MATH428 Assignment 1

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Q1

(a)

Suppose C is an open cover of X' and let $O_{x'}$ be an element of this cover including x'. Since this set includes x' it cannot be an element of τ so it must be of the form $O_{x'} = \{X' \backslash M : M \subset X \text{is compact}\}$. Consider the set of sets containing M formed by restricting each set in C to X (we will denote this restricted cover C^M). Each set in C^M is an open set, as it is either an element of τ or of the form $X\backslash C$ (where C is a compact set). Sets in τ are trivially open, while compact sets X are compact sets in a Hausdorff space and thus closed (so they have open complements). Thus C^M is a cover of M and M is a compact set so it follows that there exists a finite subcover of M F^M. Finally $F^M \cup O_{x'}$ is a cover of the whole space and is finite, so (X', τ') is compact.

(b)

Under the Euclidean topology, $\mathbb{R}^2$ is homeomorphic to $S^2\backslash\{p\}$ under the topology induced by restricting the Euclidean topology to $S^2\backslash\{p\}$. Therefore we can define the function $f : \mathbb{R}^2 \cup \{x'\} \to S^2$ by

$$f = \begin{cases} g(x) & x \in \mathbb{R}^2 \\ p & x = x' \end{cases}$$

where $g(x)$ is a homeomorphism from $\mathbb{R}^2$ to $S^2\backslash\{p\}$. It remains to show that this is a homeomorphism. If one considers an open set $O_x \subseteq S^2\backslash\{p\}$ then clearly the preimage of this set is also an open set. Considering an open set O_p containing p one can write this set as $O \cup p$ where $O = O_p\backslash\{p\}$. Furthermore since S^2 is Hausdorff, O_p is Hausdorff and so for all $x \in O$ there exists an neighborhood N of x in the induced subspace topology of O_p such that N does not contain x' and thus $N \subseteq O$. Thus O is open and so the preimage of O_p is the union of an open set in $\mathbb{R}^2$ and x'. Since $\mathbb{R}^2$ is a closed set under the Euclidean topology, $\{x'\}$ is an open set and so O_p is the union of two open sets and thus open. Thus f is continuous. Furthermore, it is by construction a bijection and thus these spaces are homeomorphic.

Q2

Clearly this annulus is a closed and bounded subset of $\mathbb{R}^2$ and must thus be compact. Furthermore S^2 is Hausdorff, so if there exists a continuous map from the annulus to S^2 then the induced identification space is homeomorphic to S^2. Without loss of generality we will set S^2 to be the unit circle centred at the origin. For convenience we will write the annulus in polar coordinates (θ, r) and the sphere in (unsurprisingly) spherical coordinates (θ, ψ, r)

$$f(\theta, r) = \left(\theta, \frac{\pi(r-1)}{3}, 1\right) \tag{1}$$

this function is obviously continuous and onto, so the identification space $A/\sim_f$ is homeomorphic to the sphere (by theorem 5.10). Furthermore, $A/\sim_f$ identifies all points at the outer boundary of the annulus to a single point (which is mapped to the South pole of the sphere under f) and all points at the inner boundary to a single point mapped to the North Pole under f. For convenience we will denote these points n and s, leading to the following diagram (Figure 1):

