Interval Analysis

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Sommaire

Interval analysis

Provides efficient techniques to

- perform guaranteed deterministic global optimization,
- evaluate all solutions of a set of nonlinear equations
- compute inner and outer approximation of the set of vectors consistent with a set of inequalities
- ...

Interval analysis

Has lead to numerous applications

- Bounded-error parameter and state estimation of nonlinear systems
- Robust bounded-error parameter and state estimation
- Parameter estimation by global optimization
- Structural identifiability study
- Distributed estimation
- ...

Interval arithmetic primer

Introduced by Sunaga in Japan and by Moore in the USA.

Limited impact until beginning of the 90s
$\Longrightarrow$ various reasons, among which implementation issues

Many books, code libraries, lists

`http://www.cs.utep.edu/interval-comp/main.html`

Ongoing standardization process IEEE P1788.

Interval of real numbers

Closed and *bounded* subset of $\mathbb{R}$

$$[x] = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} | \underline{x} \leq x \leq \overline{x}\} .$$

It is a set $\Longrightarrow$ notions such as

$$=, \in, \subset, \cap$$

are well defined.
When considering $\cup$

$$[x] \cup [y] = \left[\min(\underline{x}, \underline{y}), \max(\overline{x}, \overline{y})\right] .$$

Interval of real numbers

Other characteristics of an interval

Width

$$w\left([x]\right) = \overline{x} - \underline{x},$$

Midpoint

$$m\left([x]\right) = \frac{\underline{x} + \overline{x}}{2}.$$

Basic operations

Extended to intervals

$$\circ \in \{+,-,\times,/\},\ [x]\circ[y] = \{x \circ y | x \in [x] \text{ and } y \in [y]\}.$$

More specifically

$$\begin{cases} [x]+[y] = \left[\underline{x}+\underline{y}, \overline{x}+\overline{y}\right], \\ [x]-[y] = \left[\underline{x}-\overline{y}, \overline{x}-\underline{y}\right], \\ [x]\times[y] = \left[\min\left(\underline{x}.\underline{y}, \overline{x}.\underline{y}, \underline{x}.\overline{y}, \overline{x}.\overline{y}\right), \max\left(\underline{x}.\underline{y}, \overline{x}.\underline{y}, \underline{x}.\overline{y}, \overline{x}.\overline{y}\right)\right], \\ [x]\,/\,[y] = [x]\times\left[1/\overline{y}, 1/\underline{y}\right], \text{ if } 0 \notin [y] \text{ and undefined else..} \end{cases}$$

Inclusion function

Range of a function over an interval

$$f([x]) = \{f(x) \,|x \in [x]\}$$

$\Longrightarrow$ difficult to obtain in general
$\Longrightarrow$ sometimes even not an interval

Inclusion function $[f](.)$ of $f(.)$ satisfies

$$\forall [x] \subset \mathbb{R},\; f([x]) \subset [f]([x]).$$

Inclusion function is minimal if $\subset$ may be replaced by $=$.

Convergent inclusion function

$$\text{if } w([x]) \to 0, \text{ then } w([f]([x])) \to 0.$$

Inclusion function

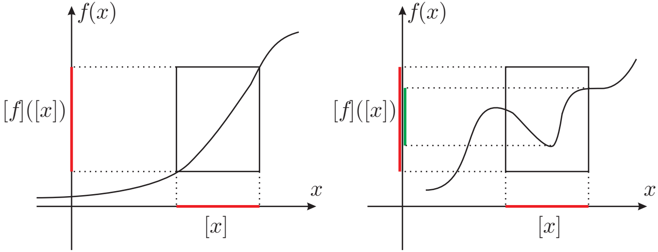
Inclusion function easy to build for monotone functions

$$
\begin{aligned}
\sqrt{[x]} &= \left[\sqrt{\underline{x}}, \sqrt{\overline{x}}\right], \text{ if } \underline{x} \geq 0, \\
\exp([x]) &= [\exp(\underline{x}), \exp(\overline{x})], \\
\tan([x]) &= [\tan(\underline{x}), \tan(\overline{x})], \text{ if } [x] \subseteq [-\pi/2, \pi/2].
\end{aligned}
$$

More complicated for other elementary functions
$\Longrightarrow$ algorithm required for $\sin, \cos, \ldots$
$\Longrightarrow$ natural inclusion function

Inclusion function

Usually, an inclusion function is not minimal



$\Longrightarrow$ some overestimation of the range (pessimism).

