Benchmark Problems for Trjectory Optimization

1. Problem formulation

1.1. Problem formulation of lunar 3-DOF landing (L3L)

The curvature of the lunar surface can be ignored, the lunar surface gravitational acceleration is constant and the vehicle can be regarded as a mass point.

The lunar soft-landing process can be treated in a two-body system. The motion of a lunar soft landing is described in three-dimensional coordinates in Fig.1. This case assumes that $\boldsymbol{oxyz}$ is the Lunar Descent Inertial Coordinate.

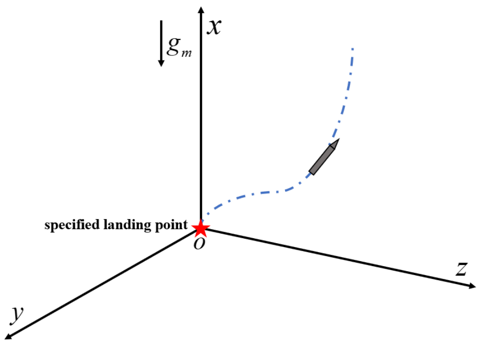

Figure 1: Lunar Descent Inertial Coordinate

Optimal control problem: The lunar soft-landing trajectory optimization problem is described as follows:

$$
\begin{aligned}
\min_{F,\boldsymbol{u}} \quad & -m_{\text{prop}}(t_f) \\
s.t. \quad & \dot{\boldsymbol{r}} = \boldsymbol{V} \\
& \dot{\boldsymbol{V}} = \frac{F}{m_{\text{prop}}+m_{\text{dry}}}\boldsymbol{u} - [g_{\text{M}}\ 0\ 0]^T \\
& \dot{m}_{\text{prop}} = \frac{F}{V_{\text{e}}} \\
& \boldsymbol{r}(0) = \boldsymbol{r}_0, \boldsymbol{V}(0) = \boldsymbol{V}_0, m_{\text{prop}}(0) = m_{\text{prop}_0} \\
& \boldsymbol{r}(t_f) = \boldsymbol{r}_f, \boldsymbol{V}(t_f) = \boldsymbol{V}_f \\
& \boldsymbol{u}^T\boldsymbol{u} = 1 \\
& F_{\text{min}} \le F \le F_{\text{max}}, \boldsymbol{r}^T\boldsymbol{e} \ge 0, \quad \boldsymbol{e} = [1 \quad 0 \quad 0]^T
\end{aligned}
\tag{1}
$$

where $\boldsymbol{r} = [x\ y\ z]^T$ is the position vector of the vehicle; $\boldsymbol{V} = [V_{\text{x}}\ V_{\text{y}}\ V_{\text{z}}]^T$ is the velocity vector; m_{prop} is the vehicle propellant mass; m_{dry} is the dry mass; $\boldsymbol{u} = [u_{\text{x}}\ u_{\text{y}}\ u_{\text{z}}]^T$ represents the direction of thrust; F denotes the thrust magnitude; V_{e} is the effective exhaust velocity of the engine; g_{M} denotes the moons gravitational acceleration.

The Hamiltonian function of the problem in Eq. (1) is a linear function of thrust F. Meanwhile, according to the optimal necessary conditions, $\boldsymbol{u}$ is not included in the switching function of the minimum Hamiltonian function.

1.2. Problem formulation of Multi-stage Rocket Launch (MSRL)

The launch mission is to carry a payload to the sun-synchronous orbit, and the launch site is assumed to be the Chinese Wenchang Satellite Launch Site.

