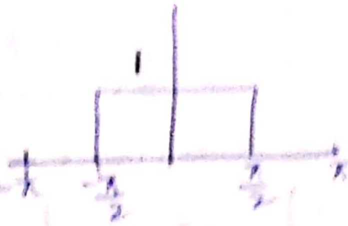
19/09/22

Assignment - 2

1) $f(x) = \begin{cases} 0 & , -\pi < x < -\frac{\pi}{2} \\ 1 & , -\frac{\pi}{2} < x < \pi/2 \\ 0 & , \frac{\pi}{2} < x < \pi \end{cases}$



$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} f(x)dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)dx + \int_{\frac{\pi}{2}}^{\pi} f(x)dx\right]$

$a_0 = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\,dx \Rightarrow a_0 = \frac{1}{\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]$

$\therefore a_0 = 1$

$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\cos\left(\frac{n\pi x}{L}\right)dx$

$\Rightarrow a_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(nx)dx$

$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\frac{\pi}{2}}^{\pi/2}$

$a_n = \frac{1}{\pi n}\left[\sin n\frac{\pi}{2} - \sin n\left(-\frac{\pi}{2}\right)\right]$

$\therefore a_n = \frac{2}{\pi n}\left(2k\pi + \frac{\pi}{2}\right) = \frac{4k}{n} + 1$

$\Rightarrow b_n = \frac{1}{l}\int_{-l}^{l} f(x).\sin\left(\frac{n\pi x}{L}\right)dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)dx.$

$= \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_{-\frac{\pi}{2}}^{\pi/2}$

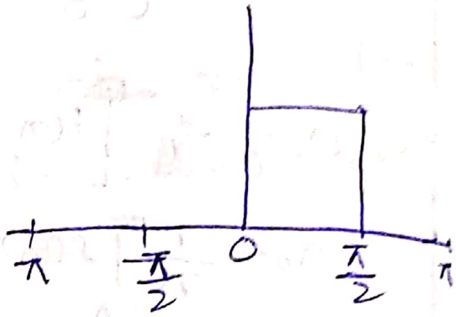
$b_n = 0$

So the fourier series is

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n x \pi}{L}\right) + b_n \sin\left(\frac{n x \pi}{L}\right)\right)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi} \sin\left(\frac{n\pi}{2}\right) \cos(nx)$$

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$



$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\,dx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_0 = \frac{1}{\pi} \left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\frac{\pi}{2}} f(x)\,dx + \int_{\frac{\pi}{2}}^{\pi} f(x)\,dx \right]$$

$$a_0 = \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} 1 \cdot dx$$

$$a_0 = \frac{1}{2}$$

$$a_n = \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} 1 \cdot \cos(nx)\,dx$$

$$a_n = \frac{1}{\pi} \left[\frac{-\sin(nx)}{n} \right]_0^{\pi/2}$$

$$= \frac{1}{n\pi} \left[(-1)^n \right] \text{----}$$

Similarly,

$$b_n = \frac{1}{\pi} \int_{0}^{}$$

2) i) $f(x) = e^{-ax}$, $-\pi \le x \le \pi$.

$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx \quad = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\frac{e^{-a\pi} - e^{a\pi}}{-a}\right]$

$= \frac{2}{a\pi}\left[\frac{e^{a\pi} - e^{-a\pi}}{2}\right] = \frac{2}{a\pi}(\sinh(a\pi))$

$\Rightarrow a_n = \frac{1}{l}\int_{-l}^{l} f(x).\cos\left(\frac{n x \pi}{l}\right)dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}.\cos(nx)dx$

$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+x^2}(-a\cos nx + x\sin nx)\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+x^2}\{-a\cos n\pi + x\sin \pi x\}\right]$

$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+x^2}\{-a\cos n\pi - n\sin n\pi\}\right.$

$\left. - \frac{e^{ax}}{a^2+x^2}[a\cos n\pi - n\sin n\pi]\right]$

$= \frac{1}{\pi}\left[\frac{e^{-a\pi}}{a^2+x^2}\{-a\cos x\pi + \frac{e^{a\pi}}{a^2+x^2}[-a\cos n\pi - n\sin n\pi]\right]$

$\Rightarrow a_n = \frac{1}{\pi}\left[a(-1)^n\left[\frac{e^{a\pi} - e^{-a\pi}}{2}\right]\times 2\right]$

$\therefore a_n = \frac{2a(-1)^n \sinh(a\pi)}{\pi(a^2+x^2)}$

$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}.\sin(nx)dx$

$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+x^2}[-a\sin nx - n\cos nx]\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\left(\frac{e^{-a\pi}}{a^2+x^2}[a\sin n\pi - b\cos n\pi\right) - \right.$

$\left.\frac{e^{a\pi}}{a^2+n^2}[a\sin n\pi - n\sin n\pi]\right]$

$= \frac{1}{\pi}\left[\frac{e^{a\pi}}{a^2+x^2}[-n\cos n\pi] - \frac{e^{a\pi}}{a^2+n^2}[-n\cos n\pi]\right]$

$$= \frac{1}{\pi}\left\{\left(\frac{e^{a\pi}-e^{a\pi}}{2}\right)\frac{n(-1)^n}{a^2+x^2}\times 2\right\}$$

$$= \frac{2n(-1)^n \sinh(a\pi)}{\pi(a^2+x^2)}.$$

So the fourier series of given function is,

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$= \frac{1}{\pi a}\sin(n\pi a) + \frac{2\sin n\pi a}{\pi(x^2+a^2)}\left[\sum_{n=1}^{\infty} a(-1)^n \cos nx + \sum_{n=1}^{\infty} n(-1)^n \sin nx\right]$$

(11) $f(x) = x + |x| \quad ; -\pi < x < \pi$

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 2x & 0 < x \leq \pi \end{cases}$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(x-x)\,dx + 2\int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{2x^2}{2}\right]_0^{\pi} = \pi$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos nx\,dx.$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx\right] + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{0}^{\pi} 2x\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n^2} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\cdot \sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 2x \sin nx\,dx$$

$$= \frac{2}{\pi}\left[-\frac{x\cos n\pi}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{1-\pi(-1)^n}{n}\right]$$

$\therefore$ Fourier series is,

$$\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[(-1)^n - 1\right]\cos nx + \sum_{n=1}^{\infty} \frac{-2}{n}\left((-1)^{n-1} - 1\right)\sin nx$$

(iii) $f(x) = \begin{cases} x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\,dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \pi x\,dx\right]$$

$$= 0 + \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_{\frac{\pi}{2}}^{3\pi/2}$$

$$= \frac{1}{\pi}\left[\pi^2 - \pi^2\right] = 0.$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cdot\cos nx\,dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)\cos nx\,dx\right]$$

$$= 0 + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \pi\cos(nx\,dx) - x\cos nx\,dx]$$

$$= 0 + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} [\pi\cos nx\,dx - x\cos nx\,dx]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{\sin nx}{n}\right)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x\cos nx\,dx$$

$$= \frac{-1}{\pi}\left\{ \pi \frac{\sin(nx)}{n} - \int (1) \cos nx \, dx \right\}_{\frac{\pi}{2}}^{2}$$

$$= \frac{-1}{\pi}\left[\frac{3\pi}{2x} \sin\left(\frac{3x\pi}{2}\right) - \frac{\pi}{2n^2} \sin\left(\frac{n\pi}{2}\right)\right]$$

$$= 0$$

$$b_n = \frac{1}{L}\int_{-l}^{l} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sin x \, dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \pi \cos nx \, dx$$

$$- \frac{1}{\pi}\int_{\frac{\pi}{2}}^{3\pi/2} x \cos nx \, dx$$

$$= \frac{2}{n}\left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2}\right] + \frac{1}{n}\left[\frac{\sin 3n\pi}{2} - \frac{\sin n\pi}{2}\right.$$

$$- \frac{1}{\pi}\left[\frac{3\pi}{2n} \sin\left(\frac{3x\pi}{2} - \frac{\pi}{2} \sin \frac{n\pi}{2}\right)\right)$$

$$= 0$$

$$f(x) = \cos x \quad , \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$a_0 = \frac{1}{L}\int_{-l}^{l} \cos x \, dx$$

$$a_0 = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos x \, dx$$

$$= \frac{4}{\pi}\int_{0}^{\frac{\pi}{2}} \cos x \, dx = \frac{4}{\pi}\left[\sin x\right]_{0}^{\pi/2} = \frac{4}{\pi}$$

$$\Rightarrow a_n = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} f(x) \cos\left(\frac{n x \pi}{L}\right) dx :$$