Natural inclusion function

$\Downarrow$

Remplace each real variable by its interval counterpart

$x \longrightarrow [x]$

Example

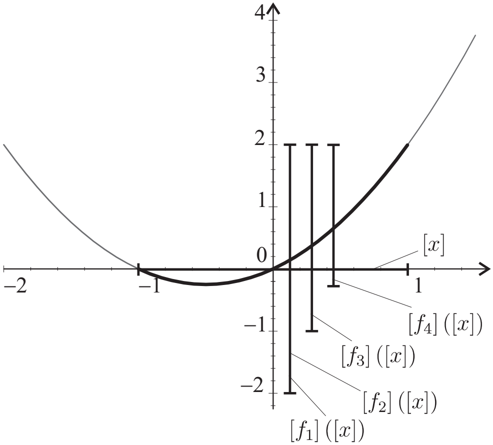
$$f_1(x) = x(x+1), \qquad f_3(x) = x^2 + x,$$
$$f_2(x) = x \times x + x, \quad f_4(x) = (x + \tfrac{1}{2})^2 - \tfrac{1}{4}.$$

Results for $[x] = [-1, 1]$

$$\begin{aligned}
[f_1]([x]) &= [x]([x]+1) = [-2, 2], \\
[f_2]([x]) &= [x] \times [x] + [x] = [-2, 2], \\
[f_3]([x]) &= [x]^2 + [x] = [-1, 2], \\
[f_4]([x]) &= \left([x] + \tfrac{1}{2}\right)^2 - \tfrac{1}{4} = \left[-\tfrac{1}{4}, 2\right].
\end{aligned}$$

Only $[f_4](.)$ is minimal $\Longleftrightarrow$ minimum number of occurences of the interval variable

Example



Centred form

For $f : \mathcal{D} \longrightarrow \mathbb{R}$, differentiable over $[x] \subset \mathcal{D}$, one has $\forall x, m \in [x]\,, \exists \xi \in [x]$ such that

$$f\left(x\right) = f\left(m\right) + \left(x - m\right) f'\left(\xi\right).$$

Then

$$f\left(x\right) \in f\left(m\right) + \left(x - m\right) f'\left([x]\right),$$

and

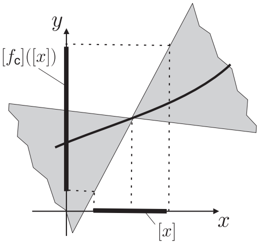
$$f\left([x]\right) \subseteq f\left(m\right) + \left([x] - m\right) \left[f'\right]\left([x]\right).$$

Centred form is the inclusion function defined by

$$[f]_c\left([x]\right) = f\left(m\right) + \left([x] - m\right) \left[f'\right]\left([x]\right)$$

Centred form

Interpretation of the centred form



Example

Consider

$$f(x) = x^2 \exp(x) - x \exp\left(x^2\right).$$

Compare the natural inclusion fonction and the centred form

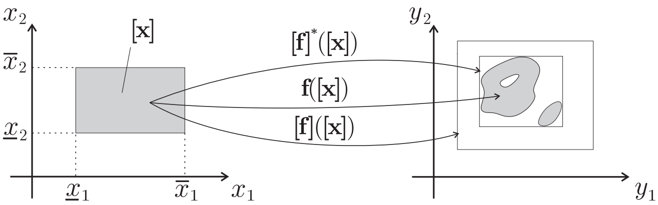
$[x]$	$f([x])$	$[f]([x])$	$[f]_c([x])$
$[0.5, 1.5]$	$[-4.148, 0]$	$[-13.82, 9.44]$	$[-25.07, 25.07]$
$[0.9, 1.1]$	$[-0.05380, 0]$	$[-1.697, 1.612]$	$[-0.5050, 0.5050]$
$[0.99, 1.01]$	$[-0.0004192, 0]$	$[-0.1636, 0.1628]$	$[-0.004656, 0.004656]$

Extension to vectors of intervals

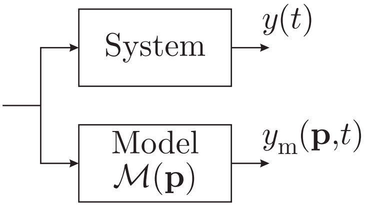
Vector of intervals or box

$$[\mathbf{x}] = [x_1] \times \cdots \times [x_n] \, .$$

Vector inclusion function



Parameter estimation



$\mathbf{y}$: vector of experimental data
$\mathbf{p}$: vector of unknown, constant parameters
$\mathbf{y}_m(\mathbf{p})$: vector of model output

Parameter estimation :
Determination of $\widehat{\mathbf{p}}$ from $\mathbf{y}$.

