Exam 2 Practice — Linear Algebra MATH 2318 (Fall 2022)

1. Consider the following linear system

$$\begin{array}{rcrcrcrcr} x_1 & & & - & x_3 & + & x_4 & = & 3 \\ 2x_1 & - & x_2 & - & x_3 & & & = & -2 \\ -x_1 & + & x_2 & + & x_3 & + & x_4 & = & 4 \\ & & x_2 & + & x_3 & + & x_4 & = & -1 \end{array}$$

   a) Write the system as a matrix equation $A\vec{x} = \vec{b}$.
   b) Compute A^{-1} using any method from class.
   c) Use A^{-1} to solve the system.

Solution.

a) We can write the system as a $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & -1 & -1 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 3 \\ -2 \\ 4 \\ -1 \end{bmatrix}$.

b) We compute A^{-1} by row reducing $[A \mid I_4]$:

$$\left[\begin{array}{rrrr|rrrr} 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 2 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \sim \\ R_2 - 2R_1 \\ R_3 + R_1 \\ \end{array} \left[\begin{array}{rrrr|rrrr} 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ \sim \\ R_3 + R_2 \\ R_4 + R_2 \end{array}$$

$$\left[\begin{array}{rrrr|rrrr} 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 2 & -1 & -2 & 1 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 + R_3 \\ -R_2 \\ \sim \\ R_4 - 2R_3 \end{array} \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 2 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & -2 & 1 \end{array}\right] \begin{array}{c} \\ R_2 + R_3 \\ \sim \\ -R_4 \end{array}$$

$$\left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 2 & -1 \end{array}\right] \begin{array}{c} R_1 - R_4 \\ R_2 - 2R_4 \\ \sim \\ \end{array} \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 & -3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 2 & -1 \end{array}\right]$$

Therefore, $A^{-1} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & -2 & -3 & 2 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix}$.

c) The solution to the system is $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & -2 & -3 & 2 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ -1 \\ 7 \end{bmatrix}$.

2. Compute the determinant of $A = \begin{bmatrix} 3 & -1 & 2 & 1 \\ 4 & 3 & 0 & -2 \\ -1 & 0 & 2 & 3 \\ 6 & 2 & 5 & 2 \end{bmatrix}$ using any method from class.

Solution.

First, we elementary row operations to create obtain three zeros in the second column, and then we apply a cofactor expansion along the second column

$$\det A = \begin{vmatrix} 3 & -1 & 2 & 1 \\ 4 & 3 & 0 & -2 \\ -1 & 0 & 2 & 3 \\ 6 & 2 & 5 & 2 \end{vmatrix} \begin{matrix} R_2 + 3R_1 \\ = \\ R_4 + 2R_1 \end{matrix} \begin{vmatrix} 3 & -1 & 2 & 1 \\ 13 & 0 & 6 & 1 \\ -1 & 0 & 2 & 3 \\ 12 & 0 & 9 & 4 \end{vmatrix} = (-1)(-1)^{1+2} \begin{vmatrix} 13 & 6 & 1 \\ -1 & 2 & 3 \\ 12 & 9 & 4 \end{vmatrix}$$

To compute this 3×3 determinant, we again apply elementary row operations first, and then a cofactor expansion along the first column:

$$\det A = \begin{vmatrix} 13 & 6 & 1 \\ -1 & 2 & 3 \\ 12 & 9 & 4 \end{vmatrix} \begin{matrix} R_1 + 13R_2 \\ = \\ R_3 + 12R_2 \end{matrix} \begin{vmatrix} 0 & 32 & 40 \\ -1 & 2 & 3 \\ 0 & 33 & 40 \end{vmatrix} = (-1)(-1)^{2+1} \begin{vmatrix} 32 & 40 \\ 33 & 40 \end{vmatrix}$$

Using the formula for 2×2 determinants, we have

$$\det A = (32)(40) - (40)(33) = 40(32 - 33) = -40.$$

3. Let A, B and C be $n \times n$ matrices such that $\det A = 2$, $\det B = -1$ and $\det C = 3$. Find $\det\left(2(A^{-1})^2 B^T C^3\right)$.

Solution.

