Chapter 3: Determinants

Goals: sections 3.1 - 3.2 - 3.3

1) Introduction to determinants

Reminder: in section 2.2, we saw that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$

$\Rightarrow$ we define (for 2×2 matrix): $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

so A invertible $\Leftrightarrow$ $\det A \neq 0$

⚠ This notion can be extended to $n\times n$ matrix.

Definitions:

Let A be a $n\times n$ matrix (with $n \geq 2$) with $A = (a_{ij})$.

⊛ We denote by A_{ij} the $(n-1)\times(n-1)$ matrix obtained from A by deleting the ith row and j-th column of A.

⊛ The determinant of A is defined by:

$$\left.\begin{aligned}\det A &= a_{11}\det A_{11} - a_{12}\det(A_{12}) + \dots + (-1)^{1+n}\det(A_{1n}) \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j}\det(A_{1j})\end{aligned}\right\}$$ cofactor expansion across first row of A.

⊛ $C_{ij} = (-1)^{i+j}\det(A_{ij})$ is called the (i,j) cofactor of A.

Remark:

The determinant of 1×1 matrix $A=(a_{11})$ is equal to a_{11}.

(definitions are consistent with $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} = a_{11}a_{22}-a_{12}a_{21}$)

Example

$A=\begin{pmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \\ -7 & 8 & 9 \end{pmatrix}$. Compute $\det A$.

$$\det(A) = 1\det\underbrace{\begin{pmatrix} -5 & 6 \\ 8 & 9\end{pmatrix}}_{A_{11}} - (2)\det\begin{pmatrix} 4 & 6 \\ -7 & 9\end{pmatrix} + (-3)\det\begin{pmatrix} 4 & -5 \\ -7 & 8\end{pmatrix}$$

$$= \big((-5)(9)-(6)(8)\big) - 2\big(4\times 9 - 6(-7)\big) - 3\big(4\times 8 - (-5)(-7)\big)$$

$$= -240$$

$\neq 0$ (we will see that it means A is invertible).

Theorem:

The determinant of a $n\times n$ matrix A can be computed by a cofactor expansion across any row or down any column of A. Meaning that:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \qquad \text{for any } 1\le i\le n$$

and

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \qquad \text{———} \quad 1\le j\le n$$

<u>Example:</u> Compute determinant of $A = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}$ ← many zero so better to use 3rd row cofactor expansion ⚠

$$\det A = 0(-1)^{3+1}\det\begin{pmatrix} 5 & 1 \\ 4 & -1 \end{pmatrix} + (-2)(-1)^{3+2}\det\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} + 0(-1)^{3+3}\det\begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix}$$

$$= 2\det\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= 2\left(1(-1) - (1)(2)\right)$$

$$= -6$$

⚠ $\det\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ can also be written $\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$ (other notation)

→ try to use 1st row cofactor expansion to compute det A ...
(same result but more computations)

<u>Theorem.</u>

If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.

(i) $A = \begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \rightarrow \det A = a_{11} a_{22} \cdots a_{nn}$

$A = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ * & & a_{nn} \end{pmatrix} \rightarrow \det A = a_{11} a_{22} \cdots a_{nn}$

Exercise:

Compute det(A) and det(B) with

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & 6 \\ 5 & 6 & -7 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ -1 & 1 & 2 & 1 \end{pmatrix}$$

← lot of zero
⇒ nice row to expand.

Solutions: det $A = -15$, det $B = 9$

2) Properties of determinants

Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 4 \\ 1 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 4 \\ 1 & 4 \end{pmatrix}$

Notice that $\det A = -\det B = \det C = 2 = 2\det(B)$

and $A \underset{R_1 \leftrightarrow R_2}{\sim} B$, $A \underset{R_2 + 2R_1}{\sim} \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} = C$, $A \underset{2R_1}{\sim} \begin{pmatrix} 2 & 4 \\ 1 & 4 \end{pmatrix} = D$

⇒ elementary row operation only impact determinant of matrix if there is a scalar multiplication of a row or interchanging row.

⇒ can compute determinant of matrix by putting it in echelon form (not REF) without using scalar multiplication of a row.

⇒ echelon form matrix is triangular

⇒ easy to compute its determinant.

