Using Gaussian elimination, find the set $\mathcal{S}$ of all solutions of the following inhomogeneous linear systems $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where **A** and **b** are defined as follows:

a

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \tag{1}$$

considering an augmented matrix

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \tag{2}$$

note that we can produce the particular solution by considering

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \leftarrow -R_0 + R_2 \tag{3}$$

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0 & -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \leftarrow -R_1 + R_0 \tag{4}$$

note we have $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as pivots, so the particular solution is

$$\begin{bmatrix} 0 & 2 & 0 & -1 & 0 & -1 \end{bmatrix}^T \tag{5}$$

and we can easily find the general solutions , note there's no need to find the coordinates for the zero vector

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{6}$$

so the general solution, remember that must follow this form $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \lambda_1 x$

$$\begin{bmatrix} 0 & 1 & 0 & 1 & -1 & 0 \end{bmatrix}^T \tag{7}$$

so our final solution is

$$\left\{ (x \in \mathbb{R}^5, \lambda_1 \in \mathbb{R}), \boldsymbol{x} := \lambda_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} \right\} \tag{8}$$

b

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \tag{9}$$