Marmara University
Faculty of Engineering

TRAJECTORY CONTROL OF REMUS100 AUV USING UDWADIA-KALABA CONSTRAINT EQUATIONS

BERKAN CANDAN

GRADUATION PROJECT REPORT
Department of Mechanical Engineering

Supervisor
Associate Professor Mehmet Berke GÜR

ISTANBUL, 2026

Marmara University

Faculty of Engineering

Developing Motion Controller for a Torpedo Type Autonomous Underwater Robot

by

Berkan Candan

February 12 , 2026, Istanbul

SUBMITTED TO THE DEPARTMENT OF MECHANICAL ENGINEERING IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF

BACHELOR OF SCIENCE
AT

MARMARA UNIVERSITY

Signature of Author(s) …..
Department of Mechanical Engineering

Certified By ....
Project Supervisor, Department of Mechanical Engineering

Accepted By ....
Head of the Department of Mechanical Engineering

ACKNOWLEDGEMENT

I would like to thank my supervisor Associate Professor Mehmet Berke Gür for the valuable guidance and advice on preparing this thesis and giving me moral and material support. Even though I hesitated at some certain stages of the thesis, he never gave up relying on me and helping me through many theoretical and applicational details. It was a privilege for me to study with him not only in terms of academic studies but also to have a grasp of the vision about life he has.

February, 2026 **Berkan Candan**

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ABBREVIATIONS

AUV	: Autonomous Underwater Vehicle
UK	: Udwadia – Kalaba Approach
NDI	: Non-linear Dynamic Inversion
MPC	: Model Predictive Control
RL	: Reinforcement Learning
LOS	: Line of Sight

ABSTRACT

Developing Motion Controller for REMUS100 AUV Using Udwadia-Kalaba Approach

Autonomous underwater vehicles (AUVs) are increasingly used nowadays for operations like long-range oceanographic surveys, oil & gas exploration, security & defensive, marine data collection and so on. Improving the maneuvering precision and energy efficiency of such vehicles requires accurate dynamic modeling to support systematic control system design. However, the highly nonlinear hydrodynamic characteristics of underwater vehicles and the uncertainties in environmental disturbances make modeling and control particularly challenging. Therefore, advanced modeling approaches are essential to achieve reliable trajectory tracking and robust performance across different operating conditions.

This study covers a Udwadia-Kalaba dynamic approach where the REMUS100 AUV is controlled to follow a user defined path. The natural dynamics of the vehicle is firstly accepted as its unconstrained motion, then several modeling constraints based on the behavior and properties of the vehicle is added on this unconstrained motion to make the system behave accordingly to the proper physical rules. Then to make the vehicle track the desired trajectory, control constraints are added on the system so that the system becomes controlled system. The important point in Udwadia Kalaba Approach is that, this method treats constraints as forces acting on the system at all times so that system behaves accordingly to the physical rules and tracks the trajectory.

ÖZET

REMUS100 Otonom Sualtı Aracı için Udwadia-Kalaba Yaklaşımı Kullanarak Hareket Kontrolcüsü Geliştirmesi

Otonom sualtı araçları (AUV'ler), günümüzde uzun menzilli okyanus araştırmaları, petrol ve gaz aramaları, güvenlik ve savunma uygulamaları, denizcilik verileri toplama gibi birçok operasyonda giderek daha yaygın olarak kullanılmaktadır. Bu tür araçların manevra hassasiyetinin ve enerji verimliliğinin artırılması, sistematik kontrol sistemi tasarımını destekleyecek doğru dinamik modellemeleri gerektirmektedir. Ancak sualtı araçlarının son derece doğrusal olmayan hidrodinamik karakteristikleri ve çevresel bozuculara bağlı belirsizlikler, modelleme ve kontrol süreçlerini oldukça zorlaştırmaktadır. Bu nedenle, farklı çalışma koşulları altında güvenilir yörünge takibi ve sağlam performans elde edebilmek için gelişmiş modelleme yaklaşımlarına ihtiyaç duyulmaktadır.

Bu çalışmada, REMUS100 AUV'nin kullanıcı tarafından tanımlanan bir yörüngeyi takip etmesini sağlayan Udwadia–Kalaba dinamik yaklaşımı ele alınmaktadır. İlk olarak aracın serbest dinamikleri onun kısıtsız sistemi olarak belirlenmiş, ardından aracın davranışı ve fiziksel özellikleri dikkate alınarak çeşitli modelleme kısıtları bu kısıtsız harekete eklenmiştir. Böylece sistemin, uygun fiziksel kurallara göre davranması sağlanmıştır. Daha sonra, aracın istenen yörüngeyi takip edebilmesi için sisteme kontrol kısıtları eklenerek sistem kontrollü bir yapıya dönüştürülmüştür. Udwadia–Kalaba yaklaşımının en önemli özelliği, kısıtların sistem üzerinde her an etki eden kuvvetler olarak ele alınmasıdır; bu sayede sistem hem fiziksel kurallara uygun şekilde hareket etmekte hem de istenen yörüngeyi uygun olarak takip etmektedir.

LIST OF FIGURES

1. INTRODUCTION

Approximately three out of four of the Earth's surface is covered with water, and most of this surface water consists of ocean water. Despite their vastness, nearly 95 percent of the world's oceans remain unexplored. These unexplored regions are believed to contain rich ecosystems and significant untapped resources, including new sources of food, renewable energy, and valuable materials. However, exploring these areas is extremely challenging due to the harsh and unpredictable conditions of the underwater environment. [1][2]

In recent years, autonomous underwater vehicles (AUVs) have emerged as a key technology for underwater exploration. These robots can navigate and perform complex missions without human intervention, making them ideal for tasks in dangerous or inaccessible areas. AUVs are used for applications such as 3D seafloor imaging, geomorphologic mapping, rapid environmental assessments, underwater pipeline inspections, and various military operations. Their ability to operate autonomously makes them a critical tool for advancing our understanding of underwater ecosystems and developing new resources.[3][4][5][6]

However, the underwater environment presents unique challenges that make controlling AUVs difficult. These vehicles are nonlinear mechanical systems affected by complex interactions between hydrodynamic forces, external disturbances like currents and waves, and internal dynamics such as actuator limitations. Additionally, the lack of GPS-like positioning systems underwater forces AUVs to rely on dead-reckoning and inertial navigation, which are prone to errors over long distances. Developing reliable motion control systems that can handle these challenges is essential for ensuring that AUVs can perform their missions successfully.

This project focuses on designing advanced motion controllers for a torpedo-type AUV, specifically the REMUS100 model. By implementing and testing control strategy Udwadia-Kalaba (UK) approach, the aim is to address the challenges posed by nonlinear dynamics and external disturbances. The performance of these controllers will be evaluated using simulations, with the goal of improving the guidance and stability of AUVs in complex underwater environments.

2. LITERATURE REVIEW

2.1. Challenges in AUV Dynamics and Control

AUVs play an important role in many underwater tasks, but their operation is made difficult by the nonlinear nature of their dynamics and the unpredictable underwater environment. AUVs are typically underactuated systems, meaning they have more degrees of freedom than the number of independent control inputs. For example, while AUVs can move in six degrees of freedom—surge, sway, heave, roll, pitch, and yaw—most designs only have control inputs for three or four of these motions. This under actuation, combined with the effects of ocean currents, waves, and sensor noise, makes precise control challenging. Accurate mathematical models of AUV dynamics are critical for designing effective control strategies, but these models often struggle to capture all the complexities of real-world conditions [7] [8].

2.2. Udwadia-Kalaba Approach

The Udwadia-Kalaba (UK) method is a more recent approach that offers a unified framework for modeling and controlling complex mechanical systems. Instead of relying on linearization or approximations, this method treats control objectives as constraints on the system's behavior. By solving these constraints explicitly, the UK approach calculates the required control forces in closed form, ensuring that both system dynamics and control objectives are satisfied. This makes it particularly useful for underactuated systems like AUVs, where traditional control methods often struggle to handle inconsistent constraints.

The UK approach is highly flexible and can adapt to situations where the constraints on the system change dynamically. For AUVs, it is well-suited for tasks such as trajectory tracking and path following, especially in environments with strong currents or other disturbances. Although the method is computationally intensive, its ability to handle complex constraints and nonlinear dynamics makes it a promising option for AUV control [10].

2.3. Nonlinear Dynamic Inversion (NDI)

Nonlinear Dynamic Inversion (NDI) is a well-established control method that simplifies the control of nonlinear systems by transforming their dynamics into a linear form. This is achieved through feedback linearization, which cancels out the nonlinear terms in the system's equations. NDI has been successfully applied in various fields, including aerospace and robotics, where it is used for tasks such as trajectory tracking and disturbance rejection. In the context of AUVs, NDI can improve stability and responsiveness, allowing the vehicle to follow predefined paths more accurately.

However, NDI has some limitations. Its performance depends heavily on the accuracy of the dynamic model, and it is sensitive to changes in parameters such as hydrodynamic coefficients and environmental conditions. These issues can reduce its robustness, particularly in environments with high levels of uncertainty [9]. Despite these challenges, NDI remains a popular choice for controlling nonlinear systems due to its effectiveness in a wide range of applications.

2.4.Path Following and Guidance Systems

Accurate path following is essential for AUVs to complete their missions successfully. Over the years, various control strategies have been developed to improve the robustness and accuracy of AUV guidance systems. Model Predictive Control (MPC), for example, uses optimization techniques to calculate control inputs that minimize errors while respecting system constraints. Sliding mode control is another popular approach, known for its robustness against disturbances and modeling inaccuracies. Heuristic methods, such as fuzzy logic controllers, have also been used to adapt control strategies dynamically based on environmental conditions [11] [12].

Each of these methods has its advantages and limitations. While MPC offers high accuracy, it requires significant computational resources, which can be a limitation for real-time applications. Sliding mode control is robust but can suffer from chattering, which may affect the longevity of the actuators. Fuzzy logic controllers are flexible but depend on expert knowledge to define the rules. These trade-offs highlight the need for continued research to identify the best control strategies for specific AUV applications.

2.5.Line of Sight and Orientation Control

It is an important aspect of all autonomous motions to move along the defined path, which is governed by the vehicle's dynamics and its orientation with the desired endpoint of the trajectory. To combine a relationship related to vehicle's orientation, the methodology of line of sight is used. This technique adjusts the vehicle's head to the desired target continuously with the help of an appropriate controller.

In [14] the geometric aspects of the line-of-sight methodology is provided for marine crafts and all the formulations are detailly provided. In terms of REMUS100 AUV, [15] provides more specific results by applying this methodology on the vehicle and most importantly, combining this technique with a PID algorithm to adjust the head of the vehicle with a discrete step size.

2.6.Applications of AUVs

The REMUS100 AUV, a torpedo-shaped vehicle, has been widely used in both research and operational contexts. Its compact size and modular design make it suitable for a variety of tasks, from mine countermeasures to scientific surveys and underwater inspections [13]. Despite its capabilities, further improvements in its control systems are needed to enhance its performance in challenging environments.

3. DESCRIPTION OF THE MODEL

When we consider the motion of the AUVs, the approach must be based on two subjects; the kinematics which focuses on the positions and velocities the system has (also these kinematic variables are states of the REMUS100 AUV as will be explained later on) and the kinetics which focuses on the forces and moments acting on the system, including actuator forces and natural dynamic forces of the system.

Besides the positions, velocities and forces; there references that these terms are defined on are also important in the subject of AUVs, where two different coordinate systems are used; the inertial frame of coordinates also called as earth frame, where the reference point is taken to be some fixed point and the motion of the AUV is considered relative to this fixed point. In general, this fixed point is taken to be the initial point where the AUV starts its motion or the docked position. The positions and orientations of the AUV are defined according to this frame.

The other coordinate system is the body fixed coordinate system, which the origin is taken as the center of buoyancy of the AUV. As the AUV continues its motion, the origin and the coordinate system also moves with the AUV. This coordinate system eventually has no position terms due to this condition, but the velocities and the forces/moments usually defined relative to this reference.

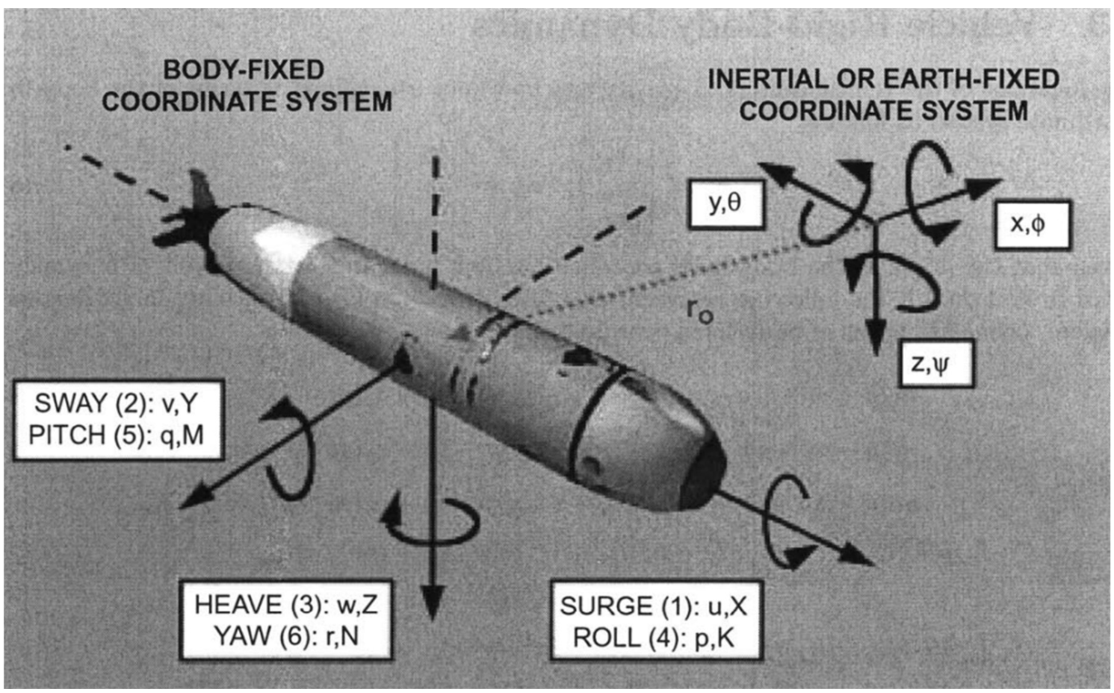

Figure 1: Coordinate Axes and Velocities of REMUS100 AUV

https://www.researchgate.net/publication/37913361_Verification_of_a_six-degree_of_freedom_simulation_model_for_the_REMUS_autonomous_underwater_vehicle

The figure 1 above shows the coordinate systems and the variables that these coordinate systems have. The definition and relations of these variables will be explained later. When the REMUS100 AUV itself is considered, it can be observed that there are three actuators (inputs) that enables AUV to perform its motion. It has a propeller that rotates so that AUV has a force applied along the direction of u (surge) and a moment applied along p (roll). n in the figure is given in terms of revolution per minutes (RPM) and will be converted into radians per second to be used in the equations.

On the other hand, REMUS100 AUV has two fins that applies forces and moments along other axes. The first fin is the stern, demonstrated as δ_s, and the other fin is the rudder, demonstrated as δ_r. These angles are defined in terms of degrees but will be converted into radians to be used in equations.

The input vector can be shown as;

$$u_n = [n \quad \delta_n \quad \delta_r]^T \tag{1}$$

As the AUV moves, the change of these inputs determine the trajectory that it follows. The important point here is that REMUS100 AUV has a motion in six degrees of freedom, which are three forces and three moments, as it can be inspected in figure 1 on the body fixed coordinate frame. However, there are only three actuators, which means there is no actuation mechanism for some degrees of freedom. Which makes the system underactuated.

