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Design and Analysis of Experiments

06 - Simple Comparisons

Version 2.11

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"*Science is simply common sense at its best, that is, rigidly accurate in observation, and merciless to fallacy in logic.*"

Thomas H. Huxley
1825-1895
English biologist

Image: http://www.iep.utm.edu/huxley/

Simple Comparative Experiments

Statistical inference for two samples

The concepts of comparison between two populations based on information obtained from their samples follow the same principles used for testing hypotheses about a single population;

Inferences for two samples frequently arise when comparing the effect of a technique (treatment) against a *control group*: placebo, classical technique, random search, etc;

Usual questions involve:

- Comparison of means;
- Comparison of variances;
- Comparison of proportions;
- etc.

Comparison of two means

Example: Length of steel rods

One of the critical aspects of manufacturing steel rods is cutting the bars with a precise length, which is expected by the customers.

This process is prone to errors, which result in additional costs for standardizing and reprocessing the rods.

An engineer is interested in comparing the current controller of the cutting scissors with a new method that could potentially improve the performance of the process.

Adapted from D.F. Carvalho Jr.'s course project for the Design and Analysis of Experiments Course, PPGEE-UFMG, June 2012. The data used in this example is not necessarily the original one.

Image: `http://www.shutterstock.com/pic-73207399/`

Comparison of two means

Example: Length of steel rods

A possible statistical model for this kind of data would be:

$$y_{ij} = \mu_i + \epsilon_{ij} \begin{cases} i = 1, 2 \\ j = 1, \ldots, n_i \end{cases}$$

Lets initially assume that the residuals ϵ_{ij} are iid $\mathcal{N}\left(0, \sigma_i^2\right)$, which implies:

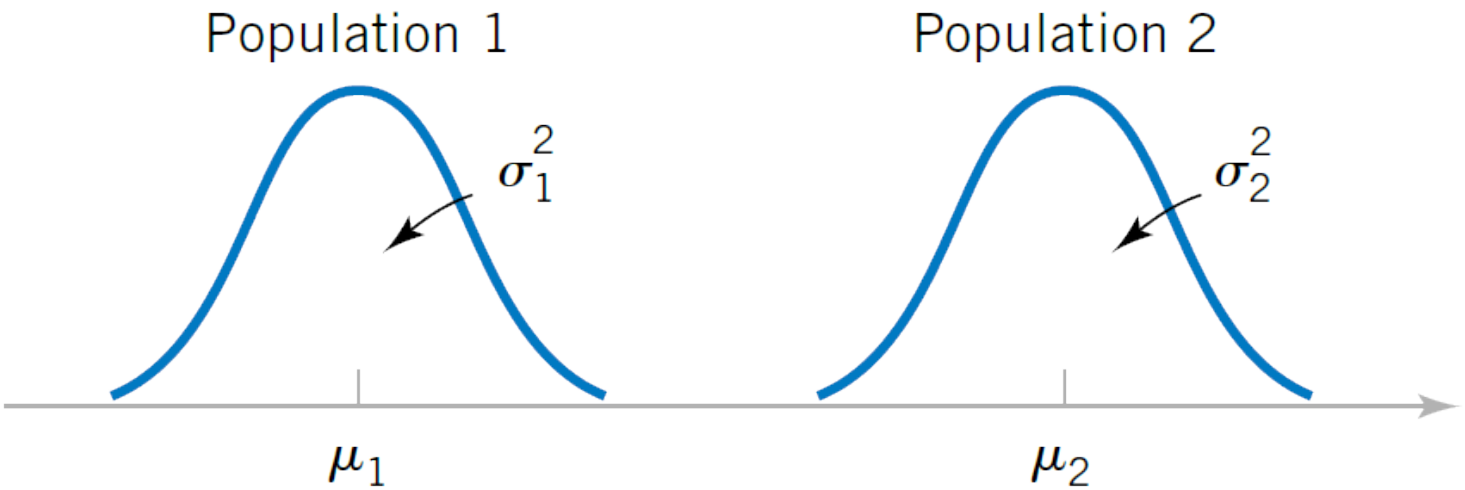

Image: D.C.Montgomery,G.C. Runger, *Applied Statistics and Probability for Engineers*,Wiley 2003.

Comparison of two means

Definitions

What we wish is to perform an inference about the difference in the mean values of constructive deviations for the two controllers. In this case, a reasonable response variable would be the *absolute error*, from which we could test our hypotheses.

The statistical hypotheses can be stated as:

$$\begin{cases} H_0 : \mu_1 - \mu_2 = 0 \\ H_1 : \mu_1 - \mu_2 \neq 0 \end{cases} \quad \textbf{or, equivalently,} \quad \begin{cases} H_0 : \mu_1 = \mu_2 \\ H_1 : \mu_1 \neq \mu_2 \end{cases}$$

Suppose a desired significance level $\alpha = 0.05$, and that the engineer is interested in detecting any difference larger than $15mm$ in the mean absolute error with a power $(1 - \beta) = 0.8$.

