APPLICATIONS OF DIFFERENTIATION II

Contents

1 Introduction

In this second section of Applications of differentiation, we will start by discussing **vertical and horizontal asymptotes** again. We will take more examples on this topic. Then we will do some **curve sketching**. We will use all the information we have so far from Applications of differentiation I to sketch the curves. We will discuss the use of **differentials in Approximation** specifically **Newton's method**. Then we will consider **related rates as an Application of differentiation**.

2 Vertical and Horizontal asymptotes

The behavior of any function at infinity both in terms of the domain and range is of importance. We are interested in describing what happens to the function as $x \to \pm\infty$ or when $f(x) \to \pm\infty$. This is done through the consideration of asymptotes.

2.1 Vertical asymptotes

The line $x = c$ is a vertical asymptote of the graph of $f(x)$ if

$$\boxed{\begin{array}{c}\lim\limits_{x\to c^-} f(x) = +\infty \text{ (or } -\infty) \\ \text{or} \\ \lim\limits_{x\to c^+} f(x) = +\infty \text{ (or } -\infty)\end{array}}$$

In general, the rational $R(x) = \dfrac{p(x)}{q(x)}$ has a vertical asymptote $x = c$ whenever $q(c) = 0$ and $p(c) \neq 0$.

2.2 Horizontal asymptotes

The horizontal line $y = b$ is called a horizontal asymptote of the graph of $y = f(x)$ if

$$\boxed{\lim_{x\to -\infty} f(x) = b \quad \text{or} \quad \lim_{x\to +\infty} f(x) = b}$$

Example 1. Find all the vertical and the horizontal asymptotes of

$$f(x) = \frac{x^2 - 9}{x^2 + 3x}$$

if they exist.

Solution
Let us consider the vertical asymptotes first. A consideration of the denominator of $f(x)$ gives

$$x^2 + 3x = x(x+3) = 0 \text{ if } x = -3 \text{ or } x = 0$$

We need to find the one-sided limits at these points (if they exist). If the limits exist and are finite at the given point, then there will be no vertical asymptote.

$$\text{At } x = 0: \lim_{x\to 0^-} \frac{x^2 - 9}{x^2 + 3x} = \lim_{x\to 0^-} \frac{(x-3)(x+3)}{x(x+3)}$$

We can get rid of the factor '$x+3$' i.e. divide through by '$x+3$'. So our answer is infinite at $x=0$. We can conclude that $\lim_{x\to 0^-} \dfrac{x^2-9}{x^2+3x} = +\infty$, since both the numerator and denominator are always negative when we approach zero from the left.

For the right-hand side limit, we have $\lim_{x\to 0^+} \dfrac{x^2-9}{x^2+3x} = -\infty$, since the numerator is negative for value very close to zero and the denominator is always positive when we approach zero from the right.

In this case, we have a vertical asymptote at $x=0$.

At $x=-3$,

$$\lim_{x\to -3^-} \frac{(x-3)(x+3)}{x(x+3)} = \frac{-3-3}{-3} = 2$$

and

$$\lim_{x\to -3^+} \frac{(x-3)(x+3)}{x(x+3)} = \frac{-6}{-3} = 2$$

Therefore, the limit exists at $x=-3$. So $x=-3$ is not a vertical asymptote.

To find the horizontal asymptote of

$$f(x) = \frac{x^2-9}{x^2+3x}$$

We consider (Recall L'Hôpital's rule: $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ if indeterminate $(\frac{0}{0}, \frac{\infty}{\infty})$)

$$\lim_{x\to -\infty} \frac{x^2-9}{x^2+3x} = \lim_{x\to -\infty} \frac{2x}{2x+3} \quad \text{(L'Hôpital's rule)}$$

Therefore

$$\lim_{x\to -\infty} \frac{2x}{2x+3} = \lim_{x\to -\infty} \frac{2}{2} = 1$$

Similarly

$$\lim_{x\to \infty} \frac{x^2-9}{x^2+3x} = 1$$

Using the method for limits at infinity, we divide by the highest power in x so that

$$\lim_{x\to -\infty} \frac{x^2-9}{x^2+3x} = \lim_{x\to -\infty} \frac{1-\frac{9}{x^2}}{1+\frac{3}{x}} = 1$$

Therefore $y=1$ is a horizontal asymptote of $y=f(x)$.

Example 2. Find the horizontal and vertical asymptotes of the graph of the function $f(x) = \dfrac{\sqrt{2x^2+1}}{3x-5}$.

Solution

We desire $\lim\limits_{x\to\infty} \dfrac{\sqrt{2x^2+1}}{3x-5}$ and $\lim\limits_{x\to-\infty} \dfrac{\sqrt{2x^2+1}}{3x-5}$ for horizontal asymptotes and the value of x for which $\dfrac{\sqrt{2x^2+1}}{3x-5} \to \infty$ for vertical asymptote.

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}}$$

$$\frac{\lim\limits_{x\to\infty} \sqrt{2+\dfrac{1}{x^2}}}{\lim\limits_{x\to\infty} \left(3-\dfrac{5}{x}\right)} = \frac{\sqrt{2+0}}{3-5\cdot 0} = \frac{\sqrt{2}}{3}$$

$\therefore$ The line $y = \dfrac{\sqrt{2}}{3}$ is a horizontal asymptote.

Considering the limit as $x \to -\infty$, note that $\sqrt{x^2} = \pm x$. Since we are considering $f(x)$ as x gets large in the $-$ve, $\sqrt{x^2}$ in this case is $-x$.

$$\frac{1}{x}\sqrt{2x^2+1} = -\frac{1}{\sqrt{x^2}}\sqrt{2x^2+1} = -\sqrt{2+\frac{1}{x^2}} \qquad \text{[For numerator]}$$

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = -\frac{\sqrt{2}}{3}$$

$\therefore$ The line $y = -\dfrac{\sqrt{2}}{3}$ is also a horizontal asymptote.

