Math 136 Review Note

Lecture 1

Addition of Vectors: $\vec{u}+\vec{v} = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}$ Scalar Multiplication: Let $c \in \mathbb{R}$, $c\vec{u} = \begin{bmatrix} c\cdot u_1 \\ \vdots \\ c\cdot u_n \end{bmatrix}$ Note: Same line different mag. direct.

Properties of Vectors Addition:

1. Commutative: $\vec{u}+\vec{w} = \vec{w}+\vec{u}$
2. Associative: $(\vec{u}+\vec{w})+\vec{v} = \vec{u}+(\vec{w}+\vec{v})$
3. Additive Identity: $\vec{u} + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ (zero vector) $= \vec{u}$
4. Additive Inverse: $-\vec{u} = \begin{bmatrix} -u_1 \\ \vdots \\ -u_n \end{bmatrix}$

Properties of Scalar Multiplication

1. Distributive: $c(\vec{u}+\vec{w}) = c\vec{u}+c\vec{w}$, $\vec{u}(c+d) = c\vec{u}+d\vec{u}$
2. Associative: $(cd)\vec{v} = c(d\vec{v})$
3. $\vec{u}\cdot 0 = 0$
4. $c\vec{u}=0$ ($c=0$ or $\vec{u}=0$)

$\vec{e_i}$ i^{th} component is 1. others are 0

Standard Basis: the set of $\vec{e_i}$ $\mathcal{E}=\{\vec{e_1}, \vec{e_2} \dots \vec{e_n}\}$ Ex) $\mathbb{R}^3 \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Lecture 2

$\mathbb{C}^n$ set of complex vectors, denotes as $\left\{\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, z_n \in \mathbb{C}\right\}$ Note: no dot product, but inner.

Dot product $\vec{u}\cdot\vec{v} = u_1\cdot v_1 + u_2\cdot v_2 \dots u_n\cdot v_n$ Note: result is scalar, Not in $\mathbb{C}^n$

Properties of Dot product

1. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ (symmetry)
2. $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$
3. $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v})$

linearity

4. non-negativity $\vec{v}\cdot\vec{v} \geq 0$, $\vec{v}\cdot\vec{v}=0 \Leftrightarrow \vec{v}=0$

Length/norm/magnitude $\sqrt{\vec{v}\cdot\vec{v}} = \|\vec{v}\|$

Note: always non-negative, length $=0 \Leftrightarrow \vec{v}=0$

Unit Vector $\|\vec{v}\|=1$. $\vec{v}\in\mathbb{R}^n$

Normalization $\vec{v}\in\mathbb{R}^n$, $\vec{v}\neq\vec{0}$, $\hat{v} = \dfrac{\vec{v}}{\|\vec{v}\|}$ $\dfrac{\text{vector}}{\text{length}}$

Find Angles between Vectors

$\vec{u}, \vec{v} \in \mathbb{R}^n$, both non-negative, then angle $\theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)$ ← dot product ← length

and $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$

Note: between $\theta \in [0, \pi]$

Orthogonal/perpendicular $\vec{u}\cdot\vec{v} = 0$

Note: $\vec{0}$ is orthogonal to every vector $\in \mathbb{R}^n$

Projection ($\vec{v}$ onto $\vec{w}$)

$\text{proj}_{\vec{w}}\vec{v} = \left(\frac{\vec{w}\cdot\vec{v}}{\|\vec{w}\|^2}\right)\vec{w}$ $\left(\frac{\text{dot product}}{(\text{length})^2}\right)$ vector

Perpendicular ($\vec{v}$ onto $\vec{w}$)

$\text{perp}_{\vec{w}}\vec{v} = \vec{v} - \text{proj}_{\vec{w}}\vec{v}$

Fields / $\mathbb{F}$

Ex) $\mathbb{C}, \mathbb{R}$ can apply math operations

Cross Product

$$\vec{u}\times\vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

Note: Only applies in $\mathbb{R}^3$

Properties of Cross Product

1. $\vec{z}\cdot\vec{u} = 0 \;\|\; \vec{z}\cdot\vec{v} = 0$ $(\vec{z} = \vec{u}\times\vec{v})$
2. $\vec{v}\times\vec{u} = -\vec{z}$
3. if $\vec{u}, \vec{v} \neq \vec{0}$, then $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta$

Note: it is the Area of parallelogram → where θ is the angle between $\vec{u}, \vec{v}$.

Linearity of Cross Product

1. $(\vec{u}+\vec{v})\times\vec{w} = (\vec{u}\times\vec{w}) + (\vec{v}\times\vec{w})$
2. $(c\vec{u})\times\vec{w} = c(\vec{u}\times\vec{w})$
2. $\vec{u}\times(\vec{v}+\vec{w}) = (\vec{u}\times\vec{v}) + (\vec{u}\times\vec{w})$
4. $\vec{u}\times c\vec{w} = c(\vec{u}\times\vec{w})$

Note: when calculating length $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$, the number that can be squared must write $|n|$ (absolute value)

Proposition 1.5.7

Let $c \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^n$, then $\|c\vec{v}\| = |c|\,\|\vec{v}\|$

Proposition 1.6.9

Let $\vec{v}, \vec{w} \in \mathbb{R}^n$, the projection and perpendicular of $\vec{v}$ onto $\vec{w}$ is orthogonal

$$\text{proj}_{\vec{w}}\vec{v}\cdot\text{perp}_{\vec{w}}\vec{v} = 0$$

Lecture 4.