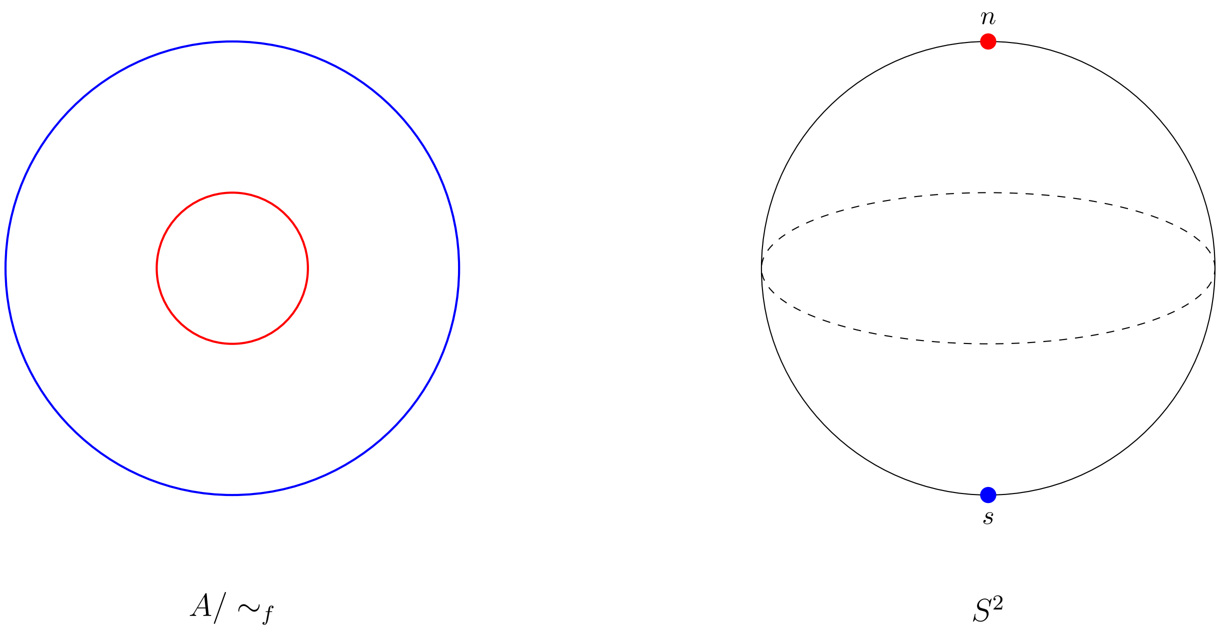

Figure 1: The identification space induced by f and the sphere. Points in blue (red) on $A/\sim_f$ are mapped to the blue (red) point on S^2.

For convenience, let g be the homeomorphism between $A/\sim_f$ and S^2 such that the point identified with the outer boundary is mapped to the South pole and the point identified with the inner boundary is mapped to the North pole. Then the pinched sphere obtained by identifying $n \sim s$ is homeomorphic to the space obtained by identifying $g^{-1}(n) \sim g^{-1}(s)$ as g is a homeomorphism. By composing this identification with the identification inducing $A/\sim_f$ we identify all points on the boundary to a single point, the precise identification given in the question. Thus $A/\sim$ is homemorphic to the pinched sphere.

Q3

(a)

This space is connected (see Figure 2)

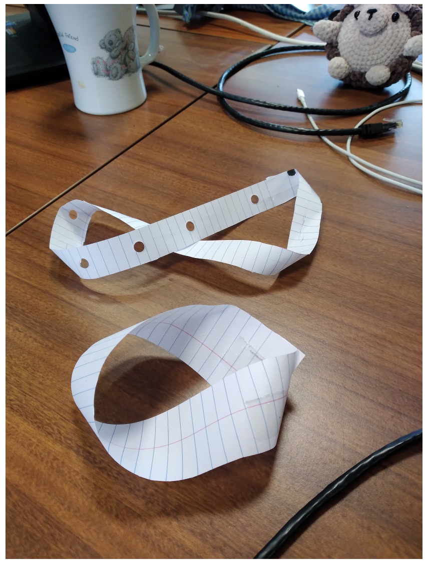

Figure 2: The Möbius strip before and after cutting down the centerline.

More formally, consider the standard representation of the Möbius strip as an identification space in $\mathbb{R}^2$. If one cuts down the centerline of the identification space and then reconnects any two of the identified sides, it is easy to see that this results in an identification space homeomorphic to the cylinder (see Figure 3)

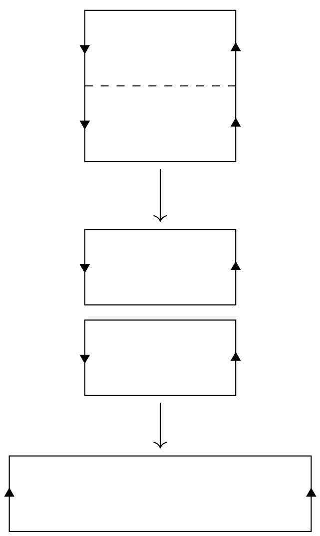

Figure 3: Cutting down the dotted line, one can rearrange the two separate pieces of the Mobius strip to obtain an identification space equivalent to the cylinder.

(b)

Consider the representation of the Möbius strip in as an identification space in $\mathbb{R}^2$ given in Figure 4. In this case, the single zero cell is drawn in blue, two one cells are drawn in red and green respectively, and a two cell is shown in red.

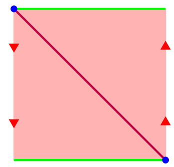

Figure 4: The Möbius strip as a cell complex.

It is hopefully clear that this representation uses a single 2-cell, so $k = 1$.

Q4

(a)

Calculating the Euler characteristic of $T \times \mathbb{P}^2$ one has

$$\chi(T \times \mathbb{P}^2) = \chi(T)\chi(\mathbb{P}^2) = \chi(S^1 \times S^1)\chi(\mathbb{P}^2) = \chi(S^1)\chi(S^1)\chi(\mathbb{P}^2) = 0$$

as $\chi(S^1) = 0$. Furthermore $\chi(S^4) = 2$ (in both cases, we use the results of exercise 5.22). Since the Euler characteristic is fixed under homeomorphism this shows that these spaces are not homeomorphic.

