

# Baseball: An Application of Markov Chains

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# History

- Andrei Andreyevich Markov<sup>1</sup>
  - Russian Mathematician
  - "On the integration of differential equations by means of continued fractions"
  - Studied under Chebychev at St. Petersburg University

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- Markov Property

- The probability of getting to state  $n$  is only determined by the previous state, and not any of the previously visited states<sup>3</sup>
- For  $\{X_n | n = 0, 1, 2, 3, \dots\}$ ,  $P(X_n = i_n | X_{n-1} = i_{n-1}) = P(X_n = i_n | X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1})$ .

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- Markov Chains

- Sequences of random variables in which the process of getting to a state does not determine the future states, only the current state determines the future states.<sup>3</sup>

# Proof 1

Theorem: Suppose  $P(X_0 = i) = P_i, i = 0, 1, 2, 3, \dots$  and

$P(X_{n+1} = j, X_n = i) = P_{ij} \forall i, j = 0, 1, 2, 3, \dots$

Then  $P(X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) = P_{i_{n-1}, i_n} P_{i_{n-2}, i_{n-1}} \cdots P_{i_0, i_1} P_{i_0}.$ <sup>4</sup>



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Proof.

We proceed by induction on  $n \in \mathbb{Z}$ . Let  $n = 1$ .

Then  $Pr(X_0 = i_0, X_1 = i_1) = Pr(X_1 = i_1 | X_0 = i_0) Pr(X_0 = i_0) = P_{i_0, i_1} P_{i_0}$   
(by the definition of conditional probability).

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By the definition of conditional probability,

$$\begin{aligned} & \Pr(X_0 = i_0, X_n = i_n, X_{n+1} = i_{n+1}) = \\ & \Pr(X_{n+1} = i_{n+1} | X_0 = i_0, \dots, X_n = i_n) \cdot \Pr(X_0 = i_0, \dots, X_n = i_n). \end{aligned}$$

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By using the definition of Markov process and applying the induction hypothesis, the above equation becomes

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Since we know that  $\Pr(X_{n+1} = i_{n+1} | X_n) = P_{i_n, i_{n+1}}$ , the above equation reduces to  $\Pr(X_{n+1} = i_{n+1} | X_n) = P_{i_n, i_{n+1}} P_{i_{n-1}, i_n} \cdots P_{i_0, i_1} P_{i_0}$ .

This completes the induction and the proof. □

# Chapman-Komolgorov Equation

Theorem: The transition matrix composed of  $n$  steps is equal to the transition matrix of  $r$  steps multiplied by the transition matrix of  $s$  steps where  $n = r + s$ . Stated simply  $P^n = P^r P^s$ .

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## Proof.

The law of total probability states that if  $A$  is an event and if  $B_1, B_2, B_3, \dots$  are mutually exclusive events, then

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$$\begin{aligned} &= \sum_{k=0}^{\infty} P(X_n = j | X_r = k) \cdot P(X_r = k | X_0 = i) \\ &= \sum_{k=0}^{\infty} P_{ik}^{(r)} P_{kj}^{(s)} \end{aligned}$$

Hence  $P^{(n)} = P^{(r)} P^{(s)}$ !



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Theorem: Let  $T$  be a  $n \times n$  transition matrix and consider an absorbing Markov chain. The matrix  $I - T$  has an inverse  $Q$ , where  $Q = I + T + T^2 + \dots$ .<sup>6</sup>



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Let  $(I - T)x = 0$ . This is equivalent to  $x = Tx$ .  
Multiplying by  $T$  on the left, we get  $Tx = T^2x$ ,  
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and find that  $x = 0$ .

Because  $x = 0$ , we know that there is only the trivial solution to  
 $(I - T)x = 0$  and thus  $I - T$  is invertible.

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Notice that the following equation holds,  
$$(I - T)(I + T + T^2 + \cdots + T^n) = I - T^{n+1}.$$

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If we multiply by the matrix  $Q$  on the left, we get  
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Taking the limit of both sides, we get  
$$Q = I + T + T^2 + \cdots.$$

Therefore  $(I - T)^{-1} = I + T + T^2 + \cdots.$





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- Expected Runs!
- Remember the proof we just did? That makes it much easier to calculate the number of expected runs from a certain state to the remainder of the inning.
- Once we calculate the transition matrix  $T$ , and the average run matrix  $R$ , we can calculate the expected number of runs from a certain state to the end of the inning.<sup>7</sup>

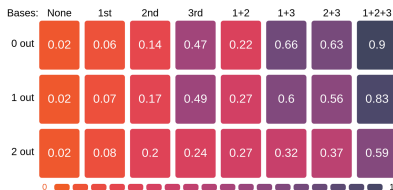


Figure: Expected Runs from 1921-2017<sup>8</sup>

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