Defn: The Derivative Fcn

The derivative fcn f' is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We say f is differentiable on interval I if $f'(a)$ exists $\forall\, a \in I$.

So, the derivative fcn is the derivative of f at each $x \in I$.

This 'primed' notation is courtesy of Newton.

Leibniz separately and simultaneously built up the concepts of calculus, with his own notation:

Given $y = f(x)$ the derivative is:

$$\frac{dy}{dx} = \frac{d}{dx}(y) = \frac{df}{dx} = \frac{d}{dx}(f)$$

differential operator

We write $f'(a)$ as $\left.\frac{dy}{dx}\right|_{x=a}$

Defn: Nth Derivative

If f is n times diff'ble, we note the n^{th} derivative as: $f^{(n)} = \frac{d^n}{dx^n}(f) = \frac{d}{dx}\left(f^{(n-1)}\right)$

Ex.

a) If f is 2 times diff'ble, the second derivative is:

$$f'' = \frac{d^2}{dx^2}(f) = \frac{d}{dx}(f')$$

b) If f is 17 times diff'ble, the 17^{th} deriv is:

$$f^{(17)} = \frac{d^{17}}{dx^{17}}(f) = \frac{d}{dx}\left(f^{(16)}\right)$$

Derivative of a Constant Fcn

Let $c \in \mathbb{R}$, $f(x) = c$. Then,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h}$$

$$= \lim_{h\to 0} 0 = 0$$

That is, for a const fcn, $f(x) = c$, we have $f'(x) = 0$. ∎

Derivative of a Linear Fcn

Let $f(x) = mx + b$, $m, b \in \mathbb{R}$. Then $f'(x) = m$.

(Exercise: show this by defn).

Derivative of a Quadratic Fcn

Let $f(x) = ax^2 + bx + c$, $a, b, c \in \mathbb{R}$. Then, $f'(x) = 2ax + b$.

(Exercise: show by defn → see last Friday's displacement fcn.)

Derivative of $\cos(x)$

We are trying to find:

$$f'(x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

Trig Identity

$$= \lim_{h \to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

if both lims exist

$$\stackrel{?}{=} \lim_{h \to 0} \left[\cos(x)\left(\frac{\cos(h) - 1}{h}\right) - \sin(x)\left(\frac{\sin(h)}{h}\right) \right]$$

$$\stackrel{?}{=} \lim_{h \to 0} \left[\cos(x)\left(\frac{\cos(h) - 1}{h}\right) \right] - \underbrace{\lim_{h \to 0} \sin(x)\left(\frac{\sin(h)}{h}\right)}$$

$$= \sin(x) \cdot (1)$$

fund. trig lim

$$= \sin(x)$$

We need to work on:

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} \cdot \frac{\cos(h) + 1}{\cos(h) + 1}$$

$$= \lim_{h \to 0} \frac{\cos^2(h) - 1}{h(\cos(h) + 1)}$$

$$= \lim_{h\to 0} \frac{(1-\sin^2(h)) - 1}{h(\cos(h)+1)}$$ ↙ trig ID

$$= \lim_{h\to 0} \frac{-\sin^2(h)}{h(\cos(h)+1)}$$

$$= \lim_{h\to 0} \frac{\sin(h)}{h} \cdot \frac{-\sin(h)}{\cos(h)+1}$$

FTL

$$= (1)\cdot\left(\frac{-0}{2}\right) = (1)(0) = 0$$

Returning to the main problem:

$$f'(x) = \lim_{h\to 0}\left[\cos(x)\left(\frac{\cos(h)-1}{h}\right) - \sin(x)\left(\frac{\sin(h)}{h}\right)\right]$$

$$= \cos(x)(0) - \sin(x)(1)$$

$$= -\sin(x)$$

That is, for $f(x) = \cos(x)$, we have $f'(x) = -\sin(x)$. $\boxtimes$

Derivative of $\sin(x)$

For $f(x) = \sin(x)$, we have $f'(x) = \cos(x)$.

(Exercise: Pr by defn → similar to $\cos(x)$ above)

Derivative of e^x

There are several ways to define 'e'.

Most common: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$$\lim_{n \to 0} (1+n)^{1/n} = e$$

For our purposes, we define e to be the unique value for which a fcn of form $f(x) = a^x$, $a \in \mathbb{R}^+$, has a tangent line of slope 1 thru $(0,1)$.

That is, we've defined for $f(x) = e^x$ that $f'(0) = 1$.

$$f'(0) = 1 = \lim_{h \to 0} \frac{e^{0+h} - e^0}{h}$$

$$1 = \lim_{h \to 0} \frac{e^h - 1}{h}$$

Then, for $f(x) = e^x$, we have:

$$f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} e^x \left(\frac{e^h - 1}{h} \right)$$

$$= e^x (1) = e^x$$

That is, for $f(x) = e^x$, we have $f'(x) = e^x$. ∎

As with limits, we have arithmetic rules for differentiation, as it would be crazy to do all derivs by defn.

Thm 7: Arithmetic Rules for Differentiation

Assume f & g are diff'ble at $x=a$.

1) Constant Multiple Rule: Let $h(x) = cf(x)$. Then h is diff'ble at $x=a$ and $h'(a) = cf'(a)$.

2) Sum Rule: Let $h(x) = f(x) + g(x)$. Then h is diff'ble at $x=a$ and $h'(a) = f'(a) + g'(a)$

3) Product Rule: Let $h(x) = f(x)g(x)$. Then h is diff'ble at $x=a$, and

$$h'(a) = f'(a)g(a) + f(a)g'(a)$$

$uv: vdu + udv$

4) Reciprocal Rule: Let $h(x) = 1/g(x)$. If $g(a) \neq 0$, then h is diff'ble at $x=a$ and

$$h'(a) = \frac{-g'(a)}{[g(a)]^2}$$

5) Quotient Rule: Let $h(x) = f(x)/g(x)$. If $g(a) \neq 0$, then h is diff'ble at $x=a$ and

$\frac{u}{v}: \frac{vdu - udv}{v^2}$

$$h'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{[g(a)]^2}$$

Pfs of Arithmetic Diff'n Rules

1) & 2) are left as exercises and follow almost directly from lim laws.

3) [Product Rule]

By defn, $$(fg)'(a) = \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a+h)g(a) + f(a+h)g(a) - f(a)g(a)}{h}$$

$$= \lim_{h\to 0} \left(\underbrace{f(a+h)}_{(*)} \left[\underbrace{\frac{g(a+h)-g(a)}{h}}_{\text{defn deriv}} \right] + \underbrace{g(a)}_{\text{const}} \left[\underbrace{\frac{f(a+h)-f(a)}{h}}_{\text{defn deriv}} \right] \right)$$

lim laws

$$= f(a)g'(a) + g(a)f'(a) \quad \blacksquare$$

We can do (*) b/c f is diff'ble, which implies that f is cts.

Being cts means by defn, $\lim_{x\to a} f(x) = f(a)$

or $\lim_{h\to 0} f(a+h) = f(a)$

4) [Reciprocal Rule].

By defn, $$\left(\frac{1}{g}\right)'(a) = \lim_{h\to 0} \frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h}$$

$$= \lim_{h \to 0} \frac{g(a) - g(a+h)}{h\,[g(a)g(a+h)]}$$

$$= \lim_{h \to 0} \Bigg[- \underbrace{\frac{g(a+h)-g(a)}{h}}_{\text{defn deriv}} \cdot \underbrace{\frac{1}{g(a)g(a+h)}}_{\text{const}} \Bigg]$$

$$= -g'(a) \cdot \frac{1}{g(a)g(a)} = \frac{-g'(a)}{[g(a)]^2} \qquad \blacksquare$$

b/c g is diff'ble
$\Rightarrow$ cts
$\Rightarrow \lim_{h \to 0} g(a+h) = g(a)$

5) [Quotient Rule]

We can combine 3) & 4) here.

$$\left(\frac{f}{g}\right)'(a) = \left(f \cdot \frac{1}{g}\right)'(a) \overset{\text{prod}}{=} f'(a)\left(\frac{1}{g}\right)(a) + f(a)\left(\frac{1}{g}\right)'(a)$$

recip.

$$= \frac{f'(a)}{g(a)} - \frac{f(a)\,g'(a)}{[g(a)]^2}$$

$$= \frac{f'(a)g(a) - g'(a)f(a)}{[g(a)]^2} \qquad \blacksquare$$

There are a few more rules we present without pf.

Thm 8: The Power Rule

Assume that $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and $f(x) = x^{\alpha}$.

Then, f is diff'ble and $f'(x) = \alpha x^{\alpha-1}$, wherever $x^{\alpha-1}$ is defined.

Ex.

$$f(x) = x^{\pi} \longrightarrow f'(x) = \pi x^{\pi-1}$$

Thm 9: The Chain Rule

Assume $y = f(x)$ is diff'ble at $x = a$ & $z = g(y)$ is diff'ble at $y = f(a)$. Then, $h(x) = g \circ f(x) = g(f(x))$ is diff'ble at $x = a$ and $h'(a) = g'(f(a)) \cdot f'(a)$.

Note: Leibniz notation pays off here:

For $z = g(y)$ and $y = f(x)$ we get:

$g'(y) = \frac{dz}{dy}$ and $f'(x) = \frac{dy}{dx}$.

Then, for $z = g(y) = g(f(x))$, we have that

$$\frac{dz}{dx} = g'(f(x)) \cdot f'(x) = \left.\frac{dz}{dy}\right|_{f(x)} \left.\frac{dy}{dx}\right|_{x}$$

That is: $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$

Ex.

$$f(x) = \sin(x^2) \Rightarrow f'(x) = \cos(x^2) \cdot \frac{d}{dx}(x^2)$$

$$= \cos(x^2) \cdot (2x)$$

$$= 2x\cos(x^2)$$

We can find some other key derivatives by applying our rules:

Ex. 1

$f(x) = \tan(x)$. Find $f'(x)$.

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \frac{u}{v} = \frac{v\,du - u\,dv}{v^2}$$

$$\frac{d}{dx}[\tan(x)] = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right]$$

Quotient Rule

$$= \frac{\cos(x)\frac{d}{dx}[\sin(x)] - \sin(x)\frac{d}{dx}[\cos(x)]}{[\cos(x)]^2}$$

$$= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x).$$

$$\therefore \frac{d}{dx}[\tan(x)] = \sec^2(x).$$

Ex. 2

$f(x) = \csc(x)$. Find $f'(x)$.

$$f(x) = \frac{1}{\sin(x)}$$

Reciprocal Rule

$$f'(x) = \frac{-\frac{d}{dx}(\sin(x))}{[\sin(x)]^2}$$

$$= \frac{-\cos(x)}{\sin^2(x)} = \frac{-\cos(x)}{\sin(x)} \cdot \frac{1}{\sin(x)}$$

$$= -\cot(x)\csc(x)$$

$$\therefore \frac{d}{dx}[\csc(x)] = -\cot(x)\csc(x).$$

You can find $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$ and

$$\frac{d}{dx}[\cot(x)] = -\csc^2(x)$$

in a similar manner.

Ex. 3

$f(x) = a^x$, $a > 0$. Find $f'(x)$.

$$a^x = e^{\ln(a^x)} = e^{x\ln(a)}$$

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{x\ln(a)}\right)$$

Chain rule

$$= e^{x\ln(a)} \cdot \frac{d}{dx}[x\ln(a)]$$

$$= e^{x\ln(a)} \cdot \ln(a)$$ ↙ const mult rule

$$= a^x \ln(a)$$

$$\therefore \frac{d}{dx}[a^x] = a^x \ln(a)$$

We now do more practice with our rules:

Ex.

