MATH411-19S2 Topics in Algebra
Homework 4:

4.1. For the following polynomials compute the Galois group and describe its action on the roots.

(1) $f = x^3 - 3x + 1 \in \mathbb{Q}[x]$;
(2) $f = x^3 + 3x + 1 \in \mathbb{Q}[x]$;
(3) $f = (x^2 - 2)(x^2 - 3) \in \mathbb{Q}[x]$;
(4) $f = (x^2 - 2)(x^2 - 3) \in \mathbb{F}_5[x]$;
(5) $f = (x^2 - 2)(x^2 - 3) \in \mathbb{Q}(\sqrt{6})[x]$.

4.2. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial with both real and non-real roots. Show that the Galois group of $f(x)$ is not abelian. Can the condition that $f(x)$ is irreducible be dropped?

4.3. Suppose E/F is a Galois extension of degree 27000. Prove that there is an intermediate field M of degree 1000 over F.

4.4. Suppose E/F is a Galois extension of degree 625. Determine the degrees of the intermediate fields $F \subset M \subset E$.

4.5. Let η be a primitive 11th root of 1. Determine the subfield lattice of $\mathbb{Q}(\eta)$. For each field, give the minimal polynomial of a primitive element for each as an extension of $\mathbb{Q}$.

4.6. Use Galois theory to construct a Galois extension E of $\mathbb{Q}$ with $\mathrm{Aut}(E|\mathbb{Q}) \simeq C_2 \times C_4$.

4.7. How many monic irreducible factors does $x^{255} - 1 \in \mathbb{F}_2[x]$ have and what are their degrees?

4.8. Prove that every polynomial in $\mathbb{Q}[x]$ of degree ≤ 4 is solvable in radicals.