Linear Combination $c_1\vec{v}_1 + c_2\vec{v}_2 \dots c_n\vec{v}_n$ (scalar: c, vector: $\vec{v}$) $\quad C \in \mathbb{F}^n \quad \vec{v}_n \in \mathbb{R}^n$

Span/ Spanning Set $\underbrace{\text{Span}\{\vec{v}_1, \vec{v}_2 \dots \vec{v}_n\}}_{\text{spanning set}} = \underbrace{\{c_1\vec{v}_1 + c_2\vec{v}_2 \cdots c_n\vec{v}_n,\ c_1, c_2 \dots c_n \in \mathbb{F}^n\}}_{\text{Set of linear combination}}$

Note: $\text{Span}\{\vec{v}_1, \vec{v}_2 .. \vec{v}_n\} \neq c_1\vec{v}_1 + c_2\vec{v}_2 \cdots c_n\vec{v}_n$

Span of one vector a line $\qquad$ **Span of two vector** a plane

Parametric Equation of a Line

Let p, q be fixed number $\in \mathbb{R}$, $q \neq 0$, then the equation through the point (x_1, y_1) with slope p/q is

$$x = x_1 + qt \qquad y = y_1 + pt$$

Note: Each value of t gives a point on the line

$q = 0$, $x = x_1$, $y = y_1 + pt$ => Vertical line

Vector Equation of a line in $\mathbb{R}^2$

$$\vec{\ell} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + t\begin{bmatrix} q \\ p \end{bmatrix},\ t \in \mathbb{R}$$

Non-zero, direction

line that goes through this vector terminal point of $\begin{bmatrix} x \\ y \end{bmatrix}$

terminal point on the line

Line in $\mathbb{R}^2$ $\quad \mathcal{L} = \{\vec{u} + t\vec{v} : t \in \mathbb{R}\} \quad \vec{v} \neq 0,$

a line through $\vec{u}$ with direction $\vec{v}$

lecture 5

Line in $\mathbb{R}^n$

$l = \vec{u} + t\vec{v}$, $t \in \mathbb{R}$ (vector equation)

Note: l_1 has same direction of $l_2 \iff c\vec{v}_1 = \vec{v}_2$

$\mathcal{L} = \{\vec{u} + t\vec{v}, t \in \mathbb{R}\}$ lines in $\mathbb{R}^n$ through $\vec{u}$ with direction $\vec{v}$

Parametric Equations

$\vec{l} = \vec{u} + t\vec{v}$, $t \in \mathbb{R}$ (vector equation)

$l_1 = u_1 + tv_1$, $l_2 = u_2 + tv_2 \cdots l_n = u_n + tv_n$, $t \in \mathbb{R}$

Vector Equation of a plane in $\mathbb{R}^n$

(through the Origin) $\vec{p} = s\vec{v} + t\vec{w}$, $s, t \in \mathbb{R}$ ($v \neq c\vec{w}$) (v, w are non-zero)

Note: if $\vec{v} = c\vec{w}$, then $\vec{p}$ is only one vector

Terminal points of $\vec{v}, \vec{w}$ are in the plane

This is the span of 2 vectors

plane in $\mathbb{R}^n$ Through the origin

$P = \text{Span}(\vec{v}, \vec{w}) = \{s\vec{v} + t\vec{w}, s, t \in \mathbb{R}\}$ through origin with direction $\vec{v}, \vec{w}$

Vector Equation of Plane

$\vec{u} \in \mathbb{R}^n$, $\vec{v}, \vec{w}$ are non-zero, $\vec{v} \neq c\vec{w}$ $\quad \vec{p} = \vec{u} + s\vec{v} + \vec{w}$

$P = \{\vec{u} + s\vec{v} + t\vec{w} : s, t \in \mathbb{R}\}$ plane through $\vec{u}$ with direction $\vec{v}, \vec{w}$

Note: Always remember that they are vector!

to calculate $\vec{v}, \vec{w}$, use 2 points!

Normal Form, Scalar Equation of Plane in $\mathbb{R}^3$

$\vec{u} \in P$,

normal vector: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \vec{0}$ $\quad$ Normal form: $\vec{n} \cdot (\vec{p} - \vec{u}) = 0$

$\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in P$, $\neq \vec{u}$ $\quad$ Scalar equation: $ax + by + cz = d$ ($d = \vec{n} \cdot \vec{u}$)

Note: only for $\mathbb{R}^3$. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{n} =$ norm

Lecture 6

Linear Equation. Coefficient. Constant Term

$a_1x_1 + a_2x_2 \cdots + a_nx_n = b$. $a_1 \cdots a_n \in \mathbb{F}$ (Coefficient)

$\uparrow$ constant

System of Linear Equation

a collection of m linear equation in n variables

$a_{11}x_1 + \cdots a_{1n}x_n = b_1$

$a_{21}x_1 + \cdots a_{2n}x_n = b_2 \quad \cdots \cdots$

Note: a_{ij} is the coefficient.

Solution Set

$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ when $x_1 = y_1 \cdots x_n = y_n$, the systems of equations are satisfied

$\{\vec{y}\}$ set of solution

Equivalent Systems 2 linear systems with same solution set.

Elementary Operations

1. Equation swap
   $e_i \leftrightarrow e_j$
2. Equation Scale
   $e_i \to me_i$, $m \in \mathbb{F}/\{0\}$
3. Equation addition
   $e_j \to ce_i + e_j$

Elementary Operations Theorem

if a single elementary operation of any type is performed on a system of equation, then the system will be equivalent

Inconsistent Systems no solution

Consistent System Unique / Infinite solutions

Lecture 7

matrix m(row) x n(col) rectangular array of scalars

a_{ij} ← row, col

$$A = \left[\begin{array}{ccc|c} a & \cdots & & x \\ b & \cdots & & y \\ c & \cdots & & z \end{array}\right]$$

$\underbrace{}_{\text{coefficient}}$ $\vec{b}$ — Augmented

Note: can apply ERO
B = A if apply single ERO
(row equivalent)

Zero/ Leading Entry

row with all zeros → zero row

$$\begin{bmatrix} a & b & c & \cdots & d \\ 0 & 0 & 0 & & 0 \end{bmatrix} \leftarrow \text{zero row}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

(leftmost non-zero #)
leading one
leading entry (4)
zero row

REF (row echelon form)

1. All zero rows occur as the final rows in matrix
2. the leading entry in any non-zero row appears in a col to the right of the col with leading entries above it

Pivot leading entry
position of ↑ = pivot positions

pivot colum → column with pivot
pivot row → row with pivot

RREF (reduced REF)

1. in REF 2. all pivots are 1 3. only non-zero entry in Pc is pivot

Unique RREF

all matrix has a unique RREF

Basic Variables column with pivot. x_i is basic
Free Variables Column without pivot

Lecture 8

$M_{m\times n}(\mathbb{R}/\mathbb{C}/\mathbb{F})$

real matrix (all real entries)
complex matrix (complex entries) $\Big]\ M_{m\times n}(\mathbb{F})$

