MATH428 Assignment 3

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Q1

(a)

Consider the map

$$R(x, y, z) = (x, y, -t + (1 - t)z)$$

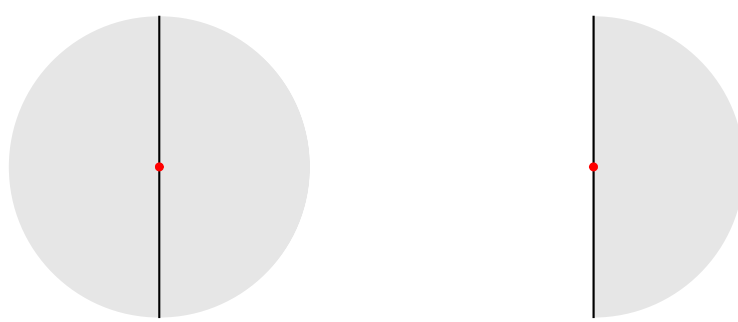
defined on both these spaces. It is hopefully clear that this is a deformation retraction from W to $E^2 \backslash \{p\}$ (for some p in the interior of E^2) and from W^+ to E^2. However $E^2 \backslash \{p\} \sim S^1$ and so

$$W \sim E^2 \backslash \{p\} \sim S^1 \nsim E^2 \sim W^+$$

Since these spaces are not homotopic, they cannot be homeomorphic.

(b)

It is relatively straightforward to construct these spaces as CW complexes (see Figure 1).



(a) $\mathring{E}^2$ (b) $\mathring{E}^2_+$

Figure 1: $\mathring{E}^2$ and $\mathring{E}^2_+$ as cell complexes.

It is hopefully now obvious that $\chi(\mathring{E}^2) = 1$ and $\chi(\mathring{E}^2_+) = 0$ and so these spaces cannot be homeomorphic.

Q2

(a)

For $f : T \to K$ and $g : K \to T$ continuous functions on path connected spaces, if $f \circ g \sim 1_K$ then it follows from Lemma 2 that

$$(f \circ g)_* = h^* \circ 1_K = h^*$$

for some isomorphism h^*. However, g_* is an isomorphism from $\pi_1(K) = \langle g, k : gk = kg^{-1} \rangle$ to $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$ so we obtain

$$g_*(gk) = g_*(kg^{-1}) \implies g_*(g)g_*(k) = g_*(k)g_*(g^{-1}) = g_*(g^{-1})g_*(k) \implies g^*(g) = g_*(g)^{-1}$$

and thus $g_*(g)$ must have order one or two. Since there is only such element in $\mathbb{Z} \times \mathbb{Z}$ (the identity element) it is hopefully obvious that $g_*(\pi_1(K)) = \mathbb{Z}$. Thus $f_* \circ g_*(\pi_1(K))$ is a quotient group of $\mathbb{Z}$. Since the quotient groups of $\mathbb{Z}$ are Abelian it follows that $f_* \circ g_*(\pi_1(K)) < \pi_1(K)$. Thus $(f \circ g)_*$ is not an isomorphism and so $f \circ g \nsim 1_K$.

(b)

Since for $f : T \to K$ and $g : K \to T$ continuous functions, $f \circ g \nsim 1_K$ it follows that these spaces cannot be homotopic and thus cannot be homeomorphic.

Q3

(a)

It is straightforward to construct a retraction from $S^1 \times S^1$ to C homeomorphic to S^1. Considering $S^1 \times S^1$ expressed in polar coordinates, the map

$$r(\theta, r, \phi) = (\theta, 1, \pi/2)$$

is clearly continuous and maps all points in $S^1 \times S^1$ to $\{(\theta, 1, \pi/2) : \theta \in [0, 2\pi)\}$. Since this subspace is also fixed under r this is a retraction.

By the second implication of Corollary 3, C homeomorphic to S^1 is a deformation retraction of $S^1 \times S^1$ if and only if their fundamental groups are isomorphic. However, since $\pi_1(S^1) = \mathbb{Z}$ is cyclic and $\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$ is not cyclic this is clearly not the case. Consequently there is deformation retraction from $S^1 \times S^1$ to a subspace C which is homeomorphic to the circle.

(b)

The subset $G = \{[\gamma] : \exists \gamma' \in [\gamma], \text{Image}(\gamma') \subseteq A\}$ of elements in $\pi_1(X, x)$ is clearly a subgroup as if $[\gamma_1], [\gamma_2] \in G$ then $[\gamma_1 \cdot \gamma_2]$ is trivially also in G.

