Consider $\mathbb{R}^2$ with $< \cdot, \cdot >$ defined for all $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^2$ as

$$< \boldsymbol{x}, \boldsymbol{y} > := \boldsymbol{x}^T \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \boldsymbol{y}$$

Is $< \cdot, \cdot >$ an inner product?

Again we check for the properties: bilinearity, positive definiteness and symmetry.

$$[x_1, x_2] \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \boldsymbol{y} = [2x_1 + x_2 \;\; 2x_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 2x_1 y_1 + x_2 y_1 + 2x_2 y_2$$

verifying the positive definiteness, we have

$$< \boldsymbol{x}, \boldsymbol{x} > = 2x_1^2 + x_2 x_1 + 2x_2^2 = 0$$

$$x_1^2 + \frac{x_2 x_1}{2} + x_2^2 = 0$$

$$(x_1 + x_2)^2 + \frac{x_2 x_1}{2} = 0$$

$$|x_1 + x_2| = |\frac{x_2 x_1}{2}|$$

$$|1 + \frac{x_2}{x_1}| = |\frac{x_2}{2}|$$

$$|\frac{1}{x_2} + \frac{1}{x_1}| = |\frac{1}{2}|$$

and we can see that $x_2 = 4, x_1 = -2$ satisfy the requirement, thus it must be the case that is not positive definite and as such not a valid inner product.