

**Wich of the following sets are subspaces of  $\mathbb{R}^3$ ?**

**a.**

$$A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) \mid \lambda, \mu \in \mathbb{R}\} \quad (1)$$

to check such set is a subspace we need to verify if it has the properties of a vector space

**identity**

first we check for identity element, defined as follows

$$x + \text{id}(A) = x \quad (2)$$

$$(\lambda, \lambda + \mu^3, \lambda - \mu^3) + (x, y, z) = (\lambda, \lambda + \mu^3, \lambda - \mu^3) \quad (3)$$

$$(\lambda + x, \lambda + \mu^3 + y, \lambda - \mu^3 + z) = (\lambda, \lambda + \mu^3, \lambda - \mu^3) \quad (4)$$

by inspection we deduce this element must be  $(0, 0, 0)$ , and indeed if we set  $\mu = 0, \lambda = 0$  we have such element, so this structure has the identity element.

**inverse element**

now we check if each element has an inverse

$$x + x^{-1} = 0 \quad (5)$$

$$(\lambda, \lambda + \mu^3, \lambda - \mu^3) + (x, y, z) = (0, 0, 0) \quad (6)$$

$$(\lambda + x, \lambda + \mu^3 + y, \lambda - \mu^3 + z) = (0, 0, 0) \quad (7)$$

and we deduce that  $x = -\lambda, y = -(\lambda + \mu^3), z = -\lambda + \mu^3$

and indeed as both variables are elements of  $\mathbb{R}$  and  $(\mathbb{R}, +)$  is a group, then it must be the case that both values are contained in  $\mathbb{R}$ . so this structure satisfy the inverse property.

closure is guaranteed due to the fact that  $(\mathbb{R}, +)$  is a group.

**associativity**

now we check for associativity

$$\begin{aligned} &v_1 + (v_2 + v_3) \\ &(\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3) + ((\lambda_2 + \lambda_3), (\lambda_2 + \lambda_3) + (\mu_2^3 + \mu_3^3), (\lambda_2 + \lambda_3) - (\mu_2^3 + \mu_3^3)) \end{aligned} \quad (8)$$

$$(\lambda_1 + \lambda_2 + \lambda_3, (\lambda_1 + \lambda_2 + \lambda_3) + (\mu_1^3 + \mu_2^3 + \mu_3^3), (\lambda_1 + \lambda_2 + \lambda_3) - (\mu_1^3 + \mu_2^3 + \mu_3^3))$$

$$\begin{aligned} &(v_1 + v_2) + v_3 \\ &(\lambda_1 + \lambda_2, (\lambda_1 + \lambda_2) + (\mu_1^3 + \mu_2^3), (\lambda_1 + \lambda_2) - (\mu_1^3 + \mu_2^3)) + (\lambda_3, \lambda_3 + \mu_3^3, \lambda_3 - \mu_3^3) \end{aligned} \quad (9)$$

$$(\lambda_1 + \lambda_2 + \lambda_3, (\lambda_1 + \lambda_2 + \lambda_3) + (\mu_1^3 + \mu_2^3 + \mu_3^3), (\lambda_1 + \lambda_2 + \lambda_3) - (\mu_1^3 + \mu_2^3 + \mu_3^3))$$

so associativity is preserved

**group is abelian**

now we need to check if the group is abelian

$$v_1 + v_2 = v_2 + v_1 \quad (10)$$

$$\begin{aligned} & (\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3) + (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3) \\ & (\lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \mu_1^3 + \mu_2^3, \lambda_1 + \lambda_2 - (\mu_1^3 + \mu_2^3)) \end{aligned} \quad (11)$$

$$\begin{aligned} & (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3) + (\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3) \\ & (\lambda_2 + \lambda_1, \lambda_2 + \lambda_1 + \mu_2^3 + \mu_1^3, \lambda_2 + \lambda_1 - (\mu_2^3 + \mu_1^3)) \end{aligned} \quad (12)$$

note that  $(\mathbb{R}, +)$  is abelian, so  $\lambda_1 + \lambda_2 = \lambda_2 + \lambda_1$ ,  $\mu_1^3 + \mu_2^3 = \mu_2^3 + \mu_1^3$  that implies the last coordinate has the form  $x - y$  in both (11) and (12), so the group is abelian