Problem formulation

1. Minimisation of a cost function, *e.g.*,

$$\widehat{\mathbf{p}} = \arg\min_{\mathbf{p}} j(\mathbf{p}) = (\mathbf{y} - \mathbf{y}_{\mathrm{m}}(\mathbf{p}))^{\mathrm{T}} (\mathbf{y} - \mathbf{y}_{\mathrm{m}}(\mathbf{p}))$$

   - Local techniques : Gauss-Newton, Levenberg-Marquardt...
   - Random search : simulated annealing, genetic algorithms...
   - Global guaranteed techniques : Hansen's algorithm

2.

Parameter bounding

Experimental data : $y(t_i)$,

t_i, $i = 1 \ldots, N$, known measurement times

$[\varepsilon_i] = [\underline{\varepsilon}_i, \overline{\varepsilon}_i]$, $i = 1, \ldots, N$, known acceptable errors

$\mathbf{p} \in \mathcal{P}_0$ deemed acceptable if for all $i = 1, \ldots, N$,

$$\underline{\varepsilon}_i \leqslant y(t_i) - y_{\mathrm{m}}(\mathbf{p}, t_i) \leqslant \overline{\varepsilon}_i.$$

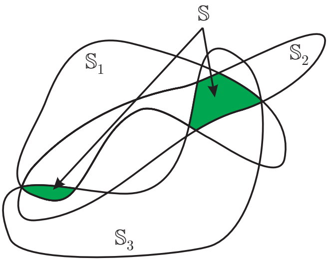
$\Longrightarrow$ Bounded-error parameter estimation :

Characterize $\mathbb{S} = \{\mathbf{p} \in \mathcal{P}_0 \mid y(t_i) - y_{\mathrm{m}}(\mathbf{p}, t_i) \in [\underline{\varepsilon}_i, \overline{\varepsilon}_i], \ i = 1, \ldots, N\}$

Parameter bounding

- When $y_m(\mathbf{p}, t_i)$ is linear in $\mathbf{p}$
  - exact description by polytopes (Walter and Piet-Lahanier, 1989...)
  - outer approximation by ellipsoids, polytopes, ... (Schweppe, 1973 ; Fogel ang Huang, 1982...)
- When $y_m(\mathbf{p}, t_i)$ is non-linear in $\mathbf{p}$
  - outer approximation by polytopes, ellipsoids... (Norton, 1987 ; Clément and Gentil, 1988 ; Cerone, 1991...)
  - approximate but guaranteed enclosure of $\mathbb{S}$ by SIVIA (Moore, 1992 ; Jaulin and Walter 1993)

Robust parameter bounding



$$\mathbb{S} = \bigcap_{\ell=1\ldots N} \mathbb{S}_\ell,$$

with

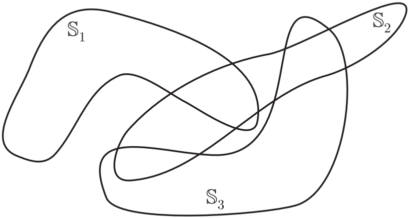
$$\mathbb{S}_\ell = \{\mathbf{p} \in \mathcal{P}_0 \mid y_\ell^{\mathrm{m}}(\mathbf{p}) - y_\ell \in [\underline{\varepsilon}_\ell, \overline{\varepsilon}_\ell]\}$$

Interval analysis [2, 3], [1] allows to get

$$\underline{\mathbb{S}} \subset \mathbb{S} \subset \overline{\mathbb{S}}$$

No consistent $\mathbf{p}$ is missed $\implies$ guaranteed set estimate.

Robust parameter bounding

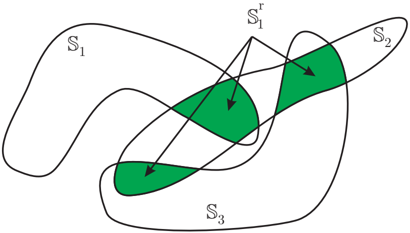


When the solution set is empty

$$\mathbb{S} = \bigcap_{\ell=1\ldots N} \mathbb{S}_\ell = \emptyset.$$

Hypothesis on model or noise violated

Robust parameter bounding



Estimator robust against n outliers

$$\mathbb{S}_n^{\mathrm{r}} = \bigcup_{1 \leqslant \ell_1 < \cdots < \ell_n \leqslant N} \; \bigcap_{\ell \neq \ell_1, \ldots, \ell \neq \ell_N} \mathbb{S}_\ell .$$

