3 points 1. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -5 & 0 & 1 \\ 1 & 3 & 1 & 0 \end{bmatrix}.$$

Do the columns of A span $\mathbb{R}^3$? Explain.

Solution.

Let us row reduce A to an echelon form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -5 & 0 & 1 \\ 1 & 3 & 1 & 0 \end{bmatrix} \begin{matrix} \sim \\ R_2 + R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & 1 & 2 \\ 0 & 2 & 0 & -1 \end{bmatrix} \begin{matrix} \sim \\ R_3 + (1/2)R_2 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & 1 & 2 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}$$

This is an echelon form, and we see that every row of A has a pivot. Therefore, by a theorem we saw in class, we can conclude that the columns of A span $\mathbb{R}^3$

6 points 2. Let $A = \begin{bmatrix} 1 & -3 & -1 & 1 & -1 \\ 2 & -6 & 1 & -3 & -9 \\ -2 & 6 & 3 & 2 & 11 \end{bmatrix}$.

a) Solve the equation $A\vec{x} = \vec{b}$, where $\vec{b} = \begin{bmatrix} -1 \\ 9 \\ 0 \end{bmatrix}$. Write the solution in parametric form and in vector parametric form.

b) Multiply A by the vector parametric form of the solution that you found in part a) and verify that the result is the vector $\vec{b}$.

c) **Without performing any row operations**, find the vector parametric form of the solution of $A\vec{x} = \vec{0}$. If you perform row operations again, no points will be awarded. *Hint*: Use part a).

Solution.

a)

$$\left[\begin{array}{ccccc|c} 1 & -3 & -1 & 1 & -1 & -1 \\ 2 & -6 & 1 & -3 & -9 & 9 \\ -2 & 6 & 3 & 2 & 11 & 0 \end{array}\right] \begin{matrix} \sim \\ R_2 - 2R_1 \\ R_3 + 2R_1 \end{matrix} \left[\begin{array}{ccccc|c} 1 & -3 & -1 & 1 & -1 & -1 \\ 0 & 0 & 3 & -5 & -7 & 11 \\ 0 & 0 & 1 & 4 & 9 & -2 \end{array}\right] \begin{matrix} \sim \\ R_2 \leftrightarrow R_3 \\ \end{matrix}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -3 & -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 4 & 9 & -2 \\ 0 & 0 & 3 & -5 & -7 & 11 \end{array}\right] \begin{matrix} R_1 + R_2 \\ \sim \\ R_3 - 3R_2 \end{matrix} \left[\begin{array}{ccccc|c} 1 & -3 & 0 & 5 & 8 & -3 \\ 0 & 0 & 1 & 4 & 9 & -2 \\ 0 & 0 & 0 & -17 & -34 & 17 \end{array}\right] \begin{matrix} \\ \sim \\ -\frac{1}{17}R_3 \end{matrix}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -3 & 0 & 5 & 8 & -3 \\ 0 & 0 & 1 & 4 & 9 & -2 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{array}\right] \begin{matrix} R_1 - 5R_3 \\ R_2 - 4R_3 \\ \sim \end{matrix} \left[\begin{array}{ccccc|c} 1 & -3 & 0 & 0 & -2 & 2 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

(This is the REF of the matrix)
Basic variables: x_1, x_3, x_4
Free variables: x_2, x_5

Parametric form:

$$\begin{aligned} x_1 &= 2 + 3x_2 + 2x_5 \\ x_2 &\text{ is free} \\ x_3 &= 2 - x_5 \\ x_4 &= -1 - 2x_5 \\ x_5 &\text{ is free} \end{aligned}$$

Vector parametric form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix}.$$

b)

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} 1 & -3 & -1 & 1 & -1 \\ 2 & -6 & 1 & -3 & -9 \\ -2 & 6 & 3 & 2 & 11 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & -3 & -1 & 1 & -1 \\ 2 & -6 & 1 & -3 & -9 \\ -2 & 6 & 3 & 2 & 11 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 & -3 & -1 & 1 & -1 \\ 2 & -6 & 1 & -3 & -9 \\ -2 & 6 & 3 & 2 & 11 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &\quad + x_5 \begin{bmatrix} 1 & -3 & -1 & 1 & -1 \\ 2 & -6 & 1 & -3 & -9 \\ -2 & 6 & 3 & 2 & 11 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 9 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 9 \\ 0 \end{bmatrix} = \vec{b}. \end{aligned}$$

c) Since $A\vec{x} = \vec{b}$ is consistent and the solution is the set of vectors $\vec{x} = \vec{p} + x_3\vec{v}_1 + x_5\vec{v}_2$, where

$$\vec{p} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix},$$

then the solution of the associated homogeneous system $A\vec{x} = \vec{0}$ is given by

$$\vec{x} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix}.$$

4 points

3. For each of the following, determine if the statement is true or false. Provide a short reasoning (one or two sentences).

   a) If a matrix has m rows and n columns, with $m > n$, the columns of the matrix cannot span $\mathbb{R}^m$.

   b) A consistent equation $A\vec{x} = \vec{b}$ where A has more columns than rows can have a unique solution.

   c) If the REF of the augmented matrix of a consistent equation $A\vec{x} = \vec{b}$ has a row of zeros, then the equation has infinitely many solutions.

   d) If a consistent equation $A\vec{x} = \vec{b}$, where A is a square matrix, has infinitely many solutions, then the REF of the augmented matrix has a row of zeros.

Solution.

a) TRUE. If A has more rows than columns, then there cannot be a pivot in every row. For example: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b) FALSE. Since A has more columns than rows, not every column can have a pivot. Therefore, there must be at least one free variable, so there are always infinitely many solutions.

c) FALSE. If the matrix A has more rows than columns, then all the variables can be basic and there will be a row of zeros (see the same example as in part a)).

d) TRUE. If the equation has infinitely many solutions, then at least one of the variables is free, meaning that at least one of the columns does not have a pivot. Since there are the same number of rows as columns, this means that there will be a row of zeros as well.