**distributivity of vectors**

now we check for distributivity property

$$\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2 \quad (13)$$

$$\begin{aligned} & \lambda((\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3) + (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3)) \\ & \lambda((\lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \mu_1^3 + \mu_2^3, \lambda_1 + \lambda_2 - (\mu_1^3 + \mu_2^3))) \\ & (\lambda(\lambda_1 + \lambda_2), \lambda(\lambda_1 + \lambda_2) + \lambda(\mu_1^3 + \mu_2^3), \lambda(\lambda_1 + \lambda_2) - \lambda(\mu_1^3 + \mu_2^3)) \end{aligned} \quad (14)$$

$$\begin{aligned} & (\lambda\lambda_1, \lambda\lambda_1 + \lambda\mu_1^3, \lambda\lambda_1 - \lambda\mu_1^3) + (\lambda\lambda_2, \lambda\lambda_2 + \lambda\mu_2^3, \lambda\lambda_2 - \lambda\mu_2^3) \\ & (\lambda(\lambda_1 + \lambda_2), \lambda(\lambda_1 + \lambda_2) + \lambda(\mu_1^3 + \mu_2^3), \lambda(\lambda_1 + \lambda_2) - \lambda(\mu_1^3 + \mu_2^3)) \end{aligned} \quad (15)$$

so distributivity property holds

**distributivity of scalars**

$$(\eta + \psi)v = \eta v + \psi v \quad (16)$$

$$\begin{aligned} & (\eta + \psi)(\lambda, \lambda + \mu^3, \lambda - \mu^3) \\ & (\eta\lambda + \psi\lambda, \eta\lambda + \psi\lambda + \eta\mu^3 + \psi\mu^3, \eta\lambda + \psi\lambda - \eta\mu^3 - \psi\mu^3) \end{aligned} \quad (17)$$

$$\begin{aligned} & \eta(\lambda, \lambda + \mu^3, \lambda - \mu^3) + \psi(\lambda, \lambda + \mu^3, \lambda - \mu^3) \\ & (\eta\lambda + \psi\lambda, \eta\lambda + \psi\lambda + \eta\mu^3 + \psi\mu^3, \eta\lambda + \psi\lambda - \eta\mu^3 - \psi\mu^3) \end{aligned} \quad (18)$$

**associativity of scalars**

we now prove associativity

$$\eta(\psi v_1) = (\eta\psi)v_1$$

$$\begin{aligned}
& \eta(\psi(\lambda, \lambda + \mu^3, \lambda - \mu^3)) \\
& \eta(\psi\lambda, \psi\lambda + \psi\mu^3, \psi\lambda - \psi\mu^3) \\
& (\eta\psi\lambda + \eta\psi\mu^3, \eta\psi\lambda - \eta\psi\mu^3)
\end{aligned} \tag{19}$$

$$\begin{aligned}
& (\eta\psi)v_1 \\
& (\eta\psi\lambda, \eta\psi\lambda + \eta\psi\mu^3, \eta\psi\lambda - \eta\psi\mu^3)
\end{aligned} \tag{20}$$

now we prove identity element of scalar multipliation

**scalar identity**

$$\eta v = v \tag{21}$$

$$\begin{aligned}
& \eta(\lambda, \lambda + \mu^3, \lambda - \mu^3) \\
& (\eta\lambda, \eta\lambda + \eta\mu^3, \eta\lambda - \eta\mu^3)
\end{aligned} \tag{22}$$

this means  $\eta = 1, 1 \in \mathbb{R}$

so this set is indeed a subset of  $\mathbb{R}^3$

**b.**

$$B = \{(\lambda^2, -\lambda, 0) \mid \lambda \in \mathbb{R}\} \tag{23}$$

first we check if this structure is an abelian group

**identity**

$$x + \text{id}(x) = x \tag{24}$$

$$\begin{aligned}
& (\lambda^2, -\lambda, 0) + (x, y, z) = (\lambda^2, -\lambda, 0) \\
& (\lambda^2 + x, -\lambda + y, z) = (\lambda^2, -\lambda, 0)
\end{aligned} \tag{25}$$

we deduce the identity element is  $(0, 0, 0)$  and indeed because  $(\mathbb{R}, +)$  is a group 0 must belong to it

**inverse**

$$x + x^{-1} = 0 \tag{26}$$

$$\begin{aligned}
& (\lambda^2, -\lambda, 0) + (x, y, z) = (0, 0, 0) \\
& (\lambda^2 + x, -\lambda + y, z) = (0, 0, 0)
\end{aligned} \tag{27}$$

we deduce that  $x = -\lambda^2, y = \lambda, z = 0$  and  $\lambda \in \mathbb{R}$  so indeed this structure has an inverse

**closure**

$$x + y \in \mathbb{R}^3 \tag{28}$$

$$\begin{aligned}
&(\lambda_0^2, -\lambda_0^1, 0) + (\lambda_1^2, -\lambda_1^1, 0) \\
&(\lambda_0^2 + \lambda_1^2, -(\lambda_0^1 + \lambda_1^1), 0)
\end{aligned} \tag{29}$$

remembering that  $\lambda \in \mathbb{R}$  and  $(\mathbb{R}, +)$  is a group, then indeed closure is a valid property for this structure.