$\Rightarrow$ Fourier series is,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty} b_n \left(\sin\left(\frac{n\pi x}{2}\right)\right)$$

$$= \frac{2}{\pi}$$

(4) i) $f(x) = 1 - x^2$; $-1 < x < 1$

$$a_0 = \frac{1}{1}\int_{-1}^{1} (1-x^2)\,dx$$

$$a_0 = \left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{4}{3}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} (1-x^2)\cos(n\pi x)\,dx$$

$$= \left[(1-x^2)\sin\left(\frac{n\pi x}{n\pi}\right) - \int (-2x)\frac{\sin(n\pi x)}{n^2\pi^2}\right]_{-1}^{1}$$

$$= \frac{4}{n^2\pi^2}(-1)^{n+1}$$

$$\Rightarrow b_n = \int_{-1}^{1} (1-x^2)\sin(n\pi x)\,dx = 0$$

$\therefore$ Fourier series is,

$$\frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(-1)^{n+1}\cos(n\pi x)$$

ii) $f(x) = \begin{cases} \frac{1}{2} + x, & -\frac{1}{2} < x < 0 \\ \frac{1}{2} - x, & 0 < x < \frac{1}{2} \end{cases}$

$$a_0 = \frac{1}{1/2}\left[\int_{-\frac{1}{2}}^{0} \left(\frac{1}{2} + x\right)dx + \int_{0}^{1/2} \left(\frac{1}{2} - x\right)dx\right]$$

$$= 2\left[\left(\frac{x}{2} + \frac{x^2}{2}\right)_{-1/2}^{0}\right] + \left[\frac{x}{2} - \frac{x^2}{2}\right]_{1/2}^{0}$$

$$= 1/2$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cdot\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{1}\left[\int_{-1/2}^{0} \left(\frac{1}{2} + x\right)\cos(2n\pi x)\,dx + \int_{0}^{\frac{1}{2}} \left(\frac{1}{2} - x\right)\cos(2n\pi x)\,dx\right.$$

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$$= 2\left[\frac{\cos(2n\pi x)}{2n\pi}\right]_{-1/2}^{0} + 2\left[\frac{-\cos(2n\pi x)}{4\pi^2 n^2}\right]_{0}^{1/2}$$

$$= 2\left[\frac{1-(-1)^n}{4\pi^2 x^2}\right] - 2\left[\frac{(-1)^n - 1}{4\pi^2 x^2}\right]$$

$$= \frac{2}{4\pi^2 n^2}\left[1-(-1)^n - (-1)^n + 1\right]$$

$$= \frac{1-(-1)^n}{\pi^2 n^2}$$

$$b_n = 2\int_0^2 f(x)\sin(2n\pi x)\,dx$$

$$b_n = 0$$

$\therefore$ Fourier Series of function,

$$\frac{1}{4} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{\pi^2 n^2}\cos(n\pi x)$$

(5) $$I(t) = \begin{cases} I_0 \sin t & , 0 \le t \le \pi \\ 0 & , \pi \le t \le 2\pi \end{cases}$$

Sol:-

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{\pi}\int_{\alpha}^{\alpha+2l} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} I_0 \sin\omega t\,dt$$

$$\therefore a_0 = \frac{2I_0}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} I_0 \sin t\cos(nt)\,dt$$

$$= \frac{I_0}{2\pi}\left[\int_0^{\pi} \sin(n+1)t - \sin(n-1)t\right]dt$$

$$= \frac{I_0}{2\pi}\left[\frac{(-1)^{n+1}-1}{n^2-1}\right] \quad n \neq 1$$

$$\therefore a_1 = \frac{I_0}{\quad}\int^{\pi} \sin 2t\cos x\,dt = \frac{I_0}{2\pi}\,\text{cost.}$$

$b_n = \frac{I_0}{\pi} \int_0 \sin t \sin(n t) dt$

$= \frac{I_0}{2\pi}\left[\int_0^{\pi} \cos(n-1)t - \cos(n+1)\cdot t\right] dt$

$\Rightarrow b_n = \frac{I_0}{2\pi}(0) = 0$

$b_1 = \frac{I_0}{\pi} \times \frac{\pi}{2} = \frac{I_0}{2}$

Fourier series of function,

$\frac{I_0}{2} + \frac{I_0}{\pi} + \frac{I_0}{2\pi}\cos x + \frac{I_0}{2\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}-1}{n^2-1}\right)\cos(nx)$

$V(t) = V_0 \cos 100\pi t$

$$f(x) = \begin{cases} 0 & ; -\frac{1}{100} < t < -\frac{1}{200} \\ V_0 \cos 100\pi t & ; -\frac{1}{200} < t < \frac{1}{200} \\ 0 & ; \frac{1}{200} \le t \le \frac{1}{100}. \end{cases}$$

$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$

$= \frac{1}{1/100}\cdot\left[\int_{-\frac{1}{200}}^{+\frac{1}{100}} V_0 \cos(100\pi t)\, dt\right]$

$= \frac{100 V_0}{100\pi}\left[\sin 100\pi t\right]_{-1/200}^{1/100}$

$= \frac{V_0}{\pi}[1+1] = \frac{2V_0}{\pi}$

$\Rightarrow a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cdot\cos nx\, dx$

$= 100\left[\int_{-1/200}^{1/100} V_0 \cos(100\pi t)\cos n t\, dt\right]$

$= \frac{100 V_0}{2}\left[\int_{-\frac{1}{200}}^{-1/100} \cos(100\pi\, n)t\, dt - \cos(100\pi - n)t\, dt\right]$

$$= 50V_0 \left[\left(\frac{\sin(100\pi+x)t}{100\pi+x} \right]_{\frac{-1}{200}}^{\frac{+1}{100}} - \left[\frac{\sin(100\pi-x)}{100\pi-x} \right]_{\frac{-1}{200}}^{\frac{1}{100}}$$

$$\Rightarrow a_n = \frac{-2V_0\cos(n\pi/2)}{(n^2-1)\pi}$$

$$\Rightarrow a_n = \frac{-2V_0\cos(n\pi/2)}{(n^2-1)\pi}$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x)dx$$

$$= \frac{1}{1/100} \int_{-\frac{1}{200}}^{\frac{1}{100}} V_0 \cos(100\pi t)dt$$

$\therefore b_n = 0$

$\therefore$ Fourier Series is,

$$\frac{2V_0}{\pi} + \sum_{n=1}^{\infty} \frac{-2V_0\cos(n\pi/2)}{(n^2-1)\pi}$$

$$f(x) = \begin{cases} -2 & -\pi < x < 0 \\ 2 & 0 < x < \pi \end{cases}$$

$f(x)$ is an odd function,

$a_0 \leq 0$, a_n should be equal to '0'.

$a_0 = 0$; $a_n = 0$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin(nx)dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} -2\sin(nx)dx + \int_{0}^{\pi} 2\sin(nx)dx \right.$$

$$b_n = \frac{4}{\pi} \left(\frac{1-(-1)^n}{n} \right)$$

fourier series of function $= \frac{4}{\pi} \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n} \sin nx \right)$

~~As $n \to \infty$, the~~ .

