Linear Algebra
Final Exam Practice MATH 2318 (Fall 2022)

1. Show that the set of polynomials in $\mathbb{P}_n(\mathbb{R})$ that satisfy $p(0) = 0$ is a subspace of $\mathbb{P}_n(\mathbb{R})$.

   Solution.

   Denote by H the set of polynomials that satisfy $p(0) = 0$. We show that H satisfies the three properties that define a subspace.

   1. First, note that the zero polynomial satisfies that $p(0) = 0$. Therefore, the zero polynomial is in H.
   2. Let $p(x)$, $q(x)$ be elements of H. Then, they satisfy that $p(0) = 0$ and $q(0) = 0$. Note that their sum satisfies that $(p + q)(0) = p(0) + q(0) = 0$. Therefore, $(p + q)(x) \in H$.
   3. Finally, let $p(x)$ be an element of H and c be a real scalar. Then, $p(0) = 0$. Moreover, $(cp)(0) = cp(0) = 0$. Therefore, $(cp)(x) \in H$.

   Since H satisfies the three properties of a subspace, H is a subspace of $\mathbb{P}_n(\mathbb{R})$.

2. Show that the set $H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times2} : c = b \right\}$ is a subspace of $M_{2\times2}$.

   Solution. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H$. Then, $c = b$, thus,

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

   Therefore, $H = \text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, which implies that H is a subspace of $M_{2\times2}$.

   Alternatively, we can use the properties of a subspace:

   1. The zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is an element of H because it satisfies that $0 = 0$.
   2. Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in H$. Then they satisfy $c_1 = b_1$ and $c_2 = b_2$. We have

$$A + B = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix},$$

      which satisfies that $c_1 + c_2 = b_1 + b_2$. Therefore, $A + B \in H$.
   3. Let $k \in \mathbb{R}$, and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H$, which satisfies $c = b$. We have

$$kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix},$$

      which satisfies that $kc = kb$. Therefore, $kA \in H$.

Since H satisfies all three properties, it is a subspace of $M_{2\times 2}$.

3. Find a basis for Col(A), Null(A) and Row(A). What is rank$A$?

$$A = \begin{bmatrix} 1 & 1 & -1 & -2 \\ -1 & -2 & 1 & 3 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

Solution.

Let us row reduce A to REF:

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ -1 & -2 & 1 & 3 \\ 2 & 3 & 1 & 4 \end{bmatrix} \begin{matrix} \sim \\ R_2 + R_1 \\ R_3 - 2R_1 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 3 & 8 \end{bmatrix} \begin{matrix} \sim \\ -R_2 \\ \end{matrix} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 3 & 8 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \sim \\ R_3 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & 9 \end{bmatrix} \begin{matrix} \\ \sim \\ \frac{1}{3}R_3 \end{matrix} \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{matrix} R_1 + R_3 \\ \sim \\ \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

A basis for Col(A) is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$, and a basis for Row($A$) is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$.

To find a basis for Null(A) we solve the homogeneous system:

$$\begin{aligned} x_1 &= -2x_4 \\ x_2 &= x_4 \\ x_3 &= -3x_4 \\ x_4 &\text{ is free} \end{aligned}$$

In vector parametric form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -2 \\ 1 \\ -3 \\ 1 \end{bmatrix}.$$

Therefore, a basis for Null(A) is $\left\{ \begin{bmatrix} -2 \\ 1 \\ -3 \\ 1 \end{bmatrix} \right\}$. Since the basis we found for Col(A) has three vectors, rank$A = 3$.

4. Show that $B = \{1+x^2, x+x^2, 1+2x+x^2\}$ is a basis for $\mathbb{P}_2$. *Hint*: Recall that $\dim(\mathbb{P}_2) = 3$ and note that B has three vectors.

Solution.

Since B has three vectors and $\dim(\mathbb{P}_2) = 3$, we just have to show that B is a linearly independent set. Let c_1, c_2, $c_3 \in \mathbb{R}$ and consider

$$\begin{aligned} c_1(1+x^2) + c_2(x+x^2) + c_3(1+2x+x^2) &= 0 + 0x + 0x^2 \\ (c_1+c_3) + (c_2+2c_3)x + (c_1+c_2+c_3)x^2 &= 0 + 0x + 0x^2 \end{aligned}$$

Thus,

$$\begin{array}{rcrcrcl} c_1 & & & + & c_3 & = & 0 \\ & & c_2 & + & 2c_3 & = & 0 \\ c_1 & + & c_2 & + & c_3 & = & 0 \end{array}$$

Now we solve this linear system by writing it as a matrix equation $A\vec{c} = \vec{0}$ and row reducing A:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \underset{R_3 - R_1}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \underset{R_3 - R_2}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

This is an echelon form. We see that every column has a pivot, so all the variables are basic. This implies that the only solution is the trivial one, so B is linearly independent. By the basis theorem, B is a basis for $\mathbb{P}_2$.