$f(x) = 2^x \sin(x)$. Find $f'(x)$.

$$f'(x) = \frac{d}{dx}(2^x)\sin(x) + 2^x \frac{d}{dx}(\sin(x))$$ [Product Rule]

$$= 2^x \ln(2)\sin(x) + 2^x \cos(x).$$

Common Trig Derivative

1) $\sin(x) = \cos(x)$
3) $\tan(x) = \sec^2(x)$
5) $\sec(x) = \sec(x)\tan(x)$
2) $\cos(x) = -\sin(x)$
4) $\csc(x) = -\cot(x)\csc(x)$
6) $\cot(x) = -\csc^2(x)$.

Sidebar:

You're likely familiar with the slope-intercept form of a line: $y = mx + b$

There is another form: point-slope form:

$$y - y_1 = m(x - x_1) \Rightarrow y = m(x - x_1) + y_1$$

$m \to$ slope $\qquad (x_1, y_1) \to$ point on the line

Defn: Tangent Line

If f is differentiable at $x = a$, then the tangent line to f at $x = a$ is the line passing thru $(a, f(a))$ with slope $f'(a)$.

This has eqn:

$$y - f(a) = f'(a)(x - a)$$

$$y = f'(a)(x - a) + f(a)$$

Ex. 1

Find the instantaneous velocity of $s(t) = 5t^2 + 6t - 7$

a) at $t = 7$

b) at $t = t_0$

a) $s'(7) = \lim_{h\to 0} \frac{s(7+h) - s(7)}{h}$

$= \lim_{h\to 0} \frac{5(7+h)^2 + 6(7+h) - 7 - (5(7^2) + 6(7) - 7)}{h}$

$= \lim_{h\to 0} \frac{245 + 70h + 5h^2 + 42 + 6h - 245 - 42}{h}$

$= \lim_{h\to 0} \frac{76h + 5h^2}{h}$

$= \lim_{h\to 0} \frac{\cancel{h}(76 + 5h)}{\cancel{h}}$

$= \lim_{h\to 0} 76 + 5h = \underline{\underline{76}}$

b) $s'(t_0) = \lim_{h\to 0} \frac{s(t_0+h) - s(t_0)}{h}$

$= \lim_{h\to 0} \frac{5(t_0+h)^2 + 6(t_0+h) - 7 - 5t_0^2 - 6t_0 + 7}{h}$

$= \lim_{h\to 0} \frac{(10t_0 + 6)\cancel{h} + 5h^{\cancel{2}}}{\cancel{h}}$

$= \lim_{h\to 0} 10t_0 + 6 + 5h$

$s'(t_0) = \underline{\underline{10t_0 + 6}} = v(t_0)$

Note: velocity is the derivative of displacement.

Ex.2

Find the eqn of the tangent line at $x = 3$ for $f(x) = \frac{1}{x+5}$.

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{3+h+5} - \frac{1}{3+5}}{h}$$

$$= \lim_{h \to 0} \left[\frac{8 - (h+8)}{(h+8)8} \right] \left[\frac{1}{h} \right]$$

$$= \lim_{h \to 0} \frac{-1}{8(h+8)} = \frac{-1}{64}$$

We have $x = 3$, and $f(3) = \frac{1}{3+5} = \frac{1}{8}$.

So, the eqn of the tangent line to $f(x)$ at $x = 3$ is: $\boxed{y = -\frac{1}{64}(x-3) + \frac{1}{8}}$

Let's connect differentiability with continuity.

Thm 1: Differentiability Implies Continuity

If f is diff'ble at $x = a$, then it is cts at $x = a$.

[By contrapositive: if f is discts at $x=a$ then f is not diff'ble at $x=a$]

Pf:

Let f be diff'ble at $x=a$. Then

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \text{ exists.}$$

Then, by a thm from earlier in the course, since $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists and since $\lim_{x\to a} x-a = 0$,

we must have $\lim_{x\to a} [f(x)-f(a)] = 0$.

$$\lim_{x\to a} [f(x)] - f(a) = 0$$

defn of cty. $\longrightarrow$ $\lim_{x\to a} f(x) = f(a)$

That is $f(x)$ is cts at $x=a$. ∎

Now, does cty imply differentiability?

↳ No. Counterexample: $f(x) = |x|$, at $x=0$.

We see $\lim_{x\to 0} |x| = 0 = |0|$ so $f(x)$ is cts at $x=0$.

$$\text{But, } f'(0) = \lim_{h\to 0} \frac{|0+h|-|0|}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h}$$

Start of Wk 7: October 30

Let's revisit the defn of the derivative of $f(x)$ at $x=a$:

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

Then, for x values very close to $x=a$, we have

$$f'(a) \approx \frac{f(x)-f(a)}{x-a}$$

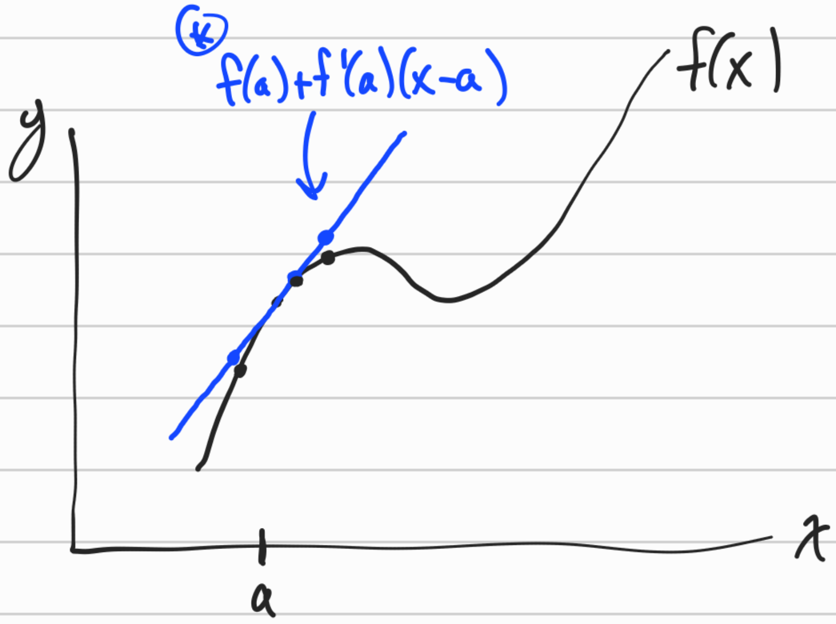
We can rearrange this formula to:

$$f(x) \approx f(a) + f'(a)(x-a) \qquad (*)$$

for x-values close to $x=a$.

Note: We recognize $(*)$ as the eqn of the tangent line to $f(x)$ at $x=a$.

Pictorially:



We can approximate the fcn values close to $x=a$ by taking a linearization or a tangent line approximation.

Defn: Linear Approximation

Let $y = f(x)$ be diff'ble at $x=a$. Then, the linear approx. to f at $x=a$ is the fcn:

$$L_a^f(x) = f(a) + f'(a)(x-a)$$

Note: We just write $L_a(x)$ if f is clear from context.

Note!

1. to see if a continuous function is diff'ible, check if the value of both sides are equal ($f'(a) = f'(a)$)

2. Read carefully! it is diff'ible for $f(x)$ like $f(x) = \frac{x^2+1}{x}$ but Becareful on "a" if $a \Rightarrow 0$ then it's not defined

3. Use limit Def and Remember Squeeze Theorem.

Defn: Linear Approximation

Week 7

Let $y = f(x)$ be diff'ble at $x=a$. Then, the linear approx. to f at $x=a$ is the fcn:

$$L_a^f(x) = f(a) + f'(a)(x-a)$$

Note: We just write $L_a(x)$ if f is clear from context.

Ex.

Use the lin appx to estimate $\sin(\sqrt{10})$.

Take $f(x) = \sin(\sqrt{x})$ and the appx is at $a = \pi^2$.

$\hookrightarrow \pi^2 \approx 10$

$\hookrightarrow$ we know $\sin(\sqrt{\pi^2})$

$$L_{\pi^2}^f = f(\pi^2) + f'(\pi^2)(x - \pi^2)$$

Now, $f(\pi^2) = \sin(\sqrt{\pi^2}) = \sin(\pi) = 0$

And $f'(x) = \cos(\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)$

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow f'(\pi^2) = \cos(\sqrt{\pi^2})\frac{1}{2\sqrt{\pi^2}} = -\frac{1}{2\pi}$$

$\therefore L_{\pi^2}^f(x) = 0 + \frac{-1}{2\pi}(x - \pi^2)$

Then $\sin(\sqrt{10}) \approx L_{\pi^2}^f(10) = -\frac{1}{2\pi}(10 - \pi^2) = \boxed{\frac{-5}{\pi} + \frac{\pi}{2}}$

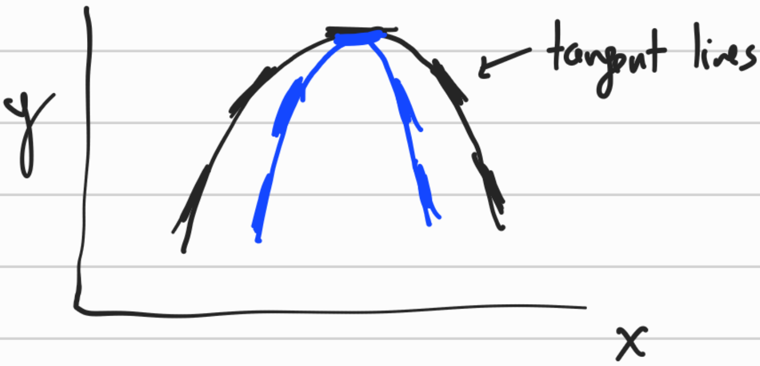
$\doteq -0.020753$

(actual $\doteq -0.020683$)

What affects how good our approx is?

- how far away we are approx. from $x=a$
  - ↳ typically: further = worse
- how curved the fcn is near $x=a$.
  - ↳ typically: more curved = worse

How do we describe curvature?



↳ we are interested in r.o.c. of r.o.c.

↳ 2nd derivative.

Thm 6: Error in Linear Appx

Assume that f is such that $|f''(x)| \leq M$ for each x in an interval I containing a point a.

Then,
$$\overbrace{\left|f(x) - L_a^f(x)\right|}^{\text{error}} \leq \frac{M}{2} \underbrace{(x-a)^2}_{\text{we will revisit this later}}$$
for each $x \in I$.

Ex.

Find an upper bound for the error on $L_{27}^{\sqrt[3]{x}}(x)$ on $[25,30]$.

Let's deal with M first.

↳ We need $f''(x)$.

$$f(x) = \sqrt[3]{x} = x^{1/3} \Rightarrow f'(x) = \frac{1}{3}x^{-2/3}$$

$$\Rightarrow f''(x) = -\frac{2}{9}x^{-5/3} = \frac{-2}{9x^{5/3}}$$

Now $|f''(x)| = \left|\frac{-2}{9x^{5/3}}\right| = \frac{2}{9x^{5/3}}$ on $[25,30]$

On $[25,30]$, $|f''(x)| = \frac{2}{9x^{5/3}}$ is maximized at $x=25$.

$$\therefore M = \frac{2}{9(25)^{5/3}} \doteq 1.03965 \times 10^{-3}$$

*Find the number that maximize the $f''(x)$ that is M

We have, on $[25,30]$

$$\left|\sqrt[3]{x} - L_{27}(x)\right| \leq \frac{1.03965\times10^{-3}}{2}\underbrace{(x-27)^2}$$

this is maximized at $x=30$ on $[25,30]$.

$$\leq \frac{1.03965\times10^{-3}}{2}(3)^2$$

Find the number that can maximize $(x-a)$

within interval

fixed

$$\boxed{\left|\sqrt[3]{x} - L_{27}(x)\right| \leq 4.67842\times10^{-3}}$$

WSIC?

