1. Solve the following linear system

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 4 \\ 2x_1 & - & 3x_2 & - & x_3 & + & 4x_4 & = & 7 \\ -2x_1 & + & 4x_2 & + & x_3 & - & 2x_4 & = & 1 \\ 5x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & -1 \end{array}$$

Solution.

We row reduce the augmented matrix:

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 4 \\ 2 & -3 & -1 & 4 & 7 \\ -2 & 4 & 1 & -2 & 1 \\ 5 & -1 & 2 & 1 & -1 \end{array}\right] \begin{array}{c} \sim \\ R_2 - 2R_1 \\ R_3 + 2R_1 \\ R_4 - 5R_1 \end{array} \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 4 \\ 0 & -5 & -3 & 2 & -1 \\ 0 & 6 & 3 & 0 & 9 \\ 0 & -6 & -3 & -4 & -21 \end{array}\right] \begin{array}{c} \sim \\ R_2 + R_3 \\ \\ R_4 + R_3 \end{array} \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 2 & 8 \\ 0 & 6 & 3 & 0 & 9 \\ 0 & 0 & 0 & -4 & -12 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_2 \\ \sim \\ R_3 - 6R_2 \\ \\ \end{array} \left[\begin{array}{rrrr|r} 1 & 0 & 1 & -1 & -4 \\ 0 & 1 & 0 & 2 & 8 \\ 0 & 0 & 3 & -12 & -39 \\ 0 & 0 & 0 & -4 & -12 \end{array}\right] \begin{array}{c} \\ \sim \\ \frac{1}{3}R_3 \\ \\ \end{array} \left[\begin{array}{rrrr|r} 1 & 0 & 1 & -1 & -4 \\ 0 & 1 & 0 & 2 & 8 \\ 0 & 0 & 1 & -4 & -13 \\ 0 & 0 & 0 & -4 & -12 \end{array}\right] \begin{array}{c} R_1 - R_3 \\ \sim \\ \\ -\frac{1}{4}R_4 \end{array} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 3 & 9 \\ 0 & 1 & 0 & 2 & 8 \\ 0 & 0 & 1 & -4 & -13 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{c} R_1 - 3R_4 \\ R_2 - 2R_4 \\ R_3 + 4R_4 \\ \sim \end{array} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

Therefore, the solution is $x_1 = 0$, $x_2 = 2$, $x_3 = -1$, and $x_4 = 3$.

2. Given the reduced echelon form of an augmented matrix of a linear system, identify the pivots, basic variables, free variables (if any) and write the solution of the linear system in parametric form.

$$\left[\begin{array}{rrrr|r} 0 & 1 & 0 & 3 & -4 \\ 0 & 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution.

The pivots are in red below:

$$\left[\begin{array}{rrrr|r} 0 & \color{red}{1} & 0 & 3 & -4 \\ 0 & 0 & \color{red}{1} & 2 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Basic variables: x_2, x_3.
Free variables: x_1, x_4.

The parametric form of the solution is:

$$\begin{array}{lcrcl} x_1 & \text{is free} & & & \\ x_2 & = & -4 & - & 3x_4 \\ x_3 & = & 9 & - & 2x_4 \\ x_4 & \text{is free} & & & \end{array}$$

3. Let $A = \begin{bmatrix} 1 & 3 & 1 & 1 \\ -2 & -6 & -1 & 0 \\ 1 & 3 & 2 & 3 \end{bmatrix}$.

a) Solve the homogeneous system $A\vec{x} = \vec{0}$.

b) **Without performing any row operations**, solve the system $A\vec{x} = \vec{b}$, where $\vec{b} = \begin{bmatrix} -4 \\ 10 \\ -2 \end{bmatrix}$.

Solution.

a) Let us row reduce the matrix:

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ -2 & -6 & -1 & 0 \\ 1 & 3 & 2 & 3 \end{bmatrix} \begin{matrix} \sim \\ R_2 + 2R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \sim \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the REF. We see that the basic variables are x_1 and x_3, while x_2 and x_4 are free. The solution of the system is:

$$\begin{array}{lcll} x_1 & = & -3x_2 & + \; x_4 \\ x_2 & \text{is free} & & \\ x_3 & = & -2x_4 & \\ x_4 & \text{is free.} & & \end{array}$$

In vector parametric form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} = x_2\vec{v}_1 + x_4\vec{v}_4.$$

b) The solution to the system $A\vec{x} = \vec{b}$ has the form $\vec{x} = \vec{p} + x_2\vec{v}_1 + x_4\vec{v}_4$, where $\vec{p}$ is a particular solution to the system which we find by setting $x_2 = x_4 = 0$ in the original system:

$$\begin{array}{rcrcr} x_1 & + & x_3 & = & -4 \\ -2x_1 & - & x_3 & = & 10 \\ x_1 & + & 2x_3 & = & -2 \end{array}$$

From the first equation we have $x_1 = -4 - x_3$, which we plug in the last equation to get $-4 - x_3 + 2x_3 = -2 \Rightarrow x_3 = -2 + 4 \Rightarrow x_3 = 2$. We plug in this value of x_3 back into the first equation to get $x_1 = -4 - 2 = -6$. Thus, the solution to $A\vec{x} = \vec{b}$ is

$$\vec{x} = \begin{bmatrix} -6 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

4. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ h \\ 4 \end{bmatrix}$. Find the values of h so that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Solution.