8) Let $f(x)$ is a even function,

$$f(-x) = f(x) \quad ; f(x+2\pi) = f(x)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(-x)\sin(-nx)\,dx$$

$$= -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$b_n = 0$$

9) i) $f(t) = t \; ; 0<t<1$

$$f(t) = t$$

$$a_0 = \frac{2}{L}\int_0^L f(t)\,dt$$

$$= \frac{2}{1}\left[\frac{t^2}{2}\right]_0^1 \Rightarrow a_0 = 1$$

$$a_n = \frac{2}{1}\int_0^1 t\cos nt\,dt$$

$$= 2\left[\frac{t\sin nt}{n} + \frac{\cos nt}{n^2}\right]_0^1$$

$$= 2\left[\frac{\sin n\pi}{n\pi^2} + \frac{\cos n\pi - 1}{n^2}\right]$$

The fourier cosine series is,

$$\frac{1}{2} + \sum_{n=1}^{\infty} 2\left[\frac{\sin n\pi}{n} + \frac{\cos n\pi - 1}{n^2}\right]\cos n t$$

$$\boxed{= \frac{1}{2} + \sum_{n=1}^{\infty} 2\left\{\frac{2[(-1)]^n - 1}{(n\pi)^2}\right\}\cos n\pi t}$$

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$$b_n = 2\int_0^1 t\sin n\pi t\,dt$$

$$= 2\left[-\frac{t\cos nt}{n} \pm \frac{\sin nt}{n^2}\right]_0^1$$

$$= 2\left[\frac{\sin n\pi}{n\pi} - \frac{\cos n\pi}{n\pi}\right]$$

fouries sine of function is

$$\sum_{n=1}^{\infty} 2\left[\frac{\sin n\pi}{n^2} - \frac{\cos n\pi}{(n\pi)^2}\right]\sin n\pi t.$$

$$\boxed{\sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi}\sin(n\pi t)}$$

(ii) $f(t) = t^2 \quad ; 0 < x < 3$

$$a_0 = \frac{2}{3}\int_0^3 t^2\,dt$$

$$a_0 = 6$$

$$a_n = \frac{2}{3}\int_0^3 t^2\cos n t\,dt$$

$$= \frac{2}{3}\left[\frac{t^2\sin n t}{n} - 2\int_0^3 t\cdot\sin t\,dt\right]$$

$$= \frac{2}{3}\left[\frac{t^2\sin n\pi t}{n\pi} - \frac{2}{\pi}\left[\frac{-t\cos n\pi t}{(n\pi)^2} + \frac{\sin n\pi t}{n\pi}\right]_0^3\right.$$

The fourier cosine series is,

$$\frac{6}{2} + \sum_{n=1}^{\infty}\frac{2}{3}\left[\frac{9\sin 3x\pi}{n\pi^2} + \frac{6\cos 3x\pi}{n\pi^2} \frac{2\sin 3n\pi}{n\pi^2}\right] \cos t$$

$$= 3 + 36\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{(n\pi)^2}\cos\left(\frac{n\pi t}{3}\right)$$

$$b_n = \frac{2}{3}\int_0^3 t^2 \sin \cancel{n} t\,dt$$

$$= \frac{2}{3}\left[\frac{-t^2 \cos nt}{n} + \int \frac{2t \cos nt}{n}\right]$$

$$= \frac{2}{3}\left[-\frac{3\cos 3n\pi}{n\pi} + 6\frac{\sin 3n\pi}{n\pi} + \frac{2\cos 3n\pi}{n\pi} - \frac{2}{n\pi}\right]$$

The fourier sine series is,

$$\sum_{n=1}^{\infty} \frac{2}{3}\left[-\frac{27(-1)^n}{n\pi} + 2\left(\frac{3}{n\pi}\right)(-1)^{n-1}\sin\left(\frac{n\pi t}{3}\right)\right]$$

(iv) $f(t) = e^t,\ 0<t<L$

$$a_0 = \frac{2}{L}\int_0^2 e^t\cdot dt \Rightarrow a_0 = \frac{2}{L}(e^2-1)$$

$$a_n = \frac{2}{L}\int_0^L e^t \cos\left(\frac{n\pi t}{L}\right)dt$$

$$\therefore a_n = \frac{2}{L}\left(\frac{e^L-(-1)^n-1}{1+\left(\frac{n\pi}{2}\right)^2}\right)$$

fourier series is,

$$\frac{1}{2}(e^2-1) + \sum_{n=1}^{\infty}\frac{2}{L}\left[\frac{e^l(-1)^n-1}{1+\left(\frac{n\pi}{L}\right)^2}\right]\cos\left(\frac{n\pi t}{2}\right)$$

$$\Rightarrow b_n = \frac{2}{L}\int_0^L e^t \sin\left(\frac{n\pi t}{L}\right)dt$$

$$b_n = \frac{2}{L}\frac{e^t}{\left(1-\left(\frac{n\pi}{L}\right)^2\right)}\left[\sin\left(\frac{n\pi t}{L}\right) - \frac{n\pi}{L}\cos\left(\frac{n\pi t}{L}\right)\right]_0^L$$

$$= \frac{2n\pi}{[L^2+(n\pi)^2]}\cdot\left[1-(-1)^n e^L\right]$$

$\Rightarrow$ The fourier sine series is.

$$\sum_{n=1}^{\infty}\frac{2n\pi}{[L^2+(n\pi)^2]}\left[1-(-1)^n e^L\right]\sin\left(\frac{n\pi t}{L}\right)$$

1) $f(x) = \frac{x^2}{2} \; ; \; -\pi < x < \pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x^2}{2} dx = \frac{2}{2\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$a_0 = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x^2}{2} \cos(nx) dx$$

$$= \frac{2}{2\pi}\int_0^{\pi} x^2 \cos(nx) dx$$

$$= \frac{1}{\pi}\left[\frac{x^2 \sin(nx)}{n} - 2\int_0^{\pi} \frac{x \sin(nx) dx}{x}\right.$$

$$= \frac{1}{\pi}\left[\frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{2\pi(-1)^n}{n^3}\right) = \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x^2}{2} \sin(nx) dx$$

$$= 0$$

Fourier series is,

$$\frac{x^2}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} \cdot \cos(nx)$$

for $x = \pi$

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} 2\frac{(-1)^n}{n^2} \cos(n\pi)$$

$$\frac{\pi^2}{6} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

for $x = 0$

$$-\frac{\pi^2}{6} = 2\left[\frac{-1}{1} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right]$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

11) $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx.$$

$$a_0 = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\,dx = \frac{1}{2\pi}\int_0^{\pi} 2\sin nx \cos x\,dx$$

$$a_n = \frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{2}{2\pi}\left[\frac{(-1)^{n-1} - 1}{n^2-1}\right]$$

$$= -\frac{1}{\pi}\left[\frac{1-(-1)^{n-1}}{n^2-1}\right]$$

$$a_n = \begin{cases} 0 & ; n = \text{odd} \quad n \neq 1 \\ \dfrac{-2}{\pi(n^2-1)} & ; n = \text{even} \end{cases}$$

$$b_n = \frac{1}{2\pi}\int_0^{\pi} \cos(n-1)x - \cos(n+1)x\,dx$$

$$b_n = \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{(n-1)} - \frac{\sin(n+1)}{(n+1)}\right]_0^{\pi}$$

$$b_n = 0$$

$\therefore$ Fourier series of function

$$\frac{1}{\pi} + \sum_{n=1}^{\infty} 1$$

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases}$$

$$a_0 = \frac{1}{5} \int_{-5}^{5} f(x)\,dx = \frac{1}{5} \int_0^5 3\,dx$$

$$a_0 = 3$$

$$a_x = \frac{1}{5} \int_{-5}^{5} f(x) \cdot \cos\left(\frac{n\pi x}{5}\right) dx$$

$$\therefore a_x = 0$$

$$b_n = \frac{1}{5} \int_{-5}^{5} f(x) \cdot \sin\left(\frac{n\pi x}{5}\right) dx$$

$$= \frac{3}{5} \int_0^5 \sin\left(\frac{n\pi x}{5}\right) dx$$

$$\therefore b_n = \frac{3}{n\pi} \cdot \left(1 - (-1)^n\right)$$

1) $$f(x) = \begin{cases} 1 & -\pi < x < 0 \\ -1 & 0 < x < \pi \end{cases}$$

complex fourier series is,

$$\sum_{n=-\infty}^{\infty} C_n e^{-in\pi x/2} dx$$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{0} e^{-inx} dx - \int_0^{\pi} e^{-inx} dx \right]$$

$$= \frac{1}{2\pi} \left[\left(\frac{e^{-inx}}{-in}\right)_{-\pi}^{0} - \left[\frac{e^{-inx}}{-in}\right]_0^{\pi} \right)$$

$$= \frac{2}{2\pi} \frac{(-1)^{2n} - 1}{in\pi} = \frac{(-1)^n - 1}{(in\pi)}$$

complex Fourier series is,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n-1}{in\pi} e^{inx} \quad n \neq 0$$

