Consider the Euclidean vector space $\mathbb{R}^5$ with the dot product. A subspace $U \subset \mathbb{R}^5$ and $x \in \mathbb{R}^5$ are given by

$$U = \operatorname{span}\left[\begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 5 \\ 0 \\ 7 \end{bmatrix}\right], \boldsymbol{x} = \begin{bmatrix} -1 \\ -9 \\ -1 \\ 4 \\ 1 \end{bmatrix}$$

a. Determine the orthogonal projection $\pi_{U(\boldsymbol{x})}$ onto U

To find the projection we need to find the basis vectors that span(U), so guassian elimination is needed

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ -1 & -3 & 4 & -3 \\ 2 & 1 & 1 & 5 \\ 0 & -1 & 2 & 0 \\ 2 & 2 & 1 & 7 \end{bmatrix} \mid 2l_1 + l_4 \rightarrow l_4$$

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ -1 & -3 & 4 & -3 \\ 2 & 1 & 1 & 5 \\ 0 & -1 & 2 & 0 \\ 0 & -4 & 9 & 1 \end{bmatrix} \mid 2l_1 + l_2 \rightarrow l_2$$

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ -1 & -3 & 4 & -3 \\ 0 & -5 & 9 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -4 & 9 & 1 \end{bmatrix} \mid l_1 \text{ switch } l_0$$

$$\begin{bmatrix} -1 & -3 & 4 & -3 \\ 0 & 1 & -3 & -1 \\ 0 & -5 & 9 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -4 & 9 & 1 \end{bmatrix} \mid l_1 + l_3 \rightarrow l_3$$

$$\begin{bmatrix} -1 & -3 & 4 & -3 \\ 0 & 1 & -3 & -1 \\ 0 & -5 & 9 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & -4 & 9 & 1 \end{bmatrix} \mid 4l_1 + l_4 \rightarrow l_4$$

$$\begin{bmatrix} -1 & -3 & 4 & -3 \\ 0 & 1 & -3 & -1 \\ 0 & -5 & 9 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -3 & -3 \end{bmatrix} \mid 4l_1 + l_4 \rightarrow l_4$$

$$\begin{bmatrix} -1 & -3 & 4 & -3 \\ 0 & 1 & -3 & -1 \\ 0 & -5 & 9 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -3 & -3 \end{bmatrix} \mid 5l_1 + l_2 \rightarrow l_2$$

$$\begin{bmatrix} -1 & -3 & 4 & -3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -6 & -6 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -3 & -3 \end{bmatrix} \mid -3l_3 + l_4 \rightarrow l_4$$

$$\begin{bmatrix} -1 & -3 & 4 & -3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -6 & -6 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid -6l_3 + l_2 \rightarrow l_2$$

$$\begin{bmatrix} -1 & -3 & 4 & -3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} | -6l_3 + l_2 \to l_2$$

$$\begin{bmatrix} -1 & -3 & 4 & -3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} | \ l_3 \text{ switch } l_2$$

$$\begin{bmatrix} -1 & -3 & 4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} | \ 3l_1 + l_0 \to l_0$$

$$\begin{bmatrix} -1 & 0 & 4 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} | \ 4l_2 + l_0 \to l_0$$

$$\begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} | \ 4l_2 + l_0 \to l_0$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so $\dim(\text{span}(U)) = 3$

