Wich of the following sets are subspaces of $\mathbb{R}^3$?

a.

$$A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) \mid \lambda, \mu \in \mathbb{R}\} \tag{1}$$

to check such set is a subspace we need to verify if it has the properties of a vector space

identitiy
first we check for identity element, defined as follows

$$x + \mathrm{id}(A) = x \tag{2}$$

$$(\lambda, \lambda + \mu^3, \lambda - \mu^3) + (x, y, z) = (\lambda, \lambda + \mu^3, \lambda - \mu^3) \tag{3}$$

$$(\lambda + x, \lambda + \mu^3 + y, \lambda - \mu^3 + z) = (\lambda, \lambda + \mu^3, \lambda - \mu^3) \tag{4}$$

by inspection we deduce this element must be $(0, 0, 0)$, and indeed if we set $\mu = 0, \lambda = 0$ we have such element, so this structure has the identity element.

inverse element
now we check if each element has an inverse

$$x + x^{-1} = 0 \tag{5}$$

$$(\lambda, \lambda + \mu^3, \lambda - \mu^3) + (x, y, z) = (0, 0, 0) \tag{6}$$

$$(\lambda + x, \lambda + \mu^3 + y, \lambda - \mu^3 + z) = (0, 0, 0) \tag{7}$$

and we deduce that $x = -\lambda, y = -(\lambda + \mu^3), z = -\lambda + \mu^3$

and indeed as both variables $\in \mathbb{R}$ and $\mathbb{R}$ is a group in itself for $+$ operator we again obtain a value $\in \mathbb{R}$ so this structure satisfy the inverse property.

closure is guaranteed due to $(R, +)$ is a group.

associativity
now we check for associativity

$$\begin{gathered} v_1 + (v_2 + v_3) \\ (\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3) + ((\lambda_2 + \lambda_3), (\lambda_2 + \lambda_3) + (\mu_2^3 + \mu_3^3), (\lambda_2 + \lambda_3) - (\mu_2^3 + \mu_3^3)) \\ (\lambda_1 + \lambda_2 + \lambda_3, (\lambda_1 + \lambda_2 + \lambda_3) + (\mu_1^3 + \mu_2^3 + \mu_3^3), (\lambda_1 + \lambda_2 + \lambda_3) - (\mu_1^3 + \mu_2^3 + \mu_3^3)) \end{gathered} \tag{8}$$

$$\begin{gathered} (v_1 + v_2) + v_3 \\ (\lambda_1 + \lambda_2, (\lambda_1 + \lambda_2) + (\mu_1^3 + \mu_2^3), (\lambda_1 + \lambda_2) - (\mu_1^3 + \mu_2^3)) + (\lambda_3, \lambda_3 + \mu_3^3, \lambda_3 - \mu_3^3) \\ (\lambda_1 + \lambda_2 + \lambda_3, (\lambda_1 + \lambda_2 + \lambda_3) + (\mu_1^3 + \mu_2^3 + \mu_3^3), (\lambda_1 + \lambda_2 + \lambda_3) - (\mu_1^3 + \mu_2^3 + \mu_3^3)) \end{gathered} \tag{9}$$

so associavity is preserved

group is abelian
now we need to check if the group is abelian

$$v_1 + v_2 = v_2 + v_1 \tag{10}$$

$$\begin{gathered}(\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3) + (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3) \\ (\lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \mu_1^3 + \mu_2^3, \lambda_1 + \lambda_2 - (\mu_1^3 + \mu_2^3))\end{gathered} \tag{11}$$

$$\begin{gathered}(\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3) + (\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3) \\ (\lambda_2 + \lambda_1, \lambda_2 + \lambda_1 + \mu_2^3 + \mu_1^3, \lambda_2 + \lambda_1 - (\mu_2^3 + \mu_1^3))\end{gathered} \tag{12}$$

note that $(\mathbb{R}, +)$ is abelian, so $\lambda_1 + \lambda_2 = \lambda_+ \lambda_1, \mu_1^3 + \mu_2^3 = \mu_2^3 + \mu_1^3$ that implies the last coordinate has the form $x - y$ in both (11) and (12), so the group is abelian

distributivity of vectors
now we check for distributivity property

$$\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2 \tag{13}$$

$$\begin{gathered}\lambda((\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3) + (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3)) \\ \lambda((\lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \mu_1^3 + \mu_2^3, \lambda_1 + \lambda_2 - (\mu_1^3 + \mu_2^3))) \\ (\lambda(\lambda_1 + \lambda_2), \lambda(\lambda_1 + \lambda_2) + \lambda(\mu_1^3 + \mu_2^3), \lambda(\lambda_1 + \lambda_2) - \lambda(\mu_1^3 + \mu_2^3)))\end{gathered} \tag{14}$$

