ELE476
DSP with Python

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Outline

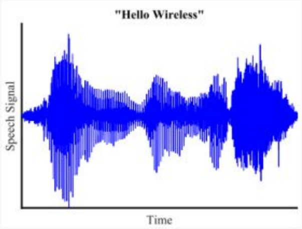
- Fourier Series
- DFT
- FFT
- Power Spectrum
- Spectrum Leakage
- Windowing
- Zero Padding

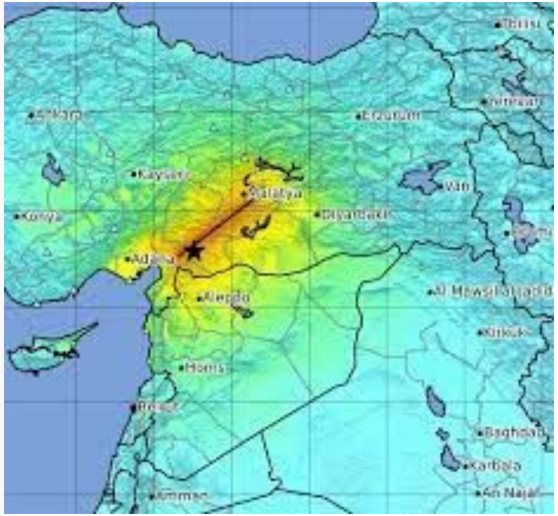
What is a signal?

A signal is any measurable quantity that varies with time

It carries or conveys information

- Speech
- GPS
- ECG
- Stock prices
- Earthquake

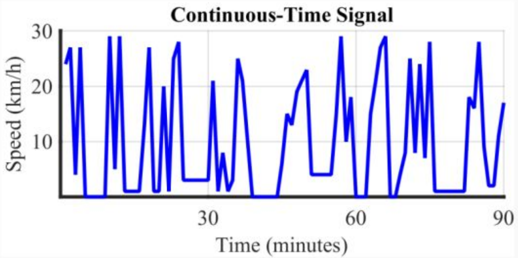




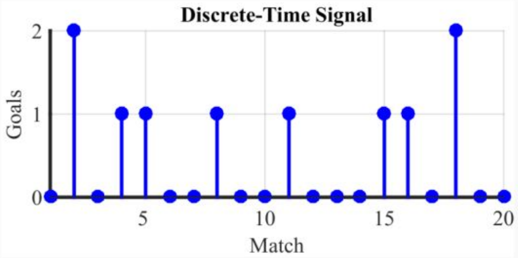
Continuous-Time vs Discrete-Time

- Continuous
  - Defined at every point
- Discrete
  - Only defined at discrete points in time

