Chapter 4: vector spaces

Goal: section 4.1 - 4.2 - 4.3 - 4.5 - 4.6

1) Vector spaces and subspaces

Definition: A nonempty set V, with an operation of addition, $\vec{x}+\vec{y}$, and an operation of scalar multiplication, $c\vec{x}$ (c in $\mathbb{R}$), is called a real vector space if for every $\vec{x}, \vec{y}, \vec{z} \in V$, and for every $c, d \in \mathbb{R}$, we have: (for all)

① $\vec{x}+\vec{y} \in V$ ($\in \leftrightarrow$ in)

② $\vec{x}+\vec{y} = \vec{y}+\vec{x}$

③ $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$

④ There exists a vector $\vec{0} \in V$, such that $\vec{x}+\vec{0} = \vec{x}$, $\forall \vec{x} \in V$

⑤ For every $\vec{x} \in V$, there exists $(-\vec{x}) \in V$ such that $\vec{x}+(-\vec{x}) = \vec{0}$

⑥ $\forall c \in \mathbb{R}, \forall \vec{x} \in V$, $c\vec{x} \in V$

⑦ $c(d\vec{x}) = (cd)\vec{x}$

⑧ $(c+d)\vec{x} = c\vec{x}+d\vec{x}$

⑨ $c(\vec{x}+\vec{y}) = c\vec{x}+c\vec{y}$

⑩ $1\vec{x} = \vec{x}$

⚠ We call the elements of V vectors

Remark: If V vector space then for each $\vec{u} \in V$, and $c \in \mathbb{R}$:

$(*)$ $0\vec{u} = \vec{0}$ $\quad (*)$ $c\vec{0} = \vec{0}$ $\quad (*)$ $-\vec{u} = (-1)\vec{u}$

The following sets are vectors spaces:

- $\mathbb{R}^n$
- $M_{m\times n}(\mathbb{R})$ (set of $m\times n$ matrices)
- set of linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$
- set of all functions $f: \mathbb{R} \to \mathbb{R}$ or $\mathbb{R}^n \to \mathbb{R}$

Definitions

The set $\mathbb{P}_n(\mathbb{R}) = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_0, \dots, a_n \in \mathbb{R}\}$
denotes the set of polynomials of degree at most n.

The zero polynomial is defined by: $0 = 0 + 0x + \cdots + 0x^n \in \mathbb{P}_n(\mathbb{R})$

$p(x) = q(x)$ if and only if $a_i = b_i$ for $i = 0, \dots, n$ where
$p(x) = a_0 + \cdots + a_n x^n$ and $q(x) = b_0 + \cdots + b_n x^n$.

addition: $(p+q)(x) = (a_0+b_0) + (a_1+b_1)x + \cdots + (a_n+b_n)x^n \in \mathbb{P}_n(\mathbb{R})$

scalar multiplication $(cp)(x) = ca_0 + ca_1 x + \cdots + ca_n x^n \in \mathbb{P}_n(\mathbb{R})$

Under the above operations, $\mathbb{P}_n(\mathbb{R})$ is a vector space.

Definition

Let $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in a vector space V.
$\text{Span}(B) = \{c_1\vec{v}_1 + \cdots + c_k\vec{v}_k \mid c_1, \dots, c_k \in \mathbb{R}\}$

Definition:

A subspace of a vector space V is a subset H of V that satisfies

1) $\vec{0} \in H$
2) For each $\vec{u}, \vec{v} \in H$, $\vec{u} + \vec{v} \in H$
3) For each $\vec{u} \in H$, each $c \in \mathbb{R}$, $c\vec{u} \in H$.

Exercise:

Prove that $H = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid x_1 + x_2 = 0 \text{ and } x_2 - x_3 = 0 \right\}$ is a subspace of $\mathbb{R}^3$.