The lin appx is useful for estimating fcn values for nasty fcns.

We care about errors to ensure our appx is not garbage.

There are also some applications of lin appx.

Estimating Change

Assume we know $f(a)$.

How does $f(x)$ change if we move to x_1 near $x=a$?

That is, what is: $\Delta y = f(x_1) - f(a)$ if $\Delta x = x_1 - a$.

Using $f(x_1) \approx L_a(x_1)$ (for Δx small):

$$\Delta y \approx \underbrace{L_a(x_1)} - f(a)$$

$$\approx \left(\cancel{f(a)} + f'(a)(x_1 - a)\right) - \cancel{f(a)}$$

$$\approx f'(a)(x_1 - a)$$

$$\boxed{\Delta y \approx f'(a)\Delta x}$$

Ex.

An ice cube of side length 3cm shrinks such that its side length reduces by 1mm.

Estimate the change in volume of the ice cube.

We have that $\Delta V \approx V'(30)(-1)$ where $V(s) = s^3$.

$$V(s) = s^3 \Rightarrow V'(s) = 3s^2 \Rightarrow V'(30) = 3(30^2) = 2700$$

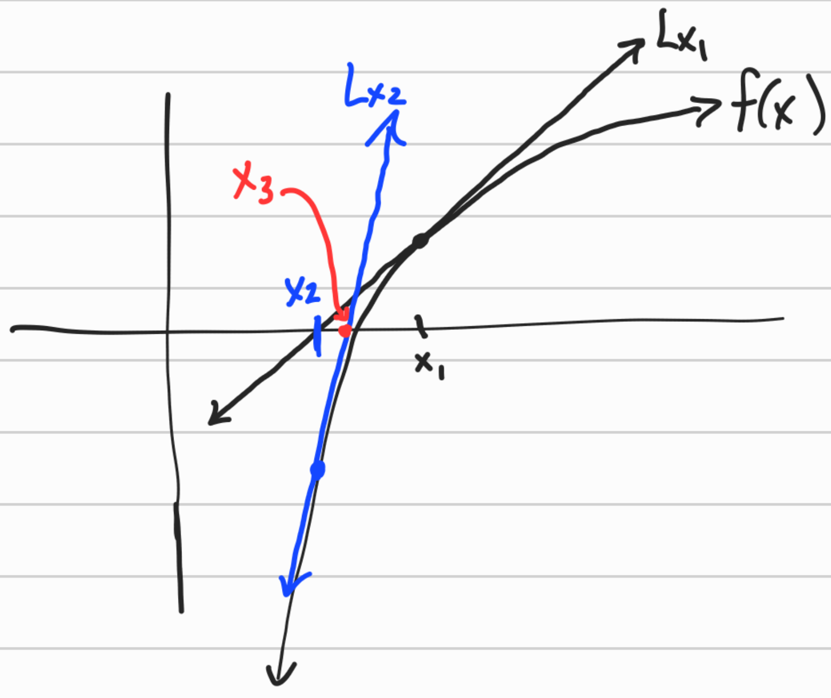
$$\therefore \Delta V \approx (2700)(-1) = \boxed{-2700\,mm^3}$$

(actual $\Delta V = V(29) - V(30) = -2611\,mm^3$)

Recall before the midterm we learned IVT and used it to come up with the Bisection Method root-finding algorithm.

We can use lin appx to motivate another, more efficient, algorithm: Newton's Method.

Pictorially:



Newton's Method:

- Make a guess, x_1, of where the root is (IVT is useful)
- Take the lin appx, $L^f_{x_1}$, and find where it intersects the x-axis, call this value x_2.
- Repeat at x_2 to find x_3 and so on...

Basic Area and Volumn formula

Cube: $A(x) = 6 \cdot x^2$ $V(x) = x^3$

Prism: $A(x) = 2 \cdot \text{basearea} + La$ $V = ba \cdot h$

Pyramid: $ba + La$ $V = ba \cdot h \times 1/3$

Sphere: $4\pi r^2$ $V = 4/3\pi r^2$

Cylinder: $2\pi rh + 2\pi r^2$ $V = \pi r^2 h$

Cone: $A = \pi rs + \pi r^2$ $V = \frac{1}{3}\pi r^2 h$

$\Delta A = A'(r)\Delta r$

πr^2

A $2\pi r$

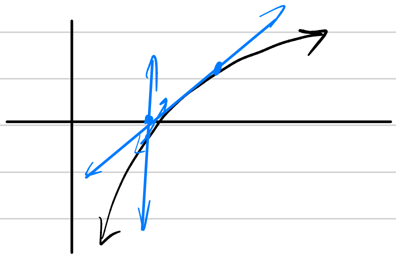
$\frac{2\pi}{2} \cdot (x - \pi)$.

$\frac{1}{3}\pi r$

Math 137 lecture Note

November 18th

Newton's Method



Examining *, we are looking for $L_{x_1}^{f}(x_2)=0$

↳ $L_{x_1}^{f}(x_2) = f(x_1)+f'(x_1)(x_2-x_1) = 0$ (looking for its root)

↳ $f'(x_1)(x_2) = f'(x_1)x_1 - f(x_1)$

↳ $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ $\quad f'(x_1) \neq 0$

∴ we can iteratively use $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$

Note:

- Unlike bisection method this does not always converge
  - ↳ f(x) dependent
  - ↳ L_{x_n} is a horizontal TL
  - ↳ pick a point too far from the root the $\{x_n\}$ might diverge or osillate
- Converge faster

Ex) Find the root of the $f(x) = x^3+5x^2-3x-17$ on $[1,3]$ accurate to 7 decimal places.

Note: that f(x) is cts everywhere, $f(1)<0$, $f(3)>0$

By IVT, root exists within $(1,3)$

Lets take $x_1 = 2$, then $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Then $x_{n+1} = x_n - \frac{x_n^3+5x_n^2-3x_n-17}{3x_n^2+10x_n-3}$

Want: $x_2 = 2 - \dfrac{2^3 + 5(2)^2 - 3(2) - 17}{3(2)^2 + 10(2) - 3}$

$= 2 - \dfrac{5}{29}$

≈ 1.82758621

(iterate again) $\rightarrow$

$x_3 \doteq 1.81486224$

$x_4 \doteq 1.81479452$

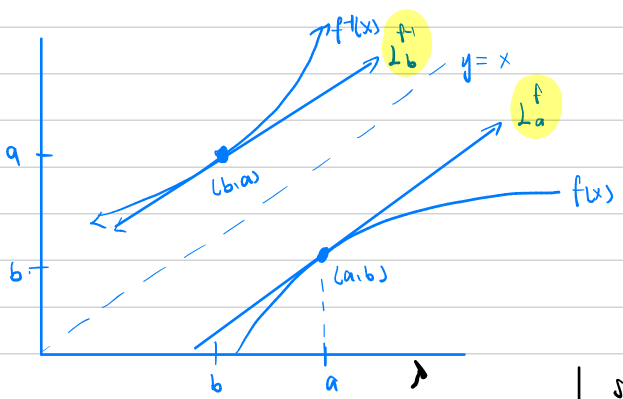
$x_5 \doteq 1.81479452$

$\therefore$ root is $x \approx 1.81479452$

Return to Derivative $\rightarrow$ Linear approximation helps find inverse f(x)

Recall: Geomatrically, Inversion = Reflection of a f(x) across the line $y=x$

Algebrically, swap x, y, isolate y.



Note: $f(a) = b$, and $f^{-1}(b) = a$

Replace with $\uparrow$

We can formulate $L_b^{f^{-1}}$ in 2 ways

1. Directly

$$L_b^{f^{-1}} = f^{-1}(b) + \left[(f^{-1})'(b)\right](x-b) \quad (*)$$

2. Algebraically find the inverse of L_a^f

$$L_a^f = y = f(a) + f'(a)(x-a)$$

Swap: $x = f(a) + f'(a)(y-a)$

Solve: $y = \dfrac{1}{f'(a)}(x - f(a)) + a \qquad f'(a) \neq 0$

$$L_b^{f^{-1}} = \frac{1}{f'(a)}(x-b) + f^{-1}(b) \quad (**)$$

$*$, $**$ for same Line

then $f^{-1\prime}(b) = \dfrac{1}{f'(a)} = \dfrac{1}{f'(f^{-1}(b))}$

Thm 10. Inverse Function Thm (IFT)

Assume that $y = f(x)$ is cts and invertible on $[c,d]$ with inverse $x = f^{-1}(y)$

and f is diff'ble at $a \in (c,d)$

if $f'(a) \neq 0$, then f^{-1} is diffible at $b = f(a)$ and

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$$

Moreover, L_a^f is also invertible and $(L_a^f)^{-1}(x) = (L_b^{f^{-1}})(x) = (L_{f(a)}^{f^{-1}})(x)$

Ex) Silly Goofy Example

Let $f(x) = 5\sqrt{x}$, let's find $f^{-1}(5)$ using IFT

By IFT, $f^{-1}(5) = \dfrac{1}{f'(f^{-1}(5))}$ ← $f'(x)$, $f^{-1}(x)$ need

Never find the Derivative of $f^{-1}(x)$

$f'(x) = 5\sqrt{x} = \dfrac{5}{2\sqrt{x}}$

$f^{-1}(x) = y = 5\sqrt{x} \rightarrow x = 5\sqrt{y}$ $\Rightarrow f^{-1}(5) = \dfrac{25}{25} = 1$

$y = \dfrac{x^2}{25}$ $\Rightarrow f'(f^{-1}(5)) = f'(1) = \dfrac{5}{2}$

Then by IFT, $f^{-1\prime}(5) = \dfrac{1}{5/2} = 2/5$

IFT via chain rule

Assume f has inverse f^{-1}, and both are diff'ble

By def, $f(f^{-1}(x)) = x$

$$\frac{d}{dx} f(f^{-1}(x)) = \frac{d}{dx}(x)$$

$$f'(f^{-1}(x)) \cdot [f^{-1\prime}(x)] = 1$$

$$(f^{-1\prime}(x)) = \frac{1}{f'(f^{-1}(x))}$$ (Same equation)

November 3rd

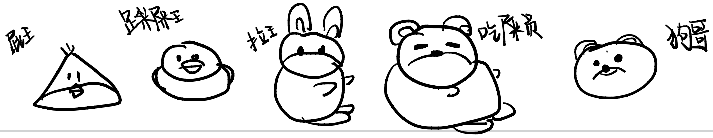
We can use IFT to find the derivative of $\ln(x)$

Ex) Let $f(x) = e^x$, we have $f^{-1}(x) = \ln(x)$

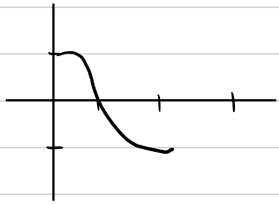
By IFT, $f^{-1\prime}(x) = \dfrac{1}{f'(f^{-1}(x))}$ $\Rightarrow (\ln(x))'(x) = \dfrac{1}{e^{\ln(x)}} = \dfrac{1}{x}$

So $\dfrac{d}{dx}[\ln(x)] = \dfrac{1}{x}$

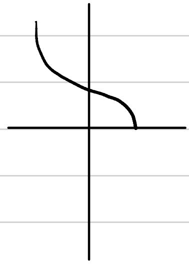
We can also use IFT/Chain rule to find derivatives of inverse Trig Functions.



