MAT224 Linear Algebra

Definitions, Lemmas, Theorems, Corollaries
and their related proofs

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The course website could be found here:
`http://www.math.toronto.edu/nhoell/MAT224/`

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Chapter 1

Vector Spaces

1.1 (Real) Vector Space

1.1.1. **Definition of real vector space:** A real vector space is a set V together with scalar

(a) **Closure under vector addition:** an operation called vector addition, which for each pair of vectors $\vec{x}, \vec{y} \in V$ produces another vector in V denoted $\vec{x} + \vec{y}$, (i.e. $\forall \vec{x}, \vec{y} \in V, \vec{x} + \vec{y} \in V$) and

(b) **Closure under scalar multiplication:** an operation called multiplication by a scalar (a real number), which for each vector $\vec{x} \in V$, an each scalar $c \in \mathbb{R}$ produces another vector in V denoted $c\vec{x}$. (i.e. $\forall \vec{x} \in V, \forall c \in \mathbb{R}, c\vec{x} \in V$)

Furthermore, the two operations must satisfy the following axioms:(important)

(a) $\forall \vec{x}, \vec{y}, \vec{z} \in V, (\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$

(b) $\forall \vec{v}, \vec{y} \in V, \vec{x} + \vec{y} = \vec{y} + \vec{x}$

(c) $\exists \vec{0} \in V s.t. \forall \vec{x} \in V, \vec{x} + \vec{0} = \vec{x}$ (Note that this property is a.k.a existence of additive identity)

(d) $\forall \vec{x} \in V, \exists (-\vec{x}) \in V \ s.t. \ \vec{x} + (-\vec{x}) = \vec{0}$ (Note that this property is a.k.a existence of additive inverse)

(e) $\forall \vec{x}, \vec{y} \in V, c \in \mathbb{R}, c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$

(f) $\forall \vec{x} \in V, c, d \in \mathbb{R}, (c + d)\vec{x} = c\vec{x} + d\vec{x}$

(g) $\forall \vec{x} \in V, c, d \in \mathbb{R}, (cd)\vec{x} = c(d\vec{x})$

(h) $\forall \vec{x} \in V, 1\vec{x} = \vec{x}$

1.1.6. **Propositions for a R-v.s.** Let V be a vector space. Then

(a) The zero vector is unique. Note that it might not necessarily be actually the zero vector in $\mathbb{R}^n$ that we are somewhat used to use.

(b) $\forall \vec{x} \in V, 0\vec{x} = 0$

(c) $\forall \vec{x} \in V$, the additive inverse is unique. Note that it might not necessarily be actually just (-1) times the vector in $\mathbb{R}^n$ that we are somewhat used to use.

(d) $\forall \vec{x} \in V,\ \forall c \in \mathbb{R},\ (-c)\vec{x} = \ -(c\vec{x})$

1.2 Sub-spaces

1.2.4. **Usual definition of subspace applied to functions in $C^0(\mathbb{R})$.** Note that by $C^n(\cdot)$ we mean the function in this set are all of $Class - n$. Let $f, g \in C^0(\mathbb{R}), letc \in \mathbb{R}$. Then,

(a) $f + g \in C^0(\mathbb{R})$, and

(b) $cf \in C(\mathbb{R})$

The proof of this lemma relies on limit theorems of calculus.

1.2.6. **(Intuitive) definition of (vector) subspace** Let V be a vector space and let $W \subseteq V$ be a subset. Then W is a (vector) subspace if W is a vector subspace itself under the operations of vector sum and scalar multiplication from V.

1.2.8. **Quick check rule for a subspace.** Let V be a vector subspace, and let W be a **non empty** subset of V. Then W is a subspace of V if and only if $\forall \vec{x}, \vec{y} \in W,\ \forall c \in \mathbb{R}$, we have $c\vec{x} + \vec{y} \in W$.

1.2.9. **Remark on the necessary condition of non-emptiness of subspace.** According to the definition of vector space that we gave in 1.1.1, a vector space must contain an additive identity element, hence it is necessary that we ensure $W \subseteq V$(from 1.2.6) is not an empty set.

1.2.13. **Theorem: Intersection of sub-spaces is a subspace.** Let V be a vector space. Then the intersection of any collection of sub-spaces of V is a subspace of V.

1.2.14. **Corollary: Hyper planes in $\mathbb{R}^n$ are sub-spaces of $\mathbb{R}^n$.** Let $a_{ij}(1 \leq i \leq m)$, let $W_i = \{(x_1, ..., x_n) \in \mathbb{R}^n | a_{i1}x_1 + ... + a_{in}x_n = 0, \ \forall 1 \leq i \leq m\}$. Then W is a subspace of $\mathbb{R}^n$.

1.3 Linear Combinations

1.3.1. **Definitions regarding L.C. and derived spans.** Let S be a subset of a vector space V, that is $S \subseteq V$.

(a) a *linear combination* of vectors in S is any sum $a_1\vec{x}_1 + ... + a_n\vec{x}_n$, where the $a_i \in \mathbb{R}$, and the $x_i \in S$.

(b) we define the *Span* of a set of vectors as follows to consider the special case of $S \stackrel{?}{=} \emptyset \in V$. **Case1:** $S \neq \emptyset$**.** In this case, we define $Span(S)$ to be all possible linear combinations using vectors in S. **Case2:** $S = \emptyset$**.** In this case, we define $Span(S = \emptyset) = \{\vec{0}\}$

(c) If $W = Span(S)$, we say S *spans(or generates)* W.

1.3.4. **Span of a subset of a vector space is a subspace.** Let V be a vector space and let S be any subset of V. Then $Span(S)$ is a subspace of V.

