MATH411 Assignment 1

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1.1.1

Let $g, h \in G'$ so by the properties of homomorphisms we have $\phi(g)\phi(h) = \phi(gh)$ and $gh \in G'$ since $G' < G$ (thus $\phi(G)$ is closed in H). Furthermore, if $g \in G'$ and 1 is the identity element of G then $\phi(g)\phi(1) = \phi(g)$ which implies that $\phi(1)$ is the identity of H so $\phi(G')$ contains the identity. If $g \in G'$ then $g^{-1} \in G$ and $\phi(g)\phi(g^{-1}) = \phi(gg^{-1}) = \phi(1)$ so for $\phi(g) \in \phi(G')$ we have $\phi(g^{-1}) = \phi(g)^{-1} \in \phi(G')$ and thus inverses exist for each element in $\phi(G')$. Finally we inherit the associative property since H is a group. Thus $\phi(G') < H$.

1.1.2

Consider $g, h \in G$. Then if $\phi(g), \phi(h) \in H'$ it follows that $\phi(gh) \in H'$ since $H' < H$. Thus $\forall g, h \in \phi^{-1}(H')$ it follows that $gh \in \phi^{-1}(H')$. Furthermore if $g \in G$ s.t. $\phi(g) \in H'$ then $\phi(g)^{-1} = \phi(g^{-1}) \in H'$ since $H' < H$ and consequently if $g \in \phi^{-1}(H')$ then $g^{-1} \in \phi^{-1}(H')$. Combining the two proceeding results trivially implies that $1 \in \phi^{-1}(H')$ and the associativity property is inherited from the larger group G. Therefore $\phi^{-1}(H') < G$.

1.1.3

From 1.9 (see below) there is a homomorphism from S_3 to some two element group (e.g. $\{1, (12)\}$) in S_4. Since $S_3 \triangleleft S_3$ but $(13)(12)(13) = (23) \notin \{1, (12)\}$, it follows that, in general, if $G' \triangleleft G$ then $\phi(G') < H$ but $\phi(G')$ may not be normal.

1.1.4

Consider the homomorphism $\varphi : H \to H/H'$ defined by $\varphi(g) = gH'$. This is a homomorphism as H' is normal and it has kernel H'. It follows therefore that $\varphi \circ \phi$ is also a homomorphism with $ker(\varphi \circ \phi) = \phi^{-1}(H')$ so $\phi^{-1}(H')$ is normal by lemma 3.10 (in J.S. Milne's notes).

1.2.1

For $h, n \in H \cap N$ we have that $hn \in H$ and $hn \in N$ so $hn \in H \cap N$. Then for $h \in H \cap N$ we have $h^{-1} \in H$ as $H < G$ and $h^{-1} \in N$ by an identical argument. Thus if $h \in H \cap N$ it follows that $h^{-1} \in H \cap N$ and as an immediate consequence that $1 \in H \cap N$. Thus $H \cap N < G \implies H \cap N < H$ (as $H \cap N \subset H$ trivially).

It remains to show that $H \cap N \triangleleft H$. Note that since N is normal, for $h, h^{-1} \in H$ and $n \in H \cap N$ we have that $hnh^{-1} \in N$ and since $h, h^{-1}, n \in H$ we also have that $hnh^{-1} \in H \cap N$. Thus $H \cap N$ is a normal subgroup of H.

1.2.2

As above it is clear that $H \cap N < G$. Consider the group $G = Sym(3)$ with $N = Sym(3)$ and $H = \{1, (12)\}$. Then clearly $H \cap N = \{1, (12)\}$ but this group is not a normal subgroup (e.g.$(13) \in G$ but $(13)(12)(31) = (23) \notin H$). Thus in general it is not the case that $H \cap N \lhd G$.