Player speed

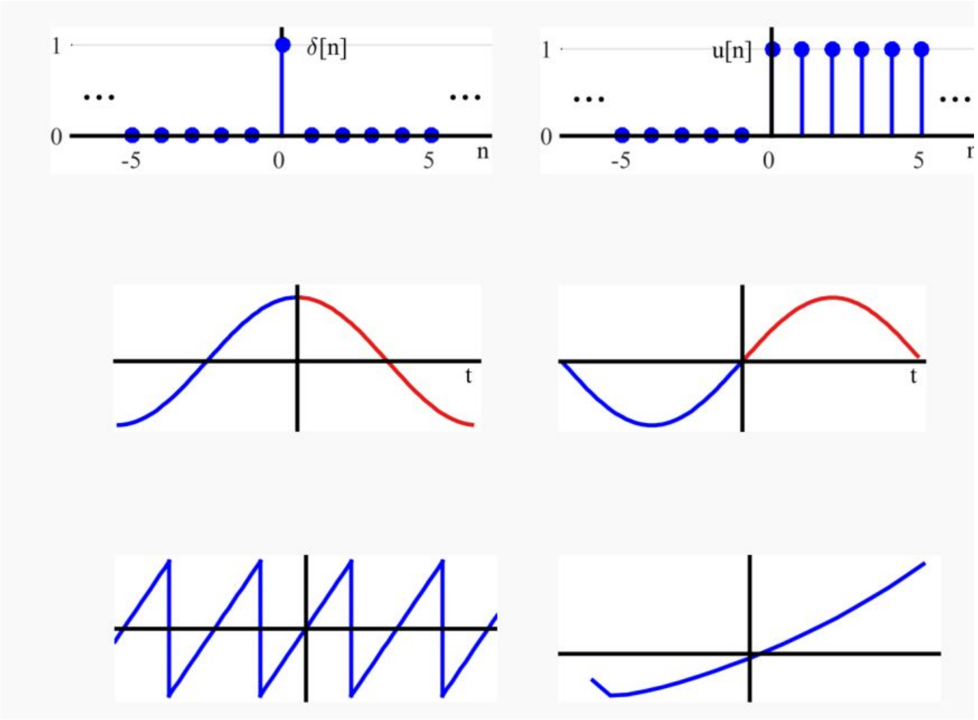


Player goals

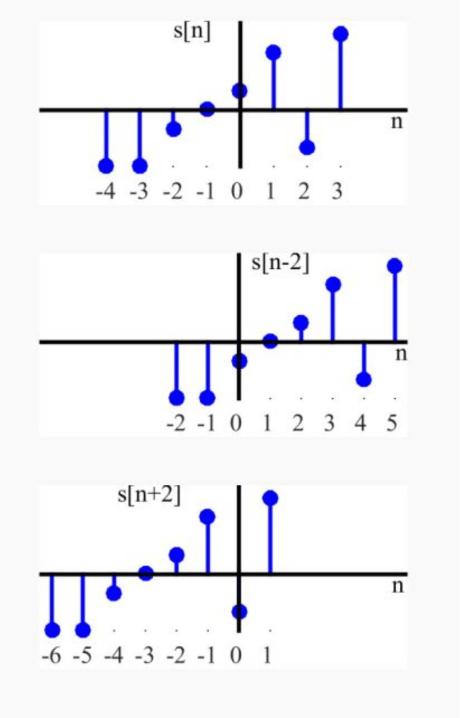


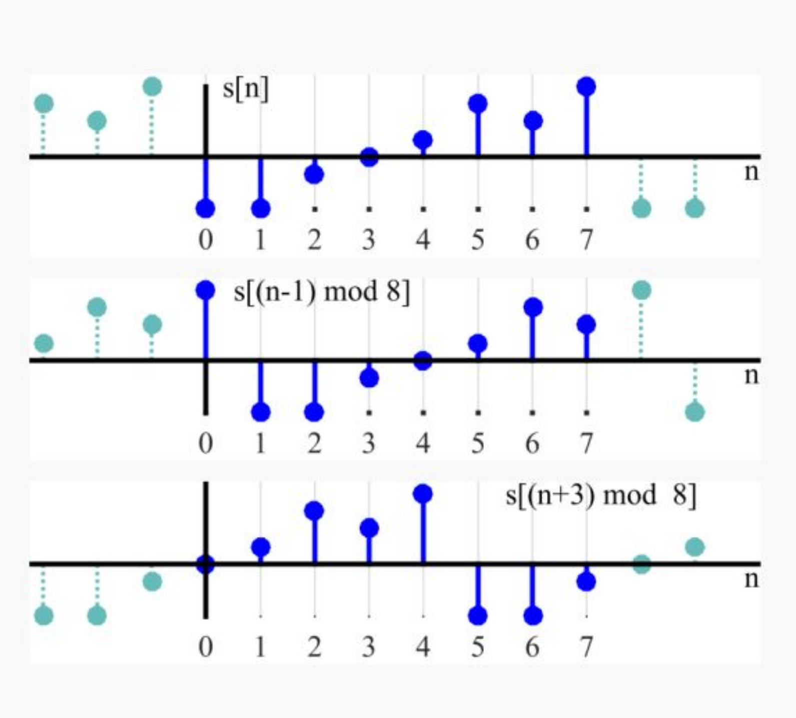
Basic Signals

- Unit impulse
- Unit step
- Even/Odd
- Periodic/Nonperiodic



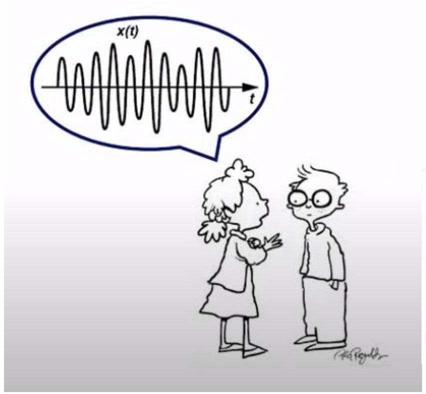
Shift in Time



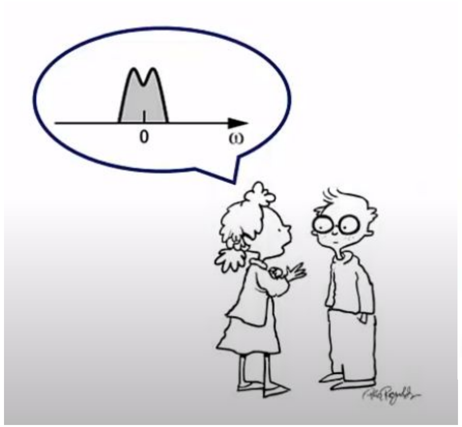


Time vs Frequency Domain