now to find the orthogonal projection, remember that we want

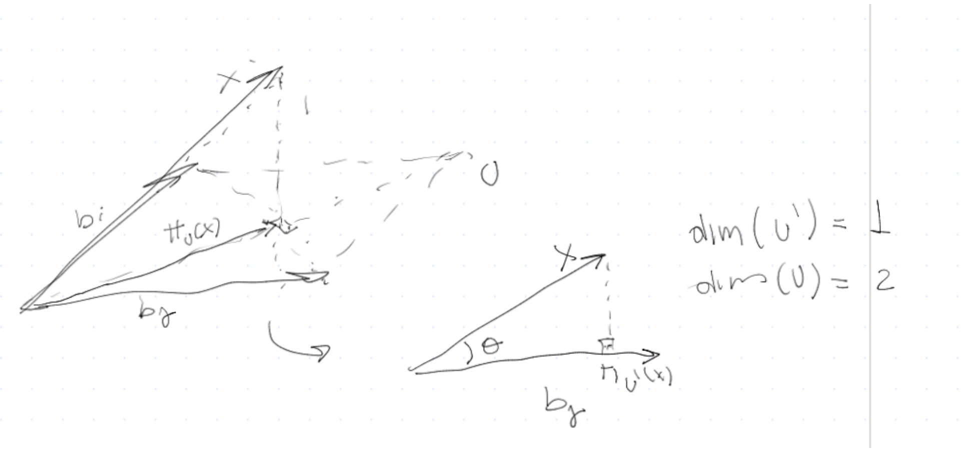

Figure 1: projection in distinct subspaces

$$\pi_{U(x)} = \Sigma_i \lambda_i \boldsymbol{b_i} = \boldsymbol{\lambda}^T \boldsymbol{B} = \boldsymbol{B}^T \boldsymbol{\lambda} = \boldsymbol{B} \boldsymbol{\lambda}^T$$

note that we have the following constraint

$$< \left(\boldsymbol{x} - \pi_{U(\boldsymbol{x})}\right), \boldsymbol{b_1} >= 0$$

$$< \left(\boldsymbol{x} - \pi_{U(\boldsymbol{x})}\right), \boldsymbol{b_2} >= 0$$

$$< \left(\boldsymbol{x} - \pi_{U(\boldsymbol{x})}\right), \boldsymbol{b_3} >= 0$$

due to being an inner product we can apply the symmetry property

$$\boldsymbol{b_1}^T \left(\boldsymbol{x} - \pi_{U(\boldsymbol{x})}\right) = 0$$

$$\boldsymbol{b_2}^T \left(\boldsymbol{x} - \pi_{U(\boldsymbol{x})}\right) = 0$$

$$\boldsymbol{b_3}^T \left(\boldsymbol{x} - \pi_{U(\boldsymbol{x})}\right) = 0$$

and the following homogeneous equation system, is homogeneous due to the fact that each $\boldsymbol{b_i}$ is a basis vector and a set of basis vector is linear independent and as such the matrix has an inverse.

from that we deduce

$$\boldsymbol{B}^T (\boldsymbol{x} - \boldsymbol{B}\boldsymbol{\lambda}^T) = 0$$

$$\boldsymbol{B}^T \boldsymbol{x} = \boldsymbol{B}^T \boldsymbol{B} \boldsymbol{\lambda}^T$$

$$\left(\boldsymbol{B}^T \boldsymbol{B}\right)^{-1} \boldsymbol{B}^T \boldsymbol{x} = \boldsymbol{\lambda}^T$$

$$\boldsymbol{\pi_U}(\boldsymbol{x}) = \boldsymbol{B}\left(\boldsymbol{B}^T \boldsymbol{B}\right)^{-1} \boldsymbol{B}^T \boldsymbol{x} =$$

the transpose of $\boldsymbol{B}$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

so $\boldsymbol{B}^T\boldsymbol{B} = \boldsymbol{I}$

and $\boldsymbol{B}\boldsymbol{B}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

the resulting multplication is the identity, so $\boldsymbol{x}_{\boldsymbol{\pi}_{\boldsymbol{U}}} = \begin{bmatrix} -1 \\ -9 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

d. Determine the distance $d(\boldsymbol{x}, U)$

as illustrated in the diagram all we have to do is find the norm of the difference between

$$\|\boldsymbol{x} - \pi_U(x)\| = \| \begin{bmatrix} -1 \\ -9 \\ -1 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ -9 \\ -1 \\ 0 \\ 0 \end{bmatrix} \| = \| \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 1 \end{bmatrix} \|$$

now we take the square root of the inner product of the resulting vector

$$\|\boldsymbol{x} - \pi_U(x)\| = \sqrt{< \boldsymbol{x}_{\textbf{dist}}, \boldsymbol{x}_{\textbf{dist}} >} = \sqrt{4^2 + 1^2} = \sqrt{\boldsymbol{x}^T\boldsymbol{x}} = \sqrt{17}$$