Geography 4203 / 5203

GIS Modeling

Class (Block) 9: Variogram & Kriging

Some Updates

- Today class + one proposal presentation
- Feb 22 Proposal Presentations
- Feb 25 Readings discussion (Interpolation)

Last Lecture

- You have seen the first part of **spatial estimation** which was about **interpolation** techniques
- You hopefully obtained an idea of what the **basic idea** of interpolation is and **when** and **why** me are supposed to make use of it
- You had some insights into the **concepts** of different techniques such as **NN**, **pycnophylactiv** I., **IDW**, **Splines**,…
- You hopefully understood the **mathematical foundation** of these approaches to better understand what you are doing in the **lab**

Today‘s Outline

- We will begin with **Geostatistics** which means we talk about **Kriging**
- It is important to understand the difference between Kriging and interpolation techniques we had so far
- We will talk about the **conceptual** basics, constraints and **methods** for Kriging without going too much into detail
- Here we need a deeper understanding of **autocorrelation** and **regionalized variable theory** as well as the incorporation of **semivariogram** models into estimations

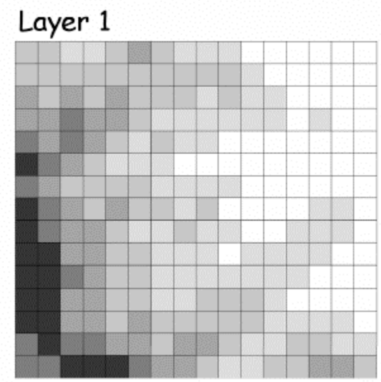
Learning Objectives

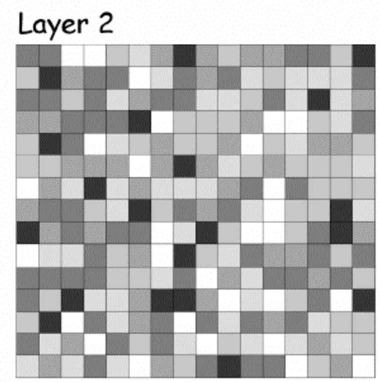
- You will understand how **Kriging** works and why it is called an **optimized local interpolator**
- You will better understand terms like **autocorrelation, semivariance, covariance**
- You will understand the **mathematical** foundation of **Kriging** and how we make use of spatial autocorrelation and semivariance to estimate **weights** for estimating

Back to How to Improve Spatial Estimation…

- Remember how we incorporated **autocorrelation** into interpolation so far
- “**Implicitly**” - we just assumed the data show **spatial dependence**, we knew this from data **exploration**
- **Parameter** settings e.g. in IDW are **arbitrary** (Radius?, # neighbors?, Weighting?) - we get more knowledge through testing different **combinations**
- How to make **theory-based** choices and parameter-settings?
- This is where **Geostatistical** methods come into the picture to help us with this **decision-making** process

- **Predictions** can be improved by incorporating the **knowledge** of **autocorrelation** (knowing the value at one location provides information of **locations nearby**)





Geostatistics

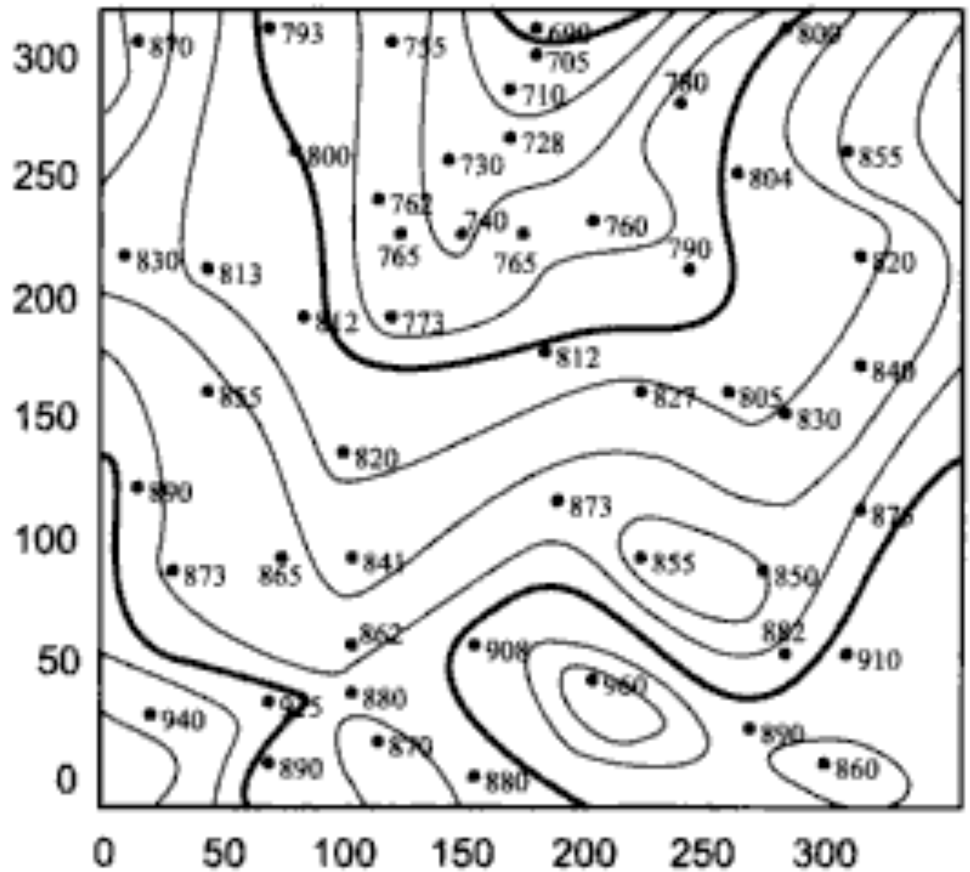
- Too many definitions to go into detail
- Basically, it is about the **analysis** and **inference** of **continuously-distributed** variables (temperature, concentrations,…)
- **Analysis**: describing the spatial variability of the phenomenon under study
- **Inference**: Estimation of the values at unknown locations (unknown values)
- Overall we need techniques for **statistical estimation** of spatial phenomena and this is what Geostatistics provides…

