Course information

School of Mathematics and Statistics

UC UNIVERSITY OF CANTERBURY
Te Whare Wānanga o Waitaha
CHRISTCHURCH NEW ZEALAND

MATH401-22S2

Dynamical systems

You may have heard of Lorenz's famous question: "Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?". The question was meant to portray one of the main concepts in chaos theory (sensitive dependence to initial conditions). This course will teach theory, techniques and applications of systems of nonlinear equations. In all cases, we are concerned with *dynamics*: the time evolution of spatial structure. We will cover the mathematics behind chaos theory, and learn techniques for analysing nonlinear systems. Since it is usually difficult or impossible to write down an exact solution to systems of nonlinear equations, the emphasis will be on qualitative techniques for classifying and understanding the behaviour of nonlinear systems. Both main types of dynamical system will be studied: discrete systems, consisting of an iterated map; and continuous systems, arising from nonlinear differential equations. The natural connections between discrete and continuous time systems will be emphasised.

This course is independent of Math363 *Dynamical systems*, although previous enrolment there is desirable.

Lecturer and course coordinator: Rua Murray (Erskine 604, email: rua.murray@canterbury.ac.nz, extn 93289)

Class times: Please check the course information system http://www.canterbury.ac.nz/courses/.

On campus and off campus delivery: We will have as many face to face classes as possible; lectures will be available via zoom and formats for tutorials and presentations will be adapted depending on circumstances. Please keep in touch if you need to be off-campus for part of the Semester.

Homework and assessment: Problem sheets will be given out every one or two weeks.

- Tutorials will be worth 10%, and will require active participation. There will be several small group activities, with presentations in weeks 3, 6, 9. Presentations will be based on a question set the previous week, and a member of each group will give a 2-3 minute presentation. Active participation in each tutorial earns 1 point, presenting earns 2 points, to a maximum of 10 points.
- There will be four assignments worth 10% each. Assignments 1, 2 and 4 will be drawn from the problem sheets, and must be done individually. These will be due on 4pm Friday 18/3, 29/4, 3/6. The third assignment is a reading project, based on reading a paper (chosen from a list provided, or negotiated alternative), and will take the form of a 1000-1500 word essay and oral presentation. The written part will be due on 27/5, and presentations will be on 2/6. This third assignment can be done in a pair.
- A three hour final exam worth 50% will be scheduled in the usual end of Semester exam period.

Learning outcomes

- To demonstrate a breadth of knowledge of dynamical systems theory traversing smooth, continuous and probabilistic settings and to apply that knowledge correctly in new situations
- To be able to articulate what it means for a dynamical system to be chaotic, including the relation to randomness

- To choose and apply appropriate theoretical and numerical tools to analyse a given dynamical system and communicate clear and correct explanations of its global asymptotic behaviour
- To exhibit mastery of both the power and limitations of standard methods of linearisation, analysis via invariant manifolds and symbolic dynamics
- To evaluate critically the findings and discussions in relevant original literature, and to exhibit familiarity with content that is relevant to the syllabus, but sits outside it
- To engage in rigorous investigation and analysis of problems in dynamical systems both independently and collaboratively

Syllabus

1. Introduction: maps and flows as dynamical systems (including return maps and interval maps).
2. Smooth local dynamics: hyperbolicity, linearisation and stability of fixed points and period orbits.
3. Smooth global dynamics: invariant manifolds; Centre manifolds and local bifurcations; global bifurcations.
4. Topological dynamics (concepts and examples).
5. Introduction to chaos.
6. Symbolic dynamics (including period 3 implies chaos).
7. Chaos in two dimensional maps.
8. Probabilistic dynamics and mass transport in two dimensions.

Recommended reading

The recommended books for this course are the texts by Alligood, Sauer and Yorke [1] and Meiss [7]. The Math363 text by Strogatz [9] is also an excellent supplement, and source of background material.

Lorenz equations: the development of the modern theory of dynamical systems has its roots in Lorenz's famous work on simplified models of atmospheric circulation. The equations which bear his name appear in the original paper [6], and subsequent decades have seen careful and beautiful unravelling of the delicate mechanisms and geometry causing the chaotic phenomena observed therein. The underlying structures were correctly identified in the 1970s by Williams and Guckenheimer (amongst others), and these ideas — interlaced with dynamical systems theory — are presented in the classic text of Guckenheimer & Holmes [4] (particularly chapters 2, 5 and 6). A comprehensive numerical and geometric study was done by Colin Sparrow in his PhD thesis, and published as [8]. More recently, computer power has advanced to such an extent that the well-founded geometric conjectures of earlier decades have been *numerically proven*: the proof of chaos in the "Lorenz attractor" was completed by Warwick Tucker (Erskine visitor to UC in 2011) in his PhD thesis, and published in [10]. The 2006 paper of Doedel, Krauskopf and Osinga [3] also outlines the source of the chaos in the Lorenz system, and describes (with amazing pictures) numerical computations of the "Lorenz manifold" which organises the phase space (both Krauskopf and Osinga are professors at the University of Auckland).

Also recommended: Alan Turing's article [11], which essentially invents the theory of pattern formation.

More advanced reading: Guckenheimer and Holmes 1983 book [4] is a standard reference for many of the ideas in this course. More recent texts, complete with proofs, are by Brin & Stuck [2] Hasselblatt & Katok [5].

References

[1] K T Alligood, T D Sauer, and J A Yorke. *Chaos: an introduction to dynamical systems*. Springer, 1997.

[2] M Brin and G Stuck. *Introduction to dynamical systems*. Cambridge University Press, 2002.

[3] Eusebius J. Doedel, Bernd Krauskopf, and Hinke M. Osinga. Global bifurcations of the Lorenz manifold. *Nonlinearity*, 19(12):2947–2972, 2006.

[4] J Guckenheimer and P Holmes. *Nonlinear oscillations, dynamical systems and bifurcations of vector fields*, volume 42 of *Appl. Math. Sc.* Springer, 1983.

[5] B Hasselblatt and A Katok. *A first course in dynamics (with a panorama of recent developments)*. Cambridge University Press, 2003.

[6] Edward N Lorenz. Deterministic nonperiodic flow. *J. Atmos. Sci*, 20:130–141, 1963.

[7] James D. Meiss. *Differential dynamical systems*, volume 14 of *Mathematical Modeling and Computation*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007.

[8] Colin Sparrow. *The Lorenz equations: bifurcations, chaos, and strange attractors*, volume 41 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1982.

[9] S.H. Strogatz. *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering*. Studies in nonlinearity. Westview Press, 1994.

[10] Warwick Tucker. A rigorous ODE solver and Smale's 14th problem. *Found. Comput. Math.*, 2(1):53–117, 2002.

[11] Alan M. Turing. The chemical basis of morphogenesis. *Phil. Trans. R. Soc. Lond. B*, 237(641):37–72, 1952.