- The real world happens in the time domain

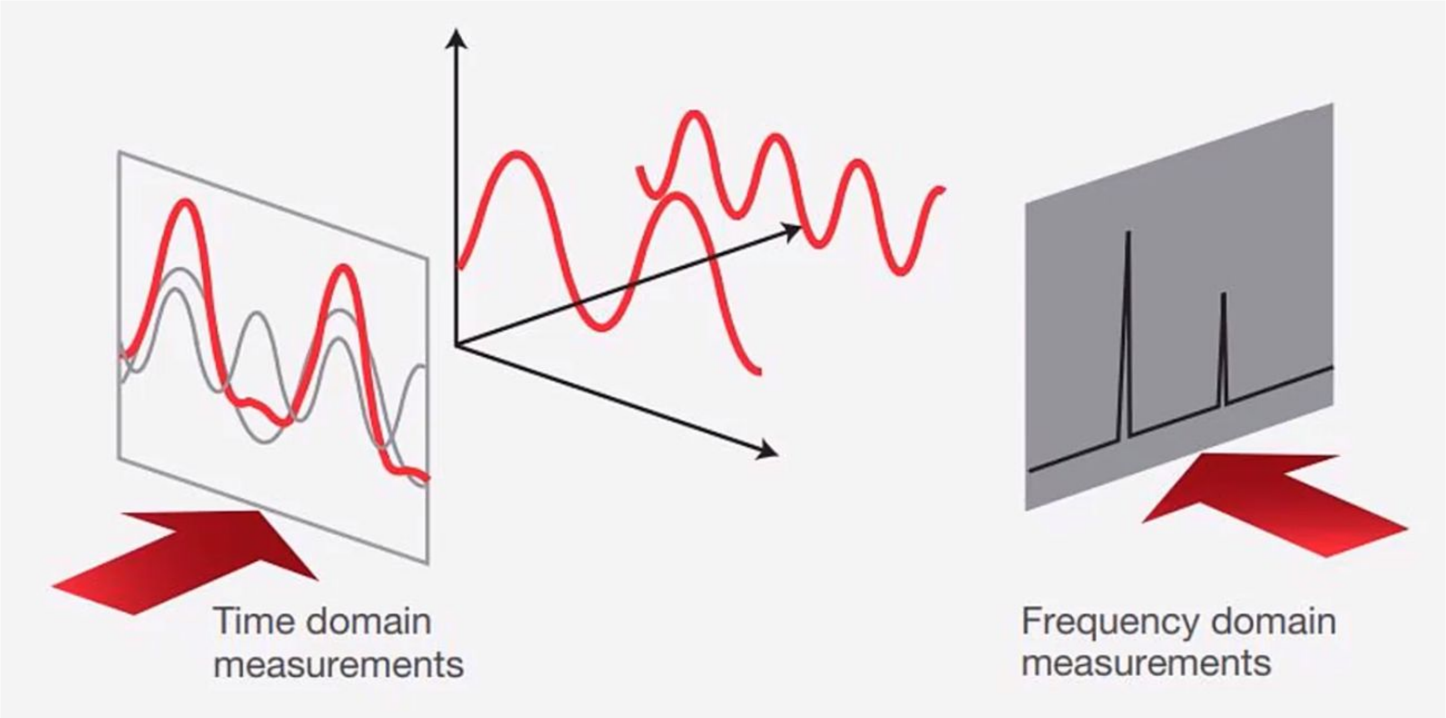


- Signals can be represented by frequency components



$$x(t) \xleftrightarrow{\text{Fourier}} X(\omega)$$

Time vs Frequency Measurements

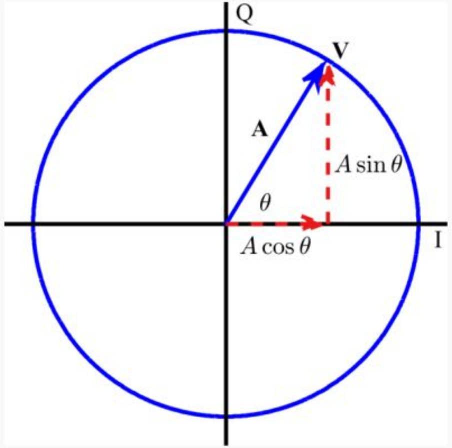


Complex Numbers

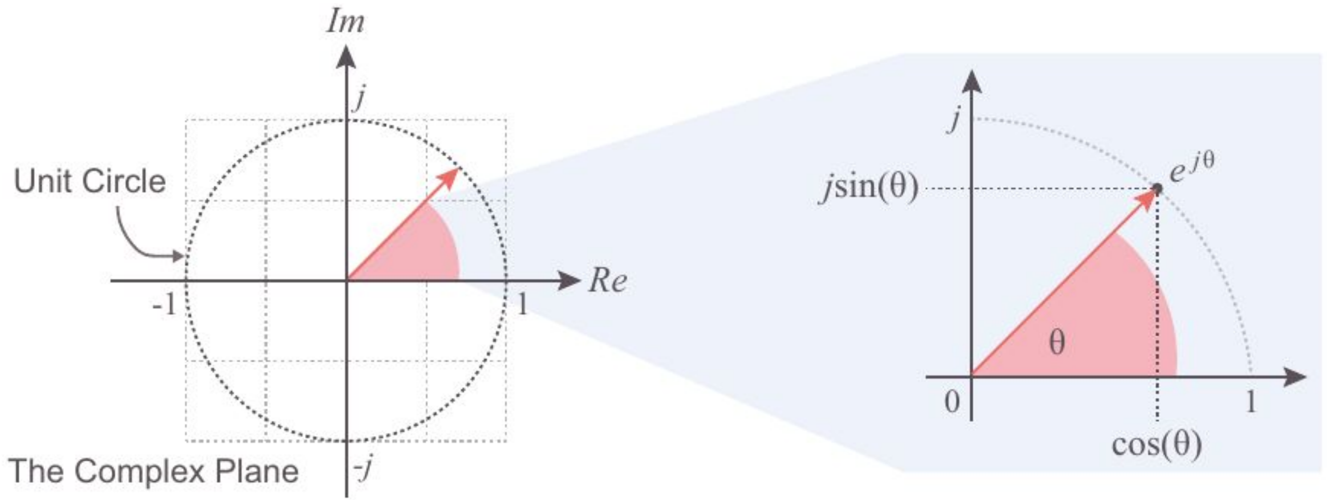
- Pair of real numbers
- I and Q parts
- Magnitude
- Phase

