MATH401 Dynamical systems (2022)

Lecture notes

1 Dynamical systems and interval maps

General idea: dynamics takes place in a *phase space* X. *Orbits* are formed by a group or semigroup T acting on X. The (semi)group is $\mathbb{Z}_+$, $\mathbb{Z}$ or $\mathbb{R}$, and is thought of as representing *time*. As time moves forward, this (semi)group action moves points around. We might write the action of "time-t" evolution on a point $x \in X$ as

$$\phi(t, x)$$

where $\phi(t, \cdot)$ is a **transformation** or **map** from X to itself. If we begin at x_0 then after 1 time unit we move to $x_1 = \phi(1, x_0)$, after 2 time units we move to $x_2 = \phi(2, x_0)$, and so on. The viewpoint in dynamical systems is to understand the following:

- **invariant sets**: are there sets which do not change under the action of the dynamics?
- **asymptotics**: what happens "long-term" to typical orbits $\{\phi(t, x_0)\}$ (*ie* as $t \to \infty$)?
- **transport**: how does the dynamics move entire subsets around X? do they "mix"?
- **stability**: do the above properties change when the dynamics is given a small or random perturbation?

1.1 Introduction to dynamical systems

We will discuss both "local properties" (what happens near an invariant set) and "global properties" (classifying the different dynamics that we see). Discussion of "nonautonomous dynamics" (where the evolution induced by ϕ is different at different times) will be limited!

Example 1. Let $X = \mathbb{R}^2$ and let ϵ be a small parameter. The FitzHugh-Nagumo equations are

$$\left.\begin{array}{rcl} \dot{v} & = & v - \frac{v^3}{3} - w + I \\ \dot{w} & = & \epsilon\,(v + 0.7 - 0.8\,w) \end{array}\right\} = F(v, w)$$

Here,

$$\begin{aligned} v &= \text{nondimensional potential (voltage) in a neuronal cell} \\ w &= \text{"inactivation variable"} \\ I &= \text{applied current} \end{aligned}$$

and the superscript $\dot{}$ is used to denote differentiation with respect to time t. The function $F(v, w)$ is called the **vector field**, or *right-hand-side (RHS)* of the equation.

Familiar idea: the usual picture of a differential equation $\dot{\mathbf{x}} = F(\mathbf{x})$ is that the vector field f is *everywhere tangent* to the solution curves of the DE. Indeed, the **orbits** are entire curves of solutions $\mathcal{O}(\mathbf{x}_0) := \{\mathbf{x}(t) \ : \ \mathbf{x}(0) = \mathbf{x}_0\}_{t\in\mathbb{R}}$.

New idea: In Example 1, the group $\mathbb{R}$ acts on $X = \mathbb{R}^2$ via *solutions of the differential equation*. In particular, if

$$(v(t), w(t)) \text{ solves the DE with initial condition } (v_0, w_0)$$

then $\varphi_t(v_0, w_0) := (v(t), w(t))$ is the **time–t map of the flow**.

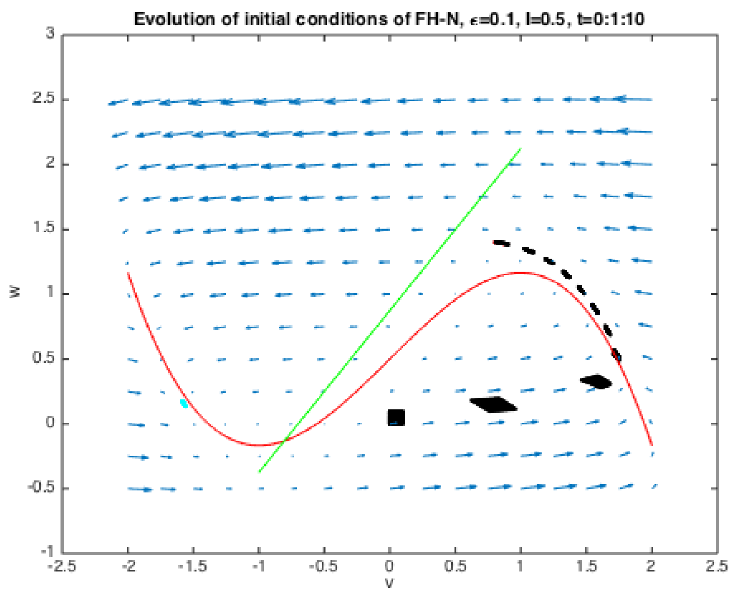

Figure 1: Evolution of a block of initial conditions under the time 1 map map of the flow induced by the FitzHugh-Nagumo equations with $\epsilon = 0.1$, $I = 0.5$.

Notice that φ_t maps initial conditions (v_0, w_0) to their evolution after t units of time. Each one of these transformations or "maps" $\{\varphi_t\}_{t\in\mathbb{R}}$ can simultaneously transform an entire subset of initial conditions to their evolved states after t units of time.

Definition. A **flow** is a family $\{\varphi_t\}_{t\in\mathbb{R}}$ of transformations of X with the properties:

- $\varphi_0 =$ id;
- $[\varphi_t]^{-1} = \varphi_{-t}$; and
- $\varphi_{s+t} = \varphi_s \circ \varphi_t = \varphi_t \circ \varphi_s$ for all $s, t \in \mathbb{R}$.

The differential equation (or generator) for a flow is $\dot{x} = F(x) := \frac{d}{dt}\varphi_t(x)|_{t=0}$.

Usually, flows are obtained by solving differential equations. Throughout this course, we will analyse flows as a **family of transformations**.

Example 2. The equation $\dot{x} = \lambda x$ can be solved by separation of variables:

$$\frac{dx}{x} = \lambda\, dt \qquad \Rightarrow \qquad \ln|x| = \lambda\, t + C \qquad \Rightarrow \qquad x = x_0\, e^{\lambda\, t}.$$

The associated flow is given by $\varphi_t(x) := x\, e^{\lambda\, t}$. As an exercise, check the flow properties.

