Linear Algebra

Assignment 1 MATH 2318 (Fall 2022)

4 points

1. Solve the linear system of equations

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & + & x_3 & = & -2 \\ 2x_1 & - & x_2 & + & 3x_3 & = & 6 \\ 3x_1 & - & 5x_2 & + & 4x_3 & = & 7 \end{array}$$

by writing out the augmented matrix of the system and carrying it to Reduced Echelon Form (REF) using Elementary Row Operations (EROs). Recall that no more than one operation should be performed on the same row at any step. Verify your answer by substituting the values you found for x_1, x_2 and x_3 in the linear system.

Solution.

$$\left[\begin{array}{rrr|r} 2 & 1 & 1 & -2 \\ 2 & -1 & 3 & 6 \\ 3 & -5 & 4 & 7 \end{array}\right] \begin{array}{c} \\ \sim \\ R_3 - R_1 \end{array} \left[\begin{array}{rrr|r} 2 & 1 & 1 & -2 \\ 2 & -1 & 3 & 6 \\ 1 & -6 & 3 & 9 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_3 \\ \sim \\ \end{array} \left[\begin{array}{rrr|r} 1 & -6 & 3 & 9 \\ 2 & -1 & 3 & 6 \\ 2 & 1 & 1 & -2 \end{array}\right] \begin{array}{c} \sim \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & -6 & 3 & 9 \\ 0 & 11 & -3 & -12 \\ 0 & 13 & -5 & -20 \end{array}\right] \begin{array}{c} \sim \\ (1/11)R_2 \\ \end{array} \left[\begin{array}{rrr|r} 1 & -6 & 3 & 9 \\ 0 & 1 & -3/11 & -12/11 \\ 0 & 13 & -5 & -20 \end{array}\right] \begin{array}{c} R_1 + 6R_2 \\ \sim \\ R_3 - 13R_2 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 15/11 & 27/11 \\ 0 & 1 & -3/11 & -12/11 \\ 0 & 0 & -16/11 & -64/11 \end{array}\right] \begin{array}{c} \\ \sim \\ (-11/16)R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 15/11 & 27/11 \\ 0 & 1 & -3/11 & -12/11 \\ 0 & 0 & 1 & 4 \end{array}\right] \begin{array}{c} R_1 - (15/11)R_3 \\ R_2 + (3/11)R_3 \\ \sim \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Therefore, $x_1 = -3$, $x_2 = 0$, $x_3 = 4$. Plugging in these values in the original linear system we find:

$$\begin{array}{rcrcrcr} 2(-3) & + & 0 & + & 4 & = & -2 \\ 2(-3) & - & 0 & + & 3(4) & = & 6 \\ 3(-3) & - & 5(0) & + & 4(4) & = & 7 \end{array}$$

3 points

2. The REF of the augmented matrix of a linear system of three equations in six variables is given by

$$\left[\begin{array}{rrrrrr|r} 0 & 1 & 0 & 3 & 0 & 2 & -5 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

State which of the variables are free and which are basic. What is the solution of the system?

Solution.

Basic variables: x_2, x_5.
Free variables: x_1, x_3, x_4, x_6.

$$\begin{array}{lcrcrcr} x_1 & \text{is free} & & & & & \\ x_2 & = & -5 & - & 3x_4 & - & 2x_6 \\ x_3 & \text{is free} & & & & & \\ x_4 & \text{is free} & & & & & \\ x_5 & = & 3 & + & x_6 & & \\ x_6 & \text{is free} & & & & & \end{array}$$

5 points 3. Consider the linear system

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & a \\ x_1 & - & x_2 & & & = & 0 \\ 3x_1 & + & x_2 & + & bx_3 & = & 0 \end{array}$$

Find the values of a and b so that the linear system has:

a) No solutions.

b) Infinitely many solutions.

c) A unique solution.

Solution.

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & a \\ 1 & -1 & 0 & 0 \\ 3 & 1 & b & 0 \end{array}\right] \begin{array}{c} \sim \\ R_2 - R_1 \\ R_3 - 3R_1 \end{array} \left[\begin{array}{rrr|r} 1 & 1 & 1 & a \\ 0 & -2 & -1 & -a \\ 0 & -2 & b-3 & -3a \end{array}\right] \begin{array}{c} \sim \\ \\ R_3 - R_2 \end{array} \left[\begin{array}{rrr|r} 1 & 1 & 1 & a \\ 0 & -2 & -1 & -a \\ 0 & 0 & b-2 & -2a \end{array}\right].$$

(This is an echelon form of the matrix.) We focus on the last row only since this row will determine if the system has any solutions.

$*$ If $b - 2 = 0$, then we have

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & a \\ 0 & -2 & -1 & -a \\ 0 & 0 & 0 & -2a \end{array}\right].$$

The only way for this system to be consistent is if $-2a = 0 \Rightarrow a = 0$. In this case, x_1 and x_2 will be basic variables, while x_3 is free, so we will have infinitely many solutions.

(This is the reduced echelon form of the matrix.) This gives us that x_1, x_2 are basic variables and that x_3 is free, which implies that we have infinitely many solutions.

Otherwise, if $-2a \neq 0$, then we have a row like $\begin{bmatrix} 0 & 0 & 0 & c \end{bmatrix}$, with $c \neq 0$.

$*$ If $b - 2 \neq 0$, then we can divide the last row by $b - 2$ to obtain $\left[\begin{array}{rrr|r} 1 & 1 & 1 & a \\ 0 & -2 & -1 & -a \\ 0 & 0 & 1 & \frac{-2a}{b-2} \end{array}\right]$,
in which case we have a unique solution regardless of the value of a, since we can do backward substitution and uniquely determine x_1, x_2, x_3.

In summary,

a) No solutions if $b = 2$, $a \neq 0$.

b) Infinitely many solutions if $b = 2$, $a = 0$.

c) Unique solution if $b \neq 2$.