Intersection of $N - n$ sets among N

Interval analysis $\Longrightarrow$ non-combinatorial solution

$$\mathbb{S}_n^{\mathrm{r}} = \left\{ \mathbf{p} \in \mathcal{P}_0 \mid \sum_{\ell=1}^{N} t_\ell\left(\mathbf{p}\right) \geq N - n \right\}$$

with

$$t_\ell\left(\mathbf{p}\right) = \left(y_\ell^{\mathrm{m}}\left(\mathbf{p}\right) - y_\ell \in \left[\underline{\varepsilon}_\ell, \overline{\varepsilon}_\ell\right]\right)$$

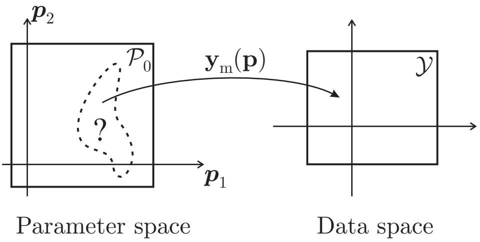
$\mathbb{S}_n^{\mathrm{r}}$ evaluated with a complexity of the order of that of $\mathbb{S}$

Set to be characterized

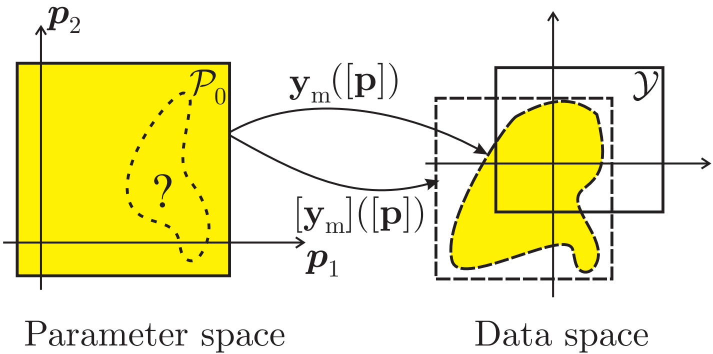
$$\begin{aligned}\mathbb{S} &= \left\{\mathbf{p} \in \mathcal{P}_0 \mid y\left(t_i\right) - y_{\mathrm{m}}\left(\mathbf{p}, t_i\right) \in \left[\underline{\varepsilon}_i, \overline{\varepsilon}_i\right], i = 1, \ldots, N\right\} \\ &= \left\{\mathbf{p} \in \mathcal{P}_0 \mid \mathbf{y}_{\mathrm{m}}\left(\mathbf{p}\right) \subset \mathcal{Y}\right\},\end{aligned}$$

with

$$\mathcal{Y} = \left[y(t_1) - \overline{\varepsilon}_1, y\left(t_1\right) - \underline{\varepsilon}_1\right] \times \cdots \times \left[y(t_N) - \overline{\varepsilon}_N, y\left(t_N\right) - \underline{\varepsilon}_N\right]$$

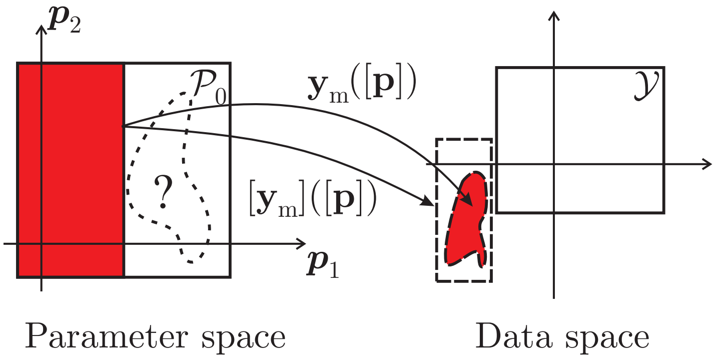


Sivia

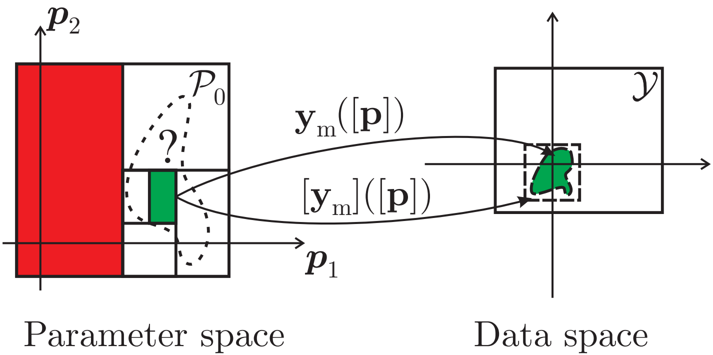


Yellow box is undetermined

Sivia



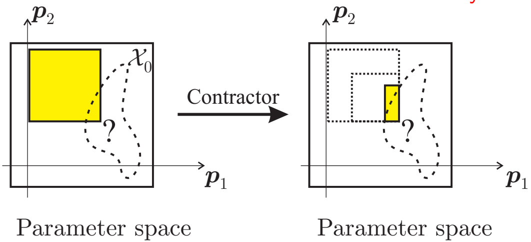
Red box proven to be outside $\mathcal{S}$



