Let V be a vector space and π an endomorphism of V.

a. Prove that π is a projection if and only if $\mathrm{id}_V - \pi$ is a projection, where id_V is the identity endomorphism of V.
a endomorphism is a function that has domain equal to the co-domain, meaning that π maps elements of V into elements of V.

suppose π is a projection, then by definition of projection applying π twice is the same as π

$$\mathrm{id}_V(\boldsymbol{x}) = \boldsymbol{x}$$

$$\pi(\boldsymbol{x}) = \pi_U(\boldsymbol{x})$$

$$\mathrm{id}_V(\boldsymbol{x}) - \pi(x) = \boldsymbol{x} - \pi_U(x)$$

$$(\mathrm{id}_V(\boldsymbol{x}) - \pi(x)) \circ (\mathrm{id}_V(\boldsymbol{x}) - \pi(\boldsymbol{x}))$$

$$\mathrm{id}_V(\boldsymbol{x} - \pi_U(\boldsymbol{x})) = \boldsymbol{x} - \pi_U(\boldsymbol{x})$$
$$\pi(\boldsymbol{x} - \pi_U(\boldsymbol{x})) = 0$$
$$(\mathrm{id}(\boldsymbol{x}) - \pi_U(\boldsymbol{x}))(\pi(\boldsymbol{x}) - \pi_U(\boldsymbol{x}))$$

$$\boldsymbol{x} - \pi_U(\boldsymbol{x})$$

as the projection of $\boldsymbol{y} = \boldsymbol{x} - \pi_U(\boldsymbol{x})$ is orthogonal to each basis vector of U, so $< \boldsymbol{e_i}, \boldsymbol{y} >= 0, \forall e_i \in U$ so indeed if we assume π as projection $\mathrm{id}_V - \pi$ is also a projection.

first half is done.

$$(\mathrm{id}_V(\boldsymbol{x}) - \pi((\boldsymbol{x}))) \circ (\mathrm{id}_V(\boldsymbol{x}) - \pi(\boldsymbol{x})) = \mathrm{id}(\boldsymbol{x}) - \pi(\boldsymbol{x})$$

is our premise

now consider $\boldsymbol{x} = 2\pi(\boldsymbol{x})$

$$\pi(\boldsymbol{x}) \circ \pi(\boldsymbol{x}) = \pi(\boldsymbol{x})$$

so $\pi(\boldsymbol{x})$ is indeed a projection

so as both halves have been proven we arrive at the desired conclusion

b. Assume now that π is a projection. Calculate $\mathrm{im}(\mathrm{id}_V - \pi)$ and $\ker(\mathrm{id}_V - \pi)$ as function of $\mathrm{im}(\pi)$ and $\ker(\pi)$
remember that image is defined as

$$\{w \in V|\ \exists v \in V : f(v) = w\}$$

and kernel

$$\{v \in V|\ f(v) = \mathbf{0}\}$$

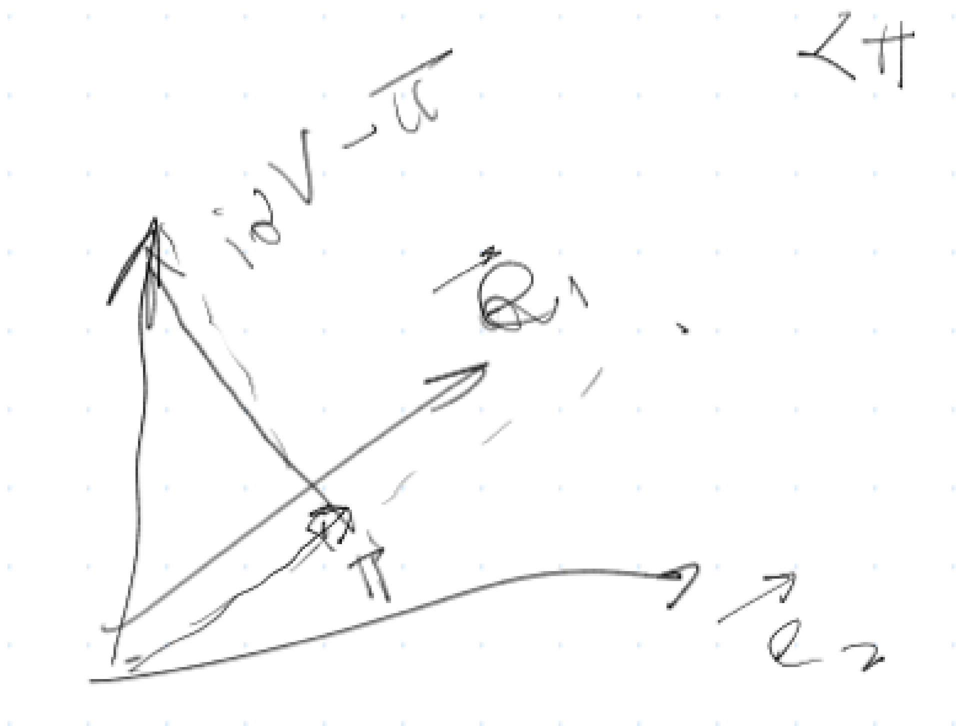

Figure 1: diagram of the problem

note that $< \mathrm{id}_V - \pi, \pi >= 0$, as such $\mathrm{id}_V - \pi$ is the orthogonal complement of π, but $\pi(\mathrm{id}_V - \pi) = \mathbf{0}$, so $\mathrm{im}(\mathrm{id}_V - \pi) \cup \{\mathbf{0}\} = \ker(\pi) \Rightarrow \mathrm{im}(\mathrm{id}_V - \pi) = \ker(\pi) \setminus \{\mathbf{0}\}$

and $(\mathrm{id}_V - \pi)(\pi) = \mathbf{0} \Rightarrow \ker(\mathrm{id}_V - \pi) = \mathrm{im}(\pi) \cup \{\mathbf{0}\}$