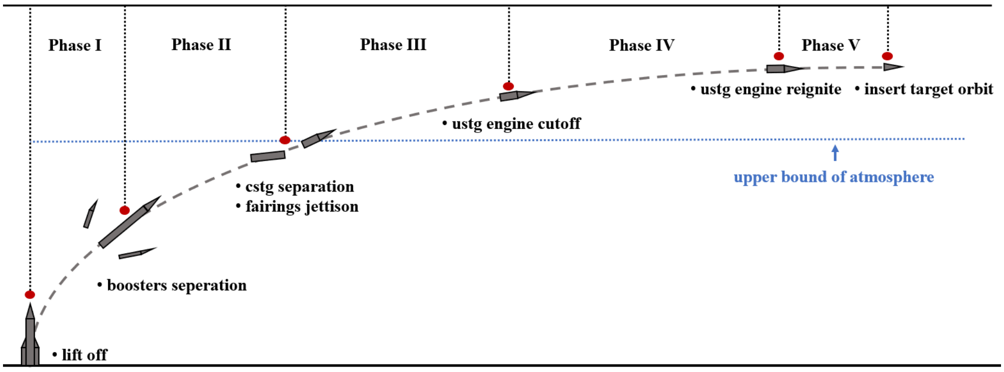

Figure 2: The phase division of the launch flight

The launch flight is divided into five phases, as shown in Fig.2 according to the structure of the rocket and flight profile. '*cstg*' and '*ustg*' are the abbreviation of the core stage and the upper stage, respectively. The other phases do not consider aerodynamics except for *Phase I* and *Phase II*. *Phase IV* is an unpowered sliding phase.

The launch can be treated in a two-body system. The motion of a multi-stage rocket is described in Earth-Centered and Earth-Fixed Coordinate (ECEF). A 3DOF dynamic model will be built where the rocket body axis is always collinear with the thrust direction by default.

Some simplifications are made that core stage separation and fairing jettison occurred at the same time, the influence of the wind field is ignored and the constraints of wreckage falling zone is not considered. We also assume that the speed of sound remains constant during the flight.

Kinematics and dynamic model:

For *Phase I*:

$$\begin{aligned}
&\dot{m}_{\text{prop}}^{\text{cstg}} = \frac{F_{\text{T}}^{\text{cstg}}}{V_{\text{e}}^{\text{rp1}}}, \dot{m}_{\text{prop}}^{\text{booster}} = \frac{F_{\text{T}}^{\text{booster}}}{V_{\text{e}}^{\text{rp1}}} \\
&\dot{\boldsymbol{r}} = \boldsymbol{V}, \dot{\boldsymbol{V}} = \frac{(F_{\text{T}}^{\text{cstg}} + F_{\text{T}}^{\text{booster}})\boldsymbol{u}_{\text{T}} + \boldsymbol{F}_{\text{aero}}}{(m_{\text{prop}} + m_{\text{dry}} + m_{\text{payload}})} + \boldsymbol{g}_{\text{E}} + \boldsymbol{a}_{\text{I}} \\
&\dot{F}_{\text{T}}^{\text{cstg}} = \chi_{\text{T}}^{\text{cstg}}, \dot{\boldsymbol{u}}_{\text{T}} = \boldsymbol{\chi}_{\boldsymbol{u}_{\text{T}}} \\
&m_{\text{prop}} = 2m_{\text{prop}}^{\text{booster}} + m_{\text{prop}}^{\text{cstg}} + m_{\text{prop}}^{\text{ustg}} \\
&m_{\text{dry}} = 2m_{\text{dry}}^{\text{booster}} + m_{\text{dry}}^{\text{cstg}} + m_{\text{dry}}^{\text{ustg}} + m_{\text{fairings}}
\end{aligned} \tag{2}$$

For *Phase II*:

$$\begin{aligned}
&\dot{m}_{\text{prop}}^{\text{cstg}} = \frac{F_{\text{T}}^{\text{cstg}}}{V_{\text{e}}^{\text{rp1}}} \\
&\dot{\boldsymbol{r}} = \boldsymbol{V}, \dot{\boldsymbol{V}} = \frac{F_{\text{T}}^{\text{cstg}}\boldsymbol{u}_{\text{T}} + \boldsymbol{F}_{\text{aero}}}{(m_{\text{prop}} + m_{\text{dry}} + m_{\text{payload}})} + \boldsymbol{g}_{\text{E}} + \boldsymbol{a}_{\text{I}} \\
&\dot{F}_{\text{T}}^{\text{cstg}} = \chi_{\text{T}}^{\text{cstg}}, \dot{\boldsymbol{u}}_{\text{T}} = \boldsymbol{\chi}_{\boldsymbol{u}_{\text{T}}} \\
&m_{\text{prop}} = m_{\text{prop}}^{\text{cstg}} + m_{\text{prop}}^{\text{ustg}}, m_{\text{dry}} = m_{\text{dry}}^{\text{cstg}} + m_{\text{dry}}^{\text{ustg}} + m_{\text{fairings}}
\end{aligned} \tag{3}$$