Solution:

1) "$\vec{0} \in H$": $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ satisfies $x_1 + x_2 = 0 + 0 = 0$ and $x_2 - x_3 = 0 - 0 = 0 \Rightarrow \vec{0} \in H$

2) "$\vec{u}, \vec{v} \in H \Rightarrow \vec{u} + \vec{v} \in H$"

Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \in H$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in H \Longrightarrow \vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$ satisfies

$$(u_1 + v_1) + (u_2 + v_2) = (u_1 + u_2) + (v_1 + v_2) = 0 + 0 = 0 \quad (\vec{u}, \vec{v} \text{ in } H)$$

$$(u_2 + v_2) - (u_3 + v_3) = (u_2 - u_3) + (v_2 - v_3) = 0 + 0 = 0$$

$\Rightarrow \vec{u} + \vec{v} \in H$

3) $\vec{u} \in H, c \in \mathbb{R} \Rightarrow c\vec{u} \in H$

$$c\vec{u} = \begin{pmatrix} cu_1 \\ cu_2 \\ cu_3 \end{pmatrix} \text{ so } \left.\begin{array}{l} cu_1 + cu_2 = c(u_1 + u_2) = c0 = 0 \\ cu_2 - cu_3 = c(u_2 - u_3) = c0 = 0 \end{array}\right\} c\vec{u} \in H$$

$\Rightarrow H$ subspace of $\mathbb{R}^3$.

Theorem:

Let $v_1, v_2, \ldots, v_k$ be vectors in a vector space V.

! Then $\text{span}\{v_1, \ldots, \vec{v}_k\}$ is a subspace of V.

Example 1:

Let $H = \left\{ \begin{pmatrix} a-3b \\ b-a \\ a \\ b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$. Show that H is a subspace of $\mathbb{R}^4$!

Solution: Notice that if $\vec{x} \in H$ then there exists a, b in $\mathbb{R}$ such that:

$$\vec{x} = \begin{pmatrix} a-3b \\ b-a \\ a \\ b \end{pmatrix} = a \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v}_1} + b \underbrace{\begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}}_{\vec{v}_2} = a\vec{v}_1 + b\vec{v}_2$$

$\Rightarrow H = \text{span}\{\vec{v}_1, \vec{v}_2\} \Rightarrow H$ subspace of $\mathbb{R}^4$

Example 2:

Is $H = \{A \in M_{2\times2}(\mathbb{R}) \mid A^2 = A\}$ a subspace of $M_{2\times2}(\mathbb{R})$?

Solution:

Notice that $I_2 \in H$ since $I_2^2 = I_2 I_2 = I_2$

However $(2I_2)^2 = (2I_2)(2I_2) = 4I_2 \neq 2I_2$

Property 3 of subspace is not satisfied $\Rightarrow$ H is NOT a subspace of $M_{2\times2}(\mathbb{R})$.

2) Null spaces, Column spaces and linear transformation

Definition: Let A be a $m\times n$ matrix. The null space of A is given by

$$\text{Null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$$

Definition: Let $A = (\vec{a}_1 \cdots \vec{a}_n)$ be a $m \times n$ matrix.

The column space of A is given by: !

$$Col(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\} = span\{\vec{a}_1, \dots, \vec{a}_n\}$$

Example: $A = \begin{pmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$. Is $\vec{b} \in Col(A)$?

Solution: $\vec{b} \in Col(A) \iff$ there exists $\vec{x} \in \mathbb{R}^3$ such that $A\vec{x} = \vec{b}$. ($\to$ REF augmented matrix)

$$\begin{pmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{pmatrix} \underset{\substack{R_2+4R_1 \\ R_3+3R_1}}{\sim} \begin{pmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{pmatrix} \underset{-\frac{R_2}{6}}{\sim} \begin{pmatrix} 1 & -3 & -4 & 3 \\ 0 & 1 & 3 & -\frac{5}{2} \\ 0 & -2 & -6 & 5 \end{pmatrix} \underset{R_3+2R_2}{\sim} \begin{pmatrix} \boxed{1} & -3 & -4 & 3 \\ 0 & \boxed{1} & 3 & -\frac{5}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow 0=0$$

free variable

Echelon form of augmented matrix $\Rightarrow$ pivot in all non zero row

$\Rightarrow$ system is consistent

$\Rightarrow \vec{b} \in Col(A)$

Theorem: Let A be a $m \times n$ matrix

- Null(A) is a subspace of $\mathbb{R}^n$
- Col(A) is a subspace of $\mathbb{R}^m$.

proof: $Col(A) = span\{\vec{a}_1, \dots, \vec{a}_n\}$ with $A = (\vec{a}_1 \cdots \vec{a}_n)$ (in $\mathbb{R}^m$) $\Rightarrow Col(A)$ subspace of $\mathbb{R}^m$.

