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Recombination, Decoupling and a few notes on the CMB

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Abstract: In this work I'll seek to give an overview on the physics behind the epoch of recombination, which made it possible for our universe to become one of atoms, and later molecules and other more complex structures, instead of a simple plasma composed by free electrons and protons. I'll proceed to try to explain the characteristics of the universe that made it possible for decoupling of electrons from photons to happen, and introduce the CMB radiation as a consequence of this process, given its important place in modern cosmology and astrophysics. For this study, I'll use the electron fraction abundance as our guiding parameter, and will calculate the relic abundance of free electrons after freeze-out, today.

Keywords: Recombination; Decoupling; Cosmic Microwave Background Radiation; Boltzmann Equation; Saha Equation; Electron Fraction Abundance; Electron Freeze-Out

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1. Introduction

First, before treating recombination and the decoupling between electrons and photons, we have to introduce the condition for decoupling between the primordial plasma (or thermal bath) and its constituent particles.
One must first understand the characteristics of the universe at the time of those events and how its participants (electrons, photons...etc) behave after an equilibrium between them and the plasma.

1.1. Decoupling Condition

Decoupling from the thermal bath happens when a particle species is no longer in thermal equilibrium with it.
The condition for thermal equilibrium is as follows:

$$\Gamma(t) \gg H(t) = \frac{\dot{a}}{a}$$

where $\Gamma(t)$ is its interaction rate, $H(t)$ the Hubble parameter and $a(t)$ the scale factor.
The interaction rate is the number of interactions per unit of time, and is given by the expression: $\Gamma = n\sigma v$, where n is the number density for the species, σ its interaction cross-section and v the relative speed between particles.
We have, then, that the condition for decoupling can be written as $\Gamma(t) \approx H(t)$.
For a relativistic fluid, as we take the system to be before decoupling, we can approximate the expansion rate of the universe as only depending on radiation energy density:

$$H = \sqrt{\frac{8\pi G}{3}\rho_r} = \sqrt{\frac{\hbar c\pi^2}{90M_{pl}^2} g_* T^4} = \pi \left(\frac{g_*}{90}\right)^{\frac{1}{2}} \frac{T^2}{M_{pl}}$$

where $M_{pl} = \sqrt{\frac{\hbar c}{8\pi G}} = 2.4 \times 10^{18} GeV$ is the Planck mass, and the final expression is written in natural units.

Also, in natural units, we have that the relativistic fluid has a relative velocity of $v \approx c = 1$, and it can be shown that the number density in equilibrium for a given particle species, i, is $n_i \approx \frac{\zeta(3) g_i T^3}{\pi^2} \propto T^3$.
Therefore, our condition for decoupling is now

$$\frac{\Gamma}{H} \approx \frac{\zeta(3) g_i T \sigma (90)^{\frac{1}{2}} M_{pl}}{\pi^3 (g_*)^{\frac{1}{2}}} \approx 1$$

Let us then develop mathematically this condition for decoupling and we'll focus in particles that, like the electron, interact via the electroweak force.
If the decoupling happened before the electroweak symmetry was broken ($T > m_{W^{\pm},Z,H} \approx 100GeV$), the electroweak interaction's cross-section is such that $\sigma = \frac{\alpha^2}{T^2}$, where α is the fine-structure constant for the electromagnetic interaction, with $\alpha = \frac{1}{137} \approx 0.01$.
On the other hand, for a much cooler universe, $\sigma = \frac{\alpha^2 T^2}{m_X^4}$, with X the massive gauge boson that mediates the interaction between the particles and the thermal bath. In this case we can use the $W^{\pm}$ boson.
Recombination and electron decoupling, which we'll treat shortly after, must happen after the point at which the universe was cool enough for the electrons to be totally bound to the hydrogen nuclei and their interactions with photons via scattering to have $\Gamma \approx H$, and so, we will treat a cooler plasma ([4], Ch. 4, pp. 2-4).

As we know, the lighter mass of the electron ($m_e \approx 0.5MeV$) means that these particles remain relativistic for temperatures below the limit at which electroweak symmetry breaks. In the same way, neutrinos, which are also coupled to the plasma through the electroweak interaction, remain relativistic up until today, as their mass is minuscule.
As the temperature of the plasma gets lower and lower, most of its constituent particle species decouple from it, until temperatures of $T \approx 10MeV$, in which the only particles that remain in the relativistic fluid are electrons, neutrinos and photons ([4], Ch. 4, p. 9).
The effective number of degrees of freedom for a bath at temperatures so low is given by a weighted average of the internal degrees of freedom of the photon (a boson) and the electron and neutrino (fermions), respectively:

$$g_* = 2 + \frac{7}{8} \cdot (2 \times 2 + 3 \times 2) = 10.75$$

The internal degrees of freedom for the electron-neutrino, for example, are $g_\nu = 2$ and $\zeta(3) \approx 1.2$. We then get:

$$\frac{\Gamma}{H} = 0.22 \frac{\alpha^2 T^3 M_{pl}}{m_W^4} \approx \frac{\alpha^2 T^3 M_{pl}}{m_W^4} \approx \left(\frac{T}{1MeV} \right)^3$$

This means that relativistic particles - neutrinos - that interact with the thermal bath via the weak force decouple after or at around $T = 1MeV$. More thorough calculations will yield that the neutrino actually decouples from the relativistic fluid at $T \approx 0.8MeV$.