For *Phase III* and *Phase V*:

$$\begin{aligned}
&\dot{m}_{\text{prop}}^{\text{ustg}} = \frac{F_{\text{T}}^{\text{ustg}}}{V_{\text{e}}^{\text{lh2}}}, \dot{\boldsymbol{r}} = \boldsymbol{V} \\
&\dot{\boldsymbol{V}} = \frac{F_{\text{T}}^{\text{ustg}}\boldsymbol{u}_{\text{T}}}{(m_{\text{prop}}^{\text{ustg}} + m_{\text{dry}}^{\text{ustg}} + m_{\text{payload}})} + \boldsymbol{g}_{\text{E}} + \boldsymbol{a}_{\text{I}}, \dot{\boldsymbol{u}}_{\text{T}} = \boldsymbol{\chi}_{\boldsymbol{u}_{\text{T}}}
\end{aligned} \tag{4}$$

For *Phase IV*:

$$\dot{m}^{\mathrm{ustg}} = 0, \dot{\boldsymbol{r}} = \boldsymbol{V}, \dot{\boldsymbol{V}} = \boldsymbol{g}_{\mathrm{E}} + \boldsymbol{a}_{\mathrm{I}} \tag{5}$$

Where $\boldsymbol{a}_{\mathrm{I}} = -2\boldsymbol{\omega}_{\mathrm{E}} \times \boldsymbol{V} - \boldsymbol{\omega}_{\mathrm{E}} \times \boldsymbol{\omega}_{\mathrm{E}} \times \boldsymbol{r}$ is the inertial acceleration vector; $\boldsymbol{r} = \left[r_{\mathrm{x}}\ r_{\mathrm{y}}\ r_{\mathrm{z}}\right]^T$ is the position vector of vehicle; $\boldsymbol{V} = [V_{\mathrm{x}}\ V_{\mathrm{y}}\ V_{\mathrm{z}}]^T$ is the velocity vector of vehicle; m_{prop} is the propellant mass of each stage; $\boldsymbol{u}_{\mathrm{T}} = [u_{\mathrm{Tx}}\ u_{\mathrm{Ty}}\ u_{\mathrm{Tz}}]^T$ represents the direction of total thrust; F_{T} denotes the thrust magnitude; V_{e} is the effective exhaust velocity of the engine; $\delta_{F_{\mathrm{T}}}$ denotes the change rate of thrust magnitude; $\delta_{\boldsymbol{u}_{\mathrm{T}}}$ denotes the change rate of thrust direction; $\boldsymbol{\omega}_{\mathrm{E}}$ is the vector of rotational angular velocity of the earth.

The $\boldsymbol{F}_{\mathrm{aero}}$ in Eq. (2) and Eq. (3) is the aerodynamic vector, which can be write as:

$$\boldsymbol{F}_{\mathrm{aero}} = -\frac{1}{2} C_{\mathrm{D}} S_{\mathrm{ref}} \rho_{air} \|\boldsymbol{V}_{\mathrm{air}}\|_2 \boldsymbol{V}_{\mathrm{air}} \tag{6}$$

Where C_{D} is the drag coefficient; S_{ref} is the reference cross-sectional area; $\boldsymbol{V}_{\mathrm{air}} = V_{\mathrm{wind}} - V$ is the vehicle velocity relative to air; $\rho_{\mathrm{air}} = \rho_{\mathrm{air0}} e^{-h/h_0}$ is the atmospheric density, $h = \|\boldsymbol{r}\|_2 - R_{\mathrm{E}}$ is the flight altitude; h_0 the reference altitude and R_E is the average Earth radius. The vector $\boldsymbol{g}_{\mathrm{E}}$ can be expressed as $\boldsymbol{g}_{\mathrm{E}} = -\frac{GM_{\mathrm{E}}}{\|\boldsymbol{r}\|_2{}^3} \boldsymbol{r}$, where G is the gravitational constant and M_{E} is the mass of Earth.

