MATH425 Assignment 5

Elliott Hughes

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W8Q4

Consider the sequence of functions f_n given by $f_1(x) = x$ and $f_{n+1}(x) = 1 + \frac{1}{f_n(x)}$. Then if $|f_n(x) - f_n(y)| < \delta$ for some $n \in \mathbb{N}$ and $\delta > 0$ it follows that

$$\begin{aligned}
|f_{n+1}(x) - f_{n+1}(y)| &= \left|1 + \frac{1}{f_n(x)} - 1 - \frac{1}{f^n(x)}\right| \\
&= \left|\frac{1}{f_n(x)} - \frac{1}{f_n(y)}\right| \\
&= \frac{|f_n(x) - f_n(y)|}{|f_n(x)f_n(y)|} \\
&< \frac{\delta}{|f_n(x)f_n(y)|}
\end{aligned}$$

Consequently if $|x - y| < \delta$ and $x, y \geq 1$ we have $|f_1(x) - f_1(y)| < \delta$, $|f_2(x) - f_2(y)| < \delta/xy$ and for f_i, $i \geq 2$ we have

$$|f_n(x) - f_n(y)| < \frac{\delta}{\Pi_{i=1}^{n-1}|f_i(x)f_i(y)|}$$

Then since $x, y \geq 1$ it follows trivially that $f_n(x), f_n(y) > 1$ for all $n > 1$ and so $|f_n(x) - f_n(y)| < \delta$. Therefore, for all $\epsilon > 0$ and for all $n \in \mathbb{N}$ one can choose $\delta = \epsilon$ such that, if $x, y > 1$ and $|x - y| < \delta$ then $|f_n(x) - f_n(y)| < \epsilon$. Thus this family is equicontinuous.

W9Q3

Consider $f \in \mathcal{F}$ and $z_0 \in B(0,1)$ and choose $D = \overline{B(0,r)}$, $r > 0$ such that $z_0 \in D^o$ (where D^o denotes the interior of D). Let $a = \min\{z - z_0\}$ for $z \in \partial D$. It follows from the Cauchy integral formula that

$$
\begin{aligned}
|f(z_0)| &= \left|\int_{\partial D} \frac{f(z)}{z-z_0} dz\right| \\
&\leq \int_0^{2\pi} \left|\frac{f(re^{i\theta})}{re^{i\theta}-z_0}\right| d\theta \\
&\leq \int_0^{2\pi} \frac{|f(re^{i\theta})|}{a} d\theta \\
&\leq \frac{1}{a}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq \frac{C}{a}
\end{aligned}
$$

Thus for every $f \in \mathcal{F}$ and $z_0 \in B(0,1)$, $|f(z_0)| \leq C/a$ and thus this family is uniformly bounded on $B(0,1)$. Since uniform boundedness on $B(0,1)$ implies uniform boundedness on every compact subset of $B(0,1)$ it follows that every sequence $\{f_i\}_{j=1}^{\infty} \subseteq \mathcal{F}$ is bounded on compact sets on $B(0,1)$. Therefore the family $\mathcal{F}' = \{f_j\}_{j=1}^{\infty}$ contains a convergent sequence $\{f_k\}_{k=1}^{\infty}$ by Montel's theorem. Since for each $j \in \mathbb{N}$ there are only a finite number of numbers in $\mathbb{N}$ less than j, one can define a new sequence $\{f_{k'}\}_{k'=1}^{\infty}$ by letting $f_1 \in \{f_{k'}\}_{k'=1}^{\infty}$ equal $f_1 \in \{f_k\}_{k=1}^{\infty}$ and for $f_i \in \{f_k'\}_{k'=1}^{\infty}$ let f_{i+1} be first term in $\{f_k\}_{k=1}^{\infty}$ with a higher index in $\{f_k\}_{k=1}^{\infty}$ and such that the matching term in $\{f_j\}_{j=1}^{\infty}$ has a higher index than the term matching f_i. To phrase this more informally, we are requiring that the the next term in $\{f_{k'}\}_{k'=1}^{\infty}$ is further along in both sequences. Then $\{f_{k'}\}_{k=1}^{\infty}$ is a subsequence of both $\{f_k\}_{k=1}^{\infty}$ and $\{f_j\}_{j=1}^{\infty}$ and since $\{f_k\}_{k=1}^{\infty}$ is convergent it must also converge on compact sets, so the desired result is obtained.