exam 2023-03-22

task 1 $\quad w = \Omega T$

using sampling theorem

$x_c(t)$ bandlimited signal

$X_c(j\Omega) = 0 \quad |\Omega| > \Omega_N$ nyquist frequency

samples $x[n] = x_c(nT)$ T-sampling period

assuming no aliasing

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\, \delta(t-nT)$$

$x_s(t)$ is sampled signal

CTFT

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-j\Omega T n}$$

from $x[n] = x_c(nT)$

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega T n}$$

DTFT of $x[n]$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$$

comparing CTFT and DTFT we obtain

$$X_s(j\Omega) = X(e^{jw})\Big|_{w=\Omega T}$$

thus we have the relation

$w = \Omega T$

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task 2

$x[n]$ $h[n]$ finite length N

a) the maximum possible length of the linear convolution of $x[n]$ with $h[n]$ is

$2N-1$

since both are of length N. Which gives $2N$.
but then they only provide "linear convolution output" when the two overlap, giving the -1

$2N-1$

b) the formula for circular convolution:

$$x[n] \circledast h[n] = \left\{ \sum_{m=0}^{N-1} x[m] \underbrace{h[((n-m))_N]}_{\text{circularly rotated } h[n]} \right\} \cdot R_N(n)$$

↑ modulo N

$R_N(n)$ is a train of 1s of length N

thus it follows from the formula that the maximum possible length of the cicular convolution of $x[n]$ with $h[n]$ is N

N

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task 3

a) it is assumed that it is supposed to say C/D (D/C) since it is an analog signal

$$x_a(t) \to \underset{\uparrow T}{\boxed{C/D}} \xrightarrow{x[n]} \boxed{H(e^{jw})} \xrightarrow{y[n]} \underset{\uparrow T}{\boxed{D/C}} \to y_a(t)$$

$x_a(t)$ analog input
$y_a(t)$ analog output
$H(e^{jw})$ filter FC
T-sampling period

convert input to digital by taking samples

this signal in frequency domain is repeated copies of the original signals spectrum

we apply filtering to extract only one repetition (if there is no aliasing) and scale it correctly by filter gain

then convert it back to analog

b)
$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}) & |\Omega| < \pi/T \\ 0 & |\Omega| \geq \pi/T \end{cases}$$

$w = \Omega T$

$H_{eff}(j\Omega)$ effective analog FC
$H(e^{j\Omega T}) = H(e^{jw})$ digital FC
T-sampling period
Ω CT frequency
w DT frequency

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task 3

c) $\frac{d^2y(t)}{dt^2} + 0.2\frac{dy(t)}{dt} + 0.16y(t) = \frac{dx(t)}{dt} + 0.1x(t)$