(*) Nul(A) (included in $\mathbb{R}^n$)

1) $A\vec{0} = \vec{0} \Rightarrow \vec{0} \in Nul(A)$

2) $\vec{u}, \vec{v} \in Nul(A) \Rightarrow A\vec{u} = A\vec{v} = \vec{0} \Rightarrow A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0} \Rightarrow \vec{u}+\vec{v} \in Nul(A)$

3) $c \in \mathbb{R}, \vec{u} \in Nul(A) \Rightarrow A\vec{u} = \vec{0} \Rightarrow cA\vec{u} = \vec{0} \Rightarrow A(c\vec{u}) = \vec{0} \Rightarrow c\vec{u} \in Nul(A)$

$\Rightarrow$ Nul(A) subspace of $\mathbb{R}^n$.

Example: Find an explicit description of Nul(A) where

$$A = \begin{pmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{pmatrix}$$

Solution:

$Nul(A) = \{\vec{x} \in \mathbb{R}^5 \mid A\vec{x} = \vec{0}\} \rightarrow$ row reduced augmented matrix.

$$\left(\begin{array}{ccccc|c} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{array}\right) \underset{\frac{R_1}{3}}{\sim} \left(\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & 3 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{array}\right) \underset{R_2-6R_1}{\sim} \left(\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{array}\right) \underset{R_1-2R_2}{\sim} \begin{array}{c} \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{array}\right) \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array}$$

Basic variables: x_1, x_3

Free variables: x_2, x_4, x_5

Substitution:

$x_1 = -2x_2 - 13x_4 - 33x_5$
x_2 free
$x_3 = 6x_4 + 15x_5$
x_4, x_5 free

$$\Rightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{pmatrix}$$

$$= x_2 \vec{v_1} + x_4 \vec{v_2} + x_5 \vec{v_3}$$

$\Rightarrow Nul(A) = span\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

Remarks:

(*) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is lin. indep. since neither vector can be written as a linear combination of the other ones (by construction REF).

(*) If $Null(A) \neq \{\vec{0}\}$ then the number of vectors in the spanning set for $Null(A)$ equals the number of free variable in $A\vec{x} = \vec{0}$. (!)

(!) (*) To find $Null(A)$ just do REF $A\vec{x} = \vec{0}$ then find vector parametric form. (!)

3) Linearly independent sets; Bases

$A = (\vec{v}_1 \dots \vec{v}_p)$

Definition: Let $\{\vec{v}_1, \dots, \vec{v}_p\}$ be a set of vectors in a space V.

(*) $\{\vec{v}_1, \dots, \vec{v}_p\}$ is said to be linearly independent if the vector equation

$$c_1 \vec{v}_1 + \dots + c_p \vec{v}_p = \vec{0}$$

has only the trivial solution $c_1 = \dots = c_p = 0$.

REF $(A|0) \leftrightarrow$ pivot in each row

(*) $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly dependent if there exists weights $c_1, \dots, c_p$, not all zero, such that:

$$c_1 \vec{v}_1 + \dots + c_p \vec{v}_p = 0$$

REF $(A|0) \leftrightarrow$ has free variable(s)

Following results from section 1.7 are still true for more general vector spaces.

① A set containing the zero vector is linearly dependent.

② A set containing a single nonzero vector is lin. indep.

(3) A set containing two or more vectors is lin. dep.
if and only if at least one vector can be written as a linear combination of other
(REF of $(\vec{u}_1 \dots \vec{u}_p)$ have free variables if $\vec{u}_i$ vector of $\mathbb{R}^n$) $\Leftrightarrow$ can't we that for $M_{n\times m}(\mathbb{R})$ or $\mathbb{P}_n(\mathbb{R})$

(4) A set containing two vectors is lin. dep. if and only if one of the vectors is a scalar multiple of the other.

Examples:

Determine if each of the following sets is lin. dep or lin. indep

(a) $\left\{ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} -3 & -6 \\ -9 & -12 \end{pmatrix} \right\}$ (in $M_{2\times 2}(\mathbb{R})$)

(b) $\{ t, t^2, 4t+2t^2 \}$ (in $\mathbb{P}_2(\mathbb{R})$)

(c) $\{3t, t^3, 2t^3-3t+2\}$ (in $\mathbb{P}_3(\mathbb{R})$)

Solution:

(a) $\begin{pmatrix} -3 & -6 \\ -9 & -12 \end{pmatrix} = 3 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow$ set is lin. <u>dep.</u>

(b) $4t+2t^2 = 4(t)+2(t^2) \Rightarrow$ set is lin. dep.

