PHL245 Modern Symbolic Logic

Tingfeng Xia

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1 Arguments

1.1 Validity

We say a deductive argument is valid iff it is not invalid. This means that we can find out if a argument is valid or not by assessing the possiblity of the case where the premise is true and conclusion is false.

1.2 Soundness

We say a deductive argument is sound iff it is

- It is valid
- All the premises are TRUE

2 Semantics in Sentential Logic

2.1 Syntax

Sentential Logic(SL) Complex (compound) statements are all built up by joining statements together using LOGICAL CONNECTIONS. we have $\wedge, \vee, \rightarrow, \leftarrow, \sim$. Where $\sim$ is the only unitary connector and others are binary.

Atomic vs Molecular A **statement** is Atomic if it has no logical connector and is molecular otherwise. We use **P-Z letters** to represent **atomic statments** Here is an example: You can have fries or salad. $\equiv P \vee Q$. Then, P is "You can have fires" and Q is "You can have salad". Notice that we are folowing the definition that P and Q are **statments**.

Informal Notation Hierachy [1] We shall see this through some examples:

- In official notation, we need $(P \vee Q)$ while it is safe to drop the parenthesis.

We can check if a statment is official or informal if it has the same number of brackets as binary connectors in the statement. If you see a bracket around a unitary connector, the sentence is not well-formed. **Right Most Rule** We say the rightmost connector in a sentence with connectors of the same level the main connector.

3 Truth Tables

3.1 Summary

First we remark that we are here talking about binary operators, so they will have and only have two operands. We have the following to consider:

- **Logical OR** $(\vee)$ is true when either one of the operands is true. It evaluates to false otherwise.
- **Logical AND** $(\wedge)$ is true when both of the operands are true and false otherwise.

[1] We use this Informal notation because the formal one is cumbersome. But in order to use the informal one, we have some conventions to follow.

- **Implication ($\rightarrow$)** is true in two cases. The first case is when the first operand is false and second case is when both operands are true.
- **Double implication, Iff ($\leftrightarrow$)** is true when either both operands are true or when both operands are false.
- **Negation ($\sim$[2])** is true when the operand is false and false otherwise.

3.2 Sementic Properties in Truth Tables

- Tautology if always true
- Contradiction if always false
- Contingent if mixed
- Consistent if there exists a row where the conclusion is all true
- Inconsistent if all rows are not all true, negation of the consistent
- Logically equivalent if two sentences have the same truth table

3.3 Full truth tables

Example: [3] evaluate $(P \wedge \sim Q) \vee R. \quad \sim R \vee Q. \quad \therefore \sim P \rightarrow Q$. Notice that this is equivalent to evaluating

$$(((P \wedge \sim Q) \vee R) \wedge (\sim R \vee Q)) \rightarrow \sim P \rightarrow Q$$

4 Addressing Ambiguity in Symboliazation

4.1 Restrictive and Non-Restrictive Clauses

We shall see this through an example

- **Cats that scratch terrifies me** 'Cats *that* scratch' is a restricted subject compared to just 'cats'.
- **Cats, which scratch, terrifies me** 'Cats, *which* scratch' is a non-restricted claim about cats, so essentially we are saying two sentences: the first one is that 'Cats terrify me' and 'Cats scratch'. (The which part is an inserted clause describing the subject before it.)

4.1.1 Symbolization of Non-Restricive Clauses

In symbolization of non-restricive sentences, we simply treat the clause and the main sentence as two seperate sentences and use logical and to connect them. (Two concurrent event)

4.1.2 Symbolization of Restrictive Clauses

We have to introduce a premise (conditional statement) to address the descriptive part.

[2] Or in this course, we may see $\sim$

[3] This example was adapted from Scharer 4.4 EG3.

4.1.3 Example

Consider the sentence "Haveing gaaoline, which is smelly, is a necessary condition for my car to run, and in that case I'm going to Vegas." and we define the following: W: Gasoline is smelly. X: I have gasoline. Y: My car runs. Z: I'm going to Vegas. **Solution:** We first extract the which clause out from the sentence since as we mentioned before this is just another concurrent sentence (use logical and to connect). The necessary condition part is easy $(Y \to X)$. The last 'in that case' part is a little bit tricky and we have rule for it (stated below), the correct interpretation is $(Y \to Z)$. Summarizing what we did, the entire sentence could be symbolized as $W \wedge ((Y \to X) \wedge (Y \to Z))$

4.1.4 "In that case" treatment

In english language, when we say 'bar foo, and in that case baz' the 'case' always refer to the thing mentioned immediately before. So in our example 'foo' would be reason for 'baz'.