Rank $M_{m\times n}(\mathbb{F})$ in RREF, $\text{rank}(A) = r$ (# of pivots)

Rank Bounds $\text{rank}(A) \leq \min\{m, n\}$

Consistent System Test (CST)

$\text{rank}(A) = \text{rank}([A\,|\,\vec{b}]) \iff$ consistent system

Lecture 9

System Rank Theorem

a) if $[A|\vec{b}]$ is cons. $\Rightarrow$ $[A|\vec{b}]$ contain $n-r$ parameter   rank/# of pivots

b) $[A|\vec{b}]$ is cons. $\iff$ $r = m$ (# of row)

Nullity # of parameter nullity(A) $(n-r)$

row vector matrix with on row $\text{Row}_i(A)$

Matrix-Vector Multiplication in Individual Entries

$$A\vec{x}\begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \\ c_1 & \cdots & c_n \\ \vdots & & \vdots \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1x_1 + a_2x_2 + \cdots a_nx_n \\ b_1x_1 + b_2x_2 + \cdots b_nx_n \\ \vdots \end{bmatrix}$$

like Dot product

Linearity of Matrix-Vector Mult.

1. $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$ 2. $A(c\vec{x}) = cA\vec{x}$

Proposition 3.10.2

$\forall \vec{e}_i \in \mathbb{F}^m$, $A\vec{x} = \vec{e}_i$ is cons $\Rightarrow$ rank$(A) = m = $ # of rows

Lecture 10

Homogeneous all the constant on the right are zero $[A|\vec{0}]$ always consistent

Non-homogeneous not all $\vec{0}$ $[A|\vec{b}]$ might be inconsistent

Nullspace solution set of a homogeneous system $(\text{Null}(A))$

Note: it's just $\text{Span}\{\vec{u}_1 \ldots \vec{u}_n\}$

Proposition 3.11.1

Let $A\vec{x} = \vec{0}$ be a homogeneous system of linear equations with solution set S.

if $\vec{x}, \vec{y} \in S$, and if $c \in \mathbb{F}$, then $\vec{x} + \vec{y} \in S$ and $c\vec{x} \in S$.

Note: S is closed under addition and scalar mult.

if $\vec{x}, \vec{y} \in S$, and $c, d \in \mathbb{F}$, then $c\vec{x} + d\vec{y} \in S$ $(\text{Span}\{\vec{x}, \vec{y}\} \in S)$

Ex) $\left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 2 & 1 & 0 & 2 \\ 0 & 1 & 2 & -1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$

① No Solution

② Infinite Solution, $\text{Null}(A) = \text{Span}\left\{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}$

Associated homogeneous system $A\vec{x} = \vec{b} \Rightarrow A\vec{x} = \vec{0}$

particular solution $\vec{x}_p$

Solution to $A\vec{x} = \vec{0}$, $A\vec{x} = \vec{b}$ $\tilde{S} = \{\vec{x}_p + \vec{x} : \vec{x} \in S\}$

Solution to $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{c}$ $\tilde{S}_c = \{(\vec{x}_c - \vec{x}_b) + \vec{z} : \vec{z} \in \vec{S}_b\}$

Lecture 11

Column Space $\text{Col}(A) = \text{Span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\} = \{A\vec{x} : \vec{x} \in \mathbb{F}^n\}$

Span of the columns of A

Ex) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\text{Col}(A) = \text{Span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$

Consistent system and Column Space

Let $A \in M_{m\times n}(\mathbb{F})$ and $\vec{b} \in \mathbb{F}^m$. The system of linear equations $A\vec{x}=\vec{b}$ is cons.
if and only if $\vec{b} \in Col(A)$ $A\vec{x}=\vec{b}$ is cons $\Leftrightarrow$ $\vec{b} \in Col(A)$

Transpose

A^T $(A^T)_{ij} = A_{ji}$, j=row, i=column)

Ex) $A=\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

Rowspace

$Row(A) = Span\{ (\overrightarrow{row}_1(A))^T, (\overrightarrow{row}_2(A))^T, \dots, (\overrightarrow{row}_m(A))^T \}$
span of the transposed rows of A

Proposition 4.1.7

Let $A, B \in M_{m\times n}(\mathbb{F})$, if B is row equivalent to A, then
$$Row(B) = Row(A)$$

Column Extraction

$A(\vec{e_i}) = \vec{a}_i \quad \forall i \in 1, \dots, n$

Equality of Matrices

Let $A, B \in M_{m\times n}(\mathbb{F})$, then $A=B \Leftrightarrow A\vec{x}=B\vec{x} \ \forall x \in \mathbb{F}^n$

Matrix Multiplication

$C=AB = A(\vec{b}_1, \vec{b}_2 \dots \vec{b}_p) = [A\vec{b}_1 \dots A\vec{b}_p]$

# of columns A = # of rows of B (not → undefined

Each column of AB is an element of Col(A)

Ex) $\begin{bmatrix} 1 & 7 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1\cdot 1+7(-1) & 1(2)+7(3) & 1(3)+7(2) \\ 2(1)+0(-1) & 0(3)+2(3) & 2(3)+2(0) \end{bmatrix}$

lecture 12 (Midterm)

Matrix sum

$A+B=C$ $C_{ij} = a_{ij} + b_{ij}$ must be the same size

$AB \neq BA$

Ex) $\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 5 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 1-1 & 3+5 \\ -2+6 & 1+2 \end{bmatrix}$

Properties of Matrix Addition

a) $A+B=B+A$ b) $A+B+C=(A+B)+C=A+(B+C)$

Zero Matrix all entries are zero $\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$

Additive Inverse matrix $-A$ $A=\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$ $-A=\begin{bmatrix} -1 & -2 \\ 2 & -2 \end{bmatrix}$

properties: 1. $0+A=A+0=A$ 2. $A+(-A)=(-A)+A=0$

Properties of Matrix Multiplication

$A,B\in M_{m\times n}(\mathbb{F})$ $C,D\in M_{n\times p}(\mathbb{F})$ $E\in M_{p\times q}\mathbb{F}$

a. $(A+B)C=AC+BC$ b. $A(C+B)=AC+BC$ c) $AC(E)=A(CE)=ACE$

Multiplication of matrix by scalar $c(A_{ij})=ca_{ij}$ $2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}=\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