A vertical asymptote will occur when $\dfrac{\sqrt{2x^2+1}}{3x-5} \to \infty$

i.e when $3x-5=0 \implies x = \dfrac{5}{3}$.

If x is close to $\dfrac{5}{3}$; and x approaches $\dfrac{5}{3}$ from the right, $f(x) \to \infty$

i.e $\lim\limits_{x\to\frac{5}{3}^+} \dfrac{\sqrt{2x^2+1}}{3x-5} = \infty$

So if x is close to $\frac{5}{3}$; and x approaches $\dfrac{5}{3}$ from the left, $f(x) \to \infty$

i.e $\lim\limits_{x\to\frac{5}{3}^-} \dfrac{\sqrt{2x^2+1}}{3x-5} = \infty$.

$\therefore$ the vertical asymptote is $x = \dfrac{5}{3}$. The sketch is shown below

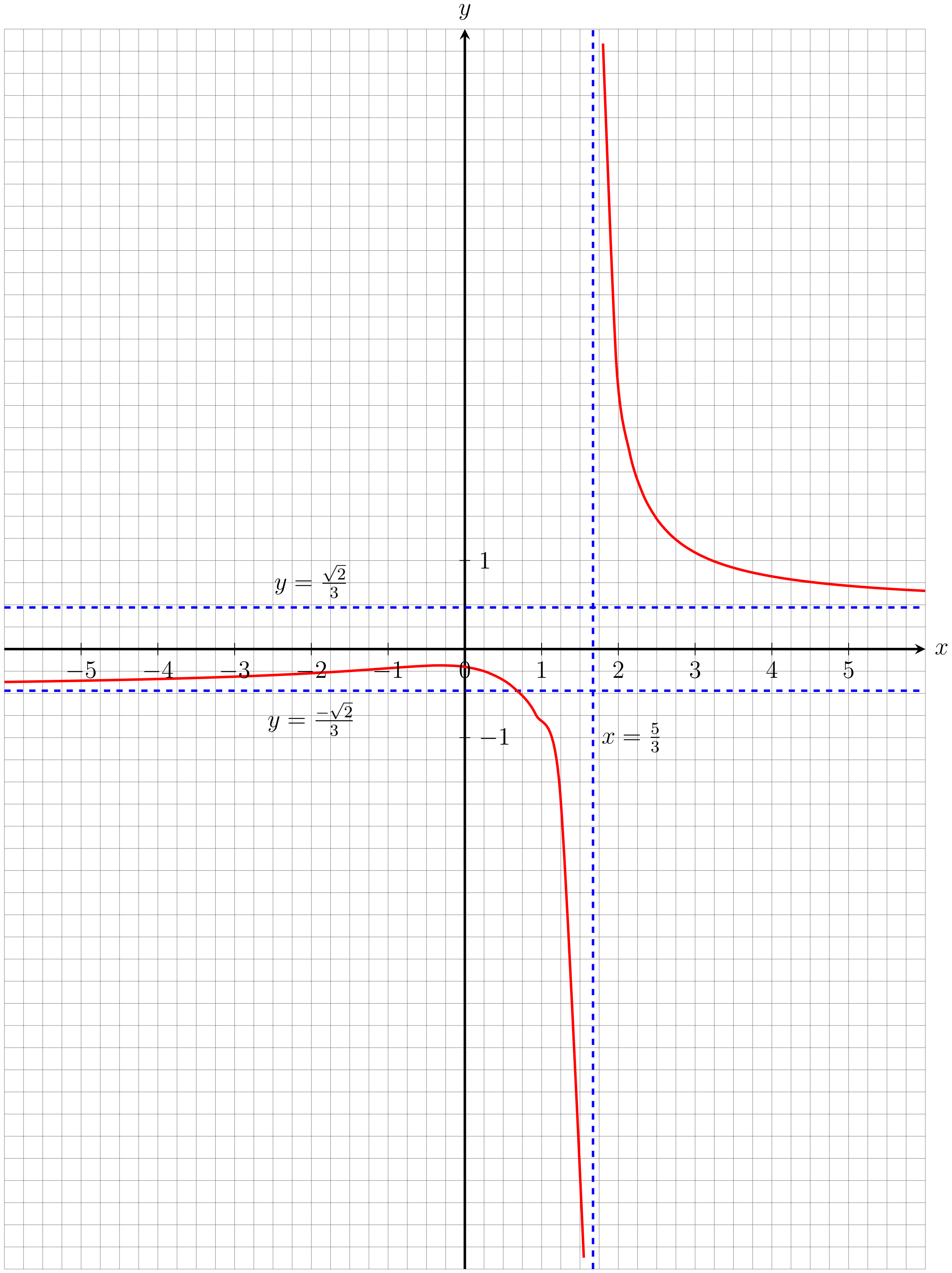

Figure 1: Graph of $y = \frac{\sqrt{2x^2+1}}{3x-5}$

3 Curve Sketching

An algorithm for sketching the graph of $y = f(x)$ requires the following steps.

Guidelines for sketching a curve

(a) Find the domain

(b) Find the $x-$ and $y-$intercepts.

(c) Find all the vertical and horizontal asymptotes.

(d) Find $f'(x)$ and critical points.

(e) Set up sign table to find intervals where f increases/decreases.

(f) Classify critical points, i.e. the relative extreme points.

(g) Find $f''(x)$ and find points of inflection.

(h) Set up sign table to find concavity of graph of f.

(i) Sketch the curve.

A useful tip would be to set up the sign table for the function $f(x)$ itself. This will help you place your graph on the coordinate system, i.e. where $f(x)$ is above or below the $x-$axis.

