MATH401 Dynamical systems

Exercise set 1 (2022)

1. Consider the differential equation

$$\begin{aligned} \dot{x} &= x - y + \alpha x(x^2 + y^2) \\ \dot{y} &= x + y + \alpha y(x^2 + y^2). \end{aligned}$$

(a) Verify that $(x, y) = (0, 0)$ is the only fixed point.

(b) Convert the system to polar coordinates, solve it and sketch solutions in the xy-plane.

2. For the Lorenz equations

$$\left.\begin{aligned} \dot{x} &= \sigma\,(y - x) \\ \dot{y} &= r\,x - x\,z - y \\ \dot{z} &= x\,y - b\,z \end{aligned}\right\} = \mathbf{F}(x, y, z)$$

with $\sigma = 10$ and $b = 8/3$. Calculate:

(a) the Jacobian matrix $D\mathbf{F}(0, 0, 0)$;

(b) the eigenvectors of the given matrix, organised according to whether the real part of the eigenvalue is negative or positive (this may depend on r);

(c) show that if $x = \sqrt{\frac{8}{3}(r-1)}$ then $(x, x, r-1)$ and $(-x, -x, r-1)$ are fixed points for $r > 1$;

(d) now calculate $D\mathbf{F}$ at this other fixed points and show that they have a purely imaginary pair of eigenvalues at $r_H = \frac{470}{19}$. *What kind of bifurcation occurs here?*

3. Verify that $\varphi_t(x) = \frac{xe^t}{xe^t - x + 1}$ is a flow for $x \in [0, 1]$ and find its associated vector field. *Is it a flow on $\mathbb{R}$?*

4. A general FitzHugh-Nagumo equation can be written as

$$\begin{aligned} \dot{v} &= F(v) - w - w_0 \\ \dot{w} &= \epsilon(v - \gamma\,w - v_0) \end{aligned}$$

where w_0, v_0, γ are constants and ϵ is a small parameter.

Show that if $v_0 = w_0 = \gamma = 0$ and $F(v) = v - v^3$ then $x = \sqrt{3}v$ satisfies the van der Pol oscillator equation

$$\ddot{x} - \mu\,(1 - x^2)\dot{x} + x = 0,$$

provided that time is scaled approriately ($\tau = c\,t$ for a suitable choice of c) and $\mu = \mu(\epsilon)$.

5. If A is a $d \times d$ matrix with eigenvalues $\{\lambda_1, \ldots, \lambda_d\}$, show that the linear flow $\varphi_t(\mathbf{x}) = \exp(At)\,\mathbf{x}$ has eigenvalues $\{e^{\lambda_1 t}, \ldots, e^{\lambda_d t}\}$. Deduce that every linear flow is orientation preserving.

Tutorial (Februrary 24, 2022)

Work in groups on problems 1–3 during the tutorial, and problems 4,5 afterwards.

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6. Consider the map $x_{n+1} = \lambda\, x_n$ where $x_n \in \mathbb{R}$, and λ is some constant. Write down the general solution for x_n in terms of x_0. Investigate the long–term dependence (*ie.* $n \to \pm\infty$) of the behaviour of the map on the value of λ. [Hint: the values $\lambda = \pm 1$ are important].

7. Redo question **6** with the two parameter family of maps $x_{n+1} = \lambda\, x_n + \mu$, [Hints: (i) if $\lambda \neq 1$, consider the change of variables: $x(y) = y + \mu/(1-\lambda)$; (ii) if $\lambda = 1$, what happens?]

8. Let $f(x) = 3x(1-x)$. Try using the Newton-Raphson method to find a fixed point of f, starting with various initial conditions $x_0 \in [0.1, 0.6]$. How rapid is the convergence? Now try to find a period–2 point of f. What is happening, and why?

9. Let $f(x) = r\, x(1-x)$ be a logistic map on the interval $I = [0,1]$.

(a) Show that 0 is an attracting fixed point when $r \in (0,1)$.

(b) Show that $p := \frac{r-1}{r}$ is a fixed point, and that it is attracting for $r \in (1,3)$.

(c) Find a formula for the fixed points of $f^2 = f \circ f$, and determine the range(s) of r for which they are attracting.

10. Let x be a periodic point of $f : X \to X$. Is the period unique?

11. Let f be the rigid rotation of the unit circle S^1 given by $f(\theta) = \theta + \omega \pmod 1$.

(a) Show that f has periodic points if and only if ω is rational. [Recall: ω is said to be *rational* if there exist integers p, q such that $\omega = p/q$, and *irrational* otherwise.]

(b) For each of $\omega = 1/5, 2/7, 22/37, 1/\sqrt{2}, \frac{\sqrt{5}-1}{2}$ compute the first 50 terms of the orbit of 0 and display them on a circle.

12. Which of the following transformations are invertible?
Write down formulae for the inverse(s): f^{-1}

(a) $f(x_n) = \lambda\, x_n,\ \lambda \in \mathbb{R}, x_n \in \mathbb{R}$;

(b) $f(x_n) = \lambda\, x_n + \mu,\ \lambda \in \mathbb{R}, x_n \in \mathbb{R}$;

(c) $f(x_n) = \lambda\, x_n(1-x_n),\ \lambda \in [0,4], x_n \in \mathbb{R}$;

(d) $f(x_n) = 2\, x_n (\mod 1),\ x_n \in \mathbb{R}$;

(e) $f(x_n, y_n) = (1 + y_n - a\, x_n^2, b\, x_n),\ (x_n, y_n) \in \mathbb{R}^2$.

13. Sketch the graph of $f^n(x)$ on the unit interval $[0,1]$ for $n = 1, 2, 3$, where $f(x) = 4x(1-x)$. In general, how many periodic points of period n or less does f have?

14. Repeat Exercise **13** for the "2X" map

$$x_{n+1} := \begin{cases} 2x_n, & 0 \le x_n \le \frac{1}{2}, \\ 2x_n - 1, & \frac{1}{2} < x_n \le 1. \end{cases}$$

Tutorial (March 4, 2022)

Attempt all problems. We will discuss question 9 and the proof of Theorem 1 from the notes in the tutorial. There will be questions for group discussion - please bring an internet capable device!
