A Factor-Augmented Vector Autoregressive (FAVAR) Approach

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- AERG -

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Contents

Introduction: Criticisms of the VAR Approach

- **Criticisms of the VAR Approach to Monetary Policy Identification:**
  - Several criticisms of the VAR approach center around its use of limited information, as low-dimensional VARs rely on a small set of variables to explain monetary policy effects.
  - **Price Puzzle**: If the Federal Reserve tightens policy in anticipation of future inflation (which is not well-captured by the data in the VAR), the VAR may misinterpret this as a policy shock, leading to misleading conclusions.

Economic Activity Impulse Responses

- **Representation of Economic Activity:**
  - Another criticism is the potential misrepresentation of "economic activity." For example, industrial production or real GDP might not fully capture the concept of economic activity that policymakers care about.
- **Limitations of Impulse Responses:**
  - Impulse responses are observable only for the variables included in the VAR. However, this is often only a small subset of the variables that researchers and policymakers are interested in.
  - The inclusion of additional variables is restricted by degrees-of-freedom issues, as adding too many variables can overfit the model.

FAVAR: Factor-Augmented VAR

- **Is It Possible to Enrich VAR Analysis with More Information?**
  - The central question is whether it's possible to condition VAR analyses of monetary policy on richer information sets while maintaining the statistical advantages of small models.
  - This paper proposes a solution by combining traditional VAR analysis with factor analysis, an approach known as **FAVAR** (Factor-Augmented VAR).
- **Stock and Watson (2002):**
  - Stock and Watson (2002) show that forecasts based on factors extracted from a larger set of data outperform those from univariate autoregressive models. This suggests that incorporating broader information can improve forecast accuracy.

Estimation Approach

- **Estimation Approach:**
  - The estimation involves:
  - (a) Two steps estimation using **Principal Component Analysis (PCA)**(which is used to extract the factors) and
  - (b) **Bayesian likelihood methods** with **Gibbs sampling** estimating both the factors and their dynamics simultaneously.
  - These methods allow us to estimate the unobservable factors F_t from the observed data series.

Unobservable Factors

- **Unobservable Factors:**
  - The relationship between the factors F_t, observed variables Y_t, and the informational time series X_t is given by:

    $$X_t = \Lambda^f F_t + \Lambda^y Y_t + e_t$$

    where Λ^f and Λ^y are the factor loadings, and e_t is the error term.
  - The key challenge in this approach is that the factors F_t are unobservable, so we rely on a large set of informational time series X_t (e.g., inflation, output, interest rates, etc.) to estimate them.

Economic Variables and Empirical Specification

- **Economic Variables Not Directly Observed:**
  - Important economic variables like potential output and cost-push shocks are not directly observed by the central bank. Instead, the bank uses information from a large number of macroeconomic indicators to infer these unobservable factors.
  - The central bank does not observe the "true" inflation rate directly either, but rather a noisy measure that could be extracted using the factors from the FAVAR model.
- **Empirical Specification:**
  - The preferred empirical specification assumes that only the policy instrument R_t (the nominal interest rate) and a large set of macroeconomic indicators X_t are observed. The factors F_t are estimated using PCA from this large set of information.

Principal Components Approach (PCA)

- **Principal Components Approach (PCA):**
  - **First Step**: The space spanned by the factors $\hat{C}(F_t, Y_t)$ is estimated using the first $K + M$ principal components of the informational time series X_t.
  - **Second Step**: The FAVAR model is estimated by standard methods, with the unobservable factors F_t replaced by the estimated factors $\hat{F}_t$.

Data

- X_t: Background Macroeconomic Time Series (110 Time Series)
- F_t: Latent Unobservable Factors (Slow Moving Principal Components)
- Y_t: Observed Variables (Federal Funds Rate, FFR)
- Time Period: January 1959 – August 2001 ($T = 511$)

Model 1: Relationship Between Information Set, Latent Factors, and Observed Variables

- How the information set (X_t) relates to unobserved factors (F_t) and the observed variable (Y_t):

$$X_t = \Lambda^f F_t + \Lambda^y Y_t + e_t$$

- Matrix Dimensions:

$$(110 \times 1) = (110 \times 3)(3 \times 1) + (110 \times 1)(1 \times 1) + (110 \times 1)$$

Model 2: Vector Autoregression (VAR)

