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Peg solitaire on graphs

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ABSTRACT

There have been several papers on the subject of traditional peg solitaire on different boards. However, in this paper we consider a generalization of the game to arbitrary boards. These boards are treated as graphs in the combinatorial sense. We present necessary and sufficient conditions for the solvability of several well-known families of graphs. In the major result of this paper, we show that the cartesian product of two solvable graphs is likewise solvable. Several related results are also presented. Finally, several open problems related to this study are given.

1. Introduction

Peg solitaire is a table game which traditionally begins with "pegs" in every space except for one which is left empty (in other words, a "hole"). If in some row or column two adjacent pegs are next to a hole (as in Fig. 1), then the peg in x can jump over the peg in y into the hole in z. The peg in y is then removed. The goal is to remove every peg but one. If this is achieved, then the board is considered solved.

There have been a number of papers regarding solving peg solitaire on various boards (see for example [1,2,4]). However, we generalize this to arbitrary boards. These boards are treated as graphs in the combinatorial sense.

A graph, $G = (V, E)$, is a set of vertices, V, and a set of edges, E. Because of the restrictions of peg solitaire, we will assume that all graphs are finite undirected graphs with no loops or multiple edges. In particular, we will *always* assume that graphs are connected. For all undefined graph theory terminology, refer to West [6].

There is a compromise inherent in this approach. For instance, the notions of "rows" and "columns" are not well defined in graph theory, as the placement of vertices is considered irrelevant. Instead, if there are pegs in vertices x and y and a hole in z, then we allow x to jump over y into z provided that $xy \in E$ and $yz \in E$. The peg in y is then removed.

In general, the game begins with a *starting state* $S \subset V$ which is a set of vertices that are empty. A *terminal state* $T \subset V$ is a set of vertices that have pegs at the end of the game. A terminal state T is *associated* with starting state S if T can be obtained from S by a series of jumps. Unless otherwise noted, we will assume that S consists of a single vertex. A graph G is *solvable* if there exists some vertex s so that, starting with $S = \{s\}$, there exists an associated terminal state consisting of a single peg. A graph G is *freely solvable* if for all vertices s so that, starting with $S = \{s\}$, there exists an associated terminal state consisting of a single peg. It is not always possible to solve a graph. A graph G is *k-solvable* if there exists some vertex s so that, starting with $S = \{s\}$, there exists an associated terminal state consisting of k nonadjacent pegs. In particular, a graph is *distance* 2*-solvable* if there exists some vertex s so that, starting with $S = \{s\}$, there exists an associated terminal state consisting of two pegs that are distance 2 apart.

The goal of this paper is to gain insight into which graphs are solvable. To show that a graph G is solvable (or freely solvable) it is sufficient to produce the required solution. However, claiming that a general graph is not solvable is more difficult. To aid in our constructions, we begin with two relatively simple, yet powerful, results.

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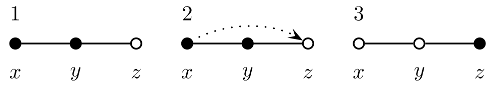

Fig. 1. A typical jump in peg solitaire.

Note that if G is a k-solvable spanning subgraph of H, then we can k-solve H using only the edges of G. Thus, the next proposition is immediate.

Proposition 1.1. *If G is a k-solvable spanning subgraph of H, then H is (at worst) k-solvable.*

The next theorem is more subtle.

Theorem 1.2. *Suppose that S is a starting state of G with associated terminal state T. Define the sets S' and T' by reversing the roles of "pegs" and "holes" in S and T, respectively. It follows that T' is a starting state of G with associated terminal state S'.*

Proof. Consider a game that is played in reverse. We begin the *reverse game* with a set of pegs in T. In the reverse game, if $xy, yz \in E(G)$ with a peg in x and holes in y and z, then x can jump over y into z, placing a new peg in y. Clearly, if S is a starting state in the original game with associated terminal state T, then the reverse game has terminal state S associated with starting state T. The reverse game is equivalent to the original game [3]. Reversing the roles of "holes" and "pegs" obtains the desired result. □

This implies that if a graph can be solved beginning with vertex u and ending in vertex v, then we can also solve the graph beginning in vertex v and ending in vertex u.

