|Y| > 1$, because f has an inverse, thus a bijection, it must be the case that $f^{-1} \circ f \Rightarrow f^{-1} \circ f(x_i) = f^{-1} \circ f(x_j), x_i = x_j$ and as we only have a single element in the kernel of the composition it must be the case that only a single vector can be in the $\ker(f^{-1} \circ f) = \{0_E\}$, so $|Y| = 1$. can't be zero because the kernel subspace is never empty.

we can apply analagous reasoning to prove $|X| = 1$, thus we conclude that $\dim(\ker(f)) = \dim(\ker(f^{-1})) \Rightarrow \dim(\mathrm{im}(f)) = \dim(\mathrm{im}(f^{-1}))$

suppose the vector is not zero, then $\ker(f) = \{v \neq 0_E\}$, then $f(v) = 0$, now consider $\ker(f^{-1}) = \{w \neq 0_E\})$ and $f^{-1}(w) = 0$

but by linearity $f(v + w) = f(v) + f(w) \land f^{-1}(v + w) = f^{-1}(v) + f^{-1}(w)$

$$f^{-1} \circ f(v + w) = (v + w) \Rightarrow f^{-1}(f(w + 0)) = (v + w) \Rightarrow w = (v + w) \tag{2}$$

that's only possible if $v = w = 0_E$

so it must be the case that $\ker(f) = \ker(f^{-1}) = \{0_E\}$

now we also know that $\ker(g \circ f) = \ker(\mathrm{id}_E) = \{0_E\} \Rightarrow \ker(f) = \ker(g \circ f)$

to prove $\mathrm{im}(g) = \mathrm{im}(g \circ f)$, we notice that g is bijective, thus surjective that implies the following

$$\mathrm{im}(g) = E \tag{3}$$

, remeber that g signature is $g : E \to E$ as an automorphism.

we also know $g \circ f(v) = E = \mathrm{id}_E \Rightarrow \mathrm{im}(g) = \mathrm{im}(g \circ f)$

the final proof is rather easy remember that $\ker(g) \subset \mathrm{im}(g)$, just repeat the same procedure applied to f to find that $\ker(g) = \ker(g \circ f) = \ker(f)$, so by definition of intersection $\mathrm{im}(g) \cap \ker(f) = \{0_E\}$