Ex) arccos(x)



$\rightarrow$



$\leftarrow$ arccos(x)
D: [-1, 1]
R: [0, π]

Note: arccos(x) is the inverse of a restricted domain cos

IFT tells that arccos(x) is diff'ble

cos(x) restricted on [0, π]
R: [-1, 1]

Derivative of arccos(x)

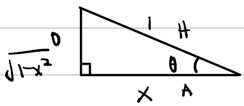
Note: $\cos^{-1}(x) \neq \text{arccos}(x)$

Now. $f(x) = \cos(x)$ and $f^{-1}(x) = \text{arccos}(x)$ then:

$$f^{-1'}(x) = \frac{1}{f'(f^{-1}(x))} = (\text{arccos}(x))'(x) = \frac{1}{-\sin(\text{arccos}(x))}$$

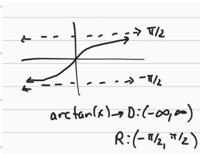
Rewrite $\frac{1}{-\sin(\text{arccos}(x))}$ Remember that arccos is an ANGLE (θ)

$\text{arccos}(x) = \theta \Rightarrow \cos(\theta) \Rightarrow x = \frac{x}{1}$



using Pytha. Theorem, we find O side is $\sqrt{1-x^2}$

then $\sin(\text{arccos}(x)) = \sin\theta = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$



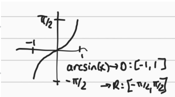
$\frac{d}{dx}[\text{arccos}(x)] = \frac{-1}{\sqrt{1-x^2}}$

$x \in [0, \pi]$

$\frac{d}{dx}[\text{arcsin}(x)] = \frac{1}{\sqrt{1-x^2}}$

$\in [-\pi/2, \pi/2]$

R: [-1, 1]



$\frac{d}{dx}[\text{arctan}(x)] = \frac{1}{1+x^2}$

$(-\pi/2, \pi/2)$ R: $[-\infty, \infty]$

Ex.1) $f(x) = \text{arcsin}(2^{5x})$

$$f'(x) = \frac{1}{\sqrt{1-(2^{10x})}} \cdot \ln 2 \cdot 2^{5x} \cdot 5$$

by chain rule

$$= \frac{5\ln 2 \cdot 2^{5x}}{\sqrt{1-2^{10x}}}$$

Ex) $f(x) = \ln(\text{arctan}(e^{\sin(x)}))$

$$f'(x) = \frac{1}{\text{arctan}(e^{\sin(x)})} \cdot \frac{1}{1+e^{2\sin(x)}} \cdot e^{\sin x}$$

$$= \frac{\cos(x)\, e^{\sin(x)}}{\text{arctan}(e^{\sin x})(1+e^{2\sin x})}$$

By chain rule

November 6^{th}

So far, we can differentiate Explicit functions → $y=f(x)$

Implicit functions:

Ex) y' for $x^3y^5+2x=y^3+4$ Note: y is an unknown function of x

$$\frac{d}{dx}[x^3y^5+2x] = \frac{d}{dx}[y^3+4]$$

$$[(3x^2)(y^5)+(x^3)(5y^4)(1)]+2 = 3y^2(1)+0 \Rightarrow \frac{dy}{dx} = \frac{-2-3x^2y^5}{5x^3y^4-3y^2}$$

$$3x^2(y^5)+x^3\cdot 5y^4\cdot\frac{dy}{dx}+2-3y^2\cdot\frac{dy}{dx}$$

Note: Given $x^2y^2=-72$, we could proceed the above steps and find $y'=\frac{-y}{x}$

↳ both positive so garbage

Extend Implicit Differentiation to find "Logarithmic differentiation"

Function: $h(x)=g(x)^{f(x)}$, $(g(x)>0)$

Ex)

$y=x^x$ $(x>0)$ find y' $y=x^x$, $y'=x^x(\ln(x)+1)$

$\ln(y)=\ln(x^x)$

$\ln(y)=x\ln(x)$ $\Rightarrow$ $\left(\frac{1}{y}\right)\frac{dy}{dx}=(\ln(x))+1$

$$\frac{dy}{dx}=y(\ln(x)+1)$$

$$=x^x(\ln(x)+1)$$

Ex) $y=\dfrac{(x-3)^3(x+4)^2(x-1)}{(x+1)^2(x^2+x+1)^3}$ ⇐ log makes it easier

$$\ln(y)=\ln\left[\frac{(x-3)^3(x+4)^2(x-1)}{(x+1)^2(x^2+x+1)^3}\right]$$

$$=[3\ln(x-3)+2\ln(x+4)+\ln(x-1)]-[2\ln(x+1)+3\ln(x^2+x+1)]$$

$$\frac{1}{y}\frac{dy}{dx}=\left(\frac{3}{x-3}\right)+\left(\frac{2}{x+4}\right)+\left(\frac{1}{x-1}\right)-\left(\frac{2}{x+1}\right)-\left(\frac{3(2x+1)}{x^2+x+1}\right)$$

$$y'=\left(\frac{(x-3)^3(x+4)^2(x-1)}{(x+1)^2(x^2+x+1)^3}\right)\left(\left(\frac{3}{x-3}\right)+\left(\frac{2}{x+4}\right)+\left(\frac{1}{x-1}\right)-\left(\frac{2}{x+1}\right)-\left(\frac{3(2x+1)}{x^2+x+1}\right)\right)$$

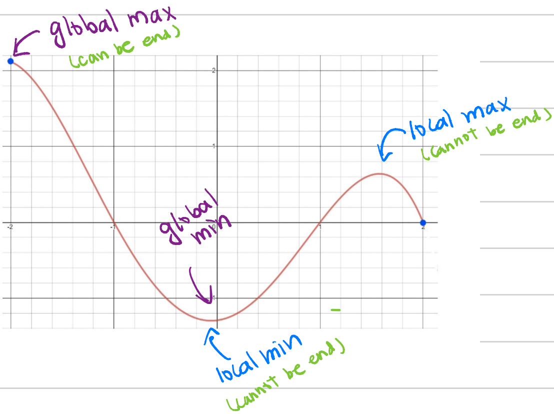
Def : Local Maxima / Minima

A point c is called a local max/min for a $f(x)$ if $\exists\,(a,b)$ such that $f(x) \leq f(c)$ / $f(x) \geq f(c)$ $\forall x \in (a,b)$

Note : that means endpoint cannot be local extrema

Global extrema that occurs at non-endpoint are also local extrema

Ex)



It's important in differential Calc to be able to identify local extrema.

November 8th

Thm 11. Local Extrema Thm (LET)

If c is a local extremum for f and $f'(c)$ exists, then $f'(c) = 0$.

Proof : ← similar proof for max

Assume, wlog, we have c is a local min. and that $f'(c)$ exists.

Since $f'(c)$ exists, we have $f'(c) = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h \to 0^{\pm}} \frac{f(c+h)-f(c)}{h}$

Since c is a local min, $\exists\,(a,b)$ s.t $c \in (a,b)$ and $f(c) \leq f(x)$ $\forall x \in (a,b)$

Then for $h > 0$ small enough that $c < c+h < b$, then $f(c+h) \geq f(c)$

This means that

$$f'(c) = \lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h} \geq 0$$

Similarly, $h < 0$ small enough that $a < c+h < c$, we have $f(c+h) \geq f(c)$ so

$$f'(c) = \lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h} \leq 0$$

Since $0 \leq f'(c) \leq 0$, then $f'(c) = 0$ ∎

Note: $f'(c)=0 \not\Rightarrow$ local max/min   local max/min $\Rightarrow f'(c)=0$

Ex) $y=|x|$

Def: Critical Points (cp's)

A point c in the domain of a f(x) is called a critical point for f if either $f'(c)=0$ or DNE

For a cts f(x) on a closed interval, EVT guaranteed a global max/min
↳ at endpoint / with (a,b)
If global extrema is on (a,b), it's also local ⇒ critical points.
↳ check Endpoint and critical points

"Closed Interval Method" for global extrema for cts f on [a,b]

1. Calculate f(a) and f(b)
2. Calculate f'(x)
3. Find f'(x)=0, DNE Must say that f'(x) exists everywhere
4. Find values at cp's.
5. Global max is the largest value, and vice versa for minima from step ①⇒④

Ex) Find the global extrema of $f(x)=x\sqrt{4-x^2}$ on $[-1,2]$

1) $f(-1)=-\sqrt{3}$, $f(2)=0$

2) $f'(x)=(1)(\sqrt{4-x^2})+(x)(\frac{1}{2}(4-x^2)^{-1/2}\cdot(-2x))$

$=\sqrt{4-x^2}-\frac{x^2}{\sqrt{4-x^2}}$

3) DNE? ⟶ $x=2$ (No -2 since it's not in the interval)

4) $f'=0 \rightarrow \sqrt{4-x^2}=\frac{x^2}{\sqrt{4-x^2}}$

$x=\pm\sqrt{2}$ (No Negative value)

$=\sqrt{2}$

Calculate: $f(2)=0$, $f(\sqrt{2})=2$ ⇒ Global max: $(\sqrt{2},2)$ Global min: $(-1,\sqrt{3})$

Next central theorem:

Consider: distance: 5km time: 5 min Speed: 60km/hr

Avg velocity: $\frac{\Delta s}{\Delta t} = \frac{5km}{1/12 hr} = 60 km/hr$

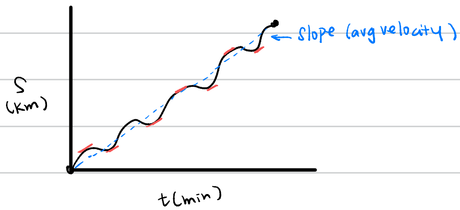
Case 1: constantly driving

Case 2: slow at some point. sped at some points.

↳ guanrantee → sometimes 60 (have to go from $speed_1 \leq 60 \leq speed_2$)

At some point, Irc = Arc

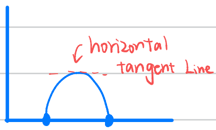
Pictice:



Motivating Ideas:

↳ Between the zeros of $f(x)$, there's a zero of $f'(x)$

"what goes up must come down theorem"



Thm 2: Rolle's Thm

Assume $f(x)$ is cts on $[a,b]$ and diff'ble on (a,b) and $f(a)$ and $f(b)=0$

then $\exists\, c \in (a,b)$ s.t $f'(c)=0$

Proof:

Case 1: $f(x)=0$

↳ then $f'(x)=0$ So $f'(c)=0\ \forall\, c\in(a,b)$

Case 2: $f(x_0)>0$ for some $x_0\in(a,b)$

↳ by EVT, since f is cts on (a,b)

$\exists$ global max on $[a,b]$ (Not end point)

which is $c\in(a,b)$

then c is also a local max

Since f is diff'ble, by LET, $f'(c)=0$

Case 3: $f(x_0)<0$ for some $x_0\in(a,b)$

↳ by EVT, since f is cts on (a,b)

$\exists$ global min on $[a,b]$ (Not end point)

which is $c\in(a,b)$

then c is also a local min

Since f is diff'ble, by LET, $f'(c)=0$

∎

November 10^{th}

Thm 1: Mean Value Theorem

Assume f(x) is cts on [a,b] and diff'ble on (a,b). Then $\exists c \in (a,b)$, s.t.

