Additional material for Chapter 2 (sections 2-8 – 2-9)

Section 2-8: Subspace of $\mathbb{R}^n$

Definition:
A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has the three following properties:

ⓐ the zero vector is in H.

ⓑ for each $\vec{u}$ and $\vec{v}$ in H, $\vec{u}+\vec{v}$ is in H

ⓒ for each $\vec{u}$ in H and each scalar c, $c\vec{u}$ is in H.

subspace $\Leftrightarrow$ non empty space (has zero vector) that is stable under addition and scalar multiplication.

Examples:

1) If $\vec{v}_1, \vec{v}_2$ are in $\mathbb{R}^n$ then span$\{\vec{v}_1, \vec{v}_2\}$ is a subspace of $\mathbb{R}^n$.

2) More generally, if $\vec{v}_1, \dots, \vec{v}_p$ in $\mathbb{R}^n$ then span$\{\vec{v}_1, \dots, \vec{v}_p\}$ is a subspace of $\mathbb{R}^n$.

3) $H = \{0\}$ $(= \text{span}\{\vec{0}\})$ is a subspace of $\mathbb{R}^n$ called the zero subspace.

Definitions:

(*) The column space of a matrix A is the set Col(A) that contains all linear combinations of the column of A.

(*) If $A = (\vec{a}_1 \dots \vec{a}_n)$ then $Col(A) = span\{\vec{a}_1, \dots, \vec{a}_n\}$

(*) The null space of a matrix A is the set Nul(A) that contains all the solution of the homogeneous equation $A\vec{x} = \vec{0}$

Remarks: If A is a $m \times n$ matrix, then $\begin{cases} Col(A) \subseteq \mathbb{R}^m \text{ (include in } \mathbb{R}^m) \\ Nul(A) \subseteq \mathbb{R}^n \end{cases}$

Example:

Let $A = \begin{pmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$. Is $\vec{b}$ in Col(A)?

$\vec{b}$ linear combination column of A $\Leftrightarrow$ there exist some vector $\vec{x}$ such that $A\vec{x} = \vec{b}$

→ row reduced the augmented matrix $(A|\vec{b})$

$$\begin{pmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{pmatrix} \underset{\substack{R_2+4R_1 \\ R_3+3R_1}}{\sim} \begin{pmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{pmatrix} \underset{R_3 + \frac{R_2}{3}}{\sim} \begin{pmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

consistent with x_3 free variable (so infinite number of solution)

$\Rightarrow \vec{b}$ is in Col(A).

Theorem:

The null space of a $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

(equivalently: the set of solution of a system of m linear homogeneous equations in n unknowns in a subspace of $\mathbb{R}^n$.)

⚠ Notion of basis for subspace ⚠

subspace can contain infinite number of vector
(if $\vec{v} \neq \vec{0}$ in H then all $c\vec{v}$ are in H)

goal/idea find small number of vectors that span the subspace.

Definition:

A basis for a subspace H of $\mathbb{R}^n$ is a linearly independant set in H that spans H.

Examples:

1) Column of an invertible matrix form a basis for all $\mathbb{R}^n$ because they are linearly independent and span $\mathbb{R}^n$ by the invertible matrix theorem.

⚠ True for $A = I_n \Rightarrow$ The set $\{e_1, \dots, e_n\}$ is called the standard basis for $\mathbb{R}^n$.

2) Find a basis for the null space of the matrix $A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$

⚠ Idea write $A\vec{x} = \vec{0}$ in parametric vector form ⚠ (after REF) or echelon form.

$$[A \; 0] \sim \dots \begin{pmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\leftrightarrow \left| \begin{array}{l} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \end{array} \right. \qquad (x_2, x_4, x_5 \text{ free})$$

$$\leftrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_2 + x_4 + 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\vec{u}} + x_4 \underbrace{\begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v}} + x_5 \underbrace{\begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}}_{\vec{w}}$$

basis Nul(A) is $\{\vec{u}, \vec{v}, \vec{w}\}$ because they span Nul(A) and are linearly independent (by construction using REF of (A|0)).

⚠ Try to remind above method if asked "<u>find basis of null space</u>"

3) Find a basis for Col(B) with

$$B = \begin{pmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\quad \vec{b_1} \quad \vec{b_2} \quad \vec{b_3} \quad \vec{b_4} \quad \vec{b_5}$$

by construction $Col(B) = span\{\vec{b_1}, \dots, \vec{b_5}\}$. However vectors $\vec{b_1}, \dots, \vec{b_5}$ may not be linearly independant $\Rightarrow$ need to find pivot columns (all other columns of B are linear combination of its pivot column).

⚠ B already in REF.

$\Rightarrow$ pivot are $\vec{b_1}, \vec{b_2}, \vec{b_5}$ $\Rightarrow$ linearly independant and spans Col(B)

$\Rightarrow$ basis for Col(B).

(indeed $\vec{b_3} = -3\vec{b_1} + 2\vec{b_2}$, $\vec{b_4} = 5\vec{b_1} - \vec{b_2}$)

⚠ Property: pivot column of A are a basis of Col(A).

⚠ can not obtained column pivot by performing ROW operations (may lead to a REF whose columns are not in Col(A))

⚠ can do column elementary operation to find column pivot of A.

Section 2.9: Dimension and rank

Remark: It can be shown that if a subspace H has a basis of p vectors then every basis of H must contains p vectors.
(see exercise 27-28 page 161 of textbook)

Definitions

(*) The dimension of a nonzero subspace H, denoted by dim H, is the number of vectors in any basis for H.
The dimension of the zero subspace $\{\vec{0}\}$ is defined to be zero.

(*) The rank of a matrix A, denoted by rank A, is the dimension of the column space of A.

Example:

Determine the rank of the matrix $A = \begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{pmatrix}$

$$A \underset{\text{REF/echelon form}}{\sim} \begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

...

pivot columns

The matrix A has 3 pivot columns so rank $A = 3$.
(attention pivot column of its echelon form may not be in Col(A) so they are not a basis for Col(A)).

⚠ How to find basis Nul(A) → we do echelon form and find "free variables"

$$\Rightarrow \dim \mathrm{Nul}(A) = \#\text{ free variables}$$
$$= \#\text{ columns} - \#\text{ column pivot}$$

Theorem (The rank theorem)

If a matrix A has n columns then

$$\operatorname{rank} A + \dim \mathrm{Nul}(A) = n$$

Theorem (The basis theorem)

Let H be a p-dimensional subspace of $\mathbb{R}^n$.
Any linearly independent set of exactly p elements in H is a basis for H.
Any set a p elements of H that spans H is a basis for H.

Theorem (extension invertible matrix theorem)

Let A be a $n \times n$ matrix. Then following statements are equivalent to the statement A is an invertible matrix.

(m) Columns of A form a basis of $\mathbb{R}^n$.
(n) $\mathrm{Col}(A) = \mathbb{R}^n$
(o) $\dim(\mathrm{Col}(A)) = n$
(p) rank $A = n$
(q) $\mathrm{Nul}\, A = \{\vec{0}\}$
(r) $\dim(\mathrm{Nul}\, A) = 0$