2. Families of graphs

In this section, we present necessary and sufficient conditions for the solvability of several well-known families of graphs. As usual, P_n and C_n will denote the path and the cycle on n vertices, respectively. The complete bipartite graph with $V = X \cup Y$, $|X| = n$, and $|Y| = m$ is denoted $K_{n,m}$. If $n = 1$, the complete bipartite graph is typically called a *star* [6].

Proposition 2.1. *The star $K_{1,n}$ is $(n-1)$-solvable.*

Proof. Note that there are essentially two choices for the initial hole. Namely, the initial hole can be placed in the center or in one of the pendant vertices. If the hole begins in the center vertex, then there are no two adjacent pegs. Thus there are no possible moves. If the hole is placed in one of the pendant vertex, then it is possible to jump from one of the other peripheral vertices over the central vertex into the hole. However, this reduces to the previous case. Hence, $K_{1,n}$ is $(n-1)$-solvable. □

We begin by determining when it is possible to solve paths. In such discussions, it is convenient to label the vertices of the path (or the cycle) with elements of the set $\{0, 1, \ldots, n-1\}$ in the obvious way.

Lemma 2.2. *If $n > 1$, then P_{2n} is not freely solvable and P_{2n+1} is not solvable.*

Proof. Note that with a path, any jump will result in an edge that has holes in both endpoints. Such an edge will be referred to as an *empty bridge*. Suppose that i and $i+1$ are the endpoints of an empty bridge with pegs on either side of the empty bridge. To solve the path, it must be possible to jump into both endpoints of the empty bridge. That is, it must be possible to jump from $i-2$ over $i-1$ into i and it must be possible to jump from $i+3$ over $i+2$ into $i+1$. Hence if there is one peg on one side of the empty bridge, then the graph cannot be solved from this state.

If the initial hole is in vertex 0, then the first jump will be from 2 over 1 into 0. This results in a single peg on one side of an empty bridge. Hence P_{2n} is not freely solvable if $n > 1$.

Note that for P_{2n+1}, the initial hole divides the $2n$ pegs into two groups. These groups must necessarily be both even or both odd. In both cases, the first jump removes two pegs from one group and adds one peg to the other group. This results in an odd number of pegs, followed by an empty bridge, followed by an even number of pegs. Suppose that there are $2k+1$ pegs in the odd group of pegs. A single jump from this side into the empty bridge forms a new empty bridge and leaves two fewer pegs in its odd group. After k such jumps, there is a single peg on one side of an empty bridge. Hence, P_{2n+1} is not solvable. □

Theorem 2.3. *The following hold for the path on n vertices: P_n is freely solvable if and only if $n = 2$; P_n is solvable if and only if n is even or $n = 3$; P_n is distance 2-solvable in all other cases.*

Proof. Note that P_3 is not freely solvable, as there are no available moves when the initial hole is placed in vertex 1. The other necessary conditions are given in Lemma 2.2. It is sufficient to give the required solutions in the remaining cases. The fact that P_2 is freely solvable is trivial. The solution for P_3 is to begin with the hole in 0, then jump from 2 over 1 into 0.

We first show that P_4 is solvable and that P_5 is distance 2-solvable. In each case, place the initial hole in position $n-2$. Jump from $n-4$ over $n-3$ into $n-2$. Finally, jump from $n-1$ over $n-2$ into $n-3$.

Assume that the following hold for some $n \geq 4$: If n is odd, then P_n is distance 2-solvable with the initial hole in position $n-2$; if n is even, then P_n is solvable with the initial hole in position $n-2$.

Now consider the case of P_{n+1}. The initial hole is placed in position $n-1$. Begin by jumping from $n-3$ over $n-2$ into $n-1$. Then jump from n over $n-1$ into $n-2$. Ignoring the holes in n and $n-1$, the graph is a path on $n-1$ vertices with a hole in $n-3$. The claim then follows by induction. $\square$

Using this we can obtain results about other graphs. In particular, the Petersen graph has a spanning path and an even number of vertices. Thus it is solvable by Theorem 2.3 and Proposition 1.1. Because it is also vertex transitive, it is freely solvable. Similar observations will be useful for the cycle.

Theorem 2.4. *The following hold for the cycle on n vertices: C_n is freely solvable if and only if n is even or $n = 3$; C_n is distance 2-solvable in all other cases.*

Proof. We begin by showing C_{2n+1} is not solvable for $n \geq 2$. Since the cycle is vertex transitive, the placement of the initial vertex is irrelevant. Further, the first two jumps are forced (up to automorphisms of the vertices). Hence, without loss of generality, the first three holes are in $2n-1$, $2n$, and 1. From this it follows that C_5 is not solvable.