Example 3. The time set $\mathbb{R}$ acts on $\mathbb{R}^3$ via solutions of the famous *Lorenz equations* [Lorenz, *J. Atmos. Sci.* 1963]:

$$\begin{aligned} \dot{x} &= \sigma\,(y - x) \\ \dot{y} &= r\,x - x\,z - y \\ \dot{z} &= x\,y - b\,z \end{aligned}$$

with parameter values $r = 28, \sigma = 10, b = 8/3$. There are three *fixed points* [invariant sets, consisting of orbits which are a single point; these are zeros of the vector field]:

- $\mathbf{x}_* = (0, 0, 0)$ — a "saddle", with two attracting directions and one repelling
- a symmetric pair at $\mathbf{x}_* = (\pm\sqrt{b(r-1)}, \pm\sqrt{b\,(r-1)}, r - 1)$ (also saddle type, with two repelling directions and one attracting)

There are special sets (invariant manifolds) associated with these fixed points that organise the dynamics on the entire phase space. It is *chaotic*, with **many** very complex orbits (see Figure 2).

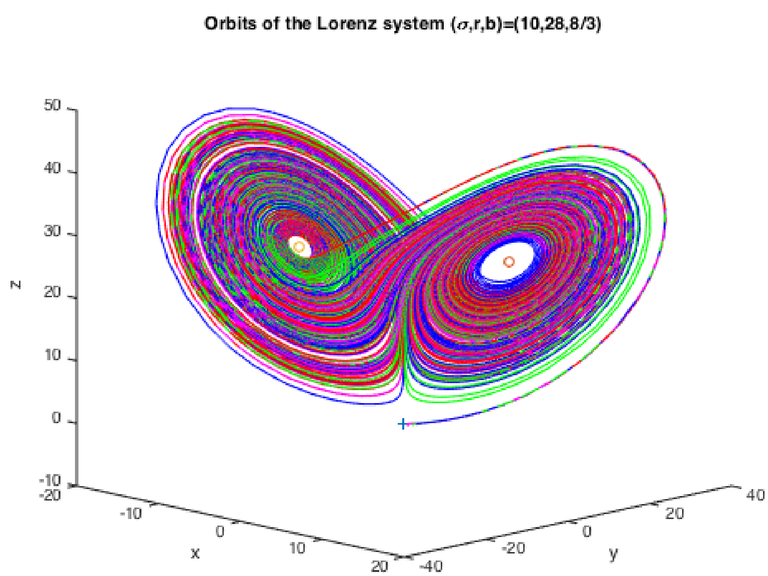

Figure 2: Evolution of several initial conditions under the Lorenz equations.

The analysis of this complex setup is done via a *Poincaré map* or *first return map* to the surface

$$\Sigma := \{(x, y, z)\ :\ z = r - 1, xy < b(r - 1)\}.$$

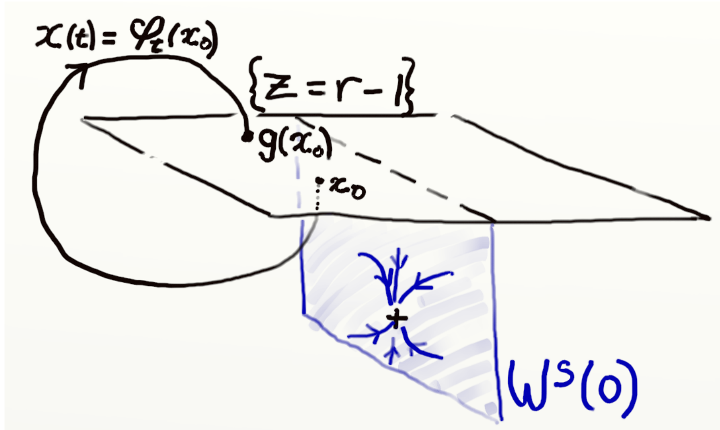

Figure 3: Construction of a Poincaré map.

The idea is to follow orbits until they return to Σ: if $\mathbf{x}_0 = (x, y, z) \in \Sigma$ let

$$\tau = \min\{t > 0 \ : \ \varphi_t(\mathbf{x}_0) \in \Sigma\} \qquad \text{and} \qquad g(\mathbf{x}_0) = \varphi_\tau(\mathbf{x}_0).$$

This manouvre achieves *two things*: the phase space is reduced by one dimension (from all of $\mathbb{R}^3$ to the two-dimensional surface Σ), and orbits (curves parametrised by $t \in \mathbb{R}$) are replaced by sequences:

$$\mathbf{x}_0, \quad \mathbf{x}_1 = g(\mathbf{x}_1), \quad \mathbf{x}_2 = g(\mathbf{x}_1), \quad \ldots, \quad \mathbf{x}_n = g^n(\mathbf{x}_0), \quad \ldots.$$

Mostly, these sequences are dense in a *Cantor-like structure* which is a “slice” through the chaotic attractor.

Looking closely, it turns out that orbits on Σ have a very strong **hyperbolic** structure, with rapid contraction transversally to the attractor, and stretching along it. For large (or even moderate) times, the contracting direction can be ignored (this called *factoring by the stable foliation*), and the “interesting” dynamics can be investigated by iterating a **piecewise $C^{1+\epsilon}$ map of an interval** $I = [-1, 1]$; see Figure 4.

For example, the Lorenz dynamics are modelled by

$$f(u) := \begin{cases} -1 + c\,u^\alpha & u > 0, \\ -f(-u) & u < 0. \end{cases}$$

Dynamics are obtained by iterating $u_1 = f(u_0)$, $u_2 = f(u_1)$, $\cdots$. The behaviour of the sequence thus obtained is **provably chaotic** for large enough c.

