Stochastic Approximation of Adaptive Voter Models

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1 Introduction

During the last decades, network science has reached maturity and now irrigates various domains. Nonetheless, the subdomain of co-evolutionary frameworks is still at its infancy. By *co-evolutionary*, we mean models where both the graph evolves as well as its nodes [1]. Modelling this co-evolution is of paramount importance since real-world networks are rarely static but on the contrary highly dynamic and open. For instance, in the context of a contagion process, infected people shall get disconnected from the others (confinement) to prevent the virus from spreading. In opinion dynamics, people sharing common opinions are more likely to get connected (homophily) and on the contrary tend to dismiss disagreeing people (selective exposure). On the case we currently focus, namely the *Adaptive Voter Model* (AVM), most of the existing results rely on approximate ordinary differential equations (ODEs) for some global parameters such as the total number of links [2], [3]. Furthermore, most of the work consider models with constant degree distribution. In contrast, the model proposed below allows each node's degree to vary along time. Second, the approximations we used keep track of the overall graph, giving some new insights into the micro-structure of the network process.

2 Model

Let us define a population of agents of size $K \geq 1$,where each agent $k \in [K]$ is endowed with a binary value $x_k = \pm 1$ called a *spin*. The spin can represent an orientation, a preference, etc. The agents interact through a directed unweighted graph A. We then define a Markov process (X^K, A^K) (for sake of conciseness, the dependence in K will be omitted when clear from the context) where $X(t) \in \{+1, -1\}^K$ is the spin profile at time t and $A(t) \in \{0,1\}^{K^2}$ is the network at time t supposed to be unweighted and directed, $X_k(t) = \pm 1$ being the orientation of agent k. The whole process (X, A) then evolves in the finite state space $\mathscr{S}_K := \{+1, -1\}^K \times \{0,1\}^{K^2}$. The following rates Φ, Γ and B entirely define the jump process, and are respectively the flipping rate, link creation rate and link deletion rate. When we say that agent k *flips*, the k^{th} component of the spin profile changes its sign.

definition of the jump rates. For all $k,l,m \in [K]$, define the *global linkage with homophily* system through its stochastic transition rates:

$$\text{flipping: } (X,A) \longrightarrow (X-2x_k e_k, A) \text{ at rate } \Phi(k;x,A) = \phi \sum_{j\in[K]} a_{kj} 1_{(x_k \neq x_j)}$$
$$\text{link creation: } (X,A) \longrightarrow (X, A+e_{lm}) \text{ at rate } \Gamma(lm;x,A) = \gamma(1-a_{lm}) 1_{(x_l = x_m)}$$
$$\text{link deletion: } (X,A) \longrightarrow (X, A-e_{lm}) \text{ at rate } B(lm;x,A) = \beta a_{lm} 1_{(x_l \neq x_m)},$$

for all $(x,A) \in \mathscr{S}_K$, $k,l,m \in [K]$, and where $\gamma, \beta, \phi > 0$ are the model parameters. By definition $e_k \in \mathbb{R}^K$ is the k^{th} vector of the canonical basis of $\mathbb{R}^K$ and $e_{lm} := e_l e_m^T \in \mathbb{R}^{K^2}$. The linkage is said *global* because two agents with same spin can get connected regardless of the current network structure. Naturally, there are many refinements and variants of this basic edge-centric model (see for instance [4] and references therein).

3 Results

For all $f \in \mathbb{R}^{\mathscr{S}_K}$, the generator $\Omega : \mathbb{R}^{\mathscr{S}_K} \mapsto \mathbb{R}^{\mathscr{S}_K}$ of the defined jump process writes

$$(\Omega f)(x,A) = \sum_k \Phi(k,x,A)\Big[f(x-2x_k e_k, A) - f(x,A)\Big] +$$
$$\sum_{lm} \Gamma(lm,x,A)\Big[f(x,A+e_{lm}) - f(x,A)\Big] + B(lm,x,A)\Big[f(x,A-e_{lm}) - f(x,A)\Big].$$

Kolmogorov-Backward claims that for all $f \in \mathbb{R}^{\mathscr{S}_K}$: $\partial_t P_t(f) = P_t \circ \Omega(f)$, where (P_t) is the semi-group of the process $P_t(f) = \mathbb{E} f\Big(X(t), A(t)\Big)$.