$$V_I = |V| \cos \measuredangle V$$
$$V_Q = |V| \sin \measuredangle V$$

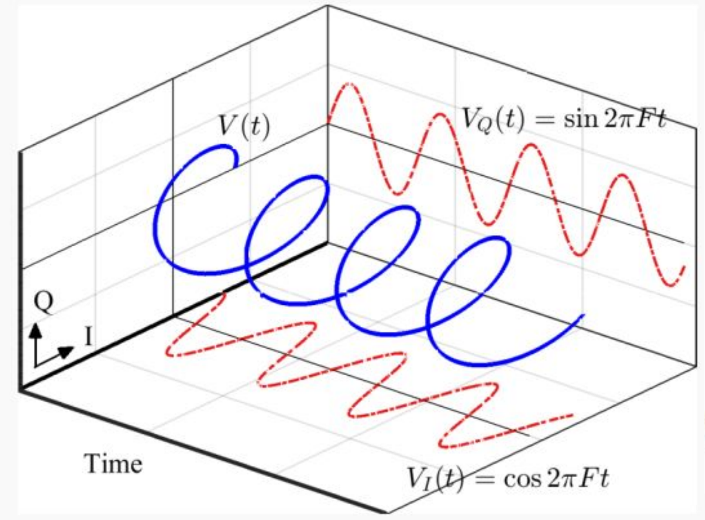
$$|V| = \sqrt{V_I^2 + V_Q^2}$$

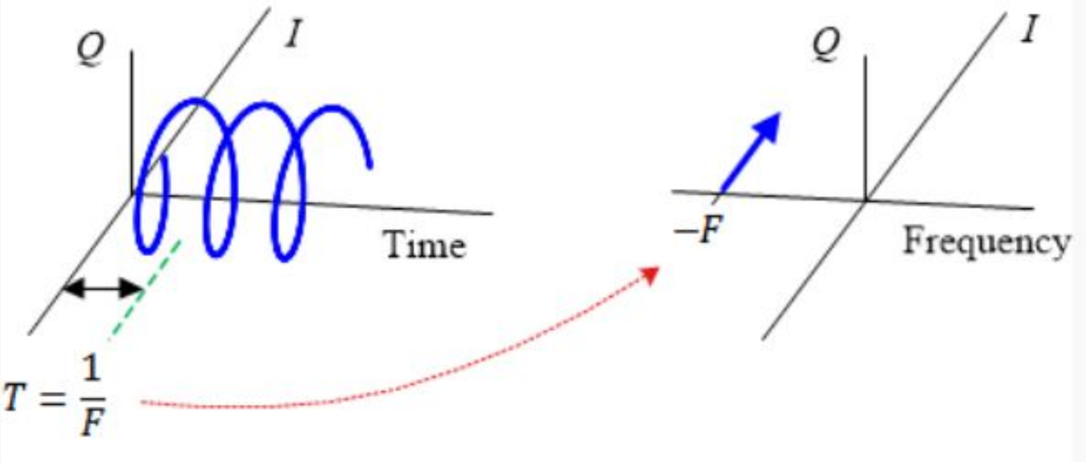


Euler’s Formula

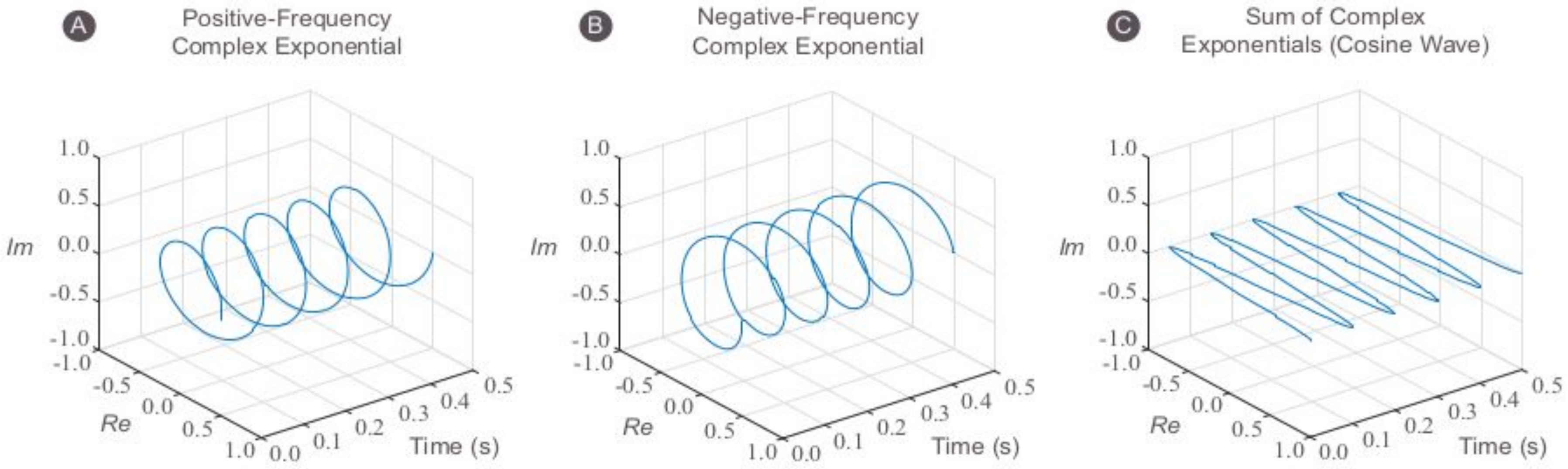


Complex Sinusoid



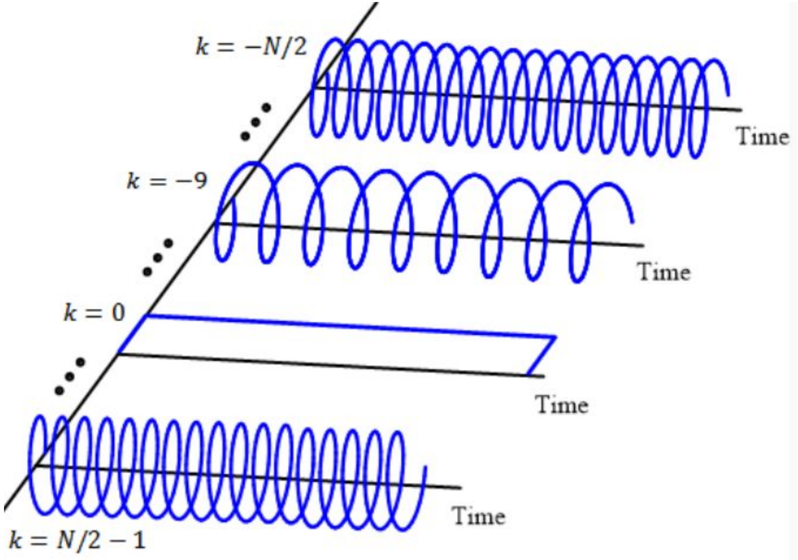


Positive/Negative Frequencies



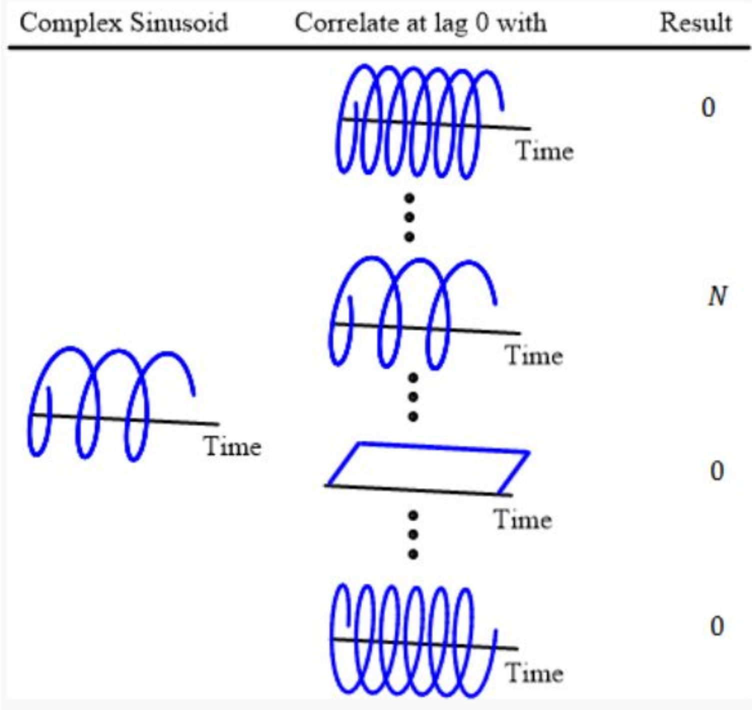
Set of Complex Sinusoids

k cycles per NTs seconds



Orthogonality

- Orthogonality is the basis for OFDM



Fourier Series

- Any periodic waveform can be expressed as sum of sine and cosine
  - varying amplitudes
  - 2nd, 3rd, 4th, … harmonics

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n cos(nx) + \sum_{n=1}^{\infty} b_n sin(nx), \qquad n = 1, 2, 3, \ldots .$$

Complex Fourier Series

$$g(x) = \sum_{n=-\infty}^{\infty} c_n e^{jnx}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) e^{-jnx} dx, \qquad n = 0, \pm 1, \pm 2, \ldots .$$

Fourier Transform

- Real world signals are never truly periodic
- Assume signal has infinite period

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt .$$

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df.$$

Discrete Fourier Transform (DFT)

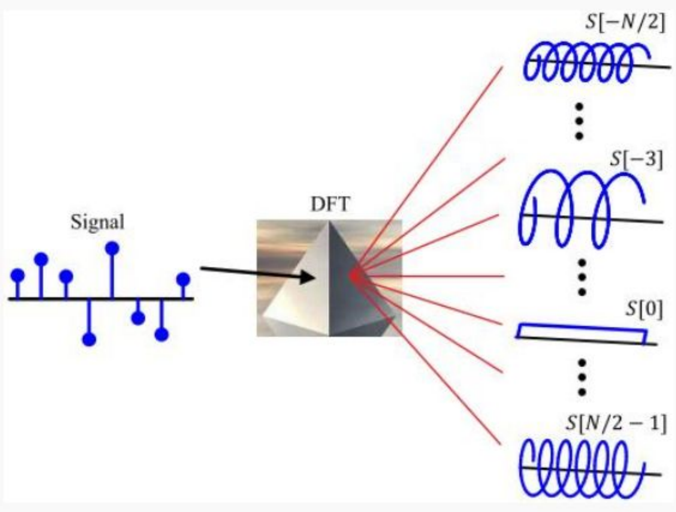
- We usually deal with samples periodically sampled as we have seen before
- n*Ts or n/Fs

$$X(f) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi fn/f_s}.$$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-2\pi kn/N}.$$