Recap: the dynamics that we study have been reduced in several stages:

DE $\xrightarrow{solution}$ flow $\xrightarrow{returns}$ (invertible) Poincaré map $\xrightarrow{factor}$ (non-invertible) reduced map

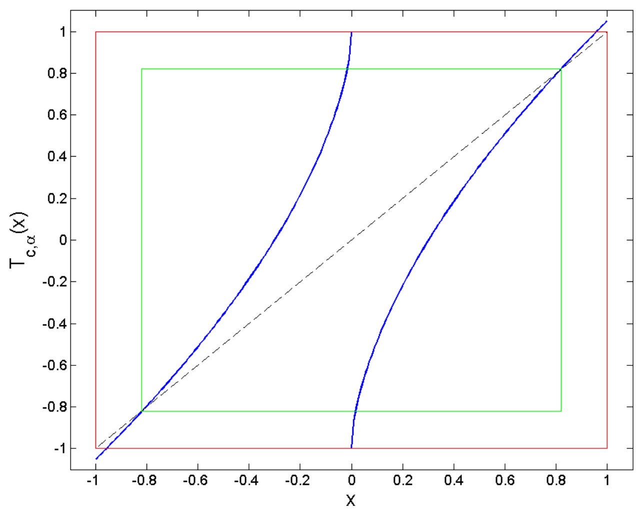

Figure 4: An "expanding factor" of a first-return Lorenz map.

1.2 The logistic family [R May, *Nature*, 1976]

We begin with a simple model, which was proposed for density dependent growth of a homogeneous population. Let x_n be the density of the population at generation n and put

$$x_{n+1} = \underbrace{r\, x_n\,(1 - x_n)}_{=:f(x_n)}.$$

This recursive relation involves a parameter r. Choosing $0 < r \leq 4$, one can show easily that $x_n \in [0,1]$ for all $n > 0$ when $x_0 \in [0,1]$.

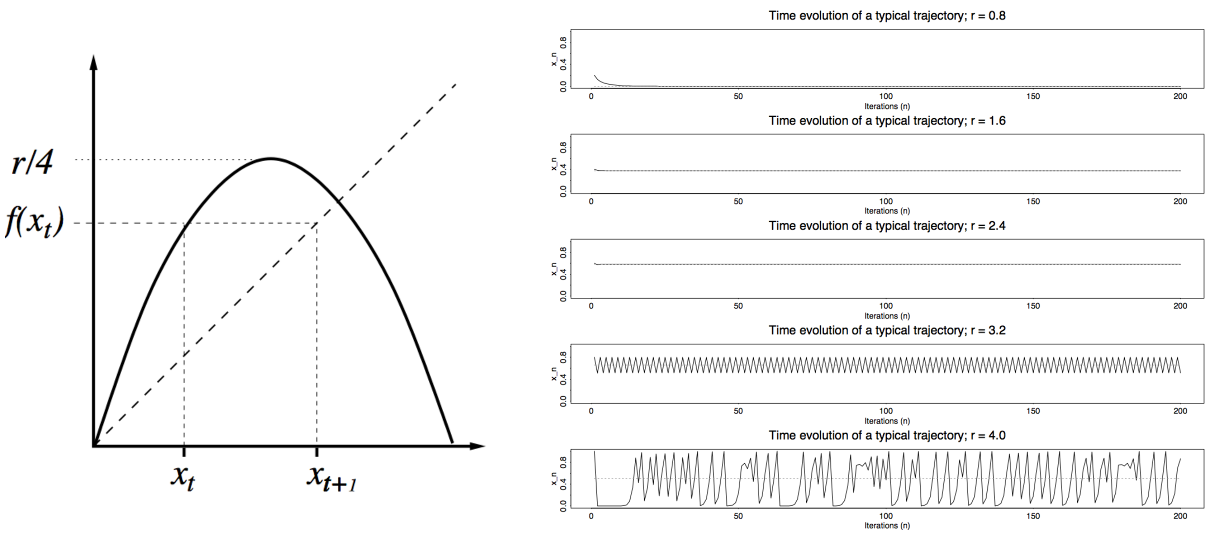

Figure 5: Left: graph of the map f generating the recurrence rule for the dynamics. Right: time series of orbits for various values of r.

The graph of f can be used to compute the evolution of $\{x_n\}_{n=0}^{\infty}$, as indicated in the above diagram. Iterating this construction, one obtains a *time series* for a typical orbit. Note that orbits are **sequences**.

Some time series plots are shown above for various values of the parameter r. For the first few parameter values, the orbit settles down to a fixed point or (in the case of $r = 3.2$) a periodic orbit. The final time series looks rather chaotic: in fact, it is!

Simple cases: let $0 < r < 1$ and $x > 0$. Then $f'(r) = r(1 - 2\,x) < r$ (since $x \geq 0$). We can use a simple trick involving the mean value theorem. Since $f(0) = 0$, we have

$$\begin{aligned}
x_{n+1} = \frac{x_{n+1}}{x_n}\, x_n &= \frac{f(x_n)}{x_n}\, x_n \\
&= \frac{f(x_n) - 0}{x_n - 0}\, x_n \\
&= \frac{f(x_n) - f(0)}{x_n - 0}\, x_n \\
&= f'(x)\, x_n \qquad \text{[for some } x \in [0, x_n]\text{, by the MVT]} \\
&< r\, x_n \\
&< x_n \qquad \text{[since } r < 1\text{]}
\end{aligned}$$

reading from left to right, we have $x_{n+1} < x_n$, so the sequence $\{x_n\}_{n=0}^{\infty}$ is **decreasing**. We already observed that each $x_n \in [0,1]$, so orbits are **decreasing and bounded below**. Therefore, there is a point x_* such that $\lim_{n\to\infty} x_n = x_*$. We can evaluate x_* rather easily! Since the function f is *continuous*,

$$x_* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \underbrace{\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)} = f(x_*).$$

The point thus satisfies $x_* = f(x_*) = rx_*(1-x_*)$. It is easy to check that $x_* = 0$ is the only non-negative solution to this equation when $r < 1$, so **every orbit converges to** $x_* = 0$. This is our first result, but it is not general enough to call it a theorem!