(c) Let c_1, c_2, c_3 real such that:

$$c_1(3t)+c_2(t^3)+c_3(2t^3-3t+2)=0 \quad (=0+0t+0t^2+0t^3)$$

$$\Rightarrow 2c_3+(3c_1-3c_3)t+0t^2+(c_2+2c_3)t^3 = 0+0t+0t^2+0t^3$$

$$\Rightarrow \left| \begin{array}{l} 2c_3=0 \\ 3c_1-3c_3=0 \\ 0=0 \\ c_2+2c_3=0 \end{array} \right. \Rightarrow \left| \begin{array}{l} c_3=0 \\ c_1=c_3=0 \\ c_2=-2c_3=0 \end{array} \right. \Rightarrow \text{lin. } \underline{\text{indep.}}$$

Notion of Bases (set that spans and are lin. indep.)

Example: Let $\vec{v_1} = \begin{pmatrix}1\\0\end{pmatrix}, \vec{v_2} = \begin{pmatrix}-1\\2\end{pmatrix}, \vec{v_3} = \begin{pmatrix}2\\-4\end{pmatrix}, \vec{v_4} = \begin{pmatrix}0\\0\end{pmatrix}$ and $A = (\vec{v_1}\ \vec{v_2}\ \vec{v_3}\ \vec{v_4})$

Δ $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^2$.

$$\begin{pmatrix} 1 & -1 & 2 & 0 & b_1 \\ 0 & 2 & -4 & 0 & b_2 \end{pmatrix} \underset{\frac{R_2}{2}}{\sim} \begin{pmatrix} 1 & -1 & 2 & 0 & b_1 \\ 0 & 1 & -2 & 0 & \frac{b_2}{2} \end{pmatrix} \underset{R_1+R_2}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 & b_1+\frac{b_2}{2} \\ 0 & 1 & -2 & 0 & \frac{b_2}{2} \end{pmatrix}$$

2 rows, 2 pivot $\Rightarrow$ consistent for all $\vec{b} = \begin{pmatrix}b_1\\b_2\end{pmatrix}$.

Thus:

$$\begin{aligned}
\mathbb{R}^2 &= \text{span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\} \\
&= \{c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} + \cancel{c_4\vec{v_4}}^{\,\vec{0}} \mid c_1, \dots, c_4 \in \mathbb{R}\} \\
&= \{c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} \mid c_1, c_2, c_3 \in \mathbb{R}\} \qquad \vec{v_3} = -2\vec{v_2} \\
&= \{c_1\vec{v_1} + (c_2 - 2c_3)\vec{v_2} \mid c_1, c_2, c_3 \in \mathbb{R}\} \\
&= \{c_1\vec{v_1} + K\vec{v_2} \mid c_1, K \in \mathbb{R}\} \\
&= \text{span}\{\vec{v_1}, \vec{v_2}\}
\end{aligned}$$

$\leftrightarrow$ What is the smallest set of vectors that have the same span?

Definition (basis) A set $B = \{\vec{v_1}, \dots, \vec{v_p}\}$ of vectors in a subspace H of a vector space V is a basis for H if

(1) B is lin. indep.

(2) B spans H, i.e. $H = \text{span}(B)$

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Remarks:

- Above definition holds for vector space ($H = V$)
- Bases are NOT unique
- Any element of H can be written as a linear combination of the vectors in B (property 2) in a unique way (property 1)

Columns of any $n \times n$ invertible matrix form a basis for $\mathbb{R}^n$ as they are lin. indep. and any vectors in $\mathbb{R}^n$ can be written as a lin. comb. of the columns of the matrix.

Particular case: columns of the $n \times n$ identity matrix $I_n = (\vec{e}_1 \cdots \vec{e}_n)$ form the standard basis for $\mathbb{R}^n$.

Example: Show that $B = \{1, x, x^2\}$ is a basis for $\mathbb{P}_2(\mathbb{R})$

Solution:

(1) Show that B is lin. indep.

Let c_1, c_2, c_3 in $\mathbb{R}$ such that $c_1(1) + c_2(x) + c_3(x^2) = 0 = 0(1) + 0(x) + 0(x^2)$

$\Rightarrow c_1 = 0,\ c_2 = 0,\ c_3 = 0$

$\Rightarrow B$ lin. indep.