1.2. Boltzmann Equation

Decoupling, then, necessitates a non-equilibrium state, for the decoupled particle species, and the evolution of these particles beyond equilibrium is well modelled by the Boltzmann equation, which seeks to calculate the evolution of a particle species', i, number density.
As this project is not about the derivation of the Boltzmann equation, we'll just present the minimum necessary for understanding the work ahead in explaining just how the density of free electrons decreased since the epoch of recombination up until today.

For a two-particle process (scattering or annihilation), which is much more likely than those between three or more particles, we may generally describe it as: $1 + 2 \rightleftharpoons 3 + 4$, where the numbers stand for the respective particle species.
The Boltzmann equation for, for example, particle species 1, is:

$$\frac{1}{a^3} \frac{d(n_1 \cdot a^3)}{dt} = -\alpha n_1 n_2 + \beta n_3 n_4$$

This expression signifies that the evolution of the number density of particle 1 depends on the rate of its destruction (paired with the destruction of particle 2) and the rate of the production of particles 3 and 4.
$\alpha = <\sigma v>$ is the thermally averaged (averaged over the relative velocity of 1 and 2) cross-section of the interaction between 1 and 2.
To determine β we need only to note that, for equilibrium, there is no variation in particle densities, so, setting the

derivative to 0, we must have $\beta = \alpha \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq}$.

The Boltzmann equation for a two-particle process can, then, be written as ([3], pp.60-61):

$$\frac{1}{a^3}\frac{d(n_1 \cdot a^3)}{dt} = -<\sigma v> \left[n_1 n_2 + n_3 n_4 \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq} \right]$$

If a particle species remained in equilibrium with the fluid as it cools, its number density would be suppressed by an exponential term, as, for lower temperatures ($T \approx m_i$), $n_i \propto e^{-m_i/T}$.
Instead, what can be observed is a relic abundance of the number density of these particles. This happens because, at $T \approx m_i$, the interaction between particles freezes-out and $\Gamma < H$.
It is useful to note that, because of how we defined our problem, $\alpha n_1 n_2 = \Gamma_1 n_1$, which means that $\Gamma_1 = n_2 <\sigma v>$, where Γ_1 is the interaction rate for particle 1.

Resolving the Boltzmann equation for any given case will enable us to calculate that relic abundance ([2], p. 119).

2. Recombination and the Saha Equation

Although neutrinos decouple from the relativistic fluid at $T \approx 1 MeV$, electrons are still coupled with photons due to Compton scattering, so we'll need to study what happens at lower temperatures.
At $T \approx 0.1 MeV$ nucleosynthesis has already begun and the electrons interact with protons and atomic nuclei via Coulomb scattering. This means that protons and nuclei are also in equilibrium with photons, which are, themselves, in equilibrium with the plasma ([4], Ch. 7, p.2).
The ionization energy for the Hydrogen atom is, as given by Rydberg's formula, 13.6 eV, and so the plasma is yet too hot for neutral matter to form in large quantities.
One of the electromagnetic interactions that keep all these particles in equilibrium is the production of Hydrogen atoms and photons from electron-proton annihilation (and the reverse process - Hydrogen ionization):

$$e^- + p \rightleftharpoons H + \gamma$$

As the universe is, at this point, cooler, only neutrinos and photons remain relativistic, and the equilibrium number densities for electrons, protons and Hydrogen atoms are given by the following expression:

$$n_i^{eq} = g_i \left(\frac{m_i T}{2\pi}\right)^{\frac{3}{2}} exp\left(\frac{\mu_i - m_i}{T}\right)$$

where μ_i is the chemical potential for the species i. In chemical equilibrium, the process must obey $\mu_e + \mu_p = \mu_H + \mu_\gamma = \mu_H$, as the photon's chemical potential is 0.
It is then possible to compute the ratio between the number densities of charged and neutral particles while in equilibrium:

$$\left(\frac{n_H}{n_e n_p}\right)_{eq} = \frac{g_H}{g_e g_p}\left(\frac{m_H\, 2\pi}{m_e m_p\, T}\right)^{\frac{3}{2}} exp\left(\frac{(\mu_H - \mu_e - \mu_p) - (m_H - m_e - m_p)}{T}\right) \approx \left(\frac{2\pi}{m_e\, T}\right)^{\frac{3}{2}} exp\left(\frac{B_H}{T}\right)$$

To arrive at this result the following relations were used: the internal degrees of freedom of the Hydrogen atom are 4, due to the 4 different spin pairings that the electron and proton may have in the atom and the internal degrees of freedom for each of them are 2, as they are fermions; the relation between chemical potentials means that that term was null; as $m_p \approx 2000 m_e$, it was assumed that $m_H \approx m_p$; and, finally, we defined $B_H = m_p + m_e - m_H$ as the binding energy of the Hydrogen atom, such that $B_H = 13.6\ eV$ ([3], p.65).

As we assume that the universe isn't electrically charged, we must have the same abundance of electrons and protons, such that $n_e = n_p$.
Besides that, we are also interested in relating this expression with the electron to baryon ratio, the electron fraction

abundance: $X_e = \frac{n_e}{n_b}$. For its part, the number density of baryons is related to the number density of photons via the baryon-photon ratio, η. Remembering that the photons are relativistic and in equilibrium with the fluid, we must have that $n_\gamma = \frac{2\zeta(3)T^3}{\pi^2}$, so $n_b = \eta\frac{2\zeta(3)T^3}{\pi^2}$.

