MATH425 TUTORIAL 1

`Your solutions to Problems 4 and 5 are to be uploaded as a pdf to the provided link on Learn by the end of the day on Friday 4 March.`

Problem 1. Suppose that a sequence of functions $f_n\colon [a,b] \to \mathbb{R}$ converges pointwise to a function f, and suppose that there exists a constant L such that

$$|f_n(x) - f_n(y)| \leq L|x - y|$$

for all $x, y \in [a,b]$ and all n. Show that f is continuous.

Problem 2. Let $g\colon [0,1] \to \mathbb{R}$ be a continuous function such that $g(1) = 0$. Show that the sequence

$$f_n(x) = x^n g(x)$$

converges to 0 uniformly on $[0,1]$.

Problem 3. Show that the series

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$$

(a) converges uniformly on $[-1,b]$ for each $b \in (0,1)$;

(b) does not converge uniformly on $[-1,1)$.

Remark: Since the supremum of $|x_n/\sqrt{n}|$ on $[-1,b]$ is $1/\sqrt{n}$, and $\sum 1/\sqrt{n}$ diverges, part (a) shows that the Weierstrass test for uniform convergence is merely a sufficient condition but not a necessary one.

Problem 4. *(part of Assignment 1)*

Determine whether or not the sequence

$$f_n(x) = \frac{\sin(nx)}{1+n^2x^2}$$

(a) converges pointwise on $\mathbb{R}$;

(b) converges uniformly on $\mathbb{R}$.

Problem 5. *(part of Assignment 1)*

(a) For each $r \in (0,1)$, show that there exists N such that

$$\frac{1}{1-r^{2n}} < 2$$

for all $n \geq N$.

(b) Determine the set $E \subseteq \mathbb{R}$ for which the series

$$S(x) = \sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}}$$

converges (pointwise), and show that the convergence is uniform on each closed sub-interval $[a,b]$ of E.

(c) Show that $\lim_{x\to\infty} S(x)$ exists, and find it.

Hint for #1: Prove that f satisfies the same inequality assumed by f_n (it is known as a Lipschitz condition.)

Hint for #2: For any given $\epsilon > 0$, show that there exists $c \in (0,1)$ such that $|f_n(x)| < \epsilon$ for all $x \in (c,1]$ and all n. Then deal with $[0,c]$ separately.

Hint for #3: Use the remainder estimate for alternating series on the subinterval where it applies.

Hint for #5(b): The set E should be contained in the common domain of all summands of the series, which consists of a number of intervals. One can reduce the amount of work by considering $S\left(\frac{1}{x}\right)$.