Discrete Fourier Transform

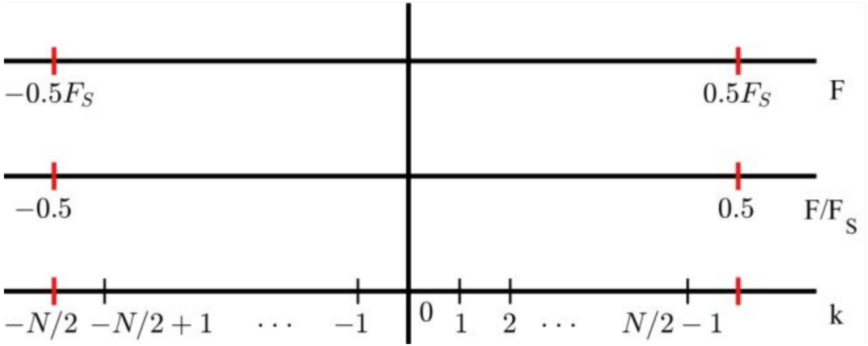
- **DFT** finds **amplitude** and **phase** contributions in a signal from each of the N discrete-time complex sinusoids
- These reference sinusoids are called **analysis frequencies**



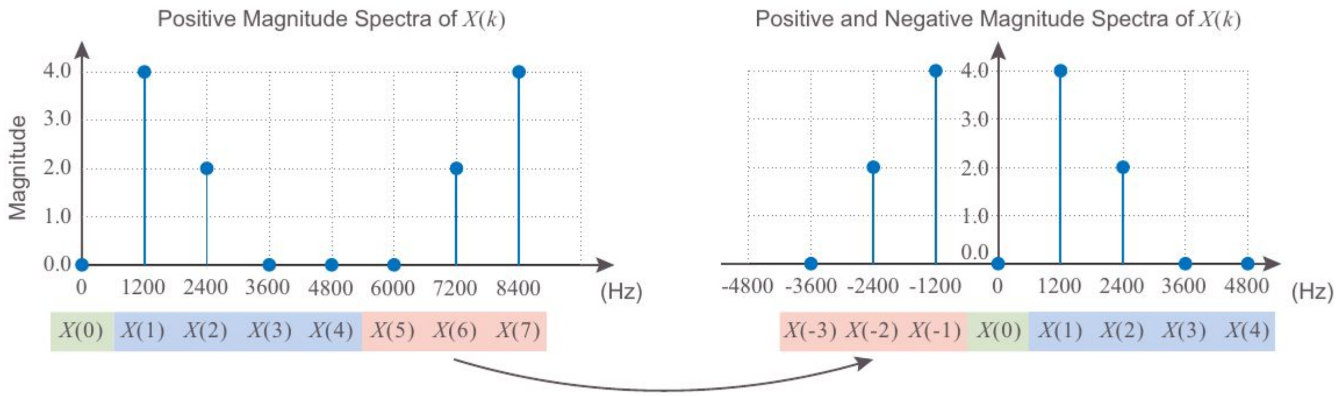
Discrete Frequencies

Suppose Fs = 100 and N = 10

- k=0 corresponds to 0Hz
- k=1 corresponds to 10Hz
- k=-2 corresponds to -20Hz



Symmetry and Periodicity



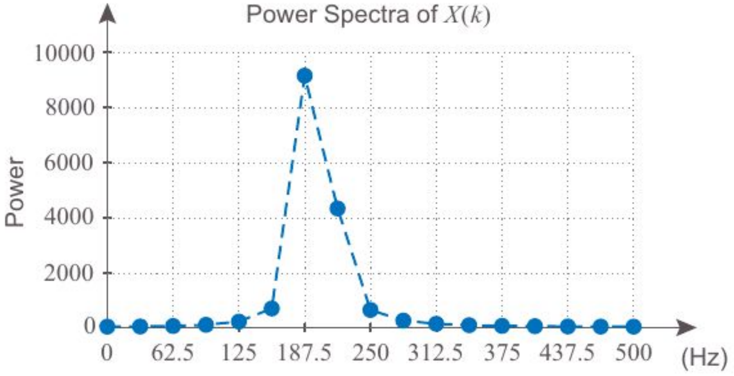
Power Spectrum

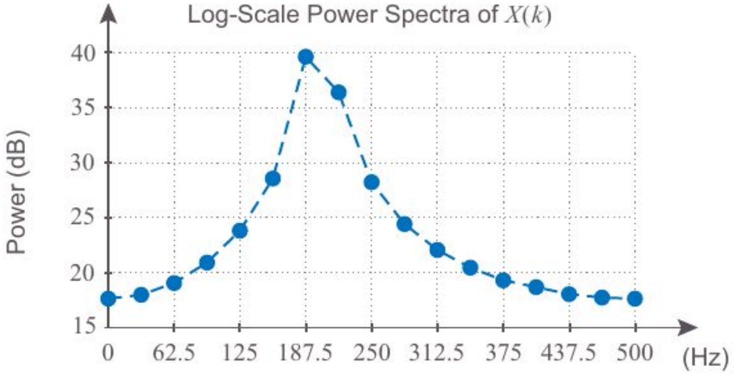
$$X'_{\text{ps}}(k) = |X'(k)|^2 .$$

$$X'_{\text{dB}}(k) = 10\log_{10}(|X'(k)|^2)\ \text{dB} \quad .$$

$$X'_{\text{dB}}(k) = 20\log_{10}(|X'(k)|)\ \text{dB}$$

Example: 200Hz Sine Signal



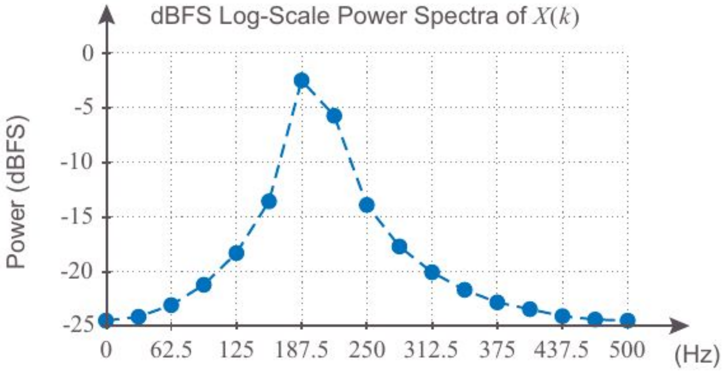


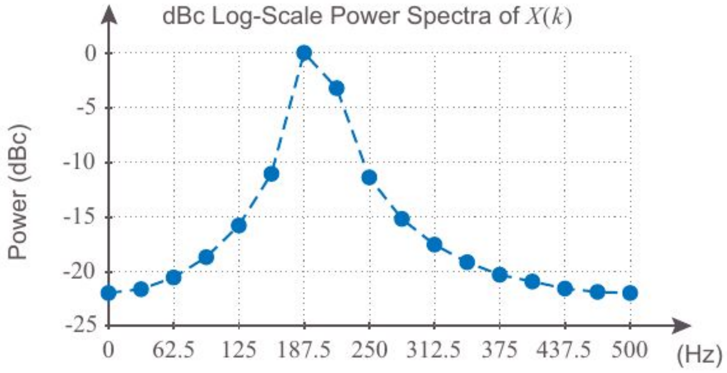
Relative decibels

