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Part I

Limits, Derivatives and Integrals

Chapter 1

Limits and Continuity of Functions

1 Introduction

A typical course in calculus comprise many topics but the most fundamental topics in the study of calculus are the concepts of limits, derivatives, and integrals. These concepts deals with functions which is one of the reasons why you studied functions in your algebra and trigonometric course.

In this chapter we will study the notion of limit of real-valued functions of single variable at a point and the continuity of a function at a point and in an interval. Under limit, we like to know how the function behaves as the independent variable, say x, approaches a given variable (*this new variable can be in the domain of the function or not*) or as x becomes very large (i.e. positively large, $+\infty$) or smaller than any other x (i.e. $-\infty$). For continuity, we are interested in knowing if a function has a *jump* or not. We begin by reminding ourselves of the following concepts of intervals :

1.1 Intervals on the real line

Let $\mathbb{R} := (-\infty, \infty)$ be the set of all real numbers and let $a, b \in \mathbb{R}$.

1. $(a, b) := \{x \in \mathbb{R} : a < x < b\}$ and it is read as an *Open Interval* ab
2. $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ and it is read as an *Close Interval* ab
3. $[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$ and it is read as an *Close-Open Interval* ab
4. $(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$ and it is read as an *Open-Close Interval* ab

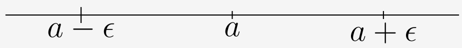
These intervals are all *bounded intervals. We should understand by ∞ as some number larger than any other number we can think of and $-\infty$ as some number* *smaller than any other number we can imagine.* $\pm\infty$ *are not numbers.* The following are *unbounded intervals*:

1. $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$
2. $(-\infty, a) := \{x \in \mathbb{R} : x < a\}$
3. $(-\infty, a] := \{x \in \mathbb{R} : x \leq a\}$
4. $(a, \infty) := \{x \in \mathbb{R} : x > a\}$
5. $[a, \infty) := \{x \in \mathbb{R} : x \geq a\}$

Let $I = (a, b)$ be an interval where $a, b \in \mathbb{R}$. Then the length of I is $|I| = b - a$ (i.e. the distance from a to b). The midpoint or the center of the interval I is $\frac{a+b}{2}$. Thus let $\epsilon > 0$ (a real positive small number), the $(a - \epsilon, a + \epsilon)$, $a \in \mathbb{R}$ is an interval whose midpoint is

$$\frac{a - \epsilon + a + \epsilon}{2} = a.$$

We say that $(a - \epsilon, a + \epsilon)$ is an interval of center a (or about a) i.e.



2 Limits of Functions

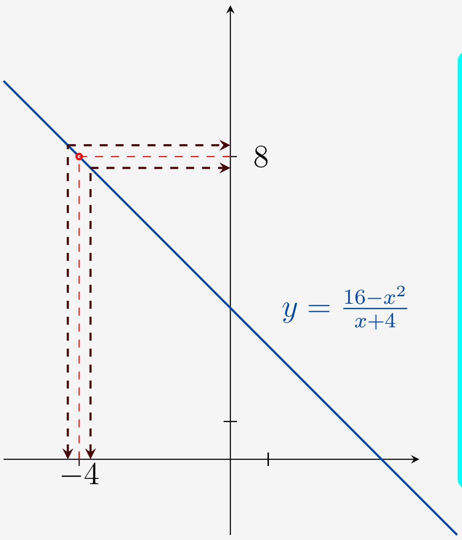
Differential and *integral* calculus are the two broad areas of calculus. These concepts are built on the foundation concept of *limits.* In this section, the approach to understanding limits will be both intuitive and analytic. Intuitive in the sense that we will concentrate on understanding what a limit is using numerical and graphical examples. The analytic approach uses algebraic methods to compute the value of a limit of a function.

2.1 Finite Limit at a Point

Let x_0 be a fixed point. We say that x approaches x_0 if x is getting closer and closer to x_0 and we write $x \to x_0$. Let us evaluate the limit of

$$f(x) = \frac{16 - x^2}{x + 4} \tag{1.1}$$

using an informal approach. First we see that the domain of f is $x \in \mathbb{R} \setminus \{-4\}$. Thus f cannot be evaluated at $x = -4$. However, f is defined for values very close to -4 on both sides. Consider the Figure 1.1 below.



$x < -4$	$f(x)$	$x > -4$	$f(x)$
-4.1	8.1	-3.9	7.9
-4.01	8.01	-3.99	7.99
-4.001	8.001	-3.999	7.999

The table indicates that as $x \to -4$ from either side, $f(x) \to 8$. We say that 8 is the limit of $f(x)$ as x approaches -4. Essentially, the graph of f is the line $y = 4 - x$ with the exception that there is a *hole* at the point that corresponds to $x = -4$.

Figure 1.1: When x is near -4, $f(x)$ is near 8.

The following definition is an informal definition of limits.

Definition 2.2 (Informal Definition)**.** Let f be a function defined on an open interval about x_0 but not necessarily at x_0 and let $L \in \mathbb{R}$. If $f(x)$ gets arbitrary close to L as $x \to x_0$, (***from either sides of*** x_0), we say the limit of $f(x)$ at x_0 is L and we write

$$\lim_{x \to x_0} f(x) = L \tag{1.2}$$

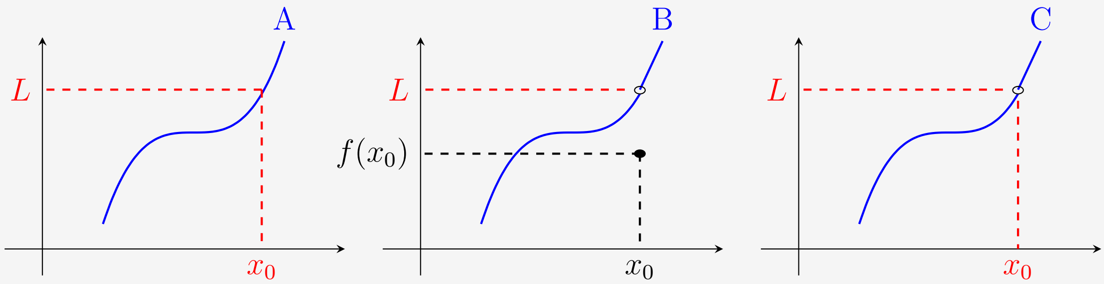
Remark 2.3. We note the following;

1. The above definition is not the precise definition of limits. We shall look at the precise definition shortly (see Definition 2.17). An alternative notation for equation (1.2) is

$$f(x) \to L \quad \text{as} \quad x \to x_0$$

which reads ***"$f(x)$ approaches L as x approaches x_0"***.

2. x_0 need not be in the domain of f. In fact we never consider $x = x_0$. The only thing that matters is how f is defined near x_0. Consider the following graphs;



For graph C, $f(x_0)$ is not defined and for graph B, $f(x_0) \neq L$. However, for graph

A, $f(x_0) = L$. But in each case, regardless of what happens at x_0, $f(x) \to L$ as $x \to x_0$.

Example 1. 1. Consider the function $f : (-1, 1) \longrightarrow \mathbb{R}$ defined by

$$f(x) = \frac{x^2(x-1)}{x}.$$

We realize that the domain of f is the set $\{x \in (-1,1) : x \neq 0\}$. Thus

$$\begin{aligned} \lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{x^2(x-1)}{x} \\ &= \lim_{x\to 0} x(x-1) = 0 \end{aligned}$$

2. Consider $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \frac{x-1}{x^2-1}.$$

Then domain of f is the set $\{x \in \mathbb{R} : x \neq \pm 1\}$. However, by definition of limit, we can find

$$\lim_{x\to 1} \frac{x-1}{x^2-1}.$$

Consider the table below, we see that $f(x) \to \frac{1}{2}$ as $x \to 1$.

$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.5	0.67	1.5	0.40
0.9	0.53	1.1	0.48
0.99	0.50	1.01	0.49
0.999	0.5	1.001	0.499
0.9999	0.5	1.0001	0.4999

3. Consider the function

$$g(x) = \begin{cases} \frac{x-1}{x^2-1} & , \quad x \neq \pm 1 \\ 2 & , \quad x = 1 \end{cases}$$

Then $\lim_{x\to 1} g(x) = 0.5$ since we are not really concerned at $x = 1$ but rather near 1.