Path constraints: Thrust magnitude constraints:

$$\begin{aligned} &F_{\mathrm{T,min}}^{\mathrm{cstg}} \le F_{\mathrm{T}}^{\mathrm{cstg}} \le F_{\mathrm{T,max}}^{\mathrm{cstg}} \\ &F_{\mathrm{T}}^{\mathrm{booster}} = F_{\mathrm{T,const}}^{\mathrm{booster}}, F_{\mathrm{T}}^{\mathrm{ustg}} = F_{\mathrm{T,const}}^{\mathrm{ustg}} \end{aligned} \tag{7}$$

The constraint for thrust direction vector writes as:

$$\boldsymbol{u}_{\mathrm{T}}{}^T \boldsymbol{u}_{\mathrm{T}} = 1 \tag{8}$$

The thrust magnitude and direction change rate constraints:

$$\left\|\chi_{\boldsymbol{u}_{\mathrm{T}}}\right\|_2 \le \sin(\omega_{\mathrm{u,max}}/2\ ) \tag{9}$$

$$\chi_{\mathrm{T}}^{\mathrm{cstg}} \le F_{\mathrm{T,max}}^{\mathrm{cstg}}/25 \tag{10}$$

The dynamic pressure constraint for the flight of *Phase I* and *Phase II* is that:

$$\frac{1}{2} \alpha \rho_{\mathrm{air}} (\boldsymbol{V}_{\mathrm{air}}^T \boldsymbol{V}_{\mathrm{air}}) \le Q_{\alpha,\mathrm{max}} \tag{11}$$

$$\frac{1}{2} \rho_{\mathrm{air}} (\boldsymbol{V}_{\mathrm{air}}^T \boldsymbol{V}_{\mathrm{air}}) \le Q_{\mathrm{max}} \tag{12}$$

Where α is the angle between $\boldsymbol{u}_{F_{\mathrm{T}}}$ and $\boldsymbol{V}_{\mathrm{air}}$.

The clear tower constraints:

$$\boldsymbol{u}_{F_{\mathrm{T}}}(t) \perp \boldsymbol{r}(0), \quad F_{\mathrm{T}}^{\mathrm{cstg}}(t) = F_{\mathrm{T,max}}^{\mathrm{cstg}}, \quad 0 \le t \le t_{\mathrm{clear}} \tag{13}$$

The working time constraints for the *cstg* engine after reignition:

$$\sum_{i=1}^{{}^{V}NE} {}^{V}h_i {}^{V}\sigma_i \ge 50 \tag{14}$$

Where the left superscript is the phase index.

Given that the flight is over the surface of the earth:

$$\|\boldsymbol{r}\|_2 \ge R_{\mathrm{E}} \tag{15}$$

Boundary constraints: The initial constraints for the whole flight are expressed as follows:

$$\begin{aligned}
&{}^{I}\boldsymbol{r}(0) = \boldsymbol{r}_0, {}^{I}\boldsymbol{V}(0) = \boldsymbol{V}_0 \\
&{}^{I}m_{\text{prop}}^{\text{cstg}}(0) = m_{\text{prop0}}^{\text{cstg}}, {}^{I}m_{\text{prop}}^{\text{booster}}(0) = m_{\text{prop0}}^{\text{booster}}, {}^{I}m_{\text{prop}}^{\text{ustg}}(0) = m_{\text{prop0}}^{\text{ustg}} \\
&{}^{I}\boldsymbol{u}_{F_{\text{T}}}(0) = \boldsymbol{u}_{F_{\text{T}}0}, {}^{I}F_{\text{T}}^{\text{cstg}}(0) = F_{\text{T0}}^{\text{cstg}}
\end{aligned} \tag{16}$$

The final constraints for the whole flight are expressed as follows:

$$^{V}inc_f = inc_{\text{tgt}}, {}^{V}ecc_f = ecc_{\text{tgt}}, {}^{V}a_f = a_{\text{tgt}} \tag{17}$$

Where inc_{tgt} is target orbital inclination; ecc_{tgt} is the target orbital eccentricity; a_{tgt} is the target orbital semi-major axis.

