Assignment 2 — Linear Algebra MATH 2318 (Fall 2022)

$\boxed{\text{4 points}}$ 1. Let

$$\left[\begin{array}{rrrrr|r} 0 & 3 & -6 & -4 & -3 & -5 \\ -1 & 3 & -10 & -4 & -4 & -2 \\ 2 & -6 & 20 & 2 & 8 & -8 \end{array}\right]$$

be the augmented matrix of a system of linear equations. Bring the matrix to REF and identify the pivots. Is the system consistent? Explain. If it is consistent, identify basic variables, free variables (if any) and write down the solution to the linear system.

Solution.

$$\left[\begin{array}{rrrrr|r} 0 & 3 & -6 & -4 & -3 & -5 \\ -1 & 3 & -10 & -4 & -4 & -2 \\ 2 & -6 & 20 & 2 & 8 & -8 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{rrrrr|r} -1 & 3 & -10 & -4 & -4 & -2 \\ 0 & 3 & -6 & -4 & -3 & -5 \\ 2 & -6 & 20 & 2 & 8 & -8 \end{array}\right] \underset{\sim}{-R_1}$$

$$\sim \left[\begin{array}{rrrrr|r} 1 & -3 & 10 & 4 & 4 & 2 \\ 0 & 3 & -6 & -4 & -3 & -5 \\ 2 & -6 & 20 & 2 & 8 & -8 \end{array}\right] \underset{R_3 - 2R_1}{\sim} \left[\begin{array}{rrrrr|r} 1 & -3 & 10 & 4 & 4 & 2 \\ 0 & 3 & -6 & -4 & -3 & -5 \\ 0 & 0 & 0 & -6 & 0 & -12 \end{array}\right] \underset{\frac{1}{3}R_2}{\sim}$$

$$\left[\begin{array}{rrrrr|r} 1 & -3 & 10 & 4 & 4 & 2 \\ 0 & 1 & -2 & -4/3 & -1 & -5/3 \\ 0 & 0 & 0 & -6 & 0 & -12 \end{array}\right] \underset{\sim}{R_1 + 3R_2} \left[\begin{array}{rrrrr|r} 1 & 0 & 4 & 0 & 1 & -3 \\ 0 & 1 & -2 & -4/3 & -1 & -5/3 \\ 0 & 0 & 0 & -6 & 0 & -12 \end{array}\right] \underset{-\frac{1}{6}R_3}{\sim}$$

$$\sim \left[\begin{array}{rrrrr|r} 1 & 0 & 4 & 0 & 1 & -3 \\ 0 & 1 & -2 & -4/3 & -1 & -5/3 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{array}\right] \underset{R_2 + \frac{4}{3}R_3}{\sim} \left[\begin{array}{rrrrr|r} \color{red}{1} & 0 & 4 & 0 & 1 & -3 \\ 0 & \color{red}{1} & -2 & 0 & -1 & 1 \\ 0 & 0 & 0 & \color{red}{1} & 0 & 2 \end{array}\right].$$

(This is the reduced echelon form of the matrix.) The system is consistent since there are no rows of the form $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & c \end{array}\right]$ with $c \neq 0$. The pivots have been highlighted in red.

Basic variables: x_1, x_2, x_4.
Free variables: x_3, x_5.

$$\begin{array}{lclcccc} x_1 & = & -3 & - & 4x_3 & - & x_5 \\ x_2 & = & 1 & + & 2x_3 & + & x_5 \\ x_3 & \text{is free} & & & & & \\ x_4 & = & 2 & & & & \\ x_5 & \text{is free} & & & & & \end{array}$$

$\boxed{\text{6 points}}$ 2. Let $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. For each of the following, determine if the given vector can be written as a linear combination of $\vec{x}, \vec{y}, \vec{z}$. If so, then find the corresponding weights.

a) $\vec{p} = \begin{bmatrix} 4 \\ 2 \\ 1 \\ 5 \end{bmatrix}$

b) $\vec{q} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

Solution.

a)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 5 \end{array}\right] \underset{\substack{R_3 - R_1 \\ R_4 - R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & -1 & 1 \end{array}\right] \underset{R_3 + R_2}{\overset{R_1 - R_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{array}\right] \underset{R_4 + R_3}{\overset{R_2 - R_3}{\sim}}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

(This is the reduced echelon form of the matrix.) This system is consistent since there are no rows of the form $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & c \end{array}\right]$ with $c \neq 0$. Moreover, all the variables are basic since there are pivots in all the columns. We obtain $x_1 = 2$, $x_2 = 3$, $x_3 = -1$. Thus, $\vec{p}$ can be written as a linear combination of $\vec{x}, \vec{y}, \vec{z}$ as $\vec{p} = 2\vec{x} + 3\vec{y} - \vec{z}$.

b)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 4 \end{array}\right] \underset{\substack{R_3 - R_1 \\ R_4 - R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & 3 \end{array}\right] \underset{R_3 + R_2}{\overset{R_1 - R_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -1 & 3 \end{array}\right] \underset{R_4 + R_3}{\overset{R_2 - R_3}{\sim}}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 7 \end{array}\right].$$

(This is an echelon form of the matrix.) This system is inconsistent since the last row reads $0x_1 + 0x_2 + 0x_3 = 7$, i.e., $0 = 7$, which is never true. Thus, $\vec{q}$ cannot be written as a linear combination of $\vec{x}, \vec{y}, \vec{z}$.

$\boxed{\text{2 points}}$ 3. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$, and $\vec{y} = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}$. For what values of h is $\vec{y}$ in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$?

Solution.

$$\left[\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{array}\right] \underset{R_3 + 2R_1}{\sim} \left[\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & -3 + 2h \end{array}\right] \underset{R_3 - 2R_2}{\sim} \left[\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 2h + 7 \end{array}\right].$$

For this system to be consistent, we require $0 = 2h + 7 \Rightarrow h = -\frac{7}{2}$. Thus $\vec{y}$ is in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$ if and only if $h = -\frac{7}{2}$.