(2) Show that $\text{span}(B) = \mathbb{P}_2(\mathbb{R})$

Let $P \in \mathbb{P}_2(\mathbb{R}) \Rightarrow$ there exists c_0, c_1, c_2 such that $P(x) = c_0 + c_1(x) + c_2(x^2) \in \text{Span}(B)$

$\Rightarrow \text{span}(B) = \mathbb{P}_2(\mathbb{R})$

(1) + (2) $\Rightarrow B$ basis for $\mathbb{P}_2(\mathbb{R})$

Definition: $B = \{1, x, \dots, x^n\}$ is called the standard basis for $\mathbb{P}_n(\mathbb{R})$

Example:
Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$. Is $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis for $\mathbb{R}^3$?

Solution:
$A = (\vec{v}_1 \vec{v}_2 \vec{v}_3) \to$ echelon form

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} \underset{R_2-2R_1}{\frown} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 3 \end{pmatrix} \underset{R_3-R_2}{\frown} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{pmatrix}$$

$\Rightarrow$ 3 pivots so A is invertible
$\Rightarrow$ columns of A are a basis for $\mathbb{R}^3$ (lin. indep. + span $\mathbb{R}^3$).

Example:
Find a basis for Null(A) with $A = \begin{pmatrix} 1 & 1 & 5 & 1 \\ 1 & 2 & 7 & 2 \\ 2 & 3 & 12 & 3 \end{pmatrix}$

Solution

$$A \underset{\substack{R_2-R_1 \\ R_3-2R_1}}{\frown} \begin{pmatrix} 1 & 1 & 5 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix} \underset{\substack{R_1-R_2 \\ R_3-R_2}}{\frown} \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{REF}$$

substitution:
$$\left| \begin{array}{l} x_1 + 3x_3 = 0 \\ x_2 + 2x_3 + x_4 = 0 \end{array} \right. \longleftrightarrow \left| \begin{array}{l} x_1 = -3x_3 \\ x_2 = -2x_3 - x_4 \\ x_3, x_4 \text{ free} \end{array} \right.$$

$$\Rightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = x_3 \vec{v}_1 + x_4 \vec{v}_2 \Rightarrow \text{Null}(A) = \text{span}\{\vec{v}_1, \vec{v}_2\}$$

Moreover $\vec{v}_1, \vec{v}_2$ are lin. indep by construction
$\Rightarrow \{\vec{v}_1, \vec{v}_2\}$ is a basis for Null(A).

Strategy to find a basis for Null(A).

(1) Row reduce A to REF

(2) Write solution in vector parametric form $\vec{x} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$ (c_i represents a x_j)

(3) $\{\vec{v}_1, \dots, \vec{v}_k\}$ = basis for Null(A)

The spanning set theorem

Idea: A basis can be constructed from a spanning set of vectors by discarding vectors that are a lin. comb. of the other ones

Theorem:

Let $\vec{v}_1, \dots, \vec{v}_k \in V$.

$\vec{v}_i$ is a lin. comb of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k$ if and only if $\text{span}\{\vec{v}_1, \dots, \vec{v}_k\}$
$= \text{span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$

(removing $\vec{v}_i$ does not change the span because $\vec{v}_i$ is a lin. comb of the other vectors.)

Example:

Find a basis for Col(A) with
$$A = \begin{array}{c} \begin{array}{cccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Solution

Recall that $Col(A) = span\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4\}$.

Notice that $\vec{a}_3 = 5\vec{a}_1 + 2\vec{a}_2$ & using spanning set theorem.

$\Rightarrow Col(A) = span\{\vec{a}_1, \vec{a}_2, \vec{a}_4\}$

(!) $\vec{a}_1, \vec{a}_2, \vec{a}_4$ lin. indep. $\Rightarrow \{\vec{a}_1, \vec{a}_2, \vec{a}_4\}$ basis for col(A).

Example:

Find a basis for Col(A) with
$$A = \begin{array}{c} \begin{array}{cccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 5 & 1 \\ 1 & 2 & 7 & 2 \\ 2 & 3 & 12 & 3 \end{pmatrix} \end{array}$$

Solution:

$Col(A) = span\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4\}$ → want to remove vectors that are lin. comb. of the other

$$A \underset{\substack{R_2 - R_1 \\ R_3 - 2R_1}}{\sim} \begin{pmatrix} 1 & 1 & 5 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix} \underset{\substack{R_1 - R_2 \\ R_3 - R_2}}{\sim} \begin{array}{c} \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{cccc} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 \end{array} \end{array} \quad \underline{\underline{REF}}$$

(!) From REF, we have $\vec{b}_3 = 3\vec{b}_1 + 2\vec{b}_2$ and $\vec{b}_4 = \vec{b}_2$

These two relations also hold for original columns $\Rightarrow$ $\vec{a}_3 = 3\vec{a}_1 + 2\vec{a}_2$ and $\vec{a}_4 = \vec{a}_2$

$\Rightarrow \ Col(A) = span\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4\}$
$\qquad\qquad = span\{\vec{a}_1, \vec{a}_2\}$

Since $\{\vec{a}_1, \vec{a}_2\}$ is lin. indep., it is a basis for $Col(A)$.