5 Derivations in Sentential Logic

Full truth tables are impratical when we have a long sentence to check validity and it turns out that derivation is a nice and simple way to achieve such goal.

5.1 Ten Basic Rules

Types of Rules

- **Elimination Rule** remove the connective (Elimination Rules are Automatic Moves)
- **Introduction** introduce the connective

5.1.1 Modus Ponens (MP)

$\phi \to \psi$, if we know ϕ, then it is reasonable to say ψ

5.1.2 Modus Tollens (MT)

$\phi \to \psi$ then $\sim \psi \to \sim \phi$, aka contrapositive

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If we deny the antecedent or affirm the consequent, that *doesn't* tell us anything!

5.1.3 Double Negation (DN)

$\sim\sim \phi \equiv \phi$. Negating a statement two times is the same as the statement itself.

5.1.4 Repetition (R)

$\phi \to \phi$ If you know ϕ then you know ϕ

5.1.5 Conjunction

Simplification (S or SL/SR) If I know $\phi \wedge \psi$ then I know ϕ and I know ψ

Adjunction (ADJ) If know ϕ and I know ψ then I know $\phi \wedge \psi$

5.1.6 Disjunction

Modus Tollendo Ponens (MTP) If I know $\phi \vee \psi$ and I know $\sim \phi$ then I know ψ. WLOG the other way around is also coorect.

Addition (ADD) If I know ϕ then I know $\phi \vee \psi$. WLOG the other way around is also true.

5.1.7 Biconditional

Biconditional-Conditional (BC) If I know $\phi \leftrightarrow \psi$ then I know that $\phi \rightarrow \psi$ and $\psi \rightarrow \phi$

Conditional-Biconditional (CB) If I knwo $\phi \rightarrow \psi$ and $\psi \rightarrow \phi$, then (by definition) I know $\phi \leftrightarrow \psi$

5.2 Basic Derivations

5.2.1 Examples of rules above

Let's see the Modus Tollens in action

$$\sim (S \vee P) \rightarrow \sim R.\ R. \text{ so } S \vee P$$

THIS IS INCOORECT! We have to take the rules "as is", since we are constructing a contrapositive, we will have $\sim\sim (S \vee P)$ as our conclusion. Here is an example of a correct usage of Modus Tollens

$$\sim P \rightarrow \sim (S \rightarrow Z).\ \sim\sim (S \rightarrow Z) \ so\ \sim\sim P$$

Take away: We can only apply rules "as they are" so the double negation, for example, only works for statements that are of the form $\sim\sim$ (complicated stuff) *we can't jump over any step, we have to write down all the steps.*

5.2.2 Justification Types

There are three things that we can do here: 1.Restate any of the given premises. 2. Use any of the aforementioned rules (write down the lines that we used the rules on and the abbreviation of the rule that we used) 3. Direct Derivation: usually an indicator of arriving at the final conclusion.

5.2.3 Example (S3.3 E2a)

Consider the problem $P \rightarrow Q$. $R \rightarrow \sim Q$. $\sim S \rightarrow R$. P. so S.

1.	~~show~~ S	
2.	$P \rightarrow Q$	Pr1
3.	P	Pr4
4.	Q	2,3,MP
5.	$R \rightarrow \sim Q$	Pr2
6.	$\sim\sim Q$	4,DN
7.	$\sim R$	5,6,MT
8.	$\sim S \rightarrow R$	Pr3
9.	$\sim\sim S$	7,8,MT
10.	S	9,DN
11.		10,DD
12.	$<$ *neglect this line* $>$	

5.2.4 Example (S3.3 E2b)

Consider the problem Y. $X \rightarrow (Y \rightarrow Z)$. $\sim X \rightarrow \sim W$. W. so $\sim\sim Z$

1.	~~show~~ $\sim\sim Z$	show conclusion
2.	$\sim\sim W$	Pr4, DN
3.	$\sim X \rightarrow \sim W$	Pr3
4.	$\sim\sim X$	2,3,MT
5.	X	4, DN
6.	$Y \rightarrow Z$	5, Pr2, MP
7.	Y	Pr1
8.	Z	6,7,MP
9.	$\sim\sim Z$	8,DN
10.		9,DD
11.	$<$ *neglect this line* $>$	

