Consider the following bivariate distribution $p(x, y)$ of two discrete random variables X and Y.

	x_1	x_2	x_3	x_4	x_5
y_1	**0.01**	**0.02**	**0.03**	**0.1**	**0.1**
y_2	**0.05**	**0.1**	**0.05**	**0.07**	**0.2**
y_4	**0.1**	**0.05**	**0.03**	**0.05**	**0.04**

Compute:

a. The marginal distribution $p(x)$ and $p(y)$

By definition the margin distribution is defined as

$$p(Z = z) = \Sigma_i p(Z = z_i)$$

where Z is a random variable and $p(X = x)$ means the probability of a x outcome being selected note that this quantity is always one, so we are instead interested in the possible values that x could assume thus

$$p\left(X = x_j\right) = \frac{n_{ij}}{N} = \frac{c_j}{N}$$

as such we have

$$p(X = x_1) = p(x_1) = \frac{c_1}{1} = \frac{0.01 + 0.05 + 0.1}{1} = 0.16$$

$$p(X = x_2) = p(x_2) = \frac{c_2}{1} = \frac{0.02 + 0.1 + 0.05}{1} = 0.17$$

$$p(X = x_3) = p(x_3) = \frac{c_3}{1} = \frac{0.03 + 0.05 + 0.03}{1} = 0.11$$

$$p(X = x_4) = p(x_4) = \frac{c_4}{1} = \frac{0.1 + 0.07 + 0.05}{1} = 0.22$$

$$p(X = x_5) = p(x_5) = \frac{c_5}{1} = \frac{0.1 + 0.2 + 0.04}{1} = 0.34$$

and as expected $(0.16 + 0.17 + 0.11 + 0.22 + 0.34) = 1$

$$p(Y = y_i) = \frac{n_{ji}}{N} = \frac{r_i}{N}$$

$$p(Y = y_1) = p(y_1) = \frac{r_1}{1} = \frac{0.01 + 0.02 + 0.03 + 0.1 + 0.1}{1} = 0.26$$

$$p(Y = y_2) = p(y_2) = \frac{r_2}{1} = \frac{0.05 + 0.1 + 0.05 + 0.07 + 0.2}{1} = 0.47$$

$$p(Y = y_3) = p(y_3) = \frac{r_3}{1} = \frac{0.1 + 0.05 + 0.03 + 0.05 + 0.04}{1} = 0.27$$

and again the sum ammounts to 1

b. The conditional distributions $p(x \mid Y = y_1)$ and $p(y \mid X = x_3)$

The conditional probability is defined as either

$$p(Y = y_i|X = x_j) = \frac{n_{ji}}{c_j}$$

$$p(X = x_j \mid Y = y_i) = \frac{n_{ij}}{r_i}$$

thus we deduce that

$$p(Y = y_i)p(X = x_j \mid Y = y_i) = p(X = x_j, Y = y_i) \Rightarrow p(X = x_j \mid Y = y_i) = \frac{p(X = x_j, Y = y_i)}{p(Y = y_i)}$$

$$\frac{1}{P(Y = y_1)}\Sigma_j p(X = x_j, Y = y_1) = \frac{0.01 + 0.02 + 0.03 + 0.1 + 0.1}{0.26} = 1$$

so if we call Y and then X and define that as $P(Y = y_1 \times X)$ as the probability of a ordered pair, then it's guaranteed that such probability must always be non zero.

so the conditional probability for each x_j is given by

$$p(X = x_1|Y = y_1) = \frac{n_{11}}{r_1} = \frac{0.01}{0.26} = 0.0384$$

$$p(X = x_2|Y = y_1) = \frac{n_{12}}{r_1} = \frac{0.02}{0.26} = 0.0769$$

$$p(X = x_3|Y = y_1) = \frac{n_{13}}{r_1} = \frac{0.03}{0.26} = 0.115$$

$$p(X = x_4|Y = y_1) = \frac{n_{14}}{r_1} = \frac{0.1}{0.26} = 0.384$$

$$p(X = x_5|Y = y_1) = \frac{n_{15}}{r_1} = \frac{0.1}{0.26} = 0.384$$

applying analogous reasoning for the other case

$$p(Y = y_1 \mid X = x_3) = \frac{n(13)}{c_3} = \frac{0.03}{0.11} = 0.272$$

$$p(Y = y_2 \mid X = x_3) = \frac{n(23)}{c_3} = \frac{0.05}{0.11} = 0.454$$

$$p(Y = y_3 \mid X = x_3) = \frac{n(33)}{c_3} = \frac{0.03}{0.11} = 0.272$$