We consider $(\mathbb{R} \setminus \{-1\}, *)$, where

$$a * b := ab + a + b, a, b \in \mathbb{R}\{-1\}$$

a . Show that $(\mathbb{R} \setminus \{-1\}, *)$ is an abelian group.
b . Solve

$$3 * x * x = 15$$

in the abelian group $(\mathbb{R} \setminus \{-1\}, *)$, where $*$ is defined in

To show $(\mathbb{R} \setminus \{-1\}, *)$ is an abelian group we must also prove the following properties

- associativity
- existence of an identity element
- existence of an inverse element
- closure property

to prove associativity:

$$a * (b * c) = (a * b) * c$$

$$a * (b * c)$$

$$a * (bc + b + c)$$

$$a(bc + b + c) + bc + b + c + a$$

$$abc + ab + ac + b + c + a \text{ , applying distribututy of multiplication}$$

now we expand the other expression

$$(a * b) * c$$

$$(ab + a + b) * c$$

$$c(ab + a + b) + ab + a + b + c$$

$$cab + ca + cb + ab + a + b + c \text{ , applying distribututy of multiplication}$$

note that by commutativity of multiplication and addition both results are the same, meaning

$$a * (b * c) = (a * b) * c$$

to prove identity element,

$$a * e = a$$

$$a * e$$

$$ae + a + e$$

$$a \text{ , set e} = 0$$

thus identity element also holds if

$$a = 0$$

now we prove inverse element,

$$a * a^{-1} = e$$

$$a * a^{-1}$$

$$aa^{-1} + a + a^{-1}$$

$$1 + a + a^{-1} \text{ , by definiton of inverse in multiplication}$$

$$a^{-1} = -a - 1 = -(a + 1) \text{ , note that a != -1, otherwise it would be the identity element}$$

now we prove closure property,

$$a * b \in \mathbb{R} \setminus \{-1\}$$

$$a * b$$

$$ab + a + b$$

$$a(b + 1) + b \text{ , rewriting the expression}$$

$$a(b + 1) + b = -1 = \frac{-1 - b}{b + 1} = -\frac{1 + b}{b + 1} = -1 \text{ , value of a required for -1 to be in } \mathbb{R}$$

impossible for $a = -1$ due to set restrictions, analogous logic arrive at the same conclusion for b, thus $(\mathbb{R} \setminus \{-1\}, *)$ is closed

it must be the case that $(\mathbb{R} \setminus \{-1\}, *)$ is a valid group due to the validity of previous properties

we now prove it's abelian, $a * b = b * a$

$$a * b$$

$$ab + a + b$$

$$b * a$$

$$b * a + a + b$$

they are equal iff $ab = ba$, but multiplication has the commutativity property, as such they are equal and the $(\mathbb{R} \setminus \{-1\}, *)$ group is abelian ■

solving b now

$$
\begin{aligned}
3 * x * x &= 15 \\
(3 * x) * x &= 15 \\
(3x + 3 + x) * x &= 15 \\
x(3x + 3 + x) + 3x + 3 + x + x &= 15 \\
3x^2 + 3x + x^2 + 3x + 3 + 2x &= 15 \\
4x^2 + 8x - 12 &= 0 \\
x^2 + 2x - 3 &= 0 \text{ , simplifying the equation} \\
x = \frac{-2 \pm \sqrt{4 - 4(-3)}}{2} &\text{ , applying Bhaskara equation} \\
x = \frac{-2 \pm 4}{2} = -1 \pm 2 &= \{1, -3\}
\end{aligned}
$$