A this point in the universe's life, the vast majority of baryonic matter was composed of protons and Hydrogen, such that we can easily assume that $n_b \approx n_p + n_H \approx n_e + n_H$. This means, therefore, that $X_e = \frac{n_e}{n_e+n_H} = \frac{1}{1+\frac{n_H}{n_e}}$, and that $\frac{n_H}{n_e^2} = \frac{1-\frac{1}{X_e}}{n_e} = \frac{X_e-1}{X_e^2 n_b}$ ([4], Ch. 7, p. 6).

We end up, finally, with the expression called the Saha equation, which is written as follows:

$$\left(\frac{1-X_e}{X_e^2}\right)_{eq} = \eta\frac{2\zeta(3)T^3}{\pi^2}\left(\frac{2\pi}{m_e\,T}\right)^{\frac{3}{2}} exp\left(\frac{B_H}{T}\right) = \eta\frac{2\zeta(3)}{\pi^2}\left(\frac{2\pi\,T}{m_e}\right)^{\frac{3}{2}} exp\left(\frac{B_H}{T}\right)$$

This, then, enables us to compute the free electron fraction.

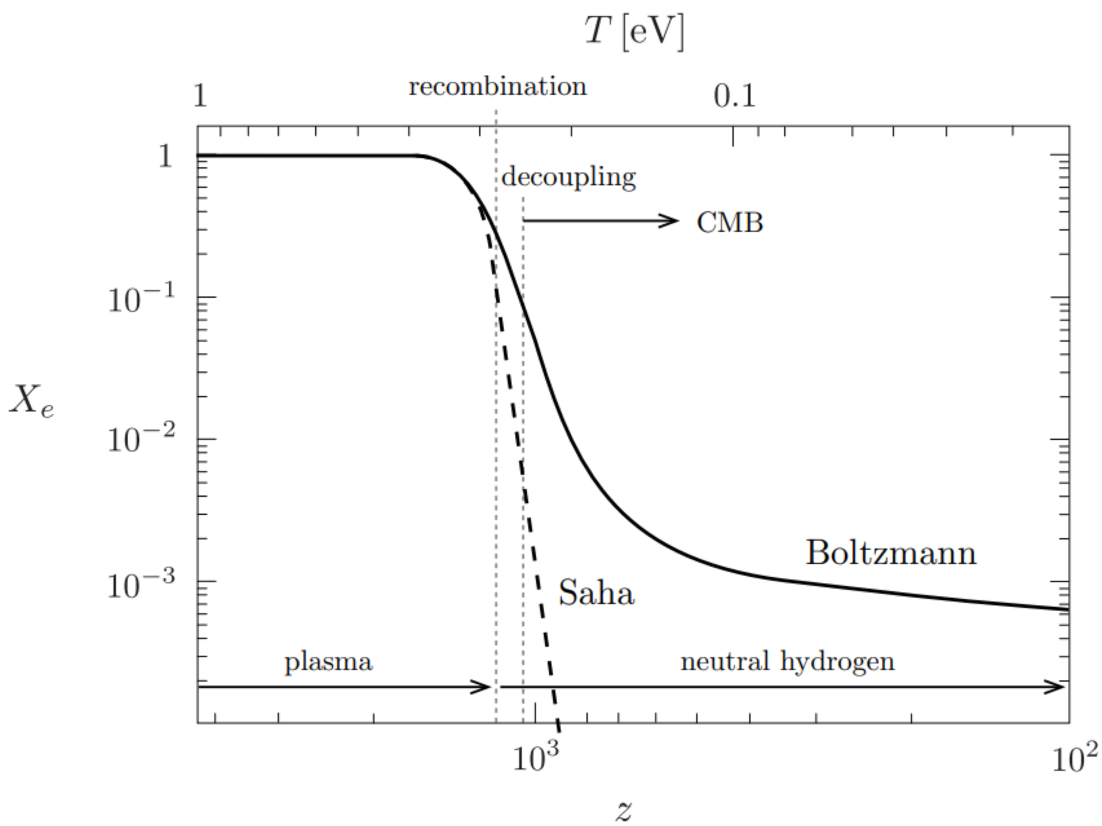

Figure 1. Free electron fraction, X_e as a function of redshift, z, and temperature, T. The dotted line represents the free electron fraction in equilibrium, given by the Saha equation, and the full line represents the same parameter in a non-equilibrium regime, given by the Boltzmann equation. Retrieved from [3]: Figure 3.8, p.66.

As we can see in figure 1, and as we've discussed earlier, as the particle - in this case the electron - decouples from the plasma and leaves thermal equilibrium, the interactions that kept it in equilibrium with the plasma freeze-out and a relic abundance persists, instead of the abundance decreasing exponentially.
It is of note that, for higher temperatures (lower redshifts), equilibrium and non-equilibrium calculations yield the same result, with the free electron fraction being basically equal to 1, as the universe isn't cool enough for electrons and protons to bind in significant quantities, resulting in $n_e = n_p \approx n_b$.