⚠ if interchange row, need to multiply by -1 to get determinant of original matrix.

Theorem

Let A be a $n\times n$ matrix. We have:

ⓐ If a multiple of one row of A is added to another row to produce B, then $\det B = \det A$.

ⓑ If two rows of A are interchanged to produce B, then $\det B = -\det A$.

ⓒ If one row of A is multiplied by scalar k to produce B, then $\det B = k \det A$

ⓓ If A has a row (or column) of zeros, then $\det A = 0$.

Example:

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}$. Compute $\det A$.

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} \underset{\substack{R_2-4R_1 \\ R_3-7R_1}}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{vmatrix} \underset{R_3-2R_2}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{vmatrix} = (1)(-3)(1) = -3$$

other way (one way (triangular)

$$\underset{-\frac{1}{3}R_2}{=} -3 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -11 \end{vmatrix}$$

$$\underset{R_3+6R_2}{=} -3 \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = (-3)(1)(1)(1) = -3$$

(triangular)

Theorem:

A square matrix A is invertible if and only if $\det A \neq 0$

Theorem:

If A is a square matrix then $\det A = \det(A^T)$

$\Longrightarrow$ to compute $\det A$, we can also perform column operations (equivalent to row operations on A^T).

⚠ DO NOT perform row and column operation at the same step ⚠

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}$$

$$\Rightarrow \det A = \begin{array}{c} \\ R_2 - 4R_1 \\ R_3 - 7R_1 \end{array} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{vmatrix} = (1)\overbrace{(-1)^{1+1}}^{1} \begin{vmatrix} -3 & -6 \\ -6 & -11 \end{vmatrix}$$

expansion

$$\underset{-\frac{1}{3}C_1}{=} (-3) \begin{vmatrix} 1 & -6 \\ 2 & -11 \end{vmatrix}$$

$$= (-3)\left((1)(-11) - (2)(-6)\right)$$

$$= (-3)(1)$$

$$= -3 \quad \text{(same result than before)}$$

Property:

If $k \in \mathbb{R}$ and A is a $n \times n$ matrix then:

$$\det(kA) = k^n \det(A)$$

Theorem:

① If A, B are $n \times n$ matrices then:

$$\det(AB) = \det(A)\det(B)$$

② If A is an invertible square matrix then:

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Remarks:

⊛ If A is $m \times n$ and B is $n \times m$ matrix then AB is $m \times m$ matrix so $\det(AB)$ makes sense but $\det(A)$ and $\det(B)$ are NOT defined.

⊛ Part 2 of above theorem: use $\det(A^{-1}A) = \det(I_n) = 1^n = 1$

⊛ In general $\det(A+B) \neq \det A + \det B$.

Example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\Rightarrow$ $\det A = (1)(0) = 0$

$\det B = (0)(1) = 0$

$\det(A+B) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = (1)(1) = 1 \neq 0$

Example:

Let A, B, C be $n\times n$ matrices such that:

$$\det A = 3, \quad \det B = -2, \quad \det C = 4.$$

Compute $\det\left(A^2 B^T C^{-1} B^2 (A^{-1})^2\right)$

$$\Rightarrow \det\left(A^2 B^T C^{-1} B^2 (A^{-1})^2\right) = \det(A^2)\det(B^T)\det(C^{-1})\det(B^2)\det\left((A^{-1})^2\right)$$

$$= (\det A)^2 \det(B) \frac{1}{\det C} (\det B)^2 (\det A^{-1})^2$$

$\det A^2 = \det(AA)$
$= \det A \det A$
$= (\det A)^2$

$$= \frac{\det(B)^3}{\det C} (\det A)^2 \left(\frac{1}{\det A}\right)^2$$

$$= \frac{(-2)^3}{4} = -2$$

Example:

Let $A = \begin{pmatrix} x & x & 1 \\ x & 1 & x \\ 1 & x & x \end{pmatrix}$ with x a scalar.