Also, lets assume that the variance of the process is unknown but similar for both controllers.

Comparison of two means

Definitions

Since the variance is unknown, it will have to be estimated from the data. As we are assuming $\sigma_1^2 \approx \sigma_2^2$, we can use the pooled variance estimator:

$$S_p^2 = \frac{(n_1 - 1)\, S_1^2 + (n_2 - 1)\, S_2^2}{n_1 + n_2 - 2} = wS_1^2 + (1 - w)\, S_2^2$$

Based on this estimator and the stated assumptions, we have that:

$$T = \frac{(\bar{y}_1 - \bar{y}_2) - (\mu_1 - \mu_2)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{(n_1+n_2-2)}$$

Comparison of two means

Rejection threshold

If we recall our working hypotheses:

$$\begin{cases} H_0 : \mu_1 - \mu_2 = 0 \\ H_1 : \mu_1 - \mu_2 \neq 0 \end{cases}$$

we have that, under H_0:

$$t_0 = \frac{(\bar{y}_1 - \bar{y}_2) - \cancelto{0}{(\mu_1 - \mu_2)}}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{(\bar{y}_1 - \bar{y}_2)}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{(n_1+n_2-2)}$$

We'll reject H_0 at the $(1 - \alpha)$ confidence level if $|t_0| \geq t_{\alpha/2;(n_1+n_2-2)}$

Comparison of two means

Sample sizes

Now recall that the process engineer was interested in some very specific characteristics for his test:

- Significance: $\alpha = 0.05$;
- Power: $(1 - \beta) = 0.8$;
- Minimally interesting effect: $\delta^* = 15mm$

From these specifications, we can obtain the required sample sizes. The derivation of the sample size formulas is not particularly difficult, but we'll concentrate only on the results. More details can be easily found in the literature[a].

[a] Check, for instance, Paul Mathews' *Sample Size Calculations*, MMB, 2010.

Comparison of two means

Sample sizes

For the general case of unequal sample sizes, we have:

$$n_1 = \left(1 + \frac{n_1}{n_2}\right)\left(\frac{s_p}{\delta^*}\right)^2 (t_{\alpha/2} + t_\beta)^2$$

where s_p is the estimated common standard deviation, and $t_{\alpha/2}$ and t_β are the $\alpha/2$ and β quantiles of the $t_{(n1+n2-2)}$ distribution. The sample size n_2 can be calculated by simply substituting (n_1/n_2) by (n_2/n_1).

For equal sample sizes ($n_1 = n_2 = n$) the expression is simplified to:

$$n = 2\left(\frac{s_p}{\delta^*}\right)^2 (t_{\alpha/2} + t_\beta)^2$$

Comparison of two means

Sample sizes

These formulas are very convenient, but leave us with a riddle: we need variance estimate in order to calculate the sample size, but we need observations to be able to estimate the variance.

There are a few ways to proceed in this case. The most practical are:

- Use process knowledge or historical data to obtain an (initial) estimate of the variance;
- Perform a pilot study and collect samples to estimate the variance.

The first method is almost always preferable since it does not imply additional costs for the experiment.

Comparison of two means

Sample sizes

If no information is available to estimate the variance, a pilot study must be performed to obtain this value. The sample size required for this pilot study is given by:

$$n_{pilot} \approx 2 \left(\frac{z_{\alpha_n/2}}{e_n} \right)^2$$

where $(1 - \alpha_n)$ is the desired confidence level for the sample size estimate of the main study, and e_n is the maximum relative error allowed for the sample size.

This calculation can yield some scarily large sample sizes for a pilot study (much larger than would be actually required for the main study itself), so use this with caution.

Comparison of two means

Sample sizes

For the steel rods experiment, suppose that the engineer uses data available from the controller manuals, as well as historical measurements, to estimate the common standard deviation for the cutting process as $\hat{\sigma} \approxeq 15mm$.

Assuming that equal sample sizes are desired, we can simply use the formula:

$$n = 2\left(\frac{\hat{\sigma}}{\delta^*}\right)^2 \left(t_{\alpha/2} + t_{\beta}\right)^2$$

Easy, right?

Comparison of two means

Sample sizes

For the steel rods experiment, suppose that the engineer uses data available from the controller manuals, as well as historical measurements, to estimate the common standard deviation for the cutting process as $\hat{\sigma} \approxeq 15mm$.

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$$n = 2\left(\frac{\hat{\sigma}}{\delta^*}\right)^2 \left(t_{\alpha/2} + t_{\beta}\right)^2$$

Easy, right?

