Consider $\mathbb{R}^3$ with the inner product

$$< \boldsymbol{x}, \boldsymbol{y} >:= \boldsymbol{x}^T \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \boldsymbol{y}$$

Furthemore, we define $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$ as the standard/canonical basis in $\mathbb{R}^3$.

a. Determine the orthogonal projection $\pi_U(\boldsymbol{e_2})$ of $\boldsymbol{e_2}$ onto

$$U = \text{span}[\boldsymbol{e_1}, \boldsymbol{e_3}]$$

the span is a plane, we note that we're not dealing with an orthogonal basis for $\mathbb{R}^3$ as a quick inspection in $\boldsymbol{A}$ show us that not all $< \boldsymbol{e_i}, \boldsymbol{e_2} >= 0$

remember that we want:

$$\pi_U(\boldsymbol{x}) = \Sigma_i b_i \lambda_i = \boldsymbol{B}\boldsymbol{\lambda}^T$$

and

$$< \boldsymbol{x} - \pi_U(\boldsymbol{x}), b_i >= b_i^T(\boldsymbol{x} - \pi_U(\boldsymbol{x})) = 0$$

$$\boldsymbol{B}^T(\boldsymbol{x} - \boldsymbol{B}\boldsymbol{\lambda}^T) = 0$$

$$\boldsymbol{B}^T\boldsymbol{x} = \boldsymbol{B}^T\boldsymbol{B}\boldsymbol{\lambda}^T$$

$$\left(\boldsymbol{B}^T\boldsymbol{B}\right)^{-1}\boldsymbol{B}^T\boldsymbol{x} = \boldsymbol{\lambda}^T$$

so the desired projection matrix is $\boldsymbol{P_\pi} = \boldsymbol{B}\left(\boldsymbol{B}^T\boldsymbol{B}\right)^{-1}\boldsymbol{B}^T$

note that $< \boldsymbol{e_1}, \boldsymbol{e_3} >= 0$ as per inspection, meaning both vectors are orthogonal to each other.

we can apply bilinearity as we have an inner product

$$< \boldsymbol{x}, \boldsymbol{e_1} > - < \boldsymbol{\pi_U}(\boldsymbol{x}), \boldsymbol{e_1} >= 0$$

$$< \boldsymbol{x}, \boldsymbol{e_3} > - < \boldsymbol{\pi_U}(\boldsymbol{x}), \boldsymbol{e_3} >= 0$$

through inspection

$$1 =< \pi_U(\boldsymbol{x}), \boldsymbol{e_1}) >$$

$$-1 =< \pi_U(\boldsymbol{x}), \boldsymbol{e_3} >$$

and we can apply the defined inner product and find a system of equations

$$[1 \;\; 0 \;\; 0] \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \pi_x \\ \pi_y \\ \pi_z \end{bmatrix} = [2 \;\; 1 \;\; 0] \begin{bmatrix} \pi_x \\ \pi_y \\ \pi_z \end{bmatrix} = 2\pi_x + \pi_y$$

$$2\pi_x + \pi_y + 0\pi_z = 1$$

$$[0\ 0\ 1]\begin{bmatrix}2 & 1 & 0\\ 1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}\begin{bmatrix}\pi_x\\ \pi_y\\ \pi_z\end{bmatrix} = [0\ -1\ 2]\begin{bmatrix}\pi_x\\ \pi_y\\ \pi_z\end{bmatrix} = -\pi_y + 2\pi_z$$

$$0\pi_x - \pi_y + 2\pi_z = -1$$

we are stuck, however we can find an orthogonal complement to $\pi_U(x)$ if we rotate $\boldsymbol{e_2}$.

to do so we find the angle between $\boldsymbol{e_2}$ and $\boldsymbol{e_1} + \boldsymbol{e_3}$

$$[1\ 0\ 1]\begin{bmatrix}2 & 1 & 0\\ 1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}\begin{bmatrix}1\\ 0\\ 1\end{bmatrix} = [2\ 0\ 2]\begin{bmatrix}1\\ 0\\ 1\end{bmatrix} = 4$$

$$\theta = \operatorname{acos}((< \boldsymbol{e_2}, \boldsymbol{e_1} + \boldsymbol{e_3} >) = \frac{\pi}{2}$$

so we don't even need to rotate

$$[0\ 1\ 0]\begin{bmatrix}2 & 1 & 0\\ 1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}\begin{bmatrix}\pi_x\\ \pi_y\\ \pi_z\end{bmatrix} = [1\ 2\ -1]\begin{bmatrix}\pi_x\\ \pi_y\\ \pi_z\end{bmatrix} = \pi_x + 2\pi_y - \pi_z$$

$$\pi_x + 2\pi_y - \pi_z = 0$$

$$\left[\begin{array}{ccc|c}2 & 1 & 0 & 1\\ 0 & -1 & 2 & -1\\ 1 & 2 & -1 & 0\end{array}\right] - 2l_2 + l_0 \to l_0$$

$$\left[\begin{array}{ccc|c}0 & -3 & 2 & 1\\ 0 & -1 & 2 & -1\\ 1 & 2 & -1 & 0\end{array}\right] 2l_1 + l_2 \to l_2$$

$$\left[\begin{array}{ccc|c}0 & -3 & 2 & 1\\ 0 & -1 & 2 & -1\\ 1 & 0 & -3 & 2\end{array}\right] - 3l_1 + l_0 \to l_0$$

$$\left[\begin{array}{ccc|c}0 & 0 & -4 & 4\\ 0 & -1 & 2 & -1\\ 1 & 0 & -3 & 2\end{array}\right] - 3l_1 + l_0 \to l_0$$

so $\pi_x = -1, \pi_y = 1, \pi_z = -1$

Compute the distance $d(\boldsymbol{e_2}, U)$

$$\|\boldsymbol{x} - \pi_U(\boldsymbol{x})\| = \|[0\ 1\ 0] - [-1\ 1\ -1]\| = \|[1\ 0\ 1]\| = \sqrt{2}$$

$$[1\ 0\ 1]\begin{bmatrix}2 & 1 & 0\\ 0 & -1 & 2\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\\ 1\end{bmatrix} = [3\ 3\ -1]\begin{bmatrix}1\\ 0\\ 1\end{bmatrix} = 3 - 1 = 2$$

Draw the scenario: standard basis vectors and $\pi_U(e_2)$