#### associativity

$$x + (y + z) = (x + y) + z \tag{30}$$

$$\begin{aligned}
&(\lambda_0^2, -\lambda_0^1, 0) + ((\lambda_1^2, -\lambda_1^1, 0) + (\lambda_2^2, -\lambda_2^1, 0)) \\
&(\lambda_0^2, -\lambda_0^1, 0) + ((\lambda_1^2 + \lambda_2^2, -(\lambda_1^1 + \lambda_2^1), 0) \\
&(\lambda_0^2 + \lambda_1^2 + \lambda_2^2, -(\lambda_0^1 + \lambda_1^1 + \lambda_2^1), 0)
\end{aligned} \tag{31}$$

$$\begin{aligned}
&((\lambda_0^2, -\lambda_0^1, 0) + (\lambda_1^2, -\lambda_1^1, 0)) + (\lambda_2^2, -\lambda_2^1, 0) \\
&(\lambda_0^2 + \lambda_1^2, -(\lambda_0^1 + \lambda_1^1), 0) + (\lambda_2^2, -\lambda_2^1, 0) \\
&(\lambda_0^2 + \lambda_1^2 + \lambda_2^2, -(\lambda_0^1 + \lambda_1^1 + \lambda_2^1), 0)
\end{aligned} \tag{32}$$

now we check if this group is abelian, that is can commute the operands

$$x + y = y + x \tag{33}$$

$$\begin{aligned}
&(\lambda_0^2, -\lambda_0^1, 0) + (\lambda_1^2, -\lambda_1^1, 0) \\
&(\lambda_0^2 + \lambda_1^2, -(\lambda_0^1 + \lambda_1^1), 0)
\end{aligned} \tag{34}$$

$$\begin{aligned}
&(\lambda_1^2, -\lambda_1^1, 0) + (\lambda_0^2, -\lambda_0^1, 0) \\
&(\lambda_1^2 + \lambda_0^2, -(\lambda_1^1 + \lambda_0^1), 0)
\end{aligned} \tag{35}$$

note that  $(\mathbb{R}, +)$  is an abelian group

now we check distributivity of vectors, associativity of scalars and identity scalar

#### distributivity of vectors

$$\mu(v_1 + v_2) = \mu v_1 + \mu v_2 \tag{36}$$

$$\begin{aligned}
&\mu((\lambda_0^2, -\lambda_0^1, 0) + (\lambda_1^2, -\lambda_1^1, 0)) \\
&\mu(\lambda_0^2 + \lambda_1^2, -(\lambda_0^1 + \lambda_1^1), 0) \\
&(\mu\lambda_0^2 + \mu\lambda_1^2, -\mu\lambda_0^1 - \mu\lambda_1^1, 0)
\end{aligned} \tag{37}$$

$$\begin{aligned}
& \mu(\lambda_0^2, -\lambda_0^1, 0) + \mu(\lambda_1^2, -\lambda_1^1, 0) \\
& (\mu\lambda_0^2, -\mu\lambda_0^1, 0) + (\mu\lambda_1^2, -\mu\lambda_1^1, 0) \\
& (\mu\lambda_0^2 + \mu\lambda_1^2, -\mu\lambda_0^1 - \mu\lambda_1^1, 0)
\end{aligned} \tag{38}$$

so this property holds

**distributivity of scalars**

$$(\mu + \psi)v = \mu v + \psi v$$

$$\begin{aligned}
& (\mu + \psi)(\lambda_0^2, -\lambda_0^1, 0) \\
& (\mu\lambda_0^2 + \psi\lambda_0^2, -\mu\lambda_0^1 - \psi\lambda_0^1, 0)
\end{aligned} \tag{39}$$

$$\begin{aligned}
& \mu(\lambda_0^2, -\lambda_0^1, 0) + \psi(\lambda_0^2, -\lambda_0^1, 0) \\
& (\mu\lambda_0^2 + \psi\lambda_0^2, -(\mu\lambda_0^1 + \psi\lambda_0^1), 0)
\end{aligned} \tag{40}$$