Quick Practice 1. 1. Use table of values to evaluate the following limits

$$\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2} \quad \text{and} \quad \lim_{x\to 1} \frac{x^2-3x+2}{x^2-x}$$

Justify your answer by an algebraic method.

2. Find the limits of the following by using table of values

$$\lim_{x\to 0}\frac{\sin(x)}{x}, \quad \lim_{x\to 0}\sin\left(\frac{\pi}{x}\right) \quad \text{and} \quad \lim_{x\to 0}\frac{\cos(x)-1}{x}.$$

We note that if x_0 ***is in the domain of*** f ***then*** $\lim_{x\to x_0} f(x) = f(x_0)$.

Example 2 (A Special Example)**.** The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & , \quad t < 0 \\ 1 & , \quad t \geq 0 \end{cases}$$

We see that $H(t) \to 0$ as $t \to 0$ from the left (i.e. $t < 0$) and $H(t) \to 1$ as $t \to 0$ from the right (i.e. $t \geq 0$). Thus there is no single number that $H(t)$ approaches as $t \to 0$. Hence $\lim_{t\to 0} H(t)$ does not exist. This leads us to the following concepts.

One-Sided Limit

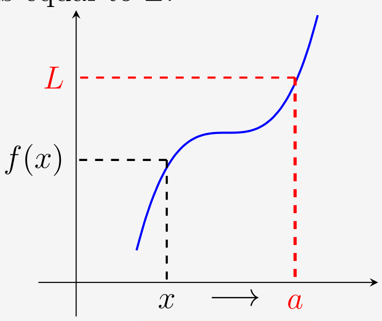
We write

$$\lim_{x\to a^-} f(x) = L$$

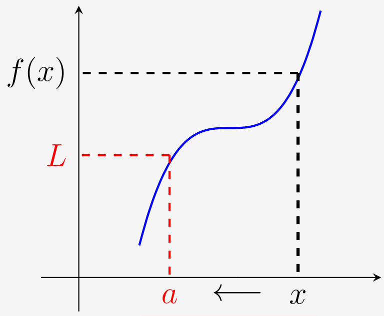
and say the ***Left-hand limit*** of $f(x)$ as $x \to a$ (or limit of $f(x)$ as $x \to a$ from the left) is equal to L if we can make the values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to a and less than a. Also, we write

$$\lim_{x\to a^+} f(x) = L$$

if we require x to be greater than a and we get the ***Right-hand limit*** of $f(x)$ as $x \to a$ is equal to L.



$\lim_{x\to a^-} f(x)$



$\lim_{x\to a^+} f(x)$

The symbol $x \to a^+$ indicates we consider only $x > a$ and $x \to a^-$ indicates we consider only $x < a$. As an example, we have from the Heaviside function defined in Example 2 that

$$\lim_{t\to 0^-} H(t) = 0 \quad \text{and} \quad \lim_{t\to 0^+} H(t) = 1.$$

2.4 Existence and Non-existence of Limits

Limits do not have to exists but it is important that we keep in mind: ***That is the existence of a limit of a function f as x approaches x_0 from both sides (or from one side) does not depend on whether f is defined at x_0 but only whether f is defined for x near x_0.*** For example, let us modify the function (1.1) as

$$f(x) = \begin{cases} \frac{16-x^2}{x+4} & , \quad x \neq -4 \\ -17 & , \quad x = -4 \end{cases}$$

Then f is defined at -4 and $f(-4) = -17$. However, the limit of $f(x)$ as x approaches -4 is still 8. If both left-hand and right-hand limits exists and has a common value L, i.e. $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L$, then we say

$$\lim_{x \to x_0} f(x) = L. \tag{1.3}$$

Limit such as (1.3) is said to be ***Two-sided limit***. The following theorem establishes the existence of limits.

Theorem 2.5 (Existence of Limits)**.** *Let f be defined on the left and on the right of a point x_0. Then the limit $\lim_{x \to x_0} f(x)$ exists if and only if $\lim_{x \to x_0^-} f(x)$ exists and $\lim_{x \to x_0^+} f(x)$ exists and*

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x).$$

In this regard, we have from equation (1.3) that

Theorem 2.6. $\lim_{x \to x_0} f(x) = L$ *if and only if* $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = L$.

Example 3. Determine whether or not the limit of the following function exists at x=4. Hence or otherwise find the limit of the function as x approaches 4.

$$f(x) = \begin{cases} \sqrt{x-4} & , \quad x > 4 \\ 8 - 2x & , \quad x < 4 \end{cases}$$

Solution

We have that

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x-4} = 0 \quad \text{and} \quad \lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} 8 - 2x = 0.$$

Hence by Theorem 2.5, $\lim_{x \to 4} f(x)$ exists and by Theorem 2.6, $\lim_{x \to 4} f(x) = 0$

We also note that existence of limit implies uniqueness. That is to say that if $\lim_{x\to x_0} f(x)$ *exists, then it is unique.*

In general the two-sided limit $\lim_{x\to x_0} f(x)$ does not exist

- if either the one-sided limits $\lim_{x\to x_0^-} f(x)$ or $\lim_{x\to x_0^+} f(x)$ fails to exists or
- if $\lim_{x\to x_0^-} f(x) = L_1$ and $\lim_{x\to x_0^+} f(x) = L_2$ but $L_1 \neq L_2$.

Example 4. We consider the limits of the following functions by first looking at their one-sided limits.

1. Let f be defined by

$$f(x) = \begin{cases} x & , \quad x \leq 0 \\ \sqrt{x} & , \quad x > 0 \end{cases}$$

   Then

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x = 0 \quad \text{and} \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{x} = 0.$$

   Thus

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0.$$

   Therefore $\lim_{x\to 0} f(x)$ exists and it is 0.

2. Let $f(x)$ be defined by

$$f(x) = \frac{x}{|x|}, \quad x \neq 0.$$

   For $x > 0$, $|x| = x$ and thus

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x}{|x|} = \lim_{x\to 0^+} \frac{x}{x} = 1.$$

   For $x < 0$, $|x| = -x$ and thus

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{-x} = -1.$$

   Hence the limit does not exist at $x = 0$ since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

3. It is also evident from the Example 2 that the Heaviside function has no limit as $t \to 0$.

2.7 Infinite Limits

Let f be a function defined on both sides of x_0, except possible at x_0 itself. Then

$$\lim_{x\to x_0} f(x) = \infty \tag{1.4}$$

means that the values of $f(x)$ can be made arbitrary large by taking x sufficiently close to x_0 but not equal to x_0. It is often read as

- limit of $f(x)$ as x approaches x_0 is infinite or
- $f(x)$ becomes infinite as x approaches x_0 or
- $f(x)$ increases without bound as x approaches x_0.

Equation (1.4) does not mean the limit exist. It expresses the particular way in which the limit does not exist.

In a similar way,

$$\lim_{x \to x_0} f(x) = -\infty \tag{1.5}$$

means that the values of $f(x)$ can be made arbitrary large negative by taken x sufficiently close to x_0 but not equal to x_0. That is $f(x)$ decreases without bound as x approaches x_0.

Example 5.

$$\lim_{x \to 0} \left(\frac{1}{x^2}\right) = \infty \quad \text{and} \quad \lim_{x \to 0} \left(-\frac{1}{x^2}\right) = -\infty$$

Definition 2.8 (Vertical Asymptote)**.** The line $x = a$ is called the vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true

$$\begin{array}{ccccc} \lim_{x \to a} f(x) = \infty & , & \lim_{x \to a^-} f(x) = \infty & , & \lim_{x \to a^+} f(x) = \infty \\ \lim_{x \to a} f(x) = -\infty & , & \lim_{x \to a^-} f(x) = -\infty & , & \lim_{x \to a^+} f(x) = -\infty \end{array}$$

Quick Practice 2. Find the vertical asymptotes of the following functions;

$$f(x) = \tan(x) \quad \text{and} \quad f(x) = \log(x)$$

2.9 Limit Theorems

We have thus far discussed informally the existence and non-existence of limits of functions at a point. However, it is not practical to reach a conclusion about the existence of limits based on a graph or a table of numerical values. In this section, we present theorems that will help us evaluate limits in a somewhat mechanical fashion. Some of the proofs of these theorems, like the squeeze principle, are provided while others are omitted. The proof of the squeeze principle is based on the formal definition of limit. The student is encouraged to provide proofs for them. They are available in many calculus textbooks. The first of these limits theorems is the following fundamental limit theorems.