Consider the homomorphism $r_* : G \to \pi_1(A, x)$ induced by the retraction r and suppose this homomorphism is not one-to-one. Then there exists γ_1 and γ_2 loops based at x contained in $A \subseteq X$ which are not homotopic in X relative $(0, 1)$, but there also exists γ_1' and γ_2' such that $\gamma_i \overset{F_i}{\sim} \gamma_i'$ in X relative $(0, 1)$ for $i = 1, 2$ and $r(\gamma_1') \sim r(\gamma_2')$ in A relative $(0, 1)$. However $r \circ F$ is a homotopy from γ_i to $r(\gamma_i')$ in A relative $(0, 1)$ for $i = 1, 2$. Therefore $\gamma_1 \sim r(\gamma_1') \sim r(\gamma_2') \sim \gamma_2$ in A relative $(0, 1)$ and so one obtains a contradiction. Thus r_* is one-to-one.

Furthermore if γ_1 is a loop based at x in A then clearly $r(\gamma_1) = \gamma_1$. If γ_1 and γ_2 are loops based at x in A that are homotopic relative $(0,1)$ in X by some function F then they are also homotopic relative $(0,1)$ by $r \circ F$. Thus if $[\gamma_1] \in \pi_1(A,x)$ then there exists $[\gamma_1]_X \supseteq [\gamma_1]$ where $[\gamma_1]_X \in G$. Furthermore r_* maps $[\gamma_1]_X \in G$ to $[\gamma_1] \in \pi_1(A,x)$ and this function is therefore onto. Consequently r_* is an isomorphism.

Q4

(a)

S^2 : It is well known that the antipodal map induces covering space action on S^2, so that for the group G composed of the antipodal map and the trivial map one has

$$|G| \times \chi(S^2/G) = \chi(S^2) \implies 2\chi(P^2) = 2 \implies \chi(P^2) = 1$$

P^2 : Given $\chi(P^2) = 1$, the formula given implies that for any G such that the elements of G induce a covering space action on P^2 one has

$$|G| \times \chi(S^2/G) = \chi(P^2) = 1$$

Since the Euler characteristic is integer valued and $|G|$ is a positive integer it follows that $|G| = 1$. Thus G must be the trivial group and there is no non-trivial group of homeomorphisms that induces a covering space action on P^2.

T : Given that $\chi(T) = 0$ we cannot rule out the existence of a non-trivial group inducing a covering space action using the formula given. Consider instead the (doubly) antipodal map

$$\phi : S^1 \times S^1 \to S^1 \times S^1, \quad \phi(x,y) = (-x,-y)$$

This map is its own inverse and is clearly continuous and onto, so it is a homeomorphism. Furthermore since the torus is Hausdorff and this map does not fix any point in T it follows that it must be a covering space action. Since ϕ is its own inverse this means that the group $G = \{1_T, \phi\}$ which is non trivial and finite induces a covering space action on T.

T_2 : It will be convenient to consider T_2 as an embedding in $\mathbb{R}^2$ centered at the origin which is symmetric about the x, y and z axes (see Figure 2).

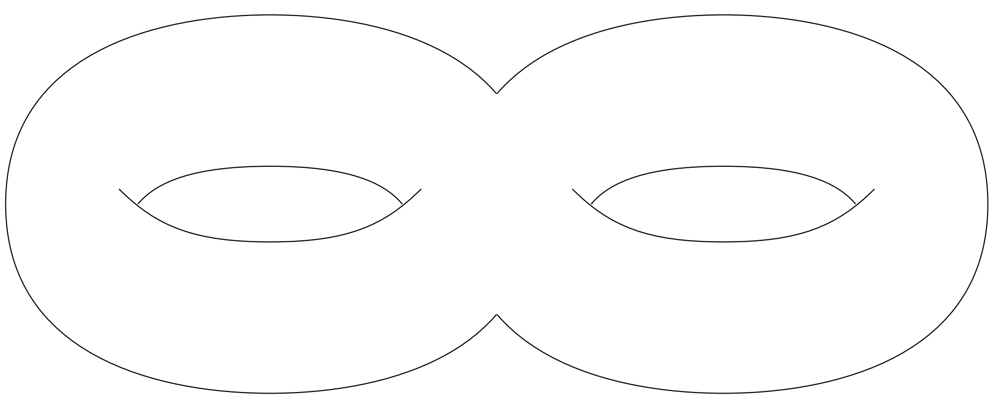


Figure 2: T_2 as an embedding in $\mathbb{R}^3$ centered which is symmetric about all axes.

In this case it is hopefully now obvious that $r(x,y,z) = (-x,-y,-z)$ is a continuous and onto function that is its own inverse on T_2. Furthermore, since this realization of T_2 does not contain the origin this map fixes no points in T_2. Thus the group $G = \{1_{T_2}, r\}$ is a non-trivial finite group that induces a covering space action on T_2.