⚠ The basis is formed by the columns in the original matrix, not the REF's columns.

⚠ When a matrix is in REF, it is easy to see relation between columns. These relations still hold for the original matrix (before applying EROs to get the REF)

Theorem:

The columns of A where A has a pivot form a basis for Col(A)

Strategy to find basis for Col(A)

1. Row reduce A to echelon form
2. Identify the pivots
3. The columns of A where there are pivots (from obtained echelon form) form a basis for Col(A).

⚠ Column of A not of echelon form (or REF)

Example:

Find a basis for Col(A) and Null(A) where $A = \begin{pmatrix} 1 & 2 & 1 & 3 & 4 \\ 3 & 6 & 2 & 6 & 9 \\ -2 & -4 & 1 & 1 & -1 \end{pmatrix}$

Solution:

$$A \underset{\substack{R_2-3R_1 \\ R_3+2R_1}}{\sim} \begin{pmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 0 & -1 & -3 & -3 \\ 0 & 0 & 3 & 7 & 7 \end{pmatrix} \underset{-R_2}{\sim} \begin{pmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 3 & 7 & 7 \end{pmatrix} \underset{\substack{R_1-R_2 \\ R_3-3R_2}}{\sim} \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -2 & -2 \end{pmatrix}$$

$$\underset{-\frac{1}{2}R_3}{\sim} \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \underset{R_2-3R_3}{\sim} \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$ REF

(*) A basis for Col(A) is $\left\{ \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} \right\}$ ($1^{st}, 3^{rd}, 4^{th}$ column of A)

(*) substitution: $\begin{cases} x_1 + 2x_2 + x_5 = 0 \\ x_3 = 0 \\ x_4 + x_5 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -2x_2 - x_5 \\ x_2 \text{ free} \\ x_3 = 0 \\ x_4 = -x_5 \\ x_5 \text{ free} \end{cases}$

$$\Rightarrow \vec{x} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} = x_2 \vec{v_1} + x_5 \vec{v_2}$$

$\Rightarrow \{\vec{v_1}, \vec{v_2}\}$ basis for Null(A).

4) Coordinate system

(section 2.4 not covered)

5) The dimension of a vector space.

In $\mathbb{R}^n$, we saw that

- Less than n vectors cannot span $\mathbb{R}^n$
- A set with more than n vectors cannot be linearly indep.

Theorem:

If a vector space V has a basis with n vectors in it, then any set in V containing more than n vectors must be lin. dep.

Theorem

If a vector space V has a basis of n vectors, then every basis for V must have n vectors

⚠ bases are not unique but the number of vectors in a basis is. ⚠

Definition:

The dimension of a nonzero subspace H, denoted $\dim H$, is the number of vectors in any basis for H. The dimension of $\{\vec{0}\}$ is 0.

Since $\{\vec{e}_1, \cdots, \vec{e}_n\}$ is a basis for $\mathbb{R}^n$, we have $\dim \mathbb{R}^n = n$.

Example:

Find a basis and the dimension of the subspace

$$W = \left\{ \begin{pmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{R} \right\}$$

Solution:

$$\begin{pmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{pmatrix} = a\begin{pmatrix}1\\2\\0\\3\end{pmatrix} + b\begin{pmatrix}1\\2\\1\\0\end{pmatrix} + c\begin{pmatrix}3\\4\\1\\3\end{pmatrix} + d\begin{pmatrix}0\\1\\1\\1\end{pmatrix}$$

$$= a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 + d\vec{v}_4$$

$\Rightarrow W = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ $\Rightarrow$ find out which vectors form a lin. ind. set

$\Leftrightarrow$ find basis $Col(A)$ with $A = (\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4)$

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 2 & 2 & 4 & 1 \\ 0 & 1 & 1 & 1 \\ 3 & 0 & 3 & 1 \end{pmatrix} \underset{\substack{R_2-R_1 \\ R_4-3R_1}}{\sim} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & -3 & 1 \end{pmatrix} \underset{R_2 \leftrightarrow R_3}{\sim} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -3 & -3 & 1 \end{pmatrix} \underset{R_4+3R_2}{\sim} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

$$\underset{R_4-4R_3}{\sim} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ pivot are on 1^{st}, 2^{nd}, 4^{th} column

$\Rightarrow$ basis for W is $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$

$\Rightarrow \dim W = 3$.