Assume that $n \geq 3$. If the holes in $2n-1$ and $2n$ are both not used, then this reduces the case of P_{2n-1} (see Lemma 2.2). If the hole in $2n$ is used, then we must jump from 3 over 2 into 1. Next, jump from 1 over 0 into $2n$. There are now holes in four adjacent vertices. Ignoring these holes reduces this problem to that of the odd path. Jumping into one of the holes results in a single peg surrounded by two empty bridges. Thus, we cannot solve the graph from this state. If we jump into $2n-1$, then we must also jump into $2n$. This reduces down to the previous case.

The solutions of these graphs follow from the fact that C_n has P_n as a spanning subgraph (see Theorem 2.3 and Proposition 1.1). Since C_n is vertex transitive, it follows that all solvable cycles are freely solvable. $\square$

Corollary 2.5. *Let G be a graph with n vertices.* (i) *If G is Hamiltonian and n is even, then G is freely solvable.* (ii) *If G is Hamiltonian and contains a triangle, then G is solvable.* (iii) *If every vertex $v \in V(G)$ has degree greater than $n/2$, then G is solvable.*

Proof. By definition, a Hamiltonian graph has a spanning cycle. Hence (i) is an immediate consequence of Theorem 2.4. If G has an odd number of vertices, then the spanning cycle is distance 2-solvable by Theorem 2.4. To obtain (ii), simply solve the spanning cycle in such a way that the final two pegs are on legs of the triangle. Finally, note that if every vertex in G has degree greater than $n/2$, then G is Hamiltonian by Dirac's Theorem [6]. Further, the number of edges in G is greater than $n^2/4$. Turàn's Theorem [6] implies that G contains a triangle. Thus, (iii) follows. $\square$

Note that Corollary 2.5 implies that several classes of graph are freely solvable. In particular, the platonic solids are Hamiltonian and have an even number of vertices, thus they are freely solvable. The same claim can be made regarding the n-dimensional hypercube. Further, the complete graph on $n \geq 3$ vertices is Hamiltonian, vertex transitive, and has a triangle. Thus, the next corollary follows immediately from this observation and Theorem 2.3.

Corollary 2.6. *The complete graph on $n \geq 2$ vertices is freely solvable.*

We now consider complete bipartite graphs.

Theorem 2.7. *Let $n \geq 2$ and $m \geq 2$. The complete bipartite graph $K_{n,m}$ is freely solvable.*

Proof. Let $V(K_{n,m}) = X \cup Y$, where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$. We begin with the hole in x_n. For $i = 1, \ldots, \lfloor (m-1)/2 \rfloor$, we define the $(2i-1)$st jump to be the jump from x_{n-1} over y_{m+2-2i} into x_n. Similarly, the $(2i)$th jump is from x_n over y_{m+1-2i} into x_{n-1}. If m is even, then we make an additional jump from x_{n-1} over y_2 into x_n. We also relabel the vertices x_n and x_{n-1} so that x_n has a hole.

We now begin a new series of jumps. In this new series of jumps, for $j = 1, \ldots, \lfloor (n-2)/2 \rfloor$ we define the $(2j-1)$st jump to be from y_1 over x_{2j-1} into y_2. Similarly, the $(2j)$th jump is from y_2 over x_{2j} into y_1. If n is odd, then we make an additional jump from y_1 over x_{n-2} into y_2. After completing this second series of jumps, there will be two pegs, one in X and one in Y. Thus, we can end with our final peg in either X or Y. From Theorem 1.2 it follows that the graph is solvable from either starting position. Hence, $K_{n,m}$ is freely solvable. $\square$

Note that if G and H both have at least two vertices, then the join of G and H has $K_{|V(G)|,|V(H)|}$ as a spanning subgraph. Hence the join is freely solvable. Further, the complete k-partite graph has either the complete graph or the complete bipartite graph as a spanning subgraph. Therefore, the complete k-partite graph is freely solvable.