Green box proven to be included in $\mathcal{S}$

Sivia with contractors

Reduce the size of undetermined boxes without any bisection



Contractors (Jaulin *et al*, 2001) based on

- interval constraint propagation (Walz)
- linear programming
- parallel linearization
- ...

Sivia with contractors

Example of interval constraint propagation

$$y_{\mathrm{m}}(\mathbf{p}) = p_1 \exp(-p_2),$$

$$p_1 \in [p_1]^0 = [-2, 2], \;\; p_2 \in [p_2]^0 = [-2, 2].$$

One want to characterize the set

$$\mathbb{S} = \left\{ \mathbf{p} \in [p_1]^0 \times [p_2]^0 \;\mid\; \mathbf{y}_{\mathrm{m}}(\mathbf{p}) \subset [1, 2] \right\}.$$

Sivia with contractors

One may write that

$$p_1 \exp(-p_2) \in [1, 2]\,,$$

thus

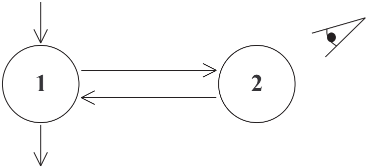
$$\begin{aligned} p_1 &\in [-2, 2] \cap \left(\frac{[1, 2]}{\exp(-[-2, 2])} \right) = [-2, 2] \cap [0.1353, 14.78] \\ &\in [0.1353, 2]\,. \end{aligned}$$

Similarly for p_2, one has

$$\begin{aligned} p_2 &\in [-2, 2] \cap \left(-\ln\left(\frac{[1, 2]}{[0.1353, 2]} \right) \right) = [-2, 2] \cap [-2.6932, 0.6932] \\ &\in [-2, 0.6932] \end{aligned}$$

Example

Estimation of the parameters of a compartmental model



State equation

$$\left\{ \begin{array}{l} x_1' = -\left(k_{01} + k_{21}\right) x_1 + k_{12} x_2 \\ x_2' = k_{21} x_1 - k_{12} x_2 \end{array} \right. \quad \text{with} \quad \left\{ \begin{array}{l} x_1(0) = 0 \\ x_2(0) = 0 \end{array} \right.$$

Observation equation

$$y(t_i) = x_2(t_i) + b(t_i), \;\; i = 1, \ldots, 16$$

Example

Model

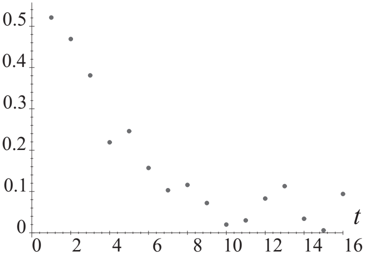
$$y_{\mathrm{m}}(\mathbf{p}, t_i) = p_1 \left(\exp(p_2 t_i) - \exp(p_3 t_i)\right), \quad i = 1, \ldots, 16,$$

where the macroparameters

$$\mathbf{p} = (p_1, p_2, p_3)^{\mathsf{T}}$$

depends on the microparameters

$$(k_{01}, k_{12}, k_{21}).$$



Simulated
noisy experimental data

Example

Macroparameter estimation with

$$\underline{\varepsilon}_i = -0.09,\ \overline{\varepsilon}_i = 0.09,\ i = 1, \dots, 16$$

Results

	SIVIA	SIVIA + ICP	ICP only
Comp. time (s)	8	6.2	0.63
Bounding box	$[0.49, 1.06]$ $[-0.293, -0.141]$ $[-5, -1.054]$	$[0.49, 1.06]$ $[-0.293, -0.141]$ $[-5, -1.054]$	$[0.52, 0.98]$ $[-0.282, -0.156]$ $[-5, -1.167]$

Conclusions

- Interval techniques provide guaranteed solution to bounded-error parameter estimation
- Robust estimation possible
- ICP or SIVIA + ICP allows more unknown parameters than SIVIA

Also possible

- Bounded-error state estimation
- Parameter estimation via deterministic global optimization

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