Theorem (The Basis Theorem)

Let H be a subspace of a vector space V such that $\dim H = p \geq 1$. Then:

① Any lin. indep. set of p vectors in H is a basis for H.

② Any set of p vectors in H that spans H is a basis for H.

Example 1: Show that $B = \left\{ \overset{M_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \overset{M_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \overset{M_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \overset{M_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$ is a basis for $\mathcal{M}_{2\times 2}(\mathbb{R})$.

(lin. indep) Let $c_1, \dots, c_4$ in $\mathbb{R}$ such that

$$c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_{\mathcal{M}_{2\times 2}(\mathbb{R})}$$

Then: $\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \implies c_1 = c_2 = c_3 = c_4 = 0 \implies B$ lin. indep

(span) Let $A \in \mathcal{M}_{22}(\mathbb{R})$. Then, there exists a, b, c, d in $\mathbb{R}^2$ such that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\implies A = aM_1 + bM_2 + cM_3 + dM_4 \in \text{Span}(B)$

$\implies \mathcal{M}_{22}(\mathbb{R}) = \text{span}(B)$

Therefore, B is a basis for $\mathcal{M}_{2\times 2}(\mathbb{R})$.

Remark:

We can construct a similar basis for $\mathcal{M}_{m\times n}(\mathbb{R})$ to get that

$$\dim \mathcal{M}_{m\times n}(\mathbb{R}) = mn$$

Remark: If H subspace of V with $\dim V < \infty$
$\Rightarrow \dim H \leq \dim V$

Example 2:

Show that $B = \{1, 1+x, 1+x+x^2\}$ is a basis for $\mathbb{P}_2(\mathbb{R})$

(lin. indep) Let c_1, c_2, c_3 in $\mathbb{R}$ such that

$$c_1(1) + c_2(1+x) + c_3(1+x+x^2) = 0_{\mathbb{P}_2(\mathbb{R})} = 0(1) + 0(x) + 0(x^2)$$

$$\Rightarrow (c_1+c_2+c_3) + (c_2+c_3)x + c_3x^2 = 0 + 0(x) + 0(x^2)$$

$$\Rightarrow \left|\begin{array}{l} c_1+c_2+c_3=0 \\ c_2+c_3=0 \\ c_3=0 \end{array}\right. \Rightarrow \left|\begin{array}{l} c_1=-c_2-c_3=0 \\ c_2=-c_3=0 \\ c_3=0 \end{array}\right. \Rightarrow B \text{ lin. indep.}$$

(span) no need
Since $\dim(\mathbb{P}_2(\mathbb{R})) = 3$ and B contains 3 vectors (lin. indep) then B is a basis for $\mathbb{P}_2(\mathbb{R})$.

6) Rank

The row space

Definition: Let $A = \begin{pmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{pmatrix}$ be a $m \times n$ matrix ($\vec{r}_i$ = i-th row of A)

The row space of A is defined by $\text{Row}(A) = \{A^T x \mid \vec{x} \in \mathbb{R}^m\} = \text{span}\{\vec{r}_1, \dots, \vec{r}_m\}$

$\rightarrow$ $\text{Row}(A)$ is a subspace of $\mathbb{R}^n$

$\rightarrow$ $\text{Col}(A^T) = \text{Row}(A)$

Theorem

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form then the nonzeros row of B form a basis for Row(B) and Row(A).

Example:

Let $A=\begin{pmatrix} 1 & 1 & 5 & 1 \\ 1 & 2 & 7 & 1 \\ 2 & 3 & 12 & 3 \end{pmatrix}$. Find a basis for Null(A), Col(A), Row(A) and state the dimension of each of these subspaces.