Theorem 2.10 (Fundamental Limits)**.** *Let c be a constant.*

- $\lim_{x\to x_0} c = c$
- $\lim_{x\to x_0} x = x_0$

The limit of a constant multiple of $f(x)$ is the constant times the limit of $f(x)$ as x approaches x_0.

Theorem 2.11 (Limit of a Constant Multiple)**.** *Let c be a constant.*

$$\lim_{x\to x_0} cf(x) = c \lim_{x\to x_0} f(x)$$

For example

$$\lim_{x\to -3} -\frac{5}{3}x = -\frac{5}{3}\lim_{x\to -3} x = -\frac{5}{3}\cdot -3 = 5$$

and also

$$\lim_{x\to 0} -x^2 = -\lim_{x\to 0} x^2 = -1\cdot(0) = 0.$$

The following theorem gives us a way of computing limits in an algebraic manner. This theorem is useful when computing the sums, products and quotients of limits of functions

Theorem 2.12 (Sums, Products and Quotients)**.** *Suppose that c is a constant and $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ exist. If $\lim_{x\to x_0} f(x) = L_1$ and $\lim_{x\to x_0} g(x) = L_2$, then*

1. *the limit of the sum (difference) of $f(x)$ and $g(x)$ is the sum (difference) of the limits of $f(x)$ and the limit of $g(x)$. That is*

$$\lim_{x\to x_0} [f(x) \pm g(x)] = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x) = L_1 \pm L_2$$

2. *then limit of the product of $f(x)$ and $g(x)$ is the product of the limit of $f(x)$ and the limit of $g(x)$. That is*

$$\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} f(x)\cdot \lim_{x\to x_0} g(x) = L_1 \cdot L_2$$

3. *the limit of the quotient of $f(x)$ and $g(x)$ is the quotient of the limit of $f(x)$ and the limit of $g(x)$ provided the denominator is not zero. That is if $g(x) \neq 0$ then*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \frac{L_1}{L_2}$$

provided $\lim_{x\to x_0} g(x) \neq 0$

We note that if all the limits exists, the Theorem 2.12 is also applicable to the one-sided limits. Moreover, Theorem 2.12 extents to sums, differences, products and quotients of more than two functions.

The second item of Theorem 2.12 can be used to calculate the limit of a positive integer power of functions. For example if $g(x) = f(x)$, then $L_1 = L_2$ and thus

$$\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} f(x) = L_1 \cdot L_2 = L_1^2 = \lim_{x\to x_0} [f(x)]^2.$$

This reasoning can be applied to the general case when $f(x)$ is a factor n times. This is stated in the Theorem below.

Theorem 2.13 (Limits of a Power)**.** *Let* $\lim_{x\to x_0} f(x)$ *exist and let* $n \in \mathbb{Z}^+$*. If* $\lim_{x\to x_0} f(x) = L$*, then*

$$\lim_{x\to x_0} [f(x)]^n = \left[\lim_{x\to x_0} f(x)\right]^n.$$

In particular, if $f(x) = x$, then

$$\lim_{x\to x_0} [f(x)]^n = \left[\lim_{x\to x_0} x\right]^n = x_0^n.$$

The limit of the nth root of a function is the nth root of the limit whenever the limit exists and has a real nth root. This fact is summarized in the theorem below.

Theorem 2.14 (Limits of a Power)**.** *Let* $\lim_{x\to x_0} f(x)$ *exist and let* $n \in \mathbb{Z}^+$*. If* $\lim_{x\to x_0} f(x) = L$*, then*

$$\lim_{x\to x_0} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to x_0} f(x)} = \sqrt[n]{L}.$$

If n *is even, we assume* $L \geq 0$.

Limit of Polynomial Functions

Some limits can be evaluated by direct substitution. Let $r \in \mathbb{Z}^+$ and consider the polynomial

$$f(x) = \sum_{r=0}^{n} \alpha_r x^r \quad \text{where } \alpha_r \in \mathbb{R} \text{ (constants)}, \ \forall n \in \mathbb{Z}^+.$$

Then applying Theorems 2.12, 2.11 and 2.13, we get that

$$\lim_{x\to x_0} f(x) = \sum_{r=0}^{n} \alpha_r \lim_{x\to x_0} x^r = \sum_{r=0}^{n} \alpha_r x_0^r = f(x_0).$$

In other words to evaluate a limit of a polynomial function of x as x approaches a real number x_0, we only need to evaluate the polynomial at $x = x_0$:

$$\lim_{x\to x_0} f(x) = f(x_0).$$

Also let $r(x) = \frac{p(x)}{q(x)}$ be a rational function. Then if x_0 is in the domain of $r(x)$ then

$$\lim_{x\to x_0} r(x) = \lim_{x\to x_0} \frac{p(x)}{q(x)} = \frac{p(x_0)}{q(x_0)}.$$

The proof of this Theorem 2.15 below is left to the student. It can be established from the precise definition of limits (see Definition 2.17 below).

Theorem 2.15. *If $f(x) \le g(x)$ whenever x is near x_0, except possibly at x_0, and the limits of f and g both exist as $x \to x_0$, the*

$$\lim_{x\to x_0} f(x) \le \lim_{x\to x_0} g(x)$$

The next theorem is very useful in evaluating some limits. It has many names: ***Squeeze Theorem, Pinching Theorem, Sandwich Theorem, Squeeze Play Theorem, Flyswatter Theorem.*** The theorem basically says that if the graph of $f(x)$ is "squeezed" between the graph of two functions $g(x)$ and $h(x)$ for all x near x_0, and if the functions g and h have a common limit L as $x \to x_0$, then there is no harm to reason that $f \to L$ as $x \to x_0$. The proof of Theorem 2.16 is given after the introduction of the precise definition of limit.

Theorem 2.16 (Squeeze/Sandwich/Pinching Theorem)**.** *If $f(x) \le g(x) \le h(x)$ when x is near x_0, except possibly at x_0 and $\lim_{xf\to x_0} f(x) = \lim_{x\to x_0} = L$ then*

$$\lim_{x\to x_0} g(x) = L.$$

Example 6. Evaluate the limit

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right).$$

Solution

Here we cannot apply the product law of limits since $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exists. (The law requires that both limits of f and g must exist). But we know that $|\sin(\alpha)| \leq 1,\ \forall\alpha \in \mathbb{R}$. Thus

$$\left|\sin\left(\frac{1}{x}\right)\right| \leq 1 \Longleftrightarrow -1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

and since $x^2 \geq 0$, we get that

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2.$$

Now

$$\lim_{x\to 0}(-x^2) = 0 = \lim_{x\to 0} x^2.$$

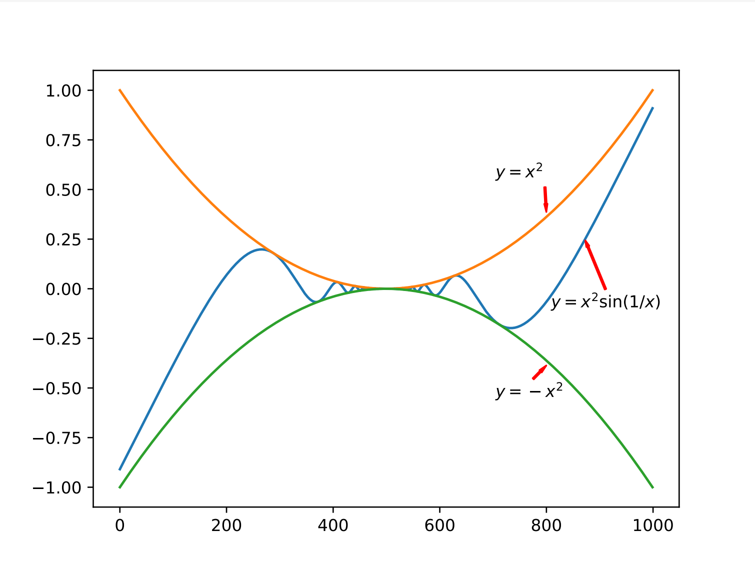
Hence by Theorem (2.15)

$$\lim_{x\to 0}(-x^2) \leq \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) \leq \lim_{x\to 0} x^2 \Longrightarrow 0 \leq \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) \leq 0.$$

Therefore

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

by Theorem (2.16). Using a graphical calculator, or by any other means, we observe the following;



Let us now look at the precise definition of limits and then prove Theorem (2.16). Students interested in the proof of Theorem (2.15) can consult any advanced calculus book.

