applied to projecting a vector onto- and rejecting it from a plane, since every plane in $\mathbb{R}^3$ has a single normal vector up to a sign (Figure 1.3).

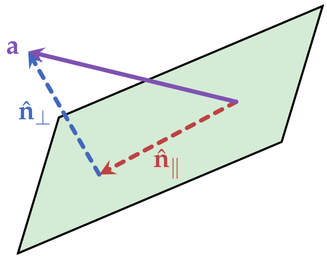

Figure 1.3: Decomposing the vector **a** into two components in respect to a plane: one parallel to the plane ($\mathbf{a}_{\parallel}$) and one orthogonal to it ($\mathbf{a}_{\perp}$). These are also called the projection and rejection of **a** on the plane, respectively.

The projection of a vector onto another vector gives rise to an important operation between two vectors: the **dot product**: given a vector **a**, its projection on the vector **b** is

$$\mathrm{proj}_{\mathbf{b}}\mathbf{a} = |\mathbf{a}|\cos(\theta), \tag{1.21}$$

where θ is the angle between the vectors. See Figure 1.4 for a visual representation. We then define the dot product between the two vectors as

$$\langle \mathbf{a}, \mathbf{b} \rangle = |\mathbf{b}|\,\mathrm{proj}_{\mathbf{b}}\mathbf{a} = |\mathbf{b}|\,|\mathbf{a}|\cos(\theta). \tag{1.22}$$

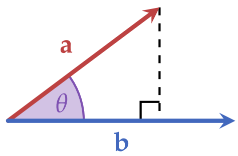

Figure 1.4: Projection of **a** onto **b**: no matter how many dimensions we use, we can always rotate our view such that we look at the plane spanned by both vectors, and **b** lies horizontally. In this way it's easy to see why the projection of **a** onto **b** is $|\mathbf{a}|\cos(\theta)$: it's simply the definitions of the cosine function ("side next to the angle divided by the hypotenuse").

It is of course convenient to have a way to calculate the dot product component-wise. To find such form, we use the same two vectors **a** and **b** from before, and their difference $\mathbf{c} = \mathbf{a} - \mathbf{b}$ (Figure 1.5).

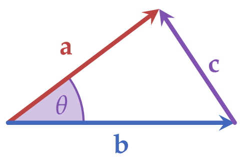

Figure 1.5: Text text text

Thanks to trigonometry we know that the following relation holds ("the law of cosines"):

$$\begin{aligned} |\mathbf{c}|^2 &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}|\cos(\theta) \\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\langle \mathbf{a}, \mathbf{b} \rangle, \end{aligned} \tag{1.23}$$

which we can rearrange into

$$\langle \mathbf{a}, \mathbf{b} \rangle = \frac{1}{2}\left(|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{c}|^2\right). \tag{1.24}$$

In $\mathbb{R}^2$ the component version of the vectors are

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_x \\ c_y \end{bmatrix} = \begin{bmatrix} a_x - b_x \\ a_y - b_y \end{bmatrix}, \tag{1.25}$$