Example 3. Sketch the graphs of the following functions.

$$\text{(a) } f(x) = \frac{x}{x-1} \qquad \text{(b) } f(x) = \frac{x}{x^2-1}$$

Solutions

(a) Consider the function $f(x) = \dfrac{x}{x-1}$.

(i) $D_f = \{x \in \mathbb{R} | x \neq 1\} = (-\infty, 1) \cup (1, \infty)$. From the domain we can easily observe that the function is undefined when $x = 1$. This is the vertical asymptote. We will formally find the vertical asymptote later.

(ii) To find $x-$intercept: Let $f(x) = 0 \implies x = 0$. So the $y-$intercept is $(0, 0)$.

(iii) For the horizontal asymptote

$$\lim_{x\to\infty} \frac{x}{x-1} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{x}} = 1$$

and

$$\lim_{x\to-\infty} \frac{x}{x-1} = \lim_{x\to-\infty} \frac{1}{1-\frac{1}{x}} = 1$$

Therefore the line $y = 1$ is the horizontal asymptote.

(iv) For the vertical asymptote we consider the point $x = 1$ where the function is undefined.

$$\lim_{x\to 1^-} \frac{x}{x-1} = -\infty \qquad \text{and} \qquad \lim_{x\to 1^+} \frac{x}{x-1} = +\infty$$

So, $x = 1$ is a vertical asymptote.

(v) To find the intervals of increase/decrease we find the derivative of $f(x)$. Therefore

$$f'(x) = \frac{1(x-1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

There are no values of x for which $f'(x) = 0$. It is, however, important to note that $f'(x)$ is undefined at $x = 1$, which is not in the domain of f.

From the sign table we can see that the function is decreasing on either side of the vertical asymptote $x = 1$. So $f(x)$ is decreasing on $(-\infty, 1)$ and on $(1, \infty)$.

Interval	$(-\infty, 1)$	$(1, \infty)$
Sign of $f'(x)$	$-$	$-$
Test value	0	2
Increasing/decreasing		

(vi) The function $f(x)$ has no extreme points.

(vii) To determine concavity, we find the second derivative so that

$$f''(x) = \frac{2}{(x-1)^3}$$

Again there is no value of x for which $f'' = 0$. So, there are no points of inflection. However, f'' is not defined at $x = 1$. We thus determine concavity on either side of the vertical asymptote.

Interval	$(-\infty, 1)$	$(1, \infty)$
Sign of $f'(x)$	$-$	$+$
Test value	0	2
Concavity	Down	Up

From the table we see that $f(x)$ is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$.

(viii) There are no inflexion points.

(ix) We can determine where the function $f(x)$ lies, with a positive sign meaning it is above the $x-$axis and negative if it is below. The important points to consider are $x = 0$ (the x intercept) and $x = 1$ (the asymptote).

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
Sign of $f(x)$	$+$	$-$	$+$
Test value	-1	0.5	2
Above/below $x-$axis	Above	Below	Above