5.2.5 Available Lines

In doing a proof, we want to keep track of what is available for us to use:

- Premises, we can always use the premises
- Unboxed lines that is not a show line
- N.B. A crossed unboxed show line is available (something that we have already shown)

5.2.6 Completeness of a Derivation

- Every show line is crossed off (goals complete)
- All lines that are not show lines are boxed off (subproof complete)
- Every line except show lines is properly justified (state the reasons for derivations of each step)

5.2.7 Abbreviations

- Donnot restate the premises
- Do multiple moves in one line

Here is an example of using the abbrevaitions in action. Consider the exmaple $(P \rightarrow\sim Q) \rightarrow (\sim R \rightarrow S). \sim S. \sim (P \rightarrow\sim Q) \rightarrow T.T \rightarrow S$ so R

1.	~~show~~ $\sim\sim Z$	show conclusion
2.	$\sim T$	P2, P4, MT
3.	$P \rightarrow\sim Q$	2, P3, MT, DN
4.	$\sim R \rightarrow S$	3, P1, MP
5.	R	4, P2, MT, DN, DD
6.	$<$ *neglect this line* $>$	

5.2.8 Conditional Derivation (Hypothetical Reasoning)

We have seen above are all direct derivations, there are two more types of derivations common. They are Conditional Derivation and Indirect Derivation. Lets take a look at conditional derivation here. The general format of this type of proofs is $\phi \rightarrow \psi$. To prove a conditional, we assume ϕ and then prve that ψ follows.

Example Consider the problem $T \rightarrow S. \sim T \rightarrow\sim R.$so $R \rightarrow S$

1.	~~show~~ $R \rightarrow S$	
2.	R	ACD (Assume antecedent)
3.	~~show~~ S	show consequence
4.	$\sim\sim R$	2, DN
5.	T	4, P2, MT, DN
6.	S	P1, 5, MP, DD
7.		3, CD(Conditional Derivation)
8.	$<$ *neglect this line* $>$	

Caveat: It is not required to write the derivation as a subproof, we can also carry the moves in just one layer and state on which line we achived a conditional derivation

5.2.9 Indirect Derivation (Reductio ad Absurdum)

The question takes the following form "Show ϕ". If we assume $\sim \phi$ and derive a contradiction, then our assumption must be false.

Example Consider the problem $P \rightarrow\sim Q.R \rightarrow Q. \sim R \rightarrow\sim P.$ so P

1.	~~show~~ $\sim P$	
2.	P	AID(Assume ID)
3.	$\sim Q$	2, P1, MP
4.	$\sim R$	P2, 3, MT
5.	$\sim P$	4, P3, MP
6.		2, 5, ID (Indirect Derivation / Reductio ad Absurdum)
7.	$<$ *neglect this line* $>$	

5.2.10 Breaking down show lines

- Look at your most recent show line
- If it's of the form $\phi \to \psi$, start a CD
- If it's any other form, start an ID

This works because our system allows mixed derivations!

Theorems

Definition: A Theorem is a tautology. And being a tautology means it could be derived from any set of premises, even including $\emptyset$ as the premise set.

5.3 Derived Rules

There are two types dericed rules: negation of blah rules (NC, NB, and DM) and situational rules.

5.3.1 Negation of Condition (NC)

If have a negation of a conditional statement, then

$$\sim (\phi \to \psi) \equiv \phi \wedge \sim \psi$$

This is essentially a 'proof by conterexample'

5.3.2 Negation of Biconditional (NB)

$$\sim (\phi \leftrightarrow \psi) \equiv (\phi \leftrightarrow \sim \psi) \equiv (\sim \phi \leftrightarrow \psi)$$

5.3.3 De Morgan's Laws (DM)

$$\sim (\phi \vee \psi) \equiv \sim \phi \wedge \sim \psi$$
$$\sim (\phi \wedge \psi) \equiv \sim \phi \vee \sim \psi$$

5.3.4 Seperation of Cases (SC)

If I know $\phi \vee \psi$, $\phi \to \chi$, and $\psi \to \chi$, then it must be the case that χ. **Special Case of SC:** Since in the generic SC, we didn't specify what our ϕ and ψ need to be, so it we can take $\phi = \xi$ and $\psi = \sim \xi$. In this case, $\psi \vee \phi \equiv \xi \vee \sim \xi$ is trivially true, and hence it must be the case that χ.

