

# Design and Analysis of Experiments

## 07 - Paired Design

Version 2.11

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*"I am driven by two main philosophies: know more today about the world than I knew yesterday and lessen the suffering of others. You'd be surprised how far that gets you."*

Neil deGrasse Tyson  
1958 -  
American astrophysicist and author.



# Comparison of two means

## Dependent populations

Suppose the following situation: a young researcher develops an optimization algorithm (A) for a given family of problems, and wants to compare its convergence speed against a method that represents the state-of-the-art (B).

The researcher implements both methods and wants to determine whether the proposed one has a better average performance for problems of that particular kind.

The measurements are made under homogeneous conditions (same computer, same operational conditions, etc.) and the time is measured in a way that is not sensitive to other processes running in the system.



# Comparison of two means

## Dependent populations

This problem has some important questions worth considering:

- What is the actual question of interest?
- What is the *population* for which that question is relevant?
- What are the independent observations for that population?
- What is the relevant sample size for the experiment?

# Comparison of two means

## Paired design

The variability due to the different test problems is a strong source of spurious variation that can and must be controlled;

An elegant solution to eliminate the influence of this nuisance parameter is the *pairing* of the measurements by problem:

- Observations are considered in pairs (A, B) for each problem;
- Hypothesis testing is done on the sample of *differences*;

# Comparison of two means

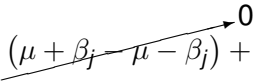
## Paired design

Let  $y_{Aj}$  and  $y_{Bj}$  denote paired observations of average time for methods A and B, for each problem instance  $j$ . The *paired differences* of the observations are simply  $d_j = y_{Aj} - y_{Bj}$ .

If we model our observations as an additive process:

$$y_{ij} = \underbrace{\mu + \tau_i}_{\mu_i} + \beta_j + \varepsilon_{ij}$$

where  $\mu$  is the grand mean,  $\tau_i$  is the effect of the  $i$ -th algorithm on the mean,  $\beta_j$  is the effect of the  $j$ -th problem, and  $\varepsilon_{ij}$  is the model residual, then:

$$\begin{aligned} d_j &= (\mu + \beta_j - \mu - \beta_j) + \tau_A - \tau_B + \varepsilon_{Aj} - \varepsilon_{Bj} \\ &= \mu_D + \varepsilon_j \end{aligned}$$


# Comparison of two means

## Paired design

The hypotheses of interest can now be defined in terms of  $\mu_D$ , e.g.:

$$\begin{cases} H_0 : \mu_D = 0 \\ H_1 : \mu_D \neq 0 \end{cases}$$

which can now be treated as a test of hypotheses for a single sample: the population of interest is the differences in average times until convergence for the problems under investigation.

# Comparison of two means

## Paired design

Some other important questions worth considering:

- In this example the minimally interesting effect size  $\delta^*$  must be expressed in terms of *average time gains across problems* (not within individual instances);
- The most important sample size to consider in this situation refers to the *number of problem instances*, and not necessarily to the number of within-problems repeated measures;
- The number of repetitions within each problem will have an impact on the uncertainty associated to each observation (that is, to each value of mean time to convergence for each algorithm on each problem), and should be selected with some care<sup>b</sup>.

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<sup>b</sup>Alternatively, we can set it as arbitrarily large, particularly in cases where the within-instance variation is small compared to the between-instances variation. As much as I hate to admit it, the lazy heuristic of setting it as  $\geq 30$  should be enough in most algorithmic studies, and much lower values are probably possible.



# Bibliography

## Required reading

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