1.3.5. **Sum of sets(with application to subs-paces).** Let $W_1 \wedge W_2$ be sub spaces of a vector space V. The sum of W_1 and W_2 is the set

$$W_1 + W_2 := \{\vec{x} \in V | \vec{x} = \vec{x_1} + \vec{x_2}, \text{for some } \vec{x_1} \in W_1, \vec{x_2} \in W_2\}$$

We think of the sum of the two sub-spaces(the two sets) as the set of vectors that can be built up from the vectors in W_1 and W_2 by linear combinations. Conversely, the vectors in the set $W_1 + W_2$ are precisely the vectors that can be broken down into the sum of a vector in W_1 and a vector in W_2. One may find it helpful to view this as an analogue to a Cartesian product of the two set with a new constraint on the result.

1.3.6. **Example.** If $W_1 = \{(a_1, a_2) \in \mathbb{R}^2 | a_2 = 0\}$ and $W_2\{(a_1, a_2) \in \mathbb{R}^2 | a_1 = 0\}$, then $W_1 + W_2 = \mathbb{R}^2$, since every vector in $\mathbb{R}^2$ can be written as the sum of vector in W_1 and a vector in W_2. For instance, we have $(5, -6) = (0, 5) + (0, -6)$, and $(5, 0) \in W_1 \wedge (0, -6) \in W_2$

1.3.8. **Proposition: The sum of spans of sets is the span of the union of the sets.** Let $W_1 = Span(S_1)$ and $W_2 = Span(S_2)$ be sub-spaces of a*(the same)* vector space V. Then $W_1 + W_2 = Span(S_1 \cup S_2)$. Notice that the proof of this gave the important idea of mutual inclusion in proving sets are equal to each other.

1.3.9. **The sum of sub-spaces is also a subspace.** Let W_1 and W_2 be sub-spaces of a vector space V. Then $W_1 + W_2$ is also a subspace of V.
Proof:
It is clear that $W_1 + W_2$ is non-empty, since neither W_1 nor W_2 is empty. Let $\vec{X}, \vec{y}$ be two vectors in $W_1 + W_2$, let $c \in \mathbb{R}$. By our choice of $\vec{x}$and $\vec{y}$, we have

$$\begin{aligned} c\vec{x} + \vec{y} &= c(\vec{x}_1 + \vec{x}_2) + (\vec{y_1} + \vec{y_2}) \\ &= (c\vec{x}_1 + \vec{y}_1) + (c\vec{x}_2 + \vec{y}_2) \in W_1 + W_2 \end{aligned}$$

Since W_1 and W_2 are sub-spaces of V, we have $(c\vec{x}_1 + \vec{y}_1) \in W_1 \wedge (c\vec{x}_2 + \vec{y}_2) \in W_2$. Then by (1.2.8), we see that indeed $W_1 + W_2$ is a subspace of V. $\mathcal{Q.E.D.}$

1.3.10. **Remark.** In general =, if W_1 and W_2 are subspaces of V, then $W_1 \cup W_2$ will not be a subspace of V. For example, consider the two sub-spaces of $\mathbb{R}^2$ given in example (1.3.6). In that case $W_1 \cup W_2$ is the union of two lines through the origin in $\mathbb{R}^2$.

1.3.11. **Proposition.** Let W_1 and W_2 be sub-spaces of vector space V and let W be a subspace of V such that $W \supseteq W_1 \cup W_2$, then $W \supseteq W_1 + W_2$

1.4 Linear (In)dependence

1.4.2. **Algebraic definition of linear dependence.** Let V be a vector space, and let $S \subseteq V$.

(a) A *linear dependence* among the vectors of S is an equation $a_1\vec{x} + ... + a_n\vec{x}_n = \vec{0}$ where the $x_i \in S$, and the $a_i \in \mathbb{R}$ are not all zero(i.e., at least one of the $a_i \neq 0$). In familiar[1] words, there exists a non-trivial solution to the equation mentioned above.

(b) the set S is said to be *linearly dependent* if there exists a linear dependence among the vectors in S.

1.4.4. **Algebraic definition of linear independence.** Let V be a vector space, and $S \subseteq V$. Then S is *linearly independent* if whenever we have $a_i \in \mathbb{R}$ and $x_i \in S$ such that $a_1\vec{x}_1 + ... + a_n\vec{x}_n = \vec{0}$, then $a_i = 0$, $\forall i$. A more conceivable way to understand this is if the aforementioned equation exists and only exists a set of trivial solution then the vectors involved in the equation are *linearly independent*

1.4.7. **Propositions regarding linear (in)dependency.**

(a) Let S be a linearly dependent subset of a vector space V, and let S' be another subset of V that contains S. Then S' is also linearly dependent.

(b) Let S be a linearly independent subset of vector space V and let S' be another subset of V that is contained in S. Then S' is also linearly independent.

Proof of (a): Since S is linearly dependent, there exists a linear dependence among the vectors in S, say, $a_1\vec{x}_1 + ... + a_n\vec{x}_n = \vec{0}$. Since S is contained in S', this is also a linear dependence among the vectors in S'. Hence S' is linear dependent. $\mathcal{Q}.\mathcal{E}.\mathcal{D}.$
Proof of (b): Consider any equation $a_1\vec{x}_1 + ... + a_n\vec{x}_n = \vec{0}$, where the $a_i \in \mathbb{R},\ \vec{x}_i \in S'$. Since S' is contained in S, we can also view this as a potential linear dependence among vectors in S. However, S is linearly independent, so it follows that all the $a_i = 0 \in \mathbb{R}$. Hence S' is also linearly independent. $\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

1.5 Interlude on solving systems of linear equations.

[1]Familiar from MAT223, Prof. Jason Siefken's IBL(Inquiry Based Learning) notes.