MATH411-22S1 Topics in Algebra
Homework 1:

1.1. Let $\phi : G \to H$ be a homomorphism of groups. Prove or disprove the following statements
(1) If $G' < G$, then $\phi(G') < H$.
(2) If $H' < H$, then $\phi^{-1}(H') < G$.
(3) If $G' \triangleleft G$, then $\phi(G') \triangleleft H$.
(4) If $H' \triangleleft H$, then $\phi^{-1}(H') \triangleleft G$.

1.2. Let $N \triangleleft G$ and $H < G$ be subgroups of G. Prove or disprove the following statements.
(1) $H \cap N \triangleleft H$
(2) $H \cap N \triangleleft G$
(3) $HN \triangleleft G$

1.3. Suppose $n = n_1 + \cdots + n_r$ for positive integers n_i. Use Lagrange's theorem to prove that $\prod n_i!$ divides $n!$.

1.4. Let G be a group and $H < G$ a subgroup of index 2. Prove that H is a normal subgroup.

1.5. Show that $\mu_4 = \{\pm 1, \pm i\} \subset \mathbb{C}^\times$ is a normal subgroup and that $\mathbb{C}^\times/\mu_4 \simeq \mathbb{C}^\times$. Is $\mathbb{R}^\times/\{\pm 1\}$ isomorphic to $\mathbb{R}^\times$?

1.6. Let $H < G$ be a subgroup. For $g \in G$ we call gHg^{-1} a "conjugate subgroup" of H. Show that this is actually a subgroup of G. Show that conjugacy determines an equivalence relation on the set of subgroups of G. Find all conjugacy classes of subgroups of S_3 (and D_4 but don't hand this in!).

1.7. Determine the order of $\mathrm{GL}_n(\mathbb{F}_q)$.

1.8. How many subgroups of $\mathrm{GL}_n(\mathbb{F}_{11})$ contain $\mathrm{SL}_n(\mathbb{F}_{11})$. How many of these are normal in $\mathrm{GL}_n(\mathbb{F}_{11})$?

1.9. How many homomorphisms $\phi : S_3 \to S_4$ are there?

1.10. Find all groups of order 8 up to isomorphism. How many groups of order 64 are there (you don't need to prove this, just google it)?