(b)

Consider the manifold $M = S^1 \times P^2 \times P^2$. This is clearly compact (by theorem 5.4) and (path) connected and its fundamental group is $\pi_1(M) = \pi_1(S^1 \times P^2 \times P^2) \cong \pi_1(S_1) \times \pi_1(P^2) \times \pi_1(P^2) \cong \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as desired.

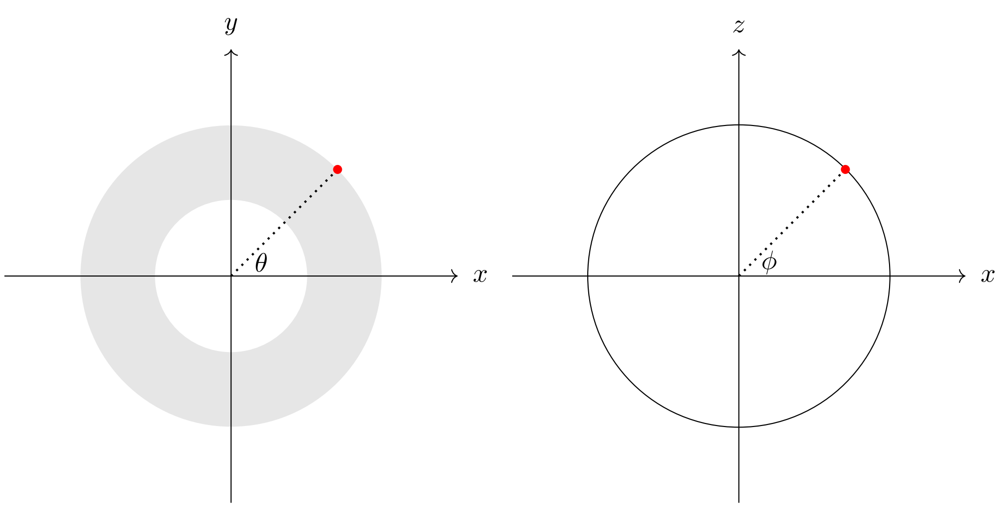
Q5

(a)

For some continuous map v that takes $s \in S^2$ to some point in the tangent plane of s, we will consider the composition of this map with the standard projection map $p : \mathbb{R}^3 \to S^2$. Since both these maps are continuous it follows that $p \circ v : S^2 \to S^2$ is a continuous map. Then, using the suggested result it follows that there must exist some $s \in S^2$ such that $p \circ v(s) = s$ or $p \circ v(s) = -s$. It is hopefully obvious that under the Euclidean metric $d(v(s), s) < d(v(s), -s)$ since v sends points to somewhere in their tangent plane. Consequently we have that $p \circ v(s) = s$ for some $s \in S^2$. Since v sends points to somewhere in their tangent plane this can only occur if $v(s) = s$ for some $s \in S^2$. Thus $v(s) = 0$ for some point in S^2, where 0 is the origin of the tangent plane.

(b)

Consider a point on the Torus given by $(\theta, \phi) \in S^1 \times S^1$, where θ measures the angle relative to the x-axis (when viewed from above) and ϕ measures the angle from the x-axis (viewed from the side). See Figure 3



(a) The Torus viewed from above (b) One ring of the Torus viewed from the side

Figure 3: A description of a point on the Torus with two rotational coordinates.

Now for some fixed angle ρ one can define the map $v_\rho : S^1 \times S^1 \to S^1 \times S^1$ which sends (θ, ϕ) to $(\theta + \rho, \phi)$. This map clearly does not fix any points in $S^1 \times S^1$. Now consider the map $r_\rho \circ v_\rho : S^1 \times S^1 \to \mathbb{R}^3$ as the map which rotates all points on the Torus by some fixed angle and then moves all points outward so that they lie on their respective tangent planes (see Figure 4).

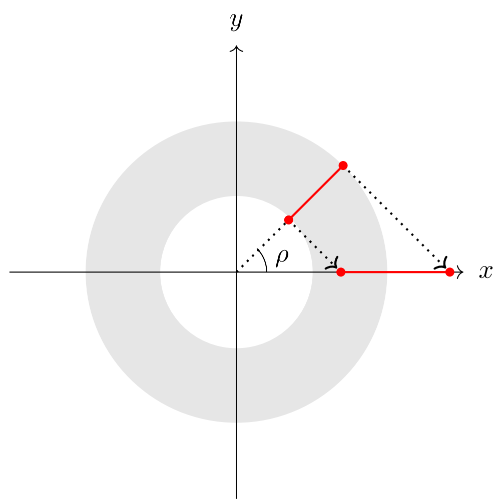


Figure 4: The action of $r_\rho \circ v_\rho$ on a circular subset of the Torus (red) viewed from above

It is hopefully clear that r_ρ is a continuous function, so $r_\rho \circ v_\rho$ is a continuous function that maps every point on the Torus to a point in its tangent plane but maps no points to themselves.