# Linear Transformations and Matrices - a short Review

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## 1 Introduction

### 1.1 Transformations on Vectors

### 1.2 Examples of Core LTs in $\mathbb{R}^2$

## 2 Definition and Properties of LTs

## 3 Vectors and Basis Sets

## 4 From LTs to Matrices

## 5 The Determinant

### 5.1 Definition and Properties

### 5.2 Calculating the Determinant

In $\mathbb{R}^2$ the determinant of two vectors $\mathbf{a}$ and $\mathbf{b}$ is simply the area of the parallelogram spanned by the two vectors (Figure 1).
Let's check several properties of this area:

1. The parallelogram spanned by the two basis vectors $\hat{\mathbf{e}}_\mathbf{1}$ and $\hat{\mathbf{e}}_\mathbf{2}$ has an area of 1: $A\left(\hat{\mathbf{e}}_\mathbf{1}, \hat{\mathbf{e}}_\mathbf{2}\right) = 1$.

2. If the two vectors are on the same line then the area equals 0, i.e. $A\left(\mathbf{a}, \lambda\mathbf{a}\right) = 0$, where $\lambda \in \mathbb{R}$.

3. The parallelogram spanned by one vector $\mathbf{a}$ and a sum of two vectors $\mathbf{b}+\mathbf{c}$ has the same area as the sum of the areas of the two parallelograms spanned by $\mathbf{a}$, $\mathbf{b}$ and by $\mathbf{a}$, $\mathbf{c}$: $A\left(\mathbf{a}, \mathbf{b}+\mathbf{c}\right) = A\left(\mathbf{a}, \mathbf{b}\right) + A\left(\mathbf{a}, \mathbf{c}\right)$ (Figure 2).

4. Scaling any of the two vectors scales the area by the same factor: $A\left(\alpha\mathbf{a}, \beta\mathbf{b}\right) = \alpha\beta A\left(\mathbf{a}, \mathbf{b}\right)$, where $\alpha, \beta \in \mathbb{R}$ (Figure 3). A special case of scaling is that of scaling by a negative scalar: for example, $A\left(-\mathbf{a}, \mathbf{b}\right) = -A\left(\mathbf{a}, \mathbf{b}\right)$.

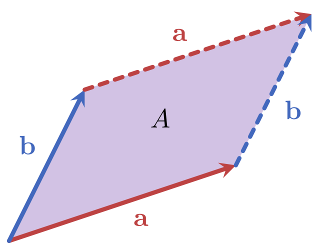


Figure 1: The area of the parallelogram spanned by two vectors $\mathbf{a}$ and $\mathbf{b}$.