$$\begin{gathered}(\lambda\lambda_1, \lambda\lambda_1 + \lambda\mu_1^3, \lambda\lambda_1 - \lambda\mu_1^3) + (\lambda\lambda_2, \lambda\lambda_2 + \lambda\mu_2^3, \lambda\lambda_2 - \lambda\mu_2^3) \\ (\lambda(\lambda_1 + \lambda_2), \lambda(\lambda_1 + \lambda_2) + \lambda(\mu_1^3 + \mu_2^3), \lambda(\lambda_1 + \lambda_2) - \lambda(\mu_1^3 + \mu_2^3))\end{gathered} \tag{15}$$

so distributivity property holds

distributivity of scalars

$$(\eta + \psi)v = \eta v + \psi v \tag{16}$$

$$\begin{gathered}(\eta + \psi)(\lambda, \lambda + \mu^3, \lambda - \mu^3) \\ (\eta\lambda + \psi\lambda, \eta\lambda + \psi\lambda + \eta\mu^3 + \psi\mu^3, \eta\lambda + \psi\lambda - \eta\mu^3 - \psi\mu^3)\end{gathered} \tag{17}$$

$$\begin{gathered}\eta(\lambda, \lambda + \mu^3, \lambda - \mu^3) + \psi(\lambda, \lambda, +\mu^3, \lambda - \mu^3) \\ (\eta\lambda + \psi\lambda, \eta\lambda + \psi\lambda + \eta\mu^3 + \psi\mu^3, \eta\lambda + \psi\lambda - \eta\mu^3 - \psi\mu^3)\end{gathered} \tag{18}$$

associativity of scalars
we now prove associativity

$\eta(\psi v_1) = (\eta\psi)v_1$

$$
\begin{gathered}
\eta(\psi(\lambda, \lambda + \mu^3, \lambda - \mu^3)) \\
\eta(\psi\lambda, \psi\lambda + \psi\mu^3, \psi\lambda - \psi\mu^3)) \\
(\eta\psi\lambda + \eta\psi\mu^3, \eta\psi\lambda - \eta\psi\mu^3))
\end{gathered} \tag{19}
$$

$$
\begin{gathered}
(\eta\psi)v_1 \\
(\eta\psi\lambda, \eta\psi\lambda + \eta\psi\mu^3, \eta\psi\lambda - \eta\psi\mu^3)
\end{gathered} \tag{20}
$$

now we prove identity element of scalar multipliation

scalar identity

$$
\eta v = v \tag{21}
$$

$$
\begin{gathered}
\eta(\lambda, \lambda + \mu^3, \lambda - \mu^3) \\
(\eta\lambda, \eta\lambda + \eta\mu^3, \eta\lambda - \eta\mu^3)
\end{gathered} \tag{22}
$$

this means $\eta = 1, 1 \in \mathbb{R}$

so this set is indeed a subset of $\mathbb{R}^3$

b.

$$
B = \{(\lambda^2, -\lambda, 0) \mid \lambda \in \mathbb{R}\} \tag{23}
$$

first we check if this structure is an abelian group

identity

$$
x + \mathrm{id}(x) = x \tag{24}
$$

$$
\begin{gathered}
(\lambda^2, -\lambda, 0) + (x, y, z) = (\lambda^2, -\lambda, 0) \\
(\lambda^2 + x, -\lambda + y, z) = (\lambda^2, -\lambda, 0)
\end{gathered} \tag{25}
$$

we deduce the identity element is $(0, 0, 0)$ and indeed because $(\mathbb{R}, +)$ is a group 0 must belong to it

inverse

$$
x + x^{-1} = 0 \tag{26}
$$

$$
\begin{gathered}
(\lambda^2, -\lambda, 0) + (x, y, z) = (0, 0, 0) \\
(\lambda^2 + x, -\lambda + y, z) = (0, 0, 0)
\end{gathered} \tag{27}
$$

we deduce that $x = -\lambda^2, y = \lambda, z = 0$ and $\lambda \in \mathbb{R}$ so indeed this structure has an inverse

