

MAT224 Linear Algebra

Definitions, Lemmas, Theorems, Corollaries
and their related proofs

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Materials in this booklet are based heavily on Prof. Nicolas Hoell's lectures as well as A COURSE IN LINEAR ALGEBRA by David B. Damiano and John B. Little

The course website could be found here:
<http://www.math.toronto.edu/nhoell/MAT224/>

Contents

1	Vector Spaces	5
1.1	(Real) Vector Space	5
1.2	Sub-spaces	6
1.3	Linear Combinations	7

Chapter 1

Vector Spaces

1.1 (Real) Vector Space

1.1.1. **Definition of real vector space:** A real vector space is a set V together with scalar

- (a) **Closure under vector addition:** an operation called vector addition, which for each pair of vectors $\vec{x}, \vec{y} \in V$ produces another vector in V denoted $\vec{x} + \vec{y}$, (i.e. $\forall \vec{x}, \vec{y} \in V, \vec{x} + \vec{y} \in V$) and
- (b) **Closure under scalar multiplication:** an operation called multiplication by a scalar (a real number), which for each vector $\vec{x} \in V$, an each scalar $c \in \mathbb{R}$ produces another vector in V denoted $c\vec{x}$. (i.e. $\forall \vec{x} \in V, \forall c \in \mathbb{R}, c\vec{x} \in V$)

Furthermore, the two operations must satisfy the following axioms:(important)

- (a) $\forall \vec{x}, \vec{y}, \vec{z} \in V, (\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
- (b) $\forall \vec{v}, \vec{y} \in V, \vec{x} + \vec{y} = \vec{y} + \vec{x}$
- (c) $\exists \vec{0} \in V$ s.t. $\forall \vec{x} \in V, \vec{x} + \vec{0} = \vec{x}$ (Note that this property is a.k.a existence of additive identity)
- (d) $\forall \vec{x} \in V, \exists (-\vec{x}) \in V$ s.t. $\vec{x} + (-\vec{x}) = \vec{0}$ (Note that this property is a.k.a existence of additive inverse)
- (e) $\forall \vec{x}, \vec{y} \in V, c \in \mathbb{R}, c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
- (f) $\forall \vec{x} \in V, c, d \in \mathbb{R}, (c + d)\vec{x} = c\vec{x} + d\vec{x}$

$$(g) \quad \forall \vec{x} \in V, c, d \in \mathbb{R}, (cd)\vec{x} = c(d\vec{x})$$

$$(h) \quad \forall \vec{x} \in V, 1\vec{x} = \vec{x}$$

1.1.6. **Propositions for a R-v.s.** Let V be a vector space. Then

(a) The zero vector is unique. Note that it might not necessarily be actually the zero vector in \mathbb{R}^n that we are somewhat used to use.

$$(b) \quad \forall \vec{x} \in V, 0\vec{x} = 0$$

(c) $\forall \vec{x} \in V$, the additive inverse is unique. Note that it might not necessarily be actually just (-1) times the vector in \mathbb{R}^n that we are somewhat used to use.

$$(d) \quad \forall \vec{x} \in V, \forall c \in \mathbb{R}, (-c)\vec{x} = -(c\vec{x})$$

1.2 Sub-spaces

1.2.4. **Usual definition of subspace applied to functions in $C^0(\mathbb{R})$.** Note that by $C^n(\cdot)$ we mean the function in this set are all of *Class* $-n$. Let $f, g \in C^0(\mathbb{R})$, let $c \in \mathbb{R}$. Then,

$$(a) \quad f + g \in C^0(\mathbb{R}), \text{ and}$$

$$(b) \quad cf \in C^0(\mathbb{R})$$

The proof of this lemma relies on limit theorems of calculus.

1.2.6. **(Intuitive) definition of (vector) subspace** Let V be a vector space and let $W \subseteq V$ be a subset. Then W is a (vector) subspace if W is a vector subspace itself under the operations of vector sum and scalar multiplication from V .

1.2.8. **Quick check rule for a subspace.** Let V be a vector subspace, and let W be a **non empty** subset of V . Then W is a subspace of V if and only if $\forall \vec{x}, \vec{y} \in W, \forall c \in \mathbb{R}$, we have $c\vec{x} + \vec{y} \in W$.