2.1. The Epoch of Recombination

As is shown in figure 1, the epoch of recombination is the start of the production of Hydrogen in big quantities, due to the binding of protons and electrons. As this recombination of free electrons and atomic nuclei happens, the fraction of free electrons in regards to baryonic matter, predictably, decreases.

Baumann defines, arbitrarily, the beginning of the recombination epoch as the stage of the universe's life when and $X_e = 0.1$ ([3], p.66). Using $\eta = 10^{-9}$, which, as we'll see, is consistent, it's easy to check via the Saha equation and Mathematica that, for $X_e = 0.1$, $T = T_{rec} \approx 0.3\ eV \approx 3600K$, much like what we see in figure 1.

This characterization makes sense, since, at this point, one can argue that the quantity of Hydrogen atoms produced is comparably larger to the quantity of free electrons (90% of free electrons and protons have by then recombined in Hydrogen atoms).
This, though, seems counterintuitive: why should we need such low temperatures, well below the energy of ionization for Hydrogen, for the epoch of recombination to start?
At this temperature, we can check using the previous expressions for number densities that $n_\gamma \approx 6.6 \times 10^{-3}\ GeV^3$; $\frac{n_H}{n_e^2} \approx 1.3 \times 10^{13}$, where it was used that $m_e = 511000\ eV$, and $\left(\frac{1-X_e}{X_e^2}\right) = 90$, which means that $n_b \approx 6.9 \times 10^{-12}\ GeV^3$. We then get a baryon to photon ratio of $\eta \approx 10^{-9} \ll 1$, which means that there are many more photons than Hydrogen atoms. This large quantity of photons means that there is still a relevant probability (as their energies follow a probability distribution with a high-energy tail) that the photons remaining in the plasma may have enough energy to ionize the atoms.
This, then, explains why only at temperatures two orders of magnitude below B_H does recombination start in large quantities: for higher temperatures, the much larger number of photons means that the atoms created are frequently ionized, and X_e is comparable to unity.

Away from mass thresholds ($T_{rec} \ll m_e$), one can establish that $T(t) \propto a(t)^{-1} = 1 + z(t)$ ([3], p.57).
Today, at $t \approx 13.9 \times 10^9$ years after the Big-Bang, the correspondent redshift is, obviously, $z_0 = 0$, and the universe's temperature is $T_0 \approx 2.7K$. So, it is easy then to compute the redshift at which we define the recombination epoch's start:

$$T_{rec} = T_0(1 + z_{rec}) \Rightarrow z_{rec} \approx 1320$$

From what we derived in A.1, this redshift is lower than the one for matter-radiation equality, which means that, at recombination, the universe was matter-dominated.
In A.2 we derived that $(1 + z) \propto t^{2/3}$, which means that the time of recombination can be estimated as

$$t_{rec} = \frac{t_0}{(1 + z_{rec})^{3/2}} = \frac{13.9 \times 10^9}{1321} \approx 290000 \text{ years}$$

3. Electron and Photon Decoupling

As the universe cools further, rampant Hydrogen production isn't the only notable development: electrons and photons will decouple and light will no longer interact with matter in the same way, leading to a transparent universe.

Electrons and photons, as explained earlier, stay in equilibrium via Compton scattering:

$$e^- + \gamma \rightleftharpoons e^- + \gamma$$

At this moment, electrons are not relativistic, so the relative velocity between them and photons is basically the speed of light.
Therefore, the interaction rate for photons in Compton scattering is $\Gamma_\gamma = n_e \sigma_T$, where $\sigma_T \approx 2 \times 10^{-3} MeV^{-2} = 2 \times 10^{-15}\ eV^{-2}$ is the cross-section of this interaction.

For decoupling to happen we need $\Gamma_\gamma(T_{dec}) \approx H(T_{dec})$.

$$\Gamma_\gamma = n_b X_e(T_{dec})\sigma_T = \frac{2\zeta(3)\eta}{\pi^2}\sigma_T X_e(T_{dec}) T_{dec}^3$$

We also know, as explained in [3], that in a matter-dominated universe, we can write H as: $H = H_0\sqrt{\Omega_{m0}}\left(\frac{T_{dec}}{T_0}\right)^{3/2}$, where, it should be noted, Baumann's notation drops the index "0" from the matter density today.

Using the same proposed model as in A.1, where $\Omega_{m0} = 0.32$, $H_0 = 70\ km \cdot s^{-1} \cdot Mpc^{-1} \approx 2 \times 10^{-33}\ eV$ and $T_0 \approx 10^{-4}\ eV$, we get:

$$X_e(T_{dec})T_{dec}^{3/2} \approx \frac{\pi^2 H_0\sqrt{\Omega_{m0}}}{2\zeta(3)\eta\sigma_T T_0^{3/2}} \approx 3.7 \times 10^{-3}\ eV^{3/2}$$

Here we used $\eta = 10^{-9}$, assuming that the ratio of baryons to photons hasn't increased considerably.