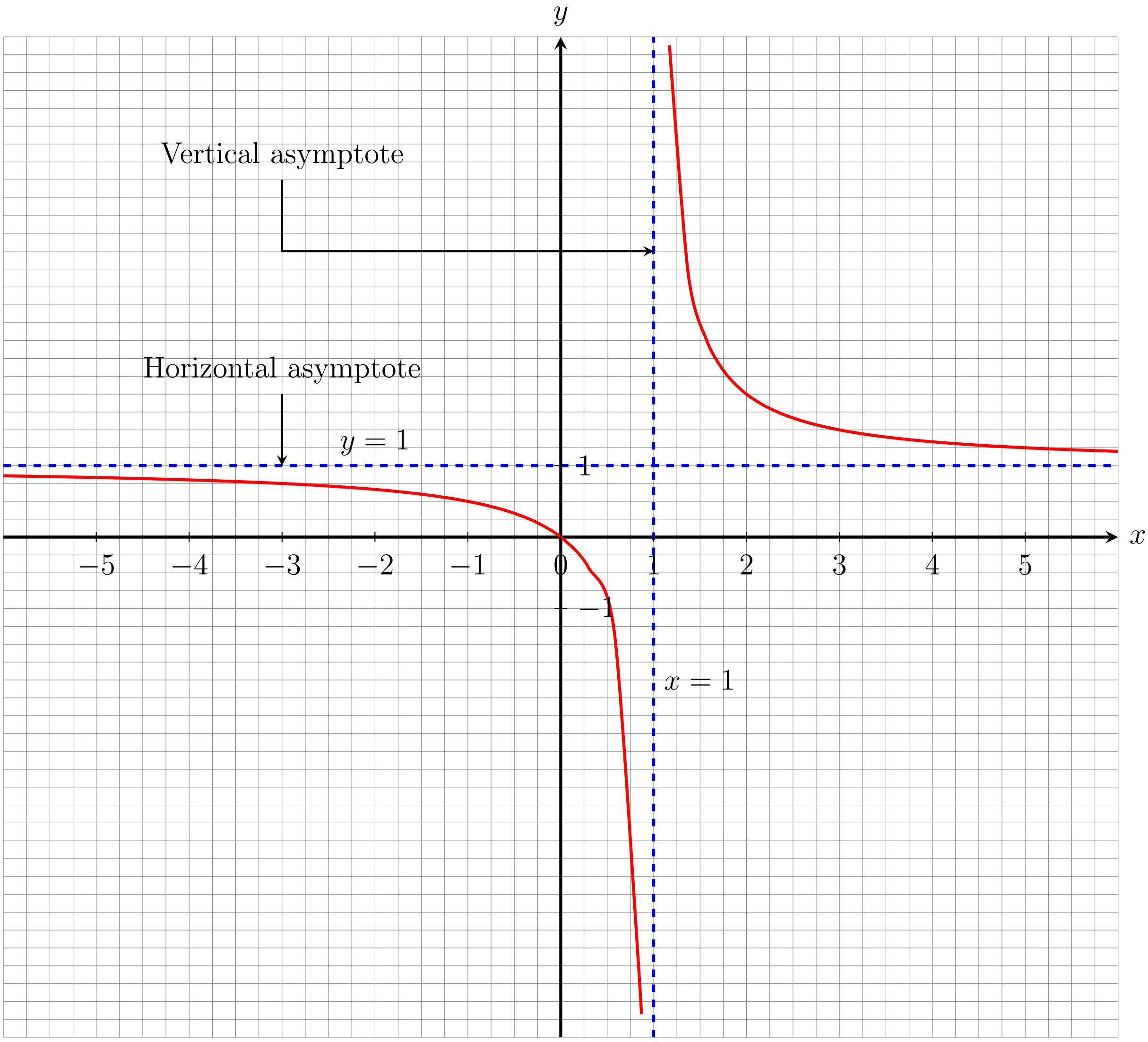

Figure 2: Graph of $y = \frac{x}{x-1}$

The graph of $f(x)$ is shown above.

(b) Consider the function

$$f(x) = \frac{x}{x^2 - 1}$$

(i) The domain

$$D_f = \{x \in \mathbb{R} | x \neq \pm 1\}$$

(ii) For the $x-$intercept, set

$$f(x) = 0 \implies \frac{x}{x^2 - 1} = 0 \implies x = 0$$

and for the $y-$intercept set

$$x = 0 \text{ so that } f(0) = \frac{0}{0 - 1} = 0$$

So the only intercept is $(0, 0)$.

(iii) For the horizontal asymptote

$$\lim_{x\to\infty} \frac{x}{x^2-1} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1-\frac{1}{x^2}} = 0$$

The horizontal asymptote is $y = 0$ and the vertical asymptotes are $x = 1$ and $x = -1$. The vertical asymptotes can be determined by considering

$$\lim_{x\to 1^+} \frac{x}{x^2-1} = +\infty \text{ and } \lim_{x\to 1^-} \frac{x}{x^2-1} = -\infty$$

Similarly

$$\lim_{x\to -1^+} \frac{x}{x^2-1} = +\infty \text{ and } \lim_{x\to -1^-} \frac{x}{x^2-1} = -\infty$$

(iv) To determine intervals of increase and decrease we find the derivative of $f(x)$.

$$f'(x) = -\frac{x^2+1}{(x^2-1)^2}$$

There are no points where $f'(x) = 0$. So, the function has no critical points. We now check on intervals of increase and decrease being guided by the vertical asymptotes.

Interval	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
Sign of $f'(x)$	$-$	$-$	$-$
Test value	-2	0	2
Increasing/decreasing	$\searrow$	$\searrow$	$\searrow$

The function is decreasing on $(-\infty,-1),(-1,1)$ and on $(1,\infty)$.

(v) To determine the concavity of $f(x)$ we determine the second derivative of the function so that

$$f''(x) = \frac{2x(x^2+3)}{(x^2-1)^3}$$

$f''(x) = 0 \implies \dfrac{2x(x^2+3)}{(x^2-1)^3} = 0 \implies x = 0$. We thus have $(0,0)$ as a point of inflection and it turns out that concavity changes at this point.

Interval	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
Sign of $f''(x)$	$-$	$+$	$-$	$+$
Test value	-2	-0.5	0.5	2
Concavity	Down	Up	Down	Up

From the table, $f(x)$ is concave up on $(-1,0)$ and on $(1,\infty)$ and concave down $(-\infty,-1)$ and on $(0,1)$.

(vi) To determine where the curve lies with respect to the $x-$axis we look for the sign of $f(x)$ on the given intervals.

Interval	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
Sign of $f(x)$	$-$	$+$	$-$	$+$
Test value	-2	-0.5	0.5	2
Above/below $x-$axis	Below	Above	Below	Above

(vii) Sketch:

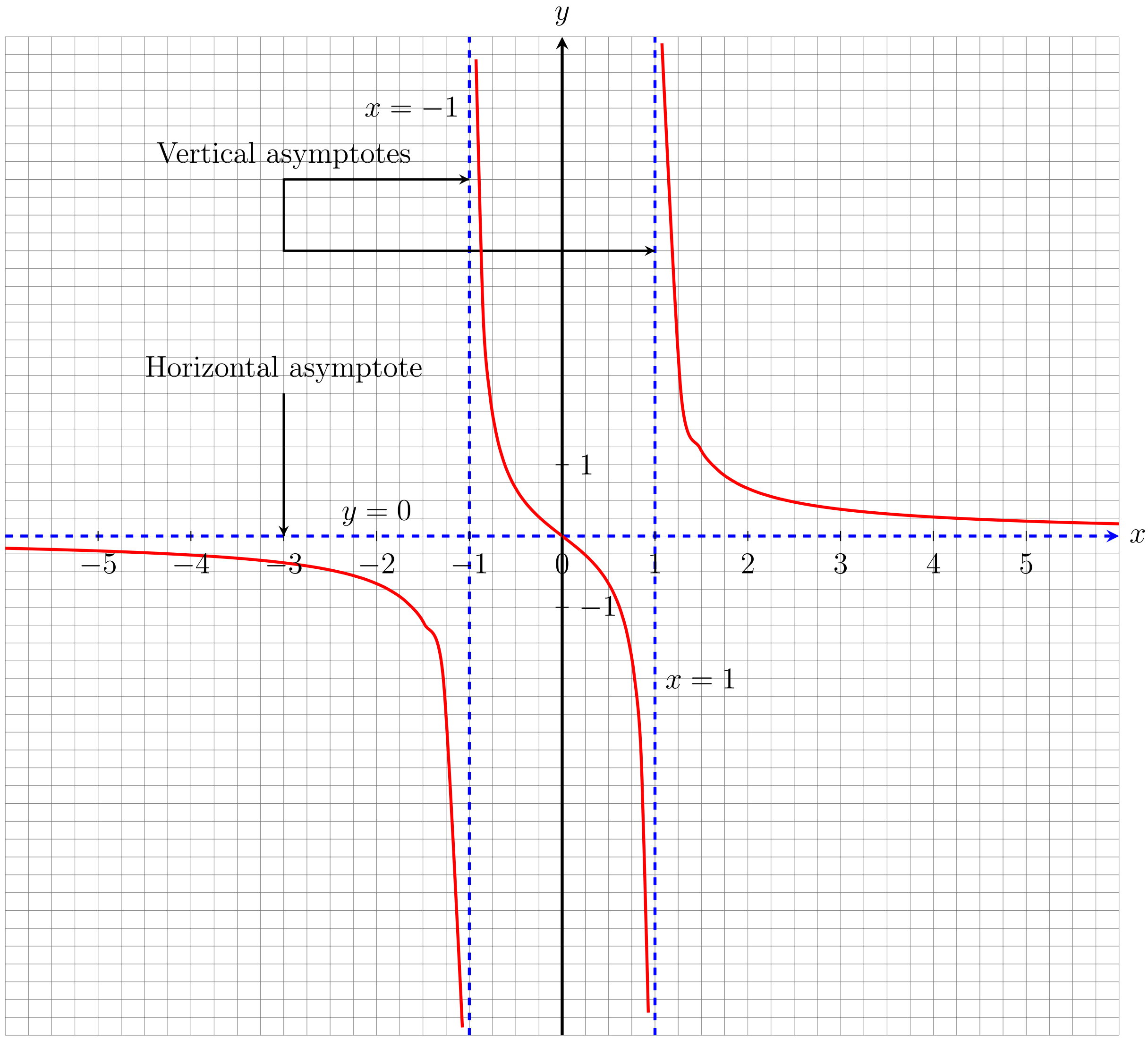

Figure 3: Graph of $y = \frac{x}{x^2-1}$

4 Graphs of Oblique asymptotes

We can redefine an asymptote to a curve as a line that the curve approaches as x tends to $\pm\infty$ or some fixed number. The line $y = mx + c$ is an asymptote to the curve $y = f(x)$ if

$$\lim_{x \to \pm\infty} (f(x) - (mx + c)) = 0.$$

If $m \neq 0$ then the line $y = mx + c$ is referred to as an oblique asymptote to the curve $y = f(x)$.

Example 4. Sketch the graph of $f(x) = \dfrac{2x^2 + 4}{x + 1}$.

Solutions

(a) The domain

$$D_f = \{x \in \mathbb{R} | x \neq -1\}$$

(b) For the intercepts:

If we set $f(x) = 0 \implies 2x^2 + 4 = 0$
There are no values of x for which $2x^2 + 4 = 0$.
For the $y-$intercept, set $x = 0$ so that $f(0) = 4$. So the only intercept is $(0, 4)$.

(c) For the asymptotes:
We note that

$$f(x) = \frac{2x^2 + 4}{x + 1} = 2x - 2 + \frac{6}{x + 1}$$

We do not have a horizontal asymptote but rather the line $y = 2x - 2$ is the oblique asymptote to the curve $f(x) = \dfrac{2x^2 + 4}{x + 1}$.

The vertical asymptote can be determined from the evaluation of the following limits:

$$\lim_{x \to -1^+} \frac{2x^2 + 4}{x + 1} = \infty \quad \text{and} \quad \lim_{x \to -1^-} \frac{2x^2 + 4}{x + 1} = \infty$$

So $x = -1$ is a vertical asymptote.

(d) To determine intervals of increase and decrease we find the derivative of $f(x)$.

$$f'(x) = \frac{2(x^2 + 2x - 2)}{(x + 1)^2}$$

The function has turning points where $f'(x) = 0$, i.e. $x^2 + 2x - 2 = 0$. We note that $x = -1 \pm \sqrt{3}$. We now check on intervals of increase and decrease, being guided by the vertical asymptotes.

Interval	$(-\infty, -2.73)$	$(-2.73, -1)$	$(-1, 0.73)$	$(0.73, \infty)$
Sign of $f'(x)$	$+$	$-$	$-$	$+$
Test value	-3	-2	0	1
Increasing/decreasing				

The function is decreasing on $(-1 - \sqrt{3}, -1)$ and on $(-1, -1 + \sqrt{3})$ and increasing on $(-\infty, -1 - \sqrt{3})$ and on $(-1 + \sqrt{3}, \infty)$.

(e) To determine the concavity of $f(x)$ we determine the second derivative of the function so that

$$f''(x) = \frac{12}{(x + 1)^3}$$

Note that $f''(0) \neq 0$ so we do not have any points of inflection. We can, however, test for concavity since $f''(x)$ is not defined at $x = -1$.

Interval	$(-\infty, -1)$	$(-1, \infty)$
Sign of $f''(x)$	$-$	$+$
Test value	-2	0
Concavity	Down	Up

From the table, we note that $f(x)$ is concave up on $(-1, \infty)$ and concave down $(-\infty, -1)$.

(f) To determine where the curve lies with respect to the $x-$axis we look for the sign of $f(x)$ on the given intervals.