Exercise: use a graphical method to investigate x_n as $n \to \infty$ when $1 < r < 3$.

More complex cases: when $r > 3.556\cdots$, dynamics are "chaotic". graphical analysis can be done via a *cobweb diagram* [see the whiteboard, or WIKIPEDIA].

The main features to observe in this example are that:

- the observed dynamical behaviour can vary fundamentally with changing a continuous parameter;
- a simple rule can generate chaotic behaviour (deterministic, but unpredictable).

1.3 Maps as dynamical systems

Let X be a *metric space* and $f : X \to X$ be continuous. Then (f, X) induces a (discrete time) dynamical system via iteration of f:

$$x_1 = f(x_0), \quad x_2 = f(x_1), \ldots, \quad x_n = f(x_{n-1}) = \cdots = \underbrace{f \circ f \circ \cdots f}_{n \text{ times}} = f^n(x_0).$$

Because

$$x_{n+m} = f^{n+m}(x_0) = \underbrace{f \circ f \circ \cdots \circ f}_{n+m \text{ times}}(x_0) = \underbrace{f \circ \cdots \circ f}_{n \text{ times}} \circ \underbrace{f \circ \cdots \circ f}_{m \text{ times}}(x_0) = f^n \circ f^m(x_0) = f^n(f^m(x_0)),$$

f induces an action of the additive semigroup $\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$. When $X = I$ is an *interval* in $\mathbb{R}$ the dynamical system is called an *interval map*.

Definition. The (forward) **orbit** of x is $\mathcal{O}^+(x) := \{f^n(x)\}_{n=0}^{\infty}$.

Note: $f^{n+m} = f^n \circ f^m$ and if f is *invertible* then $f^{-n} = (f^{-1})^n$ and f induces a *group action*, and $\mathcal{O}(x) = \{f^n(x)\}_{n\in\mathbb{Z}}$.

In the notation above, $\phi(t, x) := f^t(x)$. This kind of dynamical systems is sometimes known as a **difference equation**.

Example 1. Let $X = \mathbb{R}$ and $f(x) = x/2$. Then

$$x_1 = f(x_0) = x_0/2, x_2 = f(x_1)/2 = x_0/4, \ldots, x_n = x_0/2^n, \ldots.$$

Thus, $x_n \to 0$ as $n \to \infty$. We say that 0 **attracts** every orbit. □

Important general question: The behaviour of Example 1 is very similar to the logistic map of the previous section when $r < 1$. We naturally ask: *are these basically the same dynamical system, written in different coordinates?*

Example 2. Let $X = \mathbb{R}^d$ and let A be a $d \times d$ matrix. Then $f(x) = Ax$ generates a *linear dynamical system*. By induction,

$$f^n(x) = f^{n-1}(f(x)) = A^{n-1}f(x) = A^{n-1}Ax = A^n x,$$

so the long-term behaviour of orbits is determined by the powers A^n of the matrix. *What happens if all the eigenvalues of* A *have* $|\lambda| < 1$*? What if some* $|\lambda| = 1$ *or* $|\lambda| > 1$*?* $\square$

Example 3. Let $\frac{dx}{dt} = f(x)$ be a differential equation on $\mathbb{R}^d$ where f is smooth. For each x_0 let $x(t)$ denote the solution with initial condition $x(0) = x_0$. Let $f(x_0) = x(1)$. This is called the **time-**1 **map** of the differential equation, and is one way of extracting a discrete time dynamical system from a differential equation. $\square$

Example 4. (Chirikov's Standard Map). Consider a rotor which is confined to spin in a vertical plane. Its angle at time t is q and at integer time units it receives an impulsive burst of energy proportional to its vertical displacement. The Hamiltonian for this system is $H(q,p,t) = p^2/2 + \sum_n a\,\cos(q(t))\delta(t-n)$. Hamilton's equations are

$$\begin{aligned}\frac{dq}{dt} &= \frac{\partial H}{\partial p} = p\\ \frac{dp}{dt} &= -\frac{\partial H}{\partial q} = a\sum_n \sin(q)\,\delta(t-n).\end{aligned}$$

When $a = 0$ the equations are easy to solve: $p(t) = p_0$ and $q(t) = q_0 + t\,p_0$; that is, linear increase in angle, rotating at speed gvien by p_0. Unfortunately, when $a \neq 0$ this DE is non-autonomous, since the vector field depends on time. Formally speaking, we can recover a dynamical system by *augmenting* the (q,p) coordinates with a "dummy time variable" $\tau = t$ to give a three dimensional phase space (q,p,τ). This gives a third DE: $\frac{d\tau}{dt} = 1$, and now the the system is autonomous (albiet at the expense of increasing the dimension by one). The dynamics now has two phases: between integer times, $t \in (n, n+1)$, $p(t) = constant = p_n$ and $q(t) = q_n + p_n(t-n)$. At integer times p experiences a step jump: $p_{n+1} = p_n + a\,\sin q_{n+1}$. By looking only at these times we obtain a dynamical system

$$f(q,p) = (q + p \pmod{2\pi}, p + a\,\sin(q+p)).$$

Some orbits for $a = 1$ are shown in Figure 6. This map has a number of interesting properties. Critically, it is *area preserving*. *How might you go about showing this? and what are its consequences?* $\square$

We will begin our systematic study with something that appears much simpler!