1.2.9. **Remark on the necessary condition of non-emptiness of subspace.** According to the definition of vector space that we gave in 1.1.1, a vector space must contain an additive identity element, hence it is necessary that we ensure $W \subseteq V$ (from 1.2.6) is not an empty set.

- 1.2.13. **Theorem: Intersection of sub-spaces is a subspace.** Let V be a vector space. Then the intersection of any collection of sub-spaces of V is a subspace of V .
- 1.2.14. **Corollary: Hyper planes in \mathbb{R}^n are sub-spaces of \mathbb{R}^n .** Let a_{ij} ($1 \leq i \leq m$), let $W_i = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_{i1}x_1 + \dots + a_{in}x_n = 0, \forall 1 \leq i \leq m\}$. Then W is a subspace of \mathbb{R}^n .

1.3 Linear Combinations

- 1.3.1. **Definitions regarding L.C. and derived spans.** Let S be a subset of a vector space V , that is $S \subseteq V$.
- (a) a *linear combination* of vectors in S is any sum $a_1\vec{x}_1 + \dots + a_n\vec{x}_n$, where the $a_i \in \mathbb{R}$, and the $x_i \in S$.
 - (b) we define the *Span* of a set of vectors as follows to consider the special case of $S \stackrel{?}{=} \emptyset \in V$. **Case1:** $S \neq \emptyset$. In this case, we define $Span(S)$ to be all possible linear combinations using vectors in S . **Case2:** $S = \emptyset$. In this case, we define $Span(S = \emptyset) = \{\vec{0}\}$
 - (c) If $W = Span(S)$, we say S *spans* (or *generates*) W .
- 1.3.4. **Span of a subset of a vector space is a subspace.** Let V be a vector space and let S be any subset of V . Then $Span(S)$ is a subspace of V .
- 1.3.5. **Sum of sets(with application to sub-spaces).** Let $W_1 \wedge W_2$ be subspaces of a vector space V . The sum of W_1 and W_2 is the set

$$W_1 + W_2 := \{\vec{x} \in V \mid \vec{x} = \vec{x}_1 + \vec{x}_2, \text{ for some } \vec{x}_1 \in W_1, \vec{x}_2 \in W_2\}$$

We think of the sum of the two sub-spaces (the two sets) as the set of vectors that can be built up from the vectors in W_1 and W_2 by linear combinations. Conversely, the vectors in the set $W_1 + W_2$ are precisely the vectors that can be broken down into the sum of a vector in W_1 and a vector in W_2 . One may find it helpful to view this as an analogue to a Cartesian product of the two set with a new constraint on the result.

1.3.8. **Proposition: The sum of spans of sets is the span of the union of the sets.** Let $W_1 = \text{Span}(S_1)$ and $W_2 = \text{Span}(S_2)$ be sub-spaces of a (the same) vector space V . Then $W_1 + W_2 = \text{Span}(S_1 \cup S_2)$.

1.3.9. **The sum of sub-spaces is also a subspace.** Let W_1 and W_2 be sub-spaces of a vector space V . Then $W_1 + W_2$ is also a subspace of V .

Proof:

It is clear that $W_1 + W_2$ is non-empty, since neither W_1 nor W_2 is empty. Let \vec{x}, \vec{y} be two vectors in $W_1 + W_2$, let $c \in \mathbb{R}$. By our choice of \vec{x} and \vec{y} , we have

$$\begin{aligned} c\vec{x} + \vec{y} &= c(\vec{x}_1 + \vec{x}_2) + (\vec{y}_1 + \vec{y}_2) \\ &= (c\vec{x}_1 + \vec{y}_1) + (c\vec{x}_2 + \vec{y}_2) \in W_1 + W_2 \end{aligned}$$

Since W_1 and W_2 are sub-spaces of V , we have $(c\vec{x}_1 + \vec{y}_1) \in W_1$ and $(c\vec{x}_2 + \vec{y}_2) \in W_2$. Then by (1.2.8), we see that indeed $W_1 + W_2$ is a subspace of V . *Q.E.D.*

1.3.10. **Remark on?**

1.3.11. **Proposition.** Let W_1 and W_2 be sub-spaces of vector space V and let W be a subspace of V such that $W \supseteq W_1 \cup W_2$, then $W \supseteq W_1 + W_2$