

1. (15 pts) Shadow is writing a secret message to Harry and wants to prevent it from being understood by Thormund. He decides to use Huffman encoding to encode the message. Magically, the symbol frequencies of the message are given by the Lucas numbers, a famous sequence of integers discovered by the same person who discovered the Fibonacci numbers. The  $n$ th Lucas number is defined as  $L_n = L_{n-1} + L_{n-2}$  for  $n > 1$  with base cases  $L_0 = 2$  and  $L_1 = 1$ .
  - (a) For an alphabet of  $\Sigma = \{a, b, c, d, e, f, g, h\}$  with frequencies given by the first  $|\Sigma|$  Lucas numbers, give an optimal Huffman code and the corresponding encoding tree for Shadow to use.
  - (b) Generalize your answer to (1a) and give the structure of an optimal code when the frequencies are the first  $n$  Lucas numbers.
2. (45 pts) A good hash function  $h(x)$  behaves in practice very close to the uniform hashing assumption analyzed in class, but is a deterministic function. That is,  $h(x) = k$  each time  $x$  is used as an argument to  $h()$ . Designing good hash functions is hard, and a bad hash function can cause a hash table to quickly exit the sparse loading regime by overloading some buckets and under loading others. Good hash functions often rely on beautiful and complicated insights from number theory, and have deep connections to pseudorandom number generators and cryptographic functions. In practice, most hash functions are moderate to poor approximations of uniform hashing.

Consider the following hash function. Let  $U$  be the universe of strings composed of the characters from the alphabet  $\Sigma = [A, \dots, Z]$ , and let the function  $f(x_i)$  return the index of a letter  $x_i \in \Sigma$ , e.g.,  $f(A) = 1$  and  $f(Z) = 26$ . Finally, for an  $m$ -character string  $x \in \Sigma^m$ , define  $h(x) = ([\sum_{i=1}^m f(x_i)] \bmod \ell)$ , where  $\ell$  is the number of buckets in the hash table. That is, our hash function sums up the index values of the characters of a string  $x$  and maps that value onto one of the  $\ell$  buckets.

- (a) The following list contains US Census derived last names:  
<http://www2.census.gov/topics/genealogy/1990surnames/dist.all.last>  
Using these names as input strings, first choose a uniformly random 50% of these name strings and then hash them using  $h(x)$ .

Produce a histogram showing the corresponding distribution of hash locations when  $\ell = 200$ . Label the axes of your figure. Briefly describe what the figure shows about  $h(x)$ , and justify your results in terms of the behavior of  $h(x)$ . Do not forget to append your code.

Hint: the raw file includes information other than name strings, which will need to be removed; and, think about how you can count hash locations without building or using a real hash table.

- (b) Enumerate at least 4 reasons why  $h(x)$  is a bad hash function relative to the ideal behavior of uniform hashing.
- (c) Produce a plot showing (i) the length of the longest chain (were we to use chaining for resolving collisions under  $h(x)$ ) as a function of the number  $n$  of these strings that we hash into a table with  $\ell = 200$  buckets, (ii) the exact upper bound on the depth of a red-black tree with  $n$  items stored, and (iii) the length of the longest chain were we to use a uniform hash instead of  $h(x)$ . Include a guide of  $cn$
- Then, comment (i) on how much shorter the longest chain would be under a uniform hash than under  $h(x)$ , and (ii) on the value of  $n$  at which the red-black tree becomes a more efficient data structure than  $h(x)$  and separately a uniform hash.
3. (20 pts) Grog is struggling with the problem of making change for  $n$  cents using the smallest number of coins for his purchase of a new great sword. Grog has coin values of  $v_1 < v_2 < \dots < v_r$  for  $r$  coin types, where each coin's value  $v_i$  is a positive integer. His goal is to obtain a set of counts  $\{d_i\}$ , one for each coin type, such that  $\sum_{i=1}^r d_i = k$  and where  $k$  is minimized.

- (a) A greedy algorithm for making change is the **cashier's algorithm**, which all young wizards learn. Harry writes the following pseudocode on the whiteboard to illustrate it, where  $n$  is the amount of money to make change for and  $v$  is a vector of the coin denominations:

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wizardChange(n,v,r) :  
    d[1 .. r] = 0          // initial histogram of coin types in solution  
    while n > 0 {  
        k = 1  
        while ( k < r and v[k] > n ) { k++ }  
        if k==r { return 'no solution' }  
        else { n = n - v[k] }  
    }  
    return d
```

Thormund snorts and says Harry's code has bugs. Identify the bugs and explain why each would cause the algorithm to fail.

- (b) Sometimes the dwarves at Rocky Mountain Bank run out of coins,<sup>1</sup> and make change using whatever is left on hand. Identify a set of U.S. coin denominations for which the greedy algorithm does not yield an optimal solution. Justify your

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<sup>1</sup>It's a little known secret, but dwarven pets like to *eat* the coins. It isn't pretty for the coins, in the end.

answer in terms of optimal substructure and the greedy-choice property. (The set should include a penny so that there is a solution for every value of  $n$ .)

- (c) On the advice of wizards specializing in electricity, Rocky Mountain Bank has announced that they will be changing all coin denominations into a new set of coins denominated in powers of  $c$ , i.e., denominations of  $c^0, c^1, \dots, c^\ell$  for some integers  $c > 1$  and  $\ell \geq 1$ . (This will be done by a spell that will magically transmute old coins into new coins, before your very eyes.) Prove that the cashier's algorithm will always yield an optimal solution in this case.

Hint: first consider the special case of  $c = 2$ .

4. (20 pts) We saw on the previous problem set that the cashier's (greedy) algorithm for making change doesn't handle arbitrary denominations optimally. In this problem you'll develop a dynamic programming solution which does, but with a slight twist. Suppose we have at our disposal an arbitrary number of *cursed* coins of each denomination  $d_1, d_2, \dots, d_k$ , with  $d_1 < d_2 < \dots < d_k$ , and we need to provide  $n$  cents in change. We will always have  $d_1 = 1$ , so that we are assured we can make change for any value of  $n$ . The curse on the coins is that in any one exchange between people, with the exception of  $i = 2$ , if coins of denomination  $d_i$  are used, then coins of denomination  $d_{i-1}$  *cannot* be used. Our goal is to make change using the minimal number of these cursed coins (in a single exchange, i.e., the curse applies).

- (a) For  $i \in \{1, \dots, k\}$ ,  $n \in \mathbb{N}$ , and  $b \in \{0, 1\}$ , let  $C(i, n, b)$  denote the number of cursed coins needed to make  $n$  cents in change using only the first  $i$  denominations  $d_1, d_2, \dots, d_i$ , where  $d_{i-1}$  is allowed to be used if and only if  $i \leq 2$  or  $b = 0$ . That is,  $b$  is a Boolean "flag" variable indicating whether we are excluding denomination  $d_{i-1}$  or not ( $b = 1$  means exclude it). Write down a recurrence relation for  $C$  and prove it is correct. Be sure to include the base case.
- (b) Based on your recurrence relation, describe the order in which a dynamic programming table for  $C(i, n, b)$  should be filled in.
- (c) Based on your description in part (b), write down pseudocode for a dynamic programming solution to this problem, and give a  $\Theta$  bound on its running time (remember, this requires proving both an upper *and* a lower bound).