(b)

Let $f : Y \to X$ be a continuous function from some topological space Y to a contractible space X. Since X is contractible there exists some deformation retraction $R : X \times [0, 1] \to X$ such that $R(x, 0) = 1_X$ and $R(x, 1) = x_0$ for some $x_0 \in X$. Therefore one can define $F : X \times [0, 1] \to Y$ as $F(x, t) = R(f(x), t)$ with $F(x, 0) = f(x)$ and $F(x, 1) = x_0$. This function is continuous as f and R are continuous, so $f \overset{F}{\sim} x_0$ where x_0 is the constant function. Since our choice of f was arbitrary, for any two continuous functions $f, g : Y \to X$ it follows that $f \sim x_0 \sim g \implies f \sim g$. So both these functions are homotopic.

If $f : S^n \to S^n$ is not onto, then it can be expressed as $f : S^n \to S^n \backslash \{p\}$ for some $p \in S^n$. However $S^n \backslash \{p\}$ is homeomorphic to E^n, which is clearly contractible. From the proof above, this implies that f is homotopic to a constant function. So for $f : S^n \to S^n$, it must be either onto or homotopic to a constant function.

Q5

(a)

Let $x^* \in W$ be a point in W such that for all $y \in W$ the segment of the line that passes through both these points is in W between both of these points. Then consider $R(x, t) : W \times [0, 1] \to W$, $R(x, t) = tx^* + (1 - t)x$. For any particular direction in x_i in $x = [x_1, x_2, x_3, \ldots, x_n]$ one has

$$R(x, t)_i = tx_i^* + (1 - t)x_i$$

so for all $\epsilon > 0$ one can choose $\delta_i < \max\{(|x_i^*| + 2 + |x_i| + |t|)\epsilon, 1\}$. Then for x', t' such that $|x - x'| < \delta_i$, $|t - t'| < \delta_i$ one obtains

$$\begin{aligned}
|r(x, t) - r(x', t')| &\leq |x_i^*||t - t'| + |x_i - x_i'| + |tx_i - t'x_i'| \\
&\leq \delta_i(|x_i^*| + 1) + |x_i(t - t')| + |t'(x_i - x_i')| \\
&\leq \delta_i(|x_i^*| + 1 + |x_i| + |t'|) \\
&\leq \delta_i(|x_i^*| + 1 + |x_i| + |t| + \delta_i) \\
&\leq \delta_i(|x_i^*| + 2 + |x_i| + |t|) < \epsilon
\end{aligned}$$

If we take $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$ then this function is continuous under the infinity norm and thus in Euclidean topology. Since $R(x, 0)$ is the identity map and $R(x, 1) = x^*$ for all $x \in W$, R is a deformation retract and so W is contractible.

(b)

See Figure 5 for a sketch of S (note that S continues indefinitely in the z direction).

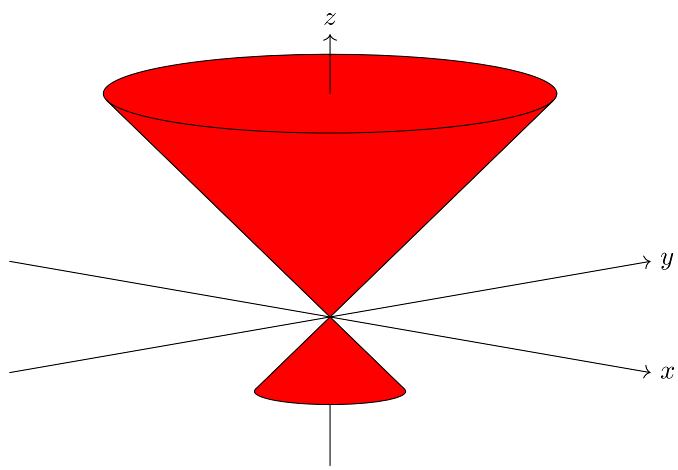

Figure 5: A sketch of S.

In cylindrical coordinates, this surface can be written as

$$S = \{(r, \theta, z) : \theta \in [0, 2\pi),\ r \leq z,\ z \geq -1\}$$

It is hopefully obvious that for any particular point $(r^*, \theta^*, z^*) \in S$ the parametric line $l(t) = ((1 - t)r^*, \theta^*, (1 - t)z^*)$ lies inside S for $t \in [0, 1]$ and connects (r^*, θ^*, z^*) to the origin. Therefore by part (a) this space is contractible.