Consider the set $\mathcal{G}$ of 3×3 matrices defined as follows:

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3} \,\middle|\, x, y, z \in \mathbb{R} \right\}$$

We define $\cdot$ as the standard matrix multiplication. Is $(\mathcal{G}, \cdot)$ a group? If yes, is it Abelian? Justify your answer.

We now prove is a group

a group must hold the following properties

- identity property
- inverse element
- closure
- associativity

we now prove identity element, $a * e = a$

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & a + x & (b + xc + z) \\ 0 & 1 & c + y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{c} \bullet \quad \text{a + x = x} \\ \bullet \ \text{b + x c +z = z} \\ \bullet \quad \text{c + y = y} \end{array}$$

$$a = 0, c = 0, b = 0$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so the identity property holds

now we prove inverse property, $a * a^{-1} = e$

$$\begin{bmatrix}1 & x & z\\0 & 1 & y\\0 & 0 & 1\end{bmatrix} \cdot \begin{bmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & a+x & (b+xc+z)\\0 & 1 & c+y\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

- a + x = 0
- b + x c +z = 0
- c + y = 0

$$a = -x, c = -y, b = -z + xy$$

$$\begin{bmatrix}1 & -x & -z+xy\\0 & 1 & -y\\0 & 0 & 1\end{bmatrix}$$

so the inverse property holds

now we prove the closure property, $a * b \in \mathbb{R}^{3\times 3}$

$$\begin{bmatrix}1 & x & z\\0 & 1 & y\\0 & 0 & 1\end{bmatrix} \cdot \begin{bmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & a+x & (b+xc+z)\\0 & 1 & c+y\\0 & 0 & 1\end{bmatrix}$$

- a + x in $\mathbb{R}$
- b + x c +z in $\mathbb{R}$
- c + y in $\mathbb{R}$

so the closure property holds

now we prove associativity, $a * (b * c) = (a * b) * c$

$$a * (b * c)$$

$$\begin{bmatrix}1 & x & z\\0 & 1 & y\\0 & 0 & 1\end{bmatrix} \cdot \left(\begin{bmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{bmatrix} \cdot \begin{bmatrix}1 & e & f\\0 & 1 & g\\0 & 0 & 1\end{bmatrix}\right)$$

$$\begin{bmatrix}1 & x & z\\0 & 1 & y\\0 & 0 & 1\end{bmatrix} \cdot \begin{bmatrix}1 & a+e & (b+ag+f)\\0 & 1 & g+c\\0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & a+e+x & ((b+ag+f)+x(g+c)+z)\\0 & 1 & g+c+y\\0 & 0 & 1\end{bmatrix}$$

$$(a * b) * c$$

$$\left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \begin{bmatrix} 1 & e & f \\ 0 & 1 & g \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & a + x & b + xc + z \\ 0 & 1 & c + y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & e & f \\ 0 & 1 & g \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & e + a + x & (f + g(a + x) + b + xc + z) \\ 0 & 1 & g + c + y \\ 0 & 0 & 1 \end{bmatrix}$$

now let's analyze the $a_{1,2}$ element of both matrices

$$((b + ag + f) + x(g + c) + z) = b + ag + f + xg + xc + z \text{ , matrix 1}$$

$$(f + g(a + x) + b + xc + z) = f + ga + gx + b + xc + z$$

note that considering commutativity of multiplication and addition both values are the same, so associativity also holds.

now we check if the group is abelian, $a * b = b * a$

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & x + a & b + xc + z \\ 0 & 1 & c + y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & a + x & z + ay + b \\ 0 & 1 & c + y \\ 0 & 0 & 1 \end{bmatrix}$$

now we compare $a_{1,2}$ again

$$b + xc + z = z + ay + b$$

that would require $ay = xc$, so is not abelian.