Are the following mappings linear?

a. Let $a, b \in \mathbb{R}$.

$$\Phi : L^1([a,b]) \to \mathbb{R} \tag{1}$$

$$f \mapsto \Phi(f) = \int_a^b f(x)dx \tag{2}$$

where $L^1([a,b])$ denotes the set of integrable functions on $[a,b]$.

by definition of linear transformation

$$\varphi(\lambda v_1 + \psi v_2) = \lambda\varphi(v_1) + \psi\varphi(v_2) \tag{3}$$

this can be seen as scale both vectors, sum then and them apply the transformation or apply the transformation and then scale.

$$\int_a^b (\lambda f(x) + \psi g(x))dx \tag{4}$$

this proof is kinda trick, bu essentially we have to see each integration as a riemman sum

$$S = \Sigma_i f(x_i)\Delta x_i \tag{5}$$

and a integration is nothing more than

$$\int_a^b f(x)dx = \lim_{\|\Delta x\| \to 0} \Sigma_i f(x_i^*)\Delta x_i \tag{6}$$

note that $f(x_i^*)$ means a point between a valid intervals, for instance $x_i^* \in [x_1, x_2]$ where the brackets denote a closed interval

now we consider the riemman sum as the linear operator instead

$$\lim_{\|\Delta x\| \to 0} \Sigma_i((\lambda f(x_i^*) + \psi g(x_i^*))\Delta x_i) \tag{7}$$

we know both functions by definition are integrable we also know that the riemman sum, similar to the integral is expected to return $x \in \mathbb{R}$, thus we can apply distributivity of multiplication here

$$\lim_{\|\Delta x\| \to 0} \Sigma_i(\lambda f(x_i^*)\Delta x_i + \psi g(x_i^*)\Delta x_i) \tag{8}$$

by a handwave argument we know we'll have n instances of $\lambda f(x^*)\Delta x$ and the same for $\psi g(x^*)\Delta x$ which implies we can do the following

$$\lim_{\|\Delta x\| \to 0} \Sigma_i \lambda f(x_i^*)\Delta x_i + \Sigma_i \psi g(x_i^*)\Delta x_i \tag{9}$$

$$\lim_{\|\Delta x\| \to 0} \lambda\Sigma_i f(x_i^*)\Delta x_i + \psi\Sigma_i g(x_i^*)\Delta x_i \tag{10}$$

remember that by definition of integration we also have the folowing

$$\lambda \int_a^b f(x)dx + \psi \int_a^b g(x)dx \tag{11}$$

so is indeed a linear transformation

b .

$$\Phi : C^1 \to C^0 \tag{12}$$

$$f \mapsto \Phi(f) = f' \tag{13}$$

where for $k \geq 1, C^k$ denotes the set of k times continuously differentiable functions, and C^0 denotes the set of continuous functions.

remember fundamental theorem of calculus

$$\int_a^b f'(x)dx = f(b) - f(a) \tag{14}$$

and if is a linear transformation then

$$(\lambda f + \psi g)' = \lambda f' + \psi g' \tag{15}$$

apply the fundamental theory of calculus once to LHS

$$\int_a^b (\lambda f + \psi g)dx = (\lambda f(b) + \psi g(b)) - (\lambda f(a) + \psi g(a)) \tag{16}$$

rearranging

$$\lambda(f(b) - f(a)) + \psi(g(b) - g(a)) \tag{17}$$

now apply the fundamental theorem of calculus once to RHS

$$\int_a^b (\lambda f' + \psi g')dx \tag{18}$$

remember the linearity of integration

$$\int_a^b \lambda f' dx + \int_a^b \psi g' dx \tag{19}$$

and apply the fundamental theorem of calculus again

$$\lambda(f(b) - f(a)) + \psi(g(b) - g(a)) \tag{20}$$

and indeed derivative is a linear operator

c.

$$\Phi : \mathbb{R} \to \mathbb{R} \tag{21}$$

$$x \mapsto \Phi(x) = \cos(x) \tag{22}$$

again we need to check

$$\cos(\lambda x + \psi y) = \lambda \cos(x) + \psi \cos(y) \tag{23}$$

remembering trig identities

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta \tag{24}$$

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta \tag{25}$$

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta \tag{26}$$

$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta \tag{27}$$

so cos is not linear as the trig identity show an extra dependency on the other operand is present when adding the arguments of cosine

d.

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^2 \tag{28}$$

$$\boldsymbol{x} \mapsto \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \boldsymbol{x} \tag{29}$$

this can be easily proven

$$\lambda \begin{bmatrix} x_0 + 2x_1 + 3x_2 \\ x_0 + 4x_1 + 3x_2 \end{bmatrix} + \psi \begin{bmatrix} y_0 + 2y_1 + 3y_2 \\ y_0 + 4y_1 + 3y_2 \end{bmatrix} = \begin{bmatrix} \lambda x_0 + \psi y_0 + 2(\lambda x_1 + \psi y_1) + 3(\lambda x_2 + \psi y_2) \\ \lambda x_0 + \psi y_0 + 4(\lambda x_1 + \psi y_1) + 3(\lambda x_2 + \psi y_2) \end{bmatrix} \tag{30}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} (\boldsymbol{x} + \boldsymbol{y}) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \left(\lambda(x_0, x_1, x_2) + \psi(y_0, y_1, y_2)^T \right) \tag{31}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} (\boldsymbol{x} + \boldsymbol{y}) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} (\lambda x_0 + \psi y_0, \lambda x_1 + \psi y_1, \lambda x_2 + \psi y_2)^T \tag{32}$$

$$\begin{bmatrix} \lambda x_0 + \psi y_0 + 2(\lambda x_1 + \psi y_1) + 3(\lambda x_2 + \psi y_2) \\ \lambda x_0 + \psi y_0 + 4(\lambda x_1 + \psi y_1) + 3(\lambda x_2 + \psi y_2) \end{bmatrix} \tag{33}$$

so is a linear operator

e. Let θ be in $[0, 2\pi[$ and

$$\boldsymbol{x} \mapsto \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \tag{34}$$

$$\lambda \begin{bmatrix} \cos(\theta)x_0 + \sin(\theta)x_1 \\ -\sin(\theta)x_0 + \cos(\theta)x_1 \end{bmatrix} + \psi \begin{bmatrix} \cos(\theta)y_0 + \sin(\theta)y_1 \\ -\sin(\theta)y_0 + \cos(\theta)y_1 \end{bmatrix} = \begin{bmatrix} \lambda\cos(\theta)x_0 + \lambda\sin(\theta)x_1 + \psi\cos(\theta)y_0 + \psi\sin(\theta)y_1 \\ -\lambda\sin(\theta)x_0 + \lambda\cos(\theta)x_1 - \psi\sin(\theta)y_0 + \psi\cos(\theta)y_1 \end{bmatrix} \tag{35}$$

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} (\boldsymbol{x} + \boldsymbol{y}) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} (\lambda x_0 + \psi y_0, \lambda x_1 + \psi y_1)^T \tag{36}$$

$$\begin{bmatrix} \lambda \cos(\theta) x_0 + \psi \cos(\theta) y_0 + \lambda \sin(\theta) x_1 + \psi \sin(\theta) y_1 \\ -\lambda \sin(\theta) x_0 - \psi \sin(\theta) y_0 + \lambda \cos(\theta) x_1 + \psi \cos(\theta)]_1 \end{bmatrix} \tag{37}$$

so is a linear transformation