3.1.Kinematics of The Vehicle

The kinematics of REMUS100 AUV inspects the velocities, positions and the orientations the vehicle has. As it was explained previously, some of these variables lay on inertial frame of coordinates while the others are defined on body-fixed coordinates. The displacements and the orientations that the vehicle has are written with respect to the inertial frame of reference. These values can be defined in vector form as follows:

$$\eta_1 = [x \quad y \quad z]^T \, and \, \eta_2 = [\phi \quad \theta \quad \psi]^T \tag{2}$$

The angles given as the orientations are also called the Euler angles. For convenience in latter applications, all these values will be explained in a single vector as follows:

$$\eta = [x \quad y \quad z \quad \phi \quad \theta \quad \psi]^T \tag{3}$$

Similarly, the body fixed coordinate system has linear and linear and angular velocities that are defined on it. These variables can also be defined in vector form as follows:

$$\nu_1 = [u \quad v \quad w]^T \, and \, \nu_2 = [p \quad q \quad r]^T \tag{4}$$

Again, for computational convenience, these values are written in a single vector.

$$\nu = [u \quad v \quad w \quad p \quad q \quad r]^T \tag{5}$$

These displacement and velocity variables form up the state vector of the REMUS100 AUV.

$$\mathcal{X} = [u \quad v \quad w \quad p \quad q \quad r \quad x \quad y \quad z \quad \phi \quad \theta \quad \psi]^T \tag{6}$$

There is a quite important point in here, one may conclude that the derivatives of displacements give the same values with body fixed velocities. However, that is not true since the vehicle changes its orientation continuously during the motion. Therefore, a coordinate transformation matrix is eventually used to make a conversion between derivatives of the states given in the inertial frame and the body fixed velocities.

This transformation matrix is dependent on the Euler angles, as these angles define the orientation of the vehicles. There are two transformation matrices, the first matrix enables a transformation between the derivatives of displacements and the body fixed translational velocities. This relation is given as:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = J_1(\eta_2) \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{7}$$

The transformation matrix here is given as:

$$J_1(\eta_2) = \begin{bmatrix} cos\psi \cdot cos\theta & -sin\psi \cdot cos\phi + cos\psi \cdot sin\theta \cdot sin\phi & sin\psi \cdot sin\phi + cos\psi \cdot sin\theta \cdot cos\phi \\ sin\psi \cdot cos\theta & cos\psi \cdot cos\phi + sin\psi \cdot sin\theta \cdot sin\phi & -cos\psi \cdot sin\phi + sin\psi \cdot sin\theta \cdot cos\phi \\ -sin\theta & cos\theta \cdot sin\phi & cos\theta \cdot cos\phi \end{bmatrix} \tag{8}$$

The transformation matrix given above is also an orthogonal matrix, which means the inverse and transpose of it are the same.

$$\left(J_1(\eta_2)\right)^{-1} = \left(J_1(\eta_2)\right)^T \tag{9}$$

This is an important relation, as we will require take the inverse of the transformation matrix to make a conversion from inertial frame to body frame in the control problem of the thesis. This property of the former matrix makes it much easier for such applications.

The second matrix provides a conversion between the derivatives of orientations in the inertial frame and the angular velocities in the body fixed coordinate system. The relation is given as:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = J_1(\eta_2) \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{10}$$

The second transformation matrix is:

$$J_2(\eta_2) = \begin{bmatrix} 1 & sin\phi \cdot tan\theta & cos\phi \cdot tan\theta \\ 0 & cos\phi & -sin\phi \\ 0 & sin\phi / cos\theta & cos\phi / cos\theta \end{bmatrix} \tag{11}$$

The inverse of this matrix is also necessary for the control problem and can be written as:

$$J_2^{\ -1}(\eta_2) = \begin{bmatrix} 1 & 0 & -sin\theta \\ 0 & cos\phi & cos\theta \cdot sin\phi \\ 0 & -sin\theta & cos\theta \cdot cos\phi \end{bmatrix} \tag{12}$$

At this point, a single relation can be written by using transformation matrices diagonally so that we could have a relation that transforms all states in body fixed coordinates to earth frame or vice versa. This relation will again prove to be useful in further sections.

$$J = \begin{bmatrix} J_1(\eta_2) & 0 \\ 0 & J_2(\eta_2) \end{bmatrix} \; and \; J^{-1} = \begin{bmatrix} J_1^{-1}(\eta_2) & 0 \\ 0 & J_2^{-1}(\eta_2) \end{bmatrix} \tag{13}$$

This relation allows us to perform:

$$\dot{\eta} = J \cdot \nu \text{ or } \nu = J^{-1} \cdot \dot{\eta} \tag{14}$$

Here, $\dot{\eta}$ is the derivative of the states in the inertial frame.

3.2.Vehicle Kinetics

Kinetics of the vehicle deals with forces and moments that are exerted on it, and these forces eventually lay on body fixed coordinate system. Inspecting the figure, the forces and moments can be written in vector form as follows:

$$\tau_1 = [X \quad Y \quad Z]^T \text{ and } \tau_2 = [K \quad M \quad N]^T \tag{15}$$

These values can be written in a singular vector:

$$\tau = [X \quad Y \quad Z \quad K \quad M \quad N]^T \tag{16}$$

There are two types of forces in kinetics which are dynamic forces and mechanical forces which their addition forms up the former vector.

3.2.1. Vehicle Rigid Body Dynamics

To write the equations for the rigid body dynamics of the REMUS100 AUV, the vehicle's gravity and buoyancy centers must be defined in vector form as follows:

$$r_g = \begin{bmatrix} x_g \\ y_g \\ z_g \end{bmatrix} \text{ and } r_b = \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} \tag{17}$$

Also, all the terms in inertia tensor are approximated as zero except the diagonal terms, since the body fixed coordinate system is centered at the buoyancy. Then the inertia tensor is given as:

$$I_O = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \tag{18}$$

Then the six degrees of freedom equations of motion is:

$$\begin{gathered}
m\left[\dot{u} - vr + wq - x_g(q^2 + r^2) + y_g(pq - \dot{r}) + z_g(pr + \dot{q})\right] = \sum X_{ext} \\
m\left[\dot{v} - wp + ur - y_g(r^2 + p^2) + z_g(qr - \dot{p}) + x_g(qp + \dot{r})\right] = \sum Y_{ext} \\
m\left[\dot{w} - uq + vp - z_g(p^2 + q^2) + x_g(rp - \dot{q}) + y_g(rq + \dot{p})\right] = \sum Z_{ext} \\
I_{xx}\dot{p} + \left(I_{zz} - I_{yy}\right)qr + m[y_g(\dot{w} - uq + vp) - z_g(\dot{v} - wp + ur)] = \sum K_{ext} \\
I_{yy}\dot{q} + (I_{xx} - I_{zz})rp + m[z_g(\dot{u} - vr + wq) - x_g(\dot{w} - uq + vp)] = \sum M_{ext} \\
I_{zz}\dot{r} + \left(I_{yy} - I_{xx}\right)pq + m[x_g(\dot{v} - wp + ur) - x_g(\dot{u} - vr + wq)] = \sum N_{ext}
\end{gathered} \tag{19}$$

3.2.2. Vehicle Mechanics

In the vehicle equations of motion, external forces and moments are written as:

$$\sum F_{ext} = F_{hydrostatic} + F_{lift} + F_{drag} + F_{control} \tag{20}$$

In this equation, each term will be observed individually, and it will be shown that these terms will be describes with vehicle coefficients using the methodology below:

$$F_d = -\left(\frac{1}{2}\rho c_d A_f\right) u|u| = X_{u|u|} u|u| \rightarrow X_{u|u|} = -\frac{1}{2}\rho c_d A_f \tag{21}$$

These coefficients will be derived for each force and moment for a better formulation of the equation of motion. All of the coefficients will be taken from the REMUS100 AUV article by Prestero. [15]

Hydrostatics

Hydrostatic forces and moments are caused by the buoyancy and weight effects of the vehicle. If the mass of the vehicle is m, the weight is W = mg and buoyancy force is B = ρ∇g considering ρ is density and ∇ is total volume displaced by the vehicle. Then the necessary coefficients for hydrostatic forces and moments are given in the REMUS100 thesis as follows:

$$X_{HS} = -(W - B)sin\theta$$

$$Y_{HS} = (W - B)cos\theta\ sin\phi$$

$$Z_{HS} = (W - B)cos\theta\ cos\phi$$

$$K_{HS} = -\left(y_g W - y_b B\right)cos\theta\ cos\phi - \left(z_g W - z_b B\right)cos\theta\ sin\phi$$

$$M_{HS} = -\left(z_g W - z_b B\right)sin\theta - \left(x_g W - x_b B\right)cos\theta\ cos\phi$$

$$N_{HS} = -\left(x_g W - x_b B\right)cos\theta\ sin\phi - \left(y_g W - y_b B\right)sin\theta \tag{22}$$

Hydrodynamic Damping

The damping force applied by the water during the vehicle's motion can be classified into three parts which are axial drag, crossflow drag and rolling drag. Note that this viscous drag force always opposes the motion of the vehicle.

Axial Drag

The axial drag equation can be expressed by the following empirical expression

$$X_{u|u|} = -\frac{1}{2}\rho c_d A_f \tag{23}$$

Where A_f is the frontal area of the vehicle and c_d is the drag force coefficient.

Crossflow Drag

The vehicle's crossflow drag is to be the sum of hull crossflow and fin crossflow drag forces. The coefficients and their corresponding formulation is given below:

$$Y_{v|v|} = Z_{w|w|} = -\frac{1}{2}\rho c_{dc} \int_{x_t}^{x_{b2}} 2R(x)dx - 2 \cdot \left(\frac{1}{2}\rho S_{fin} c_{df}\right)$$

$$M_{w|w|} = -N_{v|v|} = -\frac{1}{2}\rho c_{dc}\int_{x_t}^{x_{b2}} 2xR(x)dx - 2x_{fin}\cdot\left(\frac{1}{2}\rho S_{fin}c_{df}\right)$$

$$Y_{r|r|} = -Z_{q|q|} = -\frac{1}{2}\rho c_{dc}\int_{x_t}^{x_{b2}} 2x|x|R(x)dx - 2x_{fin}|x_{fin}|\cdot\left(\frac{1}{2}\rho S_{fin}c_{df}\right)$$

$$M_{q|q|} = N_{r|r|} = -\frac{1}{2}\rho c_{dc}\int_{x_t}^{x_{b2}} 2x^3R(x)dx - 2x_{fin}^3\cdot\left(\frac{1}{2}\rho S_{fin}c_{df}\right) \tag{24}$$

Rolling Drag

The rolling resistance come from the crossflow drag of the fins. The formulation can be approximated as follows

$$K_{p|p|} = Y_{vvf}r_{mean}^3 \tag{25}$$

Added Mass

As the REMUS100 moves throughout the trajectory, there is some amount of mass that moves with the AUV acting as a resistance to the motion. Like the drag force, added mass can be defined under the subjects of axial, crossflow, rolling and other added mass cross-terms.

Axial Added Mass

To find an appropriate axial added mass formulation, it will be assumed that the hull shape of the vehicle is an ellipsoid, which the major axis is the half of the vehicle length l and the minor axis is the half of the vehicle diameter d. Then the empirical relation for the axial added mass is:

$$X_{\dot{u}} = -\frac{4\propto\rho\pi}{3}\cdot\left(\frac{l}{2}\right)\cdot\left(\frac{d}{2}\right)^2 \tag{26}$$

Where α is an empirical parameter depending on the ratio of length and diameter.

Crossflow Added Mass

Crossflow added mass terms come from both cylindrical and cruciform hull cross sections. The formulations are given as follows:

$$Y_{\dot{v}} = -\int_{x_t}^{x_f} m_a(x)dx - \int_{x_f}^{x_{f2}} m_{af}(x)dx - \int_{x_{f2}}^{x_{b2}} m_a(x)dx$$

$$Z_{\dot{w}} = Y_{\dot{v}}$$

$$M_{\dot{w}} = \int_{x_t}^{x_f} xm_a(x)dx - \int_{x_t}^{x_f} xm_{af}(x)dx - \int_{x_{f2}}^{x_{b2}} xm_a(x)dx$$

$$\mathrm{M}_{\dot{\mathrm{v}}} = -\mathrm{M}_{\dot{\mathrm{w}}}$$

$$\mathrm{Y}_{\dot{\mathrm{r}}} = \mathrm{N}_{\dot{\mathrm{v}}}$$

$$\mathrm{Z}_{\dot{\mathrm{q}}} = \mathrm{M}_{\dot{w}}$$

$$M_{\dot{q}} = -\int_{x_{tail}}^{x_{fin}} x^2m_a(x)dx - \int_{x_{fin}}^{x_{fin2}} x^2m_{af}(x)dx - \int_{x_{fin}}^{x_{bow2}} x^2m_a(x)dx$$

$$N_{\dot{r}} = M_{\dot{q}} \tag{27}$$

Rolling Added Mass Cross-terms

Rolling added mass also comes from the added mass effects of the fins.

$$K_{\dot{p}} = -\int_{x_{fin}}^{x_{fin2}} \frac{2}{\pi} \rho a^4 dx \tag{28}$$

Added Mass Cross-terms

The cross terms are given as:

$$X_{wq} = Z_{\dot{w}} \quad X_{qq} = Z_{\dot{q}} \quad X_{vr} = -Y_{\dot{v}} \quad X_{rr} = -Y_{\dot{r}} \tag{29}$$

$$Y_{ur} = X_{\dot{u}} \quad Y_{wp} = -Z_{\dot{w}} \quad Y_{pq} = -Z_{\dot{q}} \tag{30}$$

$$Z_{uq} = -X_{\dot{u}} \quad Z_{vp} = -Y_{\dot{v}} \quad Z_{rp} = Y_{\dot{r}} \tag{31}$$

$$M_{uwa} = -(Z_{\dot{w}} - X_{\dot{u}}) \quad M_{vp} = -Y_{\dot{r}} \quad M_{rp} = \left(K_{\dot{p}} - N_{\dot{r}}\right) \quad M_{uq} = -Z_{\dot{q}} \tag{32}$$

$$N_{uva} = -(X_{\dot{u}} - Y_{\dot{v}}) \quad N_{wp} = Z_{\dot{q}} \quad N_{pq} = -\left(K_{\dot{p}} - M_{\dot{q}}\right) \quad N_{ur} = Y_{\dot{r}} \tag{33}$$

The added mass cross-terms M_{uwa} and N_{uva} are known as the Munk Moment.

Body Lift

As the vehicle moves through the water with an angle of attack, this causes flow separation and a pressure drop according to the orientation of the AUV. As a result, this pressure causes a pressure force at the center of pressure, and a moment caused by this force.

The body lift coefficients and their equation are given as:

$$Y_{uvl} = Z_{uwl} = -\frac{1}{2} \rho d^2 c_{yd\beta} \tag{34}$$

The body lift moment equation is:

$$M_{uwl} = -N_{uvl} = -\frac{1}{2} \rho d^2 c_{yd\beta} x_{cp} \tag{35}$$

Fin Lift

As it was mentioned in the body lift, there is also a force and a moment caused by the pressure acting on the fins, stern planes and rudders.