Properties of Scalar Matrix Multiplication

a) $s(A+B)=sA+sB$ c) $r(sA)=(rs)A$

b) $(r+s)A=rA+sA$ d) $s(AC)=(sA)C=A(sC)$

Properties of Matrix Transpose

a. $(A+B)^T=A^T+B^T$ c) $(AC)^T=C^TA^T$

b. $(sA)^T=sA^T$ d) $(A^T)^T=A$

Square Matrix # of rows = # of cols $M_n(\mathbb{F})$

Upper Triangular $a_{ij}=0$ for $i>j$ with $i=1\ldots n,\ j=1\ldots n$ $\begin{bmatrix} \ddots & & \\ 0 & \ddots & \\ 0 & 0 & \ddots \end{bmatrix}$ RREF are upper

Lower Triangular $\begin{bmatrix} \ddots & 0 & 0 \\ & \ddots & 0 \\ & & \ddots \end{bmatrix}$ transpose of upper <=> transpose of lower

Diagonal $n\times n$ matrix, $a_{ij}=0,\ i\neq j$ $A=\mathrm{diag}(a_{11},a_{22}\ldots a_{nn})$ both upper / lower

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}=\mathrm{diag}(1,2)$

Identity Matrix diag(1, 1, ..., 1) (I_n)

$A_{m \times n}$ $I_m A = A$ and $A I_n = A$ (multiplicative identity)

$c I_n = diag(c, ..., c)$

lecture 13

Elementary matrix obtained by performing a single ERO on the identity matrix

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Ematrix $\rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ ($2R_1$)

Proposition 4.5.3 let B = some ERO apply to $A \in M_{m \times n}(\mathbb{F})$, E = single ERO apply to I_m

then $B = EA$

Corollary 4.5.4 $B = E_k \cdots E_2 E_1 A$ $E_k \cdots E_1$ are all ERO apply to A to get B

Invertible $n \times n$ matrix, $AB = CA = I_n$

Equality of left and Right $AB = CA = I_n \Rightarrow B = C$ Inverse is unique

left Invertible iff Right invertible $AB = I_n \Leftrightarrow CA = I_n$

Inverse of a matrix $AA^{-1} = A^{-1}A = I_n$

lecture 14

Algorithm to Check Invertibility and Finding Inverse

1. Construct $[A \mid I_n]$
2. Find RREF $[R \mid B]$ of $[A \mid I_n]$
3. if $R \neq I_n$, A is not, if $R = I_n$, $A^{-1} = B$

Inverse of a 2x2 Matrix

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, if $ad - bc \neq 0$, then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

lecture 15

Function determined by the matrix $T_A : \mathbb{F}^n \rightarrow \mathbb{F}^m$ ← not the same component

\# of cols

$T_A(\vec{x}) = A\vec{x}$

input # of columns in A

Ex) let $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, $T_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T_A(\vec{x}) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x + 0y \\ 0x + 3y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix}$

Linear Transformation / linear mapping Criteria

1. $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ 2. $T(c\vec{x}) = cT(\vec{x})$

$\mathbb{F}^n \to$ domain of T $\mathbb{F}^m \to$ codomain of T

linearity over addition linearity over scalar mult

Function Determined by a Matrix is Linear

1. $T_A(\vec{x}+\vec{y}) = T_A(\vec{x}) + T_A(\vec{y})$ 2. $T_A(c\vec{x}) = cT_A(\vec{x})$ * linear transformation determined by $\vec{A}$

Alternate Characterization $T(c\vec{x}+\vec{y}) = cT(\vec{x}) + T(\vec{y})$, $T: \mathbb{F}^n \to \mathbb{F}^m$

Zero map to Zero $T: \mathbb{F}^n \to \mathbb{F}^m$, $T(\vec{0}_{\mathbb{F}^n}) = \vec{0}_{\mathbb{F}^m}$

Lecture 16

Range $T: \mathbb{F}^n \to \mathbb{F}^m$ Range$(T) \to$ set of all outputs of T

Range$(T) = \{T(\vec{x}) : \vec{x} \in \mathbb{F}^n\}$ * subset of $\mathbb{F}^m$ $\neq \emptyset$

Range of a Linear Transformation Range$(T_A) =$ Col(A)

Connection to System of Linear Equations $A\vec{x} = \vec{b}$ is consistent $\Leftrightarrow$ $\vec{b} \in$ Col(A) / $\vec{b} \in$ (Range(A))

Onto: or surjective $\to$ Range$(T) = \mathbb{F}^m$

Criteria: (satisfy one of them)

1. T_A is onto 2. Col$(A) = \mathbb{F}^m$ 3. rank$(A) = m$

Kernel Ker(T) $\to$ set of inputs of T whose output is zero

Ker$(T) = \{\vec{x} \in \mathbb{F}^n : T(\vec{x}) = \vec{0}_{\mathbb{F}^m}\}$ $\neq \emptyset$ Ker$(T_A) =$ Null(A)

subset of $\mathbb{F}^n$

One-to-One / injective $T(\vec{x}) = T(\vec{y}) \Rightarrow \vec{x} = \vec{y}$

Test: $\Leftrightarrow$ Ker$(T) = \{\vec{0}_{\mathbb{F}^n}\}$

Criteria:

1. T_A is one-to-one 2. Null$(A) = \{\vec{0}_{\mathbb{F}^n}\}$ 3. nullity$(A) = 0$ d) rank$(A) = n$

lecture 17

Let $A \in M_{n\times n}(\mathbb{F})$. The following three conditions are equivalent:

(a) A is invertible.
(b) $\text{rank}(A) = n$.
(c) $\text{RREF}(A) = I_n$.

Let $A \in M_{m\times n}(\mathbb{F})$ and let T_A be the linear transformation determined by the matrix A. The following statements are equivalent.

(a) T_A is onto.
(b) $\text{Col}(A) = \mathbb{F}^m$.
(c) $\text{rank}(A) = \cancel{m}\ n$.