(ii) $f(x) = \begin{cases} x & : 0 \leq x \leq 1 \\ 2-x & ; 1 \leq x < 2 \end{cases}$

$$C_n = \frac{1}{2} \int_0^2 f(x) \cdot e^{-in\pi x} dx$$

$$= \frac{1}{2} \left[\frac{x e^{-in\pi x}}{-in\pi} + \frac{e^{-in\pi x}}{(n\pi)^2} \right]_0^1 + \frac{1}{2} \left(\frac{(2-x) e^{-in\pi x}}{-in\pi} - \int \frac{e^{-in\pi x}}{n\pi} dx \right]$$

$$= \frac{e^{-2n\pi i}}{2(n\pi)^2}$$

$\therefore$ complex fourier series,

$$\sum_{n=\infty}^{\infty} \frac{e^{-2n\pi i}}{2(n\pi)^2} \cdot e^{in\pi x}$$

(17) $f(x) = \frac{L}{2} - x \quad , \quad 0 < x < L$

$$a_0 = \frac{2}{L} \int_0^L \left(\frac{L}{2} - x\right) dx$$

$$a_0 = 0$$

$$a_n = \frac{2}{L} \int_0^L \left(\frac{L}{2} - x\right) \cos\left(\frac{2n\pi x}{L}\right) dx$$

$$= 0$$

$$b_n = \frac{2}{L} \int_0^L \left(\frac{L}{2} - x\right) \sin\left(\frac{2n\pi x}{L}\right) dx$$

$$= \frac{2}{L} \left[\frac{L}{2} \cdot \frac{L}{2\pi n} + \left(\frac{-1}{2} \cdot \frac{L}{2\pi n} \right) \right]$$

$$= \frac{2}{L}\left[\frac{L^2}{4n\pi} + \frac{L^2}{4n\pi}\right] = \frac{L}{n\pi}$$

$$\frac{L}{2} \cdot x = \frac{L}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{2n\pi x}{L}\right)$$

$f(x) = x \quad ; \quad 0 < x < 2$

$$b_n = \frac{2}{1} \int_0^2 x \cdot \sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = 2\left[\frac{-x\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{(n\pi)^2}\right]_0^2$$

$$b_n = 2\left(\frac{-2}{n\pi}\right) = \frac{-4}{n\pi}$$

fourier sine series is,

$$\sum_{n=1}^{\infty} \left(\frac{-4}{n\pi}\right) \sin(n\pi x).$$

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Assignment-3

1) i) $f(x) = \begin{cases} 0 & x<0 \\ x & 0\le x<2 \\ 0 & x\ge 2 \end{cases}$

Fourier integral is given by:-

$$\int_0^{\infty} A(x).\cos\lambda x + B(\lambda).\sin\lambda x \, dx \, d\lambda$$

$$A(\lambda) = \frac{1}{\pi}\int_{\infty}^{\infty} f(t).\cos\lambda t \, dt \quad ; \quad B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t).\sin\lambda t \, dt.$$

$$\therefore A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\lambda t \, dt = \frac{1}{\pi}\int_0^2 t\cos\lambda t \, dt$$

$$= \frac{1}{\pi}\left[\frac{t\sin\lambda t}{\lambda} - \int \frac{\sin\lambda t}{\lambda} dt\right]_0^2$$

$$= \frac{1}{\pi}\left[\frac{t\sin\lambda t}{\lambda} + \frac{\cos\lambda t}{\lambda^2}\right]_0^2 = \frac{1}{\pi}\left[\frac{2\sin 2\lambda}{\lambda} + \frac{\cos 2\lambda}{\lambda^2} - \frac{1}{\lambda^2}\right]$$

$$\Rightarrow B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\lambda t \, dt$$

$$= \frac{1}{\pi}\int_0^2 t\sin\lambda t \, dt = \frac{1}{\pi}\left[-\frac{t\cos\lambda t}{\lambda} - \int -\frac{\cos\lambda t}{\lambda}\right]_0^2$$

$$= \frac{1}{\pi}\left[-\frac{t\cos\lambda t}{\lambda} + \frac{\sin\lambda t}{\lambda^2}\right]_0^2.$$

$$= \frac{1}{\pi}\left[-\frac{2\cos 2\lambda}{\lambda} + \frac{\sin 2\lambda}{\lambda^2}\right]$$

$\therefore$ Fourier integral is,

$$\frac{1}{\pi}\int_0^{\infty}\left[\frac{2\sin 2\lambda}{\lambda} + \frac{\cos(2\lambda)-1}{\lambda^2}\right]\cos\lambda x + \left[\frac{\sin 2\lambda - 2\lambda\cos 2\lambda}{\lambda^2}\right]\sin\lambda x \, d\lambda.$$

(ii) $f(x) = \begin{cases} 0 & , x<0 \\ 1 & , 0\le x<1 \\ 0 & x\ge 1 \end{cases}$

Fourier integral is,

$$\int_0^{\infty} A(\lambda)\cos\lambda x + B(\lambda).\sin\lambda x \, dx$$

$$A(\lambda) = \frac{1}{\pi}\int_{\infty}^{\infty} f(t).\cos\lambda t \, dt \quad ; \quad B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t).\sin\lambda t \, dt.$$

we have,

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cdot \cos\lambda t\, dt$$

$$= \frac{1}{\pi}\int_0^1 \cos\lambda t\, dt\,,$$

$$= \frac{1}{\pi}\left[\frac{\sin\lambda t}{\lambda}\right]_0^1 = \frac{1}{\pi}\frac{\sin\lambda}{\lambda}$$

$$B(\lambda) = \frac{1}{\pi}\int_0^1 \sin\lambda t\, dt$$

$$= \frac{-1}{\pi}\left[\frac{\cos\lambda t}{\lambda}\right]_0^1 = \frac{1-\cos\lambda}{\pi\lambda}$$

$\therefore$ Fourier integral is given by,

$$\int_0^{\infty} \frac{\sin\lambda}{\pi\lambda}\cos\lambda x + \frac{(1-\cos\lambda)}{\pi\lambda}\sin\lambda x\, d\lambda]$$

(iii) $$F(x) = \begin{cases} 0 & x<0 \\ e^{-x} & x>0 \end{cases}$$

Fourier integral is given by,

$$\int_0^{\infty} A(\lambda)\cdot\cos\lambda x + B(\lambda)\sin\lambda x\, d\lambda$$

$$A(\lambda) = \frac{1}{\pi}\int_0^{\infty} f(t)\cos\lambda t\, dt\ ;\ B(\lambda) = \frac{1}{\pi}\int_0^{\infty} f(t)\sin\lambda t\, dt$$

$$\Rightarrow A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cdot\cos\lambda t\, dt$$

$$= \frac{1}{\pi}\int_0^{\infty} \cancel{f}\ e^{-t}\cos\lambda t\, dt$$

$$= \frac{1}{\pi}\left[\frac{\cos\lambda t\cdot e^{-t}}{-1} + \int\frac{\lambda\cdot\sin\lambda t\, e^{-t}\,dt}{-1}\right]$$

$$= \frac{-\cos\lambda t\cdot e^{-t}}{\pi} + \frac{\lambda\sin\lambda t\, e^{-t}}{\pi} - \frac{\lambda^2}{\pi}\int\cos\lambda t\, e^{-t}dt$$