5. Let $\vec{v}$ be an eigenvector of $A \in M_{n\times n}$ with associated eigenvalue λ. Show that $k\vec{v}$, with k a nonzero scalar, is an eigenvector of A with associated eigenvalue λ.

Solution.

Since $\vec{v}$ is an eigenvector of A with associated eigenvalue λ, we know that $A\vec{v} = \lambda\vec{v}$. Multiplying this equation by k gives $k(A\vec{v}) = k(\lambda\vec{v})$. Using the arithmetic properties of matrix-vector multiplication gives $A(k\vec{v}) = \lambda(k\vec{v})$, which implies that $k\vec{v}$ is an eigenvector of A with associated eigenvalue λ.

6. Let $\vec{v}$ be an eigenvector of $A \in M_{n\times n}$ with associated eigenvalue λ. Show that $\vec{v}$ is also an eigenvector of A^2 and find the associated eigenvalue. *Hint*: Consider $A\vec{v} = \lambda\vec{v}$ and multiply by A.

Solution.

Since $\vec{v}$ is an eigenvector of A with associated eigenvalue λ, we know that $A\vec{v} = \lambda\vec{v}$. Multiplying this equation by A gives $A(A\vec{v}) = A(\lambda\vec{v})$. Using the arithmetic properties of matrix-vector multiplication gives $A^2\vec{v} = \lambda(A\vec{v})$. Substituting $A\vec{v} = \lambda\vec{v}$ gives $A^2\vec{v} = \lambda^2\vec{v}$. Therefore, $\vec{v}$ is an eigenvector of A with associated eigenvalue λ^2.

7. Let $A = \begin{bmatrix} 6 & -3 & -3 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$.

   a) Find the eigenvalues of A.
   b) Find a basis for each eigenspace of A.
   c) Is A diagonalizable? Explain. If it is, diagonalize it and find a formula for A^k.

Solution.

a) We compute $\det(A - \lambda I)$ and set it equal to 0:

$$\det(A-\lambda I) = \begin{vmatrix} 6-\lambda & -3 & -3 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = (6-\lambda)(-1)^{1+1} \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = (6-\lambda)((3-\lambda)^2-1)$$

$$= (6-\lambda)(\lambda^2 - 6\lambda + 8) = (6-\lambda)(\lambda-2)(\lambda-4) = 0$$

Therefore, the eigenvalues of A are $\lambda_1 = 6$, $\lambda_2 = 2$ and $\lambda_3 = 4$.

b) For each eigenvalue, we find a basis for $E_\lambda(A) = \text{Null}(A - \lambda I)$.

- $E_{\lambda_1}(A) = \text{Null}(A - 6I)$:

$$A - 6I = \begin{bmatrix} 0 & -3 & -3 \\ 0 & -3 & -1 \\ 0 & -1 & -3 \end{bmatrix} \overset{-\frac{1}{3}R_1}{\sim} \begin{bmatrix} 0 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & -1 & -3 \end{bmatrix} \begin{matrix} \sim \\ R_2 + 3R_1 \\ R_3 + R_1 \end{matrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{matrix} \sim \\ \frac{1}{2}R_2 \\ \end{matrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \sim \\ R_3 + 2R_2 \end{matrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution is

$$\begin{aligned} &x_1 \text{ is free} \\ &x_2 = 0 \\ &x_3 = 0 \end{aligned}$$

In vector parametric form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, a basis for $E_{\lambda_1}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

- $E_{\lambda_2}(A) = \text{Null}(A - 2I)$:

$$A - 2I = \begin{bmatrix} 4 & -3 & -3 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{matrix} R_1 + 3R_2 \\ \sim \\ R_3 + R_2 \end{matrix} \begin{bmatrix} 4 & 0 & -6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \overset{\frac{1}{4}R_1}{\sim} \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution is

$$\begin{aligned} &x_1 = \frac{3}{2}x_3 \\ &x_2 = x_3 \\ &x_3 \text{ is free} \end{aligned}$$

In vector parametric form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{3}{2} \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, a basis for $E_{\lambda_2}(A)$ is $\left\{ \begin{bmatrix} \frac{3}{2} \\ 1 \\ 1 \end{bmatrix} \right\}$.