3. Cartesian products of solvable graphs

In this section, we give results on the Cartesian product of two solvable graphs. For graphs G and H, the Cartesian product of G and H is denoted $G \square H$ [6]. For $g \in V(G)$ and $h \in V(H)$, let $(g, h) \in V(G \square H)$ denote the vertex in the Cartesian product induced by those vertices. We define G_h to be the copy of G induced by the vertex $h \in V(H)$. We will also assume that G is solvable (or distance 2-solvable) with the initial hole in g_s. In the case where G is solvable, the final peg will end in g_t. In the case where G is distance 2-solvable, the final two pegs will be in g_{t1} and g_{t3} with common neighbor g_{t2}. Analogous definitions will be used for H.

Theorem 3.1. *If G is solvable or distance 2-solvable, then $G\Box P_2$ is solvable.*

Proof. Without loss of generality, suppose that G has at least two vertices. Let $V(P_2) = \{h_1, h_2\}$. Let g'_s be any neighbor of g_s. Solve the Cartesian product by placing the initial hole in vertex (g'_s, h_1). Jump from (g_s, h_2) over (g_s, h_1) into (g'_s, h_1). Now, G_{h_1} and G_{h_2} both have holes in g_s. Solve (distance 2-solve) them independently.

If G is solvable, both copies will end with a peg in g_t. Let g'_t be any neighbor of g_t in G. Jump from (g_t, h_1) over (g_t, h_2) into (g'_t, h_2). If G is distance 2-solvable, jump from (g_{t1}, h_1) over (g_{t1}, h_2) into (g_{t2}, h_2). Jump from (g_{t3}, h_2) over (g_{t3}, h_1) into (g_{t2}, h_1). Finally, jump from (g_{t2}, h_2) over (g_{t2}, h_1) into (g_{t3}, h_1). □

Similarly, we can show that if G is freely solvable, then $G\Box P_2$ is freely solvable. Before showing more general results regarding the Cartesian product, we make the following observation.

Observation 3.2. *Suppose that G has at least three vertices and is k-solvable beginning with initial hole in g_s. Assuming that a jump is possible, then there is a first jump, say from g''_s over g'_s into g_s. It follows that if G has holes in g'_s and g''_s and pegs everywhere else, then G is k-solvable from this state. Similarly, if G is solvable with the final peg in g_t, then there is a final jump, say from g''_t over g'_t into g_t. Analogous statements can be made regarding H.*

We will use the notation defined in Observation 3.2 in our next results.

Theorem 3.3. *Suppose that G and H are graphs such that G is solvable and H is solvable. The Cartesian product $G\Box H$ is solvable.*

Proof. Note that if G or H is isomorphic to P_2, then the Cartesian product is solvable by Theorem 3.1. Hence, we may assume that both G and H have at least three vertices.

Begin with the initial hole in (g_s, h_s). Solve H_{g_s} ending in (g_s, h_t). Jump from (g'_s, h_t) over (g_s, h_t) into (g_s, h'_t). Next jump from (g''_s, h'_t) over (g'_s, h'_t) into (g'_s, h_t).

Now all copies of G have a hole in g_s and pegs elsewhere or (in the case of $G_{h'_t}$) holes in g'_s and g''_s and pegs elsewhere. In either case, solve each copy of G independently, ending with our final two pegs in g'_t and g''_t. Next jump from (g'_t, h''_s) over (g'_t, h'_s) into (g_t, h'_s). Jump from (g''_t, h_s) over (g'_t, h_s) into (g_t, h_s). Next, jump from (g_t, h'_s) over (g_t, h_s) into (g'_t, h_s).

Now, $H_{g''_t}$ has a hole in (g''_t, h_s) and pegs everywhere else. Similarly, $H_{g'_t}$ has holes in (g'_t, h'_s) and (g'_t, h''_s). In either case, we can solve $H_{g'_t}$ and $H_{g''_t}$ independently with the final two pegs in (g'_t, h_t) and (g''_t, h_t). Complete the solution by jumping from (g''_t, h_t) over (g'_t, h_t) into (g_t, h_t). □

The next two proofs follow a similar structure to above.

Theorem 3.4. *Suppose that G and H are graphs such that G is solvable and H is distance 2-solvable. The Cartesian product $G\Box H$ is solvable.*

Proof. Without loss of generality, assume that G has at least three vertices. Begin with the initial hole in (g_s, h_s). Distance 2-solve H_{g_s} with the final two pegs in (g_s, h_{t1}) and (g_s, h_{t3}). Jump from (g''_s, h_{t2}) over (g'_s, h_{t2}) into (g_s, h_{t2}). Next jump from (g'_s, h_{t1}) over (g''_s, h_{t1}) into (g''_s, h_{t2}). Finally, jump from (g_s, h_{t3}) over (g_s, h_{t2}) into (g'_s, h_{t2}).