Now, suppose the graph is always dense at all times: $\frac{\deg(k:A(t))}{K} > C_{den}\ \forall k \in [K],\ t \geq 0$. In this regime, we can assume that two agent k and j are quasi pairwise independent and the variable X_k interacts with the others only throughout the two quantities $\frac{1}{\deg(k)}\Big(\sum_j A_{kj} \mathbb{1}_{(x_k = x_j)}, \sum_j A_{kj} \mathbb{1}_{(x_k \neq x_j)}\Big)$. Hence, when taking $f(x,a) = (x_k, a_{lm})_{k,lm}$ and assume independence between particles, we get the so-called NIMFA system [5]

$$\partial_t x_k(t) = \sum_j a_{kj}(x_j - x_k) \text{ and } \partial_t a_{lm} = \gamma(1-a_{lm})H_{lm} - \beta a_{lm}(1-H_{lm}), \tag{1}$$

where $x_k(t) = \mathbb{E} X_k(t)$, $a_{lm} = \mathbb{P}(A_{lm}(t) = 1) = \mathbb{E} A_{lm}(t)$ and $H_{lm} = \frac{1+x_l x_m}{2}$.

The solution (x,a) of (1) thus evolves in $[-1,1]^K \times [0,1]^{K^2}$, and its equilibria are tractable.

Proposition 1. *Consider the global linkage with homophily. Then, its associated NIMFA system has two distinct sets of equilibria:*

- *the polarized ones* $\Big\{(x,a) : x_k = \pm 1\ \forall k \text{ and } x_l = x_m \iff a_{lm} = 1 \text{ or } x_l \neq x_m \iff a_{lm} = 0\Big\}$;
- *the consensual ones* $\Big\{(x,a) : \exists c,\ x_l = x_m = c \text{ and } a_{lm} = f(c,g) \forall l,m\Big\}$, *with* $g = \frac{\gamma}{\beta}$.

We can notice that the stationary edge $f(c,g)$ only depends on the consensus value and on the ratio g. Furthermore, it may be uncovered that in the dense case, the NIMFA is a *good* approximation in the sense of the next assertion (see also fig 1, left).

Conjecture 1. Let T be a fixed time horizon. When the flipping rate parameter is properly scaled, namely $\phi = O(\frac{1}{K})$, and the entrance law of the graph is dense, meaning $A(0)$ is dense almost surely, then we get for any Euclidian norm $||.||$,

$$\forall t \in [0,T] : ||\mathbb{E}\left[(X^K, A^K)(t) - (x,a)(t)\right]|| \to 0 \text{ as } K \longrightarrow +\infty. \tag{2}$$

Another important feature of AVMs is the emergence of *filter bubbles*, and the metric used to quantify them is the *discordance* $\mathscr{D} := \frac{1}{K^2}\sum_{l,m} a_{lm}\mathbb{1}_{(x_l \neq x_m)}$. It is widely known that voter models of finite-size converge (in finite time) toward discordance-free configurations. In this context, one of the main issues is to compute threshold values under which discordance persists (metastability) or on the contrary rapidly extincts, according to the model parameters ϕ, β, γ. We expect that for $\phi \gg \beta$, $\mathscr{D}$ persists (see fig. 1, right).

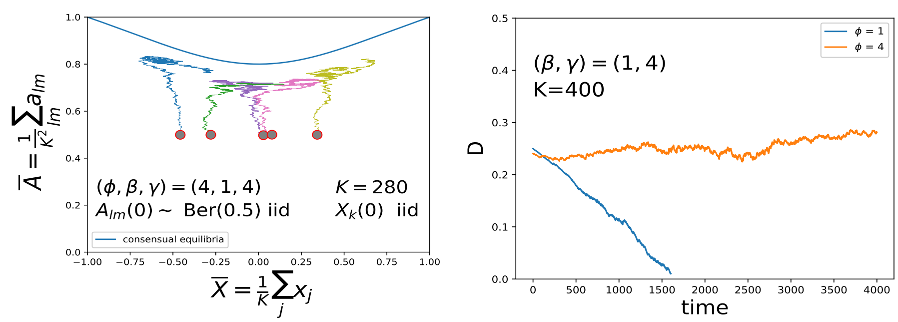

Fig. 1. On the left, we compare on the $\overline{X} - \overline{A}$ axis the consensual equilibria computed in proposition 1 with several random walks of same parameters starting from different configurations (the grey dots). On the right, the evolution of discordance $\mathscr{D}$ across time for two different values of ϕ, (β, γ) are kept constant at $(1,4)$. In one case ($\phi = 4$), discordance maintains. In the other case ($\phi = 1$), discordance vanishes.

Research Perspectives. The work in progress is dedicated to bring a rigorous answer to (2). Furthermore, some exact results can be provided using probabilistic tools such as the notion of *propagation of chaos*. Finally, the proposed methodology can be naturally applied to a wide class of adaptive networks, especially to epidemiological models.

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