Let $F = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - z = 0\}$ and $G = \{(a - b, a + b, a - 3b) \mid a, b \in \mathbb{R}\}$

a . Show that F and G are subspaces of $\mathbb{R}^3$

Again to show a structure is a subspace we need to show the following

- show that is an abelian group
- show associativity of scalar operations for multipliation
- show distributivity of scalar operations for addition
- show distributivity of vector sum operation
- show identity scalar exists in the set

Let's start by showing F and G are groups

- show that an identity element exist for addition
- show that an inverse element exist for addition
- show that associativity is a valid property
- show that any operation between two elements is contained in $\mathbb{R}^3$

identity element

$$x + \mathrm{id} = x \tag{1}$$

- proof for F

$$(x, y, z) + (a, b, c) = (x, y, z) \tag{2}$$

$$(x + a, y + b, z + c) = (x, y, z) \tag{3}$$

thus it must be the case

$$(0, 0, 0) \tag{4}$$

and indeed $0 + 0 - 0 = 0$

- proof for G

$$(a - b, a + b, a - 3b) + (x, y, z) = (a - b, a + b, a - 3b) \tag{5}$$

thus $(0, 0, 0)$ is the identity element and indeed, $0 \in \mathbb{R}$

inverse element

$$x + x^{-1} = 0 \tag{6}$$

- Proof for F

$$(x, y, z) + (a, b, c) = 0 \tag{7}$$

$$(x + a, y + b, z + c) = (0, 0, 0) \tag{8}$$

thus the inverse element

$$
\begin{aligned}
a &= -x \\
b &= -y \\
c &= -z
\end{aligned}
\tag{9}
$$

$$
-x - y + z = 0 \tag{10}
$$

notice this is the same as multipliation by -1 in the equality, so this element belongs to this subset of $\mathbb{R}^3$

- Proof for G

$$
(a - b, a + b, a - 3b) + (x, y, z) = (0, 0, 0) \tag{11}
$$

$$
\begin{aligned}
x &= b - a \\
y &= -(a + b) \\
z &= 3b - a
\end{aligned}
\tag{12}
$$

notice that $(\mathbb{R}, -)$ is closed as well as $(\mathbb{R}, +)$ so the triple $(x, y, z) \in \mathbb{R}^3$

associativity

$$
x + (y + z) = (x + y) + z \tag{13}
$$

- proof for F

$$
(x, y, z) + ((a, b, c) + (d, e, f)) \tag{14}
$$

$$
(x, y, z) + (a + d, b + e, c + f) \tag{15}
$$

$$
(x + a + d, y + b + e, z + c + f) \tag{16}
$$

$$
(x + a + d) + (y + b + e) - (z + c + f) \tag{17}
$$

now we prove the RHS

$$
((x, y, z) + (a, b, c)) + (d, e, f) \tag{18}
$$

$$
(x + a, y + b, z + c) + (d, e, f) \tag{19}
$$

$$
(x + a + d, y + b + e, z + c + f) \tag{20}
$$

$$
(x + a + d) + (y + b + e) - (z + c + f) \tag{21}
$$

- proof for G

$$
(a - b, a + b, a - 3b) + ((c - d, c + d, c - 3d) + (e - f, e + f, e - 3f)) \tag{22}
$$

$$
(a - b, a + b, a - 3b) + (c - d + e - f, c + d + e + f, c - 3d + e - 3f) \tag{23}
$$

