MATH401 Dynamical systems

Exercise set 2 (2022)

15. Consider the logistic map given by $f(x) := 3.835\,x\,(1-x)$ and let $a_0 = 0.15$.

(a) use MATLAB to calculate the first 6 points on the orbit of a_0 under f, and show that $[a_0, a_3]$ is mapped inside itself by f^3.

(b) Deduce that f has a period-3 point.

(c) Show that $|(f^3)'(x)| \leq 0.8$ for $x \in [a_0, a_3]$. What does this tell you about the orbit of the period-3 point from part (b)?

(d) Calculate the first 100 terms of the orbit of a_0. Is the result consistent with your answers to parts (a)-(c)?

Hint: you can use a simple loop with the anonymous handle `logistic = @(x,r) r*x*(1-x);`

16. The logistic map has a window of parameter values for which there is an attracting period-3 orbit. Plot the graph of $f^3(x)$ for three values of r: (i) just to the left of the period-3 window; (ii) inside the period-3 window; (iii) just to the right of the period-3 window. Comment on these graphs in terms of the existence and stability of the period-3 orbit, and describe the associated bifurcations that take place.

17. For the logistic map with $r = 3.8282$, choose an initial condition and carry out 250 iterations. Plot the iterations as a time series (x_n against n) What do you notice? Try some different initial conditions. Can you explain the behaviour by looking at the iterations as a cobweb diagram on the graph of $f^3(x)$? (Try zooming in around the turning points of f^3).

18. Consider the family of maps

$$f(x) = x + \mu\,x^3.$$

Show that 0 is a *attracting* fixed point if $\mu < 0$ but is *repelling* if $\mu > 0$. What happens when $\mu = 0$? Find the *basin of attraction* $\mathcal{B}(\{0\})$ when $\mu < 0$.

19. Consider the recurrence

$$y_{n+1} = y_n^2 - c$$

where $c > -1/4$ is a parameter.

(a) Find the fixed points of the map and determine their stability for different values of c.

(b) Use MATLAB to calculate a bifurcation diagram for this map. Identify the fixed points of the map on the diagram and analyse as much as you can of the behaviour you can see in the diagram.

(c) What happens when $c = -1/4$ and $c < -1/4$?

(d) Find a, b, r (depending on c) such that $y = h(x) = a\,x + b$ changes variables to make the dynamics of y equivalent to $x_{n+1} = f(x_n) := r\,x_n\,(1 - x_n)$.

20. Use MATLAB to generate a bifurcation diagram for the sine map

$$x_{n+1} = (r/4)\sin(\pi x_n)$$

for $0 \le r \le 4$ and $0 \le x \le 1$. Identify fixed points, periodic points, period doublings, etc. on the diagram. Compare it to the bifurcation diagram for the logistic equation.

21. For $s > 0$ the map

$$T_s = \begin{cases} s\,x & x \in [0, 1/2] \\ s\,(1-x) & x \in [1/2, 1] \end{cases}$$

is called a **tent map** (with slope s).

(a) Sketch T_s and T_s^2 for $s = 1, 1.5, 2$.

(b) Using MATLAB or otherwise, plot a bifurcation diagram for T_s where $s \in [1, 2]$.

(c) Comment on similarities and differences with diagrams for quadratic and sine families.

22. **[2017 Exam]** Let $c \le 4$ be a real parameter, and for $x \in \mathbb{R}$ let

$$f(x) = c - \frac{1}{2}x^2.$$

(a) Sketch graphs of f for $c = -1$, $c = 1$ and $c = 4$.

(b) Add the diagonal $y = x$ to each of your graphs, and point out any fixed points.

(c) Now solve for fixed points x_* of f and check that they are fixed points.

(d) Identify the parameter value c_* at which the number of fixed points changes. What kind of bifurcation is occuring here?

(e) Analyse the stability of each fixed point found in part (c). Identify the range of c for which there is at least one attracting fixed point.

(f) Find the value of c at which a period-doubling occurs, and find a formula for the period 2 orbit associated with the bifurcation.

Hint: $c - \frac{1}{2}\left(c - \frac{1}{2}x^2\right)^2 - x = \frac{1}{2}\left(c - \frac{1}{2}x^2 - x\right)\,\left(\frac{1}{2}x^2 - x + (2-c)\right).$

(g) Analyse the stability of the period-2 orbit found in the previous part; what can you say about further bifurcations as c is increased?

(h) Sketch a bifurcation diagram displaying all of the information that you know. Speculate on how the dynamics will change as c increases towards 4.

Tutorial (March 11, 2022)

We will begin with presentations on last week's "What is a ...?" questions.

Then we will discuss numbers: 17, 18, 19, 22.

Q1 [2021 Exam] This question concerns the discrete time dynamical system defined by

$$x_{n+1} = f(x_n) := \lambda\, x_n - x_n^3$$

where λ is a real parameter.

(a) **[4 marks]** Sketch the graphs $y = f(x)$ and $y = x$ for $x \in [-\sqrt{1+\lambda}, \sqrt{1+\lambda}]$ when λ takes the values 0.5, 1 and 1.5 (*ie.* make three different graphs).

(b) **[2 marks]** Find the fixed point(s) of f.

(c) **[4 marks]** Analyse the stability of the fixed points of f, depending on λ.

(d) **[2 marks]** What kind of bifurcation occurs at $\lambda = 1$?

(e) **[2 marks]** What changes about the dynamics when λ decreases through -1?

(f) **[2 marks]** What changes about the dynamics when λ passes through 2?

Hint: for (e) and (f)

$$\lambda\,(\lambda\, x - x^3) - (\lambda\, x - x^3)^3 - x = x\,(x^2 - (\lambda - 1))\,(x^2 - (\lambda + 1))\,(x^4 - \lambda\, x^2 + 1).$$

(g) **[2 marks]** Sketch a bifurcation diagram that represents all of the information derived so far.

(h) **[2 marks]** What changes about the dynamics when λ increases through 3?

Hint: You may assume without proof that $2\,(\lambda/3)^{3/2} > \sqrt{\lambda + 1}$ for $\lambda > 3$.

Assignment 1 (worth 10%, due 4pm March 18, 2022)

Problem numbers: 15, 16, 20, 21, and Q1 from the 2021 exam.