- VAR Representation:

$$\begin{pmatrix} F_t \\ Y_t \end{pmatrix} = \phi(L) \begin{pmatrix} F_t \\ Y_t \end{pmatrix} + \begin{pmatrix} v_{f,t} \\ v_{y,t} \end{pmatrix}$$

- For exampl: VAR(p)

$$\begin{pmatrix} F_t \\ Y_t \end{pmatrix} = A_1 \begin{pmatrix} F_{t-1} \\ Y_{t-1} \end{pmatrix} + \ldots + A_p \begin{pmatrix} F_{t-p} \\ Y_{t-p} \end{pmatrix} + \begin{pmatrix} v_{f,t} \\ v_{y,t} \end{pmatrix}$$

- Identification: Monetary Shock (Structural Shock)
  - Recursive VAR
  - Assumption: Latent factors do not respond to monetary policy innovations within the same month.

Estimation: Overview

- The model assumes the Fed's policy instrument R_t is observable, while other variables such as output and inflation are unobservable.
- R_t is the only observable variable included in the vector of observable variables Y_t.
- The monetary policy shock is identified using the standard recursive method by ordering the federal funds rate last and treating its innovations as policy shocks.
- The recursive ordering implies that unobserved factors do not respond to monetary policy innovations within the same period (month).

Two-Step Estimation Approach (Step 1)

- In the first step, we rely on Principal Component Analysis (PCA) when N (the number of time series) is large.
- PCA consistently recovers $K + M$ independent but arbitrary linear combinations of the unobservable factors F_t and the observed variables Y_t.
- Since Y_t is not explicitly imposed as a common component, linear combinations from $\hat{C}(F_t, Y_t)$ could include R_t, the Fed's policy instrument.
- Therefore, we cannot simply estimate a VAR in $\hat{C}(F_t, Y_t)$ and Y_t and identify the policy shock recursively.

Step 2: Removing Direct Dependence on R_t

- To address the issue, the direct dependence of $\hat{C}(F_t, Y_t)$ on R_t must first be removed.
- Since the linear combinations in $\hat{C}(F_t, Y_t)$ are unknown, we estimate their coefficients through multiple regression:

$$\hat{C}(F_t, Y_t) = b_C^* \hat{C}^*(F_t) + b_R R_t + e_t$$

- Here, $\hat{C}^*(F_t)$ represents the estimate of all common components other than R_t.

Step 3: Estimating $\hat{C}^*(F_t)$ and $\hat{F}_t$

- $\hat{C}^*(F_t)$ can be obtained by extracting principal components from a subset of slow-moving variables that are assumed not to be affected contemporaneously by R_t.
- Once we obtain $\hat{C}^*(F_t)$, the unobservable factors $\hat{F}_t$ are constructed by:

$$\hat{F}_t = \hat{C}(F_t, Y_t) - \hat{b}_R R_t$$

- A VAR model is then estimated with $\hat{F}_t$ and Y_t, and identified recursively with R_t ordered last.

How to Extend Principal Components (F_t)

- Step 1: Select "slow-moving" variables from X_t (65 series).
- Step 2: Extract 3 principal components using the 65 slow-moving variables to obtain factor components($\hat{C}^*(F_t, Y_t)$)(PCA)
- Step 3: Estimate:

$$\hat{C}(F_t, Y_t) = b_{C^*}\hat{C}^*(F_t, Y_t) + b_R R_t + e_t$$

- Step4: Back out latent factors:

$$\hat{F}_t = \hat{C}(F_t, Y_t) - \hat{b}_R R_t$$

- Step5: VAR in $\hat{F}_t$ and Y_t is estimated and identified recursively, with R_t ordered last.