$$(1+\lambda^2)A(\lambda) = \left[\frac{-\cos\lambda t\cdot e^{-t} + \lambda\sin\lambda t\, e^{-t}}{\pi}\right]_0^{\infty}$$

$$\therefore A(\lambda) = \frac{1}{\pi(1+\lambda^2)}$$

$$B(\lambda) = \frac{1}{\pi}\int_{-\infty} f(t)\sin(t)dt$$

$$= \frac{1}{\pi}\int e^{-t}\sin\lambda t\,dt \Rightarrow \frac{1}{\pi}\left[\frac{\sin\lambda t e^{-t}}{-1} - \int \frac{\lambda\cos\lambda t e^{-t}dt}{-1}\right]$$

$$= -\frac{\sin\lambda t e^{-t}}{\pi} + \frac{\lambda}{\pi}\cos\lambda t e^{-t} - \frac{\lambda^2}{\pi}\int_0^{\infty}\sin\lambda t e^{-t}dt$$

$$B(\lambda) = \left[-\frac{\sin\lambda t e^{-t}}{\pi} - \frac{\lambda\cos\lambda t e^{-t}}{\pi}\right]_0^{\infty}$$

$$= \frac{\lambda}{\pi(1+\lambda^2)}$$

$\therefore$ Integral representation is,

$$\int_0^{\infty} \frac{\cos(\lambda x)}{\pi(1+\lambda^2)} + \frac{\lambda\sin(\lambda x)}{\pi(1+\lambda^2)}\,.d\lambda$$

(iv) $f(x) = \begin{cases} e^{-|x|} & ; |x|<1 \\ 0 & \text{otherwise.} \end{cases}$

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cdot\cos\lambda t\,.dt \; ; B(\lambda) = \frac{1}{\pi}\int_{-\infty} f(t)\sin\lambda t\,dt$$

$$\therefore A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\lambda t\,dt = \frac{1}{\pi}\int_{-1}^{1} f(t)\cos\lambda t\,dt.$$

$$= \frac{2}{\pi}\left[\frac{\cos\lambda t e^{-t}}{-1} + \int\frac{\lambda\sin\lambda t e^{-t}}{-1}dt\right]_0^1$$

$$A(\lambda) = \frac{-2}{\pi}\cos\lambda t e^{-t} + \frac{2\lambda}{\pi}\sin\lambda t e^{-t}$$

$$= \frac{-2\cos(\lambda)e^{-1} + 2\lambda\sin\lambda e^{-1} + 2}{\pi(1+\lambda^2)}$$

$$B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cdot\sin\lambda t\,dt$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-|t|}\sin\lambda t\,dt, = 0$$

$\therefore$ Integral will be,

$$\int_0^{\infty}\left[\frac{-2\cos\lambda e^{-1} + 2\lambda\sin\lambda e^{-1} + 2}{\pi(1+\lambda^2)}\right]\cos\lambda x\,dx$$

2) i) $f(x) = \begin{cases} x & \\ 0 & x>5 \end{cases}$

cosine integral is given by,

$\int_0^\infty A(\lambda)\cdot\cos\lambda x\,d\lambda$; $A(\lambda) = \frac{2}{\pi}\int_0^\infty f(t)\cdot\cos(\lambda t)\,dt$

$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(t)\cdot\cos(\lambda t)\,dt$

$= \frac{2}{\pi}\int_0^5 t^2\cdot\cos(\lambda t)\,dt$

$= \frac{2}{\pi}\left[\frac{t^2\sin\lambda t}{\lambda} - \cancel{2}\int\frac{t\sin\lambda t}{\lambda}\,dt\right.$

$= \frac{2t^2\sin\lambda t}{\pi\lambda} \mp \frac{4}{\pi\lambda}\int t\sin\lambda t\,dt$

$= \frac{2t^2\sin\lambda t}{\pi\lambda} + \frac{4t}{\pi\lambda^2}\cdot\cos\lambda t - \frac{4}{\pi\lambda^2}\int\cos\lambda t\,dt$

$= \left[\frac{2t^2\sin\lambda}{\pi\lambda} + \frac{4t\cos\lambda t}{\pi\lambda^2} - \frac{4}{\pi\lambda^3}\cdot\sin\lambda t\right]_0^5$

$\therefore A(\lambda) = \frac{50\sin(5\lambda)}{\pi\lambda} + \frac{20\cos(5\lambda)}{\pi\lambda^2} - \frac{4\sin(5\lambda)}{\pi\lambda^3}$

$\therefore$ Fourier integral is,

$\int_0^\infty\left[\frac{50\sin(5\lambda)}{\pi\lambda} + \frac{20\cos(5\lambda)}{\pi\lambda^2} - \frac{4\sin(5\lambda)}{\pi\lambda^3}\right]\cos\lambda$

ii) $f(x) = \begin{cases} \sin x & 0<x<\pi \\ 0 & x>\pi \end{cases}$

cosine integral is given by,

$\int_0^\infty A(\lambda)\cos\lambda x\,d\lambda$; $A(\lambda) = \frac{2}{\pi}\int_0^\infty A(t)\cdot\cos(\lambda t)\,d\lambda$

$A(\lambda) = \frac{2}{\pi}\int_{-\pi}^{\pi} f(t)\cdot\cos\lambda t\,dt$

$= \frac{2}{\pi}\int_{\pi}^{\pi}\sin t\cos\lambda t\,dt = \frac{1}{\pi}\int \sin(1+\lambda)t + \sin(1-\lambda)t$

$$\frac{1}{\pi}\left[\frac{\quad}{(1+\lambda)} \quad \frac{\quad}{(1-\lambda)}\right]_0$$

$$= \frac{1}{\pi}\left[\frac{\cos\lambda\pi}{1+\lambda} + \frac{1}{1+\lambda} + \frac{\cos\lambda\pi}{1+\lambda} + \frac{1}{1-\lambda}\right]$$

$$= \frac{2}{\pi}\left(\frac{1+\cos\lambda\pi}{1-\lambda^2}\right)$$

$\therefore$ Fourier integral is $\displaystyle\int_0^\infty \frac{2}{\pi}\frac{(1+\cos\lambda\pi)}{(1-\lambda^2)}\cos\lambda x \, d\lambda$

3) i) $f(x) = \begin{cases} x & ; 0 < x \le 2 \\ 0 & ; x > 0 \end{cases}$

Sine Integral is given by,

$$\int_0^\infty B(\lambda)\cdot \sin\lambda x \, d\lambda$$

For $B(\lambda) = \dfrac{2}{\pi}\displaystyle\int_0^2 t\sin\lambda t \, dt$

$$= \frac{2}{\pi}\left[\frac{-t\cos\lambda t}{\lambda} + \int (1)\frac{\cos\lambda t}{\lambda} dt\right]$$

$$= \frac{2}{\pi}\left[\frac{-t\cos\lambda t}{\lambda} + \frac{1}{\lambda}\frac{\sin\lambda t}{\lambda}\right]_0^2$$

$$= \left[\frac{\sin 2\lambda}{\lambda^2} - \frac{2\cos 2\lambda}{\lambda^2}\right]\frac{2}{\pi}$$

$$= \frac{2(\sin 2\lambda - 2\lambda\cos 2\lambda)}{\pi\lambda^2}$$

$\therefore$ Integral is $\displaystyle\int_0^\infty \frac{2(\sin 2\lambda - 2\lambda\cos 2\lambda)}{\pi\lambda^2}\sin\lambda x \, d\lambda$

(ii) $f(x) = \begin{cases} \sinh x & 0 \le x \le 3 \\ 0 & x > 3 \end{cases}$

Sine Integral given by,

$$\int_0^\infty B(\lambda)\cdot \sin\lambda x \, dx$$

$$B(\lambda) = \frac{2}{\pi}\int_0^{\infty} f(t)\cdot \sin\lambda t\,dt$$

$$\therefore B(\lambda) = \frac{2}{\pi}\int_0^{3} \sinh t \sin\lambda t\,dt$$

$$= \frac{1}{\pi}\int_0^{3} e^{t}\sin\lambda t - \frac{1}{\pi}\int_0^{3} e^{-t}\sin\lambda t\,dt$$