The fin lift coefficients are:

$$Y_{uu\delta_r} = -Y_{uvf} = \rho c_{L\alpha} S_{fin}$$

$$Z_{uu\delta_s} = Z_{uwf} = -\rho c_{L\alpha} S_{fin}$$

$$Y_{urf} = -Z_{uqf} = -\rho c_{L\alpha} S_{fin} x_{fin} \tag{36}$$

The fin moment coefficients are:

$$M_{uu\ s} = M_{uwf} = \rho c_{L\alpha} S_{fin} x_{fin}$$

$$N_{uu\ r} = -N_{uvf} = \rho c_{L\alpha} S_{fin} x_{fin}$$

$$M_{urf} = N_{uqf} = -\rho c_{L\alpha} S_{fin} x_{fin} \tag{37}$$

Propulsion Model

The propeller force and moment applied are taken from the line-of-sight PID controller study [15] and the formulation for propeller force is given below:

$$X_{prop} = 1.57 \times 10^{-4} \cdot n|n| \tag{38}$$

The formulation of the propeller moment is given as:

$$K_{prop} = -2.24 \times 10^{-5} \cdot n|n| \tag{39}$$

Combined Terms

Some of the terms in equations 28-32, 33-34 and 35-36 can be combined to get easier definitions.

$$Y_{uv} = Y_{uvl} + Y_{uvf}$$

$$Y_{ur} = Y_{ur\alpha} + Y_{urf}$$

$$Z_{uw} = Z_{uwl} + Z_{uwf}$$

$$Z_{uq} = Z_{uq\alpha} + Z_{uqf}$$

$$M_{uw} = M_{uw\alpha} + M_{uwl} + M_{uwf}$$

$$M_{uq} = M_{uq\alpha} + M_{uqf}$$

$$N_{uv} = Y_{uv\alpha} + Y_{uvl} + N_{uvf}$$

$$N_{ur} = N_{ur\alpha} + N_{urf} \tag{40}$$

3.2.3. Total Vehicle Forces and Moments

Combining all the forces previously found from hydrostatics, hydrodynamic damping, added mass, body lift and moment, fin lift and moment, propeller thrust and torque; a general equation can be derived as follows:

$$\sum X_{ext} = X_{HS} + X_{u|u|}u|u| + X_{\dot{u}}\dot{u} + X_{wq}wq + X_{qq}qq + X_{vr}vr + X_{rr}rr + X_{prop}$$

$$\sum Y_{ext} = Y_{HS} + Y_{v|v|}v|v| + Y_{r|r|}r|r| + Y_{\dot{v}}\dot{v} + Y_{r}\dot{r} + Y_{ur}ur + Y_{wp}wp + Y_{pq}pq + Y_{uv}uv + Y_{uu\delta_r}u^2\delta_r$$

$$\sum Z_{ext} = Z_{HS} + Z_{w|w|}w|w| + Z_{q|q|}q|q| + Z_{\dot{w}}\dot{w} + Z_{\dot{q}}\dot{q} + Z_{uq}uq + Z_{vp}vp + Z_{rp}rp + Z_{uw}uw + Z_{uu\delta_s}u^2\delta_s$$

$$\sum K_{ext} = K_{HS} + K_{p|p|}p|p| + K_{\dot{p}}\dot{p} + K_{prop}$$

$$\sum M_{ext} = M_{HS} + M_{w|w|}w|w| + M_{q|q|}q|q| + M_{\dot{w}}\dot{w} + M_{\dot{q}}\dot{q} + M_{uq}uq + M_{vp}vp + M_{rp}rp + M_{uw}uw + M_{uu\ {}_s}u^2\delta_s$$

$$\sum N_{ext} = N_{HS} + N_{v|v|}v|v| + N_{r|r|}r|r| + N_{\dot{v}}\dot{v} + N_{\dot{r}}\dot{r} + N_{ur}ur + N_{wp}wp + N_{pq}pq + N_{uv}uv + N_{uu\ {}_r}u^2\delta_r \tag{41}$$

3.3.Combined Nonlinear Equations of Motion

In the previous chapters, the equations for the rigid body dynamics and vehicle mechanics equations have been found. To find the full model of the REMUS100 AUV equations 19 and 40 must be used together. The result six degrees of freedom equation is found to be;

In surge, or translation along the x-axis:

$$m\left[\dot{u} - vr + wq - x_g(q^2 + r^2) + y_g(pq - \dot{r}) + z_g(pr + \dot{q})\right] = X_{HS} + X_{u|u|}u|u| + X_{\dot{u}}\dot{u} + X_{wq}wq + X_{qq}qq + X_{vr}vr + X_{rr}rr + X_{prop}$$

In sway, or translation along the y-axis:

$$m\left[\dot{v} - wp + ur - y_g(r^2 + p^2) + z_g(qr - \dot{p}) + x_g(qp + \dot{r})\right] = Y_{HS} + Y_{v|v|}v|v| + Y_{r|r|}r|r| + Y_{\dot{v}}\dot{v} + Y_r\dot{r} + Y_{ur}ur + Y_{wp}wp + Y_{pq}pq + Y_{uv}uv + Y_{uu\delta_r}u^2\delta_r$$

In heave, or translation along the z-axis:

$$m\left[\dot{w} - uq + vp - z_g(p^2 + q^2) + x_g(rp - \dot{q}) + y_g(rq + \dot{p})\right] = Z_{HS} + Z_{w|w|}w|w| + Z_{q|q|}q|q| + Z_{\dot{w}}\dot{w} + Z_{\dot{q}}\dot{q} + Z_{uq}uq + Z_{vp}vp + Z_{rp}rp + Z_{uw}uw + Z_{uu\delta_s}u^2\delta_s$$

In roll, or rotation about the x-axis:

$$I_{xx}\dot{p} + \left(I_{zz} - I_{yy}\right)qr + m[y_g(\dot{w} - uq + vp) - z_g(\dot{v} - wp + ur)] = K_{HS} + K_{p|p|}p|p| + K_{\dot{p}}\dot{p} + K_{prop}$$

In pitch, or rotation about the y-axis:

$$I_{yy}\dot{q} + (I_{xx} - I_{zz})rp + m[z_g(\dot{u} - vr + wq) - x_g(\dot{w} - uq + vp)] = M_{HS} + M_{w|w|}w|w| + M_{q|q|}q|q| + M_{\dot{w}}\dot{w} + M_{\dot{q}}\dot{q} + M_{uq}uq + M_{vp}vp + M_{rp}rp + M_{uw}uw + M_{uu\delta_s}u^2\delta_s$$

In yaw, or rotation about the z-axis:

$$I_{zz}\dot{r} + \left(I_{yy} - I_{xx}\right)pq + m[x_g(\dot{v} - wp + ur) - x_g(\dot{u} - vr + wq)] = N_{HS} + N_{v|v|}v|v| + N_{r|r|}r|r| + N_{\dot{v}}\dot{v} + N_{\dot{r}}\dot{r} + N_{ur}ur + N_{wp}wp + N_{pq}pq + N_{uv}uv + N_{uu\ \ r}u^2\delta_r \qquad (42)$$

At this point, the acceleration terms are separated for a convenient approach. Then the equation are redefined as:

Translation in x-axis:

$$(m - X_{\dot{u}})\dot{u} + mz_g\dot{q} - my_g\dot{r} = X_{HS} + X_{u|u|}u|u| + \left(X_{wq} - m\right)wq + \left(X_{qq} + mx_g\right)q^2 + (X_{vr} + m)vr + \left(X_{rr} + mx_g\right)r^2 - my_gpq - mz_gpr + X_{prop}$$

Translation in y-axis:

$$(m - Y_{\dot{v}})\dot{v} - mz_g\dot{p} + \left(mx_g - Y_{\dot{r}}\right)\dot{r} = Y_{HS} + Y_{v|v|}v|v| + Y_{r|r|}r|r| + my_gr^2 + (Y_{ur} - m)ur + \left(Y_{wp} + m\right)wp + \left(Y_{pq} - mx_g\right)pq + Y_{uv}uv + my_gp^2 + mz_gqr + Y_{uu\ \ r}u^2\delta_r$$

Translation in z-axis:

$$(m - Z_{\dot{w}})\dot{w} + my_g\dot{p} - \left(mx_g + Z_{\dot{q}}\right)\dot{q} = Z_{HS} + Z_{w|w|}w|w| + Z_{q|q|}q|q| + \left(Z_{uq} + m\right)uq + \left(Z_{vp} - m\right)vp + \left(Z_{rp} - mx_g\right)rp + Z_{uw}uw + mz_g(p^2 + q^2) - my_grq + Z_{uu\ \ s}u^2\delta_s$$

Rotation about x-axis:

$$-mz_g\dot{v} + my_g\dot{w} + \left(I_{xx} - K_{\dot{p}}\right)\dot{p} = K_{HS} + K_{p|p|}p|p| - \left(I_{zz} - I_{yy}\right)qr + m(uq - vp) - mz_g(wp - ur) + K_{prop}$$

Rotation about y-axis:

$$mz_g\dot{u} - \left(mx_g + M_{\dot{w}}\right)\dot{w} + \left(I_{yy} - M_{\dot{q}}\right)\dot{q} = M_{HS} + M_{w|w|}w|w| + M_{q|q|}q|q| + \left(M_{uq} - mx_g\right)uq + \left(M_{vp} + mx_g\right)vp + \left[M_{rp} - (I_{xx} - I_{zz})\right]rp + mz_g(vr - wq) + M_{uw}uw + M_{uu\delta_s}u^2\delta_s$$

Rotation about z-axis:

$$-my_g\dot{u} + \left(mx_g - N_{\dot{v}}\dot{v}\right)\dot{v} + (I_{zz} - N_{\dot{r}})\dot{r} = N_{HS} + N_{v|v|}v|v| + N_{r|r|}r|r| + \left(N_{ur} - mx_g\right)ur + \left(N_{wp} + mx_g\right)wp + \left[N_{pq} - \left(I_{yy} - I_{xx}\right)\right]pq - my_g(vr - wq) + N_{uv}uv + N_{uu\delta_r}u^2\delta_r \tag{43}$$

Finally, all these equations can be summarized in matrix notation as given below:

$$\begin{bmatrix} m - X_{\dot{u}} & 0 & 0 & 0 & mz_g & -my_g \\ 0 & m - Y_{\dot{v}} & 0 & -mz_g & 0 & mx_g - Y_{\dot{r}} \\ 0 & 0 & m - Z_{\dot{w}} & my_g & -mx_g - Z_{\dot{q}} & 0 \\ 0 & -mz_g & my_g & I_{xx} - K_{\dot{p}} & 0 & 0 \\ mz_g & 0 & -mx_g - M_{\dot{w}} & 0 & I_{yy} - M_{\dot{q}} & 0 \\ -my_g & mx_g - N_{\dot{v}} & 0 & 0 & 0 & I_{xx} - N_{\dot{r}} \end{bmatrix} \cdot \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \sum X \\ \sum Y \\ \sum Z \\ \sum K \\ \sum M \\ \sum N \end{bmatrix} \tag{44}$$

The first matrix given here is the mass matrix of the system in the body fixed coordinates and the last vector is the remaining force terms. This relation can also be written as follows:

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} m - X_{\dot{u}} & 0 & 0 & 0 & mz_g & -my_g \\ 0 & m - Y_{\dot{v}} & 0 & -mz_g & 0 & mx_g - Y_{\dot{r}} \\ 0 & 0 & m - Z_{\dot{w}} & my_g & -mx_g - Z_{\dot{q}} & 0 \\ 0 & -mz_g & my_g & I_{xx} - K_{\dot{p}} & 0 & 0 \\ mz_g & 0 & -mx_g - M_{\dot{w}} & 0 & I_{yy} - M_{\dot{q}} & 0 \\ -my_g & mx_g - N_{\dot{v}} & 0 & 0 & 0 & I_{xx} - N_{\dot{r}} \end{bmatrix}^{-1} = \begin{bmatrix} \sum X \\ \sum Y \\ \sum Z \\ \sum K \\ \sum M \\ \sum N \end{bmatrix} \tag{45}$$

With all these definitions, that would complete analysis of kinetics and kinematics of the REMUS100 AUV.

4. UDWADIA-KALABA APPROACH

Udwadia-Kalaba approach is a methodology that handles the analytical dynamics-based control problem in a different manner by treating the modeling constraints are need to be satisfied with an applied constraint force and control constraints are need to be satisfied with an applied control force.

The control constraints specified by the user are minimized at each instant of time, the modeling constraints on the other hand, are satisfied at all times. This means that modeling constraints are required to be fulfilled at any time so that the system may behave in a proper physical manner. However, control constraint may not be exactly satisfied and errors may occur, so the proper purpose is to minimize it.

4.1. Description of The Constraints and Notation

Udwadia-Kalaba method firstly introduces the definition of unconstrained system, where a system (or a plant in control terminology) does not have any limiting conditions or constraint and it can technically make any kind of motion even though there is a corresponding actuation or not. The equation of unconstrained system is given below as:

$$M(q,t)\ddot{q} = Q(q,\dot{q},t) \tag{46}$$

Where q is the configuration vector (or state vector), M is the positive definite mass matrix that its terms may be dependent on states and time and Q is the unconstrained force vector or generalized impressed force vector that is dependent on states, first derivative of states and time.

For both controlling the system and making its proper physical definition, Udwadia-Kalaba approach requires set of constraints on the system. The first set of constraints are the modeling constraints, where these constraints determine the overall physical behavior and therefore must be satisfied at all times. Assuming the system has n_m modeling constraints that must be imposed on it, the definition of modeling constraints is:

$$\phi_m(q,\dot{q},t) = 0 \tag{47}$$

In the equation above, $\phi_m \in R^{n_m}$ is a column vector. These constraints can be holonomic or non-holonomic. It is assumed that the constraints are smooth enough to be differentiated a sufficient number of times so the they are reduced in the form given below:

$$A_m(q,\dot{q},t)\ddot{q} = b_m(q,\dot{q},t) \tag{48}$$

In the equation above A_m is an $n_m \times n$ matrix. The nature applies a force $Q_m(q,\dot{q},t)$ called “constraint force” so that these constraints are enforced. According to the Gaussian, nature applies this force in such a manner that the Gaussian G(t) given below is minimized at each instant of time:

$$G(t) = Q_m^T(q,\dot{q},t) \cdot M^{-1}(q,t) \cdot Q_m(q,\dot{q},t) \tag{49}$$

Also, the control objectives can be defined on the system as a set of new constraints where the definition is given below:

$$\phi_c(q,\dot{q},t) = 0 \tag{50}$$

Here, ϕ_c is a column vector on n_c dimensions. Again, the former expression differentiated a sufficient number of times to get:

$$A_c(q,\dot{q},t)\ddot{q} = b_c(q,\dot{q},t) \tag{51}$$

In the equation above A_c is an $n_c \times n$ matrix. To satisfy the control objectives, one needs to apply a force $Q_c(q,\dot{q},t)$ called “control force”. At each instant of time the control cost given below is minimized so that control requirements are satisfied as best as possible:

$$J_c(q,\dot{q},t) = Q_c^T(q,\dot{q},t) \cdot W(q,\dot{q},t) \cdot Q_c(q,\dot{q},t) \tag{52}$$

Where W(q,q̇,t) is a 1user defined symmetric positive definite matrix that is used as a tuning parameter.

After defining the constraint and control forces with their related costs, the goal is to find a generalized force with constraint and control forces added, which the dynamical definition is given below as:

$$M(q,t)\ddot{q} = Q(q,\dot{q},t) + Q_m(q,\dot{q},t) + Q_c(q,\dot{q},t) \tag{53}$$

Note that Q_m is found such in a manner that modeling constraints are satisfied at all times, since the proper mathematical description of the system can't be violated. On the other hand, Q_c is found such that control constraints are satisfied as best as possible.