Tobler and his Message - how to…Theory-based

- According to Tobler we expect attribute values **close together** to be **more similar** than those more distant
- This expectation drives the idea of **interpolation** and **Geostatistical** methods
- A graphical **representation** of this is the **variogram cloud**
- Presentation of the **square root of the difference between attribute pairs** against the **distance between them**

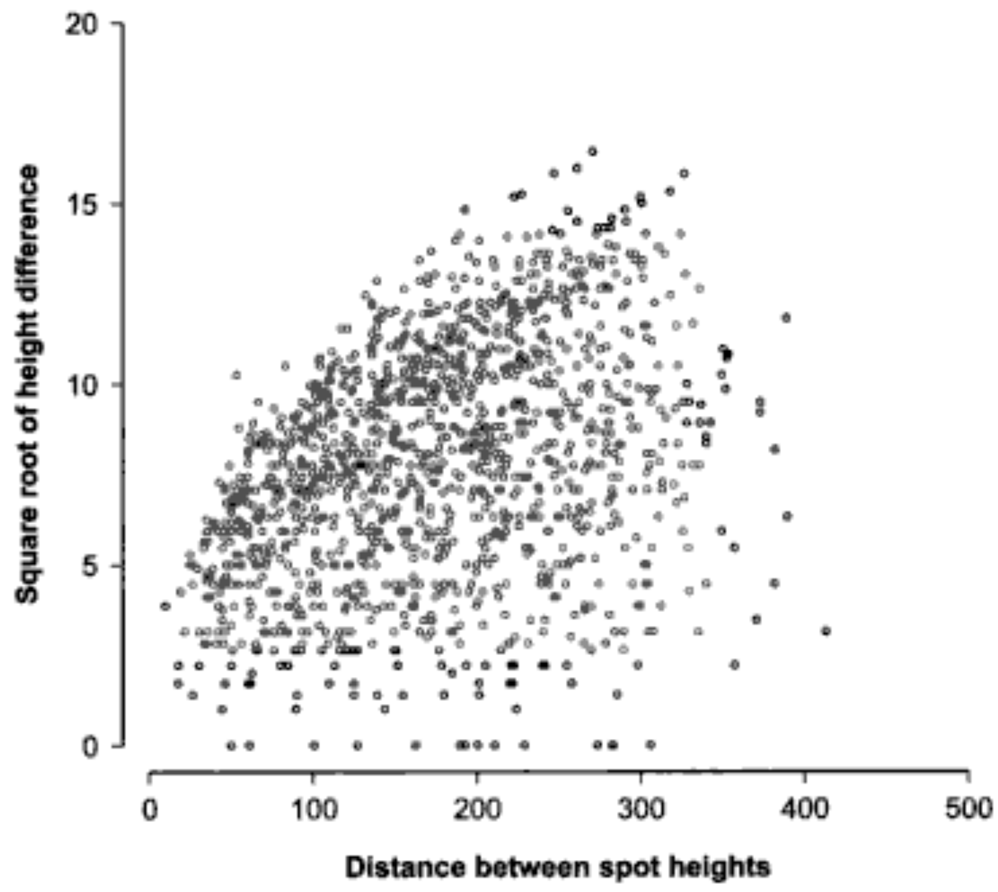
Variogram Cloudiness

- Let’s look at one example from O’Sullivan and Unwin (2003)
- Might be we understand more about **spatial autocorrelation** and how to catch it
- Do we have a **trend** in here?



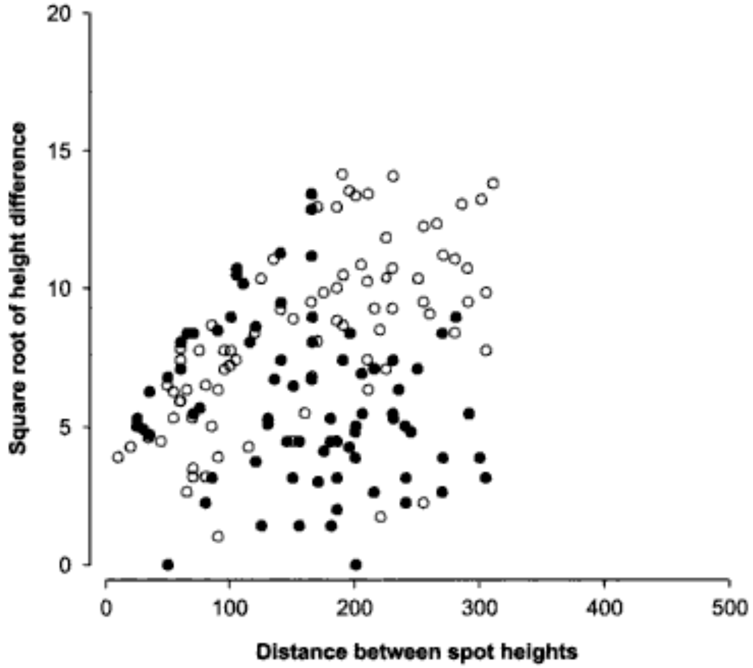
Variogram Cloudiness

- Can you see a **trend**?
- If so what does it **mean**?
- Remember the **visual trend** in the **surface**…