1.2.3

Note that HN is a subgroup as for any $h_1 n_1, h_2 n_2 \in HN$ we have that $h_1 n_1 h_2 n_2 = h_1 n_1 n_3 h_2 = h_1 h_2 n_4$ for some $n_3, n_4 \in N$ as N is normal. Thus HN is closed, inverses exist as for some $h_1 n_1 \in HN$ we have that $(h_1 n_1)^{-1} = n_1^{-1} h^{-1} = h^{-1} n_2$ for some $n_2 \in N$ (as N is normal). These first two properties together imply that $1 \in HN$ and so $HN < G$.

It is further straightforward to show that this is not a normal subgroup by considering $Sym(3)$ with $N = \{1\}$ and $H = \{1, (12)\}$. Clearly $HN = \{1, (12)\}$ which is not normal as previously discussed in 1.2.2.

1.3

For $n = n_1 + n_2 + \cdots + n_r$ we let $N = Sym(n)$. Furthermore, let $H_1 = Sym(n_1)$ be the permutations of $1, 2, \ldots, n_1$, $H_2 = Sym(n_2)$ be the permutations of $n_1 + 1, n_1 + 2, \ldots, n_1 + n_2$, etc for all $H_1, \ldots, H_r$. Then H_i is a group and since $H \subset G$ we have that $H_i < G$.

Furthermore H_1, H_2 have no elements in common except the identity, so $H_1 H_2$ must have $|H_1| \times |H_2|$ elements. Furthermore it is fairly straightforward to see that if $H_1 H_2 \ldots H_k$ has $|H_1| \times |H_2| \times \cdots \times |H_k|$ elements then $H_1 H_2 \ldots H_k$ has no elements in common with H_{k+1} except the identity so $H_1 H_2 \ldots H_k H_{k+1}$ has $|H_1| \times |H_2| \times \ldots |H_k| \times |H_{k+1}|$ elements. By induction we can see that $H_1 H_2 \ldots H_n < G$ must have $\prod_{i=1}^{r}(n_i!)$ elements and that it must be a subgroup as each constituent group has no elements in common with any of the other groups except the identity. Therefore $\prod_{i=1}^{r}(n_i!)$ divides $n!$ by Lagrange's Theorem.

1.4

Note that the number of right and left cosets are the same (so H must have two right and left cosets), so if $gh \in gH$ with $g \in G$, $h \in H$ then either $gh \in H$ which it is clearly not (as cosets disjointly partition the group G) or $gh \in Hg$. Therefore $gH = Hg$ and thus H is normal.

1.5

We have that $(\pm 1)(\pm 1) = \pm 1$, $(\pm 1)(\pm i) = \pm i$ and $(\pm i)(\pm i) = \pm 1$ so μ_4 is closed. Furthermore it contains the identity and $(-i)(i) = 1 = -1(-1)$ so inverses exist. Thus μ_4 is a subgroup. Then since $\mathbb{C}^\times$ is Abelian it follows that $\mu_4 \lhd \mathbb{C}^\times$.

We can define the function $\phi : \mathbb{C}^\times / \mu_4 \to \mathbb{C}^\times$ with $z\mu_4 = z^4$. Note that since for $u \in \mu_4$, $u^4 = 1$, this mapping is not dependent on the choice of coset leader. This is a homomorphism as for $w_1, w_2 \in \mathbb{C}^\times$ then $\phi(w_1 \mu_4)\phi(w_2 \mu_4) = w_1^4 w_2^4 = (w_1 w_2)^4 = \phi(w_1 w_2 \mu_4)$. It is also injective as if $\phi(z) = \phi(w) \implies r_z^4 = r_w^4$ and that $\theta_z = \theta_w \pmod{\pi/2}$ for $z = r_z e^{i\theta_z}$, $w = r_w e^{i\theta_w}$ so $z, w \in z\mu_4$. Finally it is surjective as for $re^{i\theta} \in \mathbb{C}^\times$ it is clear that there exists $r^{1/4} e^{i\theta/4}$ which is an element of $r^{1/4} e^{i\theta/4} \mu_4$. Therefore it follows that ϕ is an isomorphism and so $\mathbb{C}^\times / \mu_4 \cong \mathbb{C}^\times$.

