MATH401 Summary

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Flows and Interval Maps

In this course we will study two main types of dynamical systems: flows and iterated maps. Throughout we will let X be a compact metric space and I_d the identity automorphism on this space.

Definition 1 (Flows) *A flow is a parameterized family of functions defined for all X and for all $t \in$ mathbbR then $\{\psi\}_{t\in\mathbb{R}}$ is a flow if and only if*

- $\psi_0 = I_d$
- $(\psi_t)^{-1} = \psi_{-t}$
- $\psi_{s+t} = \psi_s \circ \psi_t$

This flow also defines a unique differential equation $\dot{x} = F(x) \coloneqq \frac{d}{dt}\psi_t(x)|_{t=0}$.

Definition 2 (Iterated Maps) *A dynamical system induced by a iterated map is a parameterized family of functions defined for all X and for all $n \in$ mathbbN then $\{f^n\}_{n\in\mathbb{N}}$ is a family where*

$$
\begin{aligned}
f^0 &= I_d \\
f^1 &= f \\
f^2 &= f \circ f \\
&\dots \\
f^n(x) &= f \circ f \circ \dots \circ f(x)
\end{aligned}
$$

If f is invertible then the family can be extended to be defined over $\mathbb{Z}$ in an obvious manner.

One can then define the orbit of a point x_0 under flows or iterated maps by considering the parameterized family of functions as a group or semi-group acting on X. If the orbit contains only one point, then x_0 is a fixed point. If the orbit is finite and of size q then x_0 is a q-periodic point.

Definition 3 (Attracting Fixed Points of Maps) *A fixed point p^* of an iterated map f is attracting if there exists some open neighborhood N such that*

$$x \in N qquad \lim_{n\to\infty} f^n(x) = p^*$$

N is an attracting neighborhood and the largest connected N is called the immediate basin of attraction.

There is an obvious analogous definition for the fixed points of flows. We can now make further remarks about necessary conditions for fixed points to be attracting.

Theorem 1 *If f is differentiable on X and has a fixed point at p^* then this fixed point is attracting if $|f'(p^*)| < 1$.*

A periodic orbit of size q of a map is attracting if each $x_i \in \{x_1, x_2, \ldots, x_q\}$ are attracting fixed points of f^q. Fortunately it is sufficient to instead verify the following simpler condition

Definition 4 (Attracting Periodic Orbits of Maps) *If $\Gamma = \{x_n\}_{n=1}^q$ is a periodic orbit of size q and f is differentiable at each point in Γ then Γ is attracting if*

$$\Pi_{x \in \Gamma} |f(x)'| < 1$$

It is often desirable to know when periodic points of such maps are guaranteed to exist. Fortunately this is straightforward to demonstrate

Theorem 2 *Let I be a subinterval of X and $q > 0$. Then*

1. *If $I \cap f^q(I) \neq \emptyset$ then I contains a point fixed by f^q.*
2. *If $I_0, I_1, \ldots, I_k$ is a sequence of intervals such that each $I_{j+1} \subseteq I_j$ for all $0 \leq j < q$ and $I_0 \subseteq f(I_{q-1})$ then $\bar{I_0}$ contains a point fixed by f^q.*

Hyperbolicity

It is usually more convenient to work with the linearization of maps and/or flows rather than the full non-linear dynamics. Fortunately, this simplification is permissible near hyperbolic fixed points.

Definition 5 (Hyperbolic Fixed Points) *A fixed point of a map is hyperbolic if the Jacobian derivative has no eigenvalues of modulus 1. A fixed point of a flow is hyperbolic if the Jacobian derivative of the vector field has eigenvalues with real part zero.*

Theorem 3 (Hartman-Grobman for Maps) *If x^* is a hyperbolic fixed point of f then there exists a neighborhood N of x^* and a unique 'near identity' homeomorphism $h : N \to \mathbb{R}^d$ such that $h(x^*) = 0$ and $h(f(x)) = DT(x^*)h(x)$ for all $x \in N$.*

An analogous result holds for flows. However, if we want to analyze the fixed points of flows, we will need more sophisticated machinery. In particular, it is useful to introduce the concept of variational equations.

Lemma 1 (Variational Equations) *Given a Linearization of a flow $\{\Psi_\tau(x_0)\}$ satisfy the coupled system of $d + d^2$ equations.*

$$\dot{x} = F(x) \quad x(0) = x_0$$
$$\dot{\Psi} = DF(x)\Psi \quad \Psi_0 = I_d$$

This matrix Ψ is known as the Floquet matrix. In practice, we usually wish to use the Poincare map to find out information about the stability of periodic orbits. To begin, let us choose a Poincare section

Definition 6 (Poincare Sections) *Let Σ be a $d-1$ dimensional surface (it is usually convenient to let this be the zero set of some function S). This surface is a valid Poincare section if it is transverse to the flow (that is $\nabla S \cdot F \neq 0$ for all x). Then the Poincare map is the first return map on this section.*

From this we can obtain information about the Floquet matrix and visa-versa.

