$\boxed{\text{7 points}}$ 1. Let $A = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$. The eigenvalues of $A$ are $\lambda_1 = 2$ with $a_{\lambda_1} = 2$, and $\lambda_2 = 1$, with $a_{\lambda_2} = 1$.

a) Find a basis for the corresponding eigenspaces.

b) Is $A$ diagonalizable? Justify your answer. If it is diagonalizable, find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$.

c) Give a formula for $A^k$, for any positive integer $k$.

*Solution.*

a) • $E_{\lambda_1}$:

$$A - 2I = \begin{bmatrix} -2 & -4 & -6 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{bmatrix} \overset{-1/2 R_1}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{bmatrix} \underset{\substack{R_2 + R_1 \\ R_3 - R_1}}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} x_1 &= -2x_2 - 3x_3 \\ x_2 &\text{ is free} \\ x_3 &\text{ is free,} \end{aligned}$$

so that $\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. A basis for $E_{\lambda_1}(A)$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

• $E_{\lambda_2}$:

$$A - I = \begin{bmatrix} -1 & -4 & -6 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{bmatrix} \overset{-R_1}{\sim} \begin{bmatrix} 1 & 4 & 6 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{bmatrix} \underset{\substack{R_2 + R_1 \\ R_3 - R_1}}{\sim} \begin{bmatrix} 1 & 4 & 6 \\ 0 & 3 & 3 \\ 0 & -2 & -2 \end{bmatrix} \overset{1/3 R_2}{\sim}$$

$$\begin{bmatrix} 1 & 4 & 6 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \underset{R_3 + 2R_2}{\overset{R_1 - 4R_2}{\sim}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} x_1 &= -2x_3 \\ x_2 &= -x_3 \\ x_3 &\text{ is free,} \end{aligned}$$

so that $\vec{x} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$. A basis for $E_{\lambda_2}(A)$ is $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

b) $A$ is diagonalizable because $g_{\lambda_1} = 2 = a_{\lambda_1}$, and $g_{\lambda_2} = 1 = a_{\lambda_2}$. The matrices $P$ and $D$ are given by

$$P = \begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

c) Since $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$. First, we compute $P^{-1}$:

$$\left[\begin{array}{ccc|ccc} -2 & -3 & -2 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_2 \\ \sim \\ \phantom{x} \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ -2 & -3 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \sim \\ R_2 + 2R_1 \\ \phantom{x} \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -3 & -4 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \sim \\ R_2 \leftrightarrow R_3 \\ \phantom{x} \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -3 & -4 & 1 & 2 & 0 \end{array}\right] \begin{array}{c} \phantom{x} \\ \sim \\ R_3 + 3R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 2 & 3 \end{array}\right] \begin{array}{c} \phantom{x} \\ \sim \\ -R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -2 & -3 \end{array}\right] \begin{array}{c} R_1 + R_3 \\ R_2 - R_3 \\ \sim \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & -3 \\ 0 & 1 & 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & -1 & -2 & -3 \end{array}\right]$$

Thus, $P^{-1} = \begin{bmatrix} -1 & -1 & -3 \\ 1 & 2 & 4 \\ -1 & -2 & -3 \end{bmatrix}$. Now, we calculate $A^k$:

$$\begin{aligned} A^k = PD^kP^{-1} &= \begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & -3 \\ 1 & 2 & 4 \\ -1 & -2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -2 \times 2^k & -3 \times 2^k & -2 \\ 2^k & 0 & -1 \\ 0 & 2^k & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & -3 \\ 1 & 2 & 4 \\ -1 & -2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -2^k + 2 & -4 \times 2^k + 4 & -6 \times 2^k + 6 \\ -2^k + 1 & -2^k + 2 & -3 \times 2^k + 3 \\ 2^k - 1 & 2 \times 2^k - 2 & 4 \times 2^k - 3 \end{bmatrix}. \end{aligned}$$

$\boxed{\text{6 points}}$ 2. Let $\vec{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$.

a) Show that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

b) Write $\vec{v} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$.

*Solution.*

a) Note that

- $\vec{u}_1 \cdot \vec{u}_2 = 3(2) + (-3)(2) + 0(-1) = 6 - 6 = 0$,
- $\vec{u}_1 \cdot \vec{u}_3 = 3(1) + (-3)(1) + 0(4) = 3 - 3 = 0$, and
- $\vec{u}_2 \cdot \vec{u}_3 = 2(1) + 2(1) + (-1)(4) = 2 + 2 - 4 = 0$.

Therefore, $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set of nonzero vectors, which implies that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a linearly independent set. Since there are three vectors in this set and $\dim(\mathbb{R}^3) = 3$, then $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

b) Since $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$, we know that we can write

$$\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3,$$

where $c_i = \frac{\vec{v}\cdot\vec{u}_i}{\vec{u}_i\cdot\vec{u}_i}$. Thus,

$$\begin{aligned}
c_1 &= \frac{\vec{v}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} = \frac{5(3) + (-3)(-3) + 1(0)}{3^2 + (-3)^2 + 0^2} = \frac{24}{18} = \frac{4}{3}, \\
c_2 &= \frac{\vec{v}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2} = \frac{5(2) + (-3)(2) + 1(-1)}{2^2 + 2^2 + (-1)^2} = \frac{3}{9} = \frac{1}{3}, \\
c_3 &= \frac{\vec{v}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3} = \frac{5(1) + (-3)(1) + 1(4)}{1^2 + 1^2 + 4^2} = \frac{6}{18} = \frac{1}{3}.
\end{aligned}$$

Thus,

$$\vec{v} = \frac{4}{3}\vec{u}_1 + \frac{1}{3}\vec{u}_2 + \frac{1}{3}\vec{u}_3.$$

$\boxed{\text{4 points}}$ 3. Find the closest point to $\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$ in $H = \text{Span}\{\vec{u}\}$, where $\vec{u} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$. What is the shortest distance from $\vec{v}$ to $H$?

*Solution.*

The closest point to $\vec{v}$ in $H$ is given by

$$\begin{aligned} \text{proj}_{\vec{u}}\vec{v} &= \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} \\ &= \frac{3(1)+(-1)(-2)+1(-1)+13(2)}{1^2+(-2)^2+(-1)^2+2^2}\begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} \\ &= 3\begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ -3 \\ 6 \end{bmatrix}. \end{aligned}$$

The shortest distance is given by

$$\begin{aligned} \left\|\vec{v}-\text{proj}_{\vec{u}}\vec{v}\right\| &= \left\|\begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix} - \begin{bmatrix} 3 \\ -6 \\ -3 \\ 6 \end{bmatrix}\right\| \\ &= \left\|\begin{bmatrix} 0 \\ 5 \\ 4 \\ 7 \end{bmatrix}\right\| \\ &= \sqrt{0^2+5^2+4^2+7^2} = \sqrt{90}. \end{aligned}$$