Linear Transformations and Matrices - a short Review

Peleg Bar Sapir

October 21, 2024

1 Introduction

1.1 Transformations on Vectors

1.2 Examples of Core LTs in $\mathbb{R}^2$

2 Definition and Properties of LTs

3 Vectors and Basis Sets

4 From LTs to Matrices

5 The Determinant

5.1 Definition and Properties

5.2 Calculating the Determinant

In $\mathbb{R}^2$ the determinant of two vectors $\mathbf{a}$ and $\mathbf{b}$ is simply the area of the parallelogram spanned by the two vectors (Figure 1).

Let's check several properties of this area:

1. The parallelogram spanned by the two basis vectors $\hat{\mathbf{e}}_\mathbf{1}$ and $\hat{\mathbf{e}}_\mathbf{2}$ has an area of 1: $A\left(\hat{\mathbf{e}}_\mathbf{1},\hat{\mathbf{e}}_\mathbf{2}\right)=1$.

2. If the two vectors are on the same line then the area equals 0, i.e. $A\left(\mathbf{a},\lambda\mathbf{a}\right)=0$, where $\lambda\in\mathbb{R}$.

3. The parallelogram spanned by one vector $\mathbf{a}$ and a sum of two vectors $\mathbf{b}+\mathbf{c}$ has the same area as the sum of the areas of the two parallelograms spanned by $\mathbf{a}$, $\mathbf{b}$ and by $\mathbf{a}$, $\mathbf{c}$: $A\left(\mathbf{a},\mathbf{b}+\mathbf{c}\right)=A\left(\mathbf{a},\mathbf{b}\right)+A\left(\mathbf{a},\mathbf{c}\right)$ (Figure 2).

4. Scaling any of the two vectors scales the area by the same factor: $A\left(\alpha\mathbf{a},\beta\mathbf{b}\right)=\alpha\beta A\left(\mathbf{a},\mathbf{b}\right)$, where $\alpha,\beta\in\mathbb{R}$ (Figure 3). A special case of scaling is that of scaling by a negative scalar: for example, $A\left(-\mathbf{a},\mathbf{b}\right)=-A\left(\mathbf{a},\mathbf{b}\right)$.

Using the above properties, we can derive the relation between $A\left(\mathbf{a},\mathbf{b}\right)$ and $A\left(\mathbf{b},\mathbf{a}\right)$:

$$\begin{aligned} A\left(\mathbf{a},\mathbf{b}\right) &= \cancel{A\left(\mathbf{a}+\mathbf{b},\mathbf{b}+\mathbf{a}\right)} - A\left(\mathbf{b},\mathbf{a}\right) \\ &= -A\left(\mathbf{b},\mathbf{a}\right). \end{aligned} \tag{1}$$

This means that swapping the order of vectors in the area of a parallelogram changes its sign.

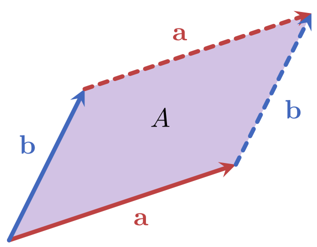

Figure 1: The area of the parallelogram spanned by two vectors $\mathbf{a}$ and $\mathbf{b}$.

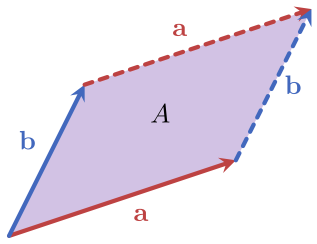

Figure 2: Adding two vectors...

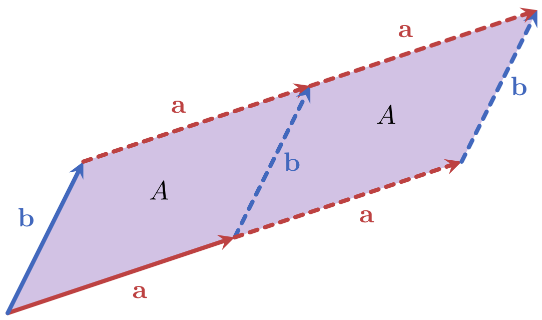

Figure 3: Scaling the vector **a** by a factor of 2 scales the area of the parallelogram by the same factor.

5.3 Matrix Rank and Null Space

6 Matrix-Matrix Products

7 Eigenvalues and Eigenvectors