The previous equation is converted into the "scaled acceleration" form, where both sides of the equation are multiplied with $M^{-1/2}$. Then the equation takes the form of

$$M^{1/2}(q,t)\ddot{q} = M^{-1/2}Q + M^{-1/2}Q_m + M^{-1/2}Q_c \tag{54}$$

The terms above can be represented with

$$\ddot{q}_s = M^{1/2}\ddot{q}, a_s = M^{-1/2}Q, \ddot{q}_s^m = \ M^{-1/2}Q_m \ and \ \ddot{q}_s^c = M^{-1/2}Q_c \tag{55}$$

Then the equation can be rewritten as:

$$\ddot{q}_s = a_s + \ddot{q}_s^m + \ \ddot{q}_s^c \tag{56}$$

Where $\ddot{q}_s$ is the scaled acceleration of the controlled system, a_s is the scaled acceleration of the unconstrained system, $\ddot{q}_s^m$ is the scaled constraint acceleration and $\ddot{q}_s^c$ is the scaled control acceleration. Using this, modeling and control constraints are written in terms of scaled accelerations as given below:

$$B_m\ddot{q}_s = b_m \ with \ B_m \coloneqq A_m M^{-1/2} \tag{57}$$

$$B_c\ddot{q}_s = b_c \ with \ B_c \coloneqq A_c M^{-1/2} \tag{58}$$

4.2.Fundamental Equation of Motion

Consider the equation of motion of the constrained system is given as

$$M(q,t)\ddot{q} = Q(q,\dot{q},t) + Q_m(q,\dot{q},t) \tag{59}$$

This equation can be written in terms of scaled accelerations as follows:

$$\ddot{q}_s = a_s + \ddot{q}_s^m \tag{60}$$

The constraint acceleration in constrained system equation can be written explicitly as follows:

$$\ddot{q}_s^m = B_m^+(b_m - B_m a_s) \tag{61}$$

Where X^+ is the Moore-Penrose inverse (pseudoinverse) of the matrix X.

The relation above makes the equation of constrained system as follows:

$$\ddot{q}_s = a_s + B_m^+(b_m - B_m a_s) = (I_n - B_m^+ B_m)a_s + B_m^+ b_m \tag{62}$$

In this equation I_n corresponds to n×n identity matrix. In the preceding equation, the scaled acceleration of the uncontrolled system is given. In other words, it is a system where only modeling constraints are enforced.

$$\ddot{u}_s = (I_n - B_m^+ B_m)a_s + B_m^+ b_m \tag{63}$$

Note that $(I_n - B_m^+ B_m)$ is an orthogonal projection matrix and it projects any scaled acceleration vector into the null space of B_m, ensuring that modeling constraints are always satisfied.

At this point, if a control force is applied on the uncontrolled system, the equation of controlled system is obtained as given below:

$$M\ddot{q} = (Q + Q_c) + Q_m \tag{64}$$

In terms of scaled accelerations;

$$\ddot{q}_s = (a_s + \ddot{q}_s^c) + \ddot{q}_s^m \tag{65}$$

If the explicit term instead of scaled modeling acceleration is used the following term is obtained:

$$\ddot{q}_s^m = B_m^+[b_m - B_m(a_s + \ddot{q}_s^c)] \tag{66}$$

Then the controlled system equation in terms of scaled accelerations becomes:

$$\ddot{q}_s = a_s + B_m^+(b_m - B_m a_s) + \ddot{q}_s^c - B_m^+ B_m \ddot{q}_s^c = (I_n - B_m^+ B_m)a_s + B_m^+ b_m + (I_n - B_m^+ B_m)\ddot{q}_s^c \tag{67}$$

Comparing this equation to the uncontrolled system scaled acceleration, one can see that another projection term is added for control scaled acceleration on to the null space of B_m in addition to the unconstrained system acceleration. This methodology ensures that modeling constraints are always satisfied, no matter the values of a_s and $\ddot{q}_s^c$.

4.3.Inconsistent Constraints

If the constraints of a dynamic system are inconsistent, there is no n dimensional vector $\ddot{q}_s$ which can satisfy both $B_m\ddot{q}_s$ and $B_c\ddot{q}_s$. However, as it was mentioned before, the modeling constraints must always be satisfied for proper mathematical description of the plant. To ensure this, Q_m will be considered as a function of Q_c. When we get an expression for Q_m as a function of Q_c and Q, it is possible to obtain Q_c using 2-norm of the error satisfying the control constraints:

$$||e|| = ||A_c\ddot{q} - b_c|| = ||B_c\ddot{q}_s - b_c|| \tag{68}$$

While minimizing the control cost at the same time. These two conditions gives the resulting scaled control acceleration as follows:

$$\ddot{q}_s^c = SB_{cms}^+(b_c - B_c\ddot{u}_s) \; and \; Q_c = M^{1/2}\, SB_{cms}^+(b_c - B_c\ddot{u}_s) \tag{69}$$

Where the variables here are:

$$S = M^{-1/2}W^{-1/2} \quad B_{cms} = B_c(I_n - B_m^+B_m)S \quad \ddot{u}_s = (I_n - B_m^+B_m)a_s + B_m^+b_m$$
$$z_c = W^{1/2}M^{1/2}\,\ddot{q}_s^c = B_{cms}^+(b_c - B_c\ddot{u}_s)$$

Then, the error relation can be rewritten as:

$$||e|| = ||B_c(I_n - B_m^+B_m)Sz_c - (b_c - B_c\ddot{u}_s)|| \tag{70}$$

With these transformations, the problem of finding $\ddot{q}_s^c$ is converted into problem of finding z_c such that it is the vector that minimizes the norm of the error.

Finding the best possible scaled control acceleration minimizing the error, the scaled equation of motion of the dynamical system can be written as:

$$\ddot{q}_s = a_s + B_m^+(b_m - B_m a_s) + (I_n + B_m^+B_m)SB_{cms}^+(b_c - B_c\ddot{u}_s) \tag{71}$$

Note that if constraints are not consistent, the controlled system acceleration found above minimizes the constraint relation in equation 67 in a least square sense. At the same time, control cost is minimized at each instant of time. For different weighting matrices W, the control cost may change but the error of the norm remains minimum.

Remark: The control force $Q_c = M^{1/2}\,SB_{cms}^+(b_c - B_c\ddot{u}_s)$ can be discontinuous when the rank of B_{cms} changes. If the control engineer desires a smoother Q_c, which in many practical applications such condition is required, an alternative cost function $\hat{J}$ is used which is an expression that includes both $||e||$ and control cost with modified weighting parameter μ. The new cost function is given below:

$$\hat{J}_c = ||A_c\ddot{q} - b_c||^2 + \mu J_c \tag{72}$$

The control force that minimizes this cost is:

$$Q_c = M^{1/2}S(B_{cms}^TB_{cms} + \mu I_n)^{-1}B_{cms}^T(b_c - B_c\ddot{u}_s) \tag{73}$$

In the preceding equation, increasing μ means smoother control force with larger error and decreasing μ means smaller control error with larger variations in control force.

4.4. Consistent Constraints

On the contrary to the previous section, the constraints given in equations 56 and 57 can also be consistent, in other words, a scaled acceleration $\ddot{q}_s$ can be found such that it can both solve $B_m\ddot{q}_s$ and $B_c\ddot{q}_s$. This condition can be inspected under two subjects, which are equal and unequal weighting matrices.

4.4.1. Unequal Weighting Matrices $W=M^{-1}$

This condition usually occurs when the control engineer desires to create a controller of a mechanical system such that the system already has some sort of modeling constraints applied on. It is shown in [UD-ARTICLE] that the control force that will both satisfy control constraints and modeling constraint is the one that obtained in the inconsistent constraints, given in the following equation:

$$Q_c = M^{1/2}\,SB_{cms}^+(b_c - B_c\ddot{u}_s) \tag{74}$$

If the constraints are consistent, equation 73 ensures that control constraints $B_c\ddot{q}_s$ and modeling constraints $B_m\ddot{q}_s$ are exactly satisfied.

4.4.2. Equal Weighting Matrices W = M^{-1}

This condition usually occurs in practical applications when some new physical limits are desired to be applied on a mechanical system, after that, these additional constraints act as additional modeling constraints in mathematical model.

As S = I_n, this condition makes some simplifications as follows:

$$B_{cms|S=I_n} \coloneqq B_{cm} = B_c(I_n - B_m^+ B_m) \ \ and \ \ \ddot{q}_s^c = B_{cm}^+(b_c - B_c\ddot{u}_s) \tag{75}$$

Also, the constraint and control forces change as follows:

$$Q_m = M^{1/2}\ddot{q}_s^m = M^{1/2}B_m^+(b_m - B_m a_s - B_m\ddot{q}_s^c) \tag{76}$$

$$Q_c = M^{1/2}\ddot{q}_s^c = M^{1/2}[B_c(I_n - B_m^+ B_m)]^+(b_c - B_c\ddot{u}_s) = M^{1/2}B_{cm}^+(b_c - B_c\ddot{u}_s) \tag{77}$$

Remark: At initial time (t=0), if a system does not satisfy the control objectives, for instance a case in which a vehicle's initial position does not lay on the desired trajectory, it is more useful to use modified constraint equations instead of standard control constraint $\phi_c(q, \dot{q}, t) = 0$ which are given below:

$$\dot{\phi}_{c_i} + \gamma_i\phi_{c_i} = 0 \quad \gamma_i > 0 \quad for\ nonholonomic\ constraints \tag{78}$$

$$\ddot{\phi}_{c_i} + \alpha_i\dot{\phi}_{c_i} + \beta_i\phi_{c_i} = 0 \quad \alpha_i, \beta_i > 0 \quad for\ holonomic\ constraints \tag{79}$$

Even when the system starts on the desired control objectives, it is numerically more stable to use modified constraint equations.[7]

5. DESIGN OF CONTROLLER

In this section, a controller design for REMUS100 AUV will be created using Udwadia-Kalaba dynamics to follow a helical path going downward with a specified radius and frequency. MATLAB will be used for algorithm design and for any part that will require integration, Euler integration is preferred for readability. The design of the controller will be investigated under three parts, which are unconstrained system design, modeling constraints and control constraints. At each part, derivations will be made and necessary matrices and vectors will be shown. At the end, these matrices will be used in a loop using MATLAB and necessary plots will be shown.

5.1. Unconstrained System Design

Unconstrained system of a plant includes the natural dynamics of the system, which includes any kind of force without input forces and the system has technically no limitations on it. For REMUS100 AUV, this means the vehicle can move at any direction, whether it has an actuator causing that motion or not, which is physically not correct. This condition will be change in latter sections by adding modeling and control constraints on the system.

The unconstrained system of REMUS100 AUV includes the model of the system, which is provided in equation 43 (and the external forces and moments are provided in equation 40).

Then, mass matrix in body fixed coordinate:

$$M_\nu = \begin{bmatrix} m - X_{\dot{u}} & 0 & 0 & 0 & mz_g & -my_g \\ 0 & m - Y_{\dot{v}} & 0 & -mz_g & 0 & mx_g - Y_{\dot{r}} \\ 0 & 0 & m - Z_{\dot{w}} & my_g & -mx_g - Z_{\dot{q}} & 0 \\ 0 & -mz_g & my_g & I_{xx} - K_{\dot{p}} & 0 & 0 \\ mz_g & 0 & -mx_g - M_{\dot{w}} & 0 & I_{yy} - M_{\dot{q}} & 0 \\ -my_g & mx_g - N_{\dot{v}} & 0 & 0 & 0 & I_{xx} - N_{\dot{r}} \end{bmatrix} \tag{80}$$

The configuration vector is:

$$\ddot{c} = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} \tag{81}$$

Note that the nomenclature for configuration vector is c instead of q, as q is also used as an angular velocity term. The force and moments vector is then:

$$Q_\nu = \begin{bmatrix} \sum X \\ \sum Y \\ \sum Z \\ \sum K \\ \sum M \\ \sum N \end{bmatrix} \tag{82}$$

Then the equation of motion for the unconstrained system is:

$$M_\nu \ddot{c} = Q_\nu \tag{83}$$

There is an important point in here though, Udwadia-Kalaba approach utilizes control constraints in trajectory problems in terms of x,y,z coordinates, which is body-fixed coordinates because the trajectory that a vehicle must follow can only be written in terms of this type coordinate system. On the other hand, the model of the vehicle is given in terms of body-fixed accelerations. This condition requires converting the model of the system; the mass matrix, the configuration vector and the force & moments vector; from body-fixed coordinate system to earth frame. This requires using the Jacobian transformation relation and its derivatives.

Taking the derivative of equation 14;

$$\dot{\nu} = \dot{J}^{-1} \cdot \dot{\eta} + J^{-1} \cdot \ddot{\eta}$$

Here, $\dot{\nu} = \ddot{c}$. Substituting into equation 82:

$$\mathrm{M}_\nu\left(\dot{J}^{-1} \cdot \dot{\eta} + \mathrm{J}^{-1} \cdot \ddot{\eta}\right) = \mathrm{Q}_\nu$$

Modifying;

$$\mathrm{M}_\nu \mathrm{J}^{-1}\ddot{\eta} = \mathrm{Q}_\nu - \mathrm{M}_\nu \dot{J}^{-1}\dot{\eta}$$

Here, if the control engineer accepts the multiplied term $\mathrm{M}_\nu \mathrm{J}^{-1}$ as the new mass matrix, it may be seen that at some point of the iteration process, this term's eigenvalues go negative, making it impossible to apply Udwadia-Kalaba dynamics, as scaled acceleration terms require square root of the mass matrices.

A different kind of approach can be made using the fact that kinetic energy of the plant is the same for both coordinate axes [16]. Using this, the previous approach is modified as follows:

$$\dot{\eta} = J\nu \quad \rightarrow \quad \ddot{\eta} = \dot{J}\nu + J\dot{\nu}$$

$$\mathrm{J}\dot{\nu} = \ddot{\eta} - \dot{\mathrm{J}}\nu \quad \rightarrow \quad \dot{\nu} = J^{-1}(\ddot{\eta} - \dot{\mathrm{J}}\nu)$$

Then the equation of unconstrained system is:

$$\mathrm{M}_\nu \left(J^{-1}(\ddot{\eta} - \dot{\mathrm{J}}\nu)\right) = \mathrm{Q}_\nu$$

Considering ν is $J^{-1}\dot{\eta}$ and modifying;

$$M_\nu J^{-1} \cdot \ddot{\eta} - M_\nu J^{-1}\dot{J}J^{-1}\dot{\eta} = Q_\nu$$

$$M_\nu J^{-1} \cdot \ddot{\eta} = Q_\nu + M_\nu J^{-1}\dot{J}J^{-1}\dot{\eta}$$

At this point, multiply the both sides of the equation with J^{-T}:

$$J^{-T}M_\nu J^{-1} \cdot \ddot{\eta} = J^{-T}Q_\nu + J^{-T}M_\nu J^{-1}\dot{J}J^{-1}\dot{\eta}$$

Now, the new equation of motion terms for earth frame is given below:

$$M_\eta = J^{-T}M_\nu J^{-1} \quad Q_\eta = J^{-T}Q_\nu + J^{-T}M_\nu J^{-1}\dot{J}J^{-1}\dot{\eta} \tag{84}$$

And the equation of motion is:

$$M_\eta \ddot{\eta} = Q_\eta \tag{85}$$

Using this equation of motion, one can see that M_η is always positive definite. Notice that this matrix changes with time as it is dependent on the transformation matrices and eventually Euler angles. By the definition of the unconstrained system of the Udwadia-Kalaba approach, this is possible and does not cause any problem throughout the design of the controller. Note that $\ddot{c} = \ddot{\eta}$ after this point.

Important point here is that there should not be any actuator forces or moments at the initial time of the controller, as the unconstrained system consists of only natural dynamics.

Then the unconstrained system scaled acceleration is simply:

$$a_s = M_\eta^{-1/2} Q_\eta \tag{86}$$

That would complete the unconstrained system part and more importantly, the coordinate system change in here make it possible to form up control constraints.

