Consider two subspaces U_1 and U_2, where U_1 is the solution space of the homogeneous equation system $\boldsymbol{A_1}\boldsymbol{x} = 0$ and U_2 is the solution space of the homogeneous equation system $\boldsymbol{A_2}\boldsymbol{x} = 0$ with

$$\boldsymbol{A_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \boldsymbol{A_2} = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix} \tag{1}$$

a. Determine the dimension of U_1, U_2

by the definiton of dimension, $|\boldsymbol{x_k}|$ where the $|\ |$ indicates cardinality of a set and $\boldsymbol{x_k}$ is one of minimal generating sets

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \leftarrow -R_1 + R_3 \tag{2}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 0 & 2 & 0 \end{bmatrix} \leftarrow -R_1 + R_0 \tag{3}$$

$$\begin{bmatrix} 0 & 2 & 0 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 0 & 2 & 0 \end{bmatrix} \leftarrow -2R_1 + R_2 \tag{4}$$

$$\begin{bmatrix} 0 & 2 & 0 \\ 1 & -2 & -1 \\ 0 & 5 & 5 \\ 0 & 2 & 0 \end{bmatrix} \leftarrow R_1 \text{ switch } R_0 \tag{5}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 0 \\ 0 & 5 & 5 \\ 0 & 2 & 0 \end{bmatrix} \leftarrow -R_1 + R_3 \tag{6}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 0 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow R_1 \text{ switch } R_2 \tag{7}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 5 & 5 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow -\alpha R_2 + R_1 \tag{8}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 5 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow R_2 \text{ switch } R_1 \tag{9}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow -R_1 + R_0 \tag{10}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow -\beta R_2 + R_0 \tag{11}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

so the dimension is 3

for $\boldsymbol{A_2}$

$$\begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix} \leftarrow -R_3 + R_0 \tag{13}$$

$$\begin{bmatrix} 0 & -2 & -2 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix} \leftarrow -3R_1 + R_3 \tag{14}$$

$$\begin{bmatrix} 0 & -2 & -2 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 0 & 5 & -7 \end{bmatrix} \leftarrow -7R_1 + R_2 \tag{15}$$

$$\begin{bmatrix} 0 & -2 & -2 \\ 1 & 2 & 3 \\ 0 & -19 & -19 \\ 0 & 5 & -7 \end{bmatrix} \leftarrow R_1 \text{ switch } R_0 \tag{16}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & -19 & -19 \\ 0 & 5 & -7 \end{bmatrix} \leftarrow \alpha R_1 + R_2 \tag{17}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 5 & -7 \end{bmatrix} \leftarrow R_3 \text{ switch } R_3 \tag{18}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & 5 & -7 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \frac{5}{2} R_1 \text{ switch } R_2 \tag{19}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & -12 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \beta R_2 + R_1 \tag{20}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & -12 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \alpha R_1 + R_0 \tag{21}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & -12 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \gamma R_2 + R_0 \tag{22}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -12 \\ 0 & 0 & 0 \end{bmatrix} \tag{23}$$

so the dimension is also 3

b. Determine basis of U_1 and U_2
check above answer

c. Determine the basis for $U_1 \cap U_2$

$$\begin{bmatrix} 1 & 0 & 1 & 3 & -3 & 0 \\ 1 & -2 & -1 & 1 & 2 & 3 \\ 2 & 1 & 3 & 7 & -5 & 2 \\ 1 & 0 & 1 & 3 & -1 & 2 \end{bmatrix} \leftarrow -R_1 + R_3 \tag{24}$$

$$\begin{bmatrix} 1 & 0 & 1 & 3 & -3 & 0 \\ 1 & -2 & -1 & 1 & 2 & 3 \\ 2 & 1 & 3 & 7 & -5 & 2 \\ 0 & 2 & 2 & 2 & -3 & -1 \end{bmatrix} \leftarrow -2R_1 + R_2 \tag{25}$$

$$\begin{bmatrix} 1 & 0 & 1 & 3 & -3 & 0 \\ 1 & -2 & -1 & 1 & 2 & 3 \\ 0 & 5 & 5 & 5 & -9 & -4 \\ 0 & 2 & 2 & 2 & -3 & -1 \end{bmatrix} \leftarrow -R_1 + R_0 \tag{26}$$

$$\begin{bmatrix} 0 & 2 & 2 & 2 & -5 & -3 \\ 1 & -2 & -1 & 1 & 2 & 3 \\ 0 & 5 & 5 & 5 & -9 & -4 \\ 0 & 2 & 2 & 2 & -3 & -1 \end{bmatrix} \leftarrow R_1 \text{ switch } R_0 \tag{27}$$

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 2 & 3 \\ 0 & 2 & 2 & 2 & -5 & -3 \\ 0 & 5 & 5 & 5 & -9 & -4 \\ 0 & 2 & 2 & 2 & -3 & -1 \end{bmatrix} \leftarrow -R_1 + R_3 \tag{28}$$

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 2 & 3 \\ 0 & 2 & 2 & 2 & -5 & -3 \\ 0 & 5 & 5 & 5 & -9 & -4 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{bmatrix} \leftarrow \frac{9}{2}R_3 + R_2 \tag{29}$$

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 2 & 3 \\ 0 & 2 & 2 & 2 & -5 & -3 \\ 0 & 5 & 5 & 5 & 0 & 5 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{bmatrix} \tag{30}$$

note that second, third and fourth column vector can be achieved through linear combination, so we can remove two of these as we please

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 2 & -5 & -3 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 2 & 2 \end{bmatrix} \tag{31}$$

notice we can obtain the last one by linear combination with following coefficients

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 1 \\ \lambda_3 &= 1 \end{aligned} \tag{32}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -5 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{33}$$

so our basis vectors are

$$\boldsymbol{x_k} = \{\boldsymbol{x_1}, \boldsymbol{x_4}, \boldsymbol{x_5}\} \tag{34}$$