Math 136 Review Note

Lecture 1

Addition of Vectors: $\vec{u}+\vec{v} = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}$

Scalar Multiplication: Let $c \in \mathbb{R}$, $c\vec{u} = \begin{bmatrix} c \cdot u_1 \\ \vdots \\ c \cdot u_n \end{bmatrix}$

Note: Same line different mag. direct.

Properties of Vectors Addition:

1. Commutative: $\vec{u}+\vec{w} = \vec{w}+\vec{u}$
2. Associative: $(\vec{u}+\vec{w})+\vec{v} = \vec{u}+(\vec{w}+\vec{v})$
3. Additive Identity: $\vec{u} + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ (zero vector) $= \vec{u}$
4. Additive Inverse: $-\vec{u} = \begin{bmatrix} -u_1 \\ \vdots \\ -u_n \end{bmatrix}$

Properties of Scalar Multiplication

1. Distributive: $c(\vec{u}+\vec{w}) = c\vec{u}+c\vec{w}$, $\vec{u}(c+d) = c\vec{u}+d\vec{u}$
2. Associative: $(cd)\vec{v} = c(d\vec{v})$
3. $\vec{u} \cdot 0 = 0$
4. $c\vec{u}=0$ ($c=0$ or $\vec{u}=0$)

$\vec{e_i}$ i^{th} component is 1. others are 0

Standard Basis: the set of $\vec{e_i}$ $\mathcal{E} = \{\vec{e_1}, \vec{e_2} \dots \vec{e_n}\}$ Ex) $R^3 \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Lecture 2

$\mathbb{C}^n$ set of complex vectors. denotes as $\left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, z_n \in \mathbb{C} \right\}$ Note: no dot product, but inner.

Dot product $\vec{u} \cdot \vec{v} = u_1 \cdot v_1 + u_2 \cdot v_2 \dots u_n \cdot v_n$ Note: result is scalar, Not in $\mathbb{C}^n$

Properties of Dot product

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (symmetry)
2. $(\vec{u}+\vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$ linearity
3. $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$
4. non-negativity $\vec{v} \cdot \vec{v} \geq 0$, $\vec{v} \cdot \vec{v} = 0 \Leftrightarrow \vec{v}=0$

Length/norm/magnitude $\sqrt{\vec{v} \cdot \vec{v}} = ||\vec{v}||$

Note: always non-negative, length $=0 \Leftrightarrow \vec{v}=0$

Unit Vector $||\vec{v}||=1$. $\vec{v} \in \mathbb{R}^n$

Normalization $\vec{v} \in \mathbb{R}^n$. $\vec{v} \neq \vec{0}$. $\hat{v} = \frac{\vec{v}}{||\vec{v}||}$ $\frac{\text{vector}}{\text{length}}$

Find Angles between Vectors

$\vec{u}, \vec{v} \in \mathbb{R}^n$, both non-negative, then angle $\theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)$ ← dot product ← length

and $\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$

Note: between $\theta \in [0, \pi]$

Orthogonal/perpendicular $\vec{u}\cdot\vec{v} = 0$ Note: $\vec{0}$ is orthogonal to every vector $\in \mathbb{R}^n$

projection ($\vec{v}$ onto $\vec{w}$) $\text{proj}_{\vec{w}}\vec{v} = \left(\frac{\vec{w}\cdot\vec{v}}{\|\vec{w}\|^2}\right)\vec{w}$ $\left(\frac{\text{dot product}}{(\text{length})^2}\right)$ vector

perpendicular ($\vec{v}$ onto $\vec{w}$) $\text{perp}_{\vec{w}}\vec{v} = \vec{v} - \text{proj}_{\vec{w}}\vec{v}$

Fields / $\mathbb{F}$ Ex) $\mathbb{C}, \mathbb{R}$ can apply math operations

Cross Product $\vec{u}\times\vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$ Note: Only applies in $\mathbb{R}^3$

Properties of Cross Product 1. $\vec{z}\cdot\vec{u} = 0 \;\|\; \vec{z}\cdot\vec{v} = 0$ $(\vec{z} = \vec{u}\times\vec{v})$ 2. $\vec{v}\times\vec{u} = -\vec{z}$

3. if $\vec{u}, \vec{v} \neq \vec{0}$, then $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta$

Note: it is the Area of parallelogram → where θ is the angle between $\vec{u}, \vec{v}$.