$$(a-b+c-d+e-f, a+b+c+d+e+f, a-3b+c-3d+e-3f) \tag{24}$$

$$((a-b, a+b, a-3b)+(c-d, c+d, c-3d))+(e-f, e+f, e-3f) \tag{25}$$

$$(a-b+c-d, a+b+c+d, a-3b+c-3d)+(e-f, e+f, e-3f) \tag{26}$$

$$(a-b+c-d+e-f, a+b+c+d+e+f, a-3b+c-3d+e-3f) \tag{27}$$

so the property also hold for G

closure

- proof for F

$x+y \in \mathbb{R}$

$$(x, y, z)+(a, b, c) \tag{28}$$

$$(x+a, y+b, z+c) \tag{29}$$

note that $x, y, z, a, b, c \in \mathbb{R}$

and $(\mathbb{R}, +)$ is group, so closure is guaranteed

- proof for G

Analagous reasoning

abelian group

$x+y=y+x$

- proof for F

$$(x, y, z)+(a, b, c) \tag{30}$$

$$(x+a, y+b, z+c) \tag{31}$$

$$(a, b, c)+(x, y, z) \tag{32}$$

$$(a+x, b+y, c+z) \tag{33}$$

note that $(\mathbb{R}, +)$ is an abelian group, so because of that $a, x, b, y, c, z \in \mathbb{R} \Rightarrow (a+x, b+y, c+z) \in \mathbb{R}^3$

- proof for G

$$(a-b, a+b, a-3b)+(c-d, c+d, c-3d) \tag{34}$$

$$(a-b+c-d, a+b+c+d, a-3b+c-3d) \tag{35}$$

$$(c-d, c+d, c-3d)+(a-b, a+b, a-3b) \tag{36}$$

$$(c - d + a - b, c + d + a + b, c - 3d + a - 3b) \tag{37}$$

note that the subtractions operands always preserve their position in the operation so their value never change and because addition is commutative, as mentioned before, this property also holds.

distributivity of vector addition

$$\psi(v_1 + v_2) = \psi v_1 + \psi v_2 \tag{38}$$

- Proof for F

$$\psi((x, y, z) + (a, b, c)) \tag{39}$$

$$\psi(x + a, y + b, z + c) \tag{40}$$

$$(\psi x + \psi a, \psi y + \psi b, \psi z + \psi c) \tag{41}$$

$$\psi(x, y, z) + \psi(a, b, c) \tag{42}$$

$$(\psi x, \psi y, \psi z) + (\psi a, \psi b, \psi c) \tag{43}$$

$$(\psi x + \psi a, \psi y + \psi b, \psi z + \psi c) \tag{44}$$

so this property holds for F

$$\psi((a - b, a + b, a - 3b) + (c - d, c + d, c - 3d)) \tag{45}$$

$$\psi((a - b + c - d, a + b + c + d, a - 3b + c - 3d) \tag{46}$$

$$(\psi(a - b) + \psi(c - d), \psi(a + b) + \psi(c + d), \psi(a - 3b) + \psi(c - 3d)) \tag{47}$$

note that this is not function application, ψ is a scalar

$$\psi(a - b, a + b, a - 3b) + \psi(c - d, c + d, c - 3d) \tag{48}$$

$$(\psi(a - b), \psi(a + b), \psi(a - 3b)) + (\psi(c - d), \psi(c + d), \psi(c - 3d)) \tag{49}$$

$$(\psi(a - b) + \psi(c - d), \psi(a + b) +)\psi(c + d), \psi(a - 3b) + \psi(c - 3d)) \tag{50}$$

so the property holds

distributivity of scalars

$$(\psi + \mu)v_1 = \psi v_1 + \mu v_1 \tag{51}$$

- Proof for F

$$(\psi + \mu)(x, y, z) \tag{52}$$

$$\omega(x, y, z) \tag{53}$$

$$(\omega x, \omega y, \omega z) \tag{54}$$

$$\psi(x, y, z) + \mu(x, y, z)) \tag{55}$$

$$(\psi x, \psi y, \psi z) + (\mu x, \mu y, \mu z)) \tag{56}$$

$$(x(\psi + \mu), y(\psi + \mu), z(\psi + \mu)) \tag{57}$$

$$(x\omega, y\omega, z\omega) \tag{58}$$

- Proof for G

$$(\psi + \mu)(a - b, a + b, a - 3b) \tag{59}$$

$$\omega(a - b, a + b, a - 3b) \tag{60}$$

$$(\omega(a - b), \omega(a + b), \omega(a - 3b)) \tag{61}$$

$$\psi(a - b, a + b, a - 3b) + \mu(a - b, a + b, a - 3b) \tag{62}$$

$$(\psi a - b, \psi a + b, \psi a - 3b) + (\mu a - b, \mu a + b, \mu a - 3b) \tag{63}$$

