Assignment 4 — Linear Algebra MATH 2318 (Fall 2022)

$\boxed{\text{6 points}}$ 1. Determine if the set is linearly dependent or linearly independent. If it is linearly dependent, find a linear dependence relation.

a) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, where $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 5 \\ 0 \\ 3 \\ -1 \end{bmatrix}$.

b) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix}$.

Solution.

a) We determine if $A\vec{x} = \vec{0}$, with $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4]$ has nontrivial solutions.

$$\begin{bmatrix} 1 & 3 & 0 & 5 \\ -2 & 0 & 4 & 0 \\ 1 & 2 & -1 & 3 \\ 1 & -2 & 1 & -1 \end{bmatrix} \begin{matrix} \sim \\ R_2 + 2R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix} \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 6 & 4 & 10 \\ 0 & -1 & -1 & -2 \\ 0 & -5 & 1 & -6 \end{bmatrix} \begin{matrix} \sim \\ R_2 \leftrightarrow R_3 \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & -1 & -1 & -2 \\ 0 & 6 & 4 & 10 \\ 0 & -5 & 1 & -6 \end{bmatrix} \begin{matrix} \\ \sim \\ R_3 + 6R_2 \\ R_4 - 5R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 6 & 4 \end{bmatrix} \begin{matrix} \\ \sim \\ \\ R_4 + 3R_3 \end{matrix} \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad \text{(this is an echelon form)}$$

Since we don't have free variables, we don't have nontrivial solutions. Hence, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly independent.

b) We determine if $A\vec{x} = \vec{0}$, with $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ has nontrivial solutions.

$$\begin{bmatrix} 1 & -1 & 4 \\ 2 & -3 & 6 \\ -1 & 2 & -2 \end{bmatrix} \begin{matrix} \sim \\ R_2 - 2R_1 \\ R_3 + R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 4 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} \\ \sim \\ R_3 + R_2 \end{matrix} \begin{bmatrix} 1 & -1 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(this is an echelon form)}$$

We see that x_3 is a free variable, so the equation has nontrivial solutions. Thus, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

To find a linear dependence relation, we finish solving $A\vec{x} = \vec{0}$:

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \sim \\ -R_2 \\ \\ \end{matrix} \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 + R_2 \\ \sim \\ \\ \end{matrix} \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(REF)}$$

We get $x_1 + 6x_3 = 0$ and $x_2 + 2x_3 = 0$, so that:

$$\begin{aligned} x_1 &= -6x_3 \\ x_2 &= -2x_3 \\ x_3 &\text{ is free} \end{aligned}$$

which is the parametric form of the solution.

Pick $x_3 = 1$, then $x_1 = -6,\ x_2 = -2,\ x_3 = 1$. Thus,

$$-6\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}.$$

4 points

2. Let $\vec{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1 \\ -3 \\ r \end{bmatrix}$. Find the value of r so that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent. For that value of r, write $\vec{v}_1$ as a linear combination of $\vec{v}_2$ and $\vec{v}_3$.

Solution.

$$\begin{bmatrix} -2 & 1 & -1 \\ 0 & 1 & -3 \\ 1 & -3 & r \end{bmatrix} \overset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} 1 & -3 & r \\ 0 & 1 & -3 \\ -2 & 1 & -1 \end{bmatrix} \underset{R_3+2R_1}{\sim} \begin{bmatrix} 1 & -3 & r \\ 0 & 1 & -3 \\ 0 & -5 & 2r-1 \end{bmatrix} \underset{R_3+5R_2}{\sim}$$

$$\sim \begin{bmatrix} 1 & -3 & r \\ 0 & 1 & -3 \\ 0 & 0 & 2r-16 \end{bmatrix}$$

If $2r - 16 = 0$, then x_3 is a free variable and the system has nontrivial solutions. Therefore, if $r = 8$, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent. To write $\vec{v}_1$ as a linear combination of the others, we finish solving the system with $r = 8$:

$$\begin{bmatrix} 1 & -3 & 8 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1+3R_2}{\sim} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(REF)}$$

We get $x_1 - x_3 = 0$ and $x_2 - 3x_3 = 0$, so that:

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= 3x_3 \\ x_3 &\text{ is free} \end{aligned}$$

Pick $x_3 = 1$, then $x_1 = 1,\ x_2 = 3,\ x_3 = 1$. Thus,

$$\vec{v}_1 + 3\vec{v}_2 + \vec{v}_3 = 0 \Rightarrow \vec{v}_1 = -3\vec{v}_2 - \vec{v}_3.$$

4 points

3. For each of the following, determine if the statement is true or false. Provide a short reasoning (one or two sentences).

   a) If the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly independent, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_1 + \vec{v}_4\}$ is linearly independent.

   b) If a set of vectors is linearly dependent then at least one of the vectors is a scalar multiple of another one.

   c) A set of 3 vectors in $\mathbb{R}^2$ is always linearly dependent.

   d) If $\{\vec{x}, \vec{y}\}$ is linearly independent, and if $\vec{z}$ is in Span$\{\vec{x}, \vec{y}\}$, then $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly dependent.

Solution.

   a) TRUE. Let c_1, c_2, c_3, c_4 be scalars and consider the vector equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4(\vec{v}_1 + \vec{v}_4) = \vec{0}$. This is equivalent to $(c_1+c_4)\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$. Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly independent, we must have that $c_1 + c_4 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, from where we conclude that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_1 + \vec{v}_4\}$ is linearly independent.

   b) FALSE. Consider the set $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$. This set is linearly dependent because the last vector is a linear combination of the first two, but neither vector is a scalar multiple of another one.

c) TRUE. When row reducing $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$, where $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are in $\mathbb{R}^2$, there will be at least one column without a pivot.

d) TRUE. Since $\vec{z}$ is in $\text{Span}\{\vec{x}, \vec{y}\}$, then $\vec{z}$ is a linear combination of $\vec{x}, \vec{y}$, thus $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly dependent.