- $E_{\lambda_3}(A) = \text{Null}(A - 4I)$:

$$A - 3I = \begin{bmatrix} 2 & -3 & -3 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{matrix} R_1 - 3R_2 \\ \sim \\ R_3 - R_2 \end{matrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \frac{1}{2}R_1 \\ -R_2 \\ \sim \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution is

$$\begin{aligned} x_1 &= 0 \\ x_2 &= -x_3 \\ x_3 &\text{ is free} \end{aligned}$$

In vector parametric form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Therefore, a basis for $E_{\lambda_3}(A)$ is $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

c) A is diagonalizable because it has three distinct eigenvalues. We can write $A = PDP^{-1}$, with

$$P = \begin{bmatrix} 1 & \frac{3}{2} & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

To find a formula for A^k, we first calculate P^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{2} & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_1 - 3/2R_2 \\ \sim \\ R_3 - R_2 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & 1 & -\frac{3}{2} & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right] \begin{matrix} \\ \sim \\ 1/2R_3 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & 1 & -\frac{3}{2} & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \begin{matrix} R_1 - 3/2R_3 \\ R_2 + R_3 \\ \sim \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{3}{4} & -\frac{3}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

Thus, $P^{-1} = \begin{bmatrix} 1 & -\frac{3}{4} & -\frac{3}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Now, we calculate A^k:

$$\begin{aligned} A^k = PD^kP^{-1} &= \begin{bmatrix} 1 & \frac{3}{2} & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 4^k \end{bmatrix} \begin{bmatrix} 1 & -\frac{3}{4} & -\frac{3}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 6^k & \frac{3}{2} \times 2^k & 0 \\ 0 & 2^k & -4^k \\ 0 & 2^k & 4^k \end{bmatrix} \begin{bmatrix} 1 & -\frac{3}{4} & -\frac{3}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 6^k & \frac{3}{4} \times 2^k - \frac{3}{4} \times 6^k & \frac{3}{4} \times 2^k - \frac{3}{4} \times 6^k \\ 0 & \frac{1}{2} \times 2^k + \frac{1}{2} \times 4^k & \frac{1}{2} \times 2^k - \frac{1}{2} \times 4^k \\ 0 & \frac{1}{2} \times 2^k - \frac{1}{2} \times 4^k & \frac{1}{2} \times 2^k + \frac{1}{2} \times 4^k \end{bmatrix}. \end{aligned}$$

8. Find the shortest distance from $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}$ to the line spanned by $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

Solution.

First, we calculate $\text{proj}_{\vec{v}}\vec{x}$:

$$\text{proj}_{\vec{v}}\vec{x} = \frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v} = \frac{1(1) + 0(-1) + 7(3)}{1^2 + (-1)^2 + 3^3} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \frac{22}{11} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix}. \tag{1}$$

The shortest distance from $\vec{x}$ to the line spanned by $\vec{v}$ is given by

$$\left\|\vec{x} - \text{proj}_{\vec{v}}\vec{x}\right\| = \left\| \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\| = \sqrt{(-1)^2 + 2^2 + 1^2} = \sqrt{6}.$$

9. Let $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$.

   a) Show that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

   b) Write $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ as a linear combination of $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$.

Solution.

a) Note that
   - $\vec{u}_1 \cdot \vec{u}_2 = 1(1) + 0(2) + 1(-1) = 1 - 1 = 0$,
   - $\vec{u}_1 \cdot \vec{u}_3 = 1(1) + 0(-1) + 1(-1) = 1 - 1 = 0$, and
   - $\vec{u}_2 \cdot \vec{u}_3 = 1(1) + 2(-1) + (-1)(-1) = 1 - 2 + 1 = 0$.

   Therefore, $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set of nonzero vectors, which implies that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a linearly independent set. Since there are three vectors in this set and $\dim(\mathbb{R}^3) = 3$, then $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

b) Since $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$, we know that we can write

$$\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3,$$

   where $c_i = \frac{\vec{v} \cdot \vec{u}_i}{\vec{u}_i \cdot \vec{u}_i}$. Thus,
   - $c_1 = \frac{\vec{v} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{3(1)+1(0)+2(1)}{1^2+0^2+1^2} = \frac{5}{2}$,
   - $c_2 = \frac{\vec{v} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} = \frac{3(1)+1(2)+2(-1)}{1^2+2^2+(-1)^2} = \frac{3}{6} = \frac{1}{2}$, and
   - $c_3 = \frac{\vec{v} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3} = \frac{3(1)+1(-1)+2(-1)}{1^2+(-1)^2+(-1)^2} = 0$.