For what values of x is A singular (i.e $\det A = 0$)?

$$\det A = \begin{vmatrix} x & x & 1 \\ x & 1 & x \\ 1 & x & x \end{vmatrix} \underset{\substack{R_1 - xR_3 \\ R_2 - xR_3}}{=} \begin{vmatrix} 0 & x(1-x) & 1-x^2 \\ 0 & 1-x^2 & x(1-x) \\ 1 & x & x \end{vmatrix} = \underbrace{(1)(-1)^{3+1}}_{1} \begin{vmatrix} x(1-x) & 1-x^2 \\ 1-x^2 & x(1-x) \end{vmatrix}$$

expand

$$\Rightarrow \det A = x(1-x)x(1-x) - (1-x^2)(1-x^2)$$
$$= x^2(1-x)^2 - (1-x^2)^2$$
$$= x^2(1-\cancel{x})^2 - \left((1-x)(1+x)\right)^2$$
$$= (1-x)^2\left(x^2 - (1+x)^2\right)$$
$$= (1-x)^2(x^2 - 1 - 2x - x^2)$$
$$= (1-x)^2(-1-2x)$$

$\det A = 0 \iff (1-x)^2 = 0$ or $1+2x = 0$

$\iff x = 1$ or $x = -\frac{1}{2}$

3) Cramer's rule, volume and linear transformation

Notation / Definition

For any $n\times n$ matrix A and any vector $\vec{b}$ in $\mathbb{R}^n$, the matrix $A_i(\vec{b})$ is the $n\times n$ matrix obtained from A by replacing its ith column by $\vec{b}$.

$$A = (\vec{a}_1 \cdots \vec{a}_n) \longrightarrow A_i(\vec{b}) = (\vec{a}_1 \cdots \underset{\uparrow\, i\text{-th column}}{\vec{b}} \cdots \vec{a}_n)$$

Theorem (Cramer's rule)

Let A be a $n\times n$ invertible matrix. For any $\vec{b}$ in $\mathbb{R}^n$, the unique solution $\vec{x}$ of $A\vec{x} = \vec{b}$ has entries given by:

$$x_i = \frac{\det A_i(\vec{b})}{\det A} \quad \text{for } i = 1, \dots, n$$

Example

Solve using Cramer's rule the system:

$$A\vec{x} = \begin{pmatrix} 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 0 & 2 & 0 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x_i = \frac{\det(A_i(\vec{b}))}{\det(A)} \quad \text{for } i = 1,2,3,4 \text{ with } \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

⊛ Compute $\det(A)$:

$$\det(A) = \begin{vmatrix} 2 & 3 & \boxed{0} & 1 \\ 0 & 0 & \boxed{0} & 3 \\ 7 & 9 & \boxed{-2} & 4 \\ 0 & 2 & \boxed{0} & 4 \end{vmatrix} = (-2)(-1)^{3+3} \begin{vmatrix} \boxed{2} & 3 & 1 \\ \boxed{0} & 0 & 3 \\ \boxed{0} & 2 & 4 \end{vmatrix}$$

expansion

expansion

$$= (-2)(2)(-1)^{1+1} \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix}$$

$$= -4\left((0)(4) - (3)(2)\right)$$

$$= -4(-6)$$

so $\det(A) = 24$ $\quad (\neq 0 \Rightarrow A \text{ invertible} \Rightarrow \text{unique } \vec{x})$

⊛ Compute $\det A_1(\vec{b})$

$$\det(A_1(\vec{b})) = \begin{vmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 3 \\ 1 & 9 & -2 & 4 \\ 0 & 2 & 0 & 4 \end{vmatrix} = (-2)(-1)^{3+3} \begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 3 \\ 0 & 2 & 4 \end{vmatrix}$$

$$= (-2)(1)(-1)^{1+1} \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = (-2)(-6) = 12$$

$$\Rightarrow x_1 = \frac{\det A_1(\vec{b})}{\det A} = \frac{12}{24} = \frac{1}{2}$$