Iroc $\rightarrow$ $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ $\leftarrow$ Average Rate of Change

Proof:

go through the point (a, f(a))

We introduce $h(x) = f(x) - \left[f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a) \right]$ $\leftarrow$ a secant line between a, b and go through (a, f(a))

$\uparrow$ slope

the height above of f(x) above the secant line.

Notice: h(x) is cts on [a,b] and diff'ble on (a,b) $\Rightarrow$ it's the difference of f and a line [cts, diff'ble everywhere]

$h(a) = h(b) = 0$ $\Rightarrow$ apply Rolle's theorem. $\Rightarrow$ $\exists c \in (a,b)$, $h'(c) = 0$

Overall, $h'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a}$, then $h'(c) = f'(c) - \dfrac{f(b)-f(a)}{b-a} = 0$ $\Rightarrow$ $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ ∎

Alway check the Hypothesis!!!

WSIC? MVT is a fundemental tool to prove further cal result

Def: Antiderivative

Given a f(x) f, an antiderivative is a f(x) F st $F'(x) = f(x)$

If $F'(x) = f(x)$ $\forall x \in I$, we say F is an antiderivative for f or I

Note: Unlike derivative, antiderivative are not unique

Ex) $f(x) = 5$: $F(x) = 5x$, $F(x) = 5x+1$ . . .

Thm 3: Constant Function Theorem

If $f'(x) = 0$, $\forall x \in I$, then $\exists \alpha \in \mathbb{R}$ such that $f(x) = \alpha$ $\forall x \in I$

Proof:

Choose $x_1, x_2 \in I$, $x_1 \neq x_2$, wlog, $x_2 > x_1$. Let $f(x_1) = \alpha$. Let $f'(x) = 0$ $\forall x \in I$

Since f is diff'ble on I, it is also cts on I. Thus, we can apply MVT.

$\exists c \in (x_1, x_2)$, s.t. $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ $\Rightarrow f(x_2) = \alpha$

$0 = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ (defined it before) $\leftarrow (x_2 \neq x_1)$

(α everywhere in the Interval since x_1, x_2 are arbitrary)

Thm 4: Antiderivative Theorem

If $f'(x)=g'(x)\ \forall x\in I$, then $\exists\, \alpha\in\mathbb{R}$ s.t. $f(x)=g(x)+\alpha\ \forall x\in I$ (all antiderivative differ by a constant)

Proof:

Consider $h(x)=f(x)-g(x)$. Notice $h(x)$ is cts and diff'ble on I since f, g are] hypothesis

Also notice $h'(x)=f'(x)-g'(x) \Rightarrow 0$

By CPT, $h(x)=\alpha\ \forall x\in I \Rightarrow f'(x)-g'(x)=\alpha \Rightarrow f(x)=g(x)+\alpha\ \forall x\in I$

Leibniz Notation for Antiderivative

$\int f(x)\,dx$ ⇐ indefinite integral of f ⇐ Denote the family of Antiderivative

$f(x)$ is the integrand

Ex) $\int 5\,dx = 5x + C$

Thm 5: Power rule for Antidifferentiation

If $\alpha \neq -1$, then $\int x^\alpha\,dx = \dfrac{x^{\alpha+1}}{\alpha+1} + C$

Pf: by differentiative

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$$

$$\int \frac{-dx}{\sqrt{1-x^2}} = \arccos(x) + C$$

Common Antiderivative

domain matches $\frac{1}{x}$

1. $\int 0\,dx = C$
2. $\int \frac{1}{x}\,dx = \ln|x| + C$
3. $\int e^x\,dx = e^x + C$
4. $\int a^x\,dx = \dfrac{a^x}{\ln(a)} + C$
5. $\int \cos x\,dx = \sin x + C$
6. $\int \sin(x)\,dx = -\cos(x) + C$
7. $\int \sec^2(x)\,dx = \tan(x) + C$

$\int \frac{dx}{1+x^2} = \arctan(x) + C$

$\ln|x|,\ x<0$

$\frac{d}{dx}\ln(-x) = \frac{1}{-x}\cdot -1$

$= \frac{1}{x}$

Another rule: $\int (\alpha_1 f(x) + \alpha_2 g(x) + \alpha_3 h(x) \cdots)\,dx = \alpha_1\int f(x)\,dx + \alpha_2\int g(x)\,dx + \alpha_3\int h(x)\,dx + \cdots$

* only use it when the antiderivative exists. * Don't forget dx, C

November 18th

using MVT to prove

Thm 6: Increasing, Decreasing Function Theorem

Let I be an interval, $x_1, x_2 \in I$, $x_1 < x_2$

1) If $f'(x)>0$, $\forall x\in I$, then $f(x_1)<f(x_2) \Rightarrow f$ is increasing on I

2) If $f'(x) \geq 0$ $\forall x \in I$, $f(x_1) \leq f(x_2) \Rightarrow f$ is non-decreasing

3) If $f'(x) < 0$ $\forall x \in I$, $f(x_1) > f(x_2) \Rightarrow f$ is decreasing

4) If $f'(x) \leq 0$ $\forall x \in I$, $f(x_1) \geq f(x_2) \Rightarrow f$ is non-increasing.

Proof (4):

Since $f'(x)$ exists on I, $f(x)$ is diff'ble on I, $f(x)$ is cts on I $[a,b]$, diff'ble (a,b)

Thus, apply MVT on f on $[a,b]$, that is, $\exists c \in (x_1, x_2)$ such that $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$

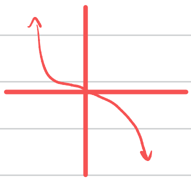
But we have $f'(x) \leq 0$, $\forall x \in I$, so $\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq 0$

But since $x_2 > x_1$, $x_2 - x_1 > 0$, so $f(x_2) - f(x_1) \leq 0 \Rightarrow f(x_2) \leq f(x_1) \Rightarrow$ Non-increasing

Note: f inc $\not\Rightarrow f' > 0$ ex) $y = x^3$

$x = 0$
$y' = 0$

f dec $\not\Rightarrow f' < 0$ ex. $y = -x^3$

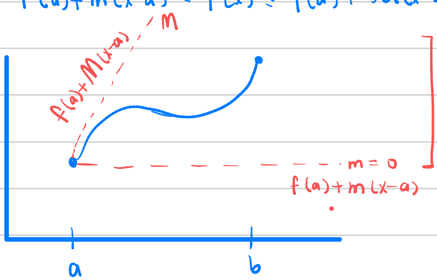


$x = 0$
$y' = 0$

Thm 7: Bounded Derivative Theorem

Assume f is cts on $[a,b]$, diff'ble on (a,b), and $m \leq f'(x) \leq M$ $\forall x \in (a,b)$ (bounded)

Then $f(a) + m(x-a) \leq f(x) \leq f(a) + M(x-a)$ $\forall x \in [a,b]$



graph trapped by these 2 Lines.

Proof:

By the hypothesis of the thm, we can apply MVT to f on $[a,b]$

Further, MVT would also apply on $[a, x_1]$ where $x_1 \in (a,b]$

$\hookrightarrow \exists c \in (a, x_1)$ s.t. $f'(c) = \dfrac{f(x_1) - f(a)}{x_1 - a}$

Then since $c \in (a,b)$, we have $m \le f'(c) \le M \Rightarrow m \le \dfrac{f(x_1) - f(a)}{x - a} \le M$ $\Big]$ $x_1 > a \Rightarrow x_1 - a > 0$

$m(x_1 - a) \le f(x_1) - f(a) \le M(x - a) \Rightarrow f(a) + m(x_1 - a) \le f(x) \le f(a) + M(x_1 - a)$

Since x_1 is arbitrary, so this holds for every x.

Ex) Prove $\sqrt{50} \in [7 + \frac{1}{16}, 7 + \frac{1}{14}]$ using BPT

Let $f(x) = \sqrt{x}$, we'll look at $f(x)$ on the interval $[49, 64]$

We have $f'(x) = \dfrac{1}{2\sqrt{x}}$, we see that $f(x)$ is cts on $[49, 64]$ and diff'ble on $[49, 64]$

Then, on $x \in (49, 64)$, we see that

$$\frac{1}{16} \le f'(x) = \frac{1}{2\sqrt{x}} \le \frac{1}{14}$$

By BPT, for $x \in [49, 64]$, we have $\sqrt{49} + \frac{1}{16}(x - 49) \le f(x) \le \sqrt{49} + \frac{1}{14}(x - 49)$

$$7 + \frac{1}{16} \le \sqrt{50} \le 7 + \frac{1}{14} \quad \leftarrow \text{Plug in } 50$$

Thm 8:

Assume that f, g are cts at $x = a$, $f(a) = g(a)$ [meet at this point]

1) if both f, g are diff'ble for $x > a$ (right) and if $f'(x) \le g'(x)$ for $x > a$, then

$$f(x) \le g(x) \text{ for } x > a$$

2) if both f, g are diff'ble for $x < a$ (left), and if $f'(x) \le g'(x)$ for $x < a$, then

$$f(x) \ge g(x) \text{ for } x < a$$

Proof:

Take all hypothesis, define $h(x) = f(x) - g(x)$, so $h(x)$ is cts, diff'ble on $x < a$

Now $h'(x) = f'(x) - g'(x)$. Since $f'(x) \le g'(x) \Rightarrow h'(x) \le 0$ for $x < a$

Note, apply MVT to the interval $[x, a]$

Then, $\exists c \in (x, a)$, s.t. $h'(c) = \dfrac{h(a) - h(x)}{a - x} \le 0$

Now, since $x<a$, $a-x>0$, also have $h(a)=f(a)-g(a)=0$

Thus, $0-h(x)\le 0 \Rightarrow h(x)\ge 0 \Rightarrow f(x)\ge g(x)\ \forall x<a$

Note: if $f'(x)<g'(x)$, we get $f(x)<g(x)$, $\forall x>a$, vice versa for $x<a$.

Ex) Prove that $x^2>\ln(1+x^2)$ for $x<0$

Proof:

polynomial

composition of cts f(x)

Let $f(x)=x^2$, $g(x)=\ln(1+x^2)$. We know that f, g are cts. $\Rightarrow f(0)=g(0)=0$

$f'(x)=2x \quad g'(x)=\frac{2x}{1+x^2} \Rightarrow f, g$ are diff'ble. $x<0$

Notice that for $x<0$, $1+x^2>1$, then for $x<0$, $\frac{2x}{1+x^2}>\frac{2x}{1}$ Since $x<0$, so $g'(x)>f'(x)$

Then by Thm, for $x<0$, $f(x)>g(x)$

November 13th

Revisit Limit:

Indeterminate Forms

- $0/0$ • $\pm\infty/\infty$ • $0\cdot\infty$ • $\infty-\infty$ • 1^∞ • ∞^0 • 0^0

To deal with ↗, we have learned derivative.

Thm 14: L'Hôpital's Rule (L'HR)

Assume $f'(x)$, $g'(x)$ near $x=a$, $g'(x)\neq 0$ near $x=a$, except possibly at $x=a$

and that $\lim_{x\to a}\frac{f(x)}{g(x)}$ is of form $\frac{0}{0}$ or $\frac{\pm\infty}{\infty}$

then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ if the latter limit exists or is $\pm\infty$

Note: L'Hopital Rule applies to $a\in\mathbb{R}$, "$a=\pm\infty$", and to one-sided limits.

You may have to apply LHR multiple time.