$\{\pm 1\}$ is a subgroup of $\mathbb{R}^\times$ as there exists a homomorphism $\phi : \mathbb{R}^\times \to \mathbb{R}_+^\times$ (where $\mathbb{R}_+^\times$ denotes the group of positive real numbers under multiplication) $\phi(x) = x^2$. This is a homomorphism as for $x, x' \in \mathbb{R}^\times$ we have $\phi(xx') = x^2 x'^2 = \phi(x)\phi(x')$. Then $ker(\phi) = \{\pm 1\}$ so this must be a normal subgroup by lemma 3.10.

It is useful to notice that $\mathbb{R}^{\times}/\pm 1$ is isomorphic to $\mathbb{R}_+^{\times}$. The map $\phi : x\{\pm 1\} \to |x|$ is well-defined as clearly ϕ maps a coset to the same element regardless of the choice of coset leader. Furthermore, it is a homomorphism as for $x, y \in \mathbb{R}^{\times}$ we have $\phi(xy\{\pm 1\}) = |xy| = |x||y| = \phi(x\{\pm 1\})\phi(y\{\pm 1\})$. It is injective as if $\phi(x\{\pm 1\}) = \phi(y\{\pm 1\}) \implies |x| = |y| \implies x = \pm y \implies x, y \in x\{\pm 1\}$. It is also surjective as for $x \in \mathbb{R}_+^{\times}$ we have $\phi(x\{\pm 1\}) = |x| = x$. Therefore these two groups are isomorphic.

Thus the problem can be reduced to the somewhat simpler question: is $\mathbb{R}_+^{\times}$ isomorphic to $\mathbb{R}^{\times}$? If an isomorphism φ from $\mathbb{R}^{\times}$ to $\mathbb{R}_+^{\times}$ does exist, this implies that $\varphi(1) = 1$ by the properties of isomorphisms. Therefore we have that $\varphi((-1)(-1)) = 1$ and so $\varphi(-1)\varphi(-1) = 1$. It follows that $\varphi(-1)^{-1} = \varphi(-1)$ and so this element of $\mathbb{R}_+^{\times}$ is its own inverse. But the only element in $\mathbb{R}_+^{\times}$ mapped to itself by $x \to x^2$ is $x = 1$. Therefore φ cannot be an isomorphism as it is not injective and so we obtain a contradiction. Thus it follows that $\mathbb{R}^{\times}/\{\pm 1\}$ is not isomorphic to $\mathbb{R}^{\times}$.

1.6

Let G be a group and $H, N < G$. Then for $h_1, h_2 \in H$ and $g \in G$ we have

$$gh_1g^{-1}gh_2g^{-1} = gh_1h_2g^{-1} \in gHg^{-1}$$

and since $h_1^{-1} \in H$ it follows that $gh_1^{-1}g^{-1} \in gHg^{-1}$ and therefore $gh_1g^{-1}gh_1^{-1}g^{-1} = gg^{-1} = 1$. Thus gHg^{-1} is a subgroup.

Define the following relation $H \sim N$ iff $H = gNg^{-1}$ for some $g \in G$. This is reflexive as $H = 1H1^{-1} = H$, symmetric as if $H = gNg^{-1}$ then $g^{-1}H = Ng^{-1}$ and thus $g^{-1}H(g^{-1})^{-1} = N$. It is also transitive as for $H_1, H_2, H_3 < G$ with $H_1 \sim H_2$ and $H_2 \sim H_3$ we have for some $g_1, g_2 \in G$

$$H_1 = g_1 H_2 g_1^{-1}, \qquad H_2 = g_2 H_3 g_2^{-1} \implies H_1 = g_1 g_2 H_3 g_2^{-1} g_1^{-1} = (g_1 g_2) H_3 (g_1 g_2)^{-1}$$

So $\sim$ is a valid equivalence relation.