$$X'_{\mathrm{dBFS}}(k) = 20\log_{10}\left(\frac{|X'(k)|}{\text{full-scale}}\right)\ \mathrm{dBFS}$$

$$X'_{\mathrm{dBc}}(k) = 10\log_{10}\left(\frac{|X'(k)|^2}{\text{carrier power}}\right)\ \mathrm{dBc}\,.$$

Example: 200Hz Sine Signal



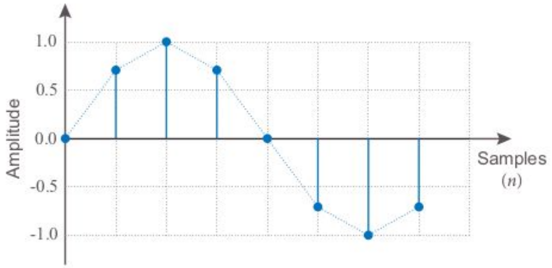


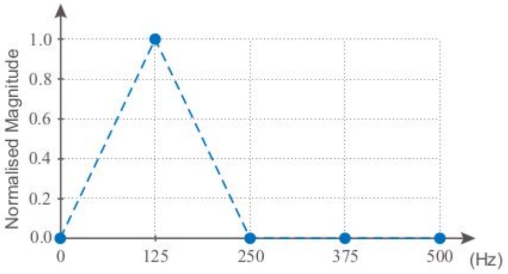
Frequency Bins

- Frequency axis divided into several discrete frequencies

$$\Delta f = \frac{f_s}{N}$$

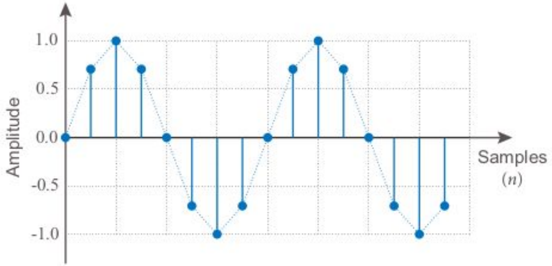
125Hz sine wave sampled at 1kHz





N = 8

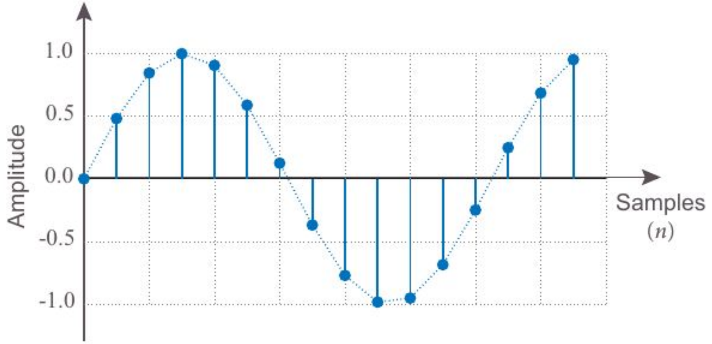
$$\Delta f = \frac{f_s}{N}$$

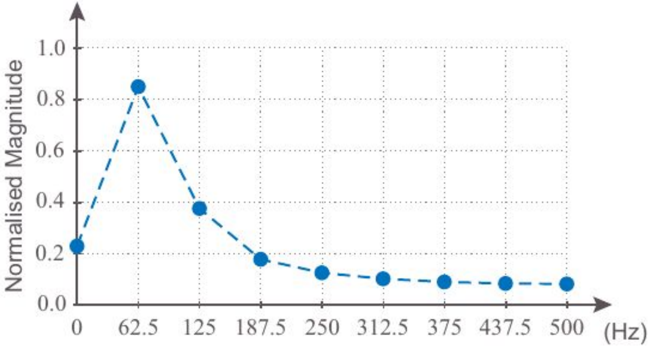


N = 16

Spectral Leakage

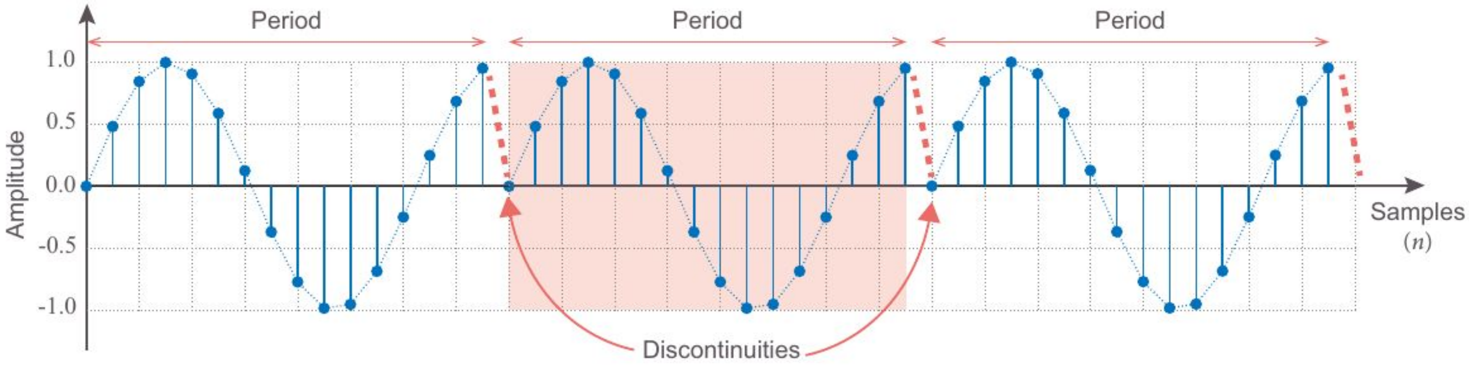
- Happens when Fs is not multiple of signal's frequency
- There is no 80Hz discrete frequency bin
- Energy spreads to neighbouring bins