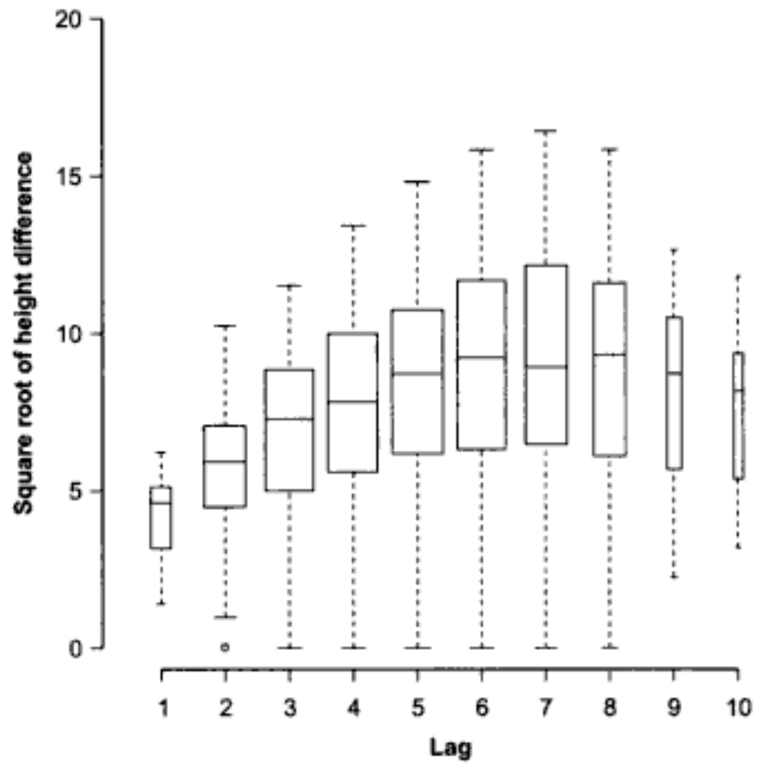
Variogram Cloudiness

- This is the representation of **N-S** (open circles) and **E-W** (filled circles) directions
- Which one shows a **clearer trend** and why?
- How do we call this **effect**?
- Why are there so few sample points now?



Variogram Cloudiness

- For only few points we have lots of **pairs** to be represented and a variogram or **semivariogram** becomes hard to be interpreted
- How to **summarize** the variogram cloud?
- One way is to use **box-plots** showing means, medians, quartiles and extremes for individual lags



Regionalized Variable Theory (RVT)

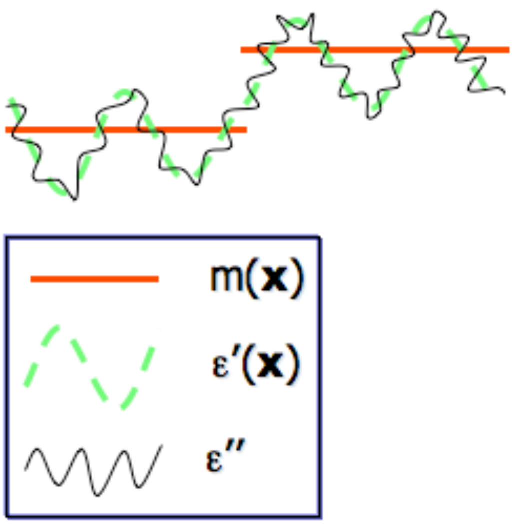
- By putting together the last slides you can understand what RVT means
- The value of a variable in space (at location x) can be expressed by summing together **three components:**

  \- A **structural component** (e.g., a spatial trend or just the mean, **m(x)**)

  \- **Random**, **spatially autocorrelated** ("**regionalized**") variable (local spatial autocorrelation = **ε'(x)**)

  \- **Random** "**noise**" (stochastic variation, not dependent on location = **ε''**)

  Z(x) = m(x) + ε'(x) + ε''



Regionalized Variable Theory (RVT)

- Combination of these three components in a **mathematical model** to develop an estimation function (function is applied to measured data to estimate values across area)
- **m(x)** is our trend (global trend surface)
- **ε'' is the random component that has nothing to do with location**
- **ε'(x)** is our regionalized component and describes the local variation of the variable in space
- This latter **relationship** can be represented by a **dispersion** measure…

Experimental Semivariance

- … here we call it the **experimental semivariance** for lag h:

$$\hat{\gamma}(\vec{h}) = \frac{1}{2N(\vec{h})} \sum_{\alpha=1}^{N(h)} \left[z(u_\alpha) - z(u_\alpha + \vec{h}) \right]^2$$

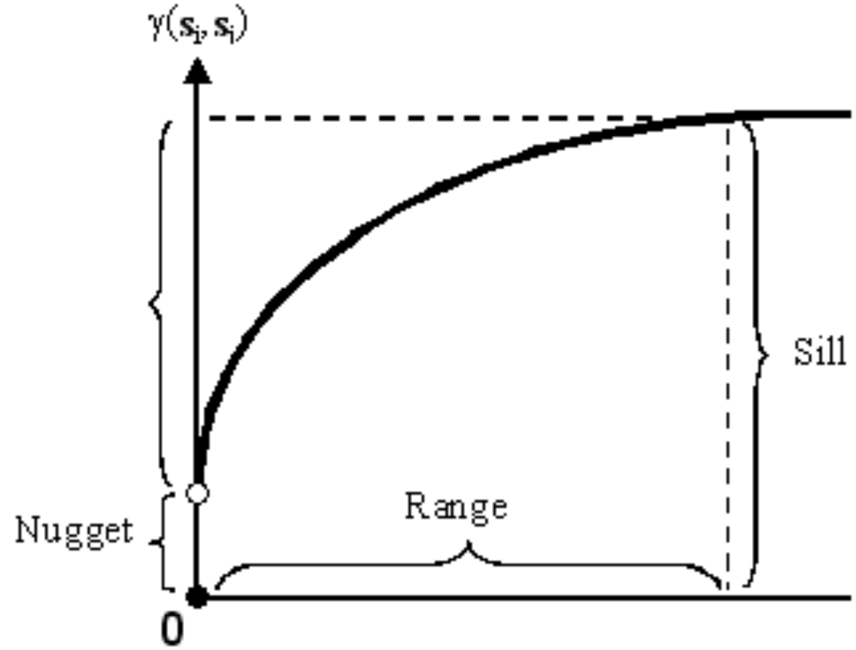
- N(h) is the number of pairs separated by lag h (+/- lag tolerance)
- h - lag width or the distance between pairs of points
- z - is the value of a point at u or u+h
- Remember what we did with the variogram cloud by “binning” the amount of data points