closure

$$
x + y \in \mathbb{R}^3 \tag{28}
$$

$$
\begin{gathered}
(\lambda_0^2, -\lambda_0^1, 0) + (\lambda_1^2, -\lambda_1^1, 0) \\
(\lambda_0^2 + \lambda_1^2, -(\lambda_0^1 + \lambda_1^1), 0))
\end{gathered} \tag{29}
$$

remembering that $\lambda \in \mathbb{R}$ and $(\mathbb{R}, +)$ is a group, then indeed the above element $\in \mathbb{R}^3$

associativity

$$
x + (y + z) = (x + y) + z \tag{30}
$$

$$
\begin{gathered}
(\lambda_0^2, -\lambda_0^1, 0) + ((\lambda_1^2, -\lambda_1^1, 0) + (\lambda_2^2, -\lambda_2^1, 0)) \\
(\lambda_0^2, -\lambda_0^1, 0) + ((\lambda_1^2 + \lambda_2^2), -(\lambda_1^1 + \lambda_2^1), 0) \\
(\lambda_0^2 + \lambda_1^2 + \lambda_2^2, -(\lambda_0^1 + \lambda_1^1 + \lambda_2^1), 0)
\end{gathered} \tag{31}
$$

$$
\begin{gathered}
((\lambda_0^2, -\lambda_0^1, 0) + (\lambda_1^2, -\lambda_1^1, 0)) + (\lambda_2^2, -\lambda_2^1, 0) \\
(\lambda_0^2 + \lambda_1^2, -(\lambda_0^1 + \lambda_1^1), 0) + (\lambda_2^2, -\lambda_2^1, 0) \\
(\lambda_0^2 + \lambda_1^2 + \lambda_2^2, -(\lambda_0^1 + \lambda_1^1 + \lambda_2^1), 0)
\end{gathered} \tag{32}
$$

now we check if this group is abelian, that is can commute the operands

$$
x + y = y + x \tag{33}
$$

$$
\begin{gathered}
(\lambda_0^2, -\lambda_0^1, 0) + (\lambda_1^2, -\lambda_1^1, 0) \\
(\lambda_0^2 + \lambda_1^2, -(\lambda_0^1 + \lambda_1^1), 0)
\end{gathered} \tag{34}
$$

$$
\begin{gathered}
(\lambda_1^2, -\lambda_1^1, 0) + (\lambda_0^2, -\lambda_1^1, 0) \\
(\lambda_1^2 + \lambda_0^2, -(\lambda_1^1 + \lambda_0^1), 0)
\end{gathered} \tag{35}
$$

note that $(\mathbb{R}, +)$ is an abelian group

now we check distributivity of vectors, associativity of scalars and identy scalar

distributivity of vectors

$$
\mu(v_1 + v_2) = \mu v_1 + \mu v_2 \tag{36}
$$

$$
\begin{gathered}
\mu((\lambda_0^2, -\lambda_0^1, 0) + (\lambda_1^2, \lambda_1^1, 0)) \\
\mu(\lambda_0^2 + \lambda_1^2, -(\lambda_0^1 + \lambda_1^1), 0) \\
(\mu\lambda_0^2 + \mu\lambda_1^2, -\mu\lambda_0^1 - \mu\lambda_1^1, 0)
\end{gathered} \tag{37}
$$

$$\begin{gathered}\mu(\lambda_0^2, -\lambda_0^1, 0) + \mu(\lambda_1^2, -\lambda_1^1, 0) \\ (\mu\lambda_0^2, -\mu\lambda_0^1, 0) + (\mu\lambda_1^2, -\mu\lambda_1^1, 0) \\ (\mu\lambda_0^2 + \mu\lambda_1^2, -\mu\lambda_0^1 - \mu\lambda_1^1, 0)\end{gathered} \tag{38}$$

so this property holds

distributivity of scalars
$(\mu + \psi)v = \mu v + \psi v$

$$\begin{gathered}(\mu + \psi)(\lambda_0^2, -\lambda_0^1, 0) \\ (\mu\lambda_0^2 + \psi\lambda_0^2, -\mu\lambda_0^1 - \psi\lambda_0^1, 0)\end{gathered} \tag{39}$$

$$\begin{gathered}\mu(\lambda_0^2, -\lambda_0^1, 0) + \psi(\lambda_0^2, -\lambda_0^1, 0) \\ (\mu\lambda_0^2 + \psi\lambda_0^2, -(\mu\lambda_0^1 + \psi\lambda_0^1), 0)\end{gathered} \tag{40}$$

