MATH425 Assignment 1

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Q2

(a)

Consider $a_n - a_{n+1}$:

$$\begin{aligned}
a_n - a_{n+1} &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} - \frac{1}{n+2} - \cdots - \frac{1}{2n+1} - \frac{1}{2n+2} \\
&= \frac{1}{n+1} - \frac{1}{2n+1} - \frac{1}{2n+2} \\
&= \frac{2}{2n+2} - \frac{1}{2n+2} - \frac{1}{2n+1} \\
&= \frac{1}{2n+2} - \frac{1}{2n+1} < 0
\end{aligned}$$

So this sequence is monotonically increasing. Since for all $n \in \mathbb{N}$

$$a_n = \sum_{j=1}^{n} \frac{1}{n+j} \leq \sum_{j=1}^{n} \frac{1}{n+1} = \frac{n}{n+1} < 1$$

this sum is bounded above and thus must converge by completeness of the Reals.

(b)

Consider the function $f(x) = 1/x$ on $[1,2]$. If we divide $[1,2]$ into intervals of length $1/n$ (so the j-th interval is $[1 + (j-1)/n, 1 + j/n]$) we define a partition P and clearly the length of each interval in this partition approaches zero as n approaches infinity. Let us evaluate the function at the right-hand of each interval, so $x_j = 1 + j/n = (n+j)/n$. Therefore we can express the n-th Riemann sum of the integral of this function (on this domain) as

$$\sum_{j=1}^{n} \frac{1}{n} \frac{n}{n+j} = \sum_{j=1}^{n} \frac{1}{n+j} = a_n$$

Since this function is Riemann integrable on this domain, it follows that

$$\lim_{n\to\infty} a_n = \int_1^2 \frac{1}{x} dx = \big[\ln(x)\big]_1^2 = \ln(2)$$

So the limit of a_n is $\ln(2)$.

Q3

(a)

It is useful to note that for all $n \in \mathbb{N}$

$$\sum_{j=1}^{2n+1} \frac{(-1)^j}{j+1} = \sum_{k=0}^{n} \left(\frac{(-1)^{2k}}{2k+1} + \frac{(-1)^{2k+1}}{2k+2} \right) = \sum_{k=0}^{n} \left(\frac{1}{2k+1} - \frac{1}{2k+2} \right)$$

By an identical argument it is straightforward to see that

$$\begin{aligned} \sum_{j=0}^{2n+1} \frac{(-1)^j}{j+1} + \sum_{j=1}^{2n+2} \frac{(-1)^j}{j+1} &= \sum_{k=0}^{n} \left(\frac{1}{2k+1} - \frac{1}{2k+2} \right) + \sum_{k=0}^{n} \left(-\frac{1}{2k+2} + \frac{1}{2k+3} \right) \\ &= \sum_{k=0}^{n} \left(\frac{1}{2k+1} - \frac{2}{2k+2} + \frac{1}{2k+3} \right) \end{aligned}$$

But clearly

$$\sum_{k=0}^{n} \left(\frac{1}{2k+1} - \frac{2}{2k+2} + \frac{1}{2k+3} \right) = 2 \sum_{j=1}^{2n+1} \frac{(-1)^j}{j+1} + \frac{1}{2n+3} - 1$$

A simple rearrangement then gives the desired result

$$1 + \sum_{k=0}^{n} \left(\frac{1}{2k+1} - \frac{2}{2k+2} + \frac{1}{2k+3} \right) = \frac{1}{2n+3} + 2 \sum_{j=1}^{2n+1} \frac{(-1)^j}{j+1}$$

It is also useful to note that, since this result holds true for any $n \in \mathbb{N}$, it will hold true in the limiting case where $n \to \infty$.