The Experimental Semiviogram

Nugget: Initial semivariance when autocorrelation is highest (intercept); or just the uncertainty where distance is close to 0 (ε'')

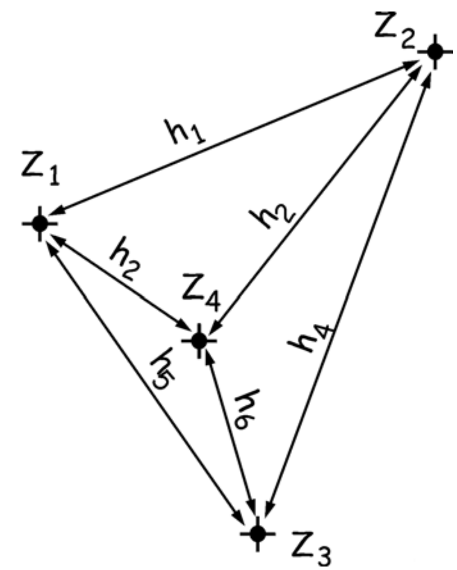
Sill: Point where the curve levels off (inherent variation where there is little autocorrelation)

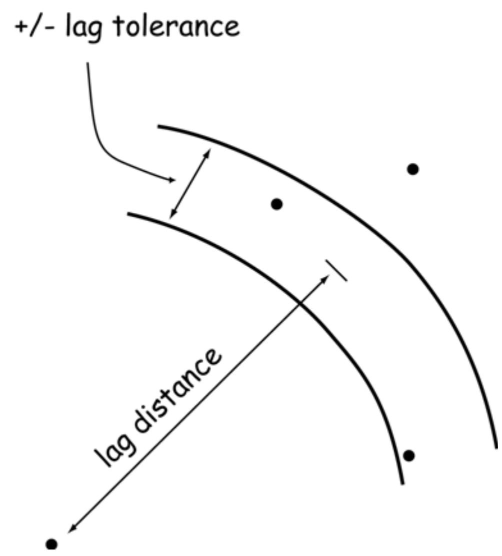
Range: Lag distance where the sill is reached (over which differences are spatially dependent)



Lag and Lag Tolerance

- This has to do with "**binning**"
- Lag distance is the direct (planar) distance between points a and b
- **Tolerances** used to define sets of values that are "**similar**" distances apart (distances rarely repeat in reality)





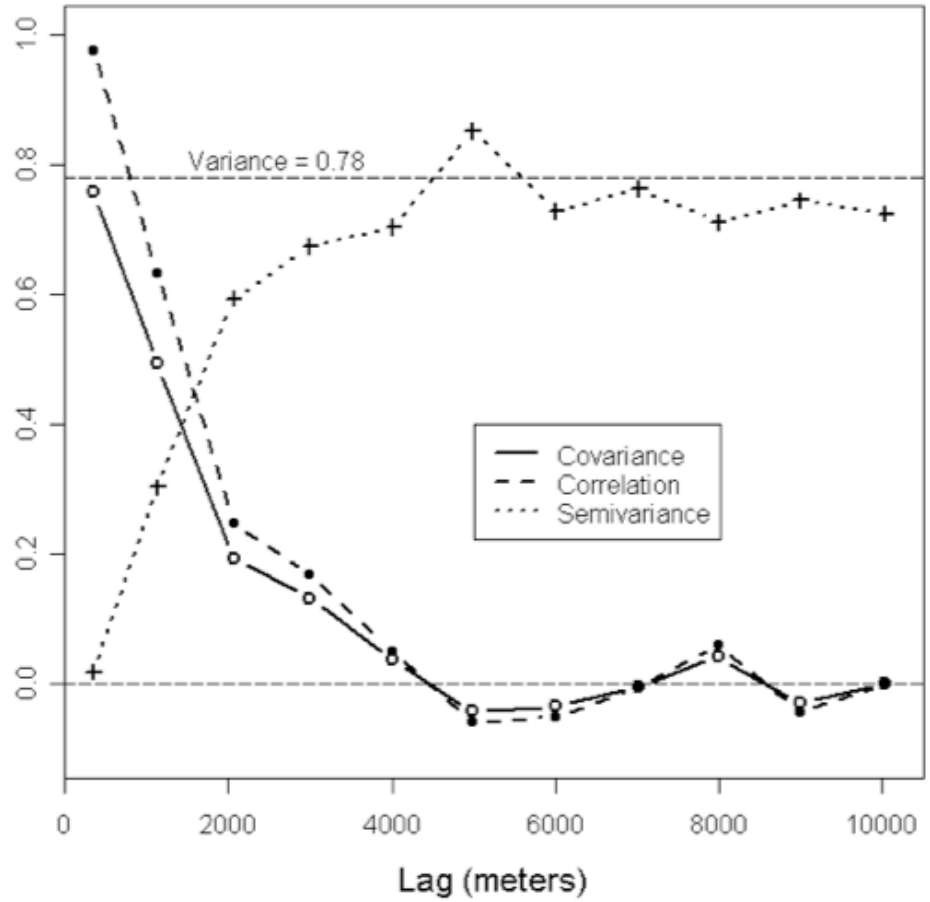
Semivariance, Covariance and Correlation

