Using the Gram-Schimdt method, turn the basis $B = (b_1, b_2)$ of a two-dimensional subspace $U \subset \mathbb{R}^3$ into an ONB $C = (c_1, c_2)$ of U, where

$$b_1 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, b_2 := \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

Remember that this method essentialy works as follows

- 1 choose a basis vector and normalize it

- 2 choose a second basis vector and project it in the span of the first, take the vector that has equal magnitude and diopposite direction to the projected vector and add to the second basis vector, this creates an orthogonal vector to the span of the first basis vector, normalize such vector

- 3 repeat 2 for a new basis vector, make sure the projected vector happens in the span of previous orthornomal basis vector, do this until every basis vector is orthornomal.

we choose b_1 as the basis for the first span.

$$\pi_{\text{span}[b_1]}(b_2)$$

to find the span remember that we want the following to be true

$$\pi_{\text{span}[b_1]}(b_2) = \Sigma_i b \lambda^T$$

$$< x - \pi_{\text{span}[b_1]}(x), b_1 > = 0$$

$$b_1{}^T (x - b\lambda^T) = 0$$

$$b_1{}^T x = b_1{}^T b \lambda^T$$

$$\lambda^T = \left(b_1{}^T b\right)^{-1} b_1{}^T x$$

$$P_\pi = b_1 \left(b_1{}^T b_1\right)^{-1} b_1{}^T$$

$$P_\pi = \frac{b_1 b_1{}^T}{b_1{}^T b_1}$$

note that the term in parenthesis is a scalar when the span is a line.

we also need to normalize $c_1 = \frac{b_1}{\|b_1\|} = \frac{b_1}{\sqrt{3}}$

$$v_1 v_1{}^T = \frac{1}{3} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [1 \ 1 \ 1] \right) = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

now the factor

$$v_1{}^T v_1 = \frac{1}{3} [1 \ 1 \ 1] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1$$

so the projection is $P_\pi b_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

and the orthogonal vector $y_2 = b_2 - \pi_{\text{span}[b_1]}(b_2) = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} \\ \frac{5}{3} \\ -\frac{1}{3} \end{bmatrix}$

now we normalize $c_2 = \frac{y_2}{\|y_2\|} = \frac{3y_2}{\sqrt{42}}$

$$< \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -\frac{4}{3} \\ \frac{5}{3} \\ -\frac{1}{3} \end{bmatrix} >= -\frac{4}{3} + \frac{5}{3} - \frac{1}{3} = 0$$

so indeed the vectors are orthogonal, the basis:

$$c_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$c_2 = \frac{1}{\sqrt{42}} \begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix}$$