Interval	$(-\infty, -1)$	$(-1, \infty)$
Sign of $f(x)$	$-$	$+$
Test value	-2	0
Above/below $x-$axis	Below	above

(g) Sketch:

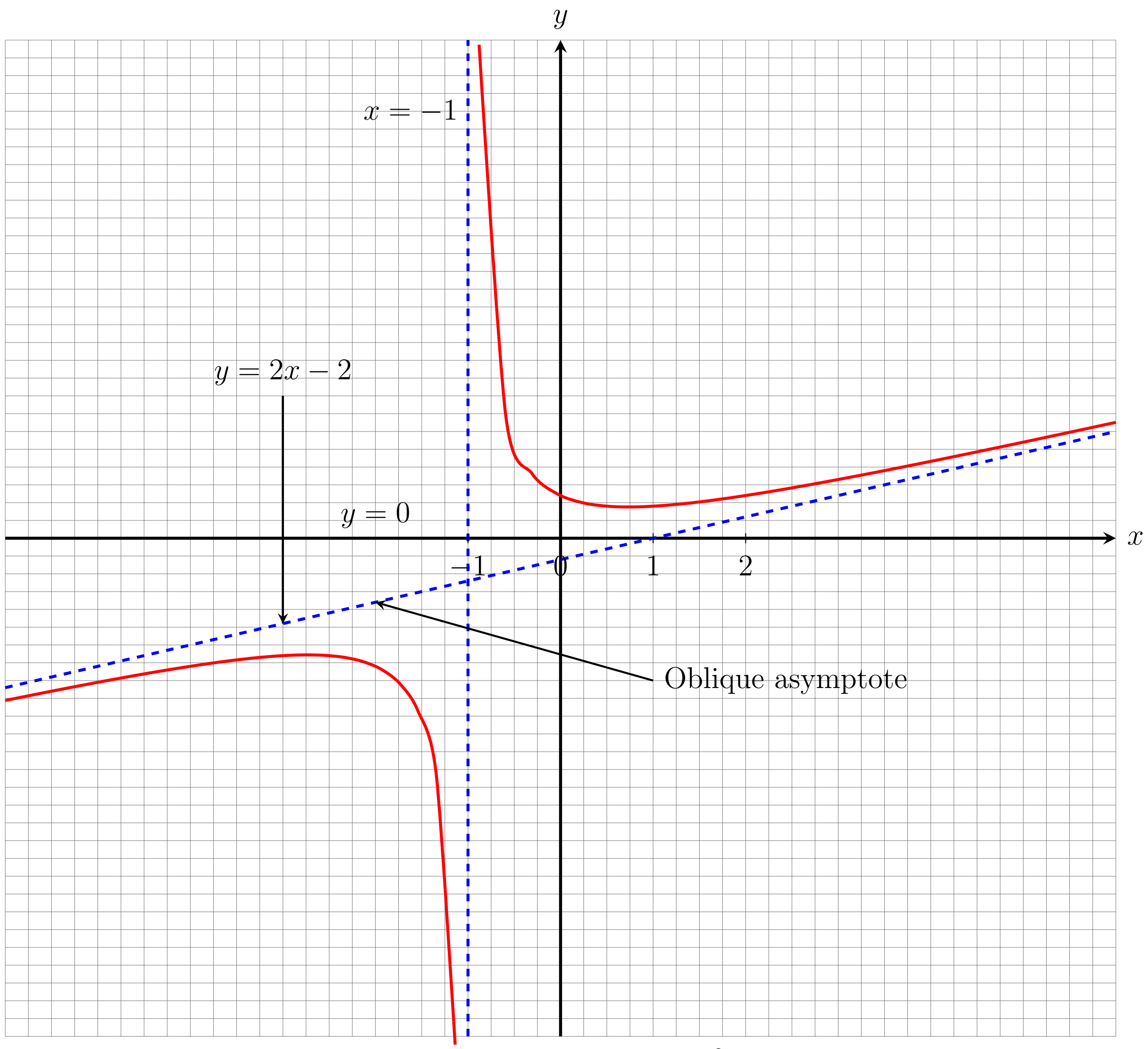

Figure 4: Graph of $y = \frac{2x^2+4}{x+1}$

5 The use of Differential in Approximation:Newton's Method

Newton's Method approximates roots to a polynomial numerically using the first derivative. Suppose that a differentiable function $y = f(x)$ has a zero at $x = c$ which is unknown. The procedure involves drawing a tangent line to $y = f(x)$ at $(x_0, f(x_0))$, and letting x_1 be the $x-$value of the point where the tangent line cuts the $x-$axis. The point x_1 is a better approximation for the zero (root) than x_0. Using x_1 as the new guess to the root, we draw another tangent at $(x_1, f(x_1))$, and let x_2 be the $x-$value of the point where the tangent line cuts the $x-$axis. Surely x_2 is a better approximation than x_1. If we continue the process, we continue to get closer and closer to the root c. It is important to note that it is not always possible to find solutions using this method. We now focus on the derivation of the formula. We take a guess and say that zero at $x = x_0$. To find x_1 given