Let $A \in M_{m\times n}(\mathbb{F})$ and let T_A be the linear transformation determined by the matrix A. The following statements are equivalent.

(a) T_A is one-to-one.
(b) $\text{Null}(A) = \{\vec{0}_{\mathbb{F}^n}\}$.
(c) $\text{nullity}(A) = 0$.
(d) $\text{rank}(A) = n$.

if $M_{n\times n}(\mathbb{F})$ then all would be equivalent

Invertibility Criteria - Second Version - pgs 132-133

Let $A \in M_{n\times n}(\mathbb{F})$ be a square matrix and let T_A be the linear transformation determined by the matrix A. The following statements are equivalent.

(a) A is invertible.
(b) T_A is one-to-one.
(c) T_A is onto.
(d) $\text{Null}(A) = \{\vec{0}\}$. That is, the only solution to the homogeneous system $A\vec{x} = \vec{0}$ is the trivial solution $\vec{x} = \vec{0}$.
(e) $\text{Col}(A) = \mathbb{F}^n$. That is, for every $\vec{b} \in \mathbb{F}^n$, the system $A\vec{x} = \vec{b}$ is consistent.
(f) $\text{nullity}(A) = 0$.
(g) $\text{rank}(A) = n$.
(h) $\text{RREF}(A) = I_n$.

any one would be invertible

Standard Matrix $[T]_\mathcal{E}$ ← using standard basis $T: \mathbb{F}^n \to \mathbb{F}^m$

$$[T]_\mathcal{E} = [T(\vec{e_1}) \quad T(\vec{e_2}) \quad \dots \quad T(\vec{e_n})]$$

Every Linear Transformation is Determined by a Matrix

$$T(\vec{x}) = [T]_\mathcal{E}\, \vec{x} \quad \text{or} \quad T = T_{[T]_\mathcal{E}}$$

Rotation about the origin

$$R_\theta(\vec{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ onto and one-to-one

Projection in standard matrix $[\text{proj}_{\vec{w}}]_\mathcal{E} = \frac{1}{w_1^2 + w_2^2}\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix}$ it is onto, not one-to-one

lecture 18

Properties of a Standard Matrix - pg 136

Let $A \in M_{m\times n}(\mathbb{F})$, let $T_A: \mathbb{F}^n \to \mathbb{F}^m$ be the linear transformation determined by A, and let $T: \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. Then

(a) $T_{[T]_\mathcal{E}} = T$.
(b) $[T_A]_\mathcal{E} = A$.
(c) T is onto if and only if $\text{rank}([T]_\mathcal{E}) = m$. # of rows
(d) T is one-to-one if and only if $\text{rank}([T]_\mathcal{E}) = n$. # of columns

a) $\forall \vec{x} \in \mathbb{F}^n$. $T_{[T]_\mathcal{E}}(\vec{x}) = [T]_\mathcal{E}\vec{x} = T(\vec{x})$

So $T_{[T]_\mathcal{E}}$ and T determine the same function/transformation

b) $[T_A]_\mathcal{E} = [T_A\vec{e_1} \dots T_A\vec{e_n}] = [A\vec{e_1} \dots A\vec{e_n}] = [\vec{a_1} \dots \vec{a_n}] = A$

column extraction

c) onto criteria

d) one-to-one criteria

Composition of linear Transformation

$T_1: \mathbb{F}^n \to \mathbb{F}^m \quad T_2: \mathbb{F}^m \to \mathbb{F}^p \qquad T_2 \circ T_1: \mathbb{F}^n \to \mathbb{F}^p$

$$(T_2 \circ T_1)(\vec{x}) = T_2(T_1(\vec{x}))$$

- it is linear if T_1, T_2 is linear

Standard Matrix of Composition of Linear Transformation

$$[T_2 \circ T_1]_\mathcal{E} = [T_2]_\mathcal{E}[T_1]_\mathcal{E}$$

Note:

calculate $\text{Ker}(T) \Rightarrow$ ① standard matrix

② Find Solution: $[T]_\mathcal{E}(\vec{x}) = \vec{0}$

lecture 19

determinant of A $\det(A) = a_{11}$ $(A \in M_{1\times1}(\mathbb{F}))$ $\det(A) = a_{11}a_{22} - a_{12}a_{21}$ $(M_{2\times2}(\mathbb{F}))$

Only works for square matrix

Ex) $\det\begin{bmatrix} 7 & -2 \\ 3 & -1 \end{bmatrix} = 7(-1) - (-2)(3) = -1$

Submatrix of A $M_{ij}(A)$ $(n-1)(n-1)$ matrix

det: minor of A

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $M_{23}(A) = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$ $\det = -6$ (minor)

(submatrix)

Det (nxn) matrix. $\det(A) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det(M_{ij}(A))$ $(n \geq 2,$ $i \in 1 \dots n$

Any row expansion or column expansion gives out same det

Ex) $\begin{bmatrix} 3 & 5 & 2 \\ 4 & 0 & 3 \\ -1 & 2 & 4 \end{bmatrix}$ $\det = -5(\det[\dots]) + 2(\det[\dots]) - 2\det[\dots]$

if a row/column is all zero $\Rightarrow \det(A) = 0$

if A is upper/lower-triangle $\Rightarrow$ $\det(A) = a_{11}a_{22}a_{33}\dots a_{nn}$

$\det(I_n) = 1$

Corollary $\det(I_n) = 1$ $\det(A) = \det(A^T)$

Elementary Matrices:

Effect of EROs on the Determinant row swap: $\det(B) = -\det(A)$ $= -1$

row scale: $\det(B) = m\det(A)$ $= m$

row addition: $\det(B) = \det(A)$ $= 1$

never zero

lecture 20

Determinant After one ERO $\det(B) = \det(E)\det(A)$

After k ERO $\det(B) = \det(E_k) \dots \det(E_2)\det(E_1)\det(A)$

Invertibility A is invertible $\Leftrightarrow$ $\det(A) \neq 0$

Determinant of a Product $\det(AB) = \det(A)\det(B)$

Determinant of Inverse $\det(A^{-1}) = \frac{1}{\det(A)}$ (A must be invertible)

Cofactor $C_{ij}(A)$ (i^{th} row j^{th} col) $= (-1)^{i+j}\det(M_{ij}(A))$

Adjucate $(\operatorname{adj}(A))_{ij} = C_{ji}(A)$ * transpose of Cofactors

Theorem, Inverse by Adjugate: $A(\operatorname{adj}(A)) = \operatorname{adj}(A)A = \det(A)I_n$

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$$

Lecture 21

Cramer's Rule Let $A\vec{x} = \vec{b}$, $\vec{b} \in \mathbb{F}^n$, $\det(A) \neq 0$ (A is invertible and square matrix)

if B is constructed by replacing j^{th} column of A by column $\vec{b}$

then the solution is:

$$x_j = \frac{\det(B_j)}{\det(A)}, \quad \forall j = 1 \dots n$$

Area of Parallelogram $\left|\det\left(\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}\right)\right|$

Scaling effect of Projection on Area Scaling effect $= 0$ (or area of the projection)

Rotation det does not change.

