Consequences:

$\Rightarrow$ Two vectors cannot span $\mathbb{R}^3$

$\Rightarrow$ Less (strictly) than n vectors in $\mathbb{R}^n$ cannot span $\mathbb{R}^n$.

Example:

Do the columns of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 9 \end{pmatrix}$ span $\mathbb{R}^3$?

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 9 \end{pmatrix} \underset{R_2-2R_1}{\sim} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 3 & 9 \end{pmatrix} \underset{R_2 \leftrightarrow R_3}{\sim} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 9 \\ 0 & 0 & 0 \end{pmatrix}$$ echelon form

number of pivot $= 2 <$ number of rows $= 3$

$\Rightarrow$ columns of A don't span $\mathbb{R}^3$.

⚠ We saw that less than 3 vectors cannot span $\mathbb{R}^3$, however not any sets of three vectors can span $\mathbb{R}^3$. We need something more for that $\Rightarrow$ more information with section 1.7 later.

5) Solution sets of linear system

Homogeneous Linear Systems

Definition: A linear system is called homogeneous if it can be written in the form $A\vec{x} = \vec{0}$, i.e the right hand side is zero.

Remark: Homogeneous system always have at least one solution, $\vec{x} = \vec{0}$, called the trivial solution.
Some homogeneous systems have nontrivial solution ($\neq \vec{0}$)

Property:
The homogeneous equation $A\vec{x} = \vec{0}$ has nontrivial solutions if and only if the equation has at least one free variable.

Examples:

1) Determine if the following linear system has nontrivial solutions.

$$\begin{cases} x_1 - 2x_2 = 0 \\ 3x_1 - 6x_2 = 0 \end{cases}$$

$$\begin{pmatrix} 1 & -2 & 0 \\ 3 & -6 & 0 \end{pmatrix} \underset{R_2 - 3_1}{\sim} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

basic variable x_1 ($= 2x_2$)
free variable x_2

$\Rightarrow$ one free variable
$\Rightarrow$ infinite number of solutions: $x_1 = 2x_2$ and x_2 free.

2) Determine if the following homogeneous system has nontrivial solutions and describe the solution set.

$$\begin{cases} 2x_1 + 4x_2 - 6x_3 = 0 \\ 4x_1 + 8x_2 - 10x_3 = 0 \end{cases}$$

(2 rows, 3 columns $\Rightarrow$ at least one free variable)

$$A = \begin{pmatrix} 2 & 4 & -6 \\ 4 & 8 & -10 \end{pmatrix} \underset{R_1/2}{\sim} \begin{pmatrix} 1 & 2 & -3 \\ 4 & 8 & -10 \end{pmatrix} \underset