Estimate VAR

- VAR Model:

$$\begin{bmatrix} F_t \\ Y_t \end{bmatrix} = \phi(L) \begin{bmatrix} F_t \\ Y_t \end{bmatrix} + \begin{bmatrix} v_{f,t} \\ v_{y,t} \end{bmatrix}$$

- Lag length: 13
- Expanded to:

$$\begin{bmatrix} PC_{1,t} \\ PC_{2,t} \\ PC_{3,t} \\ FFR_t \end{bmatrix} = \phi(L) \begin{bmatrix} PC_{1,t} \\ PC_{2,t} \\ PC_{3,t} \\ FFR_t \end{bmatrix} + \begin{bmatrix} v_{1,t} \\ v_{2,t} \\ v_{3,t} \\ v_{4,t} \end{bmatrix}$$

Impulse Response Function (Monetary Shock)

- VAR Model:

$$(F_t; Y_t) = \phi(L)(F_t; Y_t) + (v_{f,t}; v_{y,t})$$

- **Identification: Recursive VAR**

$$\epsilon_t = Av_t$$

- Structural Shock Representation:

$$\begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \\ \epsilon_{4,t} \end{bmatrix} = \begin{bmatrix} \phi_{11} & 0 & 0 & 0 \\ \phi_{21} & \phi_{22} & 0 & 0 \\ \phi_{31} & \phi_{32} & \phi_{33} & 0 \\ \phi_{41} & \phi_{42} & \phi_{43} & \phi_{44} \end{bmatrix} \begin{bmatrix} v_{1,t} \\ v_{2,t} \\ v_{3,t} \\ v_{4,t} \end{bmatrix}$$

- ϵ: Structural shocks with unit variance.
- A 1-unit ϵ_{jt} shock corresponds to a 1 standard deviation shock.
- BBE shocks need further scaling (0.25 standard deviation of FFR).

Transmission Channel

- Monetary shock (ϵ_4) affects the system:

$$\epsilon_4 \rightarrow \{PC_1, PC_2, PC_3, Y\} \rightarrow X$$

- The system is modeled in two parts:
  1. **Model 2**: The dynamic relationship among the variables:

$$\begin{pmatrix} PC_{1,t} \\ PC_{2,t} \\ PC_{3,t} \\ FFR_t \end{pmatrix} = \phi(L) \begin{pmatrix} PC_{1,t} \\ PC_{2,t} \\ PC_{3,t} \\ FFR_t \end{pmatrix} + \begin{pmatrix} \phi_{11} & 0 & 0 & 0 \\ \phi_{21} & \phi_{22} & 0 & 0 \\ \phi_{31} & \phi_{32} & \phi_{33} & 0 \\ \phi_{41} & \phi_{42} & \phi_{43} & \phi_{44} \end{pmatrix}^{-1} \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \\ \epsilon_{4,t} \end{pmatrix}$$

  2. **Model 1**: The relationship between the variables and the external factors:

$$X_t = \Lambda^f F_t + \Lambda^r Y_t$$

$$\begin{pmatrix} X_{1,t} \\ \vdots \\ X_{110,t} \end{pmatrix} = \begin{pmatrix} \Lambda^f_{1,1} & \Lambda^f_{1,2} & \Lambda^f_{1,3} \\ \vdots & \vdots & \vdots \\ \Lambda^f_{110,1} & \Lambda^f_{110,2} & \Lambda^f_{110,3} \end{pmatrix} \begin{pmatrix} PC_{1,t} \\ PC_{2,t} \\ PC_{3,t} \end{pmatrix} + \begin{pmatrix} \Lambda^y_1 \\ \vdots \\ \Lambda^y_{110} \end{pmatrix} Y_t$$

Impact of $\epsilon_{4,t}$ on $X_{20,t+i}$ (i-step ahead)

1. **Step 1**: A shock to $\epsilon_{4,t}$:
$$\Delta\epsilon_{4,t} = 1$$
2. **Step 2**: Impact on the variables $PC_{1,t+i}, PC_{2,t+i}, PC_{3,t+i}, Y_{t+i}$:
$$X_{20,t+i} = \Lambda^f_{20,1} PC_{1,t+i} + \Lambda^f_{20,2} PC_{2,t+i} + \Lambda^f_{20,3} PC_{3,t+i} + \Lambda^y_{20} Y_{t+i}$$
3. Report the results for each step ahead.
4. **Bootstrap procedure**: Resample 500 times to compute confidence intervals:
    - Resample for each t
    - Calculate 0.95, 0.05, and 0.5 quantiles (90% confidence interval).