Definition 2.17 (Precise Definition of Limit)**.** Let f be a function defined on some open interval that contains the number x_0 except possibly at x_0 itself. Then we say that the limit of $f(x)$ as $x \to x_0$ is L if $\forall\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - x_0| < \delta$ (or if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \epsilon$).

Definition 2.18 (Precise Definition of Left-Hand Limit)**.** We say that $\lim_{x \to x_0^-} f(x) = L$ if $\forall\epsilon > 0,\ \exists\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x_0 - \delta < x < x_0$

Definition 2.19 (Precise Definition of Right-Hand Limit)**.** We say that $\lim_{x \to x_0^+} f(x) = L$ if $\forall\epsilon > 0,\ \exists\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x_0 < x < x_0 + \delta$

Definition 2.20 (Precise Definition of Infinite Limit)**.** Let f be a function defined on some open interval that contains the number x_0 except possibly at x_0 itself. Then $\lim_{x \to x_0} f(x) = \infty$ means that $\forall M > 0,\ \exists\delta > 0$ such that $f(x) > M$ whenever $0 < |x - x_0| < \delta$.

These definitions will be elaborated on in your real analysis courses.

Proof of Theorem 2.16. Let $\epsilon > 0$ be given. By definition $\exists\delta_1,\ \delta_2$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - x_0| < \delta_1$ and $|h(x) - L| < \epsilon$ whenever $0 < |x - x_0| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. If $|x - x_0| < \delta$ then $|x - x_0| < \delta_1$ and $|x - x_0| < \delta_2$ so that

$$L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon$$

and the proof follows. □

2.21 Limits Involving Trigonometric Functions

We shall discuss the case where the function involve trigonometric expressions. Consider the function $f(\theta) = \frac{\sin(\theta)}{\theta}$ whose graph is the Figure 1.2 below.

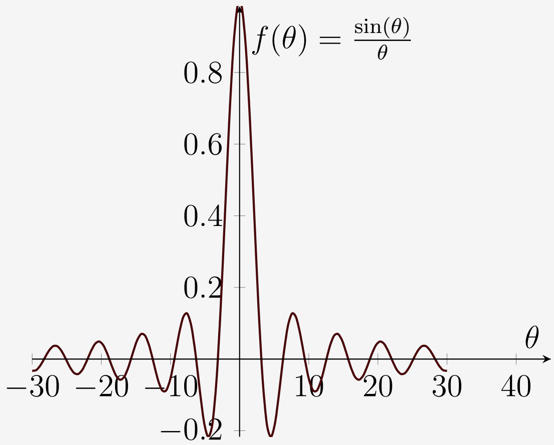


Figure 1.2: Graph of $f(\theta) = \frac{\sin(\theta)}{\theta}$

The value $\theta = 0$ is not in the domain of f. However, Figure 1.2 suggests that the limit of

$f(\theta)$ as θ approaches zero exists. We are able to provide a justification for this using the squeeze principle (see Theorem 2.16). Consider a unit circle centred at the origin such as Figure 1.3 below. AC is an arc of a unit circle. Hence $|OC| = |OA| = 1$. $\triangle ODC$ and

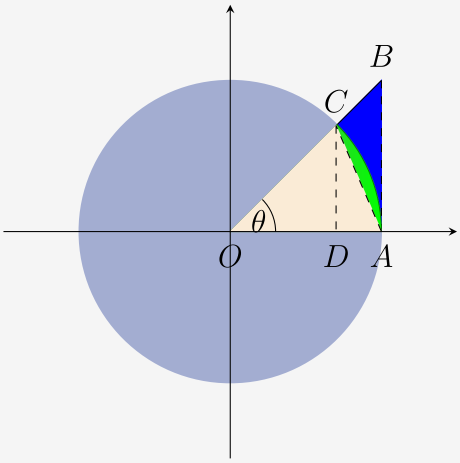


Figure 1.3: A Unit Circle

$\triangle OAB$ are right-angled. From the diagram, we see that

$$|DC| = |OC|\sin(\theta) = \sin(\theta)$$

and by similar triangle

$$\frac{|DC|}{|AB|} = \frac{|OD|}{|OA|} = |OD| = |OC|\cos(\theta) = \cos(\theta).$$

Therefore

$$|AB| = \frac{|DC|}{\cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta).$$

Now from Figure 1.3, it is evident that

$$\text{Area of } \triangle OAC \leq \text{Area of Sector } OAC \leq \text{Area of } \triangle OAB.$$

Hence by recalling the formula for area of triangle (i.e. $1/2bh$), we obtain

$$\frac{1}{2}\sin(\theta) \leq \frac{1}{2}\theta \leq \frac{1}{2}\tan(\theta) \iff 1 \leq \frac{\theta}{\sin(\theta)} \leq \frac{1}{\cos(\theta)}.$$

By Theorem (2.15), we get that

$$1 = \lim_{\theta\to 0} 1 \leq \lim_{\theta\to 0}\frac{\theta}{\sin(\theta)} \leq \lim_{\theta\to 0}\frac{1}{\cos(\theta)} = 1.$$

Therefore by applying Theorem 2.16, we have shown that

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.$$

We have thus proved the following theorem.

Theorem 2.22.

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

As a consequence, we obtain the following corollary.

Corollary 2.23.

$$\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} = 0$$

Proof. By writing $\cos(\theta) = \cos\left(\frac{\theta}{2} + \frac{\theta}{2}\right)$ we have that $\cos(\theta) - 1 = -2\sin^2\left(\frac{\theta}{2}\right)$. Therefore

$$\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} = -\lim_{\theta \to 0} \frac{2\sin^2\left(\frac{\theta}{2}\right)}{\theta} = -\lim_{\theta \to 0} \frac{\sin^2\left(\frac{\theta}{2}\right)}{\frac{\theta}{2}} = -\lim_{\theta \to 0} \left(\frac{\sin\left(\frac{\theta}{2}\right)}{\frac{\theta}{2}}\right)^2 \frac{\theta}{2}.$$

Applying laws of limits and Theorem(2.22), we obtain

$$\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} = -\lim_{\theta \to 0} \left(\frac{\sin\left(\frac{\theta}{2}\right)}{\frac{\theta}{2}}\right)^2 \lim_{\theta \to 0} \frac{\theta}{2} = 1 \cdot (0) = 0$$

$\square$

Example 7. Evaluate the following limits.

1.
$$\lim_{x \to -3} \frac{\sin(x+3)}{x^2 + 7x + 12}.$$

We observe that $x^2 + 7x + 12 = (x+3)(x+4)$. Therefore making the substitution $t = x + 3$, we obtain

$$\lim_{x \to -3} \frac{\sin(x+3)}{x^2 + 7x + 12} = \lim_{x \to -3} \frac{\sin(x+3)}{(x+3)(x+4)} = \lim_{t \to 0} \frac{\sin(t)}{t(t+1)}.$$

By limit laws

$$\lim_{t \to 0} \frac{\sin(t)}{t(t+1)} = \lim_{t \to 0} \frac{\sin(t)}{t} \cdot \lim_{t \to 0} \frac{1}{(t+1)} = 1.$$

2.

$$\lim_{x\to 0}\frac{\sin(3x)}{\sin(8x)}.$$

We observe that

$$\lim_{x\to 0}\frac{\sin(3x)}{\sin(8x)} = \frac{3}{8}\lim_{x\to 0}\frac{\sin(3x)}{3x}\frac{8x}{\sin(8x)} = \frac{3}{8}\cdot 1\cdot 1 = \frac{3}{8}$$

3.

$$\lim_{x\to 0}\frac{\sin|x|}{x}.$$

We leave this to the student to show that the limit does not exist.

2.24 Limit at Infinity

In this section we consider limits where the independent variable becomes lager than any positive number we can think of, (∞), or smaller than any negative number we can imagine, $(-\infty)$.

Definition 2.25. Let f be a function and $L \in \mathbb{R}$. We say that $f(x) \to L$ as $x \to \pm\infty$ if $f(x)$ is getting closer to L as x is becoming very large than any positive number $(x \to \infty)$ (respectively smaller than any negative number, i.e. $x \to -\infty$).

