5 points

1. Let $A = \begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 1 & 2 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & -1 \end{bmatrix}$. Find a basis for Null$(A)$ and Col(A). What is rank(A)?

Solution.

We row reduce the matrix to REF:

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 1 & 2 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & -1 \end{bmatrix} \begin{matrix} \sim \\ R_2 - R_1 \\ R_3 - 2R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -3 & -1 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \sim \\ R_3 + R_2 \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix} \begin{matrix} \\ \sim \\ -\frac{1}{2}R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} R_1 - 2R_3 \\ R_2 - R_3 \\ \sim \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The pivot columns are the first, third, and fourth one. Thus, a basis for Col(A) is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} \right\}$.

The solution to the homogeneous system is given by

$$\begin{aligned} x_1 + 2x_2 - x_5 &= 0 \\ x_3 + x_5 &= 0 \\ x_4 &= 0 \end{aligned}$$

which leads to:

$$\begin{aligned} &x_1 = -2x_2 + x_5 \\ &x_2 \text{ is free} \\ &x_3 = -x_5 \\ &x_4 = 0 \\ &x_5 \text{ is free.} \end{aligned}$$

In vector form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, a basis for Null(A) is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Finally, we have that rank$(A) = \dim \mathrm{Col}(A) = 3$.

$\boxed{\text{4 points}}$ 2. Find a basis for $H = \left\{ \begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$. *Hint*: Write H as a Span.

Solution.

Note that any element $\vec{v} \in H$ can be written as

$$\begin{aligned} \vec{v} &= a \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix} + b \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix} \\ &= a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3. \end{aligned}$$

Therefore,

$$H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}.$$

Next, we check if any of these vectors can be written as a linear combination of the others:

$$\begin{bmatrix} 3 & 6 & -1 \\ 6 & -2 & -2 \\ -9 & 5 & 3 \\ -3 & 1 & 1 \end{bmatrix} \begin{matrix} \sim \\ R_2-2R_1 \\ R_3+3R_1 \\ R_4+R_1 \end{matrix} \begin{bmatrix} 3 & 6 & -1 \\ 0 & -14 & 0 \\ 0 & 23 & 0 \\ 0 & 7 & 0 \end{bmatrix} \begin{matrix} \frac{1}{3}R_1 \\ -\frac{1}{14}R_2 \\ \frac{1}{23}R_3 \\ \frac{1}{7}R_4 \end{matrix} \begin{bmatrix} 1 & 2 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{matrix} R_1-2R_2 \\ \sim \\ R_3-R_2 \\ R_4-R_2 \end{matrix} \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We see that $\vec{v}_3 = -\frac{1}{3}\vec{v}_1$. Therefore, $\{\vec{v}_1, \vec{v}_2\}$ is a basis for H.

$\boxed{\text{4 points}}$ 3. Show that the set $B = \{2+2x+x^2, 2+2x+2x^2, x+2x^2\}$ is a basis for $\mathbb{P}_2$. *Hint*: Recall that $\dim(\mathbb{P}_2) = 3$ and note that B has three vectors.

Solution.

Since B has three vectors and $\dim(\mathbb{P}_2) = 3$, we just have to show that B is a linearly independent set. Let c_1, c_2, $c_3 \in \mathbb{R}$ and consider

$$\begin{aligned} c_1(2+2x+x^2) + c_2(2+2x+2x^2) + c_3(x+2x^2) &= 0 + 0x + 0x^2 \\ (2c_1+2c_2) + (2c_1+2c_2+c_3)x + (c_1+2c_2+2c_3)x^2 &= 0 + 0x + 0x^2 \end{aligned}$$

Thus,

$$\begin{array}{ccccccc} 2c_1 & + & 2c_2 & & & = & 0 \\ 2c_1 & + & 2c_2 & + & c_3 & = & 0 \\ c_1 & + & 2c_2 & + & 2c_3 & = & 0 \end{array}$$

Now we solve this linear system by writing it as a matrix equation $A\vec{c} = \vec{0}$ and row reducing A:

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{matrix} \frac{1}{2}R_1 \\ \sim \\ \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{matrix} \sim \\ R_2-2R_1 \\ R_3-R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} \sim \\ R_2 \leftrightarrow R_3 \\ \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

This is an echelon form. We see that every column has a pivot, so all the variables are basic. This implies that the only solution is the trivial one, so B is linearly independent. By the basis theorem, B is a basis for $\mathbb{P}_2$.

3 points

4. For each of the following, determine if the statement is true or false. Provide a short reasoning (one or two sentences).

   a) The column space of a matrix A is the set of solutions of $A\vec{x} = \vec{b}$.

   b) If a vector $\vec{v}$ is in $\mathrm{Col}(A)$, then $A\vec{x} = \vec{v}$ has a unique solution.

   c) Let A_1, A_2, A_3, A_4 be 2×2 matrices such that $M_{2\times 2} = \mathrm{Span}\{A_1, A_2, A_3, A_4\}$. The set $\{A_1, A_2, A_3, A_4\}$ is a basis for $M_{2\times 2}$.

Solution.

a) FALSE. $\mathrm{Col}(A)$ is the set of vectors that can be written as $A\vec{x}$ for some $\vec{x}$.

b) FALSE. For $\vec{v}$ to be in $\mathrm{Col}(A)$, we only require $A\vec{x} = \vec{v}$ to be consistent, so it can have infinitely many solutions.

c) TRUE. $\dim(M_{2\times 2}) = 4$, and since the given set is a spanning set containing four vectors, then, by the Basis Theorem, it is a basis for $M_{2\times 2}$.