5.2.Modeling Constraints Design

The modeling constraints consists of the constraints that make the REMUS100 AUV's motion capabilities appropriate to physical limits. For instance, when the actuator forces applied on the vehicle is observed according to the following figure, it can be seen that there is no actuator that is able apply force or moment in body fixed coordinates through the directions of sway (v), heave (w) and roll (p). As a result, the velocities in these directions must be zero. That require three modeling constraints to apply on constrained system design.

Figure 2:REMUS100 AUV

https://maritimesurveyaustralia.com.au/portfolio/autonomous-underwater-vehicle-amc-search-remus-100-nscv-gap-analysis-review/

5.2.1. Sway Constraint

Consider the relation in equation 7, which is:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}=\begin{bmatrix}cos\psi\cdot cos\theta & -sin\psi\cdot cos\phi+cos\psi\cdot sin\theta\cdot sin\phi & sin\psi\cdot sin\phi+cos\psi\cdot sin\theta\cdot cos\phi\\ sin\psi\cdot cos\theta & cos\psi\cdot cos\phi+sin\psi\cdot sin\theta\cdot sin\phi & -cos\psi\cdot sin\phi+sin\psi\cdot sin\theta\cdot cos\phi\\ -sin\theta & cos\theta\cdot sin\phi & cos\theta\cdot cos\phi\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix} \quad (87)$$

Inverting the relation above, considering the equation 8:

$$\begin{bmatrix}cos\psi\cdot cos\theta & sin\psi\cdot cos\theta & -sin\theta\\ -sin\psi\cdot cos\phi+cos\psi\cdot sin\theta\cdot sin\phi & cos\psi\cdot cos\phi+sin\psi\cdot sin\theta\cdot sin\phi & cos\theta\cdot sin\phi\\ sin\psi\cdot sin\phi+cos\psi\cdot sin\theta\cdot cos\phi & -cos\psi\cdot sin\phi+sin\psi\cdot sin\theta\cdot cos\phi & cos\theta\cdot cos\phi\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}=\begin{bmatrix}u\\ v\\ w\end{bmatrix} \quad (88)$$

The first modeling constraint enforces body fixed sway (v) velocity to be zero. From the matrix multiplication:

$$(-sin\psi\cdot cos\phi+cos\psi\cdot sin\theta\cdot sin\phi)\dot{x}+(cos\psi\cdot cos\phi+sin\psi\cdot sin\theta\cdot sin\phi)\dot{y}+(cos\theta\cdot sin\phi)\dot{z}=v \quad (89)$$

Then the term above must be zero, as there is no actuation in that direction:

$$(-sin\psi\cdot cos\phi+cos\psi\cdot sin\theta\cdot sin\phi)\dot{x}+(cos\psi\cdot cos\phi+sin\psi\cdot sin\theta\cdot sin\phi)\dot{y}+(cos\theta\cdot sin\phi)\dot{z}=0 \quad (90)$$

If the derivative of the expression above is taken with respect to time:

$$(cos\psi\cdot sin\theta\cdot sin\phi-sin\psi\cdot cos\phi)\ddot{x}+\left(-cos\psi\cdot\dot{\psi}\cdot cos\phi+sin\psi\cdot sin\phi\cdot\dot{\phi}-sin\psi\cdot\dot{\psi}\cdot sin\theta\cdot sin\phi+cos\psi\cdot cos\theta\cdot\dot{\theta}\cdot sin\phi+cos\psi\cdot sin\theta\cdot cos\phi\cdot\dot{\phi}\right)\dot{x}+(sin\psi\cdot sin\theta\cdot sin\phi+cos\psi\cdot cos\phi)\ddot{y}+\left(-sin\psi\cdot\dot{\psi}\cdot cos\phi-cos\psi\cdot sin\phi\cdot\dot{\phi}+cos\psi\cdot\dot{\psi}\cdot sin\theta\cdot sin\phi+sin\psi\cdot cos\theta\cdot\dot{\theta}\cdot sin\phi+sin\psi\cdot sin\theta\cdot cos\phi\cdot\dot{\phi}\right)\dot{y}+(cos\theta\cdot sin\phi)\ddot{z}+\left(cos\theta\cdot cos\phi\cdot\dot{\phi}-sin\theta\cdot\dot{\theta}\cdot sin\phi\right)\dot{z}=0 \quad (91)$$

Isolating the terms multiplied with second derivative of states:

$$(cos\psi \cdot sin\theta \cdot sin\phi - sin\psi \cdot cos\phi)\ddot{x} + (sin\psi \cdot sin\theta \cdot sin\phi + cos\psi \cdot cos\phi)\ddot{y} + (cos\theta \cdot sin\phi)\ddot{z} = -\big(-cos\psi \cdot \dot{\psi} \cdot cos\phi + sin\psi \cdot sin\phi \cdot \dot{\phi} - sin\psi \cdot \dot{\psi} \cdot sin\theta \cdot sin\phi + cos\psi \cdot cos\theta \cdot \dot{\theta} \cdot sin\phi + cos\psi \cdot sin\theta \cdot cos\phi \cdot \dot{\phi}\big)\dot{x} - \big(-sin\psi \cdot \dot{\psi} \cdot cos\phi - cos\psi \cdot sin\phi \cdot \dot{\phi} + cos\psi \cdot \dot{\psi} \cdot sin\theta \cdot sin\phi + sin\psi \cdot cos\theta \cdot \dot{\theta} \cdot sin\phi + sin\psi \cdot sin\theta \cdot cos\phi \cdot \dot{\phi}\big)\dot{y} - \big(cos\theta \cdot cos\phi \cdot \dot{\phi} - sin\theta \cdot \dot{\theta} \cdot sin\phi\big)\dot{z} \tag{92}$$

With these operations, previous equation takes the form of equation 47. Notice that it is only the first row, and we require two more constraints.

5.2.2. Heave Constraint

Similar to the former constraint derivation, the matrix multiplication term regarding the heave (w) velocity is given below:

$$(sin\psi \cdot sin\phi + cos\psi \cdot sin\theta \cdot cos\phi)\dot{x} + (-cos\psi \cdot sin\phi + sin\psi \cdot sin\theta \cdot cos\phi)\dot{y} + (cos\theta \cdot cos\phi)\dot{z} = w \tag{93}$$

According to the constraint, this term must be zero:

$$(sin\psi \cdot sin\phi + cos\psi \cdot sin\theta \cdot cos\phi)\dot{x} + (-cos\psi \cdot sin\phi + sin\psi \cdot sin\theta \cdot cos\phi)\dot{y} + (cos\theta \cdot cos\phi)\dot{z} = 0 \tag{94}$$

Again, as it was done in the previous section, the preceding term's derivative is taken and terms multiplied with second derivatives of states are isolated. The resulting term is:

$$(cos\psi \cdot sin\theta \cdot cos\phi + sin\psi \cdot sin\phi)\ddot{x} + \big(sin\psi \cdot sin\theta \cdot cos\phi - cos\psi \cdot \dot{\psi}\big)\ddot{y} + (cos\theta \cdot cos\phi)\ddot{z} = -\big(cos\psi \cdot \dot{\psi} \cdot sin\phi + sin\psi \cdot cos\phi \cdot \dot{\phi} - sin\psi \cdot \dot{\psi} \cdot sin\theta \cdot cos\phi + cos\psi \cdot cos\theta \cdot \dot{\theta} \cdot cos\phi - cos\psi \cdot sin\theta \cdot sin\phi \cdot \dot{\phi}\big)\dot{x} - \big(sin\psi \cdot \dot{\psi} \cdot sin\phi - cos\psi \cdot cos\phi \cdot \dot{\phi} + cos\psi \cdot \dot{\psi} \cdot sin\theta \cdot cos\phi + sin\psi \cdot cos\theta \cdot \dot{\theta} \cdot cos\phi - sin\psi \cdot sin\theta \cdot sin\phi\big)\dot{y} - \big(sin\theta \cdot \dot{\theta} \cdot cos\phi + cos\theta \cdot sin\phi \cdot \dot{\phi}\big)\dot{z} \tag{95}$$

Again, this term can take the form of equation 47.

5.2.3. Roll Constraint

Just as the transformation matrix for the derivatives of inertial positions and body fixed velocities, equation 10 and 12 are used for making a transformation from derivatives of Euler angles to body fixed angular velocities.

$$\begin{bmatrix} 1 & 0 & -sin\theta \\ 0 & cos\phi & cos\theta \cdot sin\phi \\ 0 & -sin\theta & cos\theta \cdot cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{96}$$

If the roll velocity equation is extracted from the matrix multiplication:

$$(1)\dot{\phi} + (0)\dot{\theta} + (-sin\theta)\dot{\psi} = \dot{\phi} - sin\theta \cdot \dot{\psi} = p \tag{97}$$

The roll angular velocity must be zero, then the constraint is:

$$\dot{\phi} - sin\theta \cdot \dot{\psi} = 0 \tag{98}$$

If the derivative with respect to time of this equation is taken and the second derivative terms are isolated:

$$\ddot{\phi} - sin\theta \cdot \ddot{\psi} = cos\theta \cdot \dot{\theta} \cdot \dot{\psi} \tag{99}$$

5.2.4. Forming Modeling Constraint Equation

Considering the equations 90, 94 and 98; all of these equations can be written in the form of equation 47;

$$A_m\ddot{c} = b_m$$

Substituting;

$$A_m = \begin{bmatrix} (cos\psi \cdot sin\theta \cdot sin\phi - sin\psi \cdot cos\phi) & (sin\psi \cdot sin\theta \cdot sin\phi + cos\psi \cdot cos\phi) & (cos\theta \cdot sin\phi) & 0 & 0 & 0 \\ (cos\psi \cdot sin\theta \cdot cos\phi + sin\psi \cdot sin\phi) & \left(sin\psi \cdot sin\theta \cdot cos\phi - cos\psi \cdot \dot{\psi}\right) & (cos\theta \cdot cos\phi) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -sin\theta \end{bmatrix} \quad (100)$$

$$\begin{aligned} \mathrm{b}_{\mathrm{m}_1} = -(&-cos\psi \cdot \dot{\psi} \cdot cos\phi + sin\psi \cdot sin\phi \cdot \dot{\phi} - sin\psi \cdot \dot{\psi} \cdot sin\theta \cdot sin\phi + cos\psi \cdot cos\theta \cdot \dot{\theta} \cdot sin\phi + cos\psi \\ &\cdot sin\theta \cdot cos\phi \cdot \dot{\phi})\dot{x} \\ &- \left(-sin\psi \cdot \dot{\psi} \cdot cos\phi - cos\psi \cdot sin\phi \cdot \dot{\phi} + cos\psi \cdot \dot{\psi} \cdot sin\theta \cdot sin\phi + sin\psi \cdot cos\theta \cdot \dot{\theta} \cdot sin\phi \right. \\ &\left. + sin\psi \cdot sin\theta \cdot cos\phi \cdot \dot{\phi}\right)\dot{y} - (cos\theta \cdot cos\phi \cdot \dot{\phi} - sin\theta \cdot \dot{\theta} \cdot sin\phi)\dot{z} \end{aligned}$$

$$\begin{aligned} \mathrm{b}_{\mathrm{m}_2} = -(&cos\psi \cdot \dot{\psi} \cdot sin\phi + sin\psi \cdot cos\phi \cdot \dot{\phi} - sin\psi \cdot \dot{\psi} \cdot sin\theta \cdot cos\phi + cos\psi \cdot cos\theta \cdot \dot{\theta} \cdot cos\phi - cos\psi \\ &\cdot sin\theta \cdot sin\phi \cdot \dot{\phi})\dot{x} \\ &- \left(sin\psi \cdot \dot{\psi} \cdot sin\phi - cos\psi \cdot cos\phi \cdot \dot{\phi} + cos\psi \cdot \dot{\psi} \cdot sin\theta \cdot cos\phi + sin\psi \cdot cos\theta \cdot \dot{\theta} \cdot cos\phi \right. \\ &\left. - sin\psi \cdot sin\theta \cdot sin\phi\right)\dot{y} - (sin\theta \cdot \dot{\theta} \cdot cos\phi + cos\theta \cdot sin\phi \cdot \dot{\phi})\dot{z} \end{aligned}$$

$$\mathrm{b}_{\mathrm{m}_3} = \cos\theta \cdot \dot{\theta} \cdot \dot{\psi}$$

$$b_m = \begin{bmatrix} b_{m_1} \\ b_{m_2} \\ b_{m_3} \end{bmatrix} \quad (101)$$

Then, the modeling constraint equation in scaled form is:

$$B_m = A_m M_\eta^{-1/2}$$

And the uncontrolled system acceleration is:

$$\ddot{u}_s = a_s + B_m^+(b_m - B_m a_s)$$

The scaled modeling constraint and the uncontrolled system acceleration will be used in further equations to find the new states of the system.

5.3. Control Constraints Design

The purpose of the controller is to enforce REMUS100 AUV to follow helical path where the parametric equations are desired by the user and changes with respect to time. When a helical path is considered, the necessary constraints are the radial coordinates of the helix, the vertical displacements and two more constraints to ensure that the orientation of the vehicle is converging to the slope of the helix. So, there are five constraints to be considered in the control constraint design section.

The parametric equations for the helix that the vehicle is required to follow is given below:

$$x = rad \cdot cos(t) \quad y = rad \cdot sin(t) \quad z = -k_h \cdot t \tag{102}$$

These equations will be used while determining the control constraints required. Note that "rad" is the radius to be used in the algorithm.

5.3.1. Radial Coordinates Constraints

The radial coordinates given in equation 101 can be modified as follows:

$$\phi_{c_1} = x - rad \cdot cos(t) = 0 \quad \phi_{c_2} = y - rad \cdot sin(t) = 0 \tag{103}$$

Taking the derivatives of these relations two times:

$$\dot{\phi}_{c_1} = \dot{x} + rad \cdot sin(t) = 0 \quad \dot{\phi}_{c_2} = \dot{y} - rad \cdot cos(t) = 0 \tag{104}$$

$$\ddot{\phi}_{c_1} = \ddot{x} + rad \cdot cos(t) = 0 \quad \ddot{\phi}_{c_2} = \ddot{y} + rad \cdot sin(t) = 0 \tag{105}$$

Note that these equations can be written in the form of equation 50 as follows:

$$\ddot{x} = -rad \cdot cos(t) \quad \ddot{y} = -rad \cdot sin(t) \tag{106}$$

That would complete the control constraints of the radial components of the helix.

5.3.2. Vertical Coordinates Constraints

The vertical component and its constraint versions are given simply:

$$\phi_{c_3} = z + k_h \cdot t = 0 \tag{107}$$

$$\dot{\phi}_{c_3} = \dot{z} + k_h = 0 \quad \ddot{\phi}_{c_3} = \ddot{z} = 0 \tag{108}$$

As it can be seen, the second derivative term is already equal to zero. That would complete the control constraints for the vertical component.

5.3.3. Orthogonality Constraints

Besides the position-based constraints the vehicle has, it is important that REMUS100 AUV should point its nose to the same slope the trajectory has. This constraint is relatively complex and requires long equations to obtain.