1.4 Fixed points and stability for interval maps

Let $f : I \to I$ be continuous. A point $p_* \in I$ is a **fixed point** if and only if $\mathcal{O}^+(p_*) = \{p_*\}$.

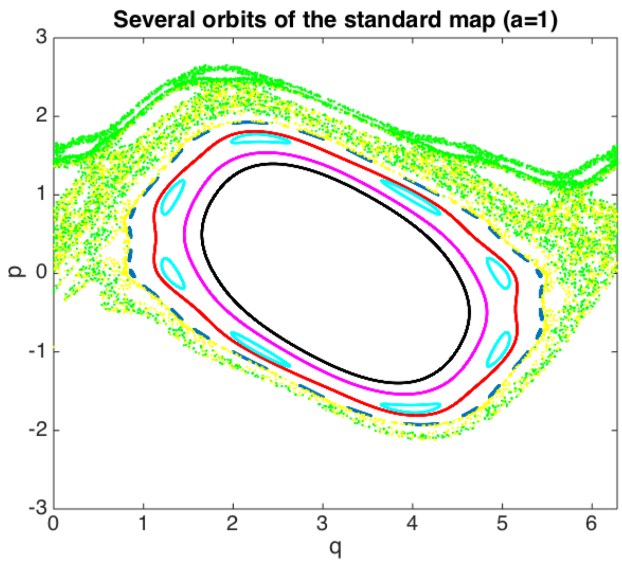

Figure 6: Sample of orbit segments of length 10000 of a standard map

Note: it is immediate that $p_* = f(p_*)$ is both necessary and sufficent for being a fixed point.

A fixed point is **attracting** (or a "sink") if it has an open neighbourhood N such that

$$x \in N \quad \Rightarrow \quad \lim_{n\to\infty} f^n(x) = p_*.$$

N is called an **attracting neighbourhood**. The largest connected N is called the **immediate basin of attraction**.

Theorem 1 *Let $f : I \to I$ be differentiable and have a fixed point at p_*. If $|f'(p_*)| < 1$ then p_* is attracting.*

Proof: The idea is to show that distances from p_* shrink at each iteration. To this end, note that

$$f'(p_*) = \lim_{h\to 0} \frac{f(p_* + h) - f(p_*)}{h}.$$

By the derivative condition, there is a $\delta > 0$ such that

$$|h| < \delta \Rightarrow \left| \frac{f(p_* + h) - p_*}{h} \right| < 1.$$

Now suppose that $x_n \in N := (p_* - \delta, p_* + \delta)$. Then, writing $x_n = p_* + h$,

$$|x_{n+1} - p_*| = |f(x_n) - p_*| = \left| \frac{f(x_n) - p_*}{x_n - p_*} \right| |x_n - p_*| < |x_n - p_*|.$$

It follows that $x_n \to p_*$ as $n \to \infty$, so that N is an attracting neigbourhood for p_*. $\square$

Exercise: find the gaps in the proof and fill them!

Exercise: Use the theorem to show that your graphical findings about orbits of the logistic map with $1 < r < 3$ are correct.

1.5 Periodic orbits for interval maps

A point x_0 is called **periodic** if there is a $q < \infty$ such that

$$x_0 \xrightarrow{f} x_1 \xrightarrow{f} x_2 \xrightarrow{f} \cdots \xrightarrow{f} x_{q-1} \xrightarrow{f} x_0.$$

This is characterised as a *fixed point of* f^q: $f^q(x_0) = x_0$ and

$$\mathcal{O}^+(x_0) = \{x_0, x_1, \dots, x_{q-1}\} = \mathcal{O}^+(x_j) \qquad \forall j = 0, \dots, q-1.$$

The number q is called a **period** of the orbit. A periodic orbit is **attracting** if the points x_j are attracting as fixed points of f^q.

Exercise: Does a periodic orbit have a unique period?

Example 1. Let $X = [-1, 1]$ and $f(x) = -x$. Then every point has period-2.

Theorem 2 *Let* $\Gamma = \{x_j\}_{j=0}^{q-1}$ *be a periodic orbit of* $f : I \to I$*. If* f *is differentiable at each point of* Γ *and* $\prod_{j=0}^{q-1} |f'(x_j)| < 1$ *then* Γ *is attracting.*

Proof: Note that $f^q(x_i) = x_i$ for each $x_i \in \Gamma$. Then, inductive application of the chain rule gives

$$\begin{aligned}
(f^q)'(x_i) &= f'(f^{q-1}(x_i)) \times f'(f^{q-2}(x_i)) \times \cdots f'(f^{q-i}(x_i)) \times f'(f^{q-i-1})(x_i) \times \cdots f'(f(x_i)) \times f'(x_i) \\
&= f'(x_{i-1}) \times f'(x_{i-2}) \times \cdots f'(x_0) \times f'(x_{q-1}) \times \cdots f'(x_{i+1}) \times f'(x_i) \\
&= \prod_{j=0}^{i-1} f'(x_j) \times \prod_{j=i}^{q-1} f'(x_i) \\
&= \prod_{j=0}^{q-1} f'(x_j).
\end{aligned}$$

Thus, the derivative of f^q is *independent of which point on the orbit is used*. By Theorem 1, each x_i is an attracting fixed point of f^q. The orbit Γ is thus attracting. $\square$

Example 2. The logistic map when $r > 3$ has fixed points at $x_* = 0, \frac{r-1}{r}$. The points

$$x_1 = \frac{1}{2r}\left[(r+1) + \sqrt{(r+1)(r-3)}\right], \qquad x_2 = \frac{1}{2r}\left[(r+1) - \sqrt{(r+1)(r-3)}\right]$$

form a period-2 orbit: $f(x_1) = x_2$ and $f(x_2) = x_1$ so that $f^2(x_1) = f(f(x_1)) = f(x_2) = x_1$.