Reflection Scaling factor: 1

Lecture 22

Eigenvector of A Square matrix A

non zero vector that $A\vec{x} = \lambda\vec{x}$

side: if A is not invertible then 0 is an eigenvalue of A

Eigenvalue of A λ **Eigenpair** $(\lambda, \vec{x})$

Fixed Point $T(\vec{x}) = \vec{x}$ / $[T]_\varepsilon \vec{x} = \vec{x}$ $\lambda = 1$

Finding Eigenvalue $A\vec{x} = \lambda\vec{x}$ or $(A - \lambda I_n)\vec{x} = \vec{0}$ (eigenvalue equation for A over $\mathbb{F}$)

$C_A(\lambda) = \det(A - \lambda I)$ (characteristic polynomial)

$C_A(\lambda) = 0$ (characteristic equation)

Ex): $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$$C_A(\lambda) = \det\begin{bmatrix} 1/2-\lambda & 1/2 \\ 1/2 & 1/2-\lambda \end{bmatrix} = \left(\tfrac{1}{2}-\lambda\right)^2 - \tfrac{1}{4} = \lambda(\lambda-1)$$

Lecture 23

Trace square matrix A $tr(A) = \sum_{i=1}^{n} a_{ii}$ sum of its diagonal entries

Features of $C_A(\lambda)$ $C_A(\lambda) = C_n\lambda^n + C_{n-1}\lambda^{(n-1)} + \cdots + C_1\lambda + C_0$

$C_n = (-1)^n$ / $C_{n-1} = (-1)^{(n-1)} tr(A)$ / $C_0 = \det(A)$

$C_A(\lambda)$, Eigenvalue over $\mathbb{C}$ $C_{n-1} = (-1)^{n-1}\sum_{i=1}^{n}\lambda_i$ / $C_0 = \prod_{i=1}^{n}\lambda_i$

include each eigenvalue even if repeated

$\sum_{i=1}^{n}\lambda_i = tr(A)$ $\prod_{i=1}^{n}\lambda_i = \det(A)$

Eigenspace $E_\lambda(A) = \text{Null}(A - \lambda I)$

Solution set to the system $(A-\lambda I)\vec{x} = \vec{0}$

$\vec{0}$ is in the eigenspace, not eigenvector

λ is an eigenvalue $\Leftrightarrow$ $E_\lambda \neq \{\vec{0}\}$, has parameter

$E_0(\lambda) \to$ A is not invertible.

Linear Combination of Eigenvector if $c\vec{x} + d\vec{y} \neq \vec{0}$, then $(\lambda_1, c\vec{x} + d\vec{y})$ is also an eigenpair of A with eqvalue λ_1

Lecture 24

Similar A is similar to B = if $\exists$ an invertible matrix $P \in M_{n\times n}(\mathbb{F})$ such that $A = PBP^{-1}$

A B have same $C_{A/B}(\lambda)$ and eqvalues in $\mathbb{F}$

Diagonalizable Matrix if $\exists$ $P \in M_{n\times n}(\mathbb{F})$ s.t. $P^{-1}AP = D$ $\Rightarrow$ A is diagonalizable over $\mathbb{F}$

↳ P diagonalizes A

All diagonal matrix are diagonalizable

Theorem ⇒ Diagonalizable if A is diagble $\Rightarrow$ $C_A(\lambda)$ has n roots (with repetition)

if P diaglize A, then diagonal entries of $D = P^{-1}AP$ are eqvalue of A

$A \in \mathbb{C}$, n roots/eqvalue

A is diagable over $\mathbb{R}$, then n roots/eqvalue

no n roots $\to$ not diagable

n Distinct Eigvalues $\Rightarrow$ Diagable   Let $A \in M_{n\times n}(\mathbb{F})$, n distinct roots, $P = [\vec{v_1} \dots \vec{v_n}]$

P is invertible / $P^{-1}AP = D = \operatorname{diag}(\lambda_1, \lambda_2 \dots \lambda_n)$

Lecture 25

Subspace $V \in \mathbb{F}^n \Rightarrow \vec{0} \in V$

$\forall \vec{x}, \vec{y} \in V$, $\vec{x}+\vec{y} \in V$ (closure under addition)

$\forall \vec{x} \in V$, $c \in \mathbb{F}$, $c\vec{x} \in V$ (closed under scalar mult)

$\{\vec{0}\} \Rightarrow$ smallest subspace $\{\mathbb{F}^n\} \Rightarrow$ largest subspace.

Ex) $\{\vec{v_1} \dots v_k\} \in \mathbb{F}^n \Rightarrow \operatorname{Span}\{v_1 \dots v_k\}$ is a subspace of $\mathbb{F}^n$

$A \in M_{m\times n}(\mathbb{F})$, $A\vec{x} = \vec{0}$ (Null(A)) is a subspace of $\mathbb{F}^n$

Subspace Test $V \in \mathbb{F}^n$, V is non-empty, $\forall \vec{x}, \vec{y} \in V$, $c \in \mathbb{F}$, $c\vec{x}+\vec{y} \in V$

Ex)

$A \in M_{m\times n}(\mathbb{F})$, Col(A) is a subspace of $\mathbb{F}^m$

$T: \mathbb{F}^n \to \mathbb{F}^m$, Range(T) is a subspace

Ker(T) is a subspace

$A \in M_{n\times n}(\mathbb{F})$, $\lambda \in \mathbb{F}$, E_λ is a subspace of $\mathbb{F}^n$

Linear Dependence $\exists c_1 \dots c_k \in \mathbb{F}$, not all zero, $c_1\vec{v_1} + \dots + c_k\vec{v_k} = \vec{0}$

if $U = \{\vec{v_1} \dots \vec{v_k}\}$, then U is lin dep set

Lecture 26

Linear Independence   all $c_1 \dots c_k$ are zero. $c_1\vec{v_1} + \dots + c_k\vec{v_k} = \vec{0}$

trivial solution

all vectors are distinct

empty set is lin indep.

