Design and Analysis of Experiments

04 - Statistical Intervals

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Felipe Campelo

http://www.cpdee.ufmg.br/~fcampelo

Graduate Program in Electrical Engineering

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"*Science is an integral part of culture. It's not this foreign thing, done by an arcane priesthood. It's one of the glories of the human intellectual tradition.*"

Stephen Jay Gould
1941-2002
American paleontologist

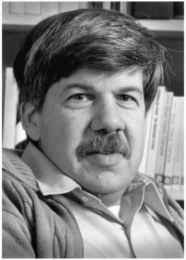


Image: Harvard University / AP

Statistical intervals are important in quantifying the uncertainty associated to a given estimate;

As an example, let's recap the coaxial cables example: *a coaxial cable manufacturing operation produces cables with a target resistance of* 50Ω *and a standard deviation of* 2Ω. *Assume that the resistance values can be well modeled by a normal distribution*.

Let us now suppose that a sample mean of $N = 25$ observations of resistance yields $\bar{x} = 48$. Given the sampling variability, it is very likely that this value is not exactly the true value of μ, but we are so far unable quantify how much uncertainty there is in this estimate.

Statistical Intervals

Definition

Statistical intervals define regions that are likely to contain the true value of an estimated parameter.

More formally, it is generally possible to quantify the level of uncertainty associated with the estimation, thereby allowing the derivation of sound conclusions at predefined levels of certainty.

Three of the most common types of interval are:

- Confidence intervals;
- Tolerance intervals;
- Prediction intervals;

Confidence intervals quantify the degree of uncertainty associated with the estimation of population parameters such as the mean or the variance.

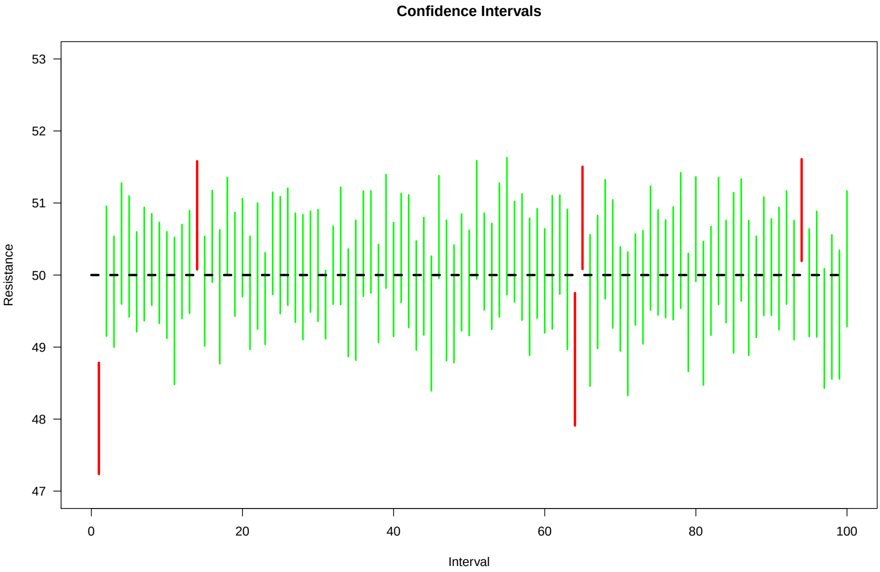
Can be defined as "*the interval that contains the true value of a given population parameter with a confidence level of* $100(1-\alpha)$";

- **Wrong**: "there is a 95% chance that the interval contains the true population mean."
- **right**: "The method used to derive the interval has a hit rate of 95%" - i.e., the interval generated has a 95% chance of "capturing" the true population parameter.

Easier to understand if you think about the confidence level as a confidence in the **method**, not in the interval.

Confidence Intervals

Example: 100 $CI_{.95}$ for a sample of 25 observations



Confidence Intervals

CI on the Mean of a Normal Variable

The two-sided $CI_{(1-\alpha)}$ for the mean of a normal population with known variance σ^2 is given by:

$$\bar{x} - \frac{\sigma}{\sqrt{N}} z_{(\alpha/2)} \leq \mu \leq \bar{x} + \frac{\sigma}{\sqrt{N}} z_{(\alpha/2)}$$

wherein $(1-\alpha)$ is the confidence level and $z_{(\alpha/2)}$ is the $(1-\alpha/2)$-quantile of the standard normal distribution.