Definition 2.26 (Horizontal Asymptote)**.** The line $y = L$ is called the horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

Example 8. Find the horizontal asymptotes of the following functions;

1. $f(x) = \frac{1}{x}$, $x \neq 0$. Letting $x \to \infty$, we get $f(x) \to 0$. Therefore the line $y = 0$ is the horizontal asymptote.

2. $f(x) = \frac{3x^2-x-2}{5x^2+4x+4}$. As $x \to \infty$ we get $f(x) \to \frac{3}{5}$. Therefore the line $y = \frac{3}{5}$ is the horizontal asymptote.

3. $f(x) = x^3$. Letting $x \to \infty$ we get $f(x) \to \infty$. Hence the function has no horizontal asymptote.

Theorem 2.27. *If $r > 0$ is a rational number, then*

$$\lim_{x\to\infty} \frac{1}{x^r} = 0.$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0.$$

Quick Practice 3. Find the horizontal asymptotes of the following functions

$$f(x) = \frac{\sin(x)}{x} \quad \text{and} \quad g(x) = \sqrt{x^2+1} - x$$

Remark 2.28. We take note of the following facts;

- If $g(x) \le f(x) \le h(x)$ and $\lim_{x\to\pm\infty} g(x) = \lim_{x\to\pm\infty} h(x) = L$. Then

$$\lim_{x\to\pm\infty} f(x) = L.$$

- Since $f(x) \le |f(x)|$, if $\lim_{x\to\pm\infty} |f(x)| = 0$, then $\lim_{x\to\pm\infty} f(x) = 0$
- $\lim_{x\to\pm\infty}[f(x) \pm g(x)] = \lim_{x\to\pm\infty} f(x) \pm \lim_{x\to\pm\infty} g(x)$ provided the sum is not $\infty - \infty$ or $-\infty + \infty$.
- $\lim_{x\to\pm\infty} f(x)g(x) = \lim_{x\to\pm\infty} f(x) \cdot \lim_{x\to\pm\infty} g(x)$ provided the product is not $0 \cdot (\pm\infty)$.
- For $n, m \in \mathbb{Z}^+$, we have that

$$\lim_{x\to\pm\infty} \sum_{r=0}^{n} a_r x^r = \lim_{x\to\pm\infty} a_n x^n \quad \text{and} \quad \lim_{x\to\pm\infty} \frac{\sum_{r=0}^{n} a_r x^r}{\sum_{r=0}^{m} b_r x^r} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m}.$$

  When the degree of the numerator of the rational function is higher than the degree of the denominator, then the graph of the rational function has an asymptote.

  Definition 2.29 (Oblique/Slant Asymptote)**.** We say that the line $y = ax + b$ is an oblique asymptote to the graph of $f(x)$ if

$$\lim_{x\to\pm\infty} (f(x) - (ax+b)) = 0.$$

  For the graph to have an oblique asymptote, $\deg(numerator) = \deg(denominator) + 1$.

- If $f(x) \le g(x)$ for all x and $\lim_{x\to\pm\infty} g(x) = -\infty$ then $\lim_{x\to\pm\infty} f(x) = -\infty$. Also

$\lim_{x\to x_0} g(x) = -\infty$ implies that $\lim_{x\to x_0} f(x) = -\infty$.

- If $g(x) \le f(x)$ for all x and $\lim_{x\to\pm\infty} g(x) = \infty$ then $\lim_{x\to\pm\infty} f(x) = \infty$. Also $\lim_{x\to x_0} g(x) = \infty$ implies that $\lim_{x\to x_0} f(x) = \infty$.

Example 9. Find the slant asymptote of the function

$$f(x) = \frac{2x^2 + x - 1}{x + 2}.$$

2.30 Limits of Functions Involving Inverse Trigonometric Expressions

Inverse Trigonometric Functions - A Brief Discussion

In this section we shall evaluate limits involving inverse of trigonometric functions. Let us briefly recall these inverse functions.

`The Sine Function`

Consider the graph in the Figure 1.4 below. We recall that the mapping $f : \mathbb{R} \longrightarrow$

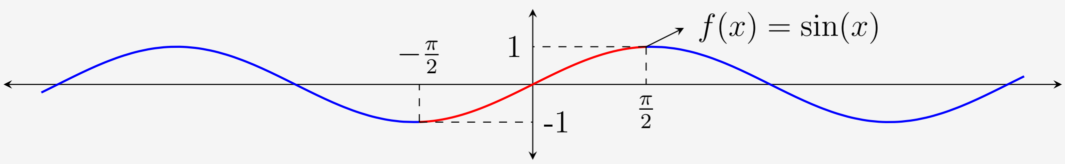


Figure 1.4: Graph of Sine function

$[-1, 1]$ defined by $f(x) = \sin(x)$ is not injective and hence has no inverse. However upon restriction the domain to the interval $[-\pi/2, \pi/2]$, it becomes a bijective mapping with an inverse. This inverse is the mapping $g : [-1, 1] \longrightarrow [-\pi/2, \pi/2]$ defined by $g(x) = \sin^{-1}(x)$. That is

$$y = \sin(x) \iff x = \sin^{-1}(y) \quad \text{whenever} \quad x \in [-\pi/2, \pi/2] \text{ and } y \in [-1, 1].$$

`The Cosine Function`

Consider the graph in the Figure 1.5 below. Similarly as in the sine function, we have for the cosine function

$$y = \cos(x) \iff x = \cos^{-1}(y) \quad \text{whenever} \quad x \in [0, \pi] \text{ and } y \in [-1, 1].$$

`The Tangent Function`

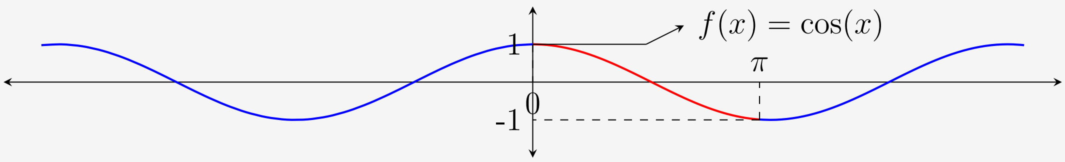


Figure 1.5: Graph of Cosine function

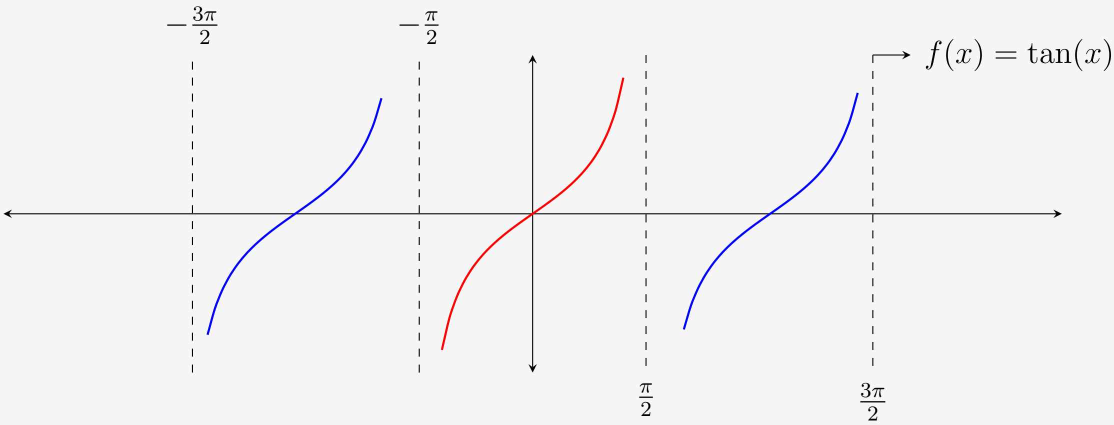


Figure 1.6: Graph of Tangent function

The tangent function is also such that

$$y = \tan(x) \iff x = \tan^{-1}(y) \quad \text{whenever} \quad x \in (-\pi/2, \pi/2) \text{ and } y \in \mathbb{R}.$$

The graph of the tangent function is the Figure 1.6 above.