5.3.5 Conditional as Disjunction (CDJ)

We have already seen this when symbolizing 'unless', but for completeness, I shall present them here.

$$\phi \to \psi \equiv \sim \phi \vee \psi$$
$$\phi \vee \psi \equiv \sim \phi \to \psi$$

6 Predicate Logic

Although our system of sentential logic is great and powerful, it is not quite strong enough since it doesn't allow assesment of logic within the sentence itself. What we need is the predicate logic, as an extension to the sentential system.

6.1 Sub-Sentential Logic

The discussion of subjects and predicates: subjects specify what we want to talk about (as its name indicates) and predicates allows to describe the subject in some certain way.

6.1.1 Subjects

- **Singular Terms** Proper names (Joe, Steve), definite descriptors (the person in the front of the class), pronouns (he/she)
- **General Terms** Universally quantifies terms and Existentially quantified terms

6.1.2 Predicates

DIfferent types of predicates are related to number of subjects that the predicate is related to.

- **Single Place** Only one subject, description of the one and only subject. For example 'Raina is evil. Raina is a student of Plato'
- **Two-Place** Usually describes the relationship between two subjects. For example 'Joe likes Marry'
- **Three-Place** Common in mathematics
- so on...

6.2 Four Types of Statements

- **Universal Affirmative (A)** All X are Y
- **Existential Affirmative (I)** There exists X that is Y
- **Universal Negative (E)** No X is ever Y
- **Existential Negative (O)** There exists X that is not Y

6.3 Deciding Universal vs Existential

6.3.1 Case where both makes sense

There is generally no right or wrong answer to this question. For example, the sentence 'Not all teachers dislike marking' could be the negation of a universally quantified sentence while we can also phrase it as there exists some teacher who likes marking. These two formulation of the sentence are both correct and in fact equivalent.

6.3.2 Case where only one way to interpret

Consider the sentence 'Some animals are dogs'. The only way that we can formulatethis is 'there exists some animal, who is a dog'. This case is rather rare.

6.4 Single-Place Predicate Logic Symbolization

6.4.1 Quantifier Symbols

Two symbols for the universal quantifier is $\forall$ while existential quantifier uses the symbol $\exists$.

6.4.2 Subject Symbols

- Inividual constants or names are marks using lower case lettes $a - h$
- Variable are written as lower case lettes $i-z$. Notice that this is different from the previous case. For example, when I don't want to specify any name for the subject (i want to pick a generic one) I will need a variable.
- Operation Letters is also $a - h$, but possibly with number places as superscript.

6.4.3 Predicate Sumbols

- Predicates are marked with upper case letters $A - O$ (may have number places as superscript in case of non-single place predicates)
- The identity sign $(=)$ is rather special. More on this latter.

6.4.4 Examples

We use lower case letters or brackets (like `format` statements) to indicate the position of insertion of the subject. Consider the sentence 'Tina Fey is amazing' and define $b :=$ Tina Fey and $A : \{1\}$ is amazing. Then the original sentence can be symbolized as Ab.

6.4.5 Canonical Form of the Universally Quantified Sentence

The sentence all Φ's are Ψ's can be represented in the following canonical form:

$$\forall\alpha(\Phi\alpha \to \Psi\alpha)$$

$$\forall\alpha(\text{Group } \alpha \to \text{Property } \alpha)$$

6.4.6 Formulation of Restricted Clause

Consider the sentence 'All Dogs with cube heads are cute', and define the following D^1: {1} is a dog. C^1: {1} is cute. F^1:{1} has a cube head. There are two way of symnbolizing it, the first one is rather straight forward

$$\forall y(Dy \land Fy \to Cy)$$

However, we can also address this using two layers restrictions, like the follow

$$\forall y(Dy \to (Fy \to Cy))$$

6.4.7 Canonical Form of the Existentially Quantified Sentence

Ways to say 'Some Φ's are Ψ's' OR 'At least one of Φ is Ψ' OR 'There is a Φ that is a Ψ'. It could be formulated as follows

$$\exists\alpha(\Phi\alpha \land \Psi\alpha)$$

i.e.,

$$\exists\alpha(\text{Group } \alpha \land \text{Property } \alpha)$$

6.5 Single Place Derivations

6.5.1 Substitution

Replace a bound variable to a free one. This transforms a synmbolic sentence to a sumbolic formula. Notice that **All instances of α are replaced wuth a singular term β** in the following puppy example

$$\forall\alpha\phi\alpha \to \phi\beta$$
$$\exists\alpha\phi\alpha \to \phi\beta$$

6.5.2 Universal Instantiation (UI)

This rule describes the subtitution of $\forall\alpha\phi\alpha$ to $\phi\beta$. This could also be described as the $\forall$ elimination rule. There is a restriction that comes with this rule which is "**Restriction:** β could not be a bound variable within $\phi\alpha$". In naive words, this means we can't substitute in a variable that is already quantified in the inner scope.

