MATH425 Assignment 1

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W3Q3

We wish to show that $\sum_{n=0}^{\infty} f_n(x)$, $f_n(x) = x^{2n}(1-x)^2$ converges uniformly on $x \in [0,1]$. For $x \in [0,1)$ this series is a geometric series with common ratio x^2 and initial term $(1-x)^2$. Since $|x^2| < 1$ on this interval this implies that the series converges to the sum

$$\frac{(1-x)^2}{1-x^2} = \frac{(1-x)(1-x)}{(1-x)(1+x)} = \frac{1-x}{1+x}$$

This function $f(x) = (1-x)/(1+x)$ is continuous on $[0,1)$. At $x = 1$ every term of the series is zero and $(1-1)/(1+1) = 0$, so the series converges pointwise to the limiting function $f(x) = (1-x)/(1+x)$ for all $x \in [0,1]$. Note that $(1-x)x^{2n} \geq 0$ for all $x \in [0,1]$ and for all $n \in N$, so the sequence of partial sums of f_n is monotone. Then, since f is continuous on $[0,1]$ and $[0,1]$ is a compact subset, it follows that convergence of the series is uniform by the partial converse presented on page 11 of the notes.

W4Q3

(a)

Consider a convergent sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathscr{C}[-T,T]$, with $f_n(T) = f_n(-T) = 0$ for all $n \in \mathbb{N}$. Since convergence in $\mathcal{C}[-T,T]$ is equivalent to uniform convergence on this interval, it is clear that $\lim_{n\to\infty} f_n(T) = 0$ as $f_n(T) = 0 \quad \forall n \in \mathbb{N}$. Therefore, since T is a limit point of $[-T,T]$ it follows that $\lim_{x\to T} f(x) = 0$ (by the result on page 9). By an identical argument it also follows that $\lim_{x\to -T} f(x) = 0$. Then since f is continuous on $[-T,T]$ it follows that $f(-T) = \lim_{x\to -T} f(x) = \lim_{x\to T} f(x) = f(T) = 0$. Thus $f \in \mathcal{C}[-T,T]$ and $f(-T) = f(T) = 0$.

Consider a sequence $\{f_n\}_{n=1}^{\infty} \subset V_T$ which is convergent in $\mathcal{C}(\mathbb{R})$. Then each f_n can be written

$$f_n(x) = \begin{cases} f_n^{V_T}(x), & x \in [-T,T] \\ 0 & x \notin [-T,T] \end{cases}$$

Where each $f_n^{V_T}(x)$ is a continuous and bounded function from $[-T,T]$ to $\mathbb{C}$ and $f_n^{V_T}(-T) = f_n^{V_T}(T) = 0$. Note this is required for continuity of f_n as if $f_n^{V_T}(T) \neq 0$ then clearly there exists some $\epsilon = |f_n^{V_T}(T)|$ such that for all $\delta > 0$ there exists some $x_* > T$ with $|x_* - T| < \delta$ but $|f_n(x_*) - f_n(T)| = \epsilon$ (an identical argument requires that $f_n^{V_T}(-T) = 0$). Since $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent in $\mathcal{C}(\mathbb{R})$ it is clear that $\{f_n^{V_T}\}_{n=1}^{\infty}$ must be uniformly convergent on $[-T,T]$. Then by the result above $\{f_n^{V_T}\}_{n=1}^{\infty}$ converges to some f^{V_T} with $f^{V_T}(-T) = f^{V_T}(T) = 0$. Therefore the function

$$f(x) = \begin{cases} f^{V_T}(x), & x \in [-T, T] \\ 0 & x \notin [-T, T] \end{cases}$$

Is in V_T. Furthermore since $\|f_n - f\|_\infty = \|f_n^{V_T} - f^{V_T}\|_\infty$ (as for $x \notin [-T,T]$, $|f_n(x) - f(x)| = 0$) and $f_n^{V_T}$ converges uniformly to f^{V_T} it follows that f_n converges uniformly to f. Thus for any sequence $f_n \in V_T$ it follows that this function converges in $\mathcal{C}(\mathbb{R})$ to $f \in V_T$. Therefore V_T is closed.

(b)

Consider the series $\sum_{n=1}^{\infty} f_n(x)$ on $\mathbb{R}$ where

$$f_n(x) = \begin{cases} 0, & x < n \\ \frac{1}{n}x - 1, & n \leq x < n + \frac{1}{2} \\ -\frac{1}{n}x + \frac{n+1}{n}, & n + \frac{1}{2} \leq x < n + 1 \\ 0, & x \geq n + 1 \end{cases}$$

Note that the maxima of the absolute value of each f_n is clearly $1/2n$. Then for $\forall \epsilon > 0$ there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$ by the Archimedean property. For $k, m \geq N$ we have

$$\left\| \sum_{n=1}^{k} f_n(x) - \sum_{n=1}^{m} f_n(x) \right\|_\infty \leq \frac{1}{2N} < \epsilon$$

So this series is Cauchy in $\mathcal{C}(\mathbb{R})$ and it consequently must converge as this space is complete. Note that each of these partial sums $\sum_{n=1}^{m} f_n(x)$ with $m \in \mathbb{N}$ is equal to zero for all $x > m + 1$ so each term in this sequence is contained in $\bigcup_{T>0} V_T$. However for any V_T there exists some $m' \in \mathbb{N}$ such that $m' > T$ and consequently $f_{m'} \notin V_T$. Therefore the limit of this series is not in $\bigcup_{T>0} V_T$ and so this space is not complete.