Spread vs. Similarity

$$\text{Covariance:}\quad C(\mathbf{h}) = \frac{1}{N(\mathbf{h})}\sum_{\alpha=1}^{N(\mathbf{h})} z(\mathbf{u}_\alpha)\cdot z(\mathbf{u}_\alpha+\mathbf{h}) - m_0\cdot m_{+\mathbf{h}}$$

$$\text{Correlation:}\quad \rho(\mathbf{h}) = \frac{C(\mathbf{h})}{\sqrt{\sigma_0\cdot\sigma_{+\mathbf{h}}}}$$

$$\text{Semivariance:}\quad \gamma(\mathbf{h}) = \frac{1}{2N(\mathbf{h})}\sum_{\alpha=1}^{N(\mathbf{h})}\left[z(\mathbf{u}_\alpha+\mathbf{h}) - z(\mathbf{u}_\alpha)\right]^2$$



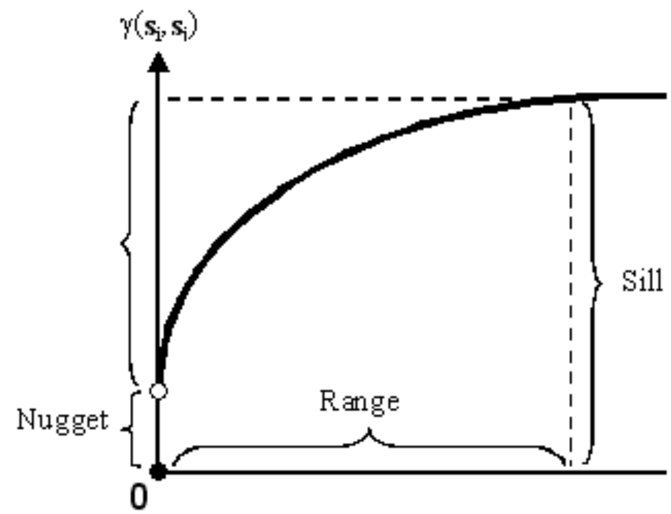
from Bohling (2005)

Semivariogram Model

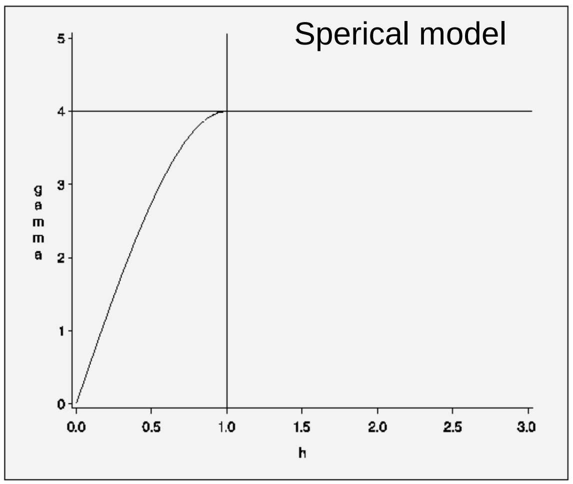
- **The empirical semivariogram** allows us to derive a **semivariogram model** to represent semivariance as a **function of separation distance**
- Thus we can infer the characteristics of the **underlying process** using the model
- Compute the semivariance between points
- **Interpolate** between sample points using an *optimal interpolator* (“kriging”)

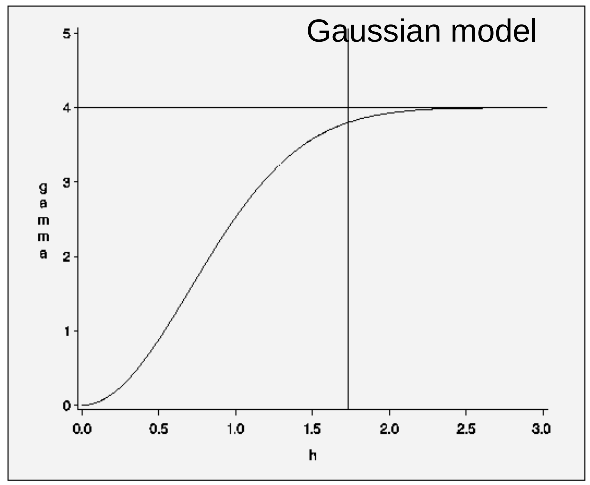
Constraints for a Semivariogram Model

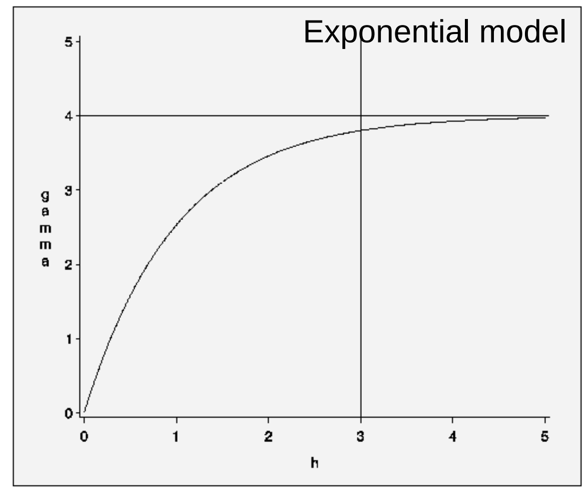
- Monotonically increasing
- Asymptotic max (sill)
- Non-negative intercept (nugget)
- Anisotropy



