Homework Assignment #2
(Due 2 April 2020)

Multivariate Time Series Analysis – Stationarity and Cointegration Tests

The objective of this homework is to study the Dividend Discount Model (DDM), i.e., the relationship between stock prices and dividends. By the way, we will also see how to compute some critical values (quantiles) of the finite-sample distribution of the test statistics for unit-root and cointegration tests.

If we are interested in testing some long-run characteristics of the DDM, the first step is to test if stock prices and dividends are non-stationary. If they are non-stationary, then the second step is to test if there is some cointegration relationship between the stock prices and dividends.

For the empirical application, we take data from the US and UK markets. We have monthly data from 1990 to 2019 for both stock markets: the price (excluding dividend payments, P_t) and the annualized dividend yield (D_t / Y_t), which you use to compute the monthly dividend payments (D_t). (Be careful, dividend yield is given in percentage.)

For instance, for the last month, in the U.S., the dividend yield is 1.82%, the price index is 3078.72, so that the monthly dividend payments are D_t = 3078.72 x 0.0182 / 12 = 4.66.94.

1. Theory

Let $r_{t+1} = (P_{t+1} + D_{t+1} - P_t)/P_t$ denote the holding return between dates *t* and *t*+1, where P_t is the stock price at date *t* and D_t is the dividend paid at date *t*.

The DDM assumes that the holding return is constant over time: $E_t[r_{t+1}] = r$. We deduce from this assumption that the price at date *t* should be driven by the following relationship, which defines the fundamental value of the stock:

$$P_t = \frac{1}{1+r} E_t[P_{t+1} + D_{t+1}]$$

Write the price P_t as a function of the sequence of all expected future dividends and of the price at the end of the times ($T \to \infty$). Show that the fundamental value of the stock writes

$$P_t = \sum_{i=1}^{\infty} \frac{E_t[D_{t+i}]}{(1+r)^i}$$

under some transversality condition that you should give. Interpret this condition.

Assume now that the dividend process is driven by an AR(1) process

$$D_{t+1} = (1+\mu) D_t + \varepsilon_{t+1}$$

where μ is the constant growth rate of the dividend process and ε_t is a white noise process.

Give the expression for the sequence of forecasts $E_t[D_{t+i}],\ \forall i > 0$, at date *t*.

Show that the fundamental value of the stock at date *t*, P_t, is a function of the dividend paid at date *t* and of the model's parameters (r and μ).

Show that, taking the log of this expression, we obtain a relationship that may be interpreted as a cointegration relation. We denote $p_t = \log(P_t)$ and $d_t = \log(D_t)$.

2. Stationarity test

2a. Computing critical values

The first step is to test if the stock price and dividend processes are stationary. The problem is that the number of observations is small and that the critical values of the stationarity test are not available for this number of observations. We therefore have to compute our own critical values for the Dickey-Fuller test. For this purpose, we use Monte-Carlo simulations.

The idea is to perform a large number of replications (take N = 10,000) of the following experiment *i*:

(1) Simulate a time series of *T* error terms $\varepsilon_t^{(i)}, t = 1,...,T$, distributed as a *N*(0,1);

(2) Compute a time series of stock price, assuming that they are driven by a random walk $p_t^{(i)} = p_{t-1}^{(i)} + \varepsilon_t^{(i)}$ (assume that the first observation is $p_0^{(i)} = 0$);

(3) Estimate the AR(1) model, under the alternative hypothesis, i.e. $\Delta p_t^{(i)} = \alpha^{(i)} + \beta^{(i)} p_{t-1}^{(i)} + u_t^{(i)}$. Compute the t-stat for $\beta^{(i)}$ denoted $t(\beta^{(i)})$.

(4) Sort the vector containing the *N* values of $t(\beta^{(i)})$, *i* = 1,…,*N*, by increasing order. Have a look at the histogram. You may have expected that the distribution of $t(\beta^{(i)})$ is symmetric around 0, since the data have been simulated under the null hypothesis $\beta^{(i)} = 0$. What do you observe?

(5) Compute the 10%, 5% and 1% critical values for the Dickey-Fuller test. They correspond to the quantiles at 10%, 5% and 1% of the distribution of $t(\beta^{(i)})$.

Remark: To check that your procedure is correct, redo the previous experiment for T = 100 and compare your critical values with those provided by Fuller (1976) (-2.58 at 10%, -2.89 at 5% and -3.51 at 1%). You should be close to these critical values (computed by Fuller using a similar procedure) at, say, two decimal places.

2b. Testing non-stationarity

For the series at hand, test the null hypothesis that the log-price has a unit root. Proceed as follows (standard Dickey-Fuller test). Run the following regression for stock prices

$$\Delta p_t = \alpha + \beta\, p_{t-1} + u_t$$

and test the null hypothesis $H_0 : \beta = 0$ using the corresponding t-stat. Of course, use the critical values of 2a. Run the same regression for the dividend process.