Using the Saha equation, and assuming $T' \approx 0.275\ eV$ for the exponential term (as evidenced by figure 1, the temperature should be between 0.25 and 0.3 eV), we get:

$$\left(\frac{1-X_e}{X_e}\right)_{eq} = \eta\frac{2\zeta(3)}{\pi^2}X_e\left(\frac{2\pi\, T_{dec}}{m_e}\right)^{\frac{3}{2}} exp\left(\frac{B_H}{T'}\right) \approx 2.1 \times 10^1 \Rightarrow X_e(T_{dec}) \approx 0.046 \Rightarrow T_{dec} \approx 0.2\ eV$$

This estimate is a bit far away from Baumann's ($T_{dec} \approx 0.27\ eV$), but it shows that, even just a bit after recombination, there was already Hydrogen production in sufficient quantities, such that the free electron fraction is now an order of magnitude less than at recombination.

Either way, using $T_{dec} \approx 0.27\ eV \approx 3100K$, we are able to arrive, by the same logic as before, to $z_{dec} \approx 1150$ and, from there, to a time of decoupling of about $t_{dec} \approx 360000$ years, not very far from the more thorough estimate in [3] of $t_{dec} \approx 380000$ years.

Decoupling of the electrons from the photons means that, finally, the universe will become transparent, as the free electrons increasingly bind to atomic nuclei forming neutral matter and photons don't have enough energy to ionize those atoms, meaning that they pass through the universe without much interaction (absorption), therefore, the universe starts getting more and more akin to an optically thin plasma.
And, from the estimate we just got, we're able to conclude that decoupling between electrons and photons - the plasma getting transparent to the photons - can only happen with a "large degree of neutrality" ([3], p.67).

3.1. Free Electron Relic Abundance

As the electron decouples from the plasma, it enters a state of non-equilibrium, in which the Saha equation no longer holds true, and, as discussed before, a relic abundance of free electrons will remain until this day.
If we are interested in understanding the level of ionization - abundance of free electrons - in the universe as of today and compare it to the abundance of baryonic matter, one needs only to use the Boltzmann equation to compute the free electron fraction at the present.

One should remember that as recombination started, the production of Hydrogen atoms from electron-proton annihilation became very efficient (photons were no longer energetic enough to maintain high levels of ionization) and X_e decreased significantly.
For a relic abundance of free electrons to last up until today, at some point the process of Hydrogen production must have become inefficient and there was an electron freeze-out.

We'll assume that the number densities of Hydrogen and photons are similar to their equilibrium abundances ([3], p. 67), as well as that $n_e \approx n_p$, so that the Boltzmann equation for this case can be written as:

$$\frac{d(n_e a^3)}{dt} = -a^3 <\sigma v> \left[n_e^2 - (n_e^{eq})^2 \right]$$

As demonstrated in A.3, we can rewrite our Boltzmann equation in terms of the free electron fraction, X_e [3][2]:

$$\frac{dX_e}{dx} = -\frac{\lambda}{x^2} \left[X_e^2 - (X_e^{eq})^2 \right] \; , \; \lambda \equiv \left[\frac{n_b <\sigma v>}{xH} \right]_{x=1} = 3.9 \times 10^3 \left(\frac{\Omega_b h}{0.03} \right) = 1.4 \times 10^5 \left(\frac{\Omega_b h}{\Omega_0^{1/2}} \right)$$

Let us define X_e^∞ as the relic abundance and T_f and x_f as the freeze out values for both parameters.
After decoupling, the fraction of free electrons given by the non-equilibrium regime will be much greater than the one for equilibrium, since that one is exponentially decreasing: $X_e \gg X_e^{eq}$.

We can, thus, obtain $\frac{dX_e}{dx} = -\frac{\lambda}{x^2} X_e^2$, which has the following solution:

$$\frac{dX_e}{dx} = -\frac{\lambda}{x^2} X_e^2 \Leftrightarrow \frac{1}{X_e^\infty} - \frac{1}{X_e(x_f)} = \frac{\lambda}{x_f}$$

$$X_e^\infty \ll X_e(x_f) \Rightarrow X_e^\infty \approx \frac{x_f}{\lambda} \approx 0.9 \times 10^{-3} \left(\frac{x_f}{x_{rec}} \right) \left(\frac{0.03}{\Omega_b h} \right)$$

If we were to do the same calculations as before to determine T_f and x_f we could get some estimates and compute X_e^∞.
Using the approximation for the thermally averaged cross-section like in A.3:

$$\Gamma_e = n_b X_e(T_f) \sigma_T \left(\frac{B_H}{T} \right)^{1/2} = \frac{2\zeta(3)\eta}{\pi^2} \sigma_T X_e T_f^{5/2}$$

$$H(T_f) = H_0 \sqrt{\Omega_{m0}} \left(\frac{T_f}{T_0} \right)^{3/2}$$

$$\frac{H}{\Gamma_e} \approx 1 \Leftrightarrow 1 = \frac{\pi^2 H_0 \sqrt{\Omega_{m0}}}{2\zeta(3)\eta\sigma_T T_0^{3/2} \, B_H^{1/2} \cdot X_e(T_f) T_f} \Leftrightarrow X_e(T_f) T_f = \frac{\pi^2 H_0 \sqrt{\Omega_{m0}}}{2\zeta(3)\eta\sigma_T T_0^{3/2} \, B_H^{1/2}} \approx 10^{-3} \, eV$$

Now we can use 2 ways for determining the parameters we want: using the Saha equation as before ([3], p.68) or using properties of the Riccati-like equation that we got through the Boltzmann equation ([2], pp.137-139).