Example 10. 1. Evaluate

$$\lim_{x \to 0} \frac{\sin^{-1}(x)}{x}.$$

Let $x = \sin(y)$. Then $y \to 0$ as $x \to 0$. Therefore

$$\lim_{x \to 0} \frac{\sin^{-1}(x)}{x} = \lim_{y \to 0} \frac{y}{\sin(y)} = 1$$

2. Evaluate

$$\lim_{x \to 1} \frac{\cos^{-1}(x)}{x - 1}.$$

Let $x = \cos(y)$. Then $y \to 0$ as $x \to 1$. Therefore

$$\lim_{x \to 1} \frac{\cos^{-1}(x)}{x - 1} = \lim_{y \to 0} \frac{y}{\cos(y) - 1} = \lim_{y \to 0} \frac{y}{-2\sin^2(y/2)}$$

since $\cos(y) - 1 = -2\sin^2(y/2)$. Let $a = y/2$ and obtain

$$\lim_{y \to 0} \frac{y}{-2\sin^2(y/2)} = \lim_{a \to 0} \frac{2a}{-2\sin^2(a)} = -\lim_{a \to 0} \frac{a^2}{\sin^2(a)} \cdot \frac{1}{a} = -\infty.$$

3. Evaluate

$$\lim_{x\to 0} \frac{x}{\tan^{-1}(x)}.$$

The answer is 1.

2.31 Exponential and Logarithmic Functions

The main focus in this section is to evaluate limits involving exponential and logarithmic expression. Before we do that let us first look at what exponential and logarithmic functions are.

Let c be a constant. Then, the general form of an exponential function is of the form

$$f(x) = ka^x + c, \quad k \neq 0,\ a \neq 1,\ \forall x \in \mathbb{R}.$$

$f(x) = a^x,\ a \neq 1$, is the simplest form of an exponential function. Let e be the number such that

$$\lim_{h\to 0} \frac{\mathrm{e}^h - 1}{h} = 1. \tag{1.6}$$

Then, the function $f(x) = \mathrm{e}^x$ is called the ***Natural Exponential Function.*** Alternatively, we can write $f(x) = \exp(x)$. The Figure 1.7 below is the graph of the natural exponential function.

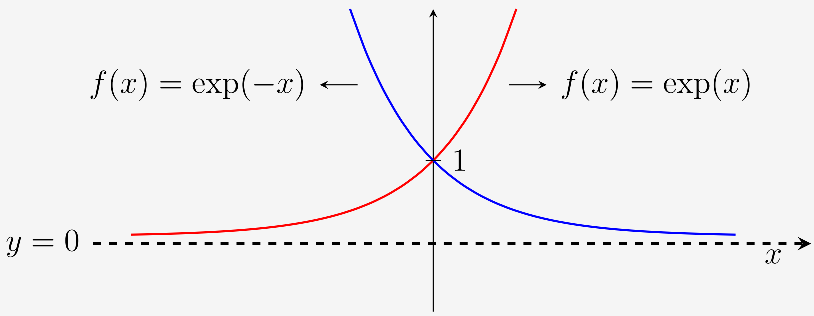


Figure 1.7: Graph of Natural Exponential Function

Some Properties of the Natural Exponential Function

1. $\exp(x) > 0$ for all $x \in \mathbb{R}$. The graph is asymptotic to the x-axis. That is the horizontal asymptote is the line $y = 0$. i.e. $\lim_{x\to-\infty} \exp(x) = \lim_{x\to\infty} \exp(-x) = 0$
2. $\exp x + y = \exp(x) \cdot \exp(y)$
3. $\exp(-x) = \frac{1}{\exp(x)}$ and $\exp(0) = 1$.

From the graph it is evident that the exponential function, $f(x) = a^x$ is either increasing or decreasing. So it is injective by horizontal line test. It therefore has an inverse function which is called the Logarithmic function with base a and it is denoted $\log_a$. That is

$$\log_a x = y \Longleftrightarrow a^y = x.$$

When $a = \mathrm{e}$, then we have the ***Natural Logarithmic Function*** and it is denoted as $\ln$. That is $\log_{\mathrm{e}} x = \ln(x)$ and thus

$$\ln(x) = y \Longleftrightarrow \mathrm{e}^y = x.$$

The graph of $\ln(x)$ is obtained by the reflection of $y = \mathrm{e}^x$ in the line $y = x$. This is shown in the Figure 1.8 below. We deduce from the graph that the domain of the logarithmic

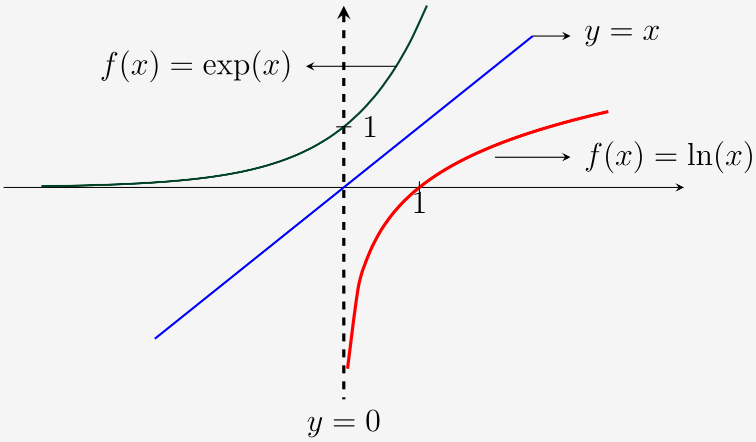


Figure 1.8: Graph of Natural Logarithmic Function

function is $(0, \infty)$ and its range if $\mathbb{R}$. The graph is asymptotic to the line $x = 0$. In other words the y-axis is a vertical asymptote of the function $f(x) = \ln(x)$. It is also obvious that $\ln(x) \geq 0$ when $x \geq 1$ and $\ln(x) < 0$ when $0 < x < 1$. $\ln(x)$ is also slowly increasing for $x > 1$. Some of its properties are as follows

1. $\ln(xy) = \ln(x) + \ln(y)$ and $\ln(x/y) = \ln(x) - \ln(y)$
2. $\ln(x^r) = r\ln(x)$ and $\ln(\mathrm{e}) = \mathrm{e}^{\ln} = 1$
3. $\ln(\mathrm{e}^x) = x$ when $x \in \mathbb{R}$ and $\mathrm{e}^{\ln(x)} = x$ when $x > 0$.

Remark 2.32. In second year calculus, you shall define the natural logarithmic function as the integral

$$\ln(x) = \int_0^x \frac{1}{t}\mathrm{d}t.$$

The following is a useful result that we will elaborate on later;
Asume that $\lim_{x\to x_0} g(x) = L$. *Then* $\lim_{x\to x_0} f(g(x)) = \lim_{x\to L} f(x)$.

Example 11. 1. Evaluate the following limits;

$$\lim_{x\to 0} \ln(1+\sin(x)) \quad \text{and} \quad \lim_{x\to\infty} \mathrm{e}^{x-x^2}.$$

We see that $\ln(1+\sin(x)) = f(g(x))$ where $f(x) = \ln(x),\ g(x) = 1+\sin(x)$. Now $\lim_{x\to 0} g(x) = 1$. Therefore

$$\lim_{x\to 0} \ln(1+\sin(x)) = \lim_{x\to 1} \ln(x) = 0.$$

Also we have

$$\mathrm{e}^{x-x^2} = \mathrm{e}^{-(x^2-x)} = f(g(x))$$

where $f(x) = \mathrm{e}^{-x}$ and $g(x) = x^2 - x$. Now $\lim_{x\to\infty} g(x) = \infty$. Hence

$$\lim_{x\to\infty} \mathrm{e}^{x-x^2} = \lim_{x\to\infty} \mathrm{e}^{-x} = 0$$

Quick Practice 4. Evaluate each of the following limits

1. $\lim_{x\to 0} \ln\left(1+\frac{\sin(x)}{x}\right)$ and $\lim_{x\to 0} \ln\left(1-\frac{\sin(x)}{x}\right)$
2. $\lim_{x\to\infty} \frac{\mathrm{e}^x}{x}$, $\lim_{x\to\infty} \frac{\ln(x)}{x}$ and $\lim_{x\to 0} \frac{\mathrm{e}^x - 1}{x}$

3 Continuous Functions

We shall study, in this section, how "nice" the graph of a function is.

3.1 Continuity at a Point

We shall study the continuity of a function only at points of the domain of the function.

