Consider two subspaces U_1 and U_2, where U_1 is spanned by the columns of $\boldsymbol{A_1}$ and U_2 is spanned by the columns of $\boldsymbol{A_2}$ with

$$\boldsymbol{A_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \boldsymbol{A_2} = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix} \tag{1}$$

a. Determine the dimensions of U_1, U_2

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \leftarrow -1R_1 + R_3 \tag{2}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 0 & 2 & 2 \end{bmatrix} \leftarrow -2R_1 + R_2 \tag{3}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 0 & 5 & 5 \\ 0 & 2 & 2 \end{bmatrix} \leftarrow -R_1 + R_0 \tag{4}$$

$$\begin{bmatrix} 0 & 2 & 2 \\ 1 & -2 & -1 \\ 0 & 5 & 5 \\ 0 & 2 & 2 \end{bmatrix} \leftarrow -R_0 + R_3 \tag{5}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & -1 \\ 0 & 5 & 5 \\ 0 & 2 & 2 \end{bmatrix} \leftarrow R_0 \text{ switch } R_3 \tag{6}$$

$$\begin{bmatrix} 0 & 2 & 2 \\ 1 & -2 & -1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow R_1 \text{ switch } R_0 \tag{7}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 2 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \alpha R_1 + R_2 \tag{8}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \alpha R_1 + R_2 \tag{9}$$

consider the following coefficients to generate the last column vector

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 1 \end{aligned} \tag{10}$$

so the final matrix is

$$\begin{bmatrix} 1 & -2 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{11}$$

and the dimension is 2

now for $\boldsymbol{A_2}$

$$\begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix} \leftarrow -R_0 + R_3 \tag{12}$$

$$\begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 0 & 5 & 2 \end{bmatrix} \leftarrow -7R_1 + R_2 \tag{13}$$

$$\begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 0 & -19 & -19 \\ 0 & 5 & 2 \end{bmatrix} \leftarrow -3R_1 + R_0 \tag{14}$$

$$\begin{bmatrix} 0 & -9 & -9 \\ 1 & 2 & 3 \\ 0 & -19 & -19 \\ 0 & 5 & 2 \end{bmatrix} \leftarrow -3R_1 + R_0 \tag{15}$$

$$\begin{bmatrix} 0 & -9 & -9 \\ 1 & 2 & 3 \\ 0 & -19 & -19 \\ 0 & 5 & 2 \end{bmatrix} \leftarrow \alpha R_2 + R_0 \tag{16}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 0 & -19 & -19 \\ 0 & 5 & 2 \end{bmatrix} \leftarrow R_3 \text{ switch } R_0 \tag{17}$$

$$\begin{bmatrix} 0 & -19 & -19 \\ 1 & 2 & 3 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow R_1 \text{ switch } R_0 \tag{18}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -19 & -19 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow R_1 \text{ switch } R_0 \tag{19}$$

we can see that no set of coefficients will allow the last column vector to be a linear combination of the other two column vectors, so it must be the case this is the final matrix and so the dimension is 3

b. Basis vector for U_1 and U_2
check above answer

c. Basis vector for $U_1 \cap U_2$
we again apply gaussian elimination to the union of basis vectors for U_1 and U_2

$$\begin{bmatrix} 1 & 0 & 3 & -3 & 0 \\ 1 & -2 & 1 & 2 & 3 \\ 2 & 1 & 7 & -5 & 2 \\ 1 & 0 & 3 & -1 & 2 \end{bmatrix} \leftarrow -R_1 + R_3 \tag{20}$$

$$\begin{bmatrix} 1 & 0 & 3 & -3 & 0 \\ 1 & -2 & 1 & 2 & 3 \\ 2 & 1 & 7 & -5 & 2 \\ 0 & 2 & 2 & -3 & -1 \end{bmatrix} \leftarrow -R_1 + R_0 \tag{21}$$

$$\begin{bmatrix} 0 & 2 & 2 & -5 & -3 \\ 1 & -2 & 1 & 2 & 3 \\ 2 & 1 & 7 & -5 & 2 \\ 0 & 2 & 2 & -3 & -1 \end{bmatrix} \leftarrow -2R_1 + R_2 \tag{22}$$

$$\begin{bmatrix} 0 & 2 & 2 & -5 & -3 \\ 1 & -2 & 1 & 2 & 3 \\ 0 & 3 & 6 & -7 & -1 \\ 0 & 2 & 2 & -3 & -1 \end{bmatrix} \leftarrow -R_0 + R_3 \tag{23}$$

$$\begin{bmatrix} 0 & 2 & 2 & -5 & -3 \\ 1 & -2 & 1 & 2 & 3 \\ 0 & 3 & 6 & -7 & -1 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix} \leftarrow R_0 \text{ switch } R_1 \tag{24}$$

$$\begin{bmatrix} 1 & -2 & 1 & 2 & 3 \\ 0 & 2 & 2 & -5 & -3 \\ 0 & 3 & 6 & -7 & -1 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix} \tag{25}$$

note that the last column vector can be obtained by the following coefficients

$$\begin{gathered} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 1 \\ \lambda_4 = 1 \end{gathered} \tag{26}$$

so the final matrix is

$$\begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 2 & 2 & -5 \\ 0 & 3 & 6 & -7 \\ 0 & 0 & 0 & 2 \end{bmatrix} \tag{27}$$

and the final dimension is 4

$$\boldsymbol{x_k} = \{\boldsymbol{x_1}, x_2, x_4, x_5\} \tag{28}$$