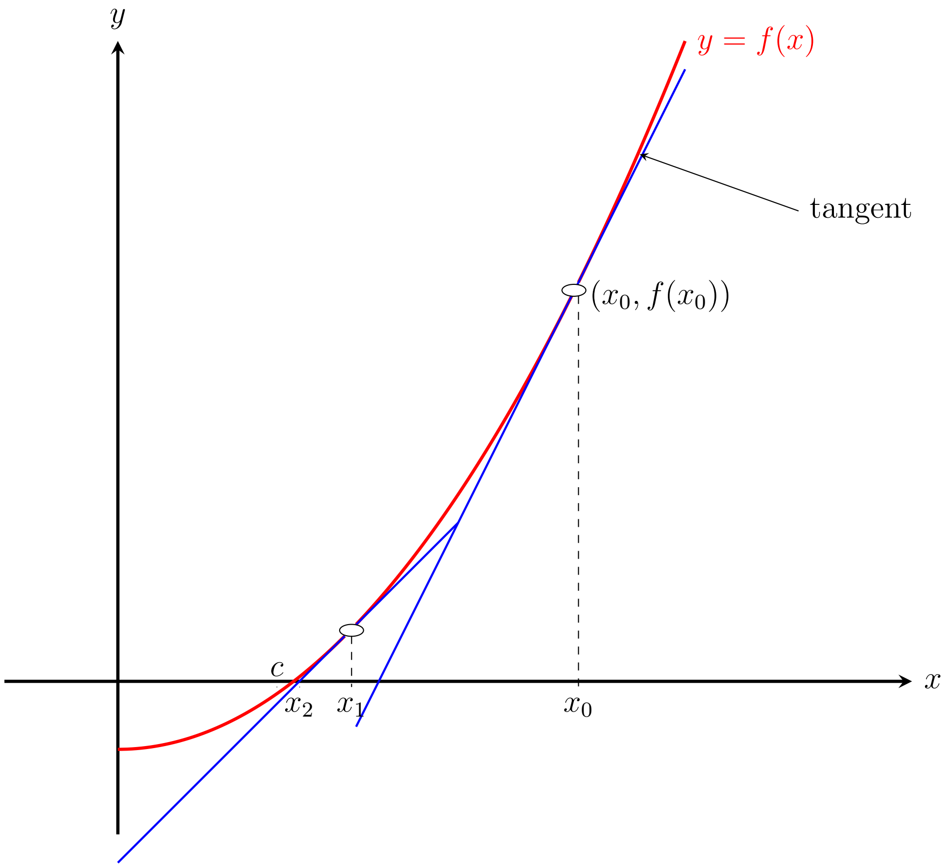

Figure 5: Shows the graph of the function $y = f(x)$

that we have a first guess x_0, we consider the slope of the tangent line at x_0 is $f'(x_0)$. Therefore the equation of the tangent is

$$y - f(x_0) = f'(x_0)(x - x_0) \tag{1}$$

The point $(x_1, 0)$ lies on the tangent line. So substitute it into (1) so that

$$\begin{aligned} 0 - f(x_0) &= f'(x_0)(x - x_0) \\ &= x_1 f'(x_0) - x_0 f'(x_0) \end{aligned}$$

Therefore $x_1 f'(x_0) = x_0 f'(x_0) - f(x_0)$.
So

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Since x_0, $f(x_0)$ and $f'(x_0)$ are known, so we can find x_1.
Now, we can use x_1 in a similar manner to find x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We then use x_2 to find x_3, and so on, iteratively. We call this approximation of the zero Newton's Method.
The general formula for Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for} \quad n = 0, 1, 2, \cdots$$

Example 5. Find a positive real root of the following

$$y = x^4 + x - 3$$

correct to four decimal places.

Solution

Since we are not given the initial guess, we can use the Intermediate Value Theorem to choose x_0. We start at $x = 0$ and determine the function value.

$$\begin{aligned} y(0) &= -3 < 0 \\ y(1) &= 1 + 1 - 3 = -1 < 0 \\ y(2) &= 2^4 + 2 - 3 = 15 > 0. \end{aligned}$$

Therefore, a positive root lies between $x = 1$ and $x = 2$. It is important to note that the root could be closer to $x = 1$ than $x = 2$ due to the magnitude of the function values.

Let us choose $x_0 = 1$. To apply Newton's Method, we require the derivative,
$\dfrac{dy}{dx} = f'(x) = 4x^3 + 1$. So,

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 1 + \frac{1}{5} = \frac{6}{5} \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 1.2 - \frac{(1.2)^4 + (1.2) - 3}{4(1.2)^3 + 1} = 1.16542 \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 1.16452 - \frac{(1.16452)^4 + 1.16452 - 3}{4(1.16452)^3 + 1} = 1.6404 \\ x_4 &= 1.6404 - \frac{(1.6404)^4 + 1.6404 - 3}{4(1.6404)^3 + 1} = 1.16404 \end{aligned}$$

Therefore $y = x^4 + x - 3$ has a root at $x = 1.1640$ (correct to four decimal places).

Example 6. Find the following correct to 3 decimal places:

(a) $\sqrt{7}$ (b) $\sqrt[4]{2}$

Solutions

(a) Set $x = \sqrt{7}$ so that

$$x^2 = 7 \implies x^2 - 7 = 0$$

We now set $f(x) = x^2 - 7$ to find x such that $f(x) = 0$.

We use the Intermediate Value Theorem to find an interval on which the positive root occurs.

$$\begin{aligned} f(0) &= -7 < 0 \\ f(1) &= 1 - 7 = -6 < 0 \\ f(2) &= 2^2 - 7 = -3 < 0 \\ f(3) &= 3^2 - 7 = 2 > 0 \end{aligned}$$

So, $f(x)$ has a zero/root on $2 \leq x \leq 3$.
Choose $x_0 = 2.5$ and use Newton's Method given that $f'(x) = 2x$.

$$\begin{aligned} x_1 &= 2.5 - \frac{f(2.5)}{f'(2.5)} = 2.5 - \frac{-0.75}{5} = \frac{53}{20} = 2.65 \\ x_2 &= 2.65 - \frac{f(2.65)}{f'(2.65)} = 2.64575 \\ x_3 &= 2.64575 - \frac{f(2.64575)}{f'(2.64575)} = 2.64575 \end{aligned}$$

Therefore $\sqrt{7} \approx 2.646$

(b) Set $x = \sqrt[4]{2}$ so that $x^4 = 2 \implies x^4 - 2 = 0$.