(b)

We know from Tutorial 2 that this series converges uniformly on this interval, so we can integrate term-by-term. Therefore (using the result from tutorial 2)

$$\int_0^1 \frac{1-x}{1+x} dx = \sum_{k=0}^{\infty} \int_0^1 x^{2k}(1-x)^2 dx$$

Considering the LHS of this equation, it is straightforward to compute the integral

$$\begin{aligned} \int_0^1 \frac{1-x}{1+x} dx &= \int_0^1 \frac{1}{1+x} dx - \int_0^1 \frac{x}{1+x} dx \\ &= \big[\ln(|1+x|)\big]_0^1 - \big[x \ln(|x+1|)\big]_0^1 + \int_0^1 \ln(|1+x|) dx \\ &= \ln(2) - \ln(2) + \big[(1+x)\ln(1+x) - x\big]_0^1 \\ &= 2\ln(2) - 1 \end{aligned}$$

Therefore we have

$$
\begin{aligned}
2\ln(2) - 1 &= \sum_{k=0}^{\infty} \int_0^1 x^{2k}(1 - 2x + x^2)dx \\
&= \sum_{k=0}^{\infty} \left[\frac{1}{2k+1}x^{2k+1} - \frac{2}{2k+2}x^{2k+2} + \frac{1}{2k+3}x^{2k+3} \right]_0^1 \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{2k+1} - \frac{2}{2k+2} + \frac{1}{2k+3} \right) \\
\implies 2\ln(2) &= 1 + \sum_{k=0}^{\infty} \left(\frac{1}{2k+1} - \frac{2}{2k+2} + \frac{1}{2k+3} \right)
\end{aligned}
$$

By the result proved in (a), we have

$$
2\ln(2) = \lim_{n\to\infty} \left(\frac{1}{2n+3} + 2\sum_{j=0}^{n} \frac{(-1)^j}{j+1} \right) = 2\sum_{j=0}^{\infty} \frac{(-1)^j}{j+1}
$$

Therefore

$$
\int_0^1 \frac{2}{1+x}dx = 2\ln(2) = 2\sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} \implies \int_0^1 \frac{1}{1+x}dx = \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1}
$$

As required.

Q1

Consider a sequence of functions $f_n(x)$ with $f_1(x) = 0$ everywhere and for $n \geq 2$

$$
f_n(x) = \begin{cases} n^{11/6}x & 0 \leq x < \frac{1}{n} \\ 2n^{5/6} - n^{11/6}x & \frac{1}{n} \leq x < \frac{2}{n} \\ 0 & x \geq \frac{2}{n} \end{cases}
$$

It is straightforward to verify that this function is continuous. Furthermore, for all $n \in \mathbb{N}$ $f_n(0) = 0$ and for $0 < x \leq 1$ there exists some N such that for all $n \geq N$, $1/n < x$. Therefore, by the Archimedean property f_n converges pointwise to zero. Thus this function fulfills the first property. It is useful to note that the segment of the function from $x = 0$ to $x = 2/n$ is symmetric about $x = 1/n$ (so the squared function on this interval will also be symmetric about the same point).

Considering the second property, we can see that for all $n \in \mathbb{N}$, $n > 1$

$$
\begin{aligned}
\int_0^1 |f_n(x)|dx &= \int_0^{1/n} n^{11/6}xdx + \int_{1/n}^{2/n} 2n^{5/6} - n^{11/6}xdx \\
&= \left[\frac{1}{2}n^{5/6}x^2 \right]_0^{1/n} + \left[2n^{5/6}x - \frac{1}{2}n^{11/6}x^2 \right]_{1/n}^{2/n} \\
&= \frac{1}{2}\left(\frac{1}{n}\right)^{1/6} + 2\left(\frac{1}{n}\right)^{1/6} - 2\left(\frac{1}{n}\right)^{1/6} + \frac{1}{2}\left(\frac{1}{n}\right)^{1/6} \\
&= \left(\frac{1}{n}\right)^{1/6}
\end{aligned}
$$

This clearly approaches zero, so the second condition also holds.

Finally we consider the third property. For all $n \in \mathbb{N}$, $n > 1$ we have

$$\begin{aligned}
\int_0^1 (f_n(x))^2 dx &= 2\int_0^{1/n} n^{121/36} x^2 dx \qquad \text{By the symmetry noted above} \\
&= \frac{2}{3}\left[n^{121/36} x^3\right]_0^{1/n} \\
&= \frac{2}{3}\left(n^{121/36} n^{-108/36}\right) \\
&= \frac{2}{3} n^{13/36}
\end{aligned}$$

Then the sequence $I_n = \frac{2}{3} n^{13/36}$ clearly approaches infinity as n approaches infinity. Therefore the third property also holds.