4. Uniform Convergence and Continuity 2

Today we consider the space of bounded continuous functions $f\colon X \to \mathbb{C}$ where X is a metric space. Notation: $\mathscr{C}(X)$. Often the subset of real-valued functions $\mathscr{C}(X;\mathbb{R})$ is useful.

Note that when X is compact, the "bounded" part is redundant. But in general we need it to be able to define the supremum norm $\|f\| = \sup_X |f|$ and subsequently a metric $d(f,g) = \|f-g\|$. The metric axioms are easy to check. The key part is the triangle inequality for norm: $\|f+g\| \le \|f\| + \|g\|$.

Now we can consider open sets, compact sets, convergent sequences, continuous functions, etc on the space $\mathscr{C}(X)$. What does a neighborhood $N(f,r) = \{g\colon \|g-f\| < r\}$ in this space look like? Closed neighborhood $\overline{N}(f,r) = \{g\colon \|g-f\| \le r\}$?

A convergent sequence in this space is just a uniformly convergent sequence.

Claim: the closed unit ball $\{f\colon \|f\| \le 1\} = \overline{N}(0,1)$ is not compact. Example in $\mathscr{C}([0,1])$ is just $f_n(x) = x^n$. The pointwise limit exists but is not continuous. Any subsequence has the same pointwise limit, which is not continuous, hence cannot converge uniformly.

The lack of compactness is an issue for arguments related to Extreme Value Theorem or subsequences. We cannot just take any sequence and extract a convergent subsequence. But at least there is a positive result: $\mathscr{C}(X)$ is complete, meaning that Cauchy sequences converge.

Proof: Suppose $\{f_n\}$ is Cauchy in $\mathscr{C}(X)$. This means the Cauchy criterion is satisfied, so the sequence converges uniformly to some function f. The function f is continuous. It is also bounded, because if we take any f_n such that $\|f - f_n\| \le 1$, by the triangle we get $|f|$ bounded by $\|f_n\| + 1$ everywhere. So, $f \in \mathscr{C}(X)$ is the limit of this sequence in the sense of convergence in $\mathscr{C}(X)$.

Examples of continuous functions on $\mathscr{C}(X)$ (often called functionals).

(1) For a fixed point $a \in X$, let $\phi(f) = f(a)$.

(2) For a fixed function $g \in X$, let $\phi(f) = \|f - g\| = d(f, g)$. (This works in any metric space).

(3) For a fixed function $g \in X$, let $\phi(f) = \sup_X |fg|$.

To illustrate some issues with the lack of compactness, here is an example. In the space $C([0,1];\mathbb{R})$ consider the set $A = \{f\colon f(0) = 1, \|f\| \leq 1\}$. This is a closed and bounded set. The function $\varphi(f) = \sup_{x\in[0,1]} |xf(x)|$ is continuous by the above. What is $\inf_{f\in A} \varphi(f)$? We have $\varphi(f) > 0$.

Functions of the form $\max(1 - nx, 0)$ have ϕ arbitrarily near 0 but the infimum of 0 is never attained.

5. Uniform Convergence and Integration

We saw on a few "triangular" examples that the integral of the pointwise limit of f_n is not always the limit of integrals. (It may even fail to exist. But the uniform convergence remedies this when the interval of integration is bounded. Recall from MATH102 that our integrals are defined as a limit of Riemann sums as follows. We divide the interval into subintervals by some points $a = x_0 \leq x_1 \leq \cdots \leq x_j = b$, pick a number $t_i \in [x_{i-1}, x_i]$ in each, and form a Riemann sum

$$\sum_{i=1}^{j} f(t_i)\Delta x_i \text{ where } \Delta x_i = x_i - x_{i-1}$$

This sum depends on the partition (call it P) of the interval and the choice of sample points t_i. Define

$$\|P\| = \max_{1\leq i\leq j} \Delta x_i.$$

We could consider whether these sums

$$S(P, f) = \sum_{i=1}^{j} f(t_i)\Delta x_i$$

approach some finite value I as $\|P\|$ tends 0 regardless of the choice of t_i. If so, we call I the Riemann integral of f on $[a, b]$, and say that f is Riemann integrable.

Notation: $\mathscr{R}$ denotes the set of all Riemann integrable functions (on a certain interval).