Rectangular Windowing

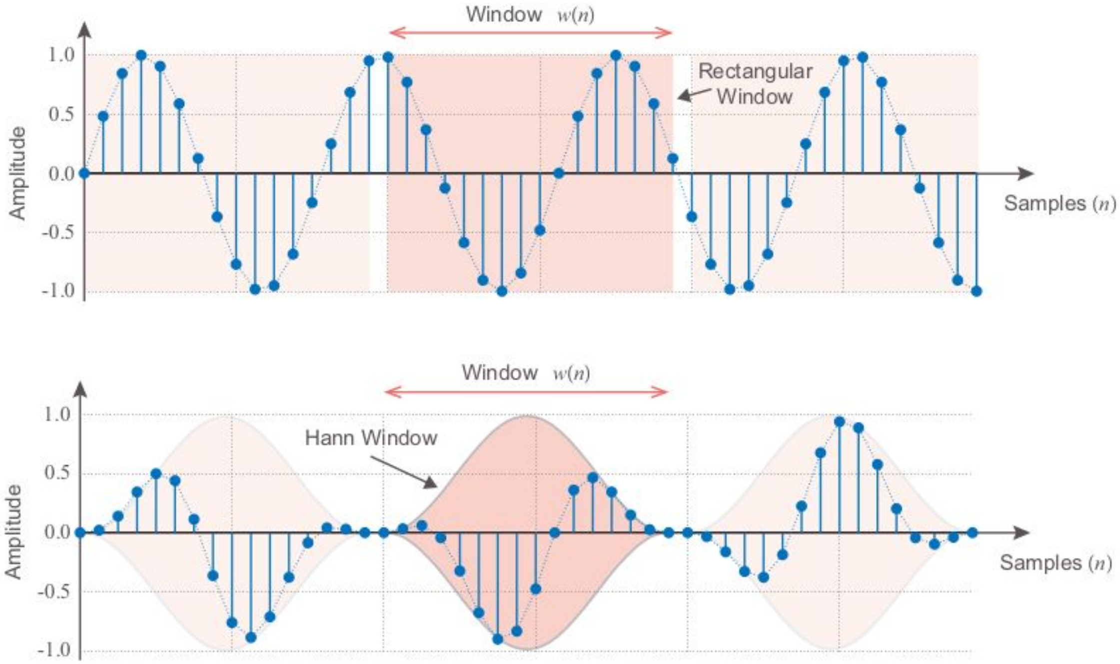
- We assume signal period of N
- Produces discontinuities



Windowing

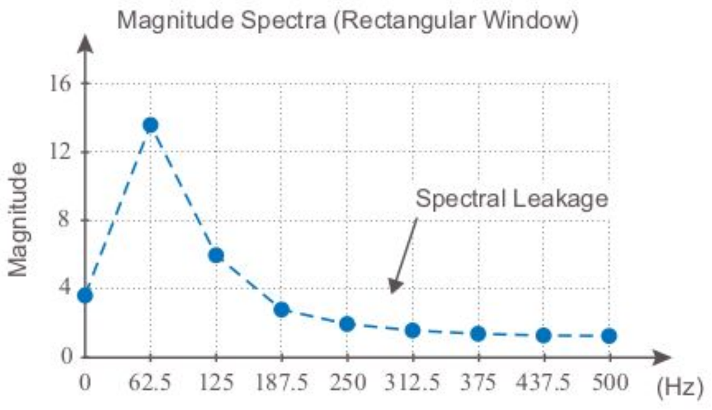
- We can reduce the effect of spectral leakage by applying particular windows to a discrete waveform before using the DFT
  - Hamming,
  - Hann,
  - Blackman-Harris and
  - Bartlett.

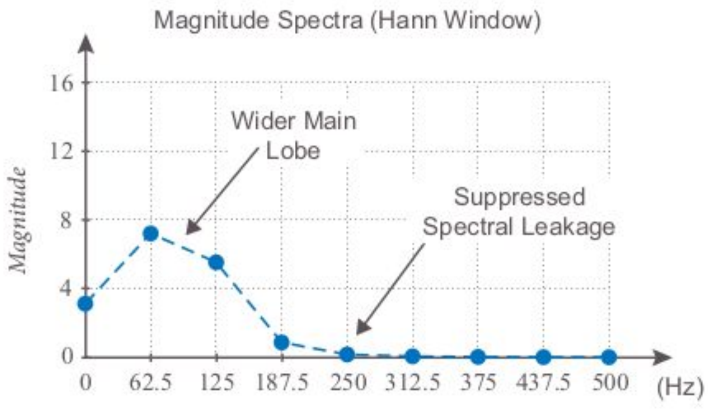
A Hann window applied to a discrete sine wave of 80Hz



A Hann window applied to a discrete sine wave of 80Hz

- Tapered windows can reduce spectral leakage in the DFT.
- However, there are some caveats.
  - Windowing has the effect of widening the main lobe of the peak frequency.
  - However, the side lobes that cause spectral leakage are reduced.