so this property holds

associativity of scalars

$$\mu(\psi v) = (\mu\psi)v \tag{41}$$

$$\begin{gathered}\mu(\psi(\lambda_0^2, -\lambda_0^1, 0)) \\ \mu(\psi\lambda_0^2, -\psi\lambda_0^1, 0) \\ (\mu\psi\lambda_0^2, -\mu\psi\lambda_0^1, 0)\end{gathered} \tag{42}$$

$$\begin{gathered}(\mu\psi)(\lambda_0^2, -\lambda_0^1, 0) \\ (\mu\psi\lambda_0^2, -\mu\psi\lambda_0^1, 0)\end{gathered} \tag{43}$$

so this property holds

identity

$$1v = v \tag{44}$$

$$\begin{gathered}1(\lambda_0^2, -\lambda_0^1, 0) = (\lambda_0^2, -\lambda_0^1, 0) \\ (1\lambda_0^2, -1\lambda_0^1, 0) = (\lambda_0^2, -\lambda_0^1, 0)\end{gathered} \tag{45}$$

so must be one this scalar.

indeed this structure is a subspace of the vector space $\mathbb{R}^3$

c.
Let γ be in $\mathbb{R}$

$$C = \{(\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3 \mid \eta_1 - 2\eta_2 + 3\eta_3 = \gamma\} \tag{46}$$

again we check if this structure is a group

identity element

$$x + 1 = x \tag{47}$$

$$\begin{gathered}(\eta_1, \eta_2, \eta_3) + (x, y, z) = (\eta_1, \eta_2, \eta_3) \\ (\eta_1 + x, \eta_2 + y, \eta_3 + z) = (\eta_1, \eta_2, \eta_3) \\ \eta_1 + x - 2(\eta_2 + y) + 3(\eta_3 + z) = \gamma\end{gathered} \tag{48}$$

that implies $(x, y, z) = (0, 0, 0) \Rightarrow 0 - 0 + 0 = 0$

so identity holds

inverse element

$$x + x^{-1} = 0 \tag{49}$$

$$\begin{gathered}(\eta_1, \eta_2, \eta_3) + (x, y, z) = (0, 0, 0) \\ (\eta_1 + x, \eta_2 + y, \eta_3 + z) = (0, 0, 0)\end{gathered} \tag{50}$$

that implies $(x, y, z) = (-\eta_1, -\eta_2, -\eta_3) \Rightarrow -\eta_1 + 2\eta_2 - 3\eta_3 = \gamma$

so inverse holds

closure property

$$x + y \in \mathbb{R}^3 \tag{51}$$

$$\begin{gathered}(\eta_1, \eta_2, \eta_3) + (\mu_1, \mu_2, \mu_3) \\ (\eta_1 + \mu_1, \eta_2 + \mu_2, \eta_3 + \mu_3)\end{gathered} \tag{52}$$

that implies $\eta_1 + \mu_1 - 2(\eta_2 + \mu_2) + 3(\eta_3 + \mu_3) = \gamma$

and indeed this expression is closed in $\mathbb{R}$

associativity

$$x + (y + z) = (x + y) + z \tag{53}$$

$$\begin{gathered}(\eta_1, \eta_2, \eta_3) + (\mu_1 + \omega_1, \mu_2 + \omega_2, \mu_3 + \omega_3) \\ (\eta_1 + \mu_1 + \omega_1, \eta_2 + \mu_2 + \omega_2, \eta_3 + \mu_3 + \omega_3)\end{gathered} \tag{54}$$

that implies $\eta_1 + \mu_1 + \omega_1 - 2(\eta_2 + \mu_2 + \omega_2) + 3(\eta_3 + \mu_3 + \omega_3) = \gamma$

$$\begin{gathered}(\eta_1 + \mu_1, \eta_2 + \mu_2, \omega_3 + \mu_3) + (\omega_1, \omega_2, \omega_3) \\ (\eta_1 + \mu_1 + \omega_1, \eta_2 + \mu_2 + \omega_2, \omega_3 + \mu_3 + \omega_3)\end{gathered} \tag{55}$$

so indeed this structure is a group, now we check if it’s abelian

$$x + y = y + x \tag{56}$$

$$\begin{gathered}(\eta_1, \eta_2, \eta_3) + (\mu_1, \mu_2, \mu_3) \\ (\eta_1 + \mu_1, \eta_2 + \mu_2, \eta_3 + \mu_3)\end{gathered} \tag{57}$$