Linearity of Cross Product

1. $(\vec{u}+\vec{v})\times\vec{w} = (\vec{u}\times\vec{w}) + (\vec{v}\times\vec{w})$ 2. $(c\vec{u})\times\vec{w} = c(\vec{u}\times\vec{w})$

2. $\vec{u}\times(\vec{v}+\vec{w}) = (\vec{u}\times\vec{v}) + (\vec{u}\times\vec{w})$ 4. $\vec{u}\times c\vec{w} = c(\vec{u}\times\vec{w})$

Note: when calculating length $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$, the number that can be squared must write $|n|$ (absolute value)

Proposition 1.5.7

Let $c \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^n$, then $\|c\vec{v}\| = |c|\,\|\vec{v}\|$

Proposition 1.6.9

Let $\vec{v}, \vec{w} \in \mathbb{R}^n$, the projection and perpendicular of $\vec{v}$ onto $\vec{w}$ is orthogonal

$\text{proj}_{\vec{w}}\vec{v} \cdot \text{perp}_{\vec{w}}\vec{v} = 0$

Lecture 4.

Linear Combination $\overset{\text{scalar}}{c_1}\overset{\text{vector}}{\vec{v}_1} + c_2\vec{v}_2 \dots c_n\vec{v}_n$ $\quad c \in \mathbb{F}^n \quad \vec{v}_n \in \mathbb{R}^n$

Span/ Spanning Set $\underbrace{\text{Span}\{\vec{v}_1, \vec{v}_2 \dots \vec{v}_n\}}_{\text{spanning set}} = \underbrace{\{c_1\vec{v}_1 + c_2\vec{v}_2 \cdots c_n\vec{v}_n,\ c_1, c_2 \dots c_n \in \mathbb{F}^n\}}_{\text{set of linear combination}}$

Note: $\text{Span}\{\vec{v}_1, \vec{v}_2 \dots \vec{v}_n\} \neq c_1\vec{v}_1 + c_2\vec{v}_2 \cdots c_n\vec{v}_n$

Span of one vector a line

Span of two vector a plane

Parametric Equation of a line

Let p, q be fixed number $\in \mathbb{R}$, $q \neq 0$, then the equation through the point (x_1, y_1) with slope p/q is

$$x = x_1 + qt \qquad y = y_1 + pt$$

Note: Each value of t gives a point on the line

$q = 0$, $x = x_1$, $y = y_1 + pt$ => Vertical line

Vector Equation of a line in $\mathbb{R}^2$

$$\vec{l} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + t\begin{bmatrix} q \\ p \end{bmatrix},\ t \in \mathbb{R}$$

Non-zero, direction

line that goes through this vector
terminal point of $\begin{bmatrix} x \\ y \end{bmatrix}$

terminal point on the line

Line in $\mathbb{R}^2$ $\quad \mathcal{l} = \{\vec{u} + t\vec{v} : t \in \mathbb{R}\}$ $\quad \vec{v} \neq 0$,

a line through $\vec{u}$ with direction $\vec{v}$

lecture 5

Line in $\mathbb{R}^n$

$l = \vec{u} + t\vec{v},\ t \in \mathbb{R}$ (vector equation)

Note: l_1 has same direction of $l_2 \Leftrightarrow c\vec{v}_1 = \vec{v}_2$

$\mathcal{L} = \{\vec{u} + t\vec{v},\ t \in \mathbb{R}\}$ lines in $\mathbb{R}^n$ through $\vec{u}$ with direction $\vec{v}$

Parametric Equations

$\vec{l} = \vec{u} + t\vec{v},\ t \in \mathbb{R}$ (vector equation)

$l_1 = u_1 + tv_1,\ l_2 = u_2 + tv_2 \cdots l_n = u_n + tv_n,\ t \in \mathbb{R}$

Vector Equation of a plane in $\mathbb{R}^n$

(through the Origin) $\vec{p} = s\vec{v} + t\vec{w},\ s,t \in \mathbb{R}$ $(v \neq c\vec{w})$ (v, w are non-zero)

Note: if $\vec{v} = c\vec{w}$, then $\vec{p}$ is only one vector

Terminal points of $\vec{v}, \vec{w}$ are in the plane

This is the span of 2 vectors

plane in $\mathbb{R}^n$ Through the origin

$P = \text{Span}(\vec{v}, \vec{w}) = \{s\vec{v} + t\vec{w},\ s,t \in \mathbb{R}\}$ through origin with direction $\vec{v}, \vec{w}$

Vector Equation of Plane

$\vec{u} \in \mathbb{R}^n$, $\vec{v}, \vec{w}$ are non-zero, $\vec{v} \neq c\vec{w}$ $\quad \vec{p} = \vec{u} + s\vec{v} + \vec{w}$

$P = \{\vec{u} + s\vec{v} + t\vec{w} : s,t \in \mathbb{R}\}$ plane through $\vec{u}$ with direction $\vec{v}, \vec{w}$

Note: Always remember that they are vector!

to calculate $\vec{v}, \vec{w}$, use 2 points!

