MATH425 Assignment 4

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W6Q3

Because f is continuous and $f(0) = 0$, we can choose a $\delta > 0$ such that $\forall x \in B(0, \delta)$ we have $|f(x)| < \epsilon$, $\epsilon < 1$. Then since all polynomials are continuous, for every $k \in \mathbb{N}$ we have that $(f(z))^k$ is continuous and $|f(z)|^k < \epsilon^k$. Therefore for all $\{a_k\}_{k=0}^{\infty}$ bounded by $M > 0$ we have

$$\left|\sum_{k=0}^{\infty} a_k (f(z))^k\right| \leq \sum_{k=0}^{\infty} |a_k||f(z)|^k \leq \sum_{k=0}^{\infty} M\epsilon^k = M\frac{\epsilon}{1-\epsilon}$$

Thus every function in this family is uniformly bounded and converges uniformly by direct application of the Weierstrass M-test. Since convergence is uniform, it follows that each function in the family is continuous by the properties of uniform convergence (each term in the series for that function will be continuous, so each partial sum will be continuous and thus the limiting function will also be continuous). Thus this family of continuous functions is uniformly bounded.

W7Q3

It will be convenient to show that $h(x) = \sqrt{x+1}$ is a contraction mapping. Assuming $\mathbb{R}$ is equipped with the usual metric we have that (for $x, y \geq 0$)

$$\begin{aligned}
|\sqrt{x+1} - \sqrt{y+1}| &= \frac{\sqrt{x+1} + \sqrt{y+1}}{\sqrt{x+1} + \sqrt{y+1}}|\sqrt{x+1} - \sqrt{y+1}| \\
&= \left|\frac{x+1-y-1}{\sqrt{x+1} + \sqrt{y+1}}\right| \\
&= \frac{|x-y|}{\sqrt{x+1} + \sqrt{y+1}} \\
&\leq \frac{1}{2}|x-y|
\end{aligned}$$

So h is a contraction mapping. Furthermore, it is useful to note that $h(x) \geq 0$ for all $x \geq 0$. Let h^n denote repeated composition of the function h so $h^2 = h \circ h$, etc (it is also convenient to let $h^0(x) = x$). These functions constructed by repeated composition are from $\mathbb{R}$ to $\mathbb{R}$ if $x \geq 0$ as $h(x) \geq 0$ for all $x \geq 0$. Then we can rewrite $|f_n(x) - f_n(y)| = |h^{n-1}(f_1(x)) - h^{n-1}(f_1(y))|$ for $x, y \in [0, 1]$, $n \in \mathbb{N}$. Then $f_1(x) = x$ so $|f_n(x) - f_n(y)| = |h^{n-1}(x) - h^{n-1}(y)| \leq |x - y|$ as h is a contraction mapping. Therefore, $\forall \epsilon > 0$ take $\delta = \epsilon$. If $x, y \in [0, 1]$ and $|x - y| < \delta = \epsilon$, $|f_n(x) - f_n(y)| \leq |x - y| < \delta = \epsilon$ for all $n \in \mathbb{N}$ and so this family of functions is equicontinuous.