Forecast Error Variance Decomposition (FEVD)

- Recall the model:

$$X_t = \Lambda F_t + \Lambda^y Y_t + e_t \quad \Rightarrow \quad X_t = \Lambda G_t + e_t, \quad G_t = \begin{pmatrix} F_t \\ Y_t \end{pmatrix}$$

- Dynamics of G_t:

$$\begin{pmatrix} F_t \\ Y_t \end{pmatrix} = \phi(L) \begin{pmatrix} F_t \\ Y_t \end{pmatrix} + \begin{pmatrix} v_{F,t} \\ v_{Y,t} \end{pmatrix}$$

$$G_t = \sum_{i=1}^{13} \phi_i G_{t-i} + v_i$$

Example: 1-Step Ahead FEVD

- The model:

$$X_t = \Lambda G_t + e_t$$

- The 1-step ahead forecast error is:

$$X_t - E[X_t|I_{t-1}] = \Lambda G_t + e_t - E[\Lambda G_t + e_t|I_{t-1}]$$

$$= \Lambda G_t + e_t - \Lambda E[G_t|I_{t-1}]$$

$$= \Lambda \left(G_t - E[G_t|I_{t-1}] \right) + e_t$$

- $G_t - E[G_t|I_{t-1}]$: 1-step ahead forecast error of G_t
- Dynamics of G_t:

$$G_t = \sum_{i=1}^{13} \phi_i G_{t-i} + v_t \quad \text{(VAR(13), assuming stationarity)}$$

- The VMA(∞) representation:

$$G_t = v_t + \phi v_{t-1} + \phi^2 v_{t-2} + \cdots$$

Example: 1-Step Ahead FEVD

- The forecast error:

$$G_t - E[G_t|I_{t-1}] = (v_t + \phi v_{t-1} + \phi^2 v_{t-2} + \cdots) - E[v_t + \phi v_{t-1} + \cdots | I_{t-1}]$$

- Since $E[v_t|I_{t-1}] = 0$, we get:

$$G_t - E[G_t|I_{t-1}] = v_t = M_0 \epsilon_t$$

- Therefore:

$$\Lambda (G_t - E[G_t|I_{t-1}]) + e_t = \Lambda_{110\times 4} v_{t4\times 1} + e_{t110\times 1}$$

$$= \Lambda M_0 \epsilon_t + e_t$$

- Final result:

$$= \psi(0)\epsilon_t + e_t$$

Example: 1-Step Ahead FEVD (Continued)

- The variance of the forecast error is given by:

$$\text{Var}[\psi(0)\epsilon_t + e_t] = \psi(0)_{110\times 4}\psi(0)^T + \text{Var}(e_t)$$

- Expanding the matrices:

$$= \begin{pmatrix} \psi_{1,1}(0) & \psi_{1,2}(0) & \psi_{1,3}(0) & \psi_{1,4}(0) \\ \vdots & \vdots & \vdots & \vdots \\ \psi_{110,1}(0) & \psi_{110,2}(0) & \psi_{110,3}(0) & \psi_{110,4}(0) \end{pmatrix} \psi(0)^T$$

$$+ \begin{pmatrix} \text{Var}(e_{1,t}) & \rho_{1,2} & \cdots & \rho_{1,110} \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{110,1} & \cdots & \cdots & \text{Var}(e_{110,t}) \end{pmatrix}$$

- The FEVD of a monetary shock (ϵ_{it}) on X_1 is:

$$\frac{\psi_{1,4}^2(0)}{\psi_{1,1}^2(0) + \psi_{1,2}^2(0) + \psi_{1,3}^2(0) + \psi_{1,4}^2(0) + var(e_{1,t})}$$

Connection Between FEVD and IRF

- The relationship between X_t and G_t:

$$X_t = \Lambda G_t + e_t \tag{1}$$

$$G_t = \sum_{i=1}^{13} \phi_i G_{t-i} + v_t \tag{2}$$

- Expanding equation (2):

$$G_t = v_t + \phi v_{t-1} + \phi^2 v_{t-2} + \cdots$$

$$G_t = M_0 \epsilon_t + \phi M_1 \epsilon_{t-1} + \phi^2 M_2 \epsilon_{t-2} + \cdots$$

- Substituting G_t into equation (1):

$$X_t = \Lambda M_0 \epsilon_t + \Lambda \phi M_1 \epsilon_{t-1} + \Lambda \phi^2 M_2 \epsilon_{t-2} + \cdots + e_t$$

$$X_t = \psi(0) \epsilon_t + \psi(1) \epsilon_{t-1} + \psi(2) \epsilon_{t-2} + \cdots + e_t$$