$s^2Y(s) + 0.2sY(s) + 0.16Y(s) = sX(s) + 0.1X(s)$

$H(s) = \frac{Y(s)}{X(s)} = \frac{s+0.1}{s^2+0.2s+0.16}$

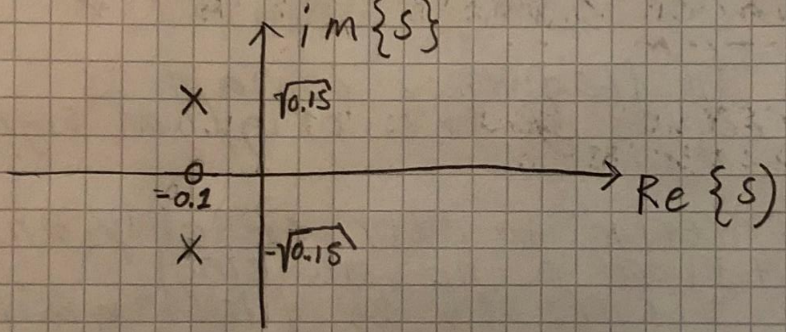
$s^2+0.2s+0.16=0$

$(s+0.1)^2-(0.1)^2+0.16=0$

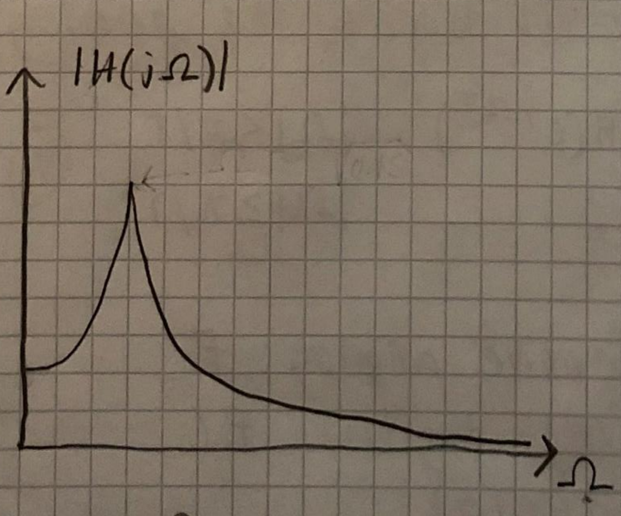
$(s+0.1)^2 = -0.15$

$s+0.1 = \pm j\sqrt{0.15}$

poles $s = -0.1 \pm j\sqrt{0.15}$ zero $s=-0.1$



d)



low pass?

possibly to sharp, difficult to say how the zero vectors lengt will affect exactly

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task 3 e) $h[n] = T \cdot h_a(nT)$

$$FT(h[n]) = H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_a\left[\frac{j\Omega}{T} + \frac{j2\pi k}{T}\right]$$

f) main advantage of impulse invariance compared to bilinear transform is

it perserves good time domain characteristics

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task 4

a) we simply multiply the desired impulse response by a window (train of 1s) of length N to make it finite

$h[n] = w[n]\, h_d[n]$

$w[n]$ window (train of 1s) of length N

$w[n] = 0 \quad n < 0, n > (N-1)$

$h_d[n]$ desired impulse response

$h[n]$ FIR

b) the window (if we assume rectangular) will be a sinc in frequency domain which will introduce ripples and a transition band which has a slope

transition band more narrow $\Rightarrow$ wider window in time domain $\Rightarrow$ longer impulse response $\Rightarrow$ higher order

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task 5 a) PCA:

- based on variance
- vectors are orthogonal
- works on gaussian where sources are dependent

ICA:

- based on statistical independence
- vectors are not necessarily orthogonal
- doesn't work on gaussian, needs independent sources

b) $X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad C = X^T X = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

1

$C - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$

$\det(C - \lambda I) = (2-\lambda)^2 - 1 = 0 \Rightarrow 2 - \lambda = \pm 1$

$\lambda = 2 \pm 1 \quad \lambda_1 = 3 \quad \lambda_2 = 1$

$S = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

2

$(C - \lambda_1 I)\vec{y} = \vec{0} \Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} / \cdot -1 \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad y_1 - y_2 = 0 \quad y_2 = t \Rightarrow y_1 = t$

$(y_1, y_2) = t(1,1) \quad \vec{v}_1 = (1,1)$

$(C - \lambda_2 I)\vec{y} = \vec{0} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad y_1 + y_2 = 0 \quad y_2 = t \Rightarrow y_1 = -t \quad y_2 = t$

$(y_1, y_2) = t(-1,1) \quad \vec{v}_2 = (-1,1)$

$V = \begin{bmatrix} \vdots & \vdots \\ \vec{v}_1 & \vec{v}_2 \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

3 didn't have time for the final step to obtain U

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task 5

c) S = singular matrix with eigenvalues (variance of the principal components) along the diagonal, zeros everywhere else

V = columns consisting of the eigenvectors (principal components) of $C = X^T X$ they create the new subspace

U = projections of X onto the subspace made from the eigenvectors (principal components) of $C = X^T X$