Definition 3.2 (Continuity at a Point)**.** Let f be a function defined on an interval I and let $x_0 \in I$. Then we say f is continuous at x_0 if and only if

$$\lim_{x\to x_0} f(x) = f(x_0).$$

If $\lim_{x\to x_0} f(x) \neq f(x_0)$, we say f is ***not continuous at*** x_0 ***for*** f ***is discontinuous at*** x_0. We also say in this case that x_0 ***is a point of discontinuity.***

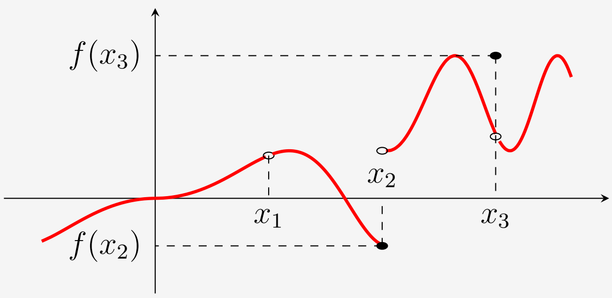
Remark 3.3. Implicitly, the definition requires three things if a function is continuous at x_0

- $f(x_0)$ is defined. That is x_0 is in the domain of the function f.

- $\lim_{x\to x_0} f(x)$ exists. So f must be defined on an open interval that contains x_0.
- $\lim_{x\to x_0} f(x) = f(x_0)$.

Thus the definition says that f is continuous at x_0 if $f(x) \to f(x_0)$ as $x \to x_0$. That is small changes in x produces only small changes in $f(x)$. More precisely, $\forall\epsilon > 0$, $\exists\delta >$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. The above three conditions are the ***Continuity Criteria.***

Example 12. 1. Consider the diagram below



(a) The function is discontinuous at x_1 because there is a break there. The reason being that $f(x_1)$ is not defined.

(b) The graph also breaks at x_2 but the reason for the discontinuity is different. Here $f(x_2)$ is defined but $\lim_{x\to x_2} f(x)$ does not exists since

$$\lim_{x\to x_2^+} f(x) \neq \lim_{x\to x_2^-} f(x)$$

(c) At x_3, $f(x_3)$ is defined and $\lim_{x\to x_3} f(x)$ exists but

$$\lim_{x\to x_3} f(x) \neq f(x_3).$$

Hence x_3 is a point of discontinuity.

2. Consider the function

$$f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & , \quad x \neq 2 \\ 1 & , \quad x = 2 \end{cases}$$

Here $f(2) = 1$ is defined and

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2 - x - 2}{x - 2} = 3$$

exists. But $\lim_{x\to 2} f(x) = 3 \neq f(2) = 1$. Therefore f is discontinuous at 2.

3. Consider the function

$$f(x) = \begin{cases} \frac{2}{x^2} & , \quad x \neq 0 \\ 1 & , \quad x = 0 \end{cases}$$

Here $f(0) = 1$ is defined but

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{2}{x^2} = \infty$$

does not exist. f is therefore discontinuous at 0.

4. Consider the function

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & , \quad x \neq 0 \\ 1 & , \quad x = 0 \end{cases}$$

We have

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin(x)}{x} = 1 = f(0).$$

Thus f is continuous at 0.

Quick Practice 5. Show that the greatest integer function is a discontinuous function.

3.4 Left and Right Continuity at a Point

Just as we defined left and right limits of functions, we do same for continuity.

Definition 3.5. Let f be a function defined on an interval I and $x_0 \in I$ be a point.

- We say f is ***right continuous*** at x_0 if $\lim_{x \to x_0^+} f(x)$ exists and $\lim_{x \to x_0^+} f(x) = f(x_0)$
- We say f is ***left continuous*** at x_0 if $\lim_{x \to x_0^-} f(x)$ exists and $\lim_{x \to x_0^-} f(x) = f(x_0)$

The simple proof of the following theorem, which basically follows from definition, is left to the student.

Theorem 3.6. *The function f defined at x_0 is continuous at x_0 if and only if f is right continuous and left continuous at x_0.*

3.7 Continuity on an Interval

Let $I = [a, b]$, $a \neq b$, and let f be a function defined on I. Then f is continuous at a if f is right continuous at a and f is continuous at b if f is left continuous at b. The reason being that f is not defined outside the interval I i.e. f is not defined to the left of a and to the right of b.

Definition 3.8 (Continuity on an Interval)**.** A function f defined on a bounded interval $I = [a, b]$ is continuous on I if f is continuous at each point of (a, b) and right continuous at a and left continuous at b. More generally, a function is continuous on a set A if f is defined on A and f is continuous at each point of A.

Example 13. Show that the function $f(x) = 1 - \sqrt{1-x^2}$ is continuous on the interval $[-1, 1]$. We show this by first assuming that $a \in (-1, 1)$. Now by laws of limits, we have

$$\lim_{x\to a} f(x) = \lim_{x\to a} 1 - \sqrt{1-x^2} = 1 - \sqrt{1-a^2} = f(a).$$

Thus by definition f is continuous on $(-1, 1)$. Now, by similar calculation, we have $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} 1 - \sqrt{1-x^2} = 1 = f(-1)$ and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1 - \sqrt{1-x^2} = 1 = f(1)$. Hence f is continuous on $[-1, 1]$.

The following are some useful results of continuous functions.

Theorem 3.9. *If f and g are continuous at a and c is a constant, then the following are also continuous functions at a.:* $f + g, \quad , f - g, \quad , cf, \quad , f \cdot g, \quad , \frac{f}{g},\ g \neq 0.$

Proof. Let f, g be continuous at a. Then, this means that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$. By limit laws, we have

$$\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = f(a) + g(a) = (f + g)(a).$$

This shows the continuity of $f + g$. The proof of the rest is left to the students. □

Theorem 3.10. *A function f is continuous at a if and only if*

$$\lim_{h\to 0} f(a + h) = f(a).$$

Proof. Suppose f is continuous at a. Then by definition, $\lim_{x\to a} f(x) = f(a)$. Let $h = x - a$. Then $h \to 0$ as $x \to a$ and $x = a + h$. Therefore

$$f(a) = \lim_{x\to a} f(x) = \lim_{h\to 0} f(a + h).$$

Conversely, suppose $\lim_{h\to 0} f(a + h) = f(a)$. Making the substitution $x = a + h$, we get that

$$f(a) = \lim_{h\to 0} f(a + h) = \lim_{x\to a} f(x)$$

and the proof is complete. □

Theorem 3.11. *The following functions are continuous at x_0 whenever they are defined at x_0.*

$$\sin(x), \quad \cos(x), \quad \tan(x).$$

Proof. By Theorem (3.10), we have that

$$\lim_{x \to x_0} \sin(x) = \lim_{h \to 0} \sin(x_0 + h) = \lim_{h \to 0} [\sin(x_0)\cos(h) + \sin(h)\cos(x_0)] = \sin(x_0).$$

Therefore $\sin(x)$ is continuous. Similarly we can show that $\cos(x)$ is continuous and since $\tan(x) = \frac{\sin(x)}{\cos(x)}$, we can apply Theorem (3.9) to show that $\tan(x)$ is continuous whenever $\cos(x) \neq 0$. □

Theorem 3.12. *The function $\ln(x)$ is continuous at every $x \in (0, \infty)$.*

Proof. By Theorem (3.10), we have that

$$\lim_{x \to x_0} \ln(x) = \lim_{h \to 0} \ln(x_0 + h).$$

Now let $x_0 + h = x_0 \mathrm{e}^t$. The $h = x_0 \mathrm{e}^t - x_0 = x_0(\mathrm{e}^t - 1)$. Since $\ln(x)$ must be defined at x_0, it implies that $x_0 \neq 0$. Hence $h \to 0$ implies that $\mathrm{e}^t \to 1$ and consequently $t \to 0$. Therefore

$$\lim_{h \to 0} \ln(x_0 + h) = \lim_{t \to 0} \ln(x_0 \mathrm{e}^t) = \lim_{t \to 0} (\ln(x_0) + \ln(\mathrm{e}^t)) = \lim_{t \to 0} (\ln(x_0) + t) = \ln(x_0)$$

and the Theorem is proved. □

Remark 3.13. We also note

$$\lim_{h \to 0} \mathrm{e}^{x_0 + h} = \mathrm{e}^{x_0} \lim_{h \to 0} \mathrm{e}^h = \mathrm{e}^{x_0}.$$

Thus the function e^x is continuous at every $x_0 \in \mathbb{R}$.