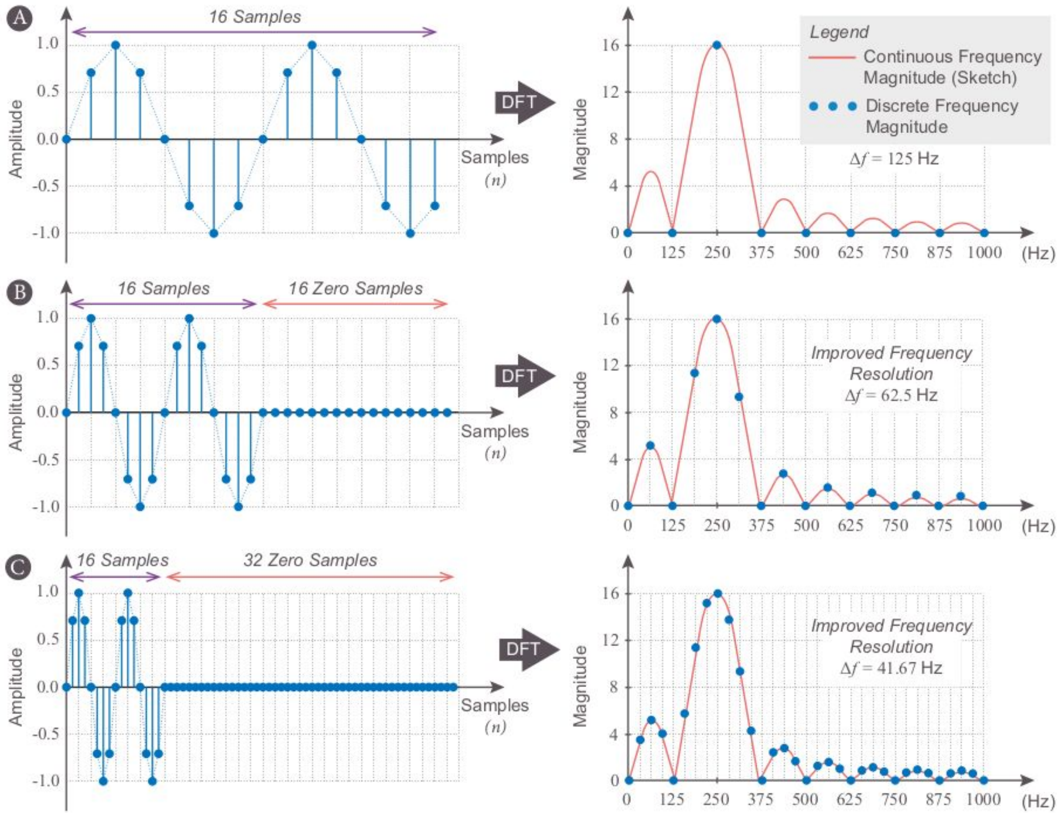




Zero Padding

- Involves inserting zero valued samples at the end
- Improves frequency resolution of DFT

250 Hz Sine wave sampled at 2k Hz

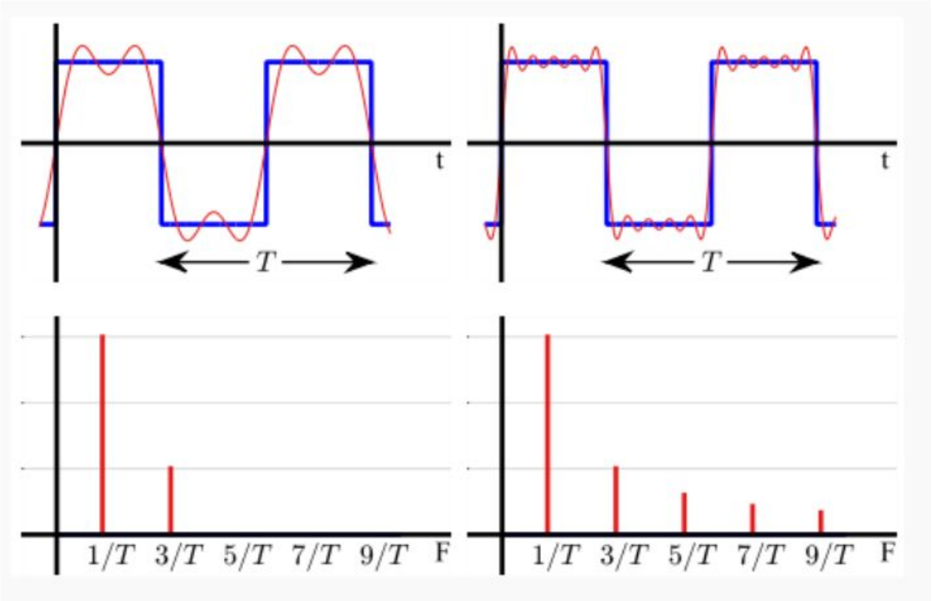


Spectral Analysis

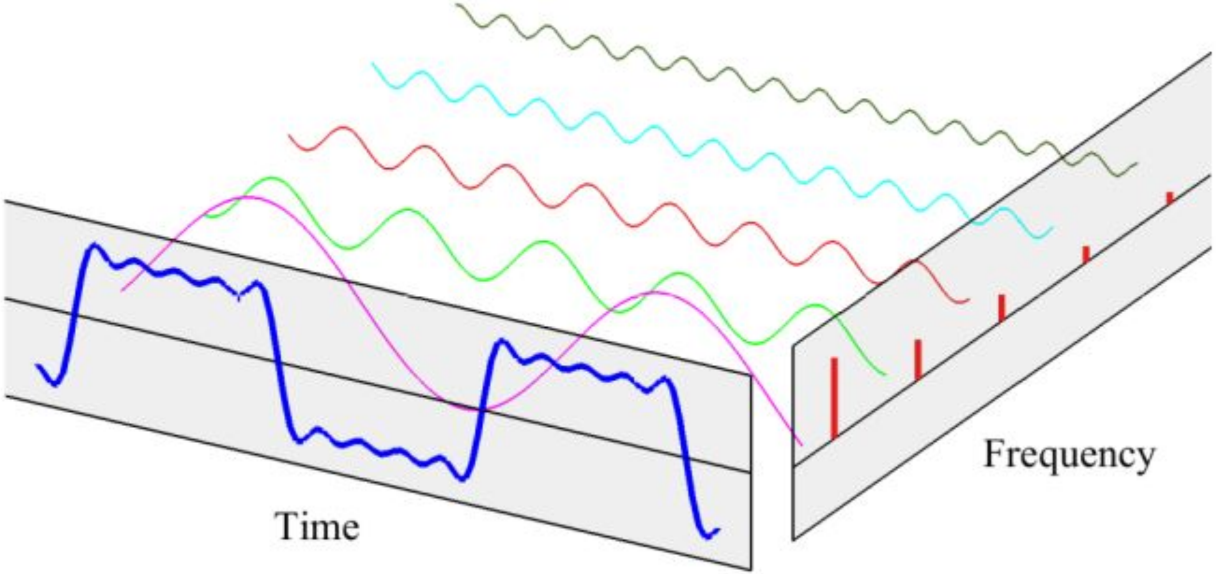
- **DFT** can be used to obtain the frequency representation of discrete-time waveforms
- **FFT** is not an approximation of the DFT; rather, it is the DFT and is effective when reducing computational complexity.
- FFT technique could only be used with DFT sizes that are a power of two.

Making Up a Signal

- Every signal is composed of sinusoids with different frequencies
- A better approximation is achieved with more sinusoids



Squarewave



GNU Radio: Making Up A Square Signal