Connection Between FEVD and IRF (Continued)

- Expanding the system for multiple time steps:

$$\begin{pmatrix} X_{1,t} \\ \vdots \\ X_{110,t} \end{pmatrix}_{110\times 1} = \begin{pmatrix} \psi_{1,1}(0) & \psi_{1,2}(0) & \psi_{1,3}(0) & \psi_{1,4}(0) \\ \vdots & \vdots & \vdots & \vdots \\ \psi_{110,1}(0) & \psi_{110,2}(0) & \psi_{110,3}(0) & \psi_{110,4}(0) \end{pmatrix} \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \\ \epsilon_{4,t} \end{pmatrix} +$$

$$\begin{pmatrix} \psi_{1,1}(1) & \psi_{1,2}(1) & \psi_{1,3}(1) & \psi_{1,4}(1) \\ \vdots & \vdots & \vdots & \vdots \\ \psi_{110,1}(1) & \psi_{110,2}(1) & \psi_{110,3}(1) & \psi_{110,4}(1) \end{pmatrix} \begin{pmatrix} \epsilon_{1,t-1} \\ \epsilon_{2,t-1} \\ \epsilon_{3,t-1} \\ \epsilon_{4,t-1} \end{pmatrix} + \cdots$$

- The impulse response of $\epsilon_{4,t}$ on $X_{1,t}$ is:

$$\psi_{1,4}(0)$$

- The impulse response of $\epsilon_{4,t}$ on $X_{1,t+i}$ is:

$$\psi_{1,4}(i)$$

Results Overview

In this section, we present the following results comparing the standard VAR model with the FAVAR approach:

- **Figure 1:** Comparison of results from a standard three-variable VAR model with FAVAR.
- **Figure 2:** Impulse responses corresponding to a 25-basis-point innovation in the Federal Funds Rate (FFR).
- **Figure 3:** Forecast Error Variance Decomposition showing the impact of different shocks.
- **Figure 4:** Comparison of Mean Squared Error (MSE) between a standard forecasting model and FAVAR.

Figure 1: Impulse Response Functions - Two-Step Estimation

- The figure1 displays the impulse response functions obtained from the two-step estimation.
- A strong price puzzle is present in the VAR specification, with the response of industrial production being very persistent.
- Adding one factor to the standard VAR dramatically changes the responses, reducing the price puzzle and bringing industrial production back toward zero in the long run.
- This suggests that the additional factors in X_t provide useful information beyond that contained in the standard VAR.

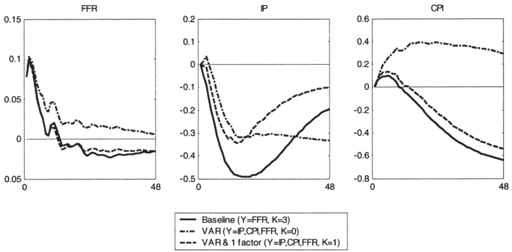

Figure 2: Impulse Response Functions with 90% Confidence Intervals

- Figure2 shows the impulse responses with 90% confidence intervals of key macroeconomic variables to a contractionary monetary policy shock.
- After the shock, real activity measures decline, prices eventually decrease, money aggregates decline, and the dollar appreciates.
- These responses are constructed for any variable in the informational data set, X_t.

