Using Gaussian elimination, find the set $\mathcal{S}$ of all solutions of the following inhomogeneous linear systems $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where **A** and **b** are defined as follows:

**a**

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 5 & -7 & -5 \\ 2 & -1 & 1 & 3 \\ 5 & 2 & -4 & 2 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 6 \end{bmatrix} \tag{1}$$

first we provide the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 2 & -1 & 1 & 3 & 4 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right] \tag{2}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 2 & -1 & 1 & 3 & 4 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right] \leftarrow -5R_0 + R_3 \tag{3}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 2 & -1 & 1 & 3 & 4 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right] \leftarrow -R_1 + R_2 \tag{4}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 0 & -6 & 8 & 8 & 7 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right] \leftarrow -R_1 + R_2 \tag{5}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & -6 & 8 & 8 & 7 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right] \leftarrow -2R_0 + R_1 \tag{6}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & -6 & 8 & 8 & 7 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right] \leftarrow R_2 \text{ switch } R_1 \tag{7}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & -6 & 8 & 8 & 7 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & -4 & 4 & -3 \end{array}\right] \leftarrow R_2 + R_3 \tag{8}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & -6 & 8 & 8 & 7 \\ 0 & 0 & -4 & 4 & -3 \end{array}\right] \leftarrow R_2 \text{ switch } R_1 \tag{9}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 0 & -4 & 4 & -3 \end{array}\right] \leftarrow 2R_1 + R_2 \tag{10}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right] \leftarrow -2R_1 + R_3 \tag{11}$$

note we can't solve this system of equations because is inconsistent, last row is impossible

**b**

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -3 & 0 \\ 2 & -1 & 0 & 1 & -1 \\ -1 & 2 & 0 & -2 & -1 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 3 \\ 6 \\ 5 \\ -1 \end{bmatrix} \tag{12}$$

again we provide the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -3 & 0 & 6 \\ 2 & -1 & 0 & 1 & -1 & 5 \\ -1 & 2 & 0 & -2 & -1 & -1 \end{array}\right] \tag{13}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -3 & 0 & 6 \\ 2 & -1 & 0 & 1 & -1 & 5 \\ -1 & 2 & 0 & -2 & -1 & -1 \end{array}\right] \leftarrow R_2 + R_3 \tag{14}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -3 & 0 & 6 \\ 2 & -1 & 0 & 1 & -1 & 5 \\ 0 & 3 & 0 & -5 & -1 & 5 \end{array}\right] \leftarrow -2R_1 + R_2 \tag{15}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -3 & 0 & 6 \\ 0 & -3 & 0 & 7 & -1 & -7 \\ 0 & 3 & 0 & -5 & -1 & 5 \end{array}\right] \leftarrow R_2 + R_3 \tag{16}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -3 & 0 & 6 \\ 0 & -3 & 0 & 7 & -1 & -7 \\ 0 & 0 & 0 & 2 & -2 & -2 \end{array}\right] \leftarrow -R_0 + R_1 \tag{17}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & -3 & 0 & 7 & -1 & -7 \\ 0 & 0 & 0 & 2 & -2 & -2 \end{array}\right] \leftarrow R_1 \text{ switch } R_2 \tag{18}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & -3 & 0 & 7 & -1 & -7 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & 0 & 0 & 2 & -2 & -2 \end{array}\right] \leftarrow \frac{2}{3}R_2 + R_3 \tag{19}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & -3 & 0 & 7 & -1 & -7 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & 0 & 0 & 0 & -2 & 0 \end{array}\right] \leftarrow R_1 \text{ switch } R_2 \tag{20}$$

from this we can find the following relationships

$$x_4 \in \mathbb{R}, x_3 = -1, x_1 = -\frac{x_4}{3}, x_2 \in \mathbb{R}, x_0 = 3 - \frac{4x_4}{3} \tag{21}$$

note that a particular solution is non-existent due to the dependency on $x_4$ and $x_2$ thus, the system has multiple solutions, we can express that in the following set

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4, \lambda_1, \lambda_2 \in R \;\middle|\; \boldsymbol{x} := \lambda_1 \begin{bmatrix} \frac{3}{\lambda_1} - \frac{4}{3} \\ -\frac{1}{3} \\ 0 \\ -\frac{1}{\lambda_1} \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \tag{22}$$