Semivariogram Models







$$\gamma_z(h) = \begin{cases} c_0\left[\frac{3}{2}\frac{h}{a_0} - \frac{1}{2}\left(\frac{h}{a_0}\right)^3\right], & \text{for } h \le a_0 \\ c_0, & \text{for } h > a_0 \end{cases}$$

$$\gamma_z(h) = c_0\left[1 - \exp\left(-\frac{h^2}{a_0^2}\right)\right]$$

$$\gamma_z(h) = c_0\left[1 - \exp\left(-\frac{h}{a_0}\right)\right]$$

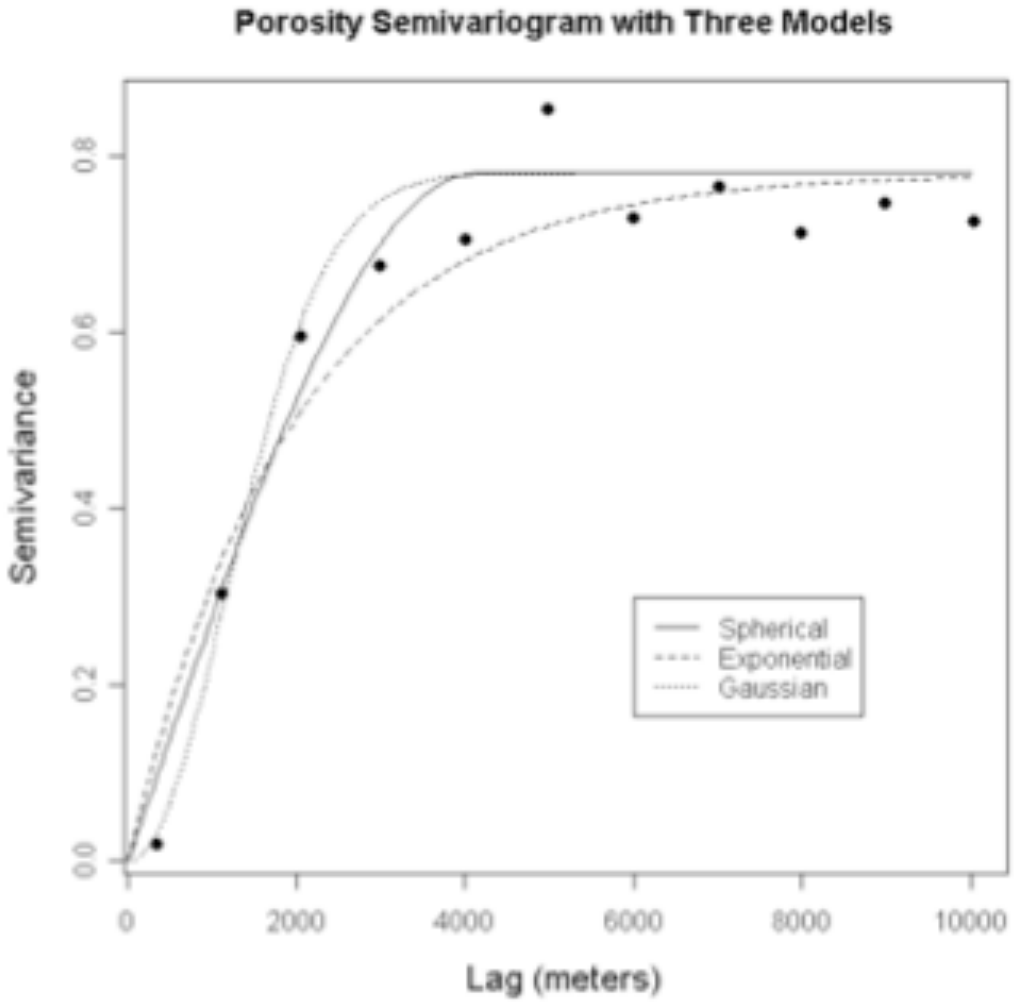
a - range
c - (nugget + sill)

…however fitting the semivariogram is complex

What can you think of would happen if there is a trend in the data?

from http://www.asu.edu/sas/sasdoc/sashtml/stat/chap34/sect12.htm

Where is the Best Fit?



from Bohling (2005)

Why is Anisotropy Important

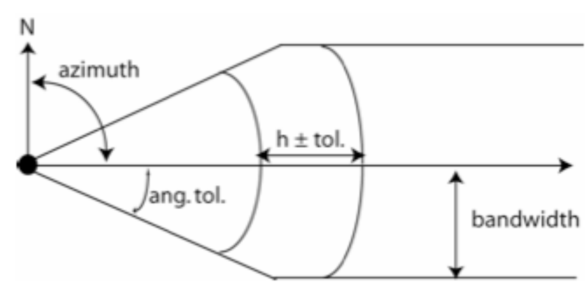
- So far an **isotropic** structure of spatial correlation has been assumed
- The semivariogram depends on the lag distance not on direction (**omnidirectional semivariogram**)
- In reality we often have differences in different directions and need an **anisotropic** semivariogram
- One model is **geometric anisotropy** (same sill in all direction but within different ranges)

Geometric Anisotropy

- Find ranges in three orthogonal principal directions and create a **three-dimensional lag vector** h = (hx,hy,hz)
- Transform this 3D vector into an equivalent **isotropic lag**:

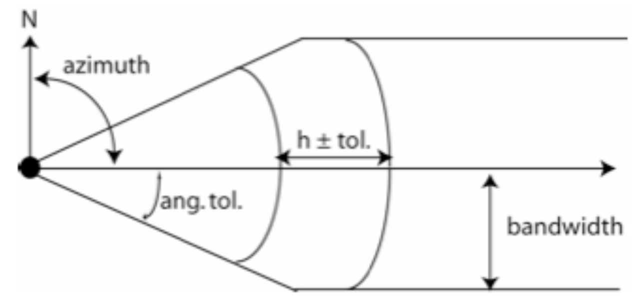
$$h = \sqrt{(h_x/a_x)^2 + (h_y/a_y)^2 + (h_z/a_z)^2}$$