1. Using the Saha equation as before, which will be only an approximation, as we are not in equilibrium:

$$\frac{1 - X_e}{X_e^{1/2}} = \frac{2\zeta(3)\eta}{\pi^2} \left(\frac{2\pi X_e T_f}{m_e} \right)^{3/2} e^{13.6/T_f}$$

   As little time since recombination has passed, but as the plasma becomes more optically thin, less photons are absorbed, so we should use a slightly smaller baryon to photon ratio. We'll assume $\eta \approx 9 \times 10^{-10}$, but, since the time of freeze-out should be after the time of decoupling, for the exponential term, we'll assume a temperature $T_f' \approx 0.26 \, eV$.
   Besides that, we'll also note that $x_e(T_f) \ll 1$, so we'll approximate $1 - X_e \approx 1$.
   Inputting these values on the equation we arrive at $X_e(T_f) \approx 4.1 \times 10^{-3}$, which means that $T_f \approx 0.24 \, eV$ and $x_f \approx 56$. At recombination, for example, we had $x_{rec} \approx 45$.

   Using the values measured by Planck in 2018 [5], we'll assume $\Omega_b h^2 = 0.02233$ and, with $H_0 = 67.37$km s^{-1} Mpc^{-1} and, therefore, $h = 0.6737$, we'll use $\Omega_b h \approx 0.03315$.
   With this value we can finally estimate the relic abundance of free electrons in the universe today:

$$X_e^{\infty} \approx 0.9 \times 10^{-3} \cdot \frac{56}{45} \cdot \frac{0.03}{0.03315} \approx 1.01 \times 10^{-3}$$

This estimate is in good accordance with Baumann's and X_e^{∞} on the order of 0.1% is consistent to what we find in figure 1.

2. The second way we can estimate this relic quantity is by doing the following analysis, as in [2]. It should be noted that Kolb uses a different definition of x, such that $x = \frac{1}{T}\, eV$.

   Our Boltzmann equation can be written as $X_e' = -\lambda x^{-2}(X_e^2 - (X_e^{eq})^2)$, where the use of « ′ » denotes a derivative with respect to x, with $\lambda = 1.4 \times 10^5 \left(\frac{\Omega_b h}{\Omega_o^{1/2}}\right)$ and $X_e^{eq} = 5.95 \times 10^7 (\Omega_b h^2)^{-1/2} x^{3/4} exp(-6.8x)$ ([2], p.137).
   If we were to define $\Delta = X_e - X_e^{eq}$ as the deviation from ionization equilibrium, one could rewrite our equation as:

$$\Delta' = -X_e^{eq\prime} - \lambda x^{-2}(\Delta + 2X_e^{eq})\Delta$$

   If we were to analyse the universe for $x > x_f$, then we could make the assumption that $X_e \gg X_e^{eq}$ and approximate $\Delta \approx X_e$, with $X_e^{eq\prime} \approx 0$, also.
   That would mean that we end up with $\Delta' \approx -\frac{\lambda}{x^2}\Delta^2$, which would again yield $X_e^{\infty} \approx \frac{x_f}{\lambda}$ when integrating from $x = x_f$ to $x = \infty$.

   Now, if we analysed the regime in which $x < x_f$, then the universe would be hotter, and it would be a fair approximation to use $\Delta' \approx 0$, as the production of neutral matter is yet slow-evolving.
   Finally we'd have $\Delta \approx \frac{-X_e^{eq\prime} x^2}{\lambda(\Delta + 2X_e^{eq})} \approx \frac{3.4x^2}{\lambda}$.

   If we're interested in determining the relic abundance, we'll have to estimate x_f.
   A fair estimate would be to say that freeze-out happens when the deviation from equilibrium is on the order of unity, such that $\Delta \approx X_e^{eq}$.

$$\frac{3.4x_f^2}{\lambda} \approx 5.95 \times 10^7 (\Omega_b h^2)^{-1/2} x_f^{3/4} exp(-6.8x_f) \Leftrightarrow$$

$$\Leftrightarrow x_f \approx \frac{1}{6.8} \ln[2.5 \times 10^{12} (\Omega_b/\Omega_0)^{1/2}] \approx 3.6 - 0.074 \ln(\Omega_b/\Omega_0)$$

   Assuming, like Kolb, that $\Omega_b/\Omega_0 \approx 0.1$, the final result will be $x_f \approx 3.8\, eV$ and $T_f \approx 0.26\, eV$, still consistent with the previous analysis and Baumann's estimate.
   Using Planck's values again, $\Omega_0^{1/2} \approx 0.7015$, and we'll arrive at an estimate for the free electron relic abundance of

$$X_e^{\infty} \approx \frac{x_f}{1.4 \times 10^5} \left(\frac{\Omega_0^{1/2}}{\Omega_b h}\right) \approx 0.85 \times 10^{-3}$$

   which is in very good agreement with what we'd expect.

Furthermore, after determining T_f and the X_e^{∞}, one shouldn't stop, but should instead translate T_f to a time interval after the Big-Bang, just like in previous sections, as one should always strive for an harmonized work.

So, just like before, and using an estimate of $T_f = 0.25\, eV \approx 2900K$, we arrive at the following redshift and time of freeze-out: $z_f \approx 1073$ and $t_f \approx 395000$ years after the Big-Bang, respectively.