Comparison of two means

Sample sizes

The last problem we have to solve is that the values of $t_{\alpha/2}$ and t_{β} are also dependent of n, which makes the equation

$$n = 2\left(\frac{\hat{\sigma}}{\delta^*}\right)^2 \left(t_{\alpha/2} + t_{\beta}\right)^2$$

transcendental in n. We'll have to iterate until we find the smallest n that satisfies:

$$n \geq 2\left(\frac{\hat{\sigma}}{\delta^*}\right)^2 \left(t_{\alpha/2} + t_{\beta}\right)^2$$

Usually $t_{\alpha/2} \approx z_{\alpha/2}$ is used for the first iteration. Easy, right?

Comparison of two means

Sample sizes

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Usually $t_{\alpha/2} \approx z_{\alpha/2}$ is used for the first iteration. Easy, right?

Image: http://vigilantmeadow.deviantart.com/art/DON-T-PANIC-165415311

Comparison of two means

Example: Length of steel rods

Required sample size:

```
> ss.calc<-power.t.test(delta=15,
                        sd=15,
                        sig.level=0.05,
                        power=0.8,
                        type="two.sample",
                        alternative="two.sided")

Two-sample t test power calculation
n = 16.71477
delta = 15
sd = 15
sig.level = 0.05
power = 0.8
alternative = two.sided

NOTE: n is number in *each* group
```

Comparison of two means

Example: Length of steel rods

Computationally, we can perform the t-test for comparing the means of two independent populations by:

```
> y<-read.table("../data files/steelrods.txt",
+               header=T)

> with(y,
+      t.test(Length.error~Process,
+             alternative = "two.sided",
+             mu = 0,
+             var.equal = TRUE,
+             conf.level = 0.95))

Two Sample t-test
data:  Length.error by Process
t = -14.312, df = 32, p-value = 1.849e-15
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -0.09272982 -0.06962312
sample estimates:
mean in group new mean in group old
       0.07782353        0.15900000
```

Comparison of two means

Example: Length of steel rods

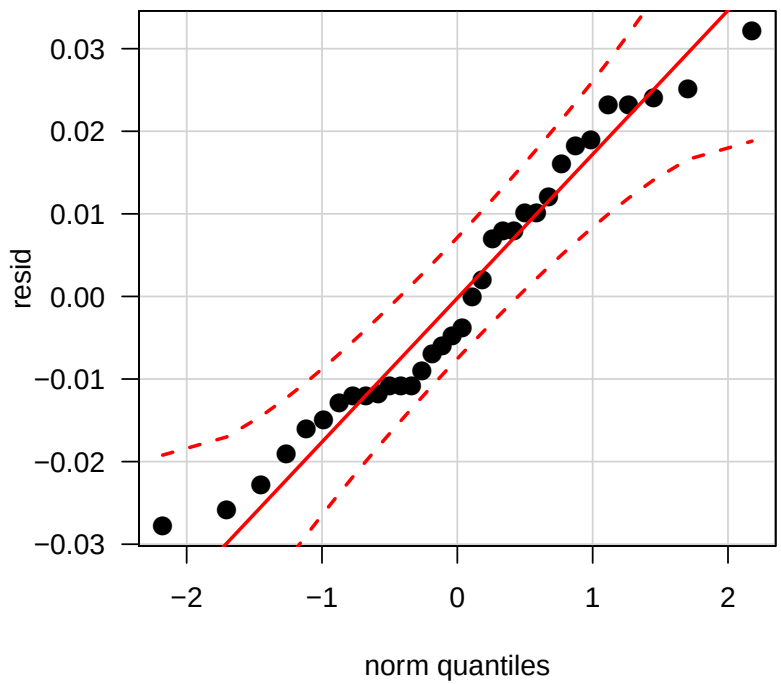
The assumptions of the test must be verified. In this particular case:

- Normality of the residuals;
- Equality of variance of the residuals;
- Independence of the residuals.

```
> resid<-y$Length.error - rep(means[2:1],
+                             each=n)

> shapiro.test(resid)
Shapiro-Wilk normality test
data:  resid
W = 0.9552, p-value = 0.176

> library(car)
> qqPlot(resid,
+        pch=16,
+        cex=1.5,
+        las=1)
```