$$(a - b(\psi + \mu), a + b(\psi + \mu), a - 3b(\psi + \mu)) \tag{64}$$

$$((a - b)\omega, (a + b)\omega, (a - 3b)\omega) \tag{65}$$

note that multiplication is commutative in $\mathbb{R}$

associativity of scalars

$$\psi(\mu v_1) = (\psi\mu)v_1 \tag{66}$$

- Proof for F

$$\psi(\mu(x, y, z)) \tag{67}$$

$$\psi(\mu x, \mu y, \mu z) \tag{68}$$

$$(\psi\mu x, \psi\mu y, \psi\mu z) \tag{69}$$

$$(\psi\mu)(x, y, z) \tag{70}$$

$$(\psi\mu x, \psi\mu y, \psi\mu z) \tag{71}$$

so the property holds

- Proof for G

$$\psi(\mu(a-b,a+b,a-3b)) \tag{72}$$

$$\psi(\mu(a-b),\mu(a+b),\mu(a-3b))) \tag{73}$$

$$(\psi\mu(a-b),\psi\mu(a+b),\psi\mu(a-3b))) \tag{74}$$

$$(\psi\mu)(a-b,a+b,a-3b) \tag{75}$$

$$(\psi\mu a-b,\psi\mu a+b,\psi\mu a-3b) \tag{76}$$

so the property holds

identity element for scalar multiplication
note that $1 \in \mathbb{R}$ for both cases so this property also holds

so both structures are vector subspaces of $\mathbb{R}^3$

b. Calculate $F \cap G$ without resorting to any basis vector
by definition the intersection is

$$U \cap A \to x \in A \wedge x \in U \tag{77}$$

where U and A are sets

$$(x,y,z) = (a-b,a+b,a-3b) \tag{78}$$

$$\begin{gathered} x = a-b \\ y = a+b \\ z = a-3b \end{gathered} \tag{79}$$

and

$$(a-b)+(a+b)-(a-3b) \tag{80}$$

$$a+3b = 0 \tag{81}$$

$$a = -3b \tag{82}$$

thus a valid basis could be

$$x_k = \left\{ \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \right\} \tag{83}$$

note that $-b$ now becomes a simple scaling factor

c. Find one basis for F and one for G, calculate $F \cap G$ using the basis vectors previously found and check your results with the previous question.
a possible basis for F

$$x_k = \left\{ \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \right\} \tag{84}$$

note that x, y are free variables and z is the bounded one, thus we use the restriction mentioned

$$x + y - z = 0 \equiv z = x + y \tag{85}$$

and remember we want $v \in \mathbb{R}^3$

and for G

$$x_k = \left\{ \begin{bmatrix} a \\ a \\ 0 \end{bmatrix}, \begin{bmatrix} -b \\ b \\ a \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3b \end{bmatrix} \right\} \tag{86}$$

$$\begin{bmatrix} 4 & 0 & 0 & a & -b & 0 \\ 0 & 2 & 0 & a & b & 0 \\ 0 & 0 & 6 & 0 & a & -3b \end{bmatrix} \tag{87}$$

note we can find a linear combination for the last three column vectors

- for the fourth one

$$\begin{aligned} \lambda_0 &= \frac{a}{4} \\ \lambda_1 &= \frac{a}{2} \\ \lambda_2 &= 0 \end{aligned} \tag{88}$$

- for the fifth one

$$\begin{aligned} \lambda_0 &= -\frac{b}{4} \\ \lambda_1 &= \frac{b}{2} \\ \lambda_2 &= \frac{a}{6} \end{aligned} \tag{89}$$

- for the last one

$$\begin{aligned} \lambda_0 &= 0 \\ \lambda_1 &= 0 \\ \lambda_2 &= -\frac{3}{6}b \end{aligned} \tag{90}$$

thus our basis can be expressed as

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \tag{91}$$

notice is the same as in the b case, so both methods are equivalent