MATH425 TUTORIAL 2

`Your solution to Problem 3, together with that to other problem(s) to be specified in the next tutorial, is to be uploaded as a pdf to the provided link on Learn by the end of the day on Friday 18 March.`

Problem 1. Let $g(x) = x(1-x)/2$. Define a sequence of functions by $f_0(x) = x$ and $f_{k+1}(x) = g(f_k(x))$ for $k = 0, 1, \ldots$. Show that for $x \in [0,1]$ the series

$$f(x) = \sum_{k=0}^{\infty} f_k(x)$$

converges and f is continuous on $[0,1]$.

Problem 2. Let $g(x) = x(1-x)$. Define a sequence of functions by $f_0(x) = x$ and $f_{n+1}(x) = g(f_n(x))$ for $n = 0, 1, \ldots$. Show that the sequence f_n converges uniformly on $[0,1]$.

Problem 3. *(part of Assignment 2)* Show that the series

$$\sum_{k=0}^{\infty} x^{2k}(1-x^2)$$

converges uniformly on $[0,1]$.

Problem 4. We say that f is *uniformly continuous* if, for each $\epsilon > 0$, there exists δ such that whenever $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. This is a stronger form of continuity, with δ depending only on ϵ but neither on x nor on y.

If $g : \mathbb{R} \to \mathbb{R}$ is a bounded uniformly continuous function, show that the function f defined by

$$f(x) = \sum_{k=1}^{\infty} \frac{g(kx)}{k^2}$$

is uniformly continuous on $\mathbb{R}$.

Hint for #1: Observe that $0 \leq g(x) \leq x/2$.

Hint for #4: First prove the following result about uniformly continuous functions: if $F, G : \mathbb{R} \to \mathbb{R}$ are uniformly continuous and c is a constant, then cF, $F + G$ and $F \circ G$ are uniformly continuous. Then estimate the remainder of the series by something like $\epsilon/3$.