Proposition   Let $S \subseteq \mathbb{F}^n$: $\vec{0} \in S$, S = lin dep

$S = \{\vec{x}\}$ (only one vec) S = lin dep $\Leftrightarrow \vec{0}$

$S = \{\vec{x}, \vec{y}\}$ (2 vecs), S = lin dep $\Leftrightarrow \vec{x} = c\vec{y}$

Linear Dependence Check

$k \geq 2$, $v_1 \cdots v_k$ are lin dep $\Leftrightarrow$ $\vec{v_i} = c\vec{v_k}$ $(i \neq k)$

$v_1 \cdots v_k$ lin indep $\Leftrightarrow$ all constant are zero

Lecture 27

Pivots and Linear Independence

let $S = \{v_1 \cdots v_k\}$ in $\mathbb{F}^n$, $A = [\vec{v_1} \cdots \vec{v_k}]$,

$\text{rank}(A) = r$, A has pivots in columns $q_1 \cdots q_r$. $U = \{\vec{v_{q_1}}, \vec{v_{q_2}} \cdots \vec{v_{q_r}}\}$

a) S = lin indep $\Leftrightarrow$ $r = k$

b) U is lin indep

c) $\vec{v} \in S$, $\vec{v} \notin U$, $\{\vec{v_{q_1}} \cdots \vec{v_{q_r}}, \vec{v}\}$ is lin dep

d) $\text{Span}(U) = \text{Span}(S)$

Bound on Number of lin indep vectors

$S = \{v_1, v_2 \cdots \vec{v_k}\}$ in $\mathbb{F}^n$, $n < k \Rightarrow$ lin dep

Lecture 28

Basis V is subspace of $\mathbb{F}^n$, $B = \{\vec{v_1} \cdots \vec{v_k}\}$. B is basis if:

B is lin indep, $V = \text{Span}(B)$

V is $\emptyset \Rightarrow \text{Span}\,\emptyset = \{\vec{0}\}$

Every Subspace has a spanning set V subspace in $\mathbb{F}^n \Rightarrow \exists v_1 \cdots v_k$ s.t.

$V = \text{Span}\{v_1 \cdots v_k\}$

Every Subspace Has a Basis V is subspace of $\mathbb{F}^n \Rightarrow V$ has a basis

Span of Subset V is subspace, $S = \{\vec{v_1} \cdots \vec{v_k}\} \subseteq V$, $\text{Span}\, S \subseteq V$

Span $\mathbb{F}^n$ iff Rank is n let $S = \{v_1 \cdots v_k\}$, $A = [v_1 \cdots v_k]$

$\text{Span}(S) = \mathbb{F}^n \Leftrightarrow \text{rank}(A) = n$

Finding Basis must be a spanning set

at least n vector (and at most)

Sizes of Basis for $\mathbb{F}^n$ S is basis $\Rightarrow k = n$

not sufficient condition

n Vectors in $\mathbb{F}^n$ Span iff lin indep. S lin indep $\Leftrightarrow$ $\text{Span}(S) = \mathbb{F}^n$

Lecture 29

Finding Basis for Col(A) Find RREF → remove redundant cols. (col no pivots)

Basis for Null(A) $\text{Null}(A) = \{t_1\vec{x_1} + \dots + t_k\vec{x_k} : t_1 \dots t_k \in \mathbb{F}\}$

$k = \text{nullity}(A) = n - \text{rank}(A)$

Find RREF(A) then solve.

Dimension is well-defined V is a subspace of $\mathbb{F}^n$, $B = \{\vec{v_1} \dots \vec{v_k}\}$. $C = \{\vec{w_1}, \dots \vec{w_\ell}\}$

$k = \ell$

Dimension # of elements in a basis for a subspace

$\dim(V)$

Bound on Dimension of Subspace $\dim(V) \leq n$

Rank and Nullity as Dimension $\text{rank}(A) = \dim(\text{Col}(A))$

$\text{nullity}(A) = \dim(\text{Null}(A))$

Rank-Nullity Theorem $n = \text{rank}(A) + \text{nullity}(A)$

$= \dim(\text{Col}(A)) + \dim(\text{Null}(A))$

Lecture 30

Unique Representation Theorem

$B = \{v_1 \dots v_n\}$. $\forall \vec{v} \in \mathbb{F}^n$, $\exists$ unique scalars $c_1 \dots c_n \in \mathbb{F}$ such that

$\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_n\vec{v_n}$

Coordinates with Respect to B / B-coordinate

Let B be a basis

$\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_n\vec{v_n} = \sum_{i=1}^{n} c_i\vec{v_i}$, $(c_i \in \mathbb{F})$

Ordered Basis $B = \{\vec{v_1} \dots v_n\}$, fixing order

Coordinate Vector Let $B = \{\vec{v_1} \dots \vec{v_n}\}$ be ordered basis $\vec{v} = \sum_{i=1}^{n} c_i\vec{v_i}$

$\hookrightarrow [\vec{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Linearity of Taking Coordinates $\forall \vec{u}, \vec{v} \in \mathbb{F}^n$, $[\vec{u}+\vec{v}]_B = [\vec{u}]_B + [\vec{v}]_B$

$\forall \vec{v} \in \mathbb{F}^n$, $c \in \mathbb{F}$, $[c\vec{v}]_B = c[\vec{v}]_B$

Change-of-Basis Matrix / Change of Coordinate

$_C[I]_B = [[\vec{v}_1]_C \dots [\vec{v}_n]_C]$ ← change of basis from B to C

$_B[I]_C = [[\vec{w}_1]_B \dots [\vec{w}_n]_B]$ from C to B

coordinate matrix of w_i consisting of vectors in B set

Changing a Basis

$[\vec{x}]_C = {}_C[I]_B[\vec{x}]_B \qquad [\vec{x}]_B = {}_B[I]_C[\vec{x}]_C$

Inverse of Change-of-Basis Matrix

$_B[I]_C = ({}_C[I]_B)^{-1} \qquad {}_B[I]_C\,{}_C[I]_B = I_n$

Lecture 31

Linear Operator linear Transformation where the domain = codomain ($T: \mathbb{F}^n \to \mathbb{F}^n$)

$[T]_\varepsilon$ be square

B matrix of T $[T]_B = [[T(\vec{v}_1)]_B \dots [T(\vec{v}_n)]_B]$

apply transformation to every member in B, then take coordinate.