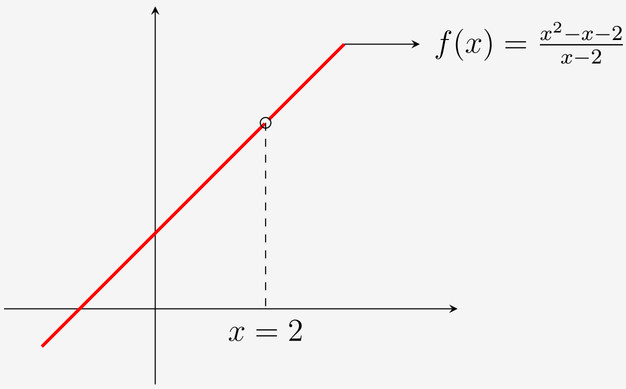
Theorem 3.14.
1. *Every polynomial is continuous at each point of $\mathbb{R}$*
2. *Every rational function is continuous at each point of its domain.*
3. *Let $u(x)$ be a continuous function at x_0. Then $\sqrt[n]{u(x)}$ is continuous at x_0 provided $u(x_0) \geq 0$ for n even.*

3.15 Discontinuity at a Point

We shall discuss the various forms of discontinuities a function could possibly have.

Removable Discontinuity

Let x_0 be a point at which the function f is `not` defined. If $\lim_{x\to x_0} f(x)$ exists, then we say that x_0 is a `removable discontinuity point.` In this case there is an extension of f which is continuous at x_0. For example the function $f(x) = \frac{x^2-x-2}{x-2}$ is discontinuous at $x = 2$



Also $\lim_{x\to 2} \frac{x^2-x-2}{x-2} = \lim_{x\to 2} x + 1 = 3$. Thus the $f(x) \to 3$ as $x \to 2$ i.e. the limit exists as $x \to 3$ but $f(x)$ is not defined at $x = 2$. Therefore $x = 2$ is a removable point of discontinuity. Now define

$$g(x) = \begin{cases} \frac{x^2-x-2}{x-2} & , \quad x \neq 2 \\ 3 & , \quad x = 2 \end{cases}$$

Then $\lim_{x\to 2} g(x) = 3 = g(2)$ which means that $g(x)$ is continuous at $x = 2$. That is $g(x)$ is an extension of $f(x)$ which is continuous at $x = 2$. As another example, consider the function $f(x) = \frac{\sin(x)}{x}$. We see that $f(x)$ is discontinuous at $x = 0$. We have seen that $f(x) \to 1$ as $x \to 0$ but $f(x)$ is not defined at $x = 0$. Therefore $x = 0$ is a removable point of discontinuity. However, a continuous extension is the function

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & , \quad x \neq 0 \\ 1 & , \quad x = 0 \end{cases}$$

Quick Practice 6. Let $f(x) = \frac{x^2+x-6}{x^2-4}$. Find an extension of $f(x)$ which is continuous at at $x = 2$.

Jump Discontinuity

Let f be a function and x_0 a point at which f is defined. If $\lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x)$ exist but

$$\lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x)$$

we say x_0 is a ***Jump Discontinuity.***

Quick Practice 7. Show that the greater integer function is an example of a function with jump discontinuities.

Infinite Discontinuity

Let f be a function and x_0 a point. We say x_0 is a point of infinite discontinuity of f if $\lim_{x \to x_0^-} f(x)$ is finite but $\lim_{x \to x_0^+} f(x) = \pm\infty$ or $\lim_{x \to x_0^+} f(x)$ is finite and $\lim_{x \to x_0^-} f(x) = \pm\infty$.

Example 14. Consider

$$f(x) = \begin{cases} \frac{1}{x} & , \quad x > 0 \\ \sqrt{|x|} & , \quad x \leq 0 \end{cases}$$

The function is defined at zero but

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = \infty.$$

Thus $f(x)$ is discontinuous at $x = 0$. On the other hand

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sqrt{|x|} = 0$$

which is finite. Hence the point of discontinuity is an infinite one.

Essential Discontinuity

This corresponds to the case where we do not have a removable, or jump, or an infinite discontinuity. As an example discuss the discontinuity of $f(x) = \sin\left(\frac{1}{x}\right)$ at $x = 0$.

The following Theorems helps to evaluate limits of continuous functions

Theorem 3.16. *If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words*

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Theorem 3.17. *If g is continuous at a and f is continuous at $g(a)$ then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a*

Proof. Let g be continuous at a. Then $\lim_{x\to a} g(x) = g(a)$. Since f is continuous at $b = g(a)$, we obtain, by Theorem (3.16),

$$\lim_{x\to a} f(g(x)) = f(g(a))$$

and therefore is continuous. □

Quick Practice 8.

1. Show that $f(x) = \sin^{-1}(x)$ is continuous on it domain. Hence evaluate the following limit.

$$\lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right)$$

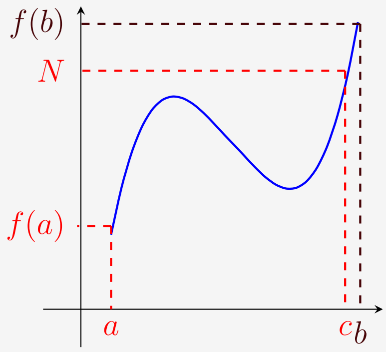
2. Where are the following functions continuous?
   (a) $f(x) = \cos(x^2 + 1)$
   (b) $f(x) = \ln(1 + \sin(x))$
   (c) $f(x) = \frac{\ln(x)+\sin^{-1}(x)+\tan(x)}{x^3-1}$
3. Evaluate the following limits
   (a) $\lim_{x\to 0} \frac{|2x-1|-|2x+1|}{x}$
   (b) $\lim_{x\to 1} \frac{\ln(x)}{x-1}$

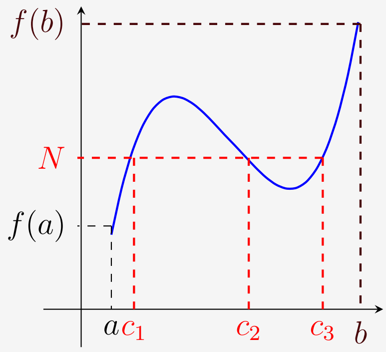
3.18 The Intermediate Value Theorem

An important property of continuous functions is expressed by the following Theorem whose proof can be found in any advanced book on calculus.

Theorem 3.19 (The Intermediate Value Theorem (IVT))**.** *Suppose that f is continuous on the closed interval $[a, b]$ and let N be a number between $f(a)$ and $f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$*

The Theorem says that if f is continuous on $[a, b]$ and y_0 is a point between $f(a)$ and $f(b)$, then the horizontal line passing through y_0 touches the graph of f at least once between a and b. That is the equation $f(x) = y_0$ has at least one solution that lies between a and b.





We see from the diagrams above that the value N can be taken on once (first diagram) or more than once (second diagram). When $N = 0$, we have the following consequence which helps in locating roots of equations.

Corollary 3.20 (Bolzano's Theorem)**.** *Let f be continuous function on $[a, b]$. If $f(a) < 0$ and $f(b) > 0$ (or $f(a) > 0$ and $f(b) < 0$), then the equation $f(x) = 0$ has a solution in the interval (a, b). That is if $f(a) \cdot f(b) < 0$, then $\exists c \in (a, b)$ such that $f(c) = 0$.*

Example 15. Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2. Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. We seek $c \in (1, 2)$ such that $f(c) = 0$. Now $f(1) = -1 < 0$ and $f(2) = 12 > 0$. Also f is a polynomial hence, it is continuous and thus by Bolzano's Theorem, there exists a $c \in (1, 2)$ such that $f(c) = 0$.

Quick Practice 9. Prove that the following equations (or functions) has a solution on the given intervals (otherwise look for a suitable interval)

1. $\sqrt{x^2+1} - 2 = 0$; $[-1, 3]$
2. $g(x) = \ln(x) - \frac{1}{4}$; $(0, 5)$
3. $h(x) = x^5 - x^2 + x - 1$. Show that $h(x) = 2$ has at least one solution.