Considering the group $Sym(3)$ there are the following subgroups $\{1\}$, $\{1, (13)\}$, $\{1, (23)\}$, $\{1, (12)\}$, $\{1, (123), (132)\}$ and $Sym(3)$. Clearly the trivial subgroup and the group itself will not be conjugate to any other subgroups, as $g1g^{-1} = 1$ for all $g \in Sym(3)$. For $\{1, (13)\}$ we have that $(23)(13)(23) = (12)$ and that $(13)(12)(13) = (23)$ so $\{1, (12)\} \sim \{1, (13)\} \sim \{1, (23)\}$ by the properties of equivalence relations. Therefore each belong to the same equivalence class.

Finally for $\{1, (123), (132)\}$ we have that for each of (12), (13), (23), the conjugacy operator returns the original subgroup $\{1, (123), (132)\}$.

$$(12)(123)(12) = (132)$$
$$(12)(132)(12) = (123)$$
$$(13)(123)(13) = (132)$$
$$(13)(132)(13) = (123)$$
$$(23)(123)(23) = (132)$$
$$(23)(132)(23) = (123)$$

Therefore this subgroup belongs in its own equivalence class. So the equivalence classes of $Sym(3)$ are the trivial group, the groups of order two, the group of order three and the group itself.

1.7

The group $GL_n(F_q)$ is the set of all invertible $n \times n$ matrices with coefficients drawn from F_q under matrix multiplication, which is equivalent to the set of all $n \times n$ matrices with coefficients drawn from F_q that have full rank. This is equivalent to the statement that all columns are linearly independent in F_q, so we can construct a matrix in this group by selecting n columns that are each linearly independent relevant criteria.

Proceeding column by column, we can see that in the first column of some matrix in this group, each element can chosen freely except that the column cannot be entirely zeros. Consequently, there are $q^n - 1$ valid choices for the coefficients in the second column. In the second column any combination of coefficients can be chosen, except those which are a scalar multiple of the first column. Therefore, there are $g^n - q$ valid choices for the second column. In fact, in the jth column, the requirement that this column be linearly independent of all previous columns implies that there are q^n choices, less all linear combinations of the previous columns (of which there are clearly q^{j-1}). This enables us to construct the following formula for $|GL_n(F_q)|$.

$$|GL_n(F_q)| = \prod_{i=1}^{n}(q^n - q^{i-1})$$

1.8

$GL_n(F_{11})$ contains $SL_n(F_{11})$ where $F_{11} \cong \mathbb{Z}/11\mathbb{Z}$. Note that $SL_n(F_{11})$ is a normal subgroup as for $A \in GL_n F_{11}$ and $B \in SL_n(F_{11})$ we have $det(ABA^{-1}) = det(A)det(B)det(A^{-1}) = det(A)det(A)^{-1} = 1$ by properties of the determinant, so $ABA^{-1} \in SL_n(F_{11})$.

Then by the correspondence theorem $\exists \Phi$ a bijection with

$$\Phi : \{H | SL_n(F_{11}) < H < GL_n(F_{11})\} \rightarrow \left\{ \tilde{H} : \tilde{H} < \frac{GL_n(F_{11})}{SL_n(F_{11})} \right\}$$

This bijection preserves normality, so that if $\tilde{H}$ is normal then it also follows that H is also normal. It is given in the notes that $GL_n(F_{11})/SL_n(F_{11})$ is isomorphic to F_{11}. Then note that there exists an isomorphism from F_{11} to $\mathbb{Z}/11\mathbb{Z}$ and that isomorphisms preserve normality (if $ghg^{-1} \in H$ for $h \in H$ then $\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g)^{-1} \in \phi(H)$) so all the normal subgroups of $GL_n(F_{11})$ which contain $SL_n(F_{11})$ correspond to a subgroup of $(\mathbb{Z}/11\mathbb{Z})^{\times}$.