The connection constraints for each phase:

$$\begin{aligned}
&{}^{q}\boldsymbol{r}({}^{q}t_f) = {}^{q+1}\boldsymbol{r}(0), {}^{q}\boldsymbol{V}({}^{q}t_f) = {}^{q+1}\boldsymbol{V}(0) \\
&{}^{I}m_{\text{prop}}^{\text{cstg}}({}^{I}t_f) = {}^{II}m_{\text{prop}}^{\text{cstg}}(0), {}^{q}m_{\text{prop}}^{\text{ustg}}({}^{q}t_f) = {}^{q+1}m_{\text{prop}}^{\text{ustg}}(0) \\
&{}^{q}\boldsymbol{u}_{F_{\text{T}}}({}^{q}t_f) = {}^{q+1}\boldsymbol{u}_{F_{\text{T}}}(0) \\
&q = I, II, III, IV
\end{aligned} \tag{18}$$

Objective function: The objective of the trajectory optimization problem is to maximize the mass of payload, i.e.,

$$J = -m_{\text{payload}} \tag{19}$$

Trajectory optimization problem: *The objective function given in Eq. (19) is minimized subject to the dynamics constraints of Eqs. (2-6), the path constraints given in Eqs. (7-15), and the boundary constraints given in Eqs. (16-18).*

2. Critical Parameters

Table 1: Parameters for L3L

Symbol	Value	Symbol	Value
g_{M}	1.62 m/s^2	m_{prop0}	694 kg
$\boldsymbol{r}_0$	[10 0.5 -1.3]^T km	$\boldsymbol{V}_0$	[-15 85 -60]^T m/s
$\boldsymbol{r}_f$	[0 0 0]^T m	$\boldsymbol{V}_f$	[-1 0 0]^T m/s
$F_{\min}$	3 000 N	$F_{\max}$	15 000 N
V_{e}	2 942 m/s	m_{dry}	3 300 kg
μ_1	1 000	μ_2	10

*The parameters value was set by the authors, according to the practical engineering.

Table 2: Parameters for MSRL

Symbol	Value	Symbol	Value
G	$6.674\,28 \times 10^{-11}$	M_{E}	$5.972\,16 \times 10^{24}$ kg
R_{E}	6 371 004 m	$\boldsymbol{\omega}_{\mathrm{E}}$	$[0\ 0\ 7.292\,1 \times 10^{-5}]^T$ rad/s
ρ_0	1.225 kg/m^3	h_0	7 200 m
S_{ref}	8.814 m^2	lat_0	19.61° N
lon_0	110.95° E	V_{wind}	$[0\ 0\ 0]^T$ m/s
$m_{\mathrm{dry}}^{\mathrm{cstg}}$	14 400 kg	$m_{\mathrm{prop}}^{\mathrm{cstg}}$	(0 to 157 500) kg
$m_{\mathrm{dry}}^{\mathrm{booster}}$	7 050 kg	$m_{\mathrm{prop}}^{\mathrm{booster}}$	(0 to 79 250) kg
$m_{\mathrm{dry}}^{\mathrm{ustg}}$	1 220 kg	$m_{\mathrm{prop}}^{\mathrm{ustg}}$	(0 to 18 740) kg
$F_{\mathrm{T,min}}^{\mathrm{cstg}}$	1 835 250 N	$F_{\mathrm{T,max}}^{\mathrm{cstg}}$	2 447 000 N
$F_{\mathrm{T,const}}^{\mathrm{booster}}$	1 223 500 N	$F_{\mathrm{T,const}}^{\mathrm{ustg}}$	166 000 N
$V_{\mathrm{e}}^{\mathrm{rp1}}$	2 942 m/s	$V_{\mathrm{e}}^{\mathrm{lh2}}$	4 315 m/s
$\omega_{\mathrm{u,max}}$	$\frac{\pi}{12}$ rad/s	$Q_{\alpha,\mathrm{max}}$	2 000
Q_{max}	180 000 Pa	inc_{tgt}	98.26°
ecc_{tgt}	0	a_{tgt}	7 071 004 m
μ_1	500	μ_2	100
$\gamma_{\mathrm{s,min}}$	0.9	$\gamma_{\mathrm{s,max}}$	1.1
ε_{σ}	0.1	$\varepsilon_{\Delta S}$	0.01
$\rho_{\sigma,\mathrm{min}}$	1	$\rho_{\sigma,\mathrm{max}}$	5
$\varepsilon_{\|S\|_\infty}$	0.2	α	1.5

*The settings of these parameters took a reference to the website created by Brügge [1] and have been checked by authors.

References

[1] N. Brügge, Long March 8 Parameters (Retrieved on 8th July 2022) URL `http://www.b14643.de/Spacerockets_1/China/CZ-8/Description/Frame.htmn`.