MATH411-22S2 Topics in Algebra
Homework 3:

3.1. Let E/F be a Galois extension of fields and $f(x) \in F[x]$ be an irreducible polynomial. Prove that $\mathrm{Gal}(E/F)$ acts transitively on the set of irreducible factors of $f(x)$ in $E[x]$.

3.2. Let E/F be an extension of fields. Show that for any subgroup $H < \mathrm{Aut}(E|F)$ the fixed subset E^H is a field.

3.3. Suppose that E/F is an algebraic extension, $\theta \in E$ and $g(x) \in F[x]$ is the minimal polynomial of θ. Show that any $\sigma \in \mathrm{Aut}(E|F)$ must send θ to a root of $g(x)$ in E. Deduce that there is a group action of $\mathrm{Gal}(f)$ on the set of roots of f in its splitting field.

3.4. Let $f \in F[x]$ and E/F the splitting field of F and suppose f factors over E as $f = (x-\theta_1)\cdots(x-\theta_n)$. Show that the homomorphism $\mathrm{Gal}(f) \hookrightarrow \mathrm{Sym}\{\theta_1, \ldots, \theta_n\}$ describing the action of $\mathrm{Gal}(f)$ on its roots $\theta_1, \ldots, \theta_n$ is injective.

3.5. The splitting field of $f = x^3 - 2$ over $\mathbb{Q}$ is $E = \mathbb{Q}(\theta, \eta)$ where θ is a cube root of 2 and η is a primitive cube root of unity. Let $\phi \in \mathrm{Gal}(f) = \mathrm{Aut}(E|F)$ be the automorphism whose action on the roots of f is given by $\phi(\eta^i\theta) = \eta^{i+1}\theta$. Write down the matrix for the $\mathbb{Q}$-linear map $\phi : E \to E$ with respect to the ordered basis $(1, \theta, \theta^2, \eta, \eta\theta, \eta\theta^2)$ of E over $\mathbb{Q}$. Find the 1-eigenspace of this matrix.

3.6. Construct a splitting field for $x^5 - 2$ over $\mathbb{Q}$. What is its degree over $\mathbb{Q}$? How many quadratic subfields does it contain (i.e., subfields that have degree 2 over $\mathbb{Q}$)?

3.7. Let E be a finite Galois extension of F with Galois group G, and let $L = E^H$ be the fixed field of a subgroup $H < G$. Show that $\mathrm{Aut}(L|F) \simeq N/H$ where N is the normalizer of H in G, i.e., $N = \{g \in G \,:\, gHg^{-1} = H\}$.