Normal Form, Scalar Equation of Plane in $\mathbb{R}^3$

$\vec{u} \in P$,

normal vector: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \vec{0}$ $\quad$ Normal form: $\vec{n} \cdot (\vec{p} - \vec{u}) = 0$

$\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in P,\ \neq \vec{u}$ $\quad$ Scalar equation: $ax + by + cz = d$ $(d = \vec{n} \cdot \vec{u})$

Note: only for $\mathbb{R}^3$. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{n} =$ norm

Lecture 6

Linear Equation. Coefficient. Constant Term

$a_1x_1 + a_2x_2 \cdots + a_nx_n = b$. $a_1 \cdots a_n \in \mathbb{F}$ (Coefficient)

$\uparrow$ constant

System of Linear Equation

a collection of m linear equation in n variables

$a_{11}x_1 + \cdots a_{1n}x_n = b_1$

$a_{21}x_1 + \cdots a_{2n}x_n = b_2$ . . . . . .

Note: a_{ij} is the coefficient.

Solution Set

$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ when $x_1 = y_1 \cdots x_n = y_n$, the systems of equations are satisfied

$\{\vec{y}\}$ set of solution

Equivalent Systems

2 linear systems with same solution set.

Elementary Operations

1. Equation swap
   $e_i \leftrightarrow e_j$
2. Equation Scale
   $e_i \to me_i$, $m \in \mathbb{F}/\{0\}$
3. Equation addition
   $e_j \to ce_i + e_j$

Elementary Operations Theorem

if a single elementary operation of any type is performed on a system of equation, then the system will be equivalent

Inconsistent Systems

no solution

Consistent System

Unique / Infinite solutions

Lecture 7

matrix m(row) x n(col) rectangular array of scalars

a_{ij} (i = row, j = col)

$$A = \left[\begin{array}{ccc|c} a & \cdots & & x \\ b & \cdots & & y \\ c & \cdots & & z \end{array}\right]$$

$\vec{b}$

Augmented

Coefficient

Note: can apply ERO
B = A if apply single ERO
(row equivalent)

Zero/ Leading Entry

row with all zeros → zero row

$$\begin{bmatrix} a & b & c & \cdots & d \\ 0 & 0 & 0 & & 0 \end{bmatrix} \leftarrow \text{zero row}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

(leftmost non-zero #)
leading one
leading entry (4)
zero row

REF (row echelon form)

1. All zero rows occur as the final rows in matrix
2. the leading entry in any non-zero row appears in a col to the right of the col with leading entries above it

Pivot leading entry
position of ↑ = pivot positions

pivot colum → column with pivot
pivot row → row with pivot

RREF (reduced REF)

1. In REF 2. all pivots are 1 3. only non-zero entry in Pc is pivot

Unique RREF

all matrix has a unique RREF

Basic Variables column with pivot. x_i is basic

Free Variables Column without pivot

Lecture 8

$M_{m \times n}(\mathbb{R} / \mathbb{C} / \mathbb{F})$

real matrix (all real entries)
complex matrix (complex entries)

$M_{m \times n}(\mathbb{F})$

Rank $M_{m\times n}(\mathbb{F})$ in RREF, $\text{rank}(A) = r$ (# of pivots)

Rank Bounds $\text{rank}(A) \leq \min\{m, n\}$

Consistent System Test (CST)

$\text{rank}(A) = \text{rank}([A \mid \vec{b}\,]) \iff$ consistent system

Lecture 9

System Rank Theorem

a) if $[A|\vec{b}\,]$ is cons. $\Rightarrow$ $[A|\vec{b}\,]$ contain $n-r$ parameter

rank/# of pivots.

b) $[A|\vec{b}\,]$ is cons. $\iff$ $r = m$ (# of row)

Nullity # of parameter nullity(A) $(n-r)$

row vector matrix with on row $\text{Row}_i(A)$

Matrix-Vector Multiplization in Individual Entries

$$A\vec{x} \begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \\ c_1 & \cdots & c_n \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1x_1 + a_2x_2 + \cdots a_nx_n \\ b_1x_1 + b_2x_2 + \cdots b_nx_n \\ \vdots \end{bmatrix}$$

like Dot product

Linearity of Matrix-Vector Mult.