Note that this group is equivalent to a cyclic group of order 10 (e.g. choose 6 as the generator) and so all subgroups must have order 1, 2, 5, or 10. Clearly there is only group of order 1 and one group of order 10 as these groups are the trivial subgroup and the entire group respectively. The groups generated by iterating one of the other elements in this group can be discerned from the multiplication table below

$$2^2 = 4, \quad 2^3 = 8, \quad 2^4 \cong 5, \quad 2^5 = 10, \quad 2^6 \cong 9, \quad 2^7 \cong 7, \quad 2^8 \cong 3, \quad 2^9 \cong 6, \quad 2^{10} \cong 1$$
$$3^2 = 9, \quad 3^3 \cong 5, \quad 3^4 \cong 4, \quad 3^5 \cong 1$$
$$4^2 \cong 5, \quad 4^3 \cong 9, \quad 4^4 \cong 3, \quad 4^5 \cong 1$$
$$5^2 \cong 3, \quad 5^3 \cong 4, \quad 5^4 \cong 9, \quad 5^5 \cong 1$$
$$6^2 \cong 3, \quad 6^3 \cong 7, \quad 6^4 \cong 9, \quad 6^5 \cong 10, \quad 6^6 \cong 5, \quad 6^7 \cong 8, \quad 6^8 \cong 4, \quad 6^9 \cong 2, \quad 6^{10} \cong 1$$
$$7^2 \cong 5, \quad 7^3 \cong 2, \quad 7^4 \cong 3, \quad 7^5 \cong 10, \quad 7^6 \cong 4, \quad 7^7 \cong 6, \quad 7^8 \cong 9, \quad 7^9 \cong 8, \quad 6^{10} \cong 1$$
$$8^2 \cong 9, \quad 8^3 \cong 6, \quad 8^4 \cong 4, \quad 8^5 \cong 10, \quad 8^6 \cong 3, \quad 8^7 \cong 2, \quad 8^8 \cong 5, \quad 8^9 \cong 7, \quad 8^{10} \cong 1$$
$$9^2 \cong 4, \quad 9^3 \cong 3, \quad 9^4 \cong 5, \quad 9^5 \cong 1$$
$$10^2 \cong 1$$

So clearly there is only one subgroup of order 2: $\{1, 10\}$ and one subgroup of order 5: $\{1, 3, 4, 5, 9\}$. Thus there are four subgroups (which are all normal as this group is Abelian). Therefore $F_{11}^{\times}$ has four normal subgroups and consequently there are four subgroups of $GL_n(F_{11})$ which contain $SL_n(F_{11})$ and all of them are normal.

1.9

We wish to find all homomorphisms from $S_3 \to S_4$. If $\phi : S_3 \to S_4$ is a homomorphism then $ker(\phi) \lhd S_3$ so all homomorphisms must have as a kernel a normal subgroup of S_3. S_3 has six subgroups (two of order two and one of order three, the trivial group and the group itself). Each group of order two is not normal as

$$(12)\{1, (23)\}(12) = \{1, (13)\}$$

Note this holds for any subgroup of order two as our choice of elements is arbitrary (e.g. it is clear that one can replace (12) and (13) with any other pair of distinct order two elements of S_3 and obtain an identical result, see 1.6 for details). Then the subgroup of 3 elements is normal as for $(12)(123)(12) = (132)$ (see 1.6 above). So it follows that each homomorphism is easily decomposable (by the homomorphism theorem) to composition of functions, one of which is an homomorphism on $S_3 : g \to g/ker(\phi)$ and one of which is an isomorphism between S_3 and $im(\phi)$. Therefore we can reduce our problem to finding all unique isomorphisms between S_3/H and $G < S_4$ where H is a normal subgroup of S_3. The groups S_3/H are as follows; $S_3/1 \cong S_3$, $S_3/S_3 \cong 1$ and $S_3/\{1, (123), (132)\} \cong \{1, a\}$ (where $a^2 = 1$).