so this property holds

**associativity of scalars**

$$\mu(\psi v) = (\mu\psi)v \tag{41}$$

$$\begin{aligned}
& \mu(\psi(\lambda_0^2, -\lambda_0^1, 0)) \\
& \mu(\psi\lambda_0^2, -\psi\lambda_0^1, 0) \\
& (\mu\psi\lambda_0^2, -\mu\psi\lambda_0^1, 0)
\end{aligned} \tag{42}$$

$$\begin{aligned}
& (\mu\psi)(\lambda_0^2, -\lambda_0^1, 0) \\
& (\mu\psi\lambda_0^2, -\mu\psi\lambda_0^1, 0)
\end{aligned} \tag{43}$$

so this property holds

**identity**

$$1v = v \tag{44}$$

$$\begin{aligned}
& 1(\lambda_0^2, -\lambda_0^1, 0) = (\lambda_0^2, -\lambda_0^1, 0) \\
& (1\lambda_0^2, -1\lambda_0^1, 0) = (\lambda_0^2, -\lambda_0^1, 0)
\end{aligned} \tag{45}$$

so must be one this scalar.

indeed this structure is a subspace of the vector space  $\mathbb{R}^3$

**c.**

Let  $\gamma$  be in  $\mathbb{R}$

$$C = \{(\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3 \mid \eta_1 - 2\eta_2 + 3\eta_3 = \gamma\} \tag{46}$$

again we check if this structure is a group

**identity element**

$$x + 1 = x \quad (47)$$

$$\begin{aligned} (\eta_1, \eta_2, \eta_3) + (x, y, z) &= (\eta_1, \eta_2, \eta_3) \\ (\eta_1 + x, \eta_2 + y, \eta_3 + z) &= (\eta_1, \eta_2, \eta_3) \\ \eta_1 + x - 2(\eta_2 + y) + 3(\eta_3 + z) &= \gamma \end{aligned} \quad (48)$$

that implies  $(x, y, z) = (0, 0, 0) \Rightarrow 0 - 0 + 0 = 0$

so identity holds

**inverse element**

$$x + x^{-1} = 0 \quad (49)$$

$$\begin{aligned} (\eta_1, \eta_2, \eta_3) + (x, y, z) &= (0, 0, 0) \\ (\eta_1 + x, \eta_2 + y, \eta_3 + z) &= (0, 0, 0) \end{aligned} \quad (50)$$

that implies  $(x, y, z) = (-\eta_1, -\eta_2, -\eta_3) \Rightarrow -\eta_1 + 2\eta_2 - 3\eta_3 = \gamma$

so inverse holds

**closure property**

$$x + y \in \mathbb{R}^3 \quad (51)$$

$$\begin{aligned} (\eta_1, \eta_2, \eta_3) + (\mu_1, \mu_2, \mu_3) \\ (\eta_1 + \mu_1, \eta_2 + \mu_2, \eta_3 + \mu_3) \end{aligned} \quad (52)$$

that implies  $\eta_1 + \mu_1 - 2(\eta_2 + \mu_2) + 3(\eta_3 + \mu_3) = \gamma$

and indeed this expression is closed in  $\mathbb{R}$

**associativity**

$$x + (y + z) = (x + y) + z \quad (53)$$

$$\begin{aligned} (\eta_1, \eta_2, \eta_3) + (\mu_1 + \omega_1, \mu_2 + \omega_2, \mu_3 + \omega_3) \\ (\eta_1 + \mu_1 + \omega_1, \eta_2 + \mu_2 + \omega_2, \eta_3 + \mu_3 + \omega_3) \end{aligned} \quad (54)$$

that implies  $\eta_1 + \mu_1 + \omega_1 - 2(\eta_2 + \mu_2 + \omega_2) + 3(\eta_3 + \mu_3 + \omega_3) = \gamma$

$$\begin{aligned} (\eta_1 + \mu_1, \eta_2 + \mu_2, \omega_3 + \mu_3) + (\omega_1, \omega_2, \omega_3) \\ (\eta_1 + \mu_1 + \omega_1, \eta_2 + \mu_2 + \omega_2, \omega_3 + \mu_3 + \omega_3) \end{aligned} \quad (55)$$

so indeed this structure is a group, now we check if it's abelian

$$x + y = y + x \quad (56)$$

$$\begin{aligned} &(\eta_1, \eta_2, \eta_3) + (\mu_1, \mu_2, \mu_3) \\ &(\eta_1 + \mu_1, \eta_2 + \mu_2, \eta_3 + \mu_3) \end{aligned} \quad (57)$$

that implies  $\eta_1 + \mu_1 - 2(\eta_2 + \mu_2) + 3(\eta_3 + \mu_3) = \gamma_1$

$$\begin{aligned} &(\mu_1, \mu_2, \mu_3) + (\eta_1, \eta_2, \eta_3) \\ &(\mu_1 + \eta_1, \mu_2 + \eta_2, \mu_3 + \eta_3) \end{aligned} \quad (58)$$

that implies  $\mu_1 + \eta_1 - 2(\mu_2 + \eta_2) + 3(\mu_3 + \eta_3) = \gamma_2$

note however that  $(\mathbb{R}, +)$  is closed, so indeed this group is abelian,  $\Rightarrow \gamma_1 = \gamma_2$

now we check for distributivity of vectors, associativity of scalars and the identity scalar