Let $I_1 = \int_0^{3} e^{at}\sin\lambda t\,dt$

$$I_1 = \frac{\sin\lambda t\cdot e^{at}}{a} - \frac{\lambda}{a^2}\cdot\cos\lambda t\, e^{at} - \frac{\lambda^2}{a^2}\int \sin\lambda t$$

$$I_1 = \frac{\frac{\sin 3\lambda\cdot e^{3a}}{a} - \frac{\lambda}{a^2}\cdot\cos 3\lambda e^{3a}}{\frac{1+\lambda^2}{a^2}} + \frac{\lambda}{a^2}$$

$\therefore$ for $a=1$

$$= \frac{\sin 3\lambda e^{3} - \lambda\cos 3\lambda e^{3}}{1+\lambda^2} + \lambda$$

and for $a=-1$

$$= \frac{-\sin 3\lambda e^{-3a} - \lambda\cos 3\lambda e^{-3a}}{1+\lambda^2} + \lambda.$$

$$= \frac{2\sin 3\lambda\cosh 3}{\pi(1+\lambda^2)} - \frac{2\lambda\cos 3\lambda\sinh 3}{\pi(1+\lambda^2)}$$

(iii) $f(x) = \begin{cases} 0 & 0\le x<1 \\ 1 & 1<x\le 2 \\ 0 & x>2 \end{cases}$

$$\int_0^{\infty} B(\lambda)\cdot\sin\lambda x\,d\lambda$$

$$B(\lambda) = \frac{-2}{\pi}\int_0^{\infty} f(t)\sin\lambda t\,dt$$

$$\therefore B(\lambda) = \frac{2}{\lambda}\int_0^{1} 0\,dt + \frac{2}{\pi}\int_1^{2} 1\sin\lambda t\,dt + \frac{2}{\pi}\int_2^{\infty} 0\,dt$$

$$= \frac{2}{\pi}\int_1^{2} \sin\lambda t\cdot dt.$$

$$= \frac{2}{\pi}\left[\frac{-\cos \lambda t}{\lambda}\right]_1$$

$$= \frac{2}{\pi}\left[\frac{-\cos 2\lambda}{\lambda} - \left(\frac{-\cos\lambda}{\lambda}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{-\cos 2\lambda}{\lambda} + \frac{\cos\lambda}{\lambda}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos\lambda - \cos 2\lambda}{\lambda}\right]$$

$\because B(0) = \frac{2}{\pi}\int_1^2 1 \sin(0) dt = 0$

$\therefore$ The sine integral is given as

$$\int_0^\infty \frac{2}{\pi}\left[\frac{\cos\lambda - \cos 2\lambda}{\lambda}\right] \sin\lambda t \cdot dt.$$

4) Consider, $f(x) = \frac{\pi}{2} e^{-x}$

Sol:- we find its cosine integral

we find $A(\lambda)$.

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(t)\cdot \cos\lambda t\, dt$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\pi}{2}\cdot e^{-t} \cos\lambda t\, dt$$

$$= \int_0^\infty e^{-t} \cos\lambda t\, dt$$

$$= \frac{\cos\lambda t \cdot e^{-t}}{-1} + \int \frac{\lambda \sin\lambda t\, e^{-t}}{-1} dt$$

$$A(\lambda) = -\cos(\lambda t) e^{-t} + \lambda \sin(\lambda t) e^{-t} - \lambda^2 \int \cos\lambda t\, e^{-t} dt$$

$$\therefore A(\lambda)[1+\lambda^2] = \left[-\cos(\lambda t) e^{-t} + \lambda \sin(\lambda t) e^{-t}\right]_0^\infty$$

$$\therefore A(\lambda) = 0 + 0 - [-1+0] / 1+\lambda^2$$

$$= \frac{1}{1+\lambda^2}$$

$\therefore$ cosine integral is,

$$\int_0^\infty A(\lambda)\cdot \cos\lambda x\, d\lambda$$

$$\Rightarrow \int_0^\infty \frac{\cos\lambda x}{1+\lambda^2}\cdot d\lambda$$

By fourier integral theorm, the integral converges to $\frac{f(x^+)+f(x^-)}{2}$, b the f^n $\frac{\pi}{2}e^{-x}$ is continous for all $x>0$, so integral function conver

so, $\int \frac{\cos\lambda x}{1+\lambda^2}\, d\lambda = \frac{\pi}{2}e^{-x}; \forall x>0$,

Hence proved.

⑤ To prove,

$$e^{-x} - e^{-2x} = \frac{6}{\pi}\int_0^\infty \frac{\lambda \sin\lambda x}{(1+\lambda^2)(4+\lambda^2)}\, d\lambda$$

consider,

$$f(x) = e^{-x} - e^{-2x}$$

$$B(\lambda) = \frac{2}{\pi}\int_0^\infty f(t)\sin\lambda t\, dt$$

$$= \frac{2}{\pi}\int_0^\infty (e^{-t} - e^{-2t})\sin\lambda t\, dt$$

$$= \frac{2}{\pi}\left[\int_0^\infty e^{-t}\sin\lambda t\, dt - \int_0^\infty e^{-2t}\sin\lambda t\, dt\right]$$

we know that,

$$L\{\sin at\} = \int_0^\infty e^{-st}\sin at\, dt$$

$$= \frac{a}{s^2+a^2}$$

we get, $B(\lambda) = \frac{2}{\pi}[F(1) - F(2)]$

$L\{\sin at\}$.

$$= \frac{2}{\pi}\left[\frac{\lambda}{1+\lambda^2} - \frac{\lambda}{\lambda^2+2^2}\right]$$

$$= \frac{2\lambda}{\pi}\left(\frac{\lambda^2+4-\lambda^2-1}{(1^2+\lambda^2)(\lambda^2+2^2)}\right)$$

$$= \frac{6\lambda}{\pi(1+\lambda^2)(\lambda^2+2^2)}$$

$\because$ Fourier sine integral is,

$$\int_0^\infty \frac{6\lambda \sin\lambda x}{\pi(1+\lambda^2)(2^2+\lambda^2)} d\lambda$$

By theorm, the value converges to $\frac{f(x^+)+f(x^-)}{2}$, but as $f(x)$ is continous, we get,

$$e^{-x} + e^{-2x} = \int_0^\infty \frac{6\lambda \sin\lambda x}{\pi(\lambda^2+1)(\lambda^2+2^2)} d\lambda$$

(6) $f(x) = \begin{cases} 0 & x<0 \\ 1/2 & x=0 \\ e^{-x} & x>0 \end{cases}$

Fourier integral is,

$$\int_0^\infty A(x)\cos\lambda x + B(\lambda)\sin\lambda x \, d\lambda$$

$$\therefore A(\lambda) = \frac{1}{\pi}\int_0^\infty f(t)\cdot\cos\lambda t\, dt$$

$$B(\lambda) = \frac{1}{\pi}\int_0^\infty f(t)\cdot\sin\lambda t\, dt$$

$$A(\lambda) = \frac{1}{\pi}\int_0^\infty e^{-t}.\cos\lambda t\, dt$$

$$= \frac{1}{\pi}\left[\frac{\cos\lambda t\, e^{-t}}{-1} + \int \frac{\lambda \sin\lambda t e^{-t}}{-1} dt\right]$$

$$= \frac{-\cos\lambda t\, e^{-t}}{\pi} - \frac{\lambda}{\pi}\left[\frac{\sin\lambda t\, e^{-t}}{-1} - \int \frac{\lambda \cos\lambda t}{-1}\right.$$

$$A(\lambda) = \left[\frac{-\cos(\lambda t)e^{-t}}{\pi} + \frac{\lambda}{\pi}\sin(\lambda t)e^{-t}\right]_0^\infty$$

$$= 0 + 0 - \left(-\frac{1}{\pi} + 0\right) \Big/ 1+\lambda^2$$

$$\therefore A(\lambda) = \frac{1}{\pi(1+\lambda^2)}$$

$$B(\lambda) = \frac{1}{\pi}\int_0^\infty e^{-t}\sin\lambda t\, dt$$

$$= \frac{1}{\pi}\left[\frac{\sin\lambda t}{-1}.e^{-t} - \int \frac{\lambda\cos\lambda t e^{-t}}{-1} dt\right]$$