Proposition 9.1.3 Let $T: \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator, $B = \{\vec{v}_1 \dots \vec{v}_n\}$ be ordered basis for $\mathbb{F}^n$

$[T(\vec{v})]_B = [T]_B[\vec{v}]_B$

Similarity of Matrix Representation

Let $T: \mathbb{F}^n \to \mathbb{F}^n$, B, C be ordered basis for $\mathbb{F}^n$, then

$$[T]_C = {}_C[I]_B[T]_B\,{}_B[I]_C$$
$$= ({}_B[I]_C)^{-1}[T]_B\,{}_B[I]_C$$

the matrices $[T]_B$ and $[T]_C$ are similar over $\mathbb{F}$

Standard matrix $[T]_\varepsilon = {}_\varepsilon[I]_B[T]_B\,{}_B[I]_\varepsilon$

$= ({}_B[I]_\varepsilon)^{-1}[T]_B\,{}_B[I]_\varepsilon$ vice versa for $[T]_B$

Lecture 32

Eigenvector, Eigenvalue, and Eigenpair of a Linear operator

Let $T: \mathbb{F}^n \to \mathbb{F}^n$, $\vec{x}$ is eigenvector $\Rightarrow T(\vec{x}) = \lambda \vec{x}$

Eigenpairs of T and $[T]_\beta$ $(\lambda, \vec{x})$ is eqpair $\Leftrightarrow$ $(\lambda, [\vec{x}]_\beta)$ is eigenpair of $[T]_\beta$

Diagonalizable $\exists$ ordered $B \in \mathbb{F}^n \Rightarrow [T]_\beta$ is a diagonal matrix

Eigenvector Basis Criterion for Diagonalizability

Let $T: \mathbb{F}^n \to \mathbb{F}^n$. T is diagable over $\mathbb{F}$ $\Leftrightarrow$ $\exists$ ordered $B = \{\vec{v_1} \ldots \vec{v_n}\}$ consisting of eqvectors of T

$A \in M_{n\times n}$, A is diagable $\Leftrightarrow$ $\exists$ basis consisting of eqvectors of A

T diagable $\Leftrightarrow$ $[T]_\beta$ diagable $T: \mathbb{F}^n \to \mathbb{F}^n$

Lecture 33

Eigenvectors Corresponding to Distinct Eigenvalues are Linearly Independent

Let $A \in M_{n\times n}(\mathbb{F})$, eqpairs $(\lambda_1, \vec{v_1}), \ldots, (\lambda_k, \vec{v_k})$ for $1 \le k \le n$

if eqvalues distinct $\Rightarrow$ set of $\{\vec{v_1} \ldots \vec{v_k}\}$ is lin indep.

Algebraic Multiplicity a_{λ_i} # of time eqvalues occur in $C_A(\lambda)$

Geometric Multiplicity g_{λ_i} $g_{\lambda_i} = \dim(E_{\lambda_i})$ dimension of eqspace E_{λ_i}

G_{λ_i} and A_{λ_i} $1 \le g_{\lambda_i} \le a_{\lambda_i}$

Proposition 9.4.9 $A \in M_{n\times n}(\mathbb{F})$, distinct eqvalues $\lambda_1 \cdots \lambda_k$

E_{λ_i} have bases B_i $\Rightarrow$ $B = B_1 \cup B_2 \cup \ldots B_k$ is lin indep

Diagonalizability Test $C_A(\lambda) = (\lambda - \lambda_1)^{a_{\lambda_1}} \cdots (\lambda - \lambda_k)^{a_{\lambda_k}} h(\lambda) \leftarrow$ irreducible in $\mathbb{F}$

$h(\lambda)$ is constant term, $a_{\lambda_i} = g_{\lambda_i}$

Equivalent to A is invertible

1. A is invertible
2. $\exists$ mxn matrix such that $AB = BA = I_n$ (A^{-1}) (only need to say $AB = I_n$, $BA = I_n$)
3. $\det(A) \neq 0$
4. $\text{rank}(A) = n$
5. $\text{nullity}(A) = 0$
6. $\text{col}(A)$ span $\mathbb{F}^n$
7. $\text{col}(A) = \mathbb{F}^n$
8. $\text{Null}(A) = \{\vec{0}\}$
11. 0 is not an eigenvalue of A ($\lambda \neq 0$) (not a root of $C_A(\lambda)$)
12. Product of eigenvalues is not 0
13. Constant term of $C_A(\lambda)$ is not 0
14. $P^{-1}AP$ is invertible $\forall$ invertible matrix P

    No connection between diagonalization and invertibility
15. T_A is one-to-one and onto
16. T_A is invertible
17. $[T_A]_\varepsilon$ is invertible $([T_A]_\varepsilon = A)$
18. $\text{Ker}(T_A) = \{\vec{0}\}$
19. $\text{range}(T_A) = \mathbb{F}^n$
20. $[T_A]_B$ is invertible $\forall$ bases B
21. The solution of $A\vec{x} = \vec{b}$ has a unique solution
22. Homogeneous system $A\vec{x} = \vec{0}$ has trivial solution
23. $\text{col}(A)$ is lin indep
24. $\text{col}(A)$ is the basis
25. A^T is invertible $(A^T)^{-1} = (A^{-1})^T$
26. $\det(A^T) = \det(A) \neq 0$
27. $\text{RREF}(A) = I_n$
28. A is a product of elementary matrices