Solution:

$$\begin{pmatrix} 1 & 1 & 5 & 1 \\ 1 & 2 & 7 & 1 \\ 2 & 3 & 12 & 3 \end{pmatrix} \underset{\substack{R_2-R_1 \\ R_3-2R_1}}{\sim} \begin{pmatrix} 1 & 1 & 5 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix} \underset{\substack{R_1-R_2 \\ R_3-R_2}}{\sim} \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ REF}$$

$\rightarrow$ basis for row(A) is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\}$, $\dim(\text{Row}(A)) = 2$

$\rightarrow$ substitution $\left| \begin{array}{l} x_1 + 3x_3 = 0 \\ x_2 + 2x_3 + x_4 = 0 \end{array} \right. \Rightarrow \left(\begin{array}{l} x_1 = -3x_3 \\ x_2 = -2x_3 - x_4 \\ x_3, x_4 \text{ free} \end{array} \right. \Rightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

$= x_3 \vec{v}_1 + x_4 \vec{v}_2$

basis for Null(A) is $\left\{ \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$, $\dim(\text{Null}(A)) = 2$

basis for Col(A) is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$, $\dim \text{Col}(A) = 2$

1st and 2nd columns of A

Strategy to find a basis for Row(A)

① Row reduce A in echelon form R.
② The nonzero rows of R form a basis for Row(A) (take transpose to get vector)

$$\underset{\text{row}}{(a\,b\,c)} \to \underset{\text{vector}}{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}$$

Rank

Notice that $\dim(\text{Null}(A)) = \#$ of free variables
$\dim(\text{Col}(A)) = \#$ of pivots

Definition:
The rank of a matrix A is the dimension of the column space of A

$$\text{rank}(A) = \dim(\text{Col}(A))$$

⚠ If $A \in M_{m\times n}(\mathbb{R})$ then $\text{rank}\,A \leq n$.

⚠ $\text{rank}\,A = \dim(\text{Col}(A)) = \#\text{ pivots} = \dim(\text{Row}(A))$

Theorem (The Rank Theorem)
If A has n columns then:

$$\text{rank}(A) + \dim(\text{Null}(A)) = n$$

$\triangleleft$ # Basic variables + # free variable $= n$.

$\triangleleft$ $\text{Row}(A) = \text{Col}(A^T) \Rightarrow \text{rank}(A) = \text{rank}(A^T)$

<u>Theorem</u> (The Invertible Matrix Theorem) (continuation)

Let A be a $n\times n$. The following are equivalent to the statement "A is invertible"

(m) Columns of A form a basis of $\mathbb{R}^n$.

(n) $\mathrm{Col}(A) = \mathbb{R}^n$

(o) $\dim(\mathrm{Col}(A)) = n$

(p) $\mathrm{rank}\, A = n$

(q) $\mathrm{Null}(A) = \{0\}$

(r) $\dim(\mathrm{Null}(A)) = 0$

<u>Example:</u>

Find a basis for $\mathrm{Null}(A)$, $\mathrm{Col}(A)$ and $\mathrm{Row}(A)$. What is $\mathrm{rank}(A)$.

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 & 4 \\ 3 & 6 & 2 & 6 & 9 \\ -2 & -4 & 1 & 1 & -1 \end{pmatrix}$$

<u>Solution:</u>

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 4 \\ 3 & 6 & 2 & 6 & 9 \\ -2 & -4 & 1 & 1 & -1 \end{pmatrix} \underset{\substack{R_2-3R_1 \\ R_3+2R_1}}{\sim} \begin{pmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 0 & -1 & -3 & -3 \\ 0 & 0 & 3 & 7 & 7 \end{pmatrix} \underset{-R_2}{\sim} \begin{pmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 3 & 7 & 7 \end{pmatrix} \underset{R_3-3R_2}{\overset{R_1-R_2}{\sim}} \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -2 & -2 \end{pmatrix}$$

$$\underset{\frac{R_2}{-2}}{\sim} \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \underset{R_2-3R_3}{\sim} \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \text{ REF}$$

A Basis for $\mathrm{col}(A)$ is $\left\{ \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} \right\}$ $\Rightarrow \dim(\mathrm{col}(A)) = 3$

$\Rightarrow \mathrm{rank}\, A = 3$

1st 3rd 4th Columns of A.

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A basis for Row(A) is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ ✓

Substitution:

$$\begin{cases} x_1 + 2x_2 + x_5 = 0 \\ x_3 = 0 \\ x_4 + x_5 = 0 \end{cases} \longleftrightarrow \begin{cases} x_1 = -2x_2 - x_5 \\ x_2 \text{ free} \\ x_3 = 0 \\ x_4 = -x_5 \\ x_5 \text{ free} \end{cases}$$

$$\longleftrightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

A basis for Null(A) is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$ ✓