MATH428 Assignment 1

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Q1

(a)

($\rightarrow$) If for all $x \in X$, $\{x\}$ is the intersection of all its neighborhoods, then $\forall y \neq x$ it must be the case that $y \notin \cap_{i\in\mathcal{I}} N_i$, where $\{N_i : i \in \mathcal{I}\}$ is the set of all neighborhoods of x. Therefore there exists N_x such that $y \notin N_x$. By relabeling one obtains that X has the property $(*)$.

($\leftarrow$) Conversely, if $\forall y \neq x$ there exists N_x a neighborhood such that $y \notin N_x$ then clearly the intersection of all neighborhoods of x must only contain x as desired. Thus (a) is equivalent to the property $(*)$.

(b)

($\rightarrow$) If any finite set is closed then $\{x\}$ is closed for all $x \in X$. Consequently this set contains all of its limit points so for $y \neq x$ is not a limit point of this set. Therefore there exists some neighborhood of y which has only trivial intersection with $\{x\}$ (e.g. x is not in this neighborhood). By relabeling it follows that this space has the property $(*)$.

($\leftarrow$) For some finite set $F \subseteq X$ one can consider the complement $O = X\backslash F$. Given that X has the property $(*)$ for any point $x \in O$ and any point $y \in F$ one can choose a neighborhood N_{xy} such that $x \in N_{xy}$ and $y \notin N_{xy}$. Each of these neighborhoods contain an open set O_{xy} such that $x \in O_{xy}$ and for a fixed x one can take the intersection of the corresponding open sets for each $y \in F$. Since this a finite intersection of open sets this intersection is itself open. Since we chose a arbitrary x it follows that for any $x \in O$ there exists an open set containing x which has only trivial intersection with F and so O is open. Since O is open it follows that F is closed as required.

(c)

Consider a topology on a set of two points $\{0, 1\}$ given by $\tau = \{\emptyset, \{0, 1\}\}$. For the only possible pair of distinct points (e.g. 0 and 1) neighborhoods of either of these points must contain an open set containing these points and thus must be supersets of $\{0, 1\}$. It is thus clear that this topology cannot have the property $(*)$.

Consider the finite complement topology on $\mathbb{R}$. Clearly for any two distinct points $x, y \in \mathbb{R}$ one can define an open set (and thus a neighborhood) $N_x = \mathbb{R}\backslash\{y\}$ which contains x and does not contain y. Consequently this topology has the property $(*)$ but it is also easy to show it is not Hausdorff. In particular, since any open set can only exclude a finite number of points all open sets are infinite and any two open sets have non-trivial intersection. Consequently for $x, y \in \mathbb{R}$ it is impossible to choose disjoint open sets O_x and O_y such that $x \in O_x$, $x \notin O_y$ and visa-versa.

Q2

(a)

Let A be an infinite subset of $\mathbb{R}$. From Exercise 2.20 it follows that every point in $\mathbb{R}$ is a limit point of A under τ_{co}. Therefore $\bar{A} = \mathbb{R}$ in this topology so A is dense in this topology.

(b)

It will be most convenient to work with closed sets in this case, so consider C a closed set induced by $(\mathbb{R}, \tau_{co})$. Then for each element in $c \in C$ consider the roots of the polynomial $p(x) - c$. Clearly if $p(x) \neq c$ then this equation has a finite number (possibly zero) of roots and if $p(x) = c$ then the set of roots is the whole space $\mathbb{R}$. So for a finite set C, $p^{-1}(C)$ is either a finite set of points or $\mathbb{R}$ if $p(x) = c$ for $c \in C$. Since a finite set of points and $\mathbb{R}$ are closed in $(\mathbb{R}, \tau_{co})$ it follows that $p^{-1}(C)$ is closed for all closed C and so p is continuous.

Consider $f(x) = |x|$. For c in a closed set C induced by $(\mathbb{R}, \tau_{co})$, the equation $f(x) - c$ has at most two roots. Therefore $f^{-1}(C)$ is also a finite set and so f is continuous in $(\mathbb{R}, \tau_{co})$.

(c)

Consider an open set in $(\mathbb{R}, \tau_{co})$ given by $O_* = \mathbb{R} \backslash \{a_1, a_2, \ldots, a_n\}$ for some $a_i \in \mathbb{R}$. For convenience we will assume without loss of generality that the complement set $\{a_1, a_2, \ldots, a_n\}$ is ordered so that $a_i < a_{i+1}$ for each $1 \leq i \leq n$. Therefore one can also write this set as

$$O_* = (-\infty, a_1) \cup (a_1, a_2) \cup \cdots \cup (a_{n-1}, a_n) \cup (a_n, \infty)$$

It is hopefully clear that O_* is open in the Euclidean topology, so its preimage under f will also be open (given f is a continuous function from $(\mathbb{R}, \tau_E)$ to $(\mathbb{R}, \tau_E)$). Since the empty set and the whole space $\mathbb{R}$ are open sets in both topologies the pre-image of both sets under f must also be open. Therefore for every open set in $O \in \tau_{co}$ the preimage of O under f is open in $(\mathbb{R}, \tau_E)$ and thus f is a continuous function from $(\mathbb{R}, \tau_E)$ to $(\mathbb{R}, \tau_{co})$.

(d)

The step function

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

The set $\{0\}$ is closed under the Euclidean topology, but its pre-image is $(-\infty, 0)$. Since this set is not the complement of a set in $(\mathbb{R}, \tau_{co})$ it is not closed. Therefore this function is not continuous.

Q3

Without loss of generality let $T = \{0, 1\}$.

(a)

$(\leftarrow)$ Since $\{0\}$ and $\{1\}$ are open sets in T, it follows that for a continuous function $f : X \to T$ the sets $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are open. These sets are trivially disjoint and their union is the whole set. Therefore if both are non-empty (e.g. the function is non-constant) then X is disconnected.

($\rightarrow$) The converse follows straightforwardly: if X is disconnected and has an open partition formed from connected components O_1 and O_2 then one can define a function f where $f(x) = 0$ if $x \in O_1$ and $f(x) = 1$ if $x \in O_2$. This function is continuous by construction and clearly non-constant. By contrapositives, if every continuous function f from X to T is constant then X is connected.

Since X is disconnected if and only if there exists a non-constant continuous function from X to T it follows that X is connected if and only if all continuous functions from X to T are constant.

(b)

Consider a continuous function from $X \cup Y$ to T. Since X and Y are connected this function must be constant on both X and Y. Without loss of generality let $f(X) = 1$. Then $f(X \cap Y) = 1$ and so $f(Y) = 1$. It follows then that $f(X \cup Y) = 1$ and so f is constant. Consequently $X \cup Y$ is connected.

(c)

Let $f : B \rightarrow T$ be continuous. Under the induced subspace topology A is a connected topological space and so the restriction of f to A must be constant. Then f must be constant on A and without loss of generality let $f(A) = 0$. Suppose that $f^{-1}(\{1\})$ is non-empty. Since f is continuous it follows that this set is open. Since B contains only limit points of A (as $B \subseteq \bar{A}$) every open set of a point in $B \backslash A$ has non-trivial intersection with A. Therefore $f^{-1}(\{1\}) \cup A \neq \emptyset$ which is a contradiction. Thus $f^{-1}(\{1\})$ empty and so f is constant on B. Since f was arbitrary it follows from (a) that B is connected.