Consider the equations of the helix given in equation 101. In terms of vector components, the REMUS100 AUV's body fixed sway and heave velocity vectors must be orthogonal to the trajectory's tangential vector. The position vector of the helix is:

$$h_p = \begin{bmatrix} rad \cdot cos(t) \\ rad \cdot sin(t) \\ -k_h \cdot t \end{bmatrix} \tag{109}$$

To find the slope of the position vector, the derivative is taken once:

$$\dot{h}_p = \begin{bmatrix} -rad \cdot sin(t) \\ rad \cdot cos(t) \\ -k_h \end{bmatrix} \tag{110}$$

This is the tangential vector belonging to the helical trajectory. This vector must be orthogonal to the sway and heave velocities of the REMUS100 AUV. The transformation matrix from inertial coordinates to body fixed coordinates is given in equation 87:

$$\begin{bmatrix} cos\psi \cdot cos\theta & sin\psi \cdot cos\theta & -sin\theta \\ -sin\psi \cdot cos\phi + cos\psi \cdot sin\theta \cdot sin\phi & cos\psi \cdot cos\phi + sin\psi \cdot sin\theta \cdot sin\phi & cos\theta \cdot sin\phi \\ sin\psi \cdot sin\phi + cos\psi \cdot sin\theta \cdot cos\phi & -cos\psi \cdot sin\phi + sin\psi \cdot sin\theta \cdot cos\phi & cos\theta \cdot cos\phi \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

This relation finds body fixed velocity vectors for the inertial velocity vector consisting of $\dot{x}$, $\dot{y}$ and $\dot{z}$. Considering that if a unit velocity vector in inertial coordinates is converted to the body frame, the body-fixed velocity will be the transformation matrices itself.

This explanation is important, as we want the body fixed velocity v and w to be orthogonal to the tangential direction of the helix. So, these two vectors' dot product must be equal to zero. Notice that the tangential vector of the helix are in inertial frame, and the coordinates of the body fixed velocities must be written in the inertial frame so that the dot product may be applied.

Eventually, for the sway to be orthogonal to the tangential direction of the helix, a dot product operation between the sway velocity in inertial coordinates (the corresponding row is taken) and the tangential vector of the helix is :

$$\begin{bmatrix} -sin\psi \cdot cos\phi + cos\psi \cdot sin\theta \cdot sin\phi & cos\psi \cdot cos\phi + sin\psi \cdot sin\theta \cdot sin\phi & cos\theta \cdot sin\phi \end{bmatrix} \begin{bmatrix} -rad \cdot sin(t) \\ rad \cdot cos(t) \\ -k_h \end{bmatrix} = 0 \tag{111}$$

The constraint can be rewritten in the form of multiplied elements as follows:

$$\phi_{c_4} = (sin\psi \cdot cos\phi + cos\psi \cdot sin\theta \cdot sin\phi)(-rad \cdot sin(t)) + (cos\psi \cdot cos\phi + sin\psi \cdot sin\theta \cdot sin\phi)(rad \cdot cos(t)) + (cos\theta \cdot sin\phi)(-k_h) = 0 \tag{112}$$

The first derivative of this relation is:

$$\dot{\phi}_{c_4} = cos\psi \cdot \dot{\psi} \cdot cos\phi \cdot \big(rad \cdot sin(t)\big) - sin\psi \cdot sin\phi \cdot \dot{\phi} \cdot \big(rad \cdot sin(t)\big) + sin\psi \cdot cos\phi \cdot \big(rad \cdot cos(t)\big) + sin\psi \cdot \dot{\psi} \cdot sin\theta \cdot sin\phi \cdot \big(rad \cdot sin(t)\big) - cos\psi \cdot cos\theta \cdot \dot{\theta} \cdot sin\phi \cdot \big(rad \cdot sin(t)\big) - cos\psi \cdot sin\theta \cdot cos\phi \cdot \dot{\phi} \cdot \big(rad \cdot sin(t)\big) - cos\psi \cdot sin\theta \cdot sin\phi \cdot \big(rad \cdot cos(t)\big) - sin\psi \cdot \dot{\psi} \cdot cos\phi \cdot \big(rad \cdot cos(t)\big) - cos\psi \cdot sin\phi \cdot \dot{\phi} \cdot \big(rad \cdot cos(t)\big) - cos\psi \cdot cos\phi \cdot \big(rad \cdot sin(t)\big) + cos\psi \cdot \dot{\psi} \cdot sin\theta \cdot sin\phi \cdot \big(rad \cdot cos(t)\big) + sin\psi \cdot cos\theta \cdot \dot{\theta} \cdot sin\phi \cdot \big(rad \cdot cos(t)\big) + sin\psi \cdot sin\theta \cdot cos\phi \cdot \dot{\phi} \cdot \big(rad \cdot cos(t)\big) - sin\psi \cdot sin\theta \cdot sin\phi \cdot \big(rad \cdot sin(t)\big) + \cdot\, sin\theta \cdot \dot{\theta} \cdot sin\phi \cdot (k_h) - cos\theta \cdot sin\phi \cdot \dot{\phi} \cdot (k_h) \tag{113}$$

The second derivative of the constraint is quite long and provided in the Appendix 1. The condition above ensures that the body fixed sway velocity of the vehicle is always orthogonal to the tangential vector of the vehicle, so that vehicle can track the trajectory. The same condition can be written for the heave body fixed velocity as follows:

$$\begin{bmatrix} sin\psi \cdot sin\phi + cos\psi \cdot sin\theta \cdot cos\phi & -cos\psi \cdot sin\phi + sin\psi \cdot sin\theta \cdot cos\phi & cos\theta \cdot cos\phi \end{bmatrix} \begin{bmatrix} -rad \cdot sin(t) \\ rad \cdot cos(t) \\ -k_h \end{bmatrix} = 0 \quad (114)$$

The multiplied form can be written as:

$$\phi_{c_5} = (sin\psi \cdot sin\phi + cos\psi \cdot sin\theta \cdot cos\phi)(-rad \cdot sin(t)) + (-cos\psi \cdot sin\phi + sin\psi \cdot sin\theta \cdot cos\phi)(rad \cdot cos(t)) + (cos\theta \cdot cos\phi)(-k_h) = 0 \quad (115)$$

The first derivative of the constraint is:

$$\dot{\phi}_{c_5} = -cos\psi \cdot \dot{\psi} \cdot sin\phi \cdot \big(rad \cdot sin(t)\big) - sin\psi \cdot cos\phi \cdot \dot{\phi} \cdot \big(rad \cdot sin(t)\big) - sin\psi \cdot sin\phi \cdot \big(rad \cdot cos(t)\big) + sin\psi \cdot \dot{\psi} \cdot sin\theta \cdot cos\phi \cdot \big(rad \cdot sin(t)\big) - cos\psi \cdot cos\theta \cdot \dot{\theta} \cdot cos\phi \cdot \big(rad \cdot sin(t)\big) + cos\psi \cdot sin\theta \cdot sin\phi \cdot \dot{\phi} \cdot \big(rad \cdot sin(t)\big) - cos\psi \cdot sin\theta \cdot cos\phi \cdot \big(rad \cdot cos(t)\big) + sin\psi \cdot \dot{\psi} \cdot sin\phi \cdot \big(rad \cdot cos(t)\big) - cos\psi \cdot cos\phi \cdot \dot{\phi} \cdot \big(rad \cdot cos(t)\big) + cos\psi \cdot sin\phi \cdot \big(rad \cdot sin(t)\big) + cos\psi \cdot \dot{\psi} \cdot sin\theta \cdot cos\phi \cdot \big(rad \cdot cos(t)\big) + sin\psi \cdot cos\theta \cdot \dot{\theta} \cdot cos\phi \cdot \big(rad \cdot cos(t)\big) - sin\psi \cdot sin\theta \cdot sin\phi \cdot \dot{\phi} \cdot \big(rad \cdot cos(t)\big) - sin\psi \cdot sin\theta \cdot cos\phi \cdot \big(rad \cdot sin(t)\big) + sin\theta \cdot \dot{\theta} \cdot cos\phi \cdot (k_h) + cos\theta \cdot sin\phi \cdot \dot{\phi} \cdot (k_h) \quad (116)$$

Again, the second derivative of the above relation is quite long and given in Appendix 1.

After obtaining the second derivatives of the constraints above, it is required to isolate terms multiplied with the second derivatives of the states to the one side of the equation and the remaining terms to the other side. Those terms will be denoted with $\ddot{\phi}_{c_4}, \ddot{\phi}_{c_5}$ *and* b_{c_4}, b_{c_5}.

5.3.4. Forming Control Constraints Equation

We have five control constraints defining the trajectory that the vehicle will follow. The important point here is that the modified constraint equations must be used since the initial state of the vehicle may not lay on the trajectory.

The constraint equation must be written in the form of equation 50:

$$A_c \ddot{c} = b_c$$

Using all the control constraints obtained in the previous sections, A_c is:

$$A_c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddot{\phi}_{c,\phi_4} & \ddot{\phi}_{c,\theta_4} & \ddot{\phi}_{c,\psi_4} \\ 0 & 0 & 0 & \ddot{\phi}_{c,\phi_5} & \ddot{\phi}_{c,\theta_5} & \ddot{\phi}_{c,\psi_5} \end{bmatrix} \quad (117)$$

And the b_c is in terms of modified constraint equations:

$$b_m = \begin{bmatrix} -rad \cdot cos(t) - \alpha_1 \dot{\phi}_{c_1} - \beta_1 \ddot{\phi}_{c_1} \\ -rad \cdot sin(t) - \alpha_2 \dot{\phi}_{c_2} - \beta_2 \ddot{\phi}_{c_2} \\ -\alpha_3 \dot{\phi}_{c_3} - \beta_1 \ddot{\phi}_{c_3} \\ b_{c_4} - \alpha_4 \dot{\phi}_{c_4} - \beta_4 \ddot{\phi}_{c_4} \\ b_{c_5} - \alpha_5 \dot{\phi}_{c_5} - \beta_5 \ddot{\phi}_{c_5} \end{bmatrix} \quad (118)$$

Then the control constraint equation in scaled form is:

$$B_c = A_c M_\eta^{-1/2}$$

5.4.Controlled System Formulation

At this point, we have all the required constraints to enforce the system to behave accordingly to the proper physical rules and force it to follow a trajectory. The plant inspected is an underactuated system where it is convenient to use the formulations given in section 4.3. Considering equation 68, if the Gaussian is chosen to be minimized, S would be equal to the identity matrix. However, equation 72 is preferred since the rank of matrix B_{cms} may change. For the equation, $\mu = 8 \times 10^{-4}$ is taken. Eventually the corresponding scaled control acceleration and force is:

$$\ddot{q}_s^c = (B_{cms}^T B_{cms} + \mu I_n)^{-1} B_{cms}^T (b_c - B_c \ddot{u}_s) \text{ and } Q_c = M^{1/2} (B_{cms}^T B_{cms} + \mu I_n)^{-1} B_{cms}^T (b_c - B_c \ddot{u}_s) \quad (119)$$

Scaled modeling acceleration and force is:

$$\ddot{q}_s^m = B_m^+ [b_m - B_m (a_s + \ddot{q}_s^c)] \text{ and } Q_m = M^{1/2}\, S B_{cms}^+ (b_m - B_m \ddot{u}_s) \quad (120)$$

At this, the system has unconstrained forces vector, modeling constraint forces vector and control forces vector. These three type of forces are added and divided to the mass matrix, so the second derivatives of the inertial states are obtained.

$$\ddot{c}_{new} = M^{-1}(Q_\eta + Q_m + Q_c) \quad (121)$$

Note the $\ddot{c}_{new}$ is a vector that includes second derivatives of all of the inertial states. Yet, this time, the derivatives of states are updated based on the constraints given to the system. Still, it is required to update the states and their first derivatives based on the new acceleration vector. For this purpose, Euler integration is used as it is a more compact and easy method. Firstly, the derivatives of the states must be updated:

$$\dot{c}_{new} = \dot{c}_{old} + \ddot{c}_{new} \cdot dt \quad (122)$$

Where dt is the time step used in the simulation. With this approach, the first derivative is updated. Now, the states can be updated as well:

$$c_{new} = c_{old} + \dot{c}_{new} \cdot dt + \frac{1}{2} \ddot{c}_{new} \cdot dt^2 \quad (123)$$

Based on these methodologies and formulas, the next section will show the results of the simulation algorithm created in MATLAB and discuss the results obtained.

5.5.Obtaining the Required Inputs

The REMUS100 AUV has three inputs which the are the two fin angles and the propeller rpm. When these input are required to be found, it is important to consider that the actuator on the vehicle act at all times to compensate the required modeling and constraint forces. So, the vehicle has natural unconstrained dynamics, but the actuator forces and moments act such that they can also provide the difference between the natural dynamics and the controlled system. Therefore, the first step would be to find the difference between constrained and unconstrained forces and moments vector.

Firstly, it is important to see that the controller operates on the inertial frame. The desired forces and moments acting on the vehicle such that it is controlled is:

$$Q_{des,\eta} = Q_\eta + Q_m + Q_c \tag{124}$$

However, to find the difference between the natural dynamic forces and moments that is acting on the body frame, the relation above must be converted back into the body frame using the reverse operations performed in equation 83:

$$Q_v = J^T Q_\eta - M_v J^{-1} \dot{J} J^{-1} \dot{\eta} \tag{125}$$

At this point, the required actuator forces and moments can be obtained by subtracting the desired vector from the unconstrained system vector:

$$Q_{act} = Q_{des} - Q \tag{126}$$

There is an important point though, all of the actuators have an effect on two of the equations of motion and their value are not identical for each of those equations. To estimate an approximate value, the least squares approach will be used.

The rudder angle has an effect on translation in y axis and rotation in z axis which the velocity coefficients are $Y_{uu\delta_r}$ and $N_{uu\delta_r}$. Instead of using only one these approaches, it is more robust to find the inputs with the least squares as follows:

$$\delta_r = \frac{\begin{bmatrix} Y_{uu\delta_r} \\ N_{uu\delta_r} \end{bmatrix}^T \begin{bmatrix} Q_{act,Y} \\ Q_{act,N} \end{bmatrix}}{u^2\left(Y_{uu\ \ r}^2 + N_{uu\ \ r}^2\right)} \tag{127}$$

Same goes for the stern angles where the motion coefficients are $Z_{uu\delta_s}$ and $M_{uu\delta_s}$. Then the angle can be approximated as follows:

$$\delta_s = \frac{\begin{bmatrix} Z_{uu\delta_s} \\ M_{uu\delta_s} \end{bmatrix}^T \begin{bmatrix} Q_{act,Z} \\ Q_{act,M} \end{bmatrix}}{u^2\left(Z_{uu\delta_s}^2 + M_{uu\delta_s}^2\right)} \tag{128}$$

Lastly, the propeller revolution can be found with same methodology considering that it has components both in translation in x-axis and rotation about x-axis:

$$n|n| = \frac{\begin{bmatrix} 1.57\times10^{-4} \\ -2.24\times10^{-5} \end{bmatrix}^T \begin{bmatrix} Q_{act,X} \\ Q_{act,K} \end{bmatrix}}{((1.57\times10^{-4})^2 + (-2.24\times10^{-5})^2)} \tag{129}$$

The value of the propeller revolution can be calculated after this step using a signum function in the according software and then it can be changed into the rpm from rad/s. It is important to mention that the least squares approach gives rise to errors since the inputs found does not fully represent the motion and they are only best fitting points between two values.

6. RESULTS AND DISCUSSIONS

In this section, the simulation file created in MATLAB will be run and necessary results shown and discussed. In the previous section, it was aimed to make the REMUS100 AUV track a helical trajectory going downward. Based on this path and the given formulations in the previous section, the simulation will be performed for 60 seconds with a time step of 0.01 seconds. The tuning parameters in the control constraints and the initial conditions are set in the script file accordingly.

The simulation will be investigated based on six outputs which are; trajectory tracking in three dimensions, control forces and moments applied, translational and angular body fixed velocities, inputs required and constraint and control errors.

