1. Let $A = \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 17 \\ 3 & 6 & 0 & 6 & 24 \end{bmatrix}$. Find a basis for Col($A$), Null($A$) and Row($A$). What is rank($A$)?

Solution.

$$\begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 17 \\ 3 & 6 & 0 & 6 & 24 \end{bmatrix} \begin{matrix} \sim \\ R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - 3R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 0 & 0 & -3 & 6 & 0 \\ 0 & 0 & -9 & 18 & 1 \\ 0 & 0 & -9 & 18 & 0 \end{bmatrix} \begin{matrix} \sim \\ -1/3R_2 \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & -9 & 18 & 1 \\ 0 & 0 & -9 & 18 & 0 \end{bmatrix} \begin{matrix} R_1 - 3R_2 \\ \sim \\ R_3 + 9R_2 \\ R_4 + 9R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 8 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 - 8R_3 \\ \sim \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for Col(A) is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \\ 17 \\ 24 \end{bmatrix} \right\}$.

A basis for Row(A) is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

A basis for Null(A) is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

rank(A) = 3.

2. Let $A = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 1 & -1 \\ 3 & 6 & -4 \end{bmatrix}$.

a) (3 points) Find the characteristic polynomial of A.
b) (2 points) Find the eigenvalues of A.

Solution.

a)

$$C_A(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} -1-\lambda & 0 & 0 \\ 3 & 1-\lambda & -1 \\ 3 & 6 & -4-\lambda \end{vmatrix} = -(1+\lambda)(-1)^{1+1} \begin{vmatrix} 1-\lambda & -1 \\ 6 & -4-\lambda \end{vmatrix}$$
$$= -(1+\lambda)((1-\lambda)(-4-\lambda)+6) = -(1+\lambda)(-4-\lambda+4\lambda+\lambda^2+6)$$
$$= -(\lambda+1)^2(\lambda+2)$$

b) $\lambda_1 = -1$, $\lambda_2 = -2$.

3. Let $A = \begin{bmatrix} -2 & -1 & -2 \\ 2 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = -1$, with $a_{\lambda_1} = 2$, and $\lambda_2 = 2$, with $a_{\lambda_2} = 1$.

a) (4 points) Find a basis for each eigenspace of A.
b) (2 point) Is A diagonalizable? Explain. If so, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. *Note*: You **do not** have to find P^{-1}.

Solution.

a)
- $E_{\lambda_1}(A) = \text{Null}(A + I_3)$:

$$A + I_3 = \begin{bmatrix} -1 & -1 & -2 \\ 2 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \underset{\sim}{-R_1} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \begin{matrix} \sim \\ R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for $E_{\lambda_1}(A)$ is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

- $E_{\lambda_1}(A) = \text{Null}(A - 2I_3)$:

$$A - 2I_3 = \begin{bmatrix} -4 & -1 & -2 \\ 2 & -1 & 4 \\ 1 & 1 & -1 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 4 \\ -4 & -1 & -2 \end{bmatrix} \begin{matrix} \sim \\ R_2 - 2R_1 \\ R_3 + 4R_1 \end{matrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 6 \\ 0 & 3 & -6 \end{bmatrix} \begin{matrix} \sim \\ -1/3R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 3 & -6 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \sim \\ R_3 - 3R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for $E_{\lambda_2}(A)$ is $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

b) A is diagonalizable because $g_{\lambda_1} = 2 = a_{\lambda_1}$ and $g_{\lambda_2} = 1 = a_{\lambda_2}$.

$$P = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

4. Let $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$. Find the closest point to $\vec{x}$ in $H = \text{Span}\{\vec{u}\}$. What is the shortest distance from $\vec{x}$ to H? You do not need to simplify your answer.

   Solution. The closest point to $\vec{x}$ in H is given by

$$\begin{aligned} \text{proj}_{\vec{u}}\vec{x} &= \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} = \frac{3(2) + 1(-1) + 1(-2)}{2^2 + (-1)^2 + (-2)^2} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \\ &= \frac{3}{9} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix} \end{aligned}$$

   The shortest distance is given by $\|\vec{x} - \text{proj}_{\vec{u}}\vec{x}\|$.

$$\vec{x} - \text{proj}_{\vec{u}}\vec{x} = \begin{bmatrix} 7/3 \\ 4/3 \\ 5/3 \end{bmatrix} \Rightarrow \|\vec{x} - \text{proj}_{\vec{u}}\vec{x}\| = \frac{1}{3}\sqrt{7^2 + 4^2 + 5^2} = \frac{1}{3}\sqrt{90}.$$

5. For each of the following, determine if the statement is true or false. You do not need to justify your answer.

   a) If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a linearly independent set of vectors in a subspace H, then $\dim(H) \geq p$.

   b) If A is an $n \times n$ matrix and the equation $A\vec{x} = \vec{b}$ is inconsistent for some $\vec{b} \in \mathbb{R}^n$, then the equation $A\vec{x} = \vec{0}$ has infinitely many solutions.

   c) If -1 is an eigenvalue of a square invertible matrix A, then -1 is also an eigenvalue of A^{-1}.

   Solution. All true.

6. (BONUS) Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be an orthogonal set of vectors in $\mathbb{R}^n$ and let t_1, t_2, t_3 be real numbers. Show that $\{t_1\vec{v}_1, t_2\vec{v}_2, t_3\vec{v}_3\}$ is an orthogonal set.

   Solution. For $i \neq j$:

$$(t_i\vec{v}_i) \cdot (t_j\vec{v}_j) = (t_i t_j)(\vec{v}_i \cdot \vec{v}_j) = 0.$$