Comparison of two means

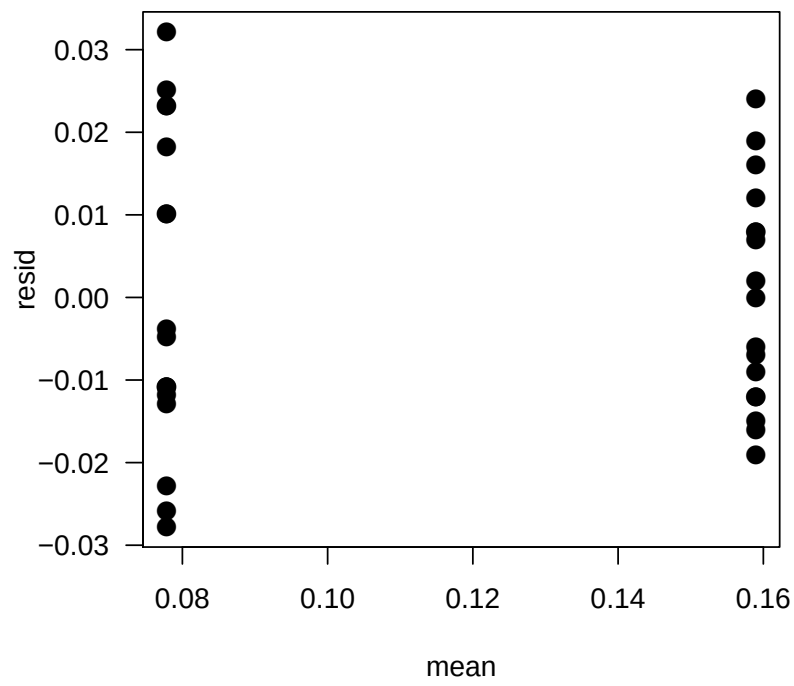
Example: Length of steel rods

The assumptions of the test must be verified. In this particular case:

- Normality of the residuals;
- Equality of variance of the residuals;
- Independence of the residuals.

```
> with(y,
+      fligner.test(Length.error~Process))
Fligner-Killeen test of homogeneity of variances
data:  Length.error by Process
Fligner-Killeen:med chi-squared = 1.6837,
df = 1, p-value = 0.1944

> plot(x=rep(means[2:1],
+            each=n),
+      y=resid,
+      xlab="mean",
+      pch=16,
+      cex=1.5,
+      las=1)
```



Comparison of two means

Example: Length of steel rods

The assumptions of the test must be verified. In this particular case:

- Normality of the residuals;
- Equality of variance of the residuals;
- Independence of the residuals.

As mentioned in an earlier lecture, there is no general test for the independence assumption, and it has to be guaranteed in the design phase.

One can at most test for serial autocorrelation in the residuals using Durbin-Watson's test, but this test is absolutely dependent on the ordering of the observations - very useful to detect ordering-related trends in the residuals, but not much more than that.

Comparison of two means

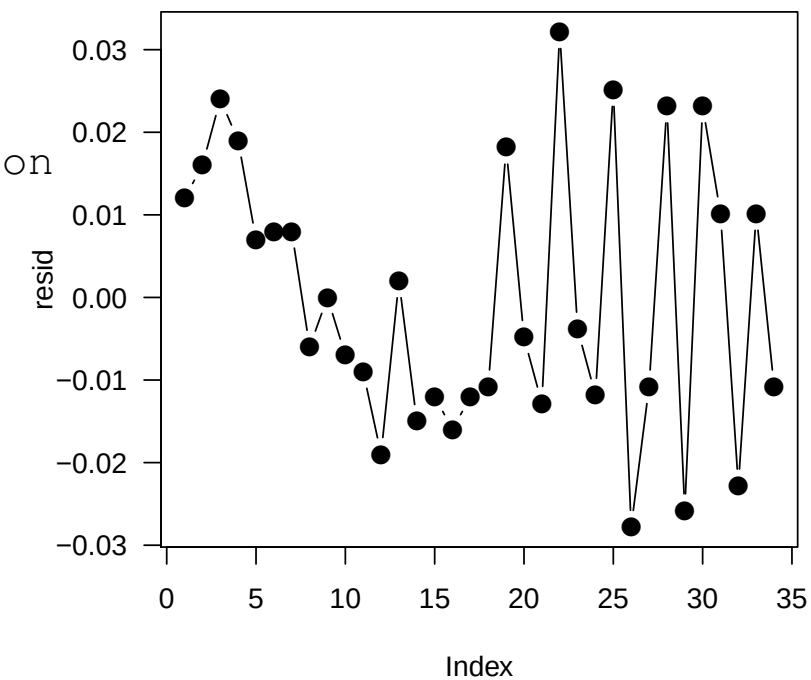
Example: Length of steel rods

The assumptions of the test must be verified. In this particular case:

- Normality of the residuals;
- Equality of variance of the residuals;
- Independence of the residuals.

```
> library(lmtest)
> with(y,
+      dwtest(Length.error~Process))
Durbin-Watson test
data:  Length.error ~ Process
DW = 2.2215, p-value = 0.6838
alternative hypothesis: true autocorrelation
is greater than 0

> plot(resid,
+      pch=16,
+      cex=1.5,
+      type="b",
+      las=1)
```



Bibliography

Required reading

1

Recommended reading

1

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  author={Felipe Campelo},
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}
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