Consider an endomorphism $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ whose transformation matrix (with respect to the standard basis in $\mathbb{R}^3$ is)

$$\boldsymbol{A_\Phi} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \tag{1}$$

- Determine $\ker(\Phi)$ and $\operatorname{im}(\Phi)$

let's apply gaussian elimination to transform into a row eschelon form matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, -R_1 + R_2 \tag{2}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, -R_1 + R_0 \tag{3}$$

$$\begin{bmatrix} 0 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, R_1 \text{ switch } R_0 \tag{4}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}, -R_1 + R_2 \tag{5}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, -R_1 + R_2 \tag{6}$$

we can see that no linear dependency is present, meaning the dimension of such matrix is 3, so the $\operatorname{im}(\Phi)$ is

$$\operatorname{im}(\Phi) = \left\{ \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \lambda_i \in \mathbb{R} \right\} \tag{7}$$

$$\ker(\Phi) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \tag{8}$$

- Determine the transformation matrix $A_{\tilde{\Phi}}$ with respect to the basis

$$B = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \tag{9}$$

we know the following commutative diagram

$$\begin{tikzcd} B \arrow[r, "A_T"] \arrow[d, "A_{\tilde{\Phi}}"] & \mathbb{R}^3_1 \arrow[dl, "A_\Phi"] \\ \mathbb{R}^3_2 & \end{tikzcd}$$

to find T we just need to consider the transformation matrix from B to $\mathbb{R}^3$, where the basis vector in $\mathbb{R}^3$ are the canonical ones.

$b_{i_B} = \lambda_{1_j} b_x + \lambda_{2_j} b_y + \lambda_{3_j} b_z \Rightarrow \lambda_i = \left[b_{i_x}, b_{i_y}, b_{i_z}\right]$

so $A_T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

and the final answer is $A_{\tilde{\Phi}} = A^T A_\Phi = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix}$