Final Review - 我还没有完全掌握

Friday, April 12, 2019 2:20 午後

1. Set theories and Topology
    a. We say a set is open iff $S = S^{int}$. Intuitively, this means every point of the set is a interior of the set.
    b. A set is closed iff $S = \overline{S}$, where we define $\overline{S} = S^{int} \cup \partial S$. A thing to notice here is that boundaries
        i. There are some weird examples worth noting, for example define $S = \{x \in \mathbb{R}^3 : 0 < |x| < 1, |x|\ irrational\}$, then we have $S^{int} = \emptyset$, and $\overline{S} = \{x \in \mathbb{R}^3 : 0 \leq |x| \leq 1, |x|\ irrational\}$. In this case, we notice that this set S is neither open nor closed, since it matched none of our definitions of openess and closedness. To see this, we note that if S is open then $\partial S \cap S = \emptyset$, and if S is closed then $\partial S \subseteq S$. So this would be the case that the boundary is an empty set.
        ii. In the case of subsets of $\mathbb{R}^1$, a open interval is open. For example $(1,2), (-\infty, 10) \subseteq \mathbb{R}$ are open subsets of the reals. A closed in $\mathbb{R}$ is always closed, for example, sets of the form $[1,2] \subseteq \mathbb{R}^1$.
    c. A set could be neither closed nor open. It could also be the case that the set is both open and closed, such as in the euclidean space, we have examples such as $\emptyset, \mathbb{R}^n$.
    d. We say that a set is bounded iff there exists an open ball centered at the origin, with a enough large radius such that the set is a subset of this ball. This is our generalization of boundedness in $\mathbb{R}^1$ since we no longer can define a unidirectional smaller or larger. (we are in a higher dimension and things can go in different directions.)
        i. An example of this sort would be the following: define $S = \{x \in \mathbb{R}^n \mid |x| = 2^{-j}, j \in \mathbb{Z}^{\geq 1}\}$. Notice that this set is bounded by the open ball with, for example, radius of 1 centered at the origin in $\mathbb{R}^n$. This particular set is "neither open nor closed", since its closure includes an extra $\{0\}$.
        ii. Notice that we say a say a set is unbounded iff it is not bounded. This iff relationship tells us that a set could not be bounded and unbounded at the same time, contrary to the open and close property of a set.
    e. Some important identities
        i. $A, B \subseteq \mathbb{R}^n$ are open $\Rightarrow A \cup B, A \cap B$ open
        ii. $A, B \subseteq \mathbb{R}^n$ are closed $\Rightarrow A \cup B, A \cap B$ closed
        iii. De Morgan's Law: $(A \cup B)^c = A^c \cap B^c$; $(A \cap B)^c = A^c \cup B^c$

iii. De Morgan's Law: $(A \cup B)^c = A^c \cap B^c \wedge (A \cap B)^c = A^c \cup B^c$

2. Limits and continuity
   a. The statement of limit, in higher dimension: $\lim_{x \to a} f(x) = L$ means, in epsilon delta notation that $\forall \epsilon > 0, \exists \delta > 0\ s.t.\ x \in S \wedge 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$. We will need to have $\forall \delta > 0, \exists x \in S, s.t.\ 0 < |x - a| < \delta$ as a condition of a. This would hold, for example in the case that $a \in S^{int}$ or $a \in \overline{(S^{int})}$.
   b. Theorem: (Limit Laws)
      Assume that $S \subseteq \mathbb{R}^n$ and that a is a point in $\mathbb{R}^n$ (in teriror point of S, where it makes sense to talk about the limit). Further assume that $f, g: S \to \mathbb{R}$ are functions and $L, M \in \mathbb{R}$ s.t. $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then, we have $\lim_{x \to a} f(x) + g(x) = L + M$ and $\lim_{x \to a} f(x) \cdot g(x) = L \cdot M$.
   c. Theorem: (Squeeze Theorem)
      Assume that $S \subseteq \mathbb{R}^n$ and $a \in \mathbb{R}^n$ at which it makes sense to talk about the limit. Also, assume that $f, g, h: S \to \mathbb{R}$ are functions and $p > 0$ s.t. $f(x) \leq g(x) \leq h(x), \forall x \in S, s.t.\ |x - a| < p$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$. Then we know that $\lim_{x \to a} g(x) = L$.
   d. Corollary: (of Squeeze Theorem)
      Assume that ..., if we have $|g(x)| \leq h(x), \forall x \in S \wedge \lim_{x \to a} h(x) = 0$, g will be squeezed to zero when the limit approaches to a.
   e. Theorem: (Relation with 1-D limits)
      Assume that $S \subseteq \mathbb{R}^n$ and that $a \in \mathbb{R}^n$ at which the limit makes sense. If $f = (f_1, \dots, f_k)$ is a vector valued function from $\mathbb{R}^n \to \mathbb{R}^k$, then $\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a} f_j(x) = L_j, \forall j = 1, \dots, k$. Where $(L_1, \dots, L_k) = L$ are the components of the vector L.