(*) Compute $\det(A_2(\vec{b}))$

$$\det(A_2(\vec{b})) = \begin{vmatrix} 2 & 1 & 0 & 1 \\ \boxed{0 \quad 0 \quad 0 \quad 3} \\ 7 & 1 & -2 & 4 \\ 0 & 0 & 0 & 4 \end{vmatrix} = 3(-1)^{2+4} \begin{vmatrix} 2 & 1 & 0 \\ 7 & 1 & -2 \\ \boxed{0 \quad 0 \quad 0} \end{vmatrix}$$

$$= 0$$

$$\Rightarrow x_2 = \frac{\det A_2(\vec{b})}{\det A} = 0$$

(*) Compute $\det A_3(\vec{b})$

$$\det(A_3(\vec{b})) = \begin{vmatrix} 2 & 3 & 1 & 1 \\ \boxed{0 \quad 0 \quad 0 \quad 3} \\ 7 & 9 & 1 & 4 \\ 0 & 2 & 0 & 4 \end{vmatrix} = (3)(-1)^{2+4} \begin{vmatrix} 2 & 3 & 1 \\ 7 & 9 & 1 \\ \boxed{0 \quad 2 \quad 0} \end{vmatrix}$$

$$= (3)(2)(-1)^{3+2} \begin{vmatrix} 2 & 1 \\ 7 & 1 \end{vmatrix}$$

$$= (-6)(2-7)$$

$$= (-6)(-5)$$

$$= 30$$

$$\Rightarrow x_3 = \frac{\det A_3(\vec{b})}{\det A} = \frac{30}{24} = \frac{15}{12} = \frac{5}{4}$$

(*) Compute $\det A_4(\vec{b})$

$$\det A_4(\vec{b}) = \begin{vmatrix} 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 7 & 9 & -2 & 1 \\ 0 & 2 & 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad x_4 = \frac{\det A_4(\vec{b})}{\det A} = \frac{0}{24} = 0$$

<u>Conclusion:</u> $\vec{x} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{5}{4} \\ 0 \end{pmatrix}$ is the unique solution of the above system

<u>Formula for A^{-1} using determinant</u>

Let A be a $n \times n$ invertible matrix.

Let $B = (\vec{b}_1 \cdots \vec{b}_n)$ be the inverse of A $(B = A^{-1})$.

$$\Rightarrow \quad AB = (A\vec{b}_1 \cdots A\vec{b}_n) = I_n = (\vec{e}_1 \cdots \vec{e}_n)$$

Thus $A\vec{b}_j = \vec{e}_j$ for $j = 1, 2, \cdots, n$

Using Cramer's rule, we obtain that:

$$i\text{-th entry of } \vec{b}_j = \frac{\det A_i(e_j)}{\det A}$$

Notice that

$$A_i(e_j) = \begin{array}{cc} \begin{array}{cccccc} & & \text{i-th column} & & & \\ \end{array} & \\ \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & a_{1n} \\ \vdots & \vdots & | & & \\ a_{j1} & a_{j2} & 1 & \cdots & a_{jn} \\ \vdots & \vdots & 0 & & \\ & & | & & \\ a_{n1} & a_{n2} & 0 & \cdots & a_{nn} \end{pmatrix} & \begin{array}{c} \\ \\ \leftarrow j\text{-th row} \\ \\ \\ \\ \end{array} \end{array}$$

$\Rightarrow$ To compute $\det A_i(\vec{e}_j)$, we can do a cofactor expansion along column i. It reads:

$$\det(A_i(\vec{e}_j)) = (1)(-1)^{j+i} \det A_{ji}$$
$$= (-1)^{i+j} \det A_{ji}$$
$$= C_{ji} \qquad (j\text{-}i \text{ cofactor})$$

! index order swap

$\Rightarrow$ (i,j) entry of A^{-1} = i-th entry of $\vec{b}_i$ = $\dfrac{C_{ji}}{\det A}$

Definition:

Let A be a $n \times n$ matrix. The cofactor matrix C of A is the $n \times n$ matrix whose (i,j) entry is the (i,j) cofactor of A, i.e,

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

The adjugate of A, denoted by $\mathrm{adj}(A)$, is defined as $\mathrm{adj}(A) = C^T$

Theorem (inverse formula)

Let A be an invertible $n \times n$ matrix. Then,

$$A^{-1} = \frac{1}{\det A} \mathrm{adj}(A) \quad \left(= \frac{1}{\det A} C^T \right)$$

Example:

Find A^{-1} using the inverse formula.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & 2 \end{pmatrix}$$

⚠ $A^{-1} = (\det A)^{-1} \operatorname{adj}(A)$ → need to compute $\det A$ and $\operatorname{adj}(A)$.