that implies $\eta_1 + \mu_1 - 2(\eta_2 + \mu_2) + 3(\eta_3 + \mu_3) = \gamma_1$

$$\begin{gathered}(\mu_1, \mu_2, \mu_3) + (\eta_1, \eta_2, \eta_3) \\ (\mu_1 + \eta_1, \mu_2 + \eta_2, \mu_3 + \eta_3)\end{gathered} \tag{58}$$

that implies $\mu_1 + \eta_1 - 2(\mu_2 + \eta_2) + 3(\mu_3 + \eta_3) = \gamma_2$

note however that $(\mathbb{R}, +)$ is closed, so indeed this group is abelian, $\Rightarrow \gamma_1 = \gamma_2$

now we check for distributivity of vectors, associativity of scalars and the identity scalar

distributivity of vectors

$$\psi(v_1 + v_2) = \psi v_1 + \psi v_2 \tag{59}$$

$$\begin{gathered}\psi((\eta_1, \eta_2, \eta_3) + (\mu_1, \mu_2, \mu_3)) \\ \psi((\eta_1 + \mu_1, \eta_2 + \mu_2, \eta_3 + \mu_3) \\ (\psi\eta_1 + \psi\mu_1, \psi\eta_2 + \psi\mu_2, \psi\eta_3 + \psi\mu_3)\end{gathered} \tag{60}$$

that implies $(\psi\eta_1 + \psi\mu_1) - 2(\psi\eta_2 + \psi\mu_2 + 3(\psi\eta_3 + \psi\mu_3) = \gamma_1$

$$\begin{gathered}\psi(\eta_1, \eta_2, \eta_3) + \psi(\mu_1, \mu_2, \mu_3) \\ (\psi\eta_1, \psi\eta_2, \psi\eta_3) + (\psi\mu_1, \psi\mu_2, \psi\mu_3) \\ (\psi\eta_1 + \psi\mu_1, \psi\eta_2 + \psi\mu_2, \psi\eta_3 + \psi\mu_3)\end{gathered} \tag{61}$$

so distributivity of vectors hold

distributivity of vectors

$$(\psi + \mu)v = \psi v + \mu v \tag{62}$$

$$\begin{gathered}(\psi + \mu)(\eta_1, \eta_2, \eta_3) \\ (\psi\eta_1 + \mu\eta_1, \psi\eta_2 + \mu\eta_2, \psi\eta_3 + \mu\eta_3)\end{gathered} \tag{63}$$

$$\begin{gathered}\psi(\eta_1, \eta_2, \eta_3) + \mu(\eta_1, \eta_2, \eta_3) \\ (\psi\eta_1, \psi\eta_2, \psi\eta_3) + (\mu\eta_1, \mu\eta_2, \mu\eta_3) \\ (\psi\eta_1 + \mu\eta_1, \psi\eta_2 + \mu\eta_2, \psi\eta_3 + \mu\eta_3)\end{gathered} \tag{64}$$

so the property holds

associativity of scalars

$$\psi(\mu v) = (\psi\mu)v \tag{65}$$

$$\begin{gathered}\psi(\mu(\eta_1, \eta_2, \eta_3)) \\ \psi((\mu\eta_1, \mu\eta_2, \mu\eta_3))((\psi\mu\eta_1, \psi\mu\eta_2, \psi\mu\eta_3))\end{gathered} \tag{66}$$

that implies $\psi\mu\eta_1 - 2\psi\mu\eta_2 + 3\psi\mu\eta_3$

$$\begin{gathered}(\psi\mu)(\eta_1, \eta_2, \eta_3) \\ (\psi\mu\eta_1, \psi\mu\eta_2, \psi\mu\eta_3)\end{gathered} \tag{67}$$

so the property holds

identitiy of scalars

$$1v = v \tag{68}$$

$$1(\eta_1, \eta_2, \eta_3) \tag{69}$$

so indeed this element is 1 and $1 \in \mathbb{R}$

so this structure is a vector subspace of $\mathbb{R}^3$

d.

$$D = \left\{(\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3 \;\middle|\; \eta_2 \in \mathbb{Z}\right\} \tag{70}$$

this can be easily disproved by considering scalar associativity

$$\psi(\mu v) = (\psi\mu)v \tag{71}$$

$$\begin{gathered}\psi(\mu\eta_1, \mu\eta_2, \mu\eta_3) \\ (\psi\mu\eta_1, \psi\mu\eta_2, \psi\mu\eta_3)\end{gathered} \tag{72}$$

if we consider that $\psi, \mu \in \mathbb{R}$ than any value $S\neg \in \mathbb{Z}$ invalidates this structure as a vector subspace of $\mathbb{R}^3$