Ex) $0/0$ $\lim_{x\to 0}\frac{e^x-x-1}{5\cos(x)+x^2-5}$

1) By sub in a, we get "$\frac{0}{0}$" $\Rightarrow$ So apply L'HR

$$\lim_{x\to 0}\frac{e^x-x-1}{5\cos(x)+x^2-5}\overset{L'HR}{=}\lim_{x\to 0}\frac{e^x-1-0}{-5\sin(x)+2x-0}\overset{L'HR}{=}\lim_{x\to 0}\frac{e^x}{-5\cos(x)+2}=-\frac{1}{3}$$

Ex) $\lim_{x\to 0} \frac{(\sin(x))}{x}$ Fundemental Trig

$\frac{\sin(x)}{x} \overset{L'HR}{=} \lim_{x\to 0} \frac{\cos(x)}{1} = 1$

Ex) $\lim_{x\to 1} \frac{\ln(x)}{1-\cos(x-1)}$

$\overset{L'HR}{=} \lim_{x\to 1} \frac{1/x}{\sin(x-1)} \overset{L'HR}{=} \lim_{x\to 1} \frac{-1/x^2}{\cos(x-1)} = -1$

Not indeterminate form we get $\frac{1}{0^{\pm}} = \pm\infty$. so limit DNE

Note: check every stage and see if it matches the hypothesis

Ex) $\frac{\pm\infty}{\infty}$ $\lim_{x\to\infty} \frac{\ln(x)}{x}$ Fundemental Log Law

$\overset{L'HR}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$

Ex) $\lim_{x\to\infty} \frac{x^2+4x+1}{3x^2-1} = \frac{1}{3}$

$\overset{L'HR}{=} \lim_{x\to\infty} \frac{2x+4}{6x} = \lim_{x\to\infty} \frac{2}{6} = \frac{1}{3}$

Ex) $0\cdot\infty$

Strategy: make) like $\pm\infty/\infty$ by dividing the reciprocal of one function

Ex) $\lim_{x\to 0^+} x\ln(x) \Rightarrow 0\cdot\infty$

1) $\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \underbrace{\frac{\ln(x)}{1/x}}_{\frac{\infty}{\infty}} \overset{L'HR}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} \frac{-x^2}{x} = 0$

Ex) $\lim_{x\to -\infty} e^x x^{5/3} \to \lim_{x\to-\infty} \frac{x^{5/3}}{e^{-x}}$ "$= \frac{-\infty}{\infty}$"

$\overset{L'HR}{=} \lim_{x\to -\infty} \frac{\frac{5}{3}x^{2/3}}{-e^{-x}} \overset{L'HR}{=} \lim_{x\to -\infty} \frac{10/9\, x^{-1/3}}{e^{-x}} = 0$ ← this is not indeterminate.

Type $\infty - \infty$ Strategy: to combine them so it looks like previous form [factor, simplify]

$\lim_{x\to\infty} \ln(3x) + \ln\left(\frac{17}{x+7}\right) = \lim_{x\to\infty} \ln\left(\frac{51x}{x+7}\right) = \ln\left(\lim_{x\to\infty} \frac{51x}{x+7}\right)$

$\overset{L'HR}{=} \ln\left(\lim_{x\to\infty} \frac{51}{1}\right) = \ln(51)$

Ex) $\lim_{x\to 0} \left[\cot(x) - \frac{1}{x}\right] = \lim_{x\to 0} \left[\frac{1}{\tan(x)} - \frac{1}{x}\right]$ ($\infty - \infty$) $= \lim_{x\to 0} \left[\frac{x-\tan(x)}{x\tan(x)}\right]$

$\overset{L'HR}{=} \lim_{x\to 0} \frac{1-\sec^2(x)}{\tan(x)+x\sec^2(x)}$ $\frac{0}{0}$ $\overset{L'HR}{=} \lim_{x\to 0} \frac{-2\sec^2(x)\tan(x)}{\sec^2(x)+\sec^2(x)+2x\sec^2(x)\tan(x)} = 0$

Types: $1^\infty, \infty^0, 0^0 \Rightarrow$ rewrite this in the form $e^{\ln[f(x)]}$

↳ log laws will make them $0\cdot\infty$. So we can apply L'HR.

Sidebar: "1^∞" $e^{\ln(1^\infty)} \leftarrow = \infty\ln(1) = 0\cdot\infty$ (indeterminate form)

Ex) $\lim_{x\to0^+} x^x$ "$=0^0$" $= \lim_{x\to0^+} e^{\ln(x^x)} = \lim_{x\to0^+} e^{x\ln x} \overset{C.4}{=} e^{\lim_{x\to0^+} x\ln(x)} \overset{L'HR}{=} e^0 = 1$

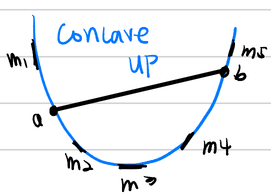
November 17th

Ex2) $\lim_{x\to\infty} (1+\frac{p}{x})^x$ "1^∞" $\Rightarrow \lim_{x\to\infty} e^{\ln(1+\frac{p}{x})^x} \Rightarrow \lim_{x\to\infty} e^{x\ln(1+\frac{p}{x})} \Rightarrow e^{\lim_{x\to\infty} x\ln(1+p/x)} \Rightarrow e^p$

Note: this gives $\lim_{x\to\infty} (1+\frac{1}{x})^x = e$ or $\lim_{x\to0} (1+x)^{1/x} = e$

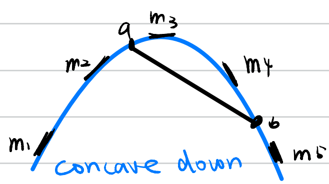
Now, the derivative of the derivative

↳ roc of the roc ↳ how the slopes of tangent lines is changing



$f'' > 0$

- $m_1 < m_2 < m_3 < m_4 < m_5$
- Secant line lies above the curve



$m_1 > m_2 > m_3 > m_4 > m_5 \leftarrow f'' < 0$

Secant line lies below the curve

Def: Concavity

The graph of f is concave up (cu) / concave down on an interval I if $\forall a, b \in I$, the secant line connecting $(a, f(a))$, $(b, f(b))$ sits above/below the graph of f

Note: Horizontal lines are neither cu / cd.

Quiz Notice:

Memorize all the theorum, all the indeterminate form and its strategy

Remember Mean Value Theorem.

November 20^{th}

Thm 10: Second derivative Test

1) If $f''(x) > 0$ $\forall x \in I$, then the graph of f is concave up on I

2) if $f''(x) < 0$ $\forall x \in I$, then the graph of f is concave down on I

Def: Inflection Point (poi)

A poi $(c, f(c))$ is called poi of f if f is cts at c and the concavity of f changes at $(c, f(c))$

these occur when $f''(x)$ changes sign at c.

Further, f'' is cts at the poi. By IVT, $f''(c) = 0$

Thm 11: Test for Inflection Points

If f'' is cts at $x=c$ and $(c, f(c))$ is a poi of f, then $f''(c) = 0$

Note: $f''(c)=0 \not\Rightarrow$ poi ex) $y = x^4$ at $x=0$

$f''(c)=0$ are candidate of poi $\rightarrow$ must check concavity on either sides.

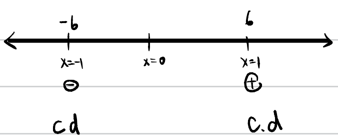
Ex) Find I of concaving and any inflection points of

a) $y = x^3$ b) $y = -1/x$

a) $y'' = 6x$

Since y'' is polynomial, so it's cts everywhere

$y''=0$ when $x=0$

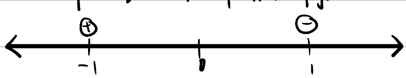


$\leftarrow$ only care if the sign on either side are different.

By Number line test, $(0,0)$ is a poi

b) $y'' = -2/x^3$ $\leftarrow$ point of discontinuity (Not a poi) (Want to know about concavity)

Since $y''=0$ DNE, must investigate concavity. y'' Never be zero



y is cu on $(-\infty, 0)$ and cd $(0, +\infty)$

No poi

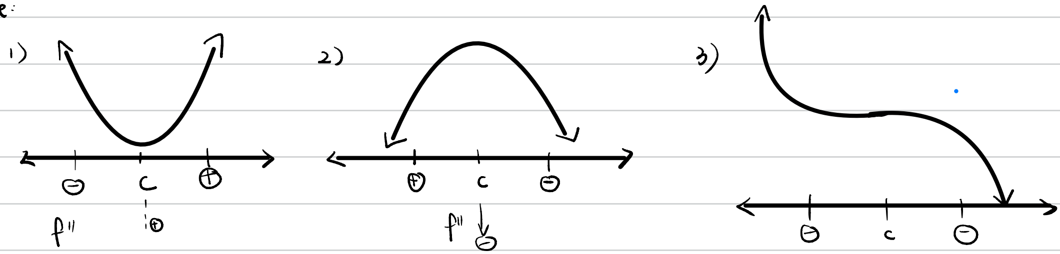
We have seen that if $x=c$ is a local extremum, then its a critical point ($f'=0$, f' DNE)
But once we found cp, we classify them.

Thm 12: First Derivative Test (FDT)

Assume c is a cp of f and f is cts at c. If $\exists\, c \in (a,b)$ s.t:

1) $f'(x)<0$ $\forall x(a,c)$ and $f'(x)>0$ $\forall x \in (c,b)$, then c is a local min.
2) $f'(x)>0$, $f'(x)<0$ $\forall x \in (c,c)$, $x \in (c,b)$, then c is a local max
3) False for 1, 2 $\Rightarrow$ c is not local max/min

Picture:



Thm 13: Second Derivative Test (SDT)

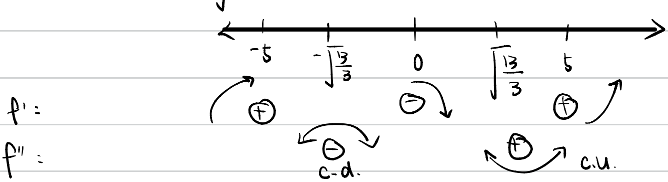
Assume $f'(c)=0$ and $f''(c)$ is cts at c if: Use only $f'(x)$ exists.