#### **distributivity of vectors**

$$\psi(v_1 + v_2) = \psi v_1 + \psi v_2 \quad (59)$$

$$\begin{aligned} &\psi((\eta_1, \eta_2, \eta_3) + (\mu_1, \mu_2, \mu_3)) \\ &\psi((\eta_1 + \mu_1, \eta_2 + \mu_2, \eta_3 + \mu_3)) \\ &(\psi\eta_1 + \psi\mu_1, \psi\eta_2 + \psi\mu_2, \psi\eta_3 + \psi\mu_3) \end{aligned} \quad (60)$$

that implies  $(\psi\eta_1 + \psi\mu_1) - 2(\psi\eta_2 + \psi\mu_2) + 3(\psi\eta_3 + \psi\mu_3) = \gamma_1$

$$\begin{aligned} &\psi(\eta_1, \eta_2, \eta_3) + \psi(\mu_1, \mu_2, \mu_3) \\ &(\psi\eta_1, \psi\eta_2, \psi\eta_3) + (\psi\mu_1, \psi\mu_2, \psi\mu_3) \\ &(\psi\eta_1 + \psi\mu_1, \psi\eta_2 + \psi\mu_2, \psi\eta_3 + \psi\mu_3) \end{aligned} \quad (61)$$

so distributivity of vectors hold

#### **distributivity of scalars**

$$(\psi + \mu)v = \psi v + \mu v \quad (62)$$

$$\begin{aligned} &(\psi + \mu)(\eta_1, \eta_2, \eta_3) \\ &(\psi\eta_1 + \mu\eta_1, \psi\eta_2 + \mu\eta_2, \psi\eta_3 + \mu\eta_3) \end{aligned} \quad (63)$$

$$\begin{aligned} &\psi(\eta_1, \eta_2, \eta_3) + \mu(\eta_1, \eta_2, \eta_3) \\ &(\psi\eta_1, \psi\eta_2, \psi\eta_3) + (\mu\eta_1, \mu\eta_2, \mu\eta_3) \\ &(\psi\eta_1 + \mu\eta_1, \psi\eta_2 + \mu\eta_2, \psi\eta_3 + \mu\eta_3) \end{aligned} \quad (64)$$

so the property holds

### associativity of scalars

$$\psi(\mu v) = (\psi \mu)v \quad (65)$$

$$\begin{aligned} & \psi(\mu(\eta_1, \eta_2, \eta_3)) \\ & \psi((\mu\eta_1, \mu\eta_2, \mu\eta_3)) \\ & (\psi\mu\eta_1, \psi\mu\eta_2, \psi\mu\eta_3) \end{aligned} \quad (66)$$

that implies  $\psi\mu\eta_1 - 2\psi\mu\eta_2 + 3\psi\mu\eta_3$

$$\begin{aligned} & (\psi\mu)(\eta_1, \eta_2, \eta_3) \\ & (\psi\mu\eta_1, \psi\mu\eta_2, \psi\mu\eta_3) \end{aligned} \quad (67)$$

so the property holds

### identity of scalars

$$1v = v \quad (68)$$

$$1(\eta_1, \eta_2, \eta_3) \quad (69)$$

so indeed this element is 1 and  $1 \in \mathbb{R}$

so this structure is a vector subspace of  $\mathbb{R}^3$

**d.**

$$D = \{(\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3 \mid \eta_2 \in \mathbb{Z}\} \quad (70)$$

this can be easily disproved by considering scalar associativity

$$\psi(\mu v) = (\psi \mu)v \quad (71)$$

$$\begin{aligned} & \psi(\mu\eta_1, \mu\eta_2, \mu\eta_3) \\ & (\psi\mu\eta_1, \psi\mu\eta_2, \psi\mu\eta_3) \end{aligned} \quad (72)$$

if we consider that  $\psi, \mu \in \mathbb{R}$  than any value  $(\psi\mu\eta_2) \notin \mathbb{Z}$  invalidates this structure as a vector subspace of  $\mathbb{R}^3$