$$= -\frac{\sin(\lambda t)e^{-t}}{\pi} + \frac{\lambda}{\pi}\left[\frac{\cos\lambda t e^{-t}}{-1}\right] + \frac{\lambda \sin\lambda t}{-1}$$

$$B(\lambda) = \left[\frac{-\frac{\sin(\lambda t)e^{-t}}{\pi} - \frac{\lambda\cos(\lambda t)e^{-t}}{\pi}}{1+\lambda^2}\right]_0^\infty$$

$$= 0 + 0 - \left[0 - \frac{\lambda}{\pi}\right] \Big/ 1+\lambda^2$$

$$B(\lambda) = \frac{\lambda}{\pi(1+\lambda^2)}$$

$\therefore$ Fourier integral is,

$$\int_0^\infty \left[\frac{\cos\lambda(x)}{\pi(1+\lambda^2)} + \frac{\lambda\sin(\lambda x)}{\pi(1+\lambda^2)}\right] d\lambda.$$

Q) 1) $f(x) = \begin{cases} |x| & -\pi \le x \le \pi \\ 0 & \text{otherwise.} \end{cases}$

complex fourier integral is given by

$$\int_{-\infty}^{\infty} C(\lambda)\cdot e^{i\lambda x}\, d\lambda \; ; \; C(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\cdot e^{-i\lambda t} dt$$

$$\therefore C(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\cdot e^{-i\lambda t} dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |t|\, e^{-i\lambda t} dt$$

$$= \frac{-1}{2\pi}\int_{-\pi}^{0} t\cdot e^{-i\lambda t} dt + \frac{1}{2\pi}\int_{0}^{\pi} t\cdot e^{-i\lambda t}\, dt$$

$$= \frac{1}{2\pi}\left[\left[\frac{t\, e^{-i\lambda t}}{-i\lambda}\right] - \int \frac{(1) e^{-i\lambda t}}{-i\lambda}\right]_0^{\pi}$$

$$= \frac{-1}{2\pi}\left[\frac{-t\cdot e^{-i\lambda t}}{-i\lambda} + \frac{e^{-i\lambda t}}{-(i\lambda)^2}\right]_{-\pi}^{0} +$$

$$\frac{1}{2\pi}\left[\frac{-t\cdot e^{-i\lambda\pi}}{i\lambda} + \frac{e^{-i\pi\lambda}}{i\lambda} - \frac{1}{\lambda^2}\right]$$

$$= \frac{-1}{2\pi}\left\{\frac{1}{\lambda^2} - \left(\frac{\pi e^{i\lambda\pi}}{i\lambda} + \frac{e^{i\lambda\pi}}{\lambda^2}\right)\right\} +$$

$$\frac{1}{2\pi}\left(\frac{-\pi e^{-i\lambda\pi}}{i\lambda} + \frac{e^{-i2\pi}}{\lambda^2} - \frac{1}{\lambda^2}\right]$$

$$= \frac{e^{i\lambda\pi} + e^{-i\lambda\pi}}{2\pi\lambda^2} + \frac{e^{i\lambda\pi} - e^{-i\lambda\pi}}{2i\lambda} - \frac{1}{2\pi\lambda^2} - \frac{1}{2\pi\lambda^2}$$

$$= \frac{\cos(\lambda\pi)}{\pi\lambda^2} + \frac{\sin\lambda\pi}{\lambda} - \frac{1}{\pi\lambda^2}$$

$\therefore$ fourier integral is,

$$\int_{-\infty}^{\infty}\left[\frac{\cos\lambda\pi}{\pi\lambda^2} + \frac{\sin\lambda\pi}{\lambda} - \frac{1}{\pi\lambda^2}\right] e^{i\lambda x}\, d\lambda)$$

10) $f(x)$ is defined on $(0,\infty)$. Fourier cosi[ne]
is $A(\lambda)$, then at pt of continuity sh[ow]
that $x^2 f(x) = \frac{2}{\pi}\int_0^\infty A^*(\lambda)\cos\lambda x\,d\lambda$,
where $A^*(\lambda) = -A''(\lambda)$

→ The fourier cosine integral of $x^2 f$

$$\int_0^\infty A^*(x)\cdot\cos(\lambda x)\,d\lambda$$

$$A^*(\lambda) = \frac{2}{\pi}\int_0^\infty t^2 f(t)\cdot\cos(\lambda t)\,dt$$

By convergence theorm,

$$x^2 f(x) = \int_0^\infty A^*(\lambda)\cdot\cos(\lambda x)\,d\lambda$$

Now prove,

$$A^* = -A''$$

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(t)\cdot\cos(\lambda t)\,dt$$

$$A'(\lambda) = \frac{2}{\pi}\frac{d}{d\lambda}\int_0^\infty f(t)\cdot\cos\lambda t\,dt$$

$$= \frac{2}{\pi}\int_0^\infty f(t)\cdot\frac{d}{d\lambda}\cos\lambda t\,dt$$

$$= -\frac{2}{\pi}\int_0^\infty t f(t)\cdot\sin\lambda t\,dt$$

$$A''(\lambda) = -\frac{2}{\pi}\cdot\frac{d}{d\lambda}\int_0^\infty t f(t)\sin\lambda t\,dt$$

$$= -\frac{2}{\pi}\int_0^\infty f(t)\cdot t\cdot\frac{d}{d\lambda}\sin\lambda t\,dt$$

$$= -\frac{2}{\pi}\int_0^\infty t^2 f(t)\cdot\cos\lambda t\,dt$$

$$\therefore A''(\lambda) = -A^*(\lambda)$$

Hence pro[ved]

(B)

1) i) e^{-at^2}

$$F(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cdot e^{-i\lambda t}dt$$

$$\therefore \hat{F}(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-at^2}\cdot e^{-i\lambda t}dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(at^2+i\lambda t)}dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(t^2+\frac{i\lambda t}{a}\right)}dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(t+\frac{i\lambda}{a}\right)^2}\cdot e^{\left(\frac{i\lambda}{4a}\right)^2}dt$$

$$=\frac{e^{-\frac{\lambda^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(t+\frac{i\lambda}{2a}\right)^2}\cdot dt$$

Let $u=\sqrt{a}\left(t+\frac{i\lambda}{2a}\right)$

$du=\sqrt{a}\cdot dt$

$t=\infty \rightarrow u=\infty$

$t=-\infty \rightarrow u=-\infty$

$$\Rightarrow \frac{e^{-\frac{\lambda^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-u^2}du}{\sqrt{a}}$$

$$\Rightarrow \because \int_{-\infty}^{\infty} e^{-u^2}du=\sqrt{\pi}$$

$$\therefore f(\lambda)=\frac{2e^{-\frac{\lambda^2}{4a}}}{\sqrt{2a}}$$

(ii) $e^{-a|t|}$

$$\hat{f}(\lambda)=\frac{1}{\sqrt{2\lambda}}\int_{-\infty}^{\infty} f(t)\cdot e^{-i\lambda t}dt$$

$$f(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|t|}\cdot e^{-i\lambda t}dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{at}\cdot e^{-i\lambda t}dt+\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-at}\cdot e^{-i\lambda t}dt$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{(a-i\lambda)t}}{(a-i\lambda)}\right]_{\infty} + \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-(a+i\lambda)t}}{-(a+i\lambda)}\right]_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a-i\lambda}\right] + \frac{1}{\sqrt{2\pi}}\left[0 - \frac{1}{-(a+i\lambda)}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a-i\lambda} + \frac{1}{a+i\lambda}\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{a+i\lambda+a-i\lambda}{a^2+\lambda^2}\right]$$

$$= \frac{2a}{\sqrt{2\pi}(a^2+\lambda^2)} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+\lambda^2}\right)$$