1) $f''(c)>0$, then c is a local min
2) $f''(c)<0$, then c is a local max
3) $f''(c)=0$, then No information Have to use FDT

Ex) Find the local extrema of $y=x^3-13x+12$ using (FDT, SDT)

$y'=3x^2-13$ $\quad$ $y''=6x$

CP at $y'=0$, $x=\pm\sqrt{\frac{13}{3}}$ $\quad$ y' is cts everywhere so no DNE situation



at $x=-\sqrt{\frac{13}{3}}$, it's a local max
$x=-\sqrt{\frac{13}{3}}$, it's a local min

Conclude: By SDT, $x=-\sqrt{\frac{13}{3}}$ is a local max and $x=\sqrt{\frac{13}{3}}$ is a local min

Note: FDT and SDT can be used alternatively

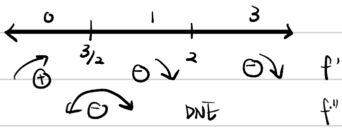
Ex) Find all max/min of $y = x^3 \sqrt[3]{2-x}$ on $[0,3]$

$y' = \dfrac{2 - 4/3x}{(2-x)^{2/3}}$ $\quad y'' = \dfrac{4/9x - 4/3}{(2-x)^{5/3}}$

1) End points: $f(0)=0$, $f(3)=-3$

2) Find c.p. $y'=0 \Rightarrow 2-4/3x = 0 \Rightarrow x = 3/2$ $\quad$ y' DNE $\rightarrow x=2$ $\quad$ SDT Does not applicable

3). Test:



By FDT, SDT, at $x = 3/2$, it's a local max

$x=2$ is nothing

Now, $f(\frac{3}{2}) = 3/2^{4/3}$ ($> f(0)$, $> f(3)$) $\quad$ So it's a global/local max at $(3/2, 3/2^{4/3})$] in $[0,3]$

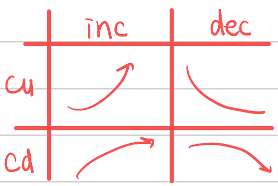
at $(3,-3)$, there is a global min

November 22nd

Curve Sketching Process

1) Identify the function domain and possibly fcn values at endpoints
2) Identify x, y intercepts
3) Identify any horizontal asymptotes ($\lim_{x \to \pm\infty} f(x)$)
4) Identify any holes and/or vertical asymptotes ($\lim_{x \to a^{\pm}} f(x)$)
5) Find all c.p's ($f'=0$, f' DNE)
6) Find where $f''=0$, DNE (concavity, possible poi)
7) Investigate the intervals divided by the points from 5), 6) for concavity/inc/dec
8) Identify any local extrema, poi from 7
9) Plot and Label

↳



Example) Sketch $f(x)=\dfrac{x^2-1}{x^2+3x}$ $\quad f'(x)=\dfrac{3x^2+2x+3}{x^2(x+3)^2}$ $\quad f''(x)=\dfrac{-6(x+1)(x^2+3)}{x^3(x+3)^3}$

1) Note that $f(x)=\dfrac{(x-1)(x+1)}{x(x+3)}$ $\quad$ Domain: $x\in(-\infty,-3)\cup(-3,0)\cup(0,\infty)$

2) Find x-int, y-int

$0=\dfrac{(x-1)(x+1)}{x(x+3)} \Rightarrow 0=(x-1)(x+1) \Rightarrow x=\pm1$

3) Find y-int: DNE, not in our domain ($x\neq0$)

$\therefore$ No y-int

4) Horizontal Asymptote

$\lim\limits_{x\to\pm\infty}\dfrac{x^2+1}{x^2+3x}=1$ (L'HR, dominating power) $\quad\therefore$ H.A at $y=1$ as $x\to\pm\infty$

5) Vertical Asymptote → Analyze $x=-3,\ x=0$

$\lim\limits_{x\to0^-}\dfrac{x^2-1}{x(x+3)}=+\infty$ ← V.A $\quad \lim\limits_{x\to0^+}\dfrac{x^2-1}{x(x+3)}=-\infty \quad \lim\limits_{x\to-3^-}\dfrac{x^2-1}{x(x+3)}=\infty$ ⇐ V.A $\quad \lim\limits_{x\to-3^+}\dfrac{x^2-1}{x(x+3)}=-\infty$

6) Find Critical Points. (from f')

$f'=0 \Rightarrow 3x^2+2x+3=0 \Rightarrow x=\dfrac{-2\pm\sqrt{-20}}{6}$ ⇐ No real root $\Rightarrow f'\neq0$

f' DNE $\Rightarrow x^2(x+3)^2=0 \Rightarrow x=0,\ x=-3$ ⇐ Not in domain, not C.P.

$\therefore$ has no local extrema

Note: Still investigate Inc/Dec on either side of $x=0,-3$

7) Find poi in f''

$f''=0 \Rightarrow -6(x+1)(x^2+3)=0 \Rightarrow x=-1$ ← Candidate

f'' DNE $\Rightarrow x^3(x+3)^3=0 \Rightarrow x=0,-3$ ← Not Candidate, $x\neq0,-3$

Note: Still investigate Concavity around $x=0,-3$

Side note: once found local extrema, poi, plug-in $f(x)$ to find coordinate

8) Plug-in and Find Inc, Dec, Concavity

a) test points: -5, -1, 1; marks: -3 DNE, 0 DNE

f': ⊕ ⊕ ⊕

b) test points: -5, -2, $-1/2$, 1; marks: -3 DNE, -1, 0 DNE

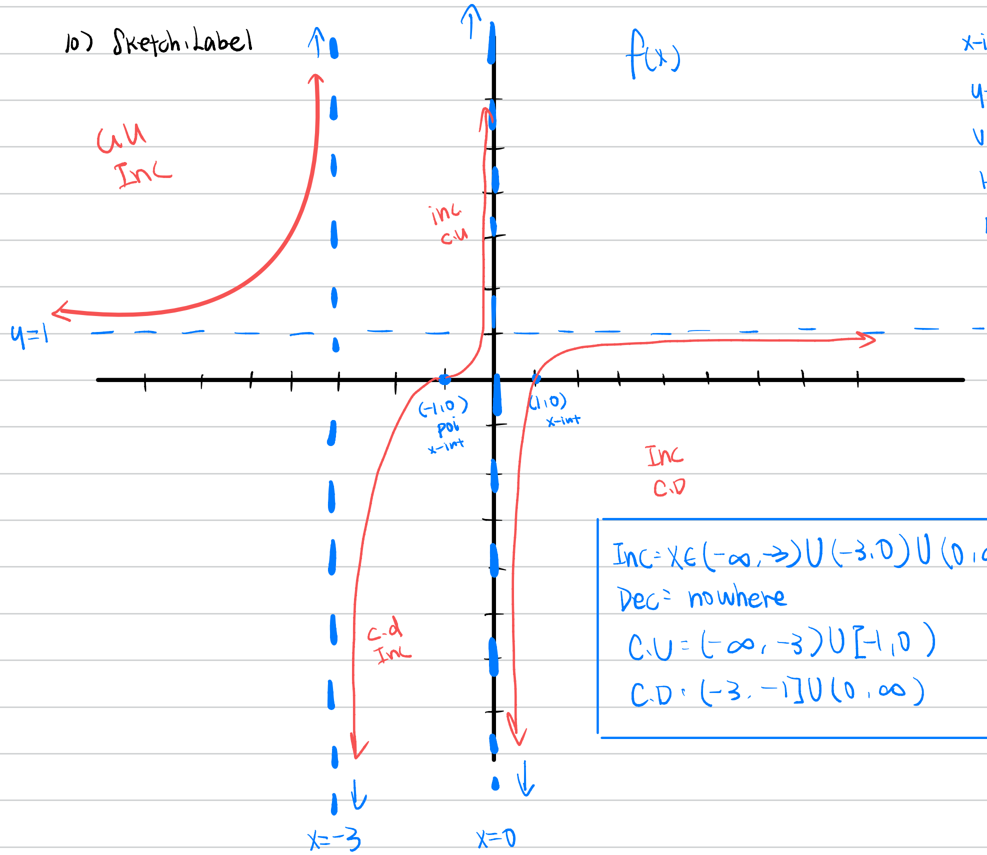
f'': ⊕ ⊖ ⊕ ⊖

poi $\quad$ (y-int)

9) Find local extrema, poi

by above steps, no local extrema, poi at y-int.

10) Sketch, Label



x-int: $(-1,0)\ (1,0)$

y-int: None

V.A: $x=0, -3$

H.A: $y=1$

LE: NO

Poi $(-1,0)$

Inc: $x \in (-\infty,-3) \cup (-3,0) \cup (0,\infty)$

Dec: nowhere

C.U: $(-\infty,-3) \cup [-1,0)$

C.D: $(-3,-1] \cup (0,\infty)$

November 24th

Ex2) Sketch $f(x) = \dfrac{e^x(x-2)}{x^2-2x}$ $\quad f'(x) = \dfrac{e^x(x-1)(x-2)}{x^3-2x^2}$ $\quad f''(x) = \dfrac{e^x(x^2-2x+2)(x-2)}{x^4-2x^3}$

1) Note: $f(x) = \dfrac{e^x(x-2)}{x(x-2)}$ $\quad$ Domain: $\{x \in \mathbb{R} \mid x \neq 0, 2\}$ or $x \in (-\infty,0) \cup (0,2) \cup (2,\infty)$

2) x-int: $y=0$

$0 = e^x(x-2) \Rightarrow x=2$

Since $x \neq 2$, so no x-int

y-int: $x=0$

No y-int $\Rightarrow$ Not in our domain

3) $\lim_{x\to\infty} \frac{e^x \cancel{(x-2)}}{x\cancel{(x-2)}} \Rightarrow \lim_{x\to\infty}\frac{e^x}{x} \overset{L'HR}{\Rightarrow} \lim_{x\to\infty}\frac{e^x}{1} = +\infty$ $\quad \frac{\infty}{\infty}$

$\lim_{x\to-\infty}\frac{e^x}{x} = 0$ $\quad \frac{0}{-\infty}$ (Not indeterminate)

∴ H.A at $y=0$ as $x\to-\infty$

4) at $x=0$

$\lim_{x\to0^-}\frac{e^x\cancel{(x-2)}}{x\cancel{(x-2)}} = \lim_{x\to0^-}\frac{e^x}{x} = -\infty$ V.A $\quad$ (e^x: ⊕, x: ⊖)

$\lim_{x\to0^+}\frac{e^x}{x} = \infty$ $\quad$ (e^x: ∞, x: ∞)

at $x=2$

$\lim_{x\to2}\frac{e^x}{x} = \frac{e^2}{2}$ $\qquad$ ∴ V.A at $x=0$, hole at $(2, \frac{e^2}{2})$

5) $f'(x) = \frac{e^x(x-1)(x-2)}{x^2(x-2)}$.

$f'=0 = e^x(x-1)(x-2) \Rightarrow x=1,2$ (not in domain), $x=1$

f' DNE $=0= x^2(x-2) \Rightarrow x=0,\cancel{2}$ ← not in domain

∴ CP at $x=1$ ($x=0,2$ still be useful for interval)

6) $f''=0 = e^x(x^2-2x+2)(x-2)=0 \Rightarrow x=2$ (not domain) ⊗

f'' DNE: $x^3(x-2) \Rightarrow x=0,2$ (not domain)

∴ Not poi.

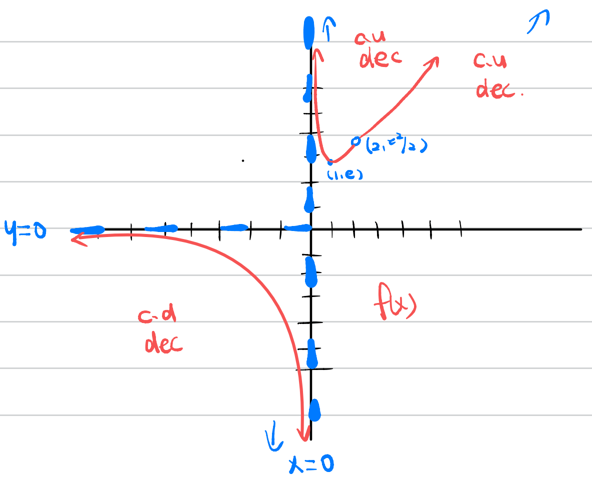
7)

	-1	0 DNE	0.5	1	1.5	2 DNE	3
f':	⊖ ↘		⊖ ↘		⊕ ↗		⊕ ↗

	-1	0 DNE	1	2 DNE	3
f''	⊖ ∩		⊕ ∪		⊕ ∪

From FDT, $x=1$ is a local min

8) local min: $(1, e)$



x-int, y-int: None

H.A: $y=0$ $\quad$ V.A: $x=0$ $\quad$ hole: $(2, \frac{e^2}{2})$

local min $(1, e)$

Interval:

Inc: $x\in[1,2)\cup(2,\infty)$

Dec: $x\in(-\infty,0)\cup(0,1]$

C.U: $x\in(0,2)\cup(2,\infty)$

C.D: $x\in(-\infty,0)$

November 27^{th}

Recall, Linear Approximation $L_a^f(x) = f(a) + f'(a)(x-a)$

This is just the tangent line to f at $(a, f(a))$

Key features:

$\hookrightarrow L_a^f(a) = f(a)$ $\quad \hookrightarrow L_a^{f'}(a) = f'(a)$ $\quad \hookrightarrow |f(x) - L_a^f(x)| \leq \frac{M}{a}(x-a)^2$ (Error)

$|f''(a)| \leq M \ \forall x \in I$

To approx. a complicated $f(x)$ with a linear one

What if we want an n^{th} degree polynomial which is a better approx which makes up to the n^{th} derivative and the $f(x)$ value at $x=a$

We construct a polynomial of the form:

$$T_{n,a}(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \dots + C_n(x-a)^n$$

Taylor degree centre

$C_0 = f(a)$ to let $\downarrow$ happen

First: we want $T_{n,a}(a) = f(a) \Rightarrow T_{n,a}(a) = C_0 + C_1\cancel{(a-a)} + \cancel{C_2(a-a)^2} + \dots \ C_n\cancel{(a-a)^n}$

Next: want $T_{n,a}'(a) = f'(a)$

$\hookrightarrow T_{n,a}'(x) = 0 + C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \dots + nC_n(x-a)^{n-1}$ $\Leftarrow C_1 = f'(a)$

$T_{n,a}'(a) = C_1 + \cancel{2C_2(a-a) + 3C_3(a-a)^2} + \dots + \cancel{nC_n(a-a)^{n-1}}$

$= C_1$

Now: want $T_{n,a}''(a) = f''(a)$

$\hookrightarrow T_{n,a}''(x) = 0 + 2C_2 + (3)(2)C_3(x-a) + \dots$ $\quad \Leftarrow$ Want $C_2 = \frac{f''(a)}{2}$

$T_{n,a}''(a) = 2C_2 + \cancel{6C_3(a-a)} + \dots$

Under the demand, $T_{n,a}'''(a) = f'''(a) \Rightarrow C_3 = \frac{f'''(a)}{6} = \frac{f'''(a)}{3!}$

The k^{th} demand leads to $C_k = \frac{f^{(k)}(a)}{k!}$

Def: Taylor Polynomial:

If f is n times diff'ble at $x=a$, we say:

the n^{th} degree Taylor Polynomial for f centred at $x=a$' is the polynomial:

$$T_{n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\hookrightarrow \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Note: If $a=0$, we call it "Maclaurin" polynomial.

Ex) Find up to $T_{5,0}(x)$ for $f(x)=\sin(x)$

1) Note $f(0)=0 \Rightarrow T_{0,0}(x)=0$ 2) $f'(x)=\cos(x) \Rightarrow f'(0)=1$ 3) $f''(x)=-\sin(x) \Rightarrow f''(0)=0$

4) $f'''(x)=-\cos(x) \Rightarrow f'''(0)=-1$ 5) $f^{(4)}(x)=\sin(x) \Rightarrow f^{(4)}(0)=0$ ... Repeats.

$\therefore T_{1,0}(x)=0+\frac{1}{1!}(x-0)^1 = x$ $T_{2,0}(x)=0+\frac{1}{1!}(x-0)^1+\frac{0}{2!}(x-0)^2 = x$

$T_{3,0}(x)=0+\frac{1}{1!}(x-0)^1+\frac{0}{2!}(x-a)^2+\frac{-1}{3!}(x-a)^3 = x-\frac{1}{6}x^3$

$T_{4,0}(x)=x-\frac{1}{6}x^3+\frac{0}{4!}(x-a)^4 = x-\frac{1}{6}x^3$

$T_{5,0}(x)=x-\frac{1}{6}x^3+\frac{1}{120}x^5$ (repeat $\uparrow$ $+\frac{1}{5!}(x-0)^5$)

Ex) Find $T_{5,0}(x)$ for $f(x)=\cos(x)$

We have $\cdot f(0)=1$ $\cdot f'(x)=-\sin(x) \Rightarrow f'(0)=0$. $f''(0)=-1$ $f'''(0)=0$ $f''''(0)=1$ $\cdot f^{(5)}(0)=0$

Then $T_{5,0}(x)=1+\frac{0}{1!}(x)^1+\frac{-1}{2!}(x)^2+\frac{0}{3!}(x)^3+\frac{1}{4!}(x)^4+\frac{0}{5!}(x)^5$

$= 1-\frac{1}{2}x^2+\frac{1}{24}x^4$

November 29th

As with any approx, we concerned about error

Def: Taylor Remainder

Assume f is n-times diff'ble at $x=a$. Then $R_{n,a}(x)=f(x)-T_{n,a}(x)$ is called the n^{th} degree Taylor remainder $f(x)$ centred at $x=a$

Note: • $|R_{n,a}(x)|$ = Error

↳ $R>0 \Rightarrow$ Under-estimate $\Rightarrow f-T>0 \Rightarrow f>T$

↳ $R<0 \Rightarrow$ Over-estimate $\Rightarrow f-T<0 \Rightarrow f<T$

Thm 1: Taylor's Thm

Assume f is $n+1$ times diff'ble on interval I containing a. Let $x \in I$.

Then $\exists$ c between x and a such that

$$f(x)-T_{n,a}(x)=R_{n,a}(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Note: For n=1, we have $T_{1,a}(x) = L_x^f(x)$

↳ By Taylor.T. $R_{1,a}(x) = \frac{f''(x)}{2!}(x-a)^2$

↳ if we have $|f''(c)| \le M \ \forall x \in I$, Then $|R_{1,a}(x)| \le \frac{M}{2}(x-a)^2$ (Error for L.Approx)

↳ For n=0, we have $T_{0,a}(x) = f(a)$

↳ By Taylor.T ⇒ $f(x) - T_{0,a}(x) = R_{0,a}(x) = f'(c)(x-a)$

↳ $f(x) - f(a) = f'(c)(x-a)$ ⇒ $\frac{f(x)-f(a)}{x-a} = f'(c)$ (MVT)

↳ Taylor's T is a higher order MVT.

Taylor's T does not say how to find, just there exists.

Now → to seek an upper bound on the error of approx.

Corolloary: Taylor's Inequality

If we have $|f^{(n+1)}(c)| \le M$, $\forall c$ between x and a, then

$$|R_{n,a}(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$ $\forall c$ between x, a

Ex) Estimating $\cos(0.1)$ with $T_{5,0}(x)$ of $\cos(x)$ ①

↳ upper bound of (a)? ②

↳ is $T_{5,0}(x)$ an over/under-estimate on $[0,1]$ ③

→ Based on 1 – 3, give an interval for the true value of $\cos(0.1)$ ④

From last lecture → $T_{5,0}(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ for $\cos(x)$ $\quad T_{3,0}(0.4) =$

① $T_{5,0}(0.1) \doteq 1 - \frac{1}{200} + \frac{1}{240000} \doteq \frac{238801}{240000}$

② By Taylor's Thm, $\exists c \in (0, 0.1) \ni R_{5,0}(0.1) = \frac{f^{(6)}(c)}{6!}(0.1-0)^6$

$|R_{5,0}(0.1)| = \left|\frac{f^{(6)}(c)}{720}|0.1-0|^6\right|$

$f^{(6)}(c) = -\cos(c)$ ⇒ $|-\cos(c)| \le 1$ for $c \in (0, 0.1)$ (∴ M=1 is a valid choice)

∴ $|R_{5,0}(0.1)| \le \frac{1}{720}|0.1|^6 = \frac{1}{720000000}$

③ Investigate $R > 0$ or $R < 0$

$R_{5,0}(x) = \frac{-\cos(c)}{720}x^6$ ⇒ for $x \in [0,1]$ ⇒ $c \in [0,1]$ ⇒ $-\cos(c) \le 0$

⇒ $x^6 \ge 0$ on $[0,1]$

∴ $R_{5,0}(x) \le 0$ ⇒ Over-estimate on $[0,1]$

④ Our approx in a) was $\frac{238801}{240000}$ and in ③, it's overestimated

By b, the worst case in $\frac{1}{720000000}$

under: [error, error + M]

Combining together, $\cos(0.1) \in \left[\frac{238801}{240000} - \frac{1}{720000000}, \frac{238801}{240000}\right]$

December 1st

Ex) 1. estimate $\sqrt[3]{30}$ using $T_{2,27}(x)$ of $\sqrt[3]{x}$

2. upper bound on the error of our approx of 1.

3. upper bound on the error of $T_{2,27}(x)$ for $x \in [20, 35]$

4. For what values of $x > 0$ is $T_{2,27}(x)$ an over/under estimate.

1. $f(x) = x^{1/3}$ $a = 27$ $n = 2$

$f(a) = 3$ $f'(a) = \frac{1}{3}(x)^{-\frac{2}{3}} = \frac{1}{27}$ $f''(a) = -\frac{2}{9}(x)^{-\frac{5}{3}} = \frac{-2}{2187}$

$T_{2,27}(x) = 3 + \frac{1}{27}(x-27) - \frac{1}{2187}(x-27)^2$

$T_{2,27}(30) = 3 + \frac{1}{27}(3) - \frac{1}{2187}(3)^2 = \frac{755}{243}$

2. By Taylor's Thm, $\exists c \in (27, 30)$ s.t. $R_{2,27}(30) = \frac{f'''(c)}{3!}(30-27)^3 \Rightarrow \underset{\text{Error}}{|R_{2,27}(30)|} = \frac{f'''(c)}{3!}(30-27)^3$

$\hookrightarrow f'''(c) = \frac{10}{27}x^{-8/3} \Rightarrow$ on $[27, 30]$, $|f'''(c)| \leq \frac{10}{27}(27)^{-8/3}$ since $f'''(c)$ is dec on I.

$\therefore |R_{2,27}(30)| \leq \frac{10(27)^{-11/3}}{3!}(30-27)^3 = \frac{5}{19683}$

Explain why plug in a number

3. By Taylor's Inequality: $|R_{2,27}(x)| \leq \frac{M|x-27|^3}{3!}$ where $|f'''(c)| \leq M$, $c \in [20, 35]$

We saw previously, that $f'''(x) = \frac{10}{27}x^{-8/3}$, then $|f'''(c)| \leq \frac{10}{27}(20)^{-8/3}$ since $|f'''(c)|$ is dec. on I.

So, we have: $|R_{2,27}(x)| \leq \frac{10}{27(3!)}(20)^{-8/3}|x-27|^3$ (plug in 35 to maximize $|R_{2,27}(x)|$)

maximize RHS to find an upper bound on the error.

$\hookrightarrow = \frac{10}{27(3!)}(20)^{-8/3}|35-27|^3 = \frac{10}{162}(20)^{-8/3}(8)^3$

$\hookrightarrow = \frac{2560}{81}(20)^{-8/3}$

$\therefore |R_{2,27}(x)| \leq \frac{2560}{81}(20)^{-8/3} = 0.0107$ on $[20, 35]$

4. $x>0 \Rightarrow$ Under/Over-estimate.

We had: $R_{2,27}(x) = \frac{f'''(c)}{3!}(x-27)^3$ for some c between x and 27

$$= \left(\frac{1}{3!}\right)\left(\frac{10}{27}c^{-8/3}\right)(x-27)^3$$

$\quad\oplus \qquad \oplus \qquad$ varies

Over-estimate: $T>f \Rightarrow R<0 : (x-27)<0 \Rightarrow 0<x<27$

Under-estimate: $T<f \Rightarrow R>0 : (x-27)>0 \Rightarrow x>27$