Figure 9 depicts the fixed points and period-2 orbits. Solid lines indicate attracting behaviour. Note that 0 is an attracting fixed point for $r \in [0, 1]$ and $p_* = \frac{r-1}{r}$ is attracting for $r \in [1, 3]$. The fixed points persist for other values of r, but are *repelling* (denoted with dotted lines). One can calculate that $f'(p_*) = -1$ when $r = 3$, and $|f'(p_*)| > 1$ for $r > 3$. By Theorem 2 (and further calculation), the period-2 orbit $\{x_1, x_2\}$ is attracting for $3 < r < 1 + \sqrt{6}$. As a r passes through 3, a **period doubling bifurcation** (PD) occurs: the fixed point p_* loses stability and, an attracting periodic point of twice the period appears. The appearance of the period-2 orbit can be seen by looking at the graphs of f^2 as r increases through 3; this is done in Figure 8.

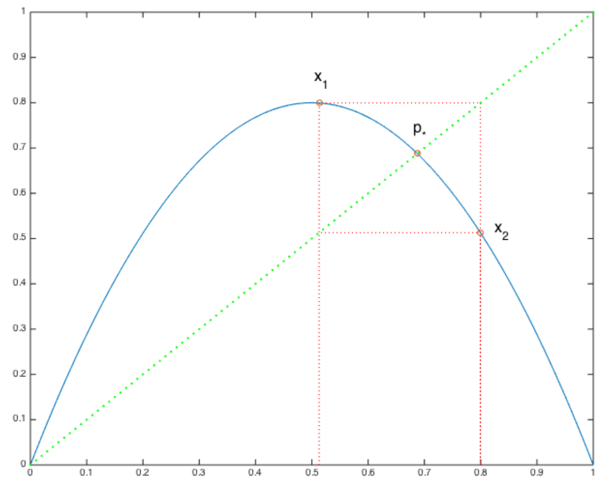

Figure 7: Period-2 orbit and fixed point of the logistic map with $r = 3.2$.

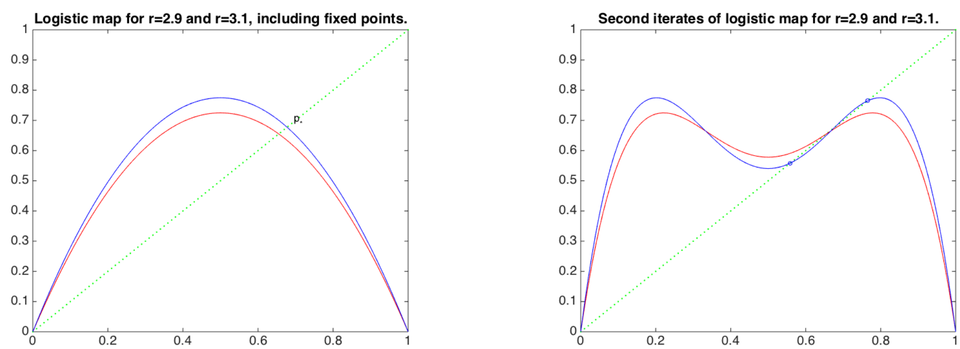

Figure 8: Graphs of first and second iterates of logistic map, before and after PD at $r = 3$; note the appearance of the period-2 points (new solutions of $x = f^2(x)$) in the right-hand diagram for $r > 3$.

By analogy with DEs, this is essentially a *pitchfork bifurcation* of f^2. Typically, a derivative -1 suggests a period doubling. Further calculation reveals that the period-2 orbit loses its atracting property in another PD (look for where $f'(x_1)f'(x_2) = -1$). The period-4 orbit that is created is attracting for a range of r, and then $\ldots$? In fact, a *bifurcation diagram* can be drawn, depicting the dependence of the fixed points and period-2 orbits on r.

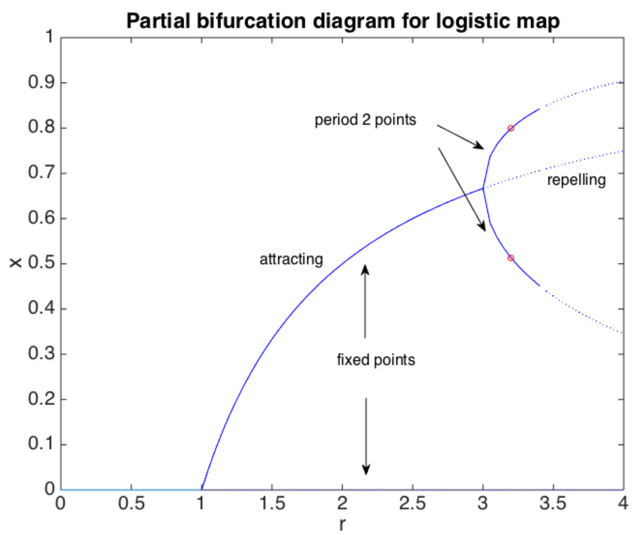

Figure 9: Fixed points and period-2 orbits of the logistic map.

1.6 A strange invariant set in the logistic family

Recall the logistic family $x_{n+1} = r\,x_n\,(1 - x_n) = f(x_n)$.