We row reduce the matrix $A = [\vec{v}_1 \ \ \vec{v}_2 \ \ \vec{v}_3]$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & h \\ 3 & 4 & 4 \end{bmatrix} \underset{\substack{R_2 - 2R_1 \\ R_3 - 3R_1}}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & h-6 \\ 0 & -2 & -5 \end{bmatrix} \underset{-\frac{1}{5}R_2}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & (6-h)/5 \\ 0 & -2 & -5 \end{bmatrix} \underset{R_3 + 2R_2}{\sim}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & (6-h)/5 \\ 0 & 0 & -(2h+13)/5 \end{bmatrix}$$

For the set to be linearly independent, we require that there is a pivot in every column. For this to occur, we need $\frac{-2h-13}{5} \neq 0 \Leftrightarrow -2h - 13 \neq 0 \Leftrightarrow h \neq -\frac{13}{2}$.

5. Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 3 & -5 & -9 \end{bmatrix}$, and define $T : \mathbb{R}^3 \to \mathbb{R}^3$ as $T(\vec{x}) = A\vec{x}$, for all $\vec{x} \in \mathbb{R}^3$. Find an $\vec{x}$ in $\mathbb{R}^3$ whose image under T is $\vec{b} = \begin{bmatrix} 6 \\ -7 \\ -9 \end{bmatrix}$. Is there more than one $\vec{x}$ in $\mathbb{R}^3$ whose image under T is $\vec{b}$?

Solution.

We must find an $\vec{x}$ in $\mathbb{R}^3$ such that $T(\vec{x}) = A\vec{x} = \vec{b}$:

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 3 & -5 & -9 & -9 \end{array}\right] \underset{R_3 - 3R_1}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 4 & -15 & -27 \end{array}\right] \underset{\substack{R_1 + 3R_2 \\ R_3 - 4R_2}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -10 & -15 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{array}\right] \underset{\substack{R_1 + 10R_3 \\ R_2 + 4R_3}}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore, $\vec{x} = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$. Since $A\vec{x} = \vec{b}$ has a unique solution, there is only one $\vec{x}$ in $\mathbb{R}^3$ whose image under T is $\vec{b}$.

6. Let $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{y} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{z} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

a) Write $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ as a linear combination of $\vec{x}, \vec{y}, \vec{z}$.

b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(\vec{x}) = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, T(\vec{y}) = \begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}, T(\vec{z}) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Find $T(\vec{b})$.

Solution.

a) We want to find scalars $c_1,\ c_2,\ c_3$ such that

$$c_1\vec{x} + c_2\vec{y} + c_3\vec{z} = \vec{b}.$$

This is equivalent to solving $A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{b}$, where $A = [\vec{x}\ \ \vec{y}\ \ \vec{z}]$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ -1 & 1 & 3 & -3 \end{array}\right] \begin{array}{c} \sim \\ R_2 - R_1 \\ R_3 + R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 4 & -2 \end{array}\right] \begin{array}{c} \\ \sim \\ R_3 - R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & -3 \end{array}\right] \begin{array}{c} \\ \sim \\ \frac{1}{3}R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{c} R_1 - R_3 \\ R_2 - R_3 \\ \sim \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Therefore, we write $\vec{b}$ as

$$\vec{b} = 2\vec{x} + 2\vec{y} - \vec{z}.$$

b) Thanks to part a), we have

$$T(\vec{b}) = T(2\vec{x} + 2\vec{y} - \vec{z}).$$

Since T is a linear transformation, we have then

$$T(\vec{b}) = 2T(\vec{x}) + 2T(\vec{y}) - T(\vec{z}) = 2\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \\ -3 \end{bmatrix}.$$

7. For each of the following, determine if the statement is true or false. Provide a short reasoning (one or two sentences).

   a) If the equation $A\vec{x} = \vec{0}$ only has the trivial solution, then the equation $A\vec{x} = \vec{b}$ has a unique solution for any $\vec{b}$.

   b) Let T be a linear transformation. If $T(\vec{x}) = T(\vec{y})$, then $\vec{x} = \vec{y}$.

   c) Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be vectors in $\mathbb{R}^n$. If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent, then each of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ can be expressed as a linear combination of the other two.

   d) If $\vec{x}$ is in Span$\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$, then $\vec{x}$ is in Span$\{\vec{v_1}, \vec{v_2}, t\vec{v_3}\}$, where t is any real number.

Solution.

a) FALSE. Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. We see that the equation $A\vec{x} = \vec{0}$ has only the trivial solution. However, the equation $A\vec{x} = \vec{b}$, with $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, is inconsistent.

b) FALSE. Take $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 \\ 2x_1 \end{bmatrix}$. We see that T is a linear transformation. Take $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then $T(\vec{x}) = T(\vec{y}) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, but $\vec{x} \neq \vec{y}$.

c) FALSE. One of the weights in the linear dependence relation might be 0, for example,

$$c_1\vec{v}_1 + c_2\vec{v}_2 + 0\vec{v}_3 = \vec{0}.$$

Then, $\vec{v}_3$ cannot be expressed as a linear combination of $\vec{v}_1,\ \vec{v}_2$.

d) FALSE. If $t = 0$, then for $\vec{x}$ to be in Span$\{\vec{v_1}, \vec{v_2}, t\vec{v_3}\}$, it must be a linear combination of $\vec{v}_1$ and $\vec{v}_2$ only, which is not necessarily true.