If $\phi : S_3 \to 1$ this is clearly a unique homomorphism. If $\phi : S_3 \to \{1, a\}$ then this homomorphism must map to a group of order 2 in S_4. This group will therefore contain an element of order 2 and the identity. There are six two-cycles in S_4, $(12), (13), (14), (23), (24), (34)$ and there are three combinations of two-cycles which have no elements in common (any combination of two-cycles where they share a vertex in common is equivalent to an element of order 3). So there are nine possible possible homomorphisms which map S_3 to an element a group of order two in S_4.

Finally a homomorphism ϕ may map S_3 to $im(\phi) \cong S_3 < S_4$. Therefore we need a subgroup with three elements of order two and two elements of order three with the relation that if a and b have order two then ab has order three. Clearly as well as S_3 there is also the following three groups $\{1, (14), (24), (12), (124), (142)\}$, $\{1, (24), (34), (23), (234), (243)\}$, $\{1, (14), (13), (14), (134), (143)\}$.

Note that it is impossible to construct a group of isomorphic to S_3 that contains an element of order two made up of two disjoint two-cycles. This is clear as the product of three two cycles in S_4 (two of which are mutually disjoint) produces an element of order four (e.g. $(12)(34)(13) = (1432)$ and $(13)(34)(12) = (1234)$) and the product of two pairs of disjoint two cycles produces an element of order two (e.g. $(12)(34)(13)(24) = (14)(23)$). By renaming vertices in this permutation (e.g. by replacing 1 with 2 and visa-versa) it is straightforward to show this holds true for any possible way to combine elements of S_4 with these properties. Consequently any subset of S_4 which contains an element comprised of two disjoint two-cycles cannot be isomorphic to S_3.

Therefore if $im(\phi) \cong S_3$ it must be equal to one of these these four groups. Since ϕ is a homomorphism it must map two-cycles in S_3 to elements of order two in $im(\phi)$. For each of the possible sets comprising $im(\phi)$ there are clearly six different ways to map the two cycles to each other (as there are 3! ways to permute three elements). Then since each element of order three is a product of two two-cycles the mapping is fully determined by the choice of which two-cycles in S_3 are mapped to which two-cycles in $im(\phi)$, so there are 24 different homomorphisms which map S_3 to $im(\phi) \cong S_4$.

Therefore there are $9 + 1 + 24 = 34$ possible homomorphisms from S_3 to S_4.

1.10

It is convenient to consider Abelian and non-Abelian groups separately.

Abelian Groups   By Corollary 8.7 of the notes, we can see that if G is an Abelian group of order 8, it must be isomorphic to a direct product of groups $H_1, H_2, \ldots, H_k$ with $H_i \cong \mathbb{Z}/n_i\mathbb{Z}$. Since the order of G is clearly the product of the orders of $H_1, H_2, \ldots, H_n$ it follows that $\prod_{i=1}^{k} n_i = 8$. Therefore, all possible Abelian groups must be isomorphic to one of

$$M_1 \cong \frac{\mathbb{Z}}{8\mathbb{Z}}$$
$$M_2 \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{4\mathbb{Z}}$$
$$M_3 \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$$

It remains to show that none of these groups are not isomorphic to each other. M_1 contains elements of order 8, but for $(a_1, a_2) \in M_2$ we have $4(a_1, a_2) = (4a_1, 4a_2) = 1$ as each element in $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$ has at most order four. Therefore M_2 contains no elements of order 4 and consequently it cannot be isomorphic to M_1. By an identical argument we can see that for $(a_1, a_2, a_3) \in M_3$ we have $2(a_1, a_2, a_3) = (2a_1, 2a_2, 2a_3) = 1$. Thus M_1 cannot be isomorphic to M_3. Furthermore, since there are elements in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ which have order four (e.g. $(1, 1)$ has order four) it follows that M_2 and M_3 are not isomorphic. Thus we can conclude that each of these groups are not isomorphic to each other and so there are three Abelian groups of order 8.

