Baseball: An Application of Markov Chains

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History

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- Andrei Andreyevich Markov[1]
  - Russian Mathematician
  - "On the integration of differential equations by means of continued fractions"
  - Studied under Chebychev at St. Petersburg University

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  - The probability of getting to state n is only determined by the previous state, and not any of the previously visited states[3]
  - For $\{X_n | n = 0, 1, 2, 3, ...\}$, $P(X_n = i_n | X_{n-1} = i_{n-1}) = P(X_n = i_n | X_0 = i_0, X_1 = i_1, \cdots, X_{n-1} = i_{n-1})$.

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- Markov Chains
  - Sequences of random variables in which the process of getting to a state does not determine the future states, only the current state determines the future states.[3]

Proof 1

Theorem: Suppose $P(X_0 = i) = P_i, i = 0, 1, 2, 3, \ldots$ and
$P(X_{n+1} = j, X_n = i) = P_{ij} \; \forall \; i, j = 0, 1, 2, 3, \ldots$
Then $P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = P_{i_{n-1}, i_n} P_{i_{n-2}, i_{n-1}} \cdots P_{i_0, i_1} P_{i_0}$.[4]

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Proof.

We proceed by induction on $n \in \mathbb{Z}$. Let $n = 1$.
Then $Pr(X_0 = i_0, X_1 = i_1) = Pr(X_1 = i_1 | X_0 = i_0) Pr(X_0 = i_0) = P_{i_0,i_1} P_{i_0}$
(by the definition of conditional probability).

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By using the definition of Markov process and applying the induction hypothesis, the above equation becomes
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Since we know that $Pr(X_{n+1} = i_{n+1} | X_n) = P_{i_n,i_{n+1}}$, the above equation reduces to $Pr(X_{n+1} = i_{n+1} | X_n) = P_{i_n,i_{n+1}} P_{i_{n-1},i_n} \cdots P_{i_0,i_1} P_{i_0}$.

This completes the induction and the proof. □

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Theorem: The transition matrix composed of n steps is equal to the transition matrix of r steps multiplied by the transition matrix of s steps where $n = r + s$. Stated simply $P^n = P^r P^s$.

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The law of total probability states that if A is an event and if $B_1, B_2, B_3, \ldots$ are mutually exclusive events, then

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$$= P((\textstyle\bigcup_{k=0}^{\infty} X_n = j, X_r = k) | X_0 = i)$$

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$$= \sum_{k=0}^{\infty} P(X_n = j | X_r = k) \cdot P(X_r = k | X_0 = i)$$

$$= \sum_{k=0}^{\infty} P_{ik}^{(r)} P_{kj}^{(s)}$$

Hence $P^{(n)} = P^{(r)} P^{(s)}$! □

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Theorem: Let T be a *nxn* transition matrix and consider an absorbing Markov chain. The matrix $I - T$ has an inverse Q, where $Q = I + T + T^2 + \ldots$.[6]

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Repeating this process over and over again, we get $x = T^nx$.
Using the fact that $lim_{n\to\infty} T^n = 0$, we can take the limit of both sides and find that $x = 0$.
Because $x = 0$, we know that there is only the trivial solution to $(I - T)x = 0$ and thus $I - T$ is invertible.

Proof (cont.)

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If we multiply by the matrix Q on the left, we get

$$I + T + T^2 + \cdots + T^n = Q(I - T^{n+1}).$$

Taking the limit of both sides, we get

$$Q = I + T + T^2 + \cdots.$$

Therefore $(I - T)^{-1} = I + T + T^2 + \cdots$. □

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- Expected Runs!
- Remember the proof we just did? That makes it much easier to calculate the number of expected runs from a certain state to the remainder of the inning.
- Once we calculate the transition matrix T, and the average run matrix R, we can calculate the expected number of runs from a certain state to the end of the inning.[7]

Bases:	None	1st	2nd	3rd	1+2	1+3	2+3	1+2+3
0 out	0.02	0.06	0.14	0.47	0.22	0.66	0.63	0.9
1 out	0.02	0.07	0.17	0.49	0.27	0.6	0.56	0.83
2 out	0.02	0.08	0.2	0.24	0.27	0.32	0.37	0.59

0 1

Figure: Expected Runs from 1921-2017[8]

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