(2) i) $y' - 4y = H(t)\cdot e^{-4t}$

Let $F\{y\} = Y$

Fourier transform on both sides,

$$F\{y' - 4y\} = F\{H(t)\cdot e^{-4t}\}$$

$$F\{y'\} - 4F\{y\} = F\{H(t)\cdot e^{-4t}\}$$

$$\Rightarrow (i\lambda)Y - 4Y = \frac{1}{\sqrt{2\pi}(4+i\lambda)}$$

$$Y = \frac{1}{\sqrt{2\pi}(4+i\lambda)(i\lambda-4)}$$

$$= \frac{1}{\sqrt{2\pi}(i\lambda^2-16)} = \frac{-1}{\sqrt{2\pi}(\lambda^2+16)}$$

$$\therefore Y = \frac{-1}{\sqrt{2\pi}(\lambda^2+16)}$$

$$= \frac{-1}{\sqrt{2\pi}(4+i\lambda)(4-i\lambda)}$$

$$= \frac{-1}{8\sqrt{2\pi}}\left[\frac{1}{4+i\lambda} + \frac{1}{4-i\lambda}\right]$$

$$= \frac{-1}{8\sqrt{2\pi}(4+i\lambda)} - \frac{1}{8\sqrt{2\pi}(4-i\lambda)}$$

$$y = F^{-1}\{Y\}$$

$$= \frac{-1}{8}\left(H(t)\cdot e^{-4t} + H(t)\, e^{4t}\right)$$

$$= -\frac{H(t)}{8}\left(e^{-4t} + e^{4t}\right)$$

(ii) $y'' + 5y' + 4y' = \delta(t-2)$

let $F\{y\} = Y$

Fourier transform,

$$F\{y'' + 5y' + 4y\} = F\{\delta(t-2)\}$$

$$F\{y''\} + 5F'\{y'\} + 4F\{y\} = F\{\delta(t-2)\}$$

$$(i\lambda)^2 Y + 5(i\lambda)Y + 4Y = \frac{e^{-i\lambda 2}}{\sqrt{2\pi}}$$

$$Y\left[(i\lambda)^2 + 5(i\lambda) + 4\right] = \frac{e^{-i2\lambda}}{\sqrt{2\pi}}$$

$$Y = \frac{e^{-i2\lambda}}{\sqrt{2\pi}\left((i\lambda)^2 + 5(i\lambda) + 4\right)}$$

$$= \frac{e^{-2i\lambda}}{3\sqrt{2\pi}}\left[\frac{1}{i\lambda+1} - \frac{1}{i\lambda+4}\right]$$

$$= \frac{e^{-2i\lambda}}{3\sqrt{2\pi}(i\lambda+1)} - \frac{e^{-2i\lambda}}{3\sqrt{2\pi}(i\lambda+4)}$$

take ($t_0 = 2$)

we get,

$$y = \frac{1}{3}\left[F^{-1}\left\{ \frac{e^{-2i\lambda}}{\sqrt{2\pi}\,(i\lambda+1)} \right\} - F^{-1}\left\{ \frac{e^{-2i\lambda}}{\sqrt{2\pi}\,(i\lambda+4)} \right\} \right]$$

$$= \frac{1}{3}\left[f(t-2) - f_1(t-2) \right]$$

$$\therefore F^{-1}\left\{ \frac{1}{\sqrt{2\pi}\,(a+i\lambda)} \right\} = H(t)\cdot e^{-at}$$

$$\therefore y = \frac{H(t-2)}{3}\left[e^{-(t-2)} - e^{-4(t-2)} \right]$$

③ Let $F\{f(t)\} = F(\omega)$ & $F(0)=0$

Then prove,

$$F\left[\int_{-\infty}^{t} f(\tau)\,d\tau \right] = \frac{1}{i\omega} F(\omega)$$

$\therefore$ By defination,

$$F\{f(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\cdot e^{-i\lambda t}\,dt$$

$$\therefore F\left\{ \int_{-\infty}^{t} f(t)\cdot d\tau \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{\infty}^{t} f(\tau)\,d\tau \cdot e^{-i\lambda t}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} f(\tau)\cdot dt\,\frac{e^{-i\lambda t}}{-i\lambda} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{f(t)\cdot e^{-i\lambda t}\,dt}{i\lambda} \Bigg\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{ 0 - 0 + \frac{1}{i\lambda}\int_{-\infty}^{\infty} f(t)\cdot e^{-i\lambda t}\,dt \right\}$$

$$= \frac{1}{i\lambda} F(\lambda) = \frac{F(\omega)}{i\omega}$$

Hence proved.

4) Let $F\{f(t)\} = F(\omega)$

To prove,

$$F[f(t) \cdot \sin(\omega_0 t)] = \frac{1}{2}[F(\omega+\omega_0) - F(\omega-\omega_0)]$$

We know that,

$$\sin(\omega_0 t) = \frac{e^{-i\omega_0 t} + e^{i\omega_0 t}}{2}$$

$$\Rightarrow F[f(t) \cdot \sin\omega_0 t]$$

$$= F\left[f(t) \cdot \left(\frac{e^{-i\omega_0 t} + e^{i\omega_0 t}}{2}\right)\right]$$

$$= \frac{1}{2\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} f(t) \cdot (e^{-i\omega_0 t} + e^{i\omega_0 t}) e^{-i\omega t} dt\right]$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{-i(\omega_0+\omega)t} dt + \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{i\omega_0 t - i\omega t} dt.$$

$$= \frac{1}{2}[F(\omega+\omega_0) + F(\omega-\omega_0)]$$

$= RHS$

Hence proved.

5) To prove,

If $F\{f(t)\} = F(\lambda)$ then $F(F(t)) = f(-\lambda)$

By basic defination,

$$f(x) = F^{-1}(F(\lambda)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\lambda) \cdot e^{i\lambda x} d\lambda$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot e^{isx} ds$$

Now let $t = -\lambda$, we get

$$f(-\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot e^{is(-\lambda)} \cdot ds$$

(q) To prove,

$$\int_{-\infty}^{\infty} \frac{dt}{(2-it+i\lambda)(2+it)} = \frac{2\pi}{4+i\lambda}$$

frequency convolution,

let 'f' and 'g' be two functions defined on $(-\infty, \infty)$ such that,

$\int_{-\infty}^{\infty} |f(t)| dt$, $\int_{-\infty}^{\infty} |g(t)| dt$ exists and

$F(f(t)) = F(\lambda)$, $F(g(t)) = G(\lambda)$, then

$F(f(t) \cdot g(t)) = (F * G)(\lambda)$

consider,

$$LHS = \int_{-\infty}^{\infty} \frac{dt}{(2-it+i\lambda)(2+it)}$$

$F(\lambda - t) = \frac{1}{2+i(\lambda-t)}$, $G(t) = \frac{1}{2+it}$

i) $F^{-1}\left[\frac{1}{a+i\lambda}\right] = \sqrt{2\pi} \cdot e^{-at} H(t)$

$F(e^{-i\lambda t} f(t)) = \hat{f}(\lambda - \lambda_0)$

$f(t) = \sqrt{2\pi}\, e^{-at} H(t) \cdot e^{-i\lambda t}$

$g(t) = \sqrt{2\pi} \cdot e^{-at} H(t)$.

we get,

$$\int_{-\infty}^{\infty} \frac{dt}{(2-it+i\lambda)(2+it)} = \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(t) e^{-i\lambda t} dt$$

$$= \int_{-\infty}^{\infty} \sqrt{2\pi} e^{-2t} (H(t)) \sqrt{2\pi}\, e^{-2t} e^{-i\lambda t} dt$$

$$= (\sqrt{2\pi})^2 \int_{-\infty}^{\infty} e^{-4t} H(t)\, e^{-i\lambda t}\, dt$$

$$= 2\pi \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2\pi}\, e^{-4t} H(t)\, e^{-i\lambda t}\, dt$$

$$= 2\pi\, F\left[\sqrt{2\pi}\, e^{-4t} H(t)\right]$$

$$= \frac{2\pi}{4+i\lambda}$$

$= R.H.S$ Hence proved.