“What is VAR and Factor Model?”

Before Presenting
“Measuring the Effects of Monetary Policy: A Factor-Augmented Vector Autoregressive (FAVAR) Approach” by Bernanke et al. (2005)

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What is Vector Autoregression (VAR)?

- A linear multivariate framework that models "dynamic interaction" between variables.
- Each variable is influenced not only by its past values but also by the past values of other variables.
- It is used for analyzing relationships between macroeconomic variables (e.g., GDP, interest rate), impulse response analysis (e.g., a shock such as a shift in monetary policy), and forecasting.

- A simple Keynesian macro model of VAR(2) with two variables:

$$\begin{bmatrix} c_t \\ y_t \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \phi_{11}(1) & \phi_{12}(1) \\ \phi_{21}(1) & \phi_{22}(1) \end{bmatrix} \begin{bmatrix} c_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \phi_{11}(2) & \phi_{12}(2) \\ \phi_{21}(2) & \phi_{22}(2) \end{bmatrix} \begin{bmatrix} c_{t-2} \\ y_{t-2} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

  which can be concisely written as

$$X_t = \mu + \Phi(1)X_{t-1} + \Phi(2)X_{t-2} + \varepsilon_t$$

- Two key assumptions:
  1. ε_t denotes the white noise s.t. $\varepsilon_t \sim (0, \Omega)$
  2. Stationarity Condition: all roots of $|I - \Phi(1)Z - \Phi(2)Z^2|$ are greater than 1 in absolute value

Impulse Response Analysis

- analyzes dynamic impacts of shocks on variables of the model.
- y_t: $(n \times 1)$ vector process following $VAR(p)$.
- Consider the inversion between a stationary VAR nad an $MA(\infty)$ - Moving Average processes.

$$VAR(p): \; y_t = \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \cdots + \Phi_p y_{t-p} + \varepsilon_t$$
$$VMA(\infty): \; y_t = \Psi_0 \varepsilon_t + \Psi_1 \varepsilon_{t-1} + \cdots + \Psi_s \varepsilon_{t-s} + \cdots$$

  - Φ_i: $(n \times n)$ coefficient matrix of the ith lagged values of y_{t-i}
  - Ψ_j: $(n \times n)$ coefficient matrix of the jth lagged shock of ε_{t-j}
  - In this case, $\Psi_0 = I$.

- Response of $y_{t+s,i}$ (the ith component of the s-step ahead value of y_t) to one unit impulse in $\varepsilon_{t,j}$ (the jth component of ε_t), is

$$\Psi_{s,ij} = \frac{\partial y_{t+s,i}}{\partial \varepsilon_{t,j}} = \frac{\partial y_{t,i}}{\partial \varepsilon_{t-s,j}}$$

- We will use this result in Identification.

Identification of Structural Shocks and Impulse Response

Decomposition of ε_t

- We want to decompose ε_t into a set of primitive shocks, each of which is orthogonal to the other.
- Let $\varepsilon_t = Au_t$, where
  - u_t: $(n \times 1)$ vector of primitive shocks
  - A: $(n \times 1)$ matrix having information on structural relations.
- Note that each component of u_t is not correlated with the other.

$$Cov(u_{it}, u_{jt}) = \begin{cases} d_{ii}(\neq 0) & for \;\; i = j \\ 0 & for \;\; i \neq j \end{cases}$$

Identification of Structural Shocks

- Let $Var(\varepsilon_t) = \Omega$.
- Ω can be decomposed as $\Omega = E[\varepsilon_t \varepsilon_t'] = E[A u_t u_t' A'] = ADA'$. ($\Omega$: symmetric and positive definite matrix)
- There are $n(n+1)/2$ number of elements identifiable in A and D. (n^2 for A, n for D, which are symmetric.)