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task 6 a) $\Delta_t \cdot \Delta_w \geq \frac{1}{2}$

it states that the product is a constant greater than or equal to $\frac{1}{2}$

which means that wider in time domain gives more narrow in frequency domain

and more narrow in time domain gives wider in frequency domain

b) $\Delta_f = \frac{\Delta_\psi}{a}$

effective width of analysis in frequency domain is inversly proportional to the scaling factor a

c) $$\Delta_f^2 = \frac{\int_{-\infty}^{\infty} (W-\omega)^2 |G(W-\omega) e^{j(W-\omega)\tau}|^2 dW}{\int_{-\infty}^{\infty} |G(W-\omega) e^{j(W-\omega)\tau}|^2 dW}$$

G is frequency domain of window

e) STFT effective width is constant, following the principal of uncertainty (see a)), and only dependent on the effective width of the window

meanwhile the effective width of CWT is inversly proportional to a which means it's different for different frequencies

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task 1 a)

$$\underbrace{R_{xx}(\tau)}_{\text{autocorrelation function}} \Leftrightarrow \underbrace{S_{xx}(j\omega)}_{\text{PSD}} = \underbrace{\lim_{T\to\infty} \frac{1}{2T} E\left[|X_T(j\omega)|^2\right]}_{\text{periodogram}}$$

the theorem states that the PSD of a random signal is equal to the ESD divided by time, when the time goes to infinity

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task 8

$x[n]$ white process $\mu_x = 0$ $R_{xx}[m] = K\delta[m]$ $K > 0$

$y[n] = A + x[n]$

A random variable $\mu_A = 0$ $\sigma_A^2 > 0$ uncorrelated with $x[\cdot]$

a) $\mu_y = E[y[n]] = E[A + x[n]] =$

(E is linear operator)

$= E[A] + E[x[n]] = \mu_A + \mu_x = 0 + 0 = 0$

b) $R_{yy}[m] = E[y[n]\,y[n-m]] = E[(A + x[n])(A + x[n-m])] =$

$= E[\underbrace{A^2}_{\sigma_A^2} + \underbrace{A\,x[n] + A\,x[n-m]}_{\text{uncorrelated}} + \underbrace{x[n]\,x[n-m]}_{\text{definition of } R_{xx}[m]}] =$

$= E[A^2] + E[A\,x[n]] + E[A\,x[n-m]] + E[x[n]\,x[n-m]] =$

$= \sigma_A^2 + 0 + 0 + R_{xx}[m] = \sigma_A^2 + K\delta[m]$ $K > 0$

c) $C_{yy}[m] = R_{yy}[m] - \mu_y \mu_y = R_{yy}[m] - 0 = R_{yy}[m] =$

$= \sigma_A^2 + K\delta[m]$ $K > 0$

d) $PSD = S_{yy}(j\omega) = CTFT(R_{yy}[m]) = 2\pi\sigma_A^2\delta(m) + K$ $K > 0$

e) $FSD = D_{yy}(e^{j\Omega}) = DTFT(C_{yy}[m]) = 2\pi\sigma_A^2\delta[m] + K$ $K > 0$

f) no since any realization of $y[n]$ might have $A \neq 0$ since it is a random variable, resulting in a mean for the realization of $y[n]$ that does not match $\mu_y = 0$

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task 9 $x[n] = y[n] + v[n]$

$y[n]$ WSS $\mu_y = 0$

causal stable LTI state space system driven by an unknown white process $w[n]$ $\sigma_w^2 > 0$

$v[n]$ unknow white noise $\sigma_v^2 > 0$ uncorrelated with $w[\cdot]$

a) $\vec{q}[n+1] = \vec{A}\vec{q}[n] + \vec{b}w[n]$

$y[n] = \vec{c}^T\vec{q}[n] + d\,x[n]$

$x[n] = y[n] + v[n]$

can not do small L

b) $\hat{\vec{q}}[n+1] = \vec{A}\hat{\vec{q}}[n] - L(x[n] - \hat{y}[n]) =$

$= (\vec{A} + L\vec{c}^T)\hat{\vec{q}}[n] + L\,x[n]$

$\hat{y}[n] = \vec{c}^T\hat{\vec{q}}[n]$

c) it can be shown that if the observer gain is chosen correctly, then it will have the same transfer function as the wiener filter which is known to be LMMSE (from the lectures), and thus the observer output is also LMMSE estimate of $y[n]$