1. $A(\vec{x} + \vec{y}\,) = A\vec{x} + A\vec{y}$
2. $A(c\vec{x}\,) = cA\vec{x}$

Proposition 3.10.2

$\forall \vec{e}_i \in \mathbb{F}^m,\ A\vec{x} = \vec{e}_i$ is cons $\Rightarrow$ rank(A) = m = # of rows

Lecture 10

Homogeneous all the constant on the right are zero $[A|\vec{0}]$ always consistent

Non-homogeneous not all $\vec{0}$ $[A|\vec{b}]$ might be inconsistent

Nullspace solution set of a homogeneous system (Null(A))

Note: it's just $\text{Span}\{\vec{u}_1 \ldots \vec{u}_n\}$

Proposition 3.11.1

Let $A\vec{x} = \vec{0}$ be a homogeneous system of linear equations with solution set S.

if $\vec{x}, \vec{y} \in S$, and if $c \in \mathbb{F}$, then $\vec{x} + \vec{y} \in S$ and $c\vec{x} \in S$.

Note: S is closed under addition and scalar mult.

if $\vec{x}, \vec{y} \in S$, and $c, d \in \mathbb{F}$, then $c\vec{x} + d\vec{y} \in S$ ($\text{Span}\{\vec{x}, \vec{y}\} \in S$)

Ex) $\left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 2 & 1 & 0 & 2 \\ 0 & 1 & 2 & -1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$

① No Solution

② Infinite Solution, $\text{Null}(A) = \text{Span}\left\{\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}$

Associated homogeneous system $A\vec{x} = \vec{b} \Rightarrow A\vec{x} = \vec{0}$

particular solution $\vec{x}_p$

Solution to $A\vec{x} = \vec{0}$, $A\vec{x} = \vec{b}$ $\tilde{S} = \{\vec{x}_p + \vec{x} : \vec{x} \in S\}$

Solution to $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{c}$ $\tilde{S}_c = \{(\vec{x}_c - \vec{x}_b) + \vec{z} : \vec{z} \in \vec{S}_b\}$

Lecture 11

Column Space $\text{Col}(A) = \text{Span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\} = \{A\vec{x} : \vec{x} \in \mathbb{F}^n\}$

Span of the columns of A

Ex) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\text{Col}(A) = \text{Span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$

Consistent system and Column Space

Let $A \in M_{m\times n}(\mathbb{F})$ and $\vec{b} \in \mathbb{F}^m$. The system of linear equations $A\vec{x}=\vec{b}$ is cons. if and only if $\vec{b} \in \operatorname{Col}(A)$ $A\vec{x}=\vec{b}$ is cons $\Leftrightarrow$ $\vec{b} \in \operatorname{Col}(A)$

Transpose

A^T $\quad (A^T)_{ij} = A_{ji}$, j=row, i=column)

Ex) $A=\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

Rowspace

$\operatorname{Row}(A) = \operatorname{Span}\{ (\overrightarrow{row}_1(A))^T, (\overrightarrow{row}_2(A))^T, \dots, (\overrightarrow{row}_m(A))^T \}$

Span of the transposed rows of A

Proposition 4.1.7

Let $A, B \in M_{m\times n}(\mathbb{F})$. if B is row equivalent to A, then

$\operatorname{Row}(B) = \operatorname{Row}(A)$

Column Extraction

$A(\vec{e_i}) = \vec{a_i}$ $\quad \forall i \in 1, \dots, n$

Equality of Matrices

Let $A, B \in M_{m\times n}(\mathbb{F})$, then $A=B \Leftrightarrow A\vec{x} = B\vec{x}$ $\forall x \in \mathbb{F}^n$

Matrix Multiplication

$C = AB = A(\vec{b_1}, \vec{b_2} \dots \vec{b_p}) = [A\vec{b_1} \dots A\vec{b_p}]$

# of columns A = # of rows of B (not → undefined

Each column of AB is an element of Col(A)

Ex) $\begin{bmatrix} 1 & 7 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 7(-1) & 1(2)+7(3) & 1(3)+7(2) \\ 2(1)+0(-1) & 0(3)+2(3) & 2(3)+2(0) \end{bmatrix}$

Lecture 12 (Midterm)

Matrix sum

$A + B = C$ $\quad C_{ij} = a_{ij} + b_{ij}$ must be the same size

$AB \neq BA$

Ex) $\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 5 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 1-1 & 3+5 \\ -2+6 & 1+2 \end{bmatrix}$