Comment your results and conclude on the stationarity/non-stationarity of the stock price and dividend processes.

2c. Power of the test

The power of a test is defined as follows: $power = \Pr(reject\ H_0 \mid H_1\ is\ true)$.

You are asked to compute the power of the stationary test when the true process follows an AR(1)-process with an autoregressive parameter of 0.96 :

$$p_t = 0.96 \cdot p_{t-1} + \epsilon_t$$

To proceed, start by simulating N = 10'000 series of this AR(1) process of the same length T as in 2a. For each simulated series, compute the test statistic for the stationary test. Finally, using the critical value for a significance level of 5% computed in 2a, compute the power of test.

Plot together the cumulative distribution function of the t-stats under the null (computed in 2a) and the cumulative distribution function of the t-stats under the AR(1) process given above. Comment.

How would the power of the test would change if we used T = 100? Would the stationary test be more or less powerful if we did the same experience with an autoregressive parameter of 0.80?

In light of this experience, comment your results in question 2b.

3. Cointegration test

Assuming that stock prices and dividends have been found to be non-stationary in the previous section, the second step is now to test if there is a cointegration relationship between the variables.

3a. Computing critical values

As before, we assume that the critical values of the cointegration test are not available for our number of observations, so that we have to compute our own critical values for the Dickey-Fuller test for cointegration. The procedure is essentially the same as in 2a.

(1) Simulate two time series of T error terms $(\varepsilon_t^{(i)}, \eta_t^{(i)}), t = 1,...,T$, distributed as two independent $N(0,1)$ variables.

(2) Compute two time series of stock prices and dividends, assuming that they are driven by independent random walks $p_t^{(i)} = p_{t-1}^{(i)} + \varepsilon_t^{(i)}$ and $d_t^{(i)} = d_{t-1}^{(i)} + \eta_t^{(i)}$ (assume that the first observation are $p_0^{(i)} = 0$ and $d_0^{(i)} = 0$);

(3) Estimate the relation between the two time series

$$p_t^{(i)} = a + b\, d_t^{(i)} + z_t^{(i)}$$

Under the null of no cointegration, the residual series $\hat{z}_t^{(i)}$ should be non-stationary. We therefore perform a standard Dickey-Fuller test on $\hat{z}_t^{(i)}$.

(4) Estimate the AR(1) model for the residuals, under the alternative hypothesis, i.e. $\Delta\hat{z}_t^{(i)} = \alpha^{(i)} + \beta^{(i)}\hat{z}_{t-1}^{(i)} + u_t^{(i)}$. Compute the t-stat for $\beta^{(i)}$ denoted $t(\beta^{(i)})$.

(5) Sort the vector containing the N values of $t(\beta^{(i)})$, $i = 1,\ldots,N$, by increasing order. Have a look at the histogram again. What do you observe now?

(6) Compute the 10%, 5% and 1% critical values for the Dickey-Fuller test for cointegration. They correspond to the quantiles at 10%, 5% and 1% of the distribution of $t(\beta^{(i)})$.

Remark: To check that your procedure is correct, redo the previous steps for $T = 100$ and compare your critical values with those provided by Phillips and Ouliaris (1988) (-3.07 at 10%, -3.37 at 5% and -3.96 at 1%). Once again, you should be close to these critical values.

3b. Testing cointegration

We now test the null hypothesis that stock prices and dividends are cointegrated. Proceed as follows.

(1) Estimate the regression for the US and UK markets:

$$p_t = a + b\, d_t + z_t$$

(2) Use the Dickey-Fuller test for testing the null of unit root in $\hat{z}_t^{(m)}$. So, estimate the regression

$$\Delta\hat{z}_t^{(m)} = \alpha + \beta\,\hat{z}_{t-1}^{(m)} + u_t$$

and test the null hypothesis $H_0 : \beta = 0$ using the corresponding t-stat. Use the critical values of 3a.

Comment your results and conclude on the cointegration/non- cointegration between the stock price and dividend processes. Looking at the values of the parameter estimates (a and b), give your conclusion about the DDM.

3c. Error-correcting model

For the U.K., estimate an error-correcting model with one lag of the form:

$$\Delta p_t = \mu_1 + \varphi_{11}\Delta p_{t-1} + \varphi_{12}\Delta d_{t-1} + \gamma_1 z_{t-1} + \varepsilon_{1,t}$$

$$\Delta d_t = \mu_2 + \varphi_{21}\Delta p_{t-1} + \varphi_{22}\Delta d_{t-1} + \gamma_2 z_{t-1} + \varepsilon_{2,t}$$

Interpret the parameter estimates and in particular the sign and significance of the γ parameters.