Show that $< \cdot, \cdot >$ defined for all $\boldsymbol{x} = [x_1, x_2]^T \in \mathbb{R}^2$ and $\boldsymbol{y} = [y1, y_2]^T \in \mathbb{R}^2$ by

$$< \boldsymbol{x}, \boldsymbol{y} >:= x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2(x_2 y_2)$$

is an inner product

Remember that an inner product is a symmetric bilinear mapping from one vector space to to the reals that has the positive definite property, such property implies only the vector $\boldsymbol{x} = 0$ has the inner product with itself equal to 0.

we start by proving the bilinearity

$$< \lambda \boldsymbol{x} + \varphi \boldsymbol{y}, \boldsymbol{z} >= \lambda < \boldsymbol{x}, \boldsymbol{z} > + \varphi < \boldsymbol{y}, \boldsymbol{z} >$$

$$< \boldsymbol{x}, \lambda \boldsymbol{y} + \varphi \boldsymbol{z} >= \lambda < \boldsymbol{x}, \boldsymbol{y} > + \varphi < \boldsymbol{x}, \boldsymbol{z} >$$

$$(\lambda x_1 + \varphi y_1) z_1 - ((\lambda x_1 + \varphi y_1) z_2 + (\lambda x_2 + \varphi y_2) z_1) + 2(\lambda x_2 \varphi y_2) z_2$$

$$(\lambda x_1 - \lambda x_1 z_2 - \lambda x_2 z_1 + 2\lambda x_2 z_2) + (\varphi y_1 z_1 - \varphi y_1 z_2 - \varphi y_2 z_1 + 2\varphi y_2 z_2)$$

$$\lambda(x_1 - x_1 z_2 - x_2 z_1 + 2x_2 z_2) + \varphi(y_1 z_1 - y_1 z_2 - y_2 z_1 + y_2 z_2)$$

$$\lambda < \boldsymbol{x}, \boldsymbol{z} > + \varphi < \boldsymbol{y}, \boldsymbol{z} >$$

$$x_1(\lambda y_1 + \varphi z_1) - (x_1(\lambda y_2 + \varphi z_2) + x_2(\lambda y_2 + \varphi z_1)) + 2x_2(\lambda y_2 + \varphi z_2)$$

$$\lambda(x_1 y_1 - x_1 y_2 - x_2 y_2 + 2x_2 y_2) + \varphi(x_1 z_1 - x_1 z_2 - x_2 z_1 + 2x_2 z_2)$$

$$\lambda < \boldsymbol{x}, \boldsymbol{y} > + \varphi < \boldsymbol{x}, \boldsymbol{z} >$$

so indeed the function is a bilinear mapping.

Now we check for positive definiteness

$$< \boldsymbol{x}, \boldsymbol{x} >= x_1^2 - (x_1 x_2 + x_2 x_1) + 2(x_2^2) = x_1^2 - 2x_1 x_2 + 2x_2^2 \Rightarrow -x_2^2 = (x_1 + x_2)^2$$

the only solution for the above equation is $(0, 0)$ thus is indeed positive definite

and now we check for the symmetric property

$$< \boldsymbol{x}, \boldsymbol{y} > - < \boldsymbol{y}, \boldsymbol{x} >= \boldsymbol{0}$$

$$x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2(x_2 y_2) - y_1 x_1 + (y_1 x_2 + y_2 x_1) - 2(y_2 x_2)$$

$$-(x_1 y_2 + x_2 y_1) + (y_1 x_2 + y_2 x_1)$$

$$\boldsymbol{0}$$

so indeed is symmetric, as the mapping is bilinear, positive definite and symmetric is an inner product for the vector space $V = \mathbb{R}^2$ assuming the field $\mathbb{F} = \mathbb{R}$.