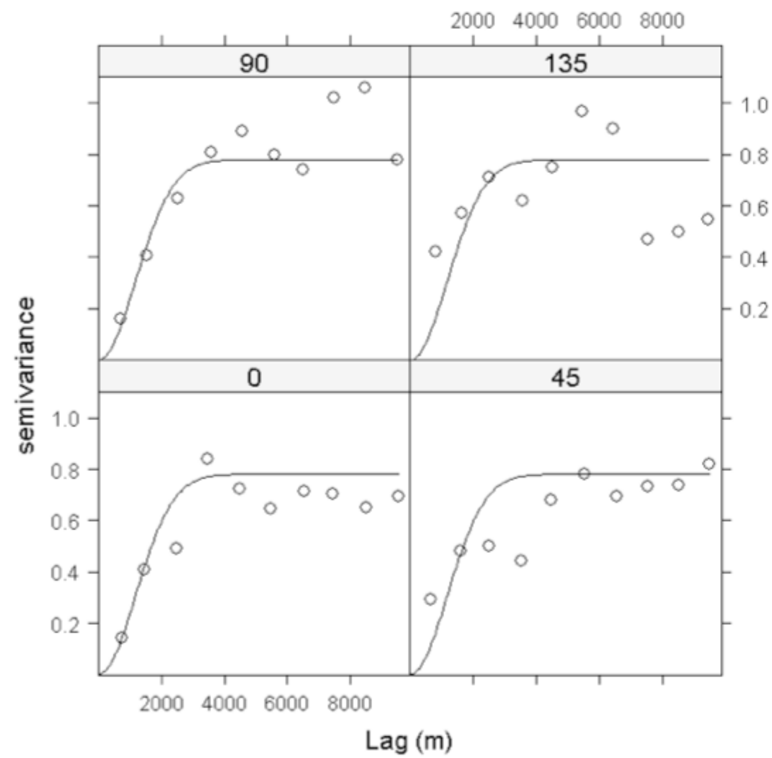
- Semivariance values computed for pairs that fall within directional bands and prescribed lag limits (searching for directional dependence)



Geometric Anisotropy

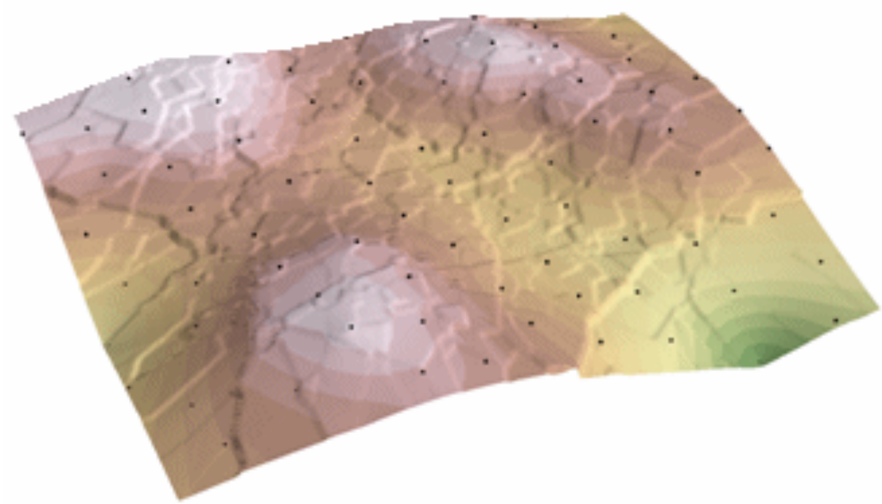
- These directional bands are defined by **azimuthal direction**, **angular tolerance** and **bandwidth**
- You did this already in the lab and hopefully knew what you were doing…





Kriging

- Statistically-based “optimal” estimator of spatial variables
- Predictions based on **regionalized variable theory - you know this now!**
- Kriging first used by Matheron (1963) in honour of D.G. **Krige** - a south African mining engineer who laid the groundwork for “geostatistics”



Kriging Principles

- Similar to IDW, **weights** are used with measured variables to estimate values at unknown locations (a **weighted average**)
- Weights are given in a statistically optimal fashion (given a **specific model**, assumptions about the **trend**, **autocorrelation** and **stochastic variation** in the predicted variable)
- In other words: We use the **semivariance model** to formulate the **weights**

Ordinary Kriging

- For a basic understanding we look at **Ordinary Kriging**
- **Weighting** of data points according to distance
- Estimated values z are the sum of a regional mean m(x) and a spatially correlated random component **ε'(x)**
- **(m(x) is implicit in the system)**
- **Assumptions**:

  no trend

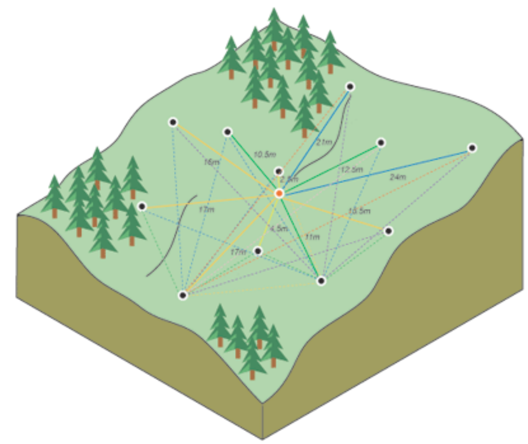
  isotropy

  variogram can be defined with **math model**

  same **semivariogram** applies for the **whole** study area (variation is a function of distance, not location + constant mean - **stationarity** of the spatial process)