Now all copies of G have a hole in g_s and pegs elsewhere or (in the case of $G_{h_{t1}}$) holes in g'_s and g''_s and pegs elsewhere. In either case, solve them independently with the final two pegs for each copy in g'_t and g''_t.

Jump from (g'_t, h''_s) over (g'_t, h'_s) into (g_t, h'_s). Then jump from (g''_t, h_s) over (g'_t, h_s) into (g_t, h_s). Next jump from (g_t, h'_s) over (g_t, h_s) into (g'_t, h_s).

Now, $H_{g''_t}$ has a hole in (g''_t, h_s) and pegs everywhere else. Similarly, $H_{g'_t}$ has holes in (g'_t, h'_s) and (g'_t, h''_s). In either case, we can distance 2-solve $H_{g'_t}$ and $H_{g''_t}$ independently with the final four pegs in (g'_t, h_{t1}), (g'_t, h_{t3}), (g''_t, h_{t1}), and (g''_t, h_{t3}).

Complete the solution by jumping from (g'_t, h_{t1}) over (g''_t, h_{t1}) into (g''_t, h_{t2}). Then jump from (g''_t, h_{t3}) over (g'_t, h_{t3}) into (g'_t, h_{t2}). Finally, jump from (g''_t, h_{t2}) over (g'_t, h_{t2}) into (g_t, h_{t2}). □

Theorem 3.5. *Suppose that G and H are graphs such that G and H are both distance 2-solvable. The Cartesian product $G\Box H$ is solvable.*

Proof. Begin with the initial hole in (g_{t1}, h_s). Jump from (g_{t2}, h'_s) over (g_{t1}, h'_s) into (g_{t1}, h_s). Next jump from (g_{t1}, h''_s) over (g_{t2}, h''_s) into (g_{t2}, h'_s). Finally, jump from (g_{t3}, h'_s) over (g_{t3}, h''_s) into (g_{t2}, h''_s).

Now $H_{g_{t1}}$ and $H_{g_{t3}}$ have holes in h'_s and h''_s and pegs elsewhere. Hence they can be distance 2-solved independently. In both cases, end with pegs for each copy in h_{t1} and h_{t3}. Jump from (g_{t2}, h_{t1}) over (g_{t1}, h_{t1}) into (g_{t1}, h_{t2}). Then jump from (g_{t2}, h_{t3}) over (g_{t3}, h_{t3}) into (g_{t3}, h_{t2}). Now jump from (g_{t3}, h_{t2}) over (g_{t3}, h_{t1}) into (g_{t2}, h_{t1}). Next jump from (g_{t1}, h_{t2}) over (g_{t1}, h_{t3}) into (g_{t2}, h_{t3}).

Now, each copy of G have holes in g_{t1} and g_{t3} and pegs elsewhere. By Theorem 1.2, these can be solved independently with the final two pegs in g'_s and g''_s. The rest of the proof is analogous that of Theorem 3.4. □

4. Open problems

We end our discussion by giving several open problems as a basis for future research. One of the main open problems is to establish necessary and sufficient conditions for an arbitrary graph to be solvable. In particular, which trees are solvable?

Define $G_{n,k}$ to be the set of all graphs on n vertices and k edges. Given a fixed n, what is the minimum k such that every graph in this set is solvable?

In the literature, one of the main concerns is the minimum number of *moves* necessary to solve the puzzle. A *move* is a series of jumps made with a single peg. Given a solvable graph G, what is the minimum number of moves necessary to solve G?

In *peg duotaire* (see for example [5]) two players take turns making peg solitaire jumps. Whoever is left without a jump loses. For which graphs does Player One have a winning strategy? For which graphs does Player Two have a winning strategy? What if we consider peg solitaire moves rather than jumps?

As a variation on traditional peg solitaire, we instead try to leave the *maximum* number of pegs possible under the caveat that we make a jump whenever one is available. For an arbitrary graph G, determine the maximum number of pegs that can be left in this variant.

In another variation of peg solitaire, it is required that the final peg is in the same location as the original hole, in other words, $S = T = \{u\}$. What graphs can be solved in this variant?

Finally, we consider a generalization of the last two problems. Given a graph G and a starting state S, determine all terminal states associated with S.

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