Now, let $f(x) = x^4 - 2$. By the Intermediate Value Theorem, we have that there is a zero on interval $1 \leq x \leq 2$ since

$$f(1) = -1 < 0 \text{ and } f(2) = 16 - 2 = 14 > 0.$$

Choose $x_0 = 1$ and use the fact that $f'(x) = 4x^3$.

$$\begin{aligned} x_1 &= 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{4} = 1.25 \\ x_2 &= 1.25 - \frac{f(1.25)}{f'(1.25)} = 1.1935 \\ x_3 &= 1.1935 - \frac{f(1.1935)}{f'(1.1935)} = 1.18923 \\ x_4 &= 1.18921 \end{aligned}$$

So $\sqrt[4]{2} \approx 1.189$

6 Related Rates

Related rate problems involve finding a rate at which a quantity changes by relating that quantity to other quantities whose rates of change are known. The rate of change is usually with respect to time. Since in real life applications, quantities relate to each other, related rates concept has broad applications.
In related rates we often use implicit differentiation.
Generally if two **related quantities** are changing **over time, the rates** at which the quantities change are **related**.

Quick Illustration: If a balloon is being filled with air, both the radius of the balloon and volume of the balloon are increasing. There is a rate at which the radius is increasing and a rate at which the volume of the balloon is increasing. These two rates are related; i.e $\dfrac{dr}{dt}$ and $\dfrac{dv}{dt}$ are related rates. We take the following examples.

Example 7. Air is being pumped into a spherical balloon at a rate of 5 cm^3/min. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 20 cm.

Solution
The first thing that we'll need to do here is to identify what information that we've been given and what we want to find. Before we do that let's notice that both the volume of the balloon and the radius of the balloon will vary with time and so are functions of time, i.e. $V(t)$ and $r(t)$.

We know that air is being pumped into the balloon at a rate of 5 cm^3/min. This is the rate at which the volume is increasing. Recall that rates of change are nothing more than derivatives and so we know that,

$$V'(t) = 5$$

We want to determine the rate at which the radius is changing. Again, rates are derivatives and so it looks like we want to determine,

$$r'(t) = ? \quad \text{when} \quad r(t) = \frac{d}{2} = 10 \text{ cm}$$

Note that we needed to convert the diameter to a radius.

Now that we've identified what we have been given and what we want to find we needed to relate these two quantities to each other. In this case we can relate the volume and the radius with the formula for the volume of a sphere.

$$V(t) = \frac{4}{3}\pi\left[r(t)\right]^3$$

Now we don't really want a relationship between the volume and the radius. What we really want is a relationship between their derivatives. We can do this by differentiating both sides with respect to t. In other words, we will need to do implicit differentiation on the above formula. Doing this gives,

$$V' = 4\pi r^2 r'$$

Note that at this point we went ahead and dropped the (t) from each of the terms. Now all that we need to do is plug in what we know and solve for what we want to find.

$$5 = 4\pi(10^2)r' \implies r' = \frac{1}{80\pi}\ \text{cm/min}$$

Recall that

$$r' = \frac{dr}{dt}$$

The units of the derivative will be the units of the numerator (cm in the previous example) divided by the units of the denominator (min in the previous example).

Example 8. Water is draining from the bottom of a cone-shaped funnel at the rate of $0.03 ft^3/sec$. The height of the funnel is 2 ft and the radius at the top of the funnel is 1 ft. At what rate is the height of the water in the funnel changing when the height of the water is $\frac{1}{2} ft$?

(The volume of the cone is $\frac{1}{3}\pi r^2 h$ where r is the radius and h is the height.)

Solution: Step 1:We draw a picture to introduce the variables. Let h denote the height

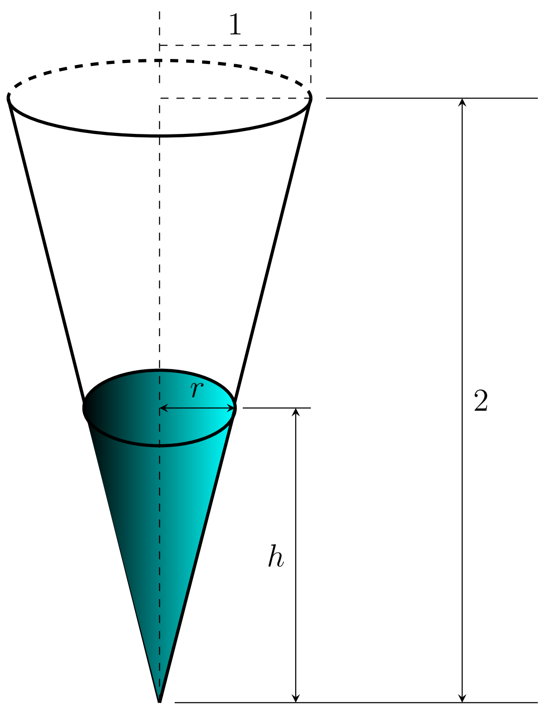

Figure 6: Show Cone

of the water in the funnel, r denote the radius of the water at its surface, and V denote the volume of the water.

Step 2: We need to determine $\frac{dh}{dt}$ when $h = \frac{1}{2} ft$. We know that $\frac{dV}{dt} = -0.03 ft/sec$.

Step 3: The volume of water in the cone is

$$V = \frac{1}{3}\pi r^2 h.$$

From the figure, we see that we have similar triangles. Therefore, the ratio of the sides in the two triangles is the same. Therefore, $\frac{r}{h} = \frac{1}{2}$ or $r = \frac{h}{2}$. Using this fact, the equation for volume can be simplified to

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

Step 4: We apply the chain rule while differentiating both sides of this equation with respect to time t, we obtain

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}.$$

Step 5: We want to find $\frac{dh}{dt}$ when $h = \frac{1}{2}ft$. Since water is leaving at the rate

of $0.03ft^3/sec$, we know that $\frac{dV}{dt} = -0.03ft^3/sec.$
Therefore

$$-0.03 = \frac{\pi}{4}\left(\frac{1}{2}\right)^2\frac{dh}{dt},$$

which implies

$$-0.03 = \frac{\pi}{16}\frac{dh}{dt}.$$

It follows that

$$\frac{dh}{dt} = -\frac{0.48}{\pi} = -0.153ft/sec.$$