Theorem 4 (Eigenvalue Equivalence Principle) *If Γ is a period τ orbit and x_0 is a point on valid Poincare section intersecting this orbit then the set of eigenvalues of the linearized Poincare Map is equal to the eigenvalues of the Floquet matrix less one eigenvalue equal to one.*

Invariant Manifolds

Theorem 5 (Invariant Manifold Theorem) *Let f be a C^r diffeomorphism and let x^* be a hyperbolic fixed point. Then there exists an open neighborhood N of x^* such that the local stable and unstable manifolds exist, have the same dimension as equivalent subspace and are tangent to that subspace.*

Definition 7 (Global Stable Manifold) *The global stable manifold is given by*

$$W^s(x^*) = \{x : f^n(x) \to x^* \quad as \quad n \to \infty\} = \cup_{n=0} f^{-n} W^s_{loc}(x^*)$$

if $W^s(x^*) \cup W^u(y^*)$ for two fixed points this implies a 'heteroclinic connection' between these fixed points. If $y^* = x^*$ then this is a homoclinic orbit. For non-hyperbolic fixed points there is a similar result to the existence for theorems for manifolds we have used above.

Theorem 6 (Center Manifold Theorem) *Let x^* be a non-hyperbolic fixed point of C^r dynamical system. To the splitting $E^s \oplus E^u \oplus E^c$ there correspond locally invariant manifolds W^s, W^u and W^c which are tangent to their respective subspaces. Furthermore W^s and W^u are unique, but W^c may not be unique.*

Topological Conjugacy

Definition 8 (Topological Conjugacy) *We say (Y, g) and (X, f) are topologically conjugate if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. Since h is invertible we can also write this as $h \circ f \circ h^{-1} = g$.*

It is also useful to define the notion of an ω limit set.

Definition 9 (ω limit set) *The ω limit set of a point x^* is*

$$\omega(x^*) = \lim_{n\to\infty} \cap_{k=n}^{\infty} \mathcal{O}^+(f^{\bar{k}}(x_0))$$

This allows us to define a notion of an attracting invariant set

Definition 10 (Attracting Invariant Sets) *An invariant set Ω is attracting if there is an open set U such that $\Omega \subset U$ and for $x \in U \implies \omega(x) \subset \Omega$. The largest set such that this prior property holds is called the basin of attraction.*

Chaos

Definition 11 (Chaos in the sense of Devaney) *A dynamical system (X, f) is called if it has an invariant set Ω such that*

1. *Periodic points are dense in Ω.*
2. *There is at least one x such that $\mathcal{O}^+(x)$ is dense in Ω.*
3. *The system restricted to Ω, f_Ω has sensitive dependence to initial conditions. There is a δ_0 such that for any $x \in \Omega$ and $\epsilon > 0$ there exists an $x' \in \Omega$ and $n \in \mathbb{N}$ such that $d(x, x') < \epsilon$ but $d(f^n(x), f^n(x')) > \delta_0$.*

There is some redundancy in the chaos conditions. In particular we have that

Theorem 7 *Let X be complete, f continuous and Ω a non-empty transitive invariant set. Then if periodic points are dense in Ω then either Ω is a single periodic orbit or f is chaotic on Ω.*

There is a useful transitivity lemma

Lemma 2 (Transitivity Lemma) *Let $\emptyset \neq \Omega \subseteq X$ be invariant, satisfy chaos conditions 1 and 2 and not be a single periodic orbit. Then*

1. *Ω has no isolated points.*
2. *If U, V with $U \cap \Omega \neq \emptyset$ and $V \cap \Omega \neq \emptyset$ there is an $n \geq 0$ such that $\Omega \cap T^n(U) \cap V \neq \emptyset$.*

Finally we require the notion of a semi-conjugacy

Definition 12 *A function $\psi : X \to Y$ is a semi-conjugacy if $\psi \circ f = g \circ \psi$ but ψ is not invertible.*

We can now construct a condition for chaos in one space implying chaos in the other

Theorem 8 *Let Ω be an chaotic invariant set for (f, X) and let ψ be a semi-conjugacy between X and Y. Then either $\psi(\Omega)$ is a single periodic orbit of g in Y or a chaotic invariant set of g in Y.*