UH honor code: *By signing my name on this test I promise that I will not commit any acts of academic dishonesty during, before, or after the administration of this assignment. Academic honesty violations include (but are not limited to) the following: sharing questions, sharing answers, using unauthorized software, getting online help, having others take the exam for me, using notes or books, or using unauthorized formula sheets during the exam. If I am caught in such an act, I understand that it will result in the cancellation of my assignment score. In addition, the Math Department and my instructor have the right to take appropriate actions in response to any violations of UH Academic Honesty Policy.*

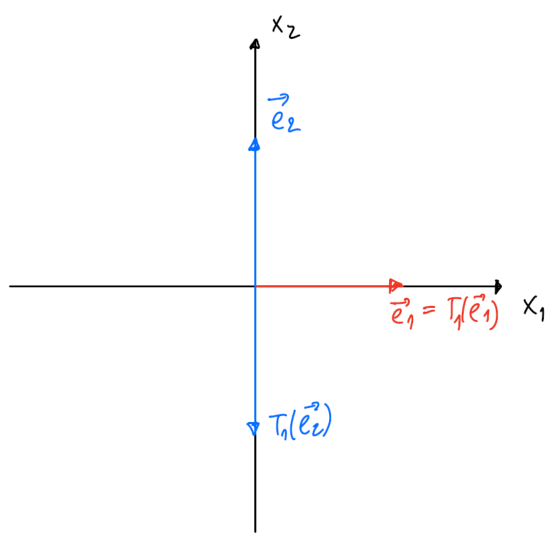
Name and ID: ______________________________

4 points

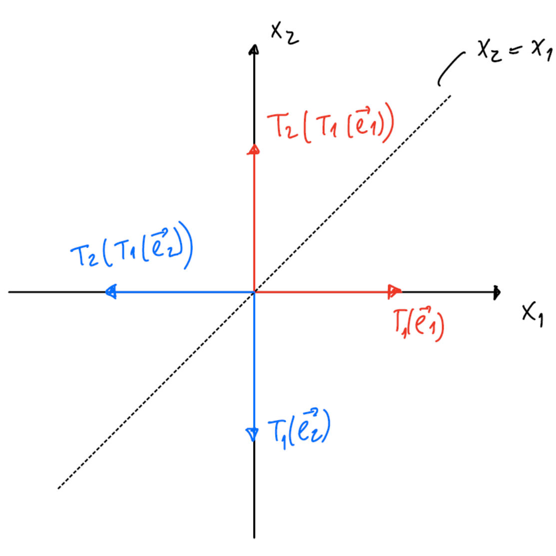
1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that first reflects points through the x_1-axis, and then reflects points through the line $x_2 = x_1$. Find the standard matrix of T. Draw a picture to support your claims.

Solution.

We apply the first transformation (we call it T_1) which reflects points through the x_1-axis:



Then, to the resulting vectors $T_1(\vec{e}_1)$ and $T_1(\vec{e}_2)$, we apply the second transformation (we call it T_2) which reflects points through the line $x_2 = x_1$:



We see that the vector $\vec{e}_1$ got transformed to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and the vector $\vec{e}_2$ got transformed to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Therefore, the standard matrix of this transformation is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

3 points

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 5x_2 + 4x_3 \\ x_2 - 6x_3 \end{bmatrix}$$

Determine if T is onto and/or one-to-one. *Hint*: Find A, the standard matrix of T, and study the number of solutions of $A\vec{x} = \vec{b}$ for any $\vec{b} \in \mathbb{R}^2$.

Solution.

The standard matrix of T is given by $A = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix}$. Note that A is already in echelon form and we can see that every row has a pivot. Therefore, $T(\vec{x}) = A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$ in $\mathbb{R}^2$. Thus, T is onto. From A we see that the third column does not have a pivot, therefore $A\vec{x} = \vec{0}$ has infinitely many solutions. Thus, T is not one-to-one.

6 points

3. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$. Compute the following

a) $CA - 3B$.

b) $A^T C + B^T$.

Solution.

a) Compute:

$$CA = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 10 & -10 & 3 \\ 10 & -15 & 7 \end{bmatrix},$$

then

$$\begin{aligned} CA - 3B &= \begin{bmatrix} 10 & -10 & 3 \\ 10 & -15 & 7 \end{bmatrix} - 3\begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 10 & -10 & 3 \\ 10 & -15 & 7 \end{bmatrix} + \begin{bmatrix} -21 & 15 & -3 \\ -3 & 12 & 9 \end{bmatrix} \\ &= \begin{bmatrix} -11 & 5 & 0 \\ 7 & -3 & 16 \end{bmatrix}. \end{aligned}$$

b) Compute:

$$A^T C = \begin{bmatrix} 2 & 4 \\ 0 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 16 \\ 5 & -15 \\ -3 & 4 \end{bmatrix}$$

then

$$A^T C + B^T = \begin{bmatrix} -2 & 16 \\ 5 & -15 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ -5 & -4 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 5 & 17 \\ 0 & -19 \\ -2 & 1 \end{bmatrix}.$$

3 points

4. For each of the following, determine if the statement is true or false. Provide a short reasoning (one or two sentences).

a) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. If $m > n$, then T cannot be onto.

b) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. If $n > m$, then T cannot be one-to-one.

c) Let T be a linear transformation. If $T(\vec{x}) = \vec{0}$, then $\vec{x} = \vec{0}$.

Solution.

a) TRUE. This means that the standard matrix of T (which is $m \times n$) has more rows than columns. Therefore, not every row can have a pivot, which means that $T(\vec{x}) = \vec{b}$ is not consistent for all $\vec{b}$.

b) TRUE. This means that the standard matrix of T (which is $m \times n$) has more columns than rows. Therefore, if $T(\vec{x}) = \vec{b}$ is consistent, there will be at least one free variable, which means that we will have infinitely many solutions.

c) FALSE. T may be the zero transformation which transform every vector (including nonzero vectors) to the zero vector.