⊛ Compute $\det(A)$

$$\det(A) = \begin{vmatrix} \boxed{3 \quad 0 \quad 0} \\ -1 \quad 1 \quad 0 \\ -2 \quad 3 \quad 2 \end{vmatrix} = 3(-1)^{1+1} \begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = (3)(1)(2) = 6$$

↑ triangular → $\det = (1)(2)$

⊛ Compute $\operatorname{adj}(A) = C^T$ → need $C_{11}, \dots, C_{33}$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = 2$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} -1 & 0 \\ -2 & 2 \end{vmatrix} = (-1)(-1)(2) = 2$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix} = \big((-1)(3) - (1)(-2)\big) = -3 + 2 = -1$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 0 \\ 3 & 2 \end{vmatrix} = 0$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 3 & 0 \\ -2 & 2 \end{vmatrix} = (1)(3)(2) = 6$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 0 \\ -2 & 3 \end{vmatrix} = -9$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 3 & 0 \\ -1 & 0 \end{vmatrix} = 0$$

$$C_{33} = (-1)^{3+3} \begin{pmatrix} 3 & 0 \\ -1 & 1 \end{pmatrix} = 3$$

$$\Rightarrow C = \begin{pmatrix} 2 & 2 & -1 \\ 0 & 6 & -9 \\ 0 & 0 & 3 \end{pmatrix} \Rightarrow Adj(A) = C^T = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ -1 & -9 & 3 \end{pmatrix}$$

Conclusion: $A^{-1} = \frac{1}{\det(A)} Adj(A) = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{1}{6} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$

⚠ do not forget to do transpose of C at the end to get Adj(A)!

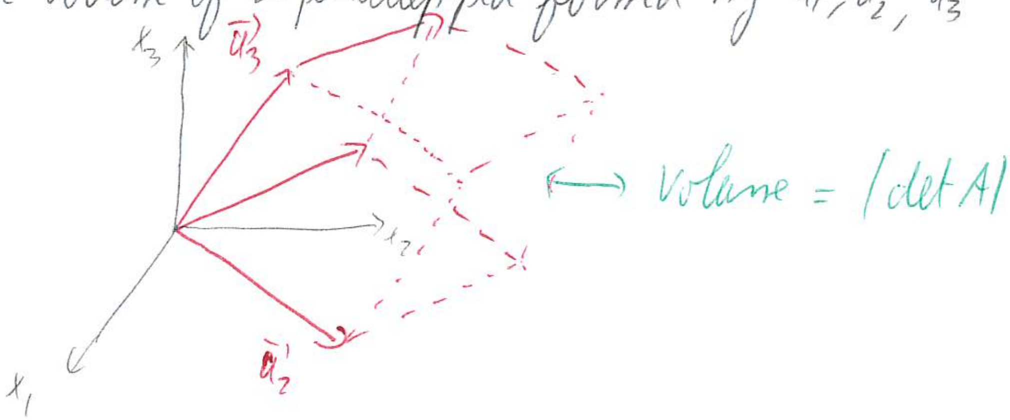
Area and Volume

<u>Theorem:</u> Let $A = (\vec{a}_1\ \vec{a}_2)$ be a 2×2 matrix.

The area of the parallelogram formed by $\vec{a}_1, \vec{a}_2$ equals $|\det A|$

<u>Theorem:</u> Let $A = (\vec{a}_1\ \vec{a}_2\ \vec{a}_3)$ be a 3×3 matrix.

The volume of the parallepiped formed by $\vec{a}_1, \vec{a}_2, \vec{a}_3$ equals $|\det A|$



⚠ absolute value

<u>Example:</u> Find the area of the parallelogram with vertices at points $(0,0)$, $(5,2)$, $(6,4)$, $(11,6)$

$\Rightarrow \vec{a}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad \vec{a}_2 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

$$\Rightarrow \text{area} = \det(\vec{a}_1\ \vec{a}_2) = \det\begin{pmatrix} 5 & 6 \\ 2 & 4 \end{pmatrix} = (5)(4) - (6)(2)$$
$$= 20 - 12$$
$$= 8$$