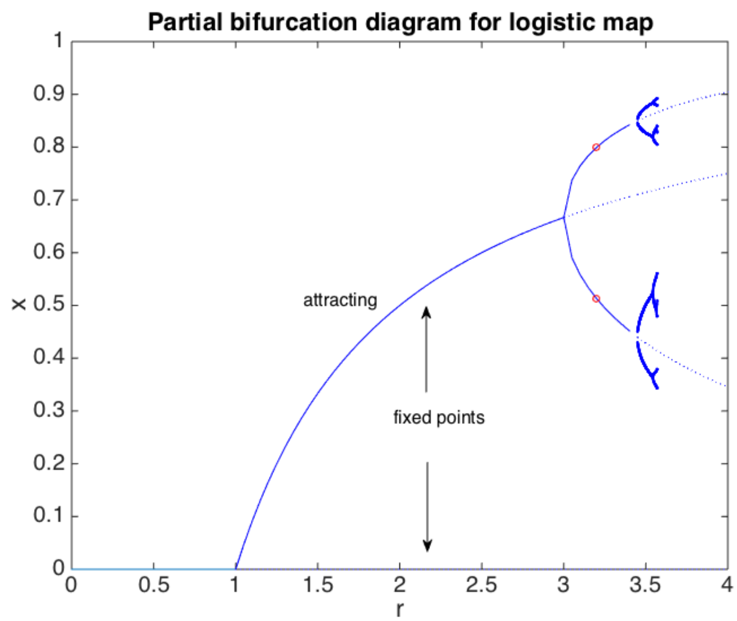

Figure 10: Several period doublings in the logistic family, culminating in a strange set.

We augment Figure 9, depicting the fixed point $p_* = \frac{r-1}{r}$ which split at $r = 3$ in a PD doubling bifurcation into a period-2 orbit.

There is a range (r_1, r_2) where the period-2 orbit is attracting. Then, at $r_2 \approx 3.449$ the derivative $(f^2)'$ has decreased to -1, and a further PD occurs, giving rise to a period-4 orbit which is attracting for $r \in (r_2, r_3)$ where $r_3 \approx 3.544$. At this latter value of r a further PD occurs, giving rise to a period-8 orbit, which is attracting for a small window. Indeed, an entire *cascade* of PDs occurs, culminating at $r_\infty \approx 3.5699$. For $r > r_\infty$ the logistic family is “chaotic”, and at r_∞ one obtains a **strange invariant set.** Amazingly, a *renormalisation theory* due to Feigenbaum allows one to show that the gaps between the PDs decay geometrically,

$$\frac{r_n - r_{n-1}}{r_{n+1} - r_n} \to 4.669\cdots.$$

This constant is called the Feigenbaum constant, and it is typical in *unimodal families*.

1.7 But wait, there's more

One-dimensional maps are easy to probe numerically. Iteration can be done with simple functions, and bifurcation diagrams can be computed quickly. Figure 11 displays the complex behaviour that can occur for $r > r_\infty$. At each r value the attracting set is approximated by running an orbit forward for a few hundred iterations. The resulting approximation to a typical ω-limit set is then displayed for each r value. Notice that sometimes the attractor seems "fat" or "dense"; at other parameter values it is a low period periodic orbit.

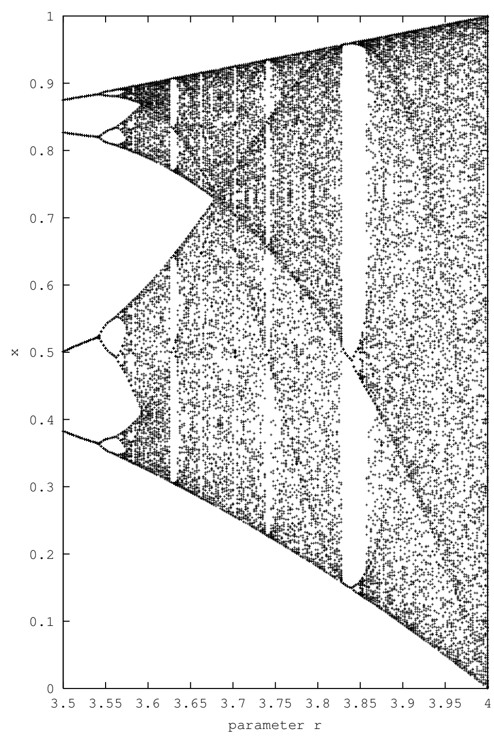

Figure 11: BIfurcation diagram of the logistic family.

Notice that there is a rather large window around $r = 3.83$ where there is an attracting period-3 orbit; we will investigate this in tutorials, and eventually learn that chaotic dynamics coexist in this window! Also

- the set of parameters with an attracting periodic orbit is dense
- there are an infinite number of period-doubling cascades
- there is a *positive measure* set of parameters r where the orbits are described by a probability density function

1.8 Existence of periodic orbits for bounded interval maps

Let $I \subset \mathbb{R}$ be a closed and bounded interval. The following lemma is really just the *intermediate value theorem* in disguise. It relies on the fact that interval has an **ordering** (so cannot be applied on circles, where the endpoints meet up), and on the fact the map is bounded.

Remarks on notation: just as f moves points around, it can also move entire sets: $f(J) = \{y = f(x) \,:\, x \in J\}$; the notation $\overline{J}$ denotes the closure of J (this is the smallest closed set containing J — if $J = (\alpha, \beta)$ is an interval then $\overline{J} = [\alpha, \beta]$.).