We use the properties of determinants to obtain

$$\det\left(2(A^{-1})^2 B^T C^3\right) = 2^n \left(\det(A^{-1})\right)^2 \left(\det B^T\right) (\det C)^3 = 2^n \left(\frac{1}{\det A}\right)^2 (\det B)(\det C)^3$$
$$= 2^n \left(\frac{1}{2}\right)^2 (-1)(3)^3 = -27(2^{n-2}).$$

4. Let $A = \begin{bmatrix} a+1 & 0 \\ 1 & -1 \end{bmatrix}$. Find all the values of a so that the matrix $A^2 + 3A$ is singular.

Solution.

First, we compute $A^2 + 3A$.

$$A^2 + 3A = \begin{bmatrix} a+1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a+1 & 0 \\ 1 & -1 \end{bmatrix} + 3 \begin{bmatrix} a+1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} (a+1)^2 & 0 \\ a & 1 \end{bmatrix} + \begin{bmatrix} 3(a+1) & 0 \\ 3 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} (a+1)^2 + 3(a+1) & 0 \\ a+3 & -2 \end{bmatrix} = \begin{bmatrix} (a+1)(a+4) & 0 \\ a+3 & -2 \end{bmatrix}$$

A matrix is singular if and only if its determinant is zero. Therefore, we compute $\det(A^2 + 3A)$ and make it equal to zero:

$$\det(A^2 + 3A) = \begin{vmatrix} (a+1)(a+4) & 0 \\ a+3 & -2 \end{vmatrix} = -2(a+1)(a+4) = 0$$

We have then that $\det(A^2 + 3A) = 0$ if and only if $a + 1 = 0$ or $a + 4 = 0$. Therefore, $A^2 + 3A$ is singular if and only if $a = -1$ or $a = -4$.

5. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$. It is known that six of the cofactors of A are:

$$C_{11} = 1, \quad C_{12} = -1, \quad C_{13} = 1, \quad C_{21} = -1, \quad C_{22} = 1, \quad C_{23} = 1.$$

a) Compute $\operatorname{adj}(A)$.

b) Compute $\det A$.

c) Find A^{-1}.

Solution.

a) To obtain $\operatorname{adj}(A)$ we need to compute the rest of the cofactors of A:

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = -2 + 1 = -1$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -(-1 + 2) = -1$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3$$

Therefore, the cofactor matrix of A is

$$C = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & -3 \end{bmatrix}$$

The adjugate matrix of A is

$$\operatorname{adj} A = C^T = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & -3 \end{bmatrix}$$

b) We do a cofactor expansion along the third row:

$$\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = 1(-1) + 1(-1) = -1 - 1 = -2$$

c) The inverse of A is given by

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = -\frac{1}{2} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & -3 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & -1/2 & 3/2 \end{bmatrix}.$$

6. For each of the following, determine if the statement is true or false. Provide a short reasoning (one or two sentences).

a) Cramer's rule can be applied to any kind of linear system.

b) If $\det A \neq 0$, for some $n \times n$ matrix A, then the columns of A are linearly independent.

c) If a square matrix has two identical columns, then its determinant is zero.

d) If a square matrix A is not invertible, then the system $A\vec{x} = \vec{b}$ is inconsistent for all $\vec{b}$.

e) If A and B are invertible, then so is $A + B$.

f) If A and B are any two matrices such that $AB = I_n$, then A and B are both invertible.

Solution.

a) FALSE. Cramer's rule can only be applied to linear systems where the associated matrix is square and invertible.

b) TRUE. From the Invertible Matrix Theorem, we know that if $\det A \neq 0$ then A is invertible which in turn implies that its columns are linearly independent.

c) TRUE. We could subtract one column from the other which does not change the determinant, and this would produce a column of zeros. Doing a cofactor expansion along the column of zeros reveals that the determinant is zero.

d) FALSE. The linear system $A\vec{x} = \vec{0}$ is always consistent regardless of the matrix A.

e) FALSE. Take $A = I_n$, $B = -I_n$. Both A and B are invertible, but $A + B$ equals the zero matrix which is not invertible.

f) FALSE. A and B might not be square. For example, take $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $AB = I_2$, but A and B are not square so they cannot be considered invertible.