For the more usual case with an unknown variance,

$$\bar{x} - \frac{s}{\sqrt{N}} t_{(\alpha/2;N-1)} \leq \mu \leq \bar{x} + \frac{s}{\sqrt{N}} t_{(\alpha/2;N-1)}$$

wherein $t_{(\alpha/2;N-1)}$ is the corresponding quantile of the t distribution with $N-1$ degrees of freedom.

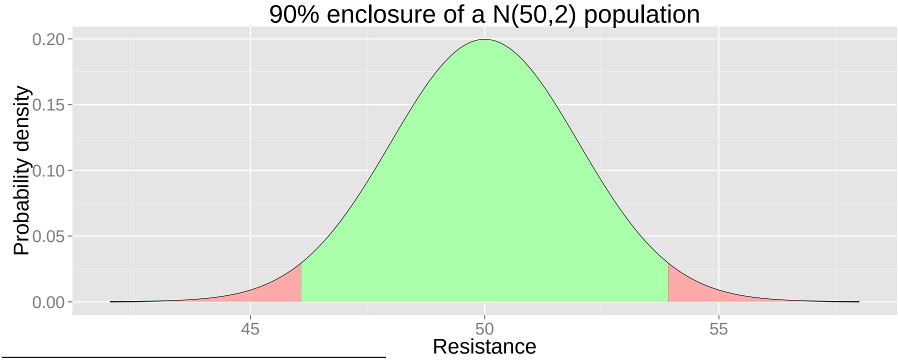
Similarly, a two-sided confidence interval on the variance of a normal variable can be easily calculated:

$$\frac{(N-1)s^2}{\chi^2_{\alpha/2;N-1}} \leq \sigma^2 \leq \frac{(N-1)s^2}{\chi^2_{1-\alpha/2;N-1}}$$

wherein $\chi^2_{\alpha/2;N-1}$ and $\chi^2_{1-\alpha/2;N-1}$ are the upper and lower $(\alpha/2)$-quantiles of the χ^2 distribution with $N-1$ degrees of freedom.

Tolerance Intervals
Definition

"*A tolerance interval is an* ***enclosure*** *interval for a specified proportion of the sampled population, not its mean or standard deviation. For a specified confidence level, you may want to determine lower and upper bounds such that a given percent of the population is contained within them.*"[1].



[1] J.G. Ramírez: `https://www.sas.com/resources/whitepaper/wp_4430.pdf`

The common practice in engineering of defining specification limits by adding $\pm 3\sigma$ to a given estimate of the mean arises from this definition - for a normally-distributed population, aproximately 99.75% of the observations will fall within these limits.

However, as in most cases the true population variance is unknown, one has to use its estimate s^2 and compensate for the uncertainty in this estimation. The two-sided tolerance interval is then given as:

$$\bar{x} \pm \sqrt{\frac{(N-1)\left(N + z^2_{(\alpha/2)}\right)}{N\,\chi^2_{(\gamma;N-1)}}}$$

wherein γ is the proportion of the population to be enclosed, and $1-\alpha$ represents the desired confidence level for the interval.

Prediction Intervals

Definition

Prediction intervals quantify the uncertainty associated with forecasting the value of a future observation;

Essentially, one is interested in obtaining an interval within which he or she can declare that the next observation will fall with a given probability;

For a normal distribution, we have:

$$\bar{x} - t_{(\alpha/2;N-1)} s \sqrt{1 + \frac{1}{N}} \leq X_{N+1} \leq \bar{x} + t_{(\alpha/2;N-1)} s \sqrt{1 + \frac{1}{N}}$$

which is similar to the confidence interval for the mean, but adding 1 to the term within the square root to account for the prediction noise.

Statistical intervals quantify the uncertainty associated with different aspects of estimation;

Reporting intervals is always better that point estimates, as it provides to you (and your readers) the necessary information to quatify the location and spread of your estimated values;

The correct interpretation is a little tricky (although not that difficult)[2], but it is essential in order to derive the correct conclusions based on the statistical interval of interest.

[2] See the table at the end of: `https://www.sas.com/resources/whitepaper/wp_4430.pdf`

Bibliography

Required reading

1. J.G. Ramírez, *Statistical Intervals: Confidence, Prediction, Enclosure*: `https://www.sas.com/resources/whitepaper/wp_4430.pdf`
2. D.C. Montgomery and G.C. Runger, *Applied Statistics and Probability for Engineers*, Chapter 8. 3rd Ed., Wiley 2005.

Recommended reading

1. Simply Statistics (blog) - `http://simplystatistics.org`
2. R. Dawkins, *Climbing Mount Improbable*, W.W.Norton&Co.,1997.

About this material

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