Lemma 3 *Let $f : I \to I$ be continuous. If $J \subseteq I$ is a subinterval such that either $J \subseteq f(J)$ or $f(J) \subseteq J$ then $\overline{J}$ contains a fixed point.*

Proof of lemma: We suppose that $J = (\alpha, \beta)$ and $J \subseteq f(J)$, since this is the harder direction. Because $f(J)$ completely covers J there must be $x, y \in \overline{J}$ such that $f(x) \leq \alpha$ and $f(y) \geq \beta$. Let $\phi(t) = f(t) - t$. Then $\phi(x) = f(x) - x \leq \alpha - x \leq \alpha - \alpha = 0$ since $x \geq \alpha$. Similarly, $\phi(y) \geq 0$. By the intermediate value theorem, there is $s \in [x, y]$ such that $\phi(s) = 0$; that is, $f(s) - s = 0$. Then, s is the required fixed point. The other case is similar. $\square$

Theorem 4 *Let $f : I \to I$ be continuous, J a subinterval of I and $q > 0$:*

1. *if $J \cap f^q(J) \neq \emptyset$ then I contains a period-q orbit;*
2. *if $J_0, J_1, \ldots, J_{q-1}$ is a sequence of intervals such that each $J_{i+1} \subseteq f(J_i)$ $0 \leq i < q$ and $J_0 \subseteq f(J_{q-1})$ then $\overline{J_0}$ contains a period q point.*

Proof: First, since f is continuous, f^q is continuous. In case 1, put $J_\infty = J \cup f^q(J) \cup f^{2q}(J) \cup \cdots$. Then J_∞ is a subinterval of I such that $f^q(J_\infty) \subseteq J_\infty$, and the fixed point of f^q exists by the lemma. In case 2, note that $J_0 \subseteq f(J_{q-1}) \subseteq f(f(J_{q-2})) \subseteq \cdots f^q(J_0)$. The result now follows from the lemma applied to f^q on J_0. $\square$

Remarks

- The boundedness of I is needed for the first condition in the theorem to be useful. Suppose that $f(x) = x + 1$ generates dynamics on $\mathbb{R}$ and $J = [0, 2]$. Then $f(J) = [1, 3]$ satisfies the intersection property, and the J_∞ constructed in the proof would be $[0, \infty)$. However, we cannot apply the lemma on J_∞, because it is unbounded. [Of course, f has no periodic points.]
- The proof uses that fact that if A, B are intervals such that $A \cap B \neq \emptyset$ then $A \cup B$ is an interval.
- The condition in the second part of the theorem will be useful later when it comes to establishing chaos. If $J_1 \subseteq f(J_0)$ we will say that J_0 f*–covers* J_1 (this makes sense: the image of J_0 under f includes J_1). The theorem says that each q-cycle of f-coverings gives rise to a period-q orbit. When it comes to the *chaos* part of the notes, we will look at *directed graphs* on collections of subintervals where f-covering corresponds to an edge. Easy to check conditions on the graph structure will be enough to prove the presence of chaotic dynamics!

1.9 Periodic orbits on the circle

The circle S^1 can be modelled as $[0,1) = \mathbb{R}/\mathbb{Z}$. Two points $x, y \in \mathbb{R}$ model the same point on the circle if $y - x \in \mathbb{Z}$; that is, $y = x + p$ for some integer p, or $x \equiv y \pmod 1$. If $x \in \mathbb{R}$, let $\psi(x) = x - \lfloor x \rfloor$. Then $\psi : \mathbb{R} \to [0,1)$ is a *quotient map* which executes the reduction of $\mathbb{R}$ to the circle S^1.

[If you prefer to think of S^1 as the unit circle in the complex plane, then $z = \psi(x) := e^{2\pi\, ix}$ provides the link between $z \in S^1$ and $x \in \mathbb{R}$. $\mathbb{R}$ is called the *universal cover* of the circle.]

Continuous, orientation preserving maps on S^1 can be modelled by continuous maps $F : \mathbb{R} \to \mathbb{R}$ with compatibility conditions

$$f(\psi(x)) = \psi(F(x)) \qquad \text{and} \qquad F(x+1) = F(x) + k \qquad \text{for some fixed } k \in \mathbb{Z}.$$

Such an F is called a **lift**. Clearly, if $x \equiv y \pmod 1$ then there is an n such that

$$F(y) = F(x+p) = F(x+p-1)+k = F(x+p-2)+k+k = \cdots = F(x)+k\,p \equiv F(x) \pmod 1.$$

For simplicity, we choose a lift F where $F(0) \in [0,1)$.

An orbit $\Gamma = \{z_0, z_1, \ldots, z_{q-1}\}$ on S^1 is q-periodic if $f^q(z_0) = z_0$.

In terms of the lift, let $\psi(x_0) = z_0$. Then

$$\psi(x_0) = z_0 = f^q(z_0) = f^q(\psi(x_0)) = \psi(F^q(x_0)).$$

Because x_0 and $F^q(x_0)$ are equivalent under ψ, $F^q(x_0) = x_0 + p$ for some $p \in \mathbb{Z}$.

Example. Fix $\rho \in [0,1)$ and consider the rotation $z \mapsto f(z) := z\, e^{2\pi\, i\, \rho}$. Suppose that z_0 is q-periodic. We can learn about ρ via a suitable lift! Let $F(x) = x + \rho$. Then $F(x+1) = x + 1 + \rho = F(x) + 1$ and using $\psi(x) = e^{2\pi\, i\, x}$,

$$f(\psi(x)) = \psi(x)\, e^{2\pi\, i\, \rho} = e^{2\pi\, i\, x}\, e^{2\pi\, i\, \rho} = e^{2\pi\, i\,(x+\rho)} = \psi(F(x)).$$

Thus, F is indeed a lift, and $F^q(x) = x + \underbrace{\rho + \rho + \cdots + \rho}_{q \text{ times}} = x + \rho\, q$. By our observation above, there is a $p \in \mathbb{Z}$ such that

$$F^q(x_0) = x_0 + p.$$

But $F^q(x_0) = x_0 + \rho\, q$. Thus, $\rho\, q = p$ and $\rho = \frac{p}{q}$, a *rational number*! By reversing this calculation, it is easy to show that if ρ is rational, then every point of the corresponding circle rotation is periodic, with period a divisor of the denominator of ρ.

Exercise: what happens when the rotation angle ρ is irrational?

Exercise: what is a **rotation number** and which circle maps have them? Are such maps “rotation-like”?