Example of Proof using UI Suppose I want to show that $\forall x(Fx \to Gx).Fa.\text{so}Ga.$. The general format is still the same as proofs in sentential logic. We will do

1. show Ga
2. $Fa \to Ga$, Pr1, UI
3. Ga Pr2, 2, MP
4. 3, DD

6.5.3 Existential Instantiation (EI)

If we have a $\exists\alpha\phi\alpha$, we can replace it with $\phi\beta$. The rule comes with a **Restriction:** β must be a *arbitrary* term[4] (i through z) that does not occyr in ANY previous line or premise. This will also be refered to ass the $\exists elimination rule$.

[4]Arbitrary Term: Weak condition is 'A term that is not free in an unboxed line', while the strong condition is 'A term that does not appear in any previous line or premise'

6.5.4 Golden Rule

EI first, UI to MATCH. Always match UI to something useful. There is one exception to this rule, which is the case of Buried Existential.

6.5.5 Example

Consider the quesiton

$$\exists x(Fx \vee \sim Gx).\forall y(Fy \rightarrow Ay).\forall zGz. \quad \text{So } \sim \forall x \sim Ax$$

1.	~~show~~ $\sim \forall x \sim Ax$	
2.	$\forall x \sim x$	AID
3.	$Fi \vee \sim Gi$	P1, EI
4.	Gi	P2, UI
5.	Fi	4, DN, 3, MTP
6.	$Fi \rightarrow Ai$	P2 UI/i
7.	Ai	5, 6, MP
8.	$\sim Ai$	2, UI
9.		7, 8, ID
10.	$< neglect\ this\ line >$	

6.5.6 Existential Generalization (EG)

If I know $\phi\beta$ then I can conclude $\exists\alpha\phi\alpha$. This rule does come with two restrictions

- **Restriction 1:** α cannot be a bound variable within $\phi\beta$
- **Restriction 2:** If α is different from β, then α cannot be free within $\phi\beta$

This rule is also refered to as the $\exists$ Introduction Rule. I puppy example would be to say 'Steven enjoys the notability app', then for sure 'there exists something, something is a person and something enjoys the notability app'. There are certain difference between generalization rules and Instantiation rules, we shall take a look at some examples on genralizing '$Fa \wedge Ga$'

- $\exists x(Fx \wedge Gx)$ generalizing both a's to x
- $\exists y(Fy \wedge Gy)$ generalziing both a's to y
- $\exists x(Fx \wedge Ga)$ generalizing the first a to x. Notice that this definately incorrect in the Instantiation settings, but in fact this is valid here in generalization.
- $\exists z(Fa \wedge Gz)$ genralizing the second a to z
- $\exists x(Fa \wedge Ga)$ generalziing to x. This is a wierd one.

6.5.7 Universal Derivation (UD)

This is a technique rather than a rule which allows the introduction of the universal quantifier. To add a universal quantifier, we can simply do a 'show' of the statement, with no premise whatsoever. The general format will be

- Show $\forall\alpha\phi\alpha$
-     Show $\phi\alpha$
-     CD/ID/DD
- $\forall\alpha\phi\alpha$, UD

The restriction of this technique is 'α cannot appear unbound in any previous available line, or in a premise used in an available line'. This adds to our list of derivation skills: DD, ID, CD. Now we have the fourth way UD.

Example on UD (Double) This is considered to be a double UD example because we have to do UD twice. The exmaple is

$$Ha.\ \exists w\exists z(Hz\wedge\sim Gw)\rightarrow\forall xAx.\ \exists i\sim Ai\ \text{ So } \forall x\forall y(Fx\rightarrow Gy)$$

6.6 Quantifier Negation (QN)

A quantifier or a negation of quantifier must be the main operator.

- $\sim\forall\alpha\phi\equiv\exists\alpha\sim\phi$
- $\sim\exists\alpha\phi\equiv\forall\alpha\sim\phi$