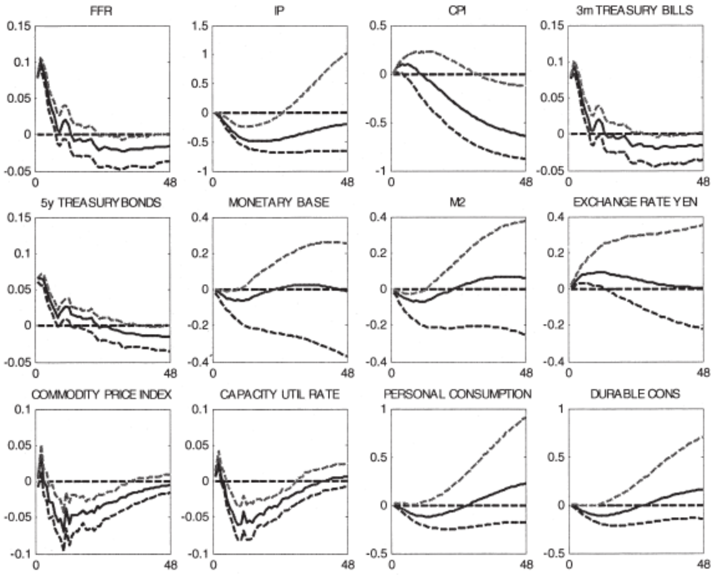

Figure: Impulse Response Functions with 90% Confidence Intervals

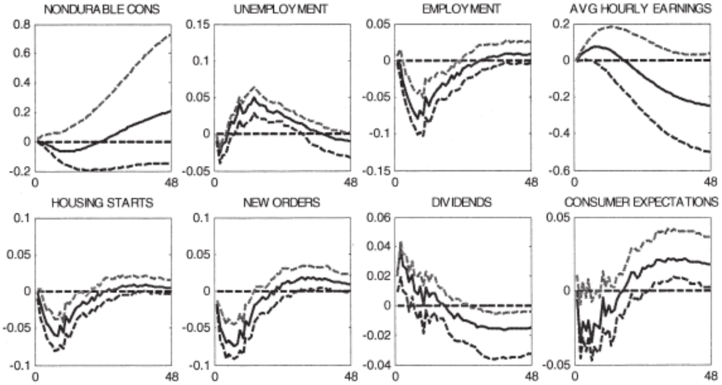

FIGURE II
Impulse Responses Generated from FAVAR with Three Factors and FFR Estimated by Principal Components with Two-Step Bootstrap

Figure: Impulse Response Functions with 90% Confidence Intervals

Figure 3: Variance Decomposition in the VAR Context

- Figure3 shows the variance decomposition in the VAR context.
- Variance decomposition determines the fraction of the forecast error of a variable attributable to a particular shock.
- A standard result is that the monetary policy shock explains a small fraction (0 to 10.3%) of the forecast error of real activity measures or inflation.
- The R^2 for the common components is particularly low for money aggregates, with values of 10.3% for the monetary base and 5.2% for M2.
- This suggests lower confidence in the impulse response estimates for money aggregates.

TABLE I

CONTRIBUTION OF THE POLICY SHOCK TO VARIANCE OF THE COMMON COMPONENT

Variables	Variance decomposition	R^2
Federal funds rate	0.454	*1.000
Industrial production	0.054	0.707
Consumer price index	0.038	0.870
3-month treasury bill	0.433	0.975
5-year bond	0.403	0.925
Monetary base	0.005	0.104
M2	0.005	0.052
Exchange rate (Yen/$)	0.007	0.025
Commodity price index	0.049	0.652
Capacity utilization	0.100	0.753
Personal consumption	0.006	0.108
Durable consumption	0.005	0.062
Nondurable cons.	0.002	0.062
Unemployment	0.103	0.817
Employment	0.066	0.707
Aver. hourly earnings	0.007	0.072
Housing starts	0.032	0.387
New orders	0.081	0.624
S&P dividend yield	0.062	0.549
Consumer expectations	0.036	0.700

The column titled Variance decomposition reports the fraction of the variance of the forecast error, at the 60-month horizon, explained by the policy shock. R^2 refers to the fraction of the variance of the variable explained by the common factors, $(\hat{F}_t', Y_t')$. See text for details.

* This is by construction.

Figure: Variance Decomposition in the VAR Context

Figure 4: Forecasting Performance Using FAVAR

- Figure4 compares forecasting performance using FAVAR to traditional forecasting methods.(Stock & Watson (2002))
- The Mean Squared Error (MSE) is reduced when FAVAR is used, relative to other traditional forecasting approaches.

Table 2. Simulated Out-of-Sample Forecasting Results Industrial Production, 12-Month Horizon

Forecast method	*Relative MSE*
Univariate autoregression	1.00
Vector autogression	.97
Leading indicators	.86
Principal components	.58
Principal components, $k = 1$	.94
Principal components, $k = 2$	.62
Principal components, $k = 3$	.55
Principal components, $k = 4$	.56
Principal components, AR	.69
Root MSE, AR model	.049

NOTE: For each forecast method, this table shows the ratio of the MSE of the forecast made by the method for that row to the MSE of a univariate autoregressive forecast with lag length selected by the BIC. The final line presents the root MSE for the autoregressive model in native (decimal growth rate) units at an annual rate.

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