Impulse Responses to Structural Shocks

- Impulse response of y_t to u_{t-s} is as follows:

$$\frac{\partial y_t}{\partial u'_{t-s}} = \frac{\partial y_t}{\partial \varepsilon'_{t-s} A^{-1'}} = \frac{\partial y_t}{\partial \varepsilon'_{t-s}} \cdot A = \Psi_s \cdot A.$$

- v_t: normalized shock s.t. $v_t = D^{-1/2} u_t$, $Var(v_t) = E[v_t v'_t] = I_n$.
- Impulse response of y_t to v_{t-s} is as follows:

$$\frac{\partial y_t}{\partial v'_{t-s}} = \frac{\partial y_t}{\partial \varepsilon'_{t-s} (AD^{1/2})^{-1'}} = \frac{\partial y_t}{\partial \varepsilon'_{t-s}} \cdot AD^{1/2} = \Psi \cdot AD^{1/2}.$$

- For Cholesky decomposition (In special case) $\varepsilon_t = Pv_t$ with $P = AD^{1/2}$, we have

$$\frac{\partial y_t}{\partial v'_{t-s}} = \Psi_s \cdot P$$

- In this case, we can express the impulse response in a simple scalar form:

$$\frac{\partial y_{t,i}}{\partial v'_{t-s,j}} = \Psi^i_s \cdot P_j = \Psi^i_s a_j \sqrt{d_{ii}}$$

What is Factor Model?

- A method for dimensional reduction. Reduces a large number of observed variables into a smaller number of *latent factors.*
  - factor: an underlying unobservable variable that influences more than one observed variable and accounts for the correlations among these observed variables
- To determine the number and nature of *latent factors* that account for the variation and covariation among a set of observed variables. Different variables measure similar movements that is governed by certain common factors.
- Which variables to use? In empirical work, the number of parameters increases very fast with the number of variables.

The Formality of Factor Model

- Let X be a $(n \times 1)$ random vector for the p observables:

$$X = \begin{pmatrix} X_1 & X_2 & \cdots & X_n \end{pmatrix}'$$

- m unobservable common factors: $f_1, \ f_2, \ \cdots, f_m$. In general, we want $m \ll p$.
- We can express $X_i = \mu_i + \lambda_{i1} f_1 + \cdots + \lambda_{im} f_m + \varepsilon_i$.
  - λ_{ij}: "factor loading"
  - ε_i: "specific factor" of X_i
- The above model can be rewritten in matrix form: $X = \Lambda f + \varepsilon$.

Assumptions

- $E[f_i] = 0 \ (i = 1, \cdots p)$, $E[\varepsilon_i] = 0 \ (i = 1, \cdots m)$
- $Var(f_i) = 1, \ i = 1, \cdots, p$, $Var(\varepsilon_i) = \psi_i \ i = 1, \cdots m$. ($\psi_i$ is called the specific variance.)
- $Cov(f_i, f_j) = 0 \ (\forall i \neq j)$, $Cov(\varepsilon_i, \varepsilon_j) = 0 \ (\forall i \neq j)$, $Cov(f_i, \varepsilon_j) = 0$ $(\forall i = 1, \cdots p, \ j = 1, \cdots, m)$
- In matrix notation, Σ (var-cov matrix of X) is written as:

$$\Sigma = \Lambda\Lambda' + \Psi$$

Estimation Methods: Principle Component Analysis (PCA)

- We can get the p number of eigenvalues($\hat{v}_1, \cdots \hat{v}_p$) and eigenvectors ($\hat{e}_1, \cdots \hat{e}_p$) of sample var-cov matrix. $\Sigma = EVE'$ for orthogonal E and Diagonal V.
- $\Sigma = EVE' = \sum_{i=1}^{p} v_i e_i e_i' \approx \sum_{i=1}^{m} v_i e_i e_i' = (\sqrt{\nu_1} e_1 \cdots \sqrt{\nu_m} e_m)(\sqrt{\nu_1} e_1 \cdots \sqrt{\nu_m} e_m)'$
- Using $\Sigma \approx \Lambda\Lambda'$, we have the estimator for the factor loadings: $\hat{\lambda}_{ij} = \sqrt{\hat{\nu}_j}\hat{e}_{ji}$
- To estimate the specific variances in Ψ, we can use the relation that $\Psi = \Sigma - \Lambda\Lambda'$