6.1. Trajectory Tracking Plots

When the simulation is performed with the configurations given through the paper, the position plot of the REMUS100 AUV is given below:

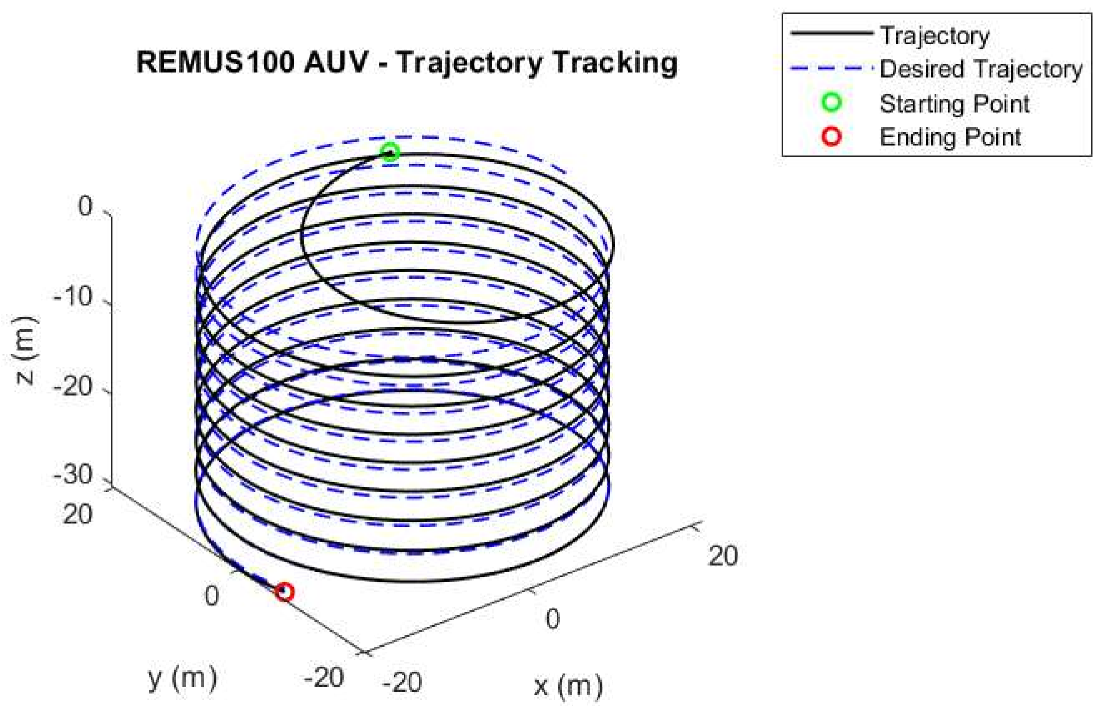

Figure 3: Trajectory Tracking 3D Plot of REMUS100 AUV

As it can be seen, at the initial times of the simulation, the vehicle is not able to follow the trajectory, which is shown in blue color, in an acceptable manner yet with the increasing time it shows a better performance. Even though at the last seconds of the simulation the motion seems almost perfect, one can see that there is still some quite low deviation from the ideal trajectory, not in circular motion of the helix but in the vertical motion of it.

Considering equation 117, one can see that the vertical control constraint does not have an acceleration term to be used in the control force applied. The modified constraint equation uses the trajectory and its first derivative, but an acceleration term would allow system to provide better response on the vertical motion.

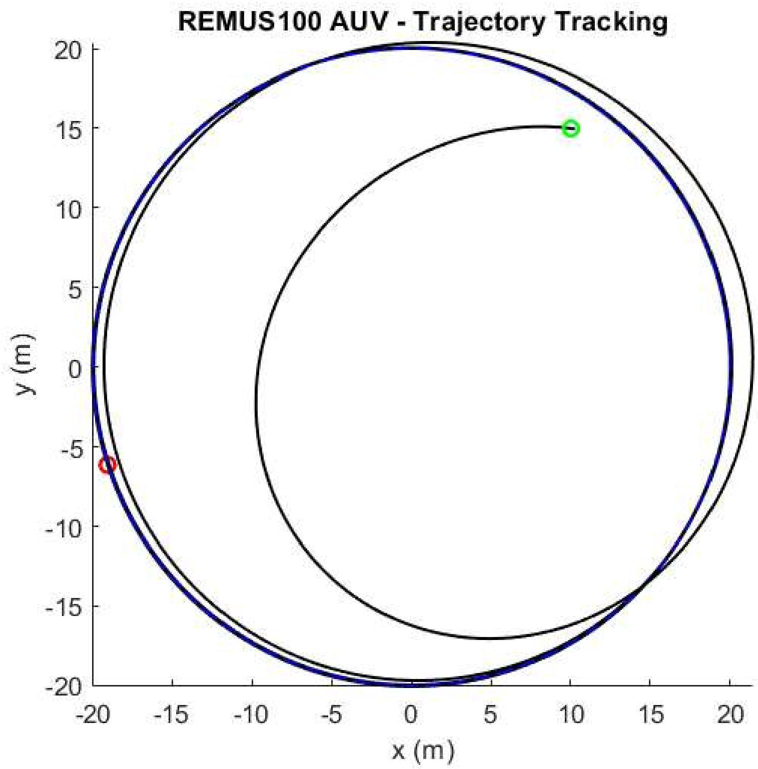

Figure 4: Trajectory Tracking 3D Plot of REMUS100 AUV Top View

On the other hand, the figure above shows that the vehicle track the circular trajectory almost in a perfect manner. It seems obvious that an acceleration term for the vertical motion is required for a better response.

6.2.Control Forces Applied

The previous configurations used in the simulation makes the control forces and moments applied on the system given in the Figure 5 below. Notice that x and y directions of the control force continuously changes like sinusoidal wave, as the vehicle moves on a circle and changes its direction every second. Besides that, there is no control force acting through the vertical direction, as there is no acceleration term in the constraint equation as it mentioned before. This conditions makes the system relatively hard to control. The moments except the moment around x axis changes continuously but since the roll motion is constrained, there is no change in moment around x axis and it is quite close to zero.

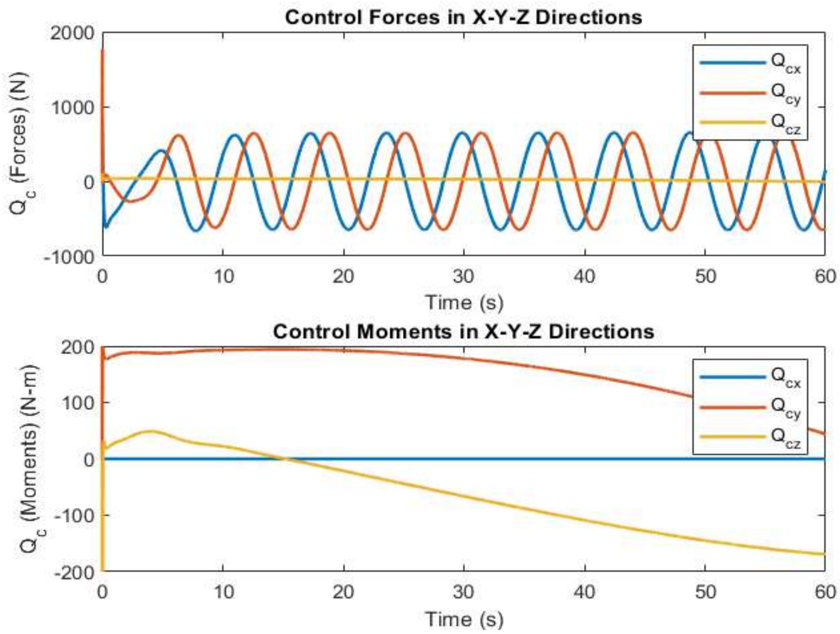

Figure 5: Control Forces and Moments Acting on the AUV

6.3.Translation and Angular Velocities

It was one of the primary concerns when defining the limitations of the system to show which speeds of the vehicle can be zero or non-zero in section 5.2. It was concluded that it is not possible to have a velocity for REMUS100 AUV unless there is an actuation on that direction. So; sway (v), heave (w) and roll (p) velocities are taken as zero and mathematical equations shown previously.

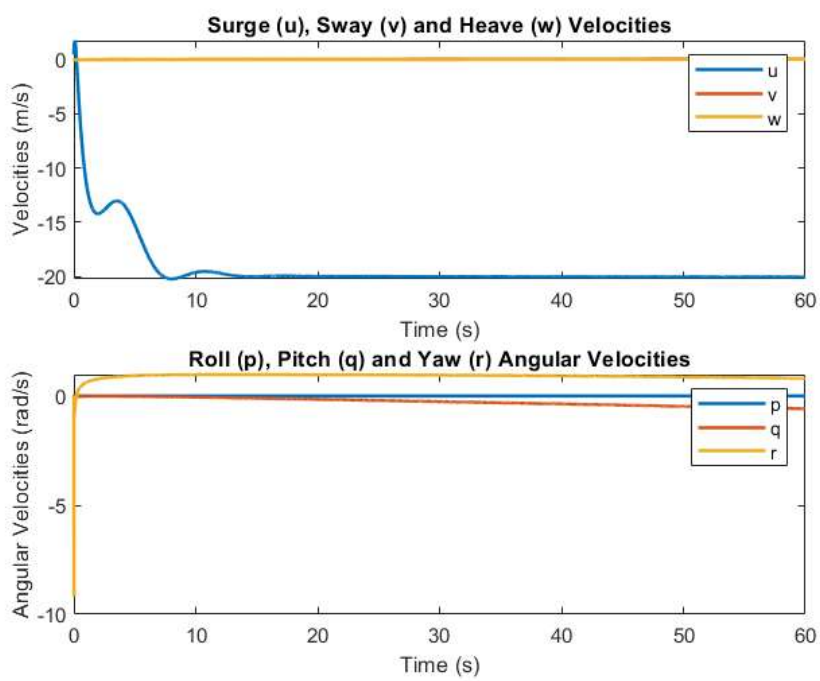

Figure 6: Body Fixed Velocities of the Vehicle

In the figure above, it can be clearly seen that the surge (u), pitch (q) and yaw (r) velocities can change so that vehicle may move accordingly but the sway (v), heave (w) and roll (p) velocities stay on zero through all times. It is correct to say that the vehicle obeys the under-actuation constraints and follow the trajectory well enough with these limitations.

6.4.Propeller RPM and Fin Angles

The input values found through the use of equations 126,127 and 128. Noting that they are approximated values using least squares, their calculation is subjected to some errors. These error on the other hand are relatively low yet still requires to be checked in real life application.

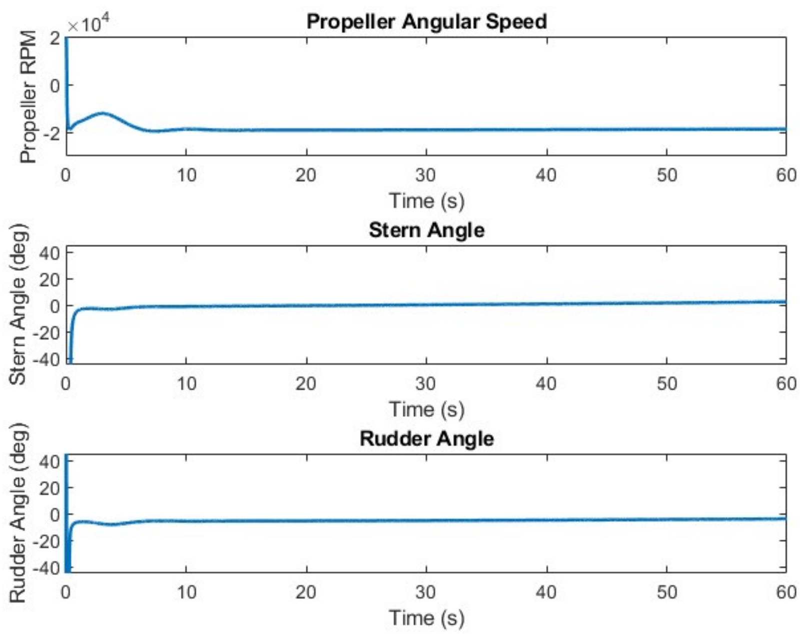

Figure 7: Propeller Angular Speed and Fin Angles

The figure above shows the inputs versus time. At the beginning, the inputs have quite high values. This is expected as the equation used to calculate them are divided to velocity which has low values at the initial times of the simulation, causing inputs to be high.

6.5.Constraint and Control Errors

Udwadia-Kalaba approach states that, no matter what the control constraints are, modeling constraints must be satisfied at all times so that the system shows appropriate behavior to the physical rules. That implies the error related to the modeling constraints must be quite small at all instants of time. The figure below shows the modeling constraints errors and as it can be seen, they are all on the order of 10^{-10} levels which is the expected and the required condition.

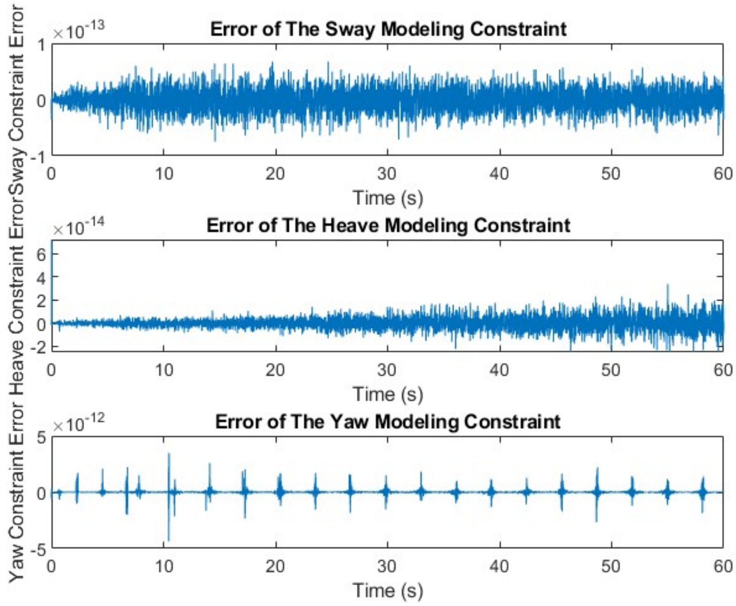

Figure 8: Modeling Constraints Error

The control constraint on the other hand, may show some error in the controller simulation. The figure below shows those errors and they expected and quite good on circular motion and orientation constraints. The circular motion shows some deviation at the initial times but it goes about zero after 10 seconds. However, the vertical component constraint of the helix shows higher errors and it deviates to higher values and comes back at later times then 60 seconds. This shows the importance of the acceleration term of constraints, which there isn't for the vertical constraint.

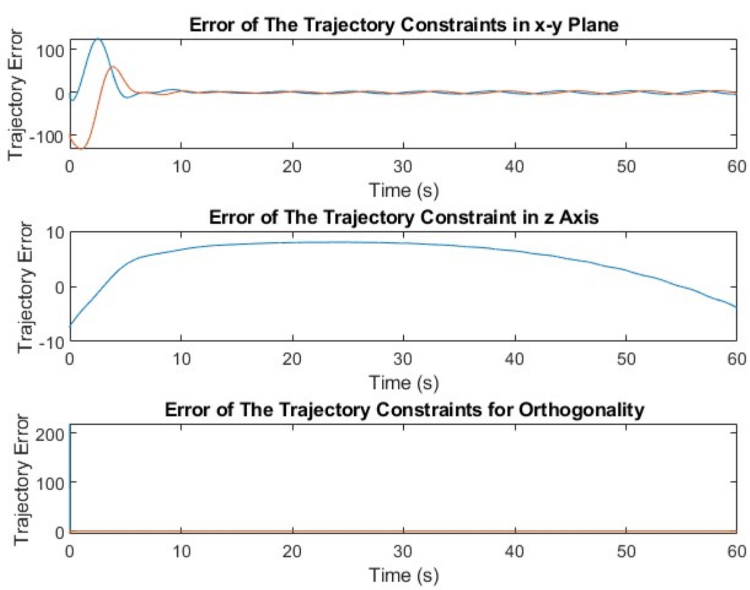

Figure 9: Control Constraints Error

7. CONCLUSION

This study provides a control methodology for REMUS100 AUV by applying constraint on the vehicle and using those constraints as forces and moments applied so that vehicle can behave properly according to the physical models and it may also track a user defined trajectory. For this purpose, UK approach is used. REMUS100 AUV has highly complex nonlinear dynamic equations, yet, in this approach provided no linearization or numerical approximation is used. Instead, the dynamic model of the system is accepted as the unconstrained system and constraint and control forces applied on this unconstrained system so that we would have a controlled plant.