Formal definition: $f \in \mathscr{R} \iff \exists I\ \forall\epsilon > 0\ \exists\delta > 0$ such that if $\|P\| < \delta$, then $|S(P, f) - I| < \epsilon$ for any choice of t_i.

Theorem: If $f_n \in \mathscr{R}$ for all n, and $f_n \to f$ uniformly, then $f \in \mathscr{R}$ and

$$\int_a^b f\,dx = \lim_{n\to\infty} \int_a^b f_n\,dx. \tag{1}$$

Example: $f_n(x) = \sqrt{x^4 + 1/n}$ on $[0, 1]$. The integral of f_n is non-elementary, but at least we can see $f_n \to x^2$ uniformly,

$$|f_n - f| = \frac{1/n}{\sqrt{x^4 + 1/n} + x^2} \leq \frac{1/n}{\sqrt{1/n}} = \frac{1}{\sqrt{n}} \to 0$$

hence

$$\lim_{n\to\infty} \int_0^1 f_n\,dx = \int_0^1 x^2\,dx = 1/3$$

Proof of theorem: Recall from the above definition that $\displaystyle\int_a^b f(x)\,dx = I \iff$

(2) $\forall\epsilon > 0\ \exists\delta > 0$ such that if $\|P\| < \delta$ then $|S(P,f) - I| < \epsilon$ for any choice of t_i.

An observation: if $|f - g| < \epsilon$, then

$$|S(P,f) - S(P,g)| < \epsilon(b-a).$$

Let's first demonstrate the existence of the limit on the RHS of (1). Let $I_n = \displaystyle\int_a^b f_n\,dx$. For any $\epsilon > 0$, there exists N such that $|f_n(x) - f_m(x)| < \frac{\epsilon}{b-a}$ for all $x \in [a,b]$ and all $n, m \geq N$. So,

$$\left|\int_a^b f_n\,dx - \int_a^b f_m\,dx\right| \leq \int_a^b |f_n - f_m|\,d\alpha < (b-a)\cdot\frac{\epsilon}{b-a} = \epsilon$$

Hence the sequence of integrals is Cauchy. Let I be its limit. It remains to show that (2) holds. Given $\epsilon > 0$, pick N_1 such that

$$\left|\int_a^b f_n\,d\alpha - I\right| < \epsilon/3$$

for $n \geq N_1$. Then pick N_2 such that $\sup|f - f_m| < \frac{\epsilon}{3(b-a)}$ for $m \geq N_2$. Let $n = \max(N_1, N_2)$. Pick a partition P such that $|S(P,f_n) - I_n| < \frac{\epsilon}{3}$. By the triangle inequality,

$$|S(P,f) - I| \leq |S(P,f) - S(P,f_n)| + |S(P,f_n) - I_n| + |I_n - I| < \epsilon$$

as desired.

Note that on an unbounded interval, uniform convergence would not have similar effect: For example, triangles with height $1/n$ and base $2n$ still have area 1 and converge uniformly to 0. But the integrals like $\displaystyle\int_a^\infty f\,dx = \lim_{b\to\infty}\int_a^b f\,dx$ are different objects (improper) so one has to be extra careful with them.

As always, a theorem for sequences gives a theorem for series: if $\sum f_k$ is a uniformly convergent series of functions in $\mathscr{R}$, then the sum is also in $\mathscr{R}$ and

$$\int \sum f_k = \sum \int f_k.$$

Examples: Recall that

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \tag{3}$$

where convergence is uniform on $[0, 1/2]$. Integrate both sides:

$$-\log(1-x)\Big|_0^{1/2} = \sum_{k=0}^{\infty} \frac{(1/2)^{k+1}}{k+1}$$

hence

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)2^{k+1}} = \log 2$$

$\left(\text{the sum can be written nicely as } \displaystyle\sum_{k=1}^{\infty} \frac{1}{k2^k}\right)$.

How about the alternating version of this sum: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{n2^k} =?$

This can be obtained from the integral of (3) over $[-1/2, 0]$, which is

$$-\log(1-x)\Big|_{-1/2}^{0} = -\sum_{k=0}^{\infty} \frac{(-1/2)^{k+1}}{k+1}$$

Hence $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k2^k} = \log(3/2)$.

Additional work is required to handle $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \log(2)$ because convergence on $[-1, 0]$ is not uniform. But by choosing c close to -1 and estimating the contributions of $[-1, -c]$ one can justify the above.