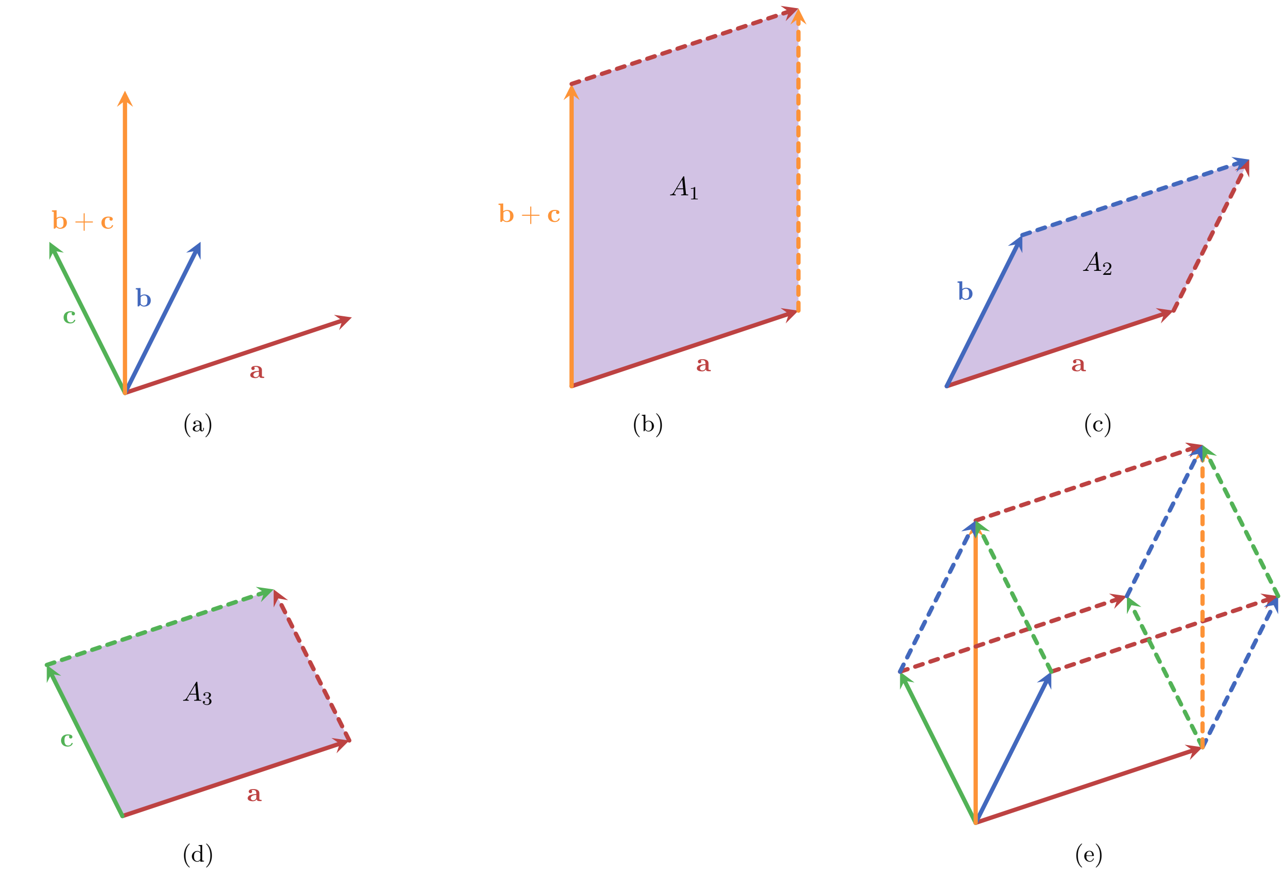


Figure 2: The area of the parallelogram spanned by the vectors $\mathbf{a}$ and $\mathbf{b}+\mathbf{c}$: (a) all four vectors, (b)-(d) the areas of the differenct parallelograms $A_1 = A\left(\mathbf{a}, \mathbf{b}+\mathbf{c}\right)$, $A_2 = A\left(\mathbf{a}, \mathbf{b}\right)$, $A_3 = A\left(\mathbf{a}, \mathbf{c}\right)$. In (e) we see that adding the areas $A_2$ and $A_3$ gives us the area $A_1$.

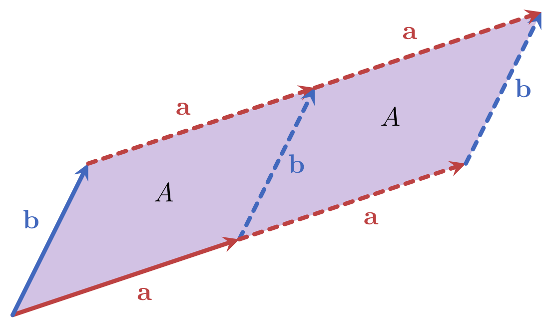


Figure 3: Scaling the vector $\mathbf{a}$ by a factor of 2 scales the area of the parallelogram by the same factor.

Using the above properties, we can derive the relation between $A(\mathbf{a}, \mathbf{b})$ and $A(\mathbf{b}, \mathbf{a})$:

$$\begin{aligned} A(\mathbf{a}, \mathbf{b}) &= \cancel{A(\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{a})} - A(\mathbf{b}, \mathbf{a}) \\ &= -A(\mathbf{b}, \mathbf{a}) . \end{aligned} \tag{1}$$

This means that swapping the order of vectors in the area of a parallelogram changes its sign.
Now we can calculate the area of the parallelogram spanned by two vectors $\mathbf{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}}$ and $\mathbf{b} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}}$:

$$\begin{aligned} A(\mathbf{a}, \mathbf{b}) &= A\left(a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}}, b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}}\right) \\ &= A\left(a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}}, b_x\hat{\mathbf{x}}\right) + A\left(a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}}, b_y\hat{\mathbf{y}}\right) \\ &= \cancel{A\left(a_x\hat{\mathbf{x}}, b_x\hat{\mathbf{x}}\right)} + A\left(a_y\hat{\mathbf{y}}, b_x\hat{\mathbf{x}}\right) + A\left(a_x\hat{\mathbf{x}}, b_y\hat{\mathbf{y}}\right) + \cancel{A\left(a_y\hat{\mathbf{y}}, b_y\hat{\mathbf{y}}\right)} \\ &= a_y b_x A(\hat{\mathbf{y}}, \hat{\mathbf{x}}) + a_x b_y A(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \\ &= a_y b_x(-1) + a_x b_y(1) \\ &= a_x b_y - a_y b_x . \end{aligned} \tag{2}$$

### 5.3 Matrix Rank and Null Space

## 6 Matrix-Matrix Products

## 7 Eigenvalues and Eigenvectors