MATH425 Assignment 1

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Q4

(a)

Note at $x = 0$, $sin(n(0)) = 0$. At $x \neq 0$ we have

$$\left|\frac{sin(nx)}{1+n^2x^2}\right| < \left|\frac{1}{1+n^2x^2}\right| < \left(\frac{1}{nx}\right)^2 \tag{1}$$

Then $\forall \epsilon > 0 \quad \exists N \in \mathbb{N}$ s.t. $\forall n \geq N \implies \frac{1}{n^2} < x^2\epsilon$. So $\forall x \in \mathbb{R}$, $\forall \epsilon > 0 \quad \exists N \in \mathbb{N}$ s.t. $\forall n \leq N$

$$\left|\frac{sin(nx)}{1+n^2x^2} - 0\right| < \epsilon$$

So $f_n(x)$ converges pointwise to zero.

(b)

Consider the sequence of points $\{1/n\}_{n=1}^{\infty}$. At each point in this sequence we have that

$$\frac{sin(1)}{1+1/n} \neq 0$$

And in the limiting case $f_n(1/n) \to sin(1) \neq 0$. So this sequence does not converge uniformly as $\exists x_* \in \mathbb{R}$ s.t. $\exists \epsilon < sin(1)/2$ s.t. $\forall n \in \mathbb{N}$, $n > 0$ we have that $|f_n(x_*)| > \epsilon$.

Q5

(a)

Note for $r \in (0,1) \quad \forall \epsilon > 0 \quad \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ we have $r^{2n} < \epsilon$. Thus $\forall r \in (0,1) \quad \exists N \in \mathbb{N}$ s.t. $\forall n \geq N \implies r^{2n} < 1/2 \implies 1 < 2(1-r^{2n}) \implies 1/(1-r^{2n}) < 2$ as required.

(b)

Note that at $x = 1, -1$ each term in the series is undefined so the infinite sum is thus also undefined. Considering the separate subdomain in turn, we will first turn to $x \in (-1, 1)$. Inside this domain, we can apply the ratio test to determine the series' convergence.

$$R_n(x) = \left| \left(\frac{x^{n+1}}{1 - x^{2n+2}} \right) / \left(\frac{x^n}{1 - x^{2n}} \right) \right| = \left| \frac{x(1 - x^{2n})}{1 - x^{2n+2}} \right| = \left| \frac{x - x^{2n+2}}{1 - x^{2n+2}} \right|$$

Since $x \in (-1, 1)$ in the limit $R_n(x)$ will clearly approach $|x| < 1$ and so by the ratio test we have that $R_n(x)$ converges pointwise when $x \in (0, 1)$. For $x > 1$ we have

$$\left| \frac{x^n}{1 - x^{2n}} \right| = \left| \frac{1}{(1 - x^n)(1 + x^n)} \right| < \left| \frac{1 + x^n}{(1 - x^n)(1 + x^n)} \right| = \left| \frac{1}{1 - x^n} \right|$$

Then we can apply the ratio test to see that

$$R_n(x) = \left| \left(\frac{1}{1 - x^{n+1}} \right) / \left(\frac{1}{1 - x^n} \right) \right| = \left| \frac{1 - x^n}{1 - x^{n+1}} \right|$$

Since $x > 1$ it follows that in the limit that $R_n(x)$ for this series will converge to $1/x < 1$. Therefore the series $\sum_{n=1}^{\infty} |1/(1 - x^n)|$ converges by the ratio test. Thus the series $|S(x)| = \sum_{n=1}^{\infty} |x^n/(1 - x^{2n})|$ is bounded above for all $x > 1$ and (since each individual term is strictly positive) it must converge to a finite limit by the completeness axiom. Since $\sum_{n=1}^{k} |x^n/(1 - x^{2n})| \geq \left| \sum_{n=1}^{k} x^n/(1 - x^{2n}) \right|$ we obtain absolute convergence for any x s.t. $|x| > 1$. Therefore the set E on which $S(x)$ converges pointwise is $E = \{x \in \mathbb{R} : |x| \neq 1\}$.

We must now consider the question of uniform convergence. Note that each individual term has the derivative

$$\begin{aligned}
f_n'(x) &= \frac{nx^{n-1}(1 - x^{2n}) + 2nx^{2n-1}x^n}{(1 - x^{2n})^2} \\
&= \frac{nx^{n-1} - nx^{3n-1} + 2nx^{3n-1}}{(1 - x^{2n})^2} \\
&= \frac{nx^{n-1} + nx^{3n-1}}{(1 - x^{2n})^2} \\
&= \frac{nx^{n-1}(1 + x^{2n})}{(1 - x^{2n})^2}
\end{aligned}$$

So for a closed interval $[a, b]$ inside the domain of the series and with $a > 0$ the absolute value of each term will be maximized either at one of the end points or at $x = 0$ (as this is the only point where $f_n' = 0$) if this is inside the closed interval. Since the series converges absolutely pointwise we can thus set M_n equal to the sum of the absolute value of each term at both endpoints and at $x = 0$ (if this is inside the interval) and obtain a series $\sum_{n=1}^{\infty} M_n$. This series must be convergent as the sum of two or more convergent series is itself convergent. Then it follows that for each point inside the closed interval we have $|x^n/(1 - x^{2n})| < M_n$ and so the Weierstrass M-test can be applied directly to show that the series converges uniformly inside this interval. Then since convergence for every interval $[a, b]$ in the domain of the series with $a > 0$ implies absolute convergence for any interval in the domain of the series (see the argument for pointwise convergence above) the series converges uniformly in every closed interval inside the domain of $S(x)$.

(c)

Note from above that we have (for $x > 1$)

$$\left| \frac{x^n}{1 - x^{2n}} \right| < \left| \frac{1}{1 - x^n} \right| < \left| \frac{1}{(1 - x)^n} \right| = \left| \frac{1}{1 - x} \right|^n = \left(\frac{1}{x - 1} \right)^n$$

So this series converges to $\delta/(1 - \delta)$ where $\delta = |1/(1 - x)|$ by the properties of geometric series. Note that the function $g(x) = 1/(x - 1)$ has a range that spans all the positive reals and that it is clearly monotonically decreasing. Then for all $\epsilon > 0$ choose $x_* > 1$ s.t. $\forall x \geq x_*$ we have that $1/(x - 1) \leq \delta_*$ where

$$\delta_* < \frac{\epsilon}{1+\epsilon}$$

Thus for every $\epsilon > 0 \quad \exists x_*$ s.t. $\forall x > x_*$ we have

$$\left|\sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}}\right| < \sum_{n=1}^{\infty} \left|\frac{x^n}{1-x^{2n}}\right| < \sum_{n=1}^{\infty} \left(\frac{1}{x-1}\right)^n \leq \frac{\delta_*}{1-\delta_*} < \epsilon$$

Therefore $\lim_{x\to\infty} S(x) = 0$.