Ordinary Kriging

- You see it looks similar to IDW
- But: Weight computation is much more complex based on the **inverted variogram**
- Weights reach the value **zero** when we reach the **range** a
- Ordinary: **no trend**, **regional mean** is estimated from sample



$$\hat{Z}(s_0) = \sum_{i=1}^{N} \lambda_i Z(s_i)$$

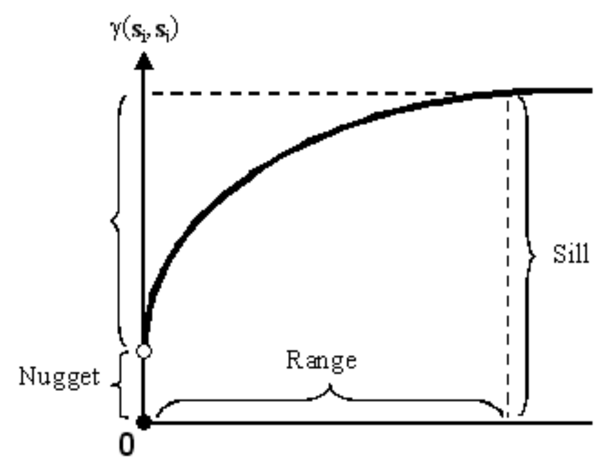
where:

$Z(s_i)$ = the measured value at the i^{th} location.

λ_i = an unknown weight for the measured value at the i^{th} location.

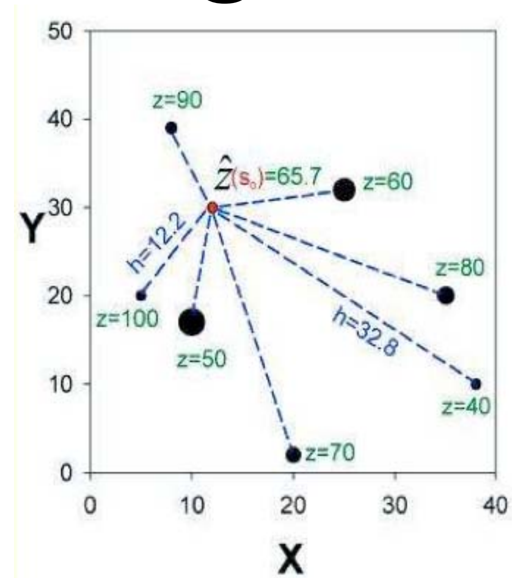
s_0 = the prediction location.

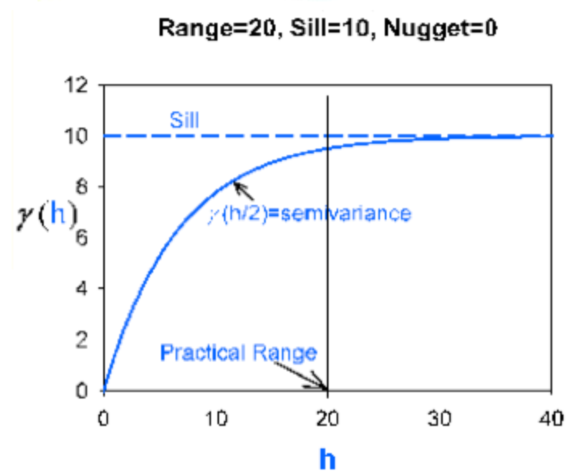
N = the number of measured values.



How Semivariance Determines Kriging Weights

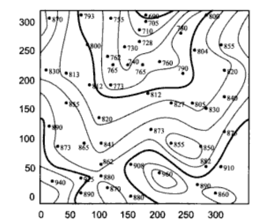
- **Weighted average** of the observed attribute values
- Weights are a function of the **variogram model** & sum to 1 (they should to be **unbiased**!)
- You see the sizes of the points proportional to their weights they get
- Points with distances > range will have a **weight** of **zero**, **WHY**?

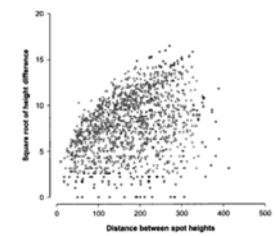




Typical Working Steps

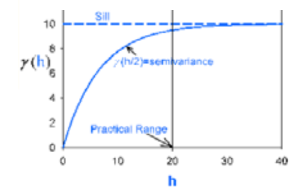
- Describing the data / identifying spatial autocorrelation and trends (**experimental semivariogram**)



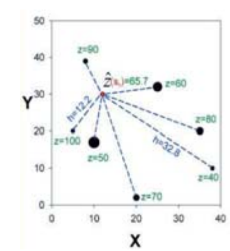


- Building the **semivariogram model** by using the **mathematical function**

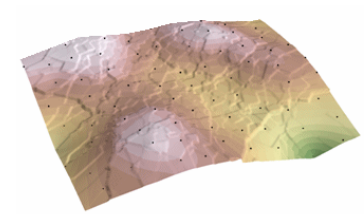
$$\gamma_z(h) = \begin{cases} c_0\left[\frac{3}{2}\frac{h}{a_0} - \frac{1}{2}\left(\frac{h}{a_0}\right)^3\right], & \text{for } h \le a_0 \\ c_0, & \text{for } h > a_0 \end{cases}$$



- Using the **semivariogram model** for defining the **weights**



- **Evaluate interpolated** surfaces



A First Summary

- Kriging is complex
- Here you obtained a first overview of the **principles** of this technique
- Weights are derived from the **semivariogram** model to intelligently infere **unmeasured** values
- You have seen the underlying **rules** of the semivariogram and hopefully understood why **autocorrelation**, **directional trends** and **stationarity** are important
- You also understood the difference to other local interpolators, don’t you?