One of the critical points was the fact that, in UK approach, the control constraints are written in terms of inertial frame states but the model of the REMUS100 AUV was given in body frame, which requires a conversion. During the process, it was a quite staggering problem to come up with a system matrix having eigenvalues lower than zero. This problem was solved using the fact that, kinetic energy of the system can't change from one coordinate system to another, providing a methodology to make conversion and accepting mass matrix having eigenvalues higher than zero. This problem was solvable for REMUS100 AUV but different plants may require altered approaches.

On the other hand, obtaining the control constraints in UK methodology is another problem since the entire control constraints is specific to that one single trajectory. Considering that the operation can be very lengthy as it was given in Appendix 1, the method may not be a suitable approach for the plants that trajectories change continuously. Moreover, an approach where the trajectory's derivative is taken automatically and substituted in the control constraint equation would provide an easier approach.

When the controller is inspected in general, it is possible to say that it provides a relatively easy approach for highly complex systems but the control engineer must have a very good knowledge of the plant so that it would be possible to determine the constraints or necessary coordinate transformations without any lack of information.

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9. APPENDIX

→ Second Derivative

$$-s3\cdot\dot\psi^2\cdot c1\cdot(r\cdot\sin(t)) + c3\cdot\ddot\psi\cdot c1\cdot(r\cdot\sin(t)) - c3\cdot\dot\psi\cdot s1\cdot\dot\phi\cdot(r\cdot\sin(t)) + c3\cdot\dot\psi\cdot c1\cdot(r\cdot\cos(t)) - c3\cdot\dot\psi\cdot s1\cdot\dot\phi\cdot(r\cdot\sin(t))$$
$$- s3\cdot c1\cdot\dot\phi^2\cdot(r\cdot\sin(t)) - s3\cdot s1\cdot\ddot\phi\cdot(r\cdot\sin(t)) - s3\cdot s1\cdot\dot\phi\cdot(r\cdot\cos(t)) + c3\cdot\dot\psi\cdot c1\cdot(r\cdot\cos(t)) - s3\cdot s1\cdot\dot\phi\cdot(r\cdot\cos(t))$$
$$- s3\cdot c1\cdot(r\cdot\sin(t)) + c3\cdot\dot\psi^2\cdot s2\cdot s1\cdot(r\cdot\sin(t)) + s3\cdot\ddot\psi\cdot s2\cdot s1\cdot(r\cdot\sin(t)) + s3\cdot\dot\psi\cdot c2\cdot\dot\theta\cdot s1\cdot(r\cdot\sin(t))$$
$$+ s3\cdot\dot\psi\cdot s2\cdot c1\cdot\dot\phi\cdot(r\cdot\sin(t)) + s3\cdot\dot\psi\cdot s2\cdot s1\cdot(r\cdot\cos(t)) + s3\cdot\dot\psi\cdot c2\cdot\dot\theta\cdot s1\cdot(r\cdot\sin(t)) + c3\cdot s2\cdot\dot\theta^2\cdot s1\cdot(r\cdot\sin(t))$$
$$- c3\cdot c2\cdot\ddot\theta\cdot s1\cdot(r\cdot\sin(t)) - c3\cdot c2\cdot\dot\theta\cdot c1\cdot\dot\phi\cdot(r\cdot\sin(t)) - c3\cdot c2\cdot\dot\theta\cdot s1\cdot(r\cdot\cos(t)) + s3\cdot\dot\psi\cdot s2\cdot c1\cdot\dot\phi\cdot(r\cdot\sin(t))$$
$$- c3\cdot c2\cdot\dot\theta\cdot c1\cdot\dot\phi\cdot(r\sin(t)) + c3\cdot s2\cdot s1\cdot\dot\phi^2\cdot(r\cdot\sin(t)) - c3\cdot s2\cdot c1\cdot\ddot\phi(r\cdot\sin(t)) - c3\cdot s2\cdot c1\cdot\dot\phi\cdot(r\cdot\cos(t))$$
$$+ s3\cdot\dot\psi\cdot s2\cdot s1\cdot(r\cdot\cos(t)) - c3\cdot c2\cdot\dot\theta\cdot s1\cdot(r\cdot\cos(t)) - c3\cdot s2\cdot c1\cdot\dot\phi\cdot(r\cdot\cos(t)) + c3\cdot s2\cdot s1\cdot(r\cdot\sin(t))$$
$$- c3\cdot\dot\psi^2\cdot c1\cdot(r\cdot\cos(t)) - s3\cdot\ddot\psi\cdot c1\cdot(r\cdot\cos(t)) + s3\cdot\dot\psi\cdot s1\cdot\dot\phi\cdot(r\cdot\cos(t)) + s3\cdot\dot\psi\cdot c1\cdot(r\cdot\sin(t))$$
$$+ s3\cdot\dot\psi\cdot s1\cdot\dot\phi\cdot(r\cdot\cos(t)) - c3\cdot c1\cdot\dot\phi^2\cdot(r\cdot\cos(t)) - c3\cdot s1\cdot\ddot\phi\cdot(r\cdot\cos(t)) + c3\cdot s1\cdot\dot\phi\cdot(r\cdot\sin(t))$$
$$+ s3\cdot\dot\psi\cdot c1\cdot(r\cdot\sin(t)) + c3\cdot s1\cdot\dot\phi\cdot(r\cdot\sin(t)) - c3\cdot c1\cdot(r\cdot\cos(t)) - s3\cdot\dot\psi^2\cdot s2\cdot s1\cdot(r\cdot\cos(t)) + c3\cdot\ddot\psi\cdot s2\cdot s1\cdot(r\cdot\cos(t))$$
$$+ c3\cdot\dot\psi\cdot c2\cdot\dot\theta\cdot s1\cdot(r\cdot\cos(t)) + c3\cdot\dot\psi\cdot s2\cdot c1\cdot\dot\phi\cdot(r\cdot\cos(t)) - c3\cdot\dot\psi\cdot s2\cdot s1\cdot(r\cdot\sin(t)) + c3\cdot\dot\psi\cdot c2\cdot\dot\theta\cdot s1\cdot(r\cdot\cos(t))$$
$$- s3\cdot s2\cdot\dot\theta^2\cdot s1\cdot(r\cdot\cos(t)) + s3\cdot c2\cdot\ddot\theta\cdot s1\cdot(r\cdot\cos(t)) + s3\cdot c2\cdot\dot\theta\cdot c1\cdot\dot\phi\cdot(r\cdot\cos(t)) - s3\cdot c2\cdot\dot\theta\cdot s1\cdot(r\cdot\sin(t))$$
$$+ c3\cdot\dot\psi\cdot s2\cdot c1\cdot\dot\phi\cdot(r\cdot\cos(t)) + s3\cdot c2\cdot\dot\theta\cdot c1\cdot\dot\phi\cdot(r\cdot\cos(t)) - s3\cdot s2\cdot s1\cdot\dot\phi^2\cdot(r\cdot\cos(t)) + s3\cdot s2\cdot c1\cdot\ddot\phi\cdot(r\cdot\cos(t))$$
$$+ s3\cdot s2\cdot c1\cdot\dot\phi\cdot(r\cdot\sin(t)) - c3\cdot\dot\psi\cdot s2\cdot s1\cdot(r\cdot\sin(t)) - s3\cdot c2\cdot\dot\theta\cdot s1\cdot(r\cdot\sin(t)) - s3\cdot s2\cdot c1\cdot\dot\phi\cdot(r\cdot\sin(t)) - s3\cdot s2\cdot s1\cdot(r\cdot\cos(t))$$
$$+ c2\cdot\dot\theta^2\cdot s1\cdot(k) + s2\cdot\ddot\theta\cdot s1\cdot(k) + s2\cdot\dot\theta\cdot c1\cdot\dot\phi\cdot(k) + s2\cdot\dot\theta\cdot s1\cdot\dot\phi\cdot(k) - c2\cdot c1\cdot\dot\phi^2\cdot(k) - c2\cdot s1\cdot\ddot\phi\cdot(k)$$

→ Second Derivative

$$s3\cdot\dot\psi^2\cdot s1\cdot(r\cdot\sin(t)) - c3\cdot\ddot\psi\cdot s1\cdot(r\cdot\sin(t)) - c3\cdot\dot\psi\cdot c1\cdot\dot\phi\cdot(r\cdot\sin(t)) - c3\cdot\dot\psi\cdot s1\cdot(r\cdot\cos(t)) - c3\cdot\dot\psi\cdot c1\cdot\dot\phi\cdot(r\cdot\sin(t))$$
$$+ s3\cdot s1\cdot\dot\phi^2\cdot(r\cdot\sin(t)) - s3\cdot c1\cdot\ddot\phi\cdot(r\cdot\sin(t)) - s3\cdot c1\cdot\dot\phi\cdot(r\cdot\cos(t)) - c3\cdot\dot\psi\cdot s1\cdot(r\cdot\cos(t)) - s3\cdot c1\cdot\dot\phi\cdot(r\cdot\cos(t))$$
$$+ s3\cdot s1\cdot(r\cdot\sin(t)) + c3\cdot\dot\psi^2\cdot s2\cdot c1\cdot(r\cdot\sin(t)) + s3\cdot\ddot\psi\cdot s2\cdot c1\cdot(r\cdot\sin(t)) + s3\cdot\dot\psi\cdot c2\cdot\dot\theta\cdot c1\cdot(r\cdot\sin(t)) - s3\cdot\dot\psi\cdot s2\cdot s1\cdot\dot\phi\cdot(r\cdot\sin(t))$$
$$+ s3\cdot\dot\psi\cdot s2\cdot c1\cdot(r\cdot\cos(t)) + s3\cdot\dot\psi\cdot c2\cdot\dot\theta\cdot c1\cdot(r\cdot\sin(t)) + c3\cdot s2\cdot\dot\theta^2\cdot c1\cdot(r\cdot\sin(t)) - c3\cdot c2\cdot\ddot\theta\cdot c1\cdot(r\cdot\sin(t))$$
$$+ c3\cdot c2\cdot\dot\theta\cdot s1\cdot\dot\phi\cdot(r\cdot\sin(t)) - c3\cdot c2\cdot\dot\theta\cdot c1\cdot(r\cdot\cos(t)) - s3\cdot\dot\psi\cdot s2\cdot s1\cdot\dot\phi\cdot(r\cdot\sin(t)) + c3\cdot c2\cdot\dot\theta\cdot s1\cdot\dot\phi\cdot(r\cdot\sin(t))$$
$$+ c3\cdot s2\cdot c1\cdot\dot\phi^2\cdot(r\cdot\sin(t)) + c3\cdot s2\cdot s1\cdot\ddot\phi\cdot(r\cdot\sin(t)) + c3\cdot s2\cdot s1\cdot\dot\phi\cdot(r\cdot\cos(t)) + s3\cdot\dot\psi\cdot s2\cdot c1\cdot(r\cdot\cos(t))$$
$$- c3\cdot c2\cdot\dot\theta\cdot c1\cdot(r\cdot\cos(t)) + c3\cdot s2\cdot s1\cdot\dot\phi\cdot(r\cdot\cos(t)) + c3\cdot s2\cdot c1\cdot(r\cdot\sin(t)) + c3\cdot\dot\psi^2\cdot s1\cdot(r\cdot\cos(t) + s3\cdot\ddot\psi\cdot s1\cdot(r\cdot\cos(t))$$
$$+ s3\cdot\dot\psi\cdot c1\cdot\dot\phi\cdot(r\cdot\cos(t)) - s3\cdot\dot\psi\cdot s1\cdot(r\cdot\sin(t)) + s3\cdot\dot\psi\cdot c1\cdot\dot\phi\cdot(r\cdot\cos(t)) + c3\cdot s1\cdot\dot\phi^2\cdot(r\cdot\cos(t)) - c3\cdot c1\cdot\ddot\phi\cdot(r\cdot\cos(t))$$
$$+ c3\cdot c1\cdot\dot\phi\cdot(r\cdot\sin(t)) - s3\cdot\dot\psi\cdot s1\cdot(r\cdot\sin(t)) + c3\cdot c1\cdot\dot\phi\cdot(r\cdot\sin(t)) + c3\cdot s1\cdot(r\cdot\cos(t)) - s3\cdot\dot\psi^2\cdot s2\cdot c1\cdot(r\cdot\cos(t))$$
$$+ c3\cdot\ddot\psi\cdot s2\cdot c1\cdot(r\cdot\cos(t)) + c3\cdot\dot\psi\cdot c2\cdot\dot\theta\cdot c1\cdot(r\cdot\cos(t)) - c3\cdot\dot\psi\cdot s2\cdot s1\cdot\dot\phi\cdot(r\cdot\cos(t)) - c3\cdot\dot\psi\cdot s2\cdot c1\cdot(r\cdot\sin(t))$$
$$+ c3\cdot\dot\psi\cdot c2\cdot\dot\theta\cdot c1\cdot(r\cdot\cos(t)) - s3\cdot s2\cdot\dot\theta^2\cdot c1\cdot(r\cdot\cos(t)) + s3\cdot c2\cdot\ddot\theta\cdot c1\cdot(r\cdot\cos(t)) + s3\cdot c2\cdot\dot\theta\cdot s1\cdot\dot\phi\cdot(r\cdot\cos(t))$$
$$- s3\cdot c2\cdot\dot\theta\cdot c1\cdot(r\cdot\sin(t)) - c3\cdot\dot\psi\cdot s2\cdot s1\cdot\dot\phi\cdot(r\cdot\cos(t)) - s3\cdot c2\cdot\dot\theta\cdot s1\cdot\dot\phi\cdot(r\cdot\cos(t)) - s3\cdot s2\cdot c1\cdot\dot\phi^2\cdot(r\cdot\cos(t))$$
$$- s3\cdot s2\cdot s1\cdot\ddot\phi\cdot(r\cdot\cos(t)) + s3\cdot s2\cdot s1\cdot\dot\phi\cdot(r\cdot\sin(t)) - c3\cdot\dot\psi\cdot s2\cdot c1\cdot(r\cdot\sin(t)) - s3\cdot c2\cdot\dot\theta\cdot c1\cdot(r\cdot\sin(t))$$
$$+ s3\cdot s2\cdot s1\cdot\dot\phi\cdot(r\cdot\sin(t)) - s3\cdot s2\cdot c1\cdot(r\cdot\cos(t)) + c2\cdot\dot\theta^2\cdot c1\cdot(k) + s2\cdot\ddot\theta\cdot c1\cdot(k) - s2\cdot\dot\theta\cdot s1\cdot\dot\phi\cdot(k) - s2\cdot\dot\theta\cdot s1\cdot\dot\phi\cdot(k)$$
$$+ c2\cdot c1\cdot\dot\phi^2\cdot(k) + c2\cdot s1\cdot\ddot\phi\cdot(k)$$

Appendix 1: Second Derivative of the Orientation Control Constraints