Non-Abelian Groups   By Langrange's theorem, we know that even non-Abelian groups of order 8 can only contain elements of order 2, 4 and 8. In fact, if a group of order 8 contains an element of order 8 this clearly implies that it is isomorphic to the cyclic group of order 8 (which is Abelian). So we can limit ourselves to considering non-Abelian groups containing only elements of 4 and 2. First consider the case where for all a, b elements of this group of order two we have that ab is also of order two. Then $abab = e$ so $(ab)^{-1} = ab$ but by definition $(ab)^{-1} = b^{-1}a^{-1} = ba$ so $ab = ba$. Therefore it follows that the group is in fact Abelian and that any non-Abelian group must in fact include an element of order four.

Let G be a non-Abelian group of order 8 with a an element of order four. Suppose that then there exists some element b of order two, which is not in $< a >$. This implies that the eight elements of this group can be written as $\{1, a, a^2, a^3, ab, ab^2, ab^3, b\}$. Since the group has eight elements in it, it follows that ba, ba^2, ba^3 must each be mapped to some element inside this set. If $ba^n = a^m$ for some $n, m \in \{1, 2, 3\}$ then $b = a^{n-m}$ by cancellation, which is a contradiction of our initial hypothesis that b is not in $< a >$. Therefore $ba^n = a^m b$ for each $n, m \in \{1, 2, 3\}$.

If $ba = ab$ then $a^2 b = aba = ba^2$ and $a^3 b = ba^3$ so the whole group is Abelian. If $ba = a^2 b$ then $a = ba^2 b$ and so $a^2 = ba^2 b b a^2 b = 1$ which is a contradiction. Therefore $ba = a^3 b$. If $ba^2 = ab$ then $a^2 = bab$ so $a^4 = ba^2 b = 1$ so $a^2 = 1$ which is a contradiction. This implies that $ba^2 = ab^2$ (as trivially $ba^2 = a^3 b$ implies that $ba = ba^2$ which is an obvious contradiction). Finally this implies that $ba^3 = ab$ and so this group can be expressed as $< a, b \mid a^4 = b^2 = 1, ab = ba^{-1} >$ (note this is isomorphic to the Dihedral group of order 4).

Now we will consider the case where all elements in G that are not in the subgroup $< a >$ have order four. Then for $b \neq a^n$ for some $n \in [0, 4]$ it follows that b has order four and that b^2 has order two. This implies that $b^2 = a^2$ by our assumption. This suggests that the group contains the following elements 1, a, a^2, a^3, b, ab, $a^2 b$, $a^3 b$. Since this group has order 8 it must also be true that ba, ba^2 and ba^3 are equivalent to elements inside this group. Clearly $a^2 b = b^3 = ba^2$. Then if $ba = ab$ and $ba^3 = a^3 b$ then this group is Abelian, but by assumption it was not. Therefore $ba = a^3 b$ and $ab = ba^3$. This forms a closed group, which can be expressed as $< a, b \mid a^4 = b^4 = 1, a^2 = b^2, ab = ba^3 >$ (note that $ab = ba^3 \implies a^3 b = ba$ as $a^3 b = a^2 ba^3 = b^3 a^3 = ba^5 = ba$ as required). This is isomorphic to the Quarternion group.

There are therefore three Abelian group and two Non-Abelian groups of order 8, for a total of five groups. Groupprops suggests there are 267 groups of order 64 [1].

References

[1] Groupprops. *Groups of Order 64*. URL: https://groupprops.subwiki.org/wiki/Groups_of_order_64.