   Thus,

$$\vec{v} = \frac{5}{2}\vec{u}_1 + \frac{1}{2}\vec{u}_2.$$

10. Find the closest point to $\vec{v} = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$ in $H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$. What is the shortest distance from $\vec{v}$ to H?

Solution.

Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$. The closest point to $\vec{v}$ in H is given by

$$\begin{aligned} \text{proj}_{\vec{u}}\vec{v} &= \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} \\ &= \frac{-1+2+0+2}{1+1+0+4} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} \\ &= \frac{3}{6} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}. \end{aligned}$$

The shortest distance is given by

$$\begin{aligned} \left\|\vec{v} - \text{proj}_{\vec{u}}\vec{v}\right\| &= \left\| \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ 3 \\ 0 \end{bmatrix} \right\| = \frac{1}{2} \left\| \begin{bmatrix} -3 \\ 3 \\ 6 \\ 0 \end{bmatrix} \right\| \\ &= \frac{1}{2}\sqrt{9+9+36+0} \\ &= \frac{\sqrt{54}}{2}. \end{aligned}$$

11. Let $\{\vec{u}, \vec{v}\}$ be an orthogonal set of vectors in $\mathbb{R}^n$, and let $\vec{x} \in \text{Span}\{\vec{u}\}$ and $\vec{y} \in \text{Span}\{\vec{v}\}$. Prove that $\vec{x}$ and $\vec{y}$ are orthogonal.

Solution.

We must show that $\vec{x} \cdot \vec{y} = 0$. Since $\vec{x} \in \text{Span}\{\vec{u}\}$, there exists a scalar c_1 such that $\vec{x} = c_1\vec{u}$. Similarly, since $\vec{y} \in \text{Span}\{\vec{v}\}$, there exists a scalar c_2 such that $\vec{y} = c_2\vec{v}$. Therefore,

$$\vec{x} \cdot \vec{y} = \big(c_1\vec{u}\big) \cdot \big(c_2\vec{v}\big) = (c_1c_2)(\vec{u} \cdot \vec{v}).$$

Since $\{\vec{u}, \vec{v}\}$ is an orthogonal set, we know that $\vec{u} \cdot \vec{v} = 0$. Therefore, $\vec{x} \cdot \vec{y} = 0$, which implies that $\vec{x}$ and $\vec{y}$ are orthogonal.

12. For each of the following, determine if the statement is true or false. Provide a short reasoning (one or two sentences).

a) If A is singular, then A is not diagonalizable.

b) If A is invertible, then A is diagonalizable.

c) Let $A \in M_{n\times n}$. If $\dim(\mathrm{Null}(A)) = 1$, then A is not invertible.

d) Any linearly independent set is an orthogonal set.

e) Any orthogonal set of vectors is a linearly independent set.

f) Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ be an orthogonal set of vectors in $\mathbb{R}^n$, and let $t_1, t_2, \ldots, t_p$ be real numbers. The set $\{t_1\vec{v}_1, t_2\vec{v}_2, \ldots, t_p\vec{v}_p\}$ is orthogonal.

Solution.

a) FALSE. Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$. This matrix is singular since its determinant is zero. However, A has two distinct eigenvalues (0 and 2), therefore A is diagonalizable.

b) FALSE. Consider $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. This matrix is invertible because its determinant is not zero (it is 1). The eigenvalue is $\lambda = 1$ with algebraic multiplicity $a_\lambda = 2$. A basis for the corresponding eigenspace is $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ (verify this), so the geometric multiplicity is $g_\lambda = 1 \neq a_\lambda$. Therefore, A is not diagonalizable.

c) TRUE. This means that $A\vec{x} = \vec{0}$ has nontrivial solutions. Alternatively, you can also see that $\mathrm{rank} A = n - 1 < n$. Therefore, A is not invertible.

d) FALSE. A counterexample is the set

$$\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

which is linearly independent, but not orthogonal.

e) FALSE. A counterexample is the set

$$\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

which is orthogonal, but not linearly independent because one of the vectors is the zero vector.

f) TRUE. $(t_i\vec{v}_i) \cdot (t_j\vec{v}_j) = (t_it_j)(\vec{v}_i \cdot \vec{v}_j) = (t_it_j)(0) = 0$, for all $i \neq j$. So the set $\{t_1\vec{v}_1, t_2\vec{v}_2, \ldots, t_p\vec{v}_p\}$ is orthogonal.