4. Overview of the Results

Table 1. Approximate temperature - in kelvin - redshift, time after the Big-Bang - in years - and free electron fraction - in percentage - at which the universe found itself at the times of recombination, electron-photon decoupling, electron freeze-out and the present.

	Recombination	Decoupling	Freeze-out	Present
T (K)	3600	3100	2900	2.7
z	1320	1150	1073	0
t (years)	290000	380000	395000	13.9×10^9
X_e (%)	10	1	0.4	0.1

In table 1 we have the significant parameters determined here, or in Baumann's lectures [3], for the times of recombination, decoupling, electron freeze-out and the present.
One can easily see the time progression of the events that is triggered by the cooling of the universe and how the beginning of an accelerated production of neutral matter leads to an ever decreasing abundance of free electrons.

5. A quick remark on the importance of the ever-frozen CMB

As the universe cooled, electrons and protons were allowed to combine into neutral atoms, leading to a decrease in free electron density.
The photon mean free path, $l = \frac{1}{n_e\sigma}$, rose considerably and surpassed the horizon distance, leaving it to decouple from matter which, for its part, meant a start to our transparent universe ([3], p.64).

The CMB - Cosmic Microwave Background - radiation is the remnant of that decoupling, the photons that have reached us from when the universe was approximately 380000 years old.
Their temperature of $T_{dec} \approx 3100K$ means that, initially, this light was on the near-Infrared (NIR) region of the spectrum: Wien's Law gives us a wavelength of peak emission of $\lambda_{dec} = 935\ nm$, assuming we can treat the primordial universe as a black body, which is a good assumption, since, up until that instant in time, it was a optically thick plasma, meaning that we are only receiving the light it emitted via thermal radiation, which was, up until then, in equilibrium with the plasma.

Given a redshift of $z \approx 1150$, one would expect that the CMB radiation was detected in the present with a wavelength of peak emission of about $\lambda_0 \approx 1.1\ mm$, which, according to the standard characterization, just barely falls in the microwave region of the spectrum. Via Wien's Law, it corresponds to a temperature of $T_0 \approx 2.7K$, for a black body, which is exactly what we see[1]: in the 1990s the American Astronomical Society could witness how well a black body with $T = T_0$ fits the observed data from the CMB radiation, something that is very clear by the comparison between the two in figure 2.

[1] As the redshift of decoupling that we determined here was calculated from the assumption that $T_0 = 2.7K$ this is nothing out of the ordinary. The exciting phenomena is how well the CMB approximates to a black body with that temperature.

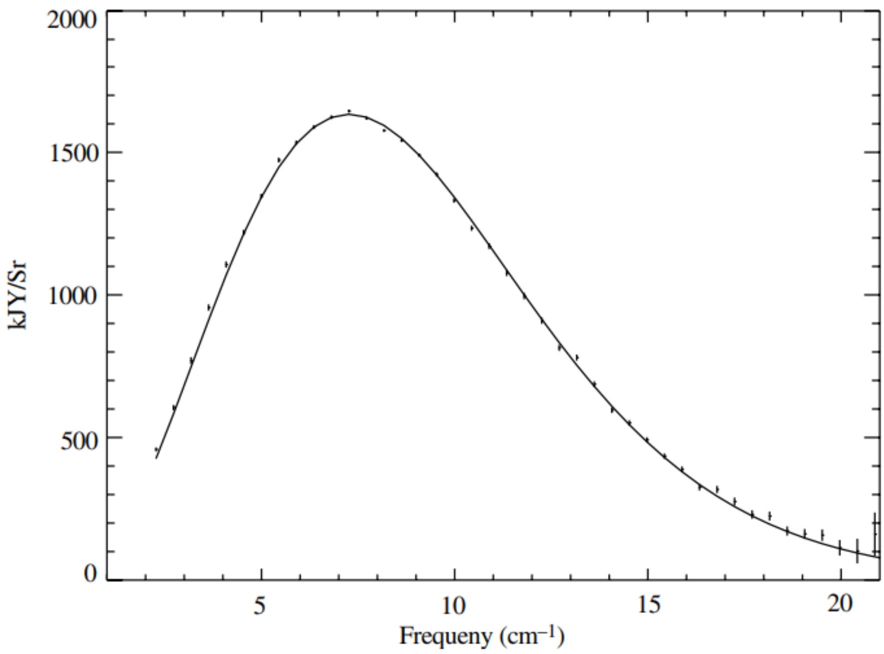

Figure 2. Comparison between the intensity of radiation observed with the FIRAS radiometer carried by COBE and a black-body spectrum with temperature of $2.728K$, from D. J. Fixsen et al., Astrophys. J. 473, 576 (1996) [astro-ph/9605054]. The intensity is measured in kiloJansky per steradian and the horizontal axis gives the reciprocal wavelength - here called frequency - in cm^{-1}. The 1σ experimental uncertainty in intensity is indicated by the tiny vertical bars; the uncertainty in wavelength is negligible. Retrieved from [1]: Figure 2.1, p.106.

This radiation, the isotropic photon background that we can today observe, is the further we'll ever be able to see: a static, frozen picture of the primordial universe. Previous instants of the universe's life are barred for us, and only cosmological models can help us then.
But we should still rejoice in the finding of this CMB, as it "is one of the pillars on which the model of a hot Big Bang rests" ([7], p.8). Besides that, Pettini remarks that the CMB also provides a solution to Olber's paradox. The proposed paradox does fail for optical light, since its main sources (stars) do not have an infinite and static observable universe to emit in, but for millimeter radiation there is no problem: the sky is bright everywhere in all directions with great isotropy[2].

Finally, the CMB, a significant, if not **the** significant part of the basis for the cosmological standard model, leads us in a few future directions for perfecting this model: determining cosmological parameters with ever more precision [5] and analysing scale anisotropies in the CMB sky, as it can be found a lot of data about structure formation for a post-recombination and post-decoupling universe in the CMB radiation - the thermal fluctuations in the observed data have much to tell us about the density inhomogeneities of the early universe which trickled down, after thousand of millions of years, into the large structures that span through our present universe.

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2 Average fluctuations in temperature along the entire background are of the order of $\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle^{\frac{1}{2}} \approx 10^{-5}$.

Appendix A. Derivations

Appendix A.1. Radiation-Matter Equality

In Baumann's lectures ([3], p.25), a model is proposed such that $\Omega_{r0} = 9.4 \times 10^{-5}$ and $\Omega_{m0} = 0.32$.

For matter-radiation equality to happen we must have $\rho_r = \rho_m$.
Using the following relations, we can arrive at the redshift at which this happens:

- $\rho_r = \rho_{r0}(\frac{a_0}{a})^4$
- $\rho_m = \rho_{m0}(\frac{a_0}{a})^3$
- $\Omega_{r0} = \rho_{r0}/H_0^2$
- $\Omega_{m0} = \rho_{m0}/H_0^2$
- $a = (1+z)\ ,\ \ a_0 \equiv 1$

$$\rho_r = \rho_m \iff \Omega_{r0}(1+z) = \Omega_{m0} \Rightarrow z \approx 3400$$

Appendix A.2. Redshift-Time relation for a matter-dominated universe

For a matter-dominated universe, we have $\rho = \rho_m = \rho_{m0}(\frac{a_0}{a})^3$.
The Friedmann equation, written as $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3}$ means that we arrive at the following relation:

$$\left(\frac{\dot{a}}{a}\right)^2 \propto a^{-3} \Leftrightarrow a^{1/2}da \propto dt \Leftrightarrow a(t) = (1+z(t))^{-1} \propto t^{2/3}$$

Appendix A.3. Electron Freeze-Out

For this case, a good approximation for the thermally averaged cross-section is $<\sigma v> \approx \sigma_T \left(\frac{B_H}{T}\right)^{1/2}$. Inputting that into the Boltzmann equation gives us:

$$\frac{d(n_e a^3)}{dt} = -a^3\sigma_T\left(\frac{B_H}{T}\right)^{1/2}\left[n_e^2 - (n_e^{eq})^2\right]$$

By definition $n_e = X_e n_b$, and using $n_b a^3 = \text{Const.}$ ([3], p. 68), we get:

$$\frac{d(n_e a^3)}{dt} = \frac{dX_e}{dt}n_b a^3 = -a^3 n_b^2\sigma_T\left(\frac{B_H}{T}\right)^{1/2}\left[X_e^2 - (X_e^{eq})^2\right] \Leftrightarrow \frac{dX_e}{dt} = -n_b\sigma_T\left(\frac{B_H}{T}\right)^{1/2}\left[X_e^2 - (X_e^{eq})^2\right]$$

If we use a change of variable $x = \frac{B_H}{T}$, we have that, away from mass thresholds, $T = Ag_{*s}^{-1/3}a^{-1}$.

$$\frac{dX_e}{dx} = \frac{dX_e}{dx}\frac{dx}{dt}$$

$$\frac{d(B_H/T)}{dt} = -\frac{B_H}{T^2}\frac{dT}{dt} = -\frac{B_H}{T^2}\frac{d(Ag_{*s}^{-1/3}a^{-1})}{dt} = \frac{B_H}{T^2}Ag_{*s}^{-1/3}a^{-1}\left(\frac{\dot{a}}{a}\right) = x \cdot H(T)$$

We, therefore, get:

$$\frac{dX_e}{dx} = -\left[\frac{n_b<\sigma v>}{xH}\right]\left[X_e^2 - (X_e^{eq})^2\right] = -\frac{\lambda}{x^2}\left[X_e^2 - (X_e^{eq})^2\right]\ ,\ \lambda \equiv \left[\frac{n_b<\sigma v>}{xH}\right]_{x=1}$$

For this final result we were motivated by the explanation in [6]: we implicitly "assumed that $<\sigma v>$ is independent of x " and λ was "chosen to be independent of x, by letting $x = 1$ and leaving the dependence of x in the factor x^{-2} ". Although this reasoning was made for cold-matter freeze-out, it still holds up in this case if λ is chosen well enough, since, like is shown in 3.1, $n_b<\sigma v> \propto T^{5/2}$ and $H \propto T^{3/2}$, which means that:

$$\left[\frac{n_b<\sigma v>}{xH}\right] = \frac{\text{Const.}}{x^2}$$

When one surpasses some of the weirdness in the notation put forward by Baumann, it is easy to see that we should end up with a solution of the sort:

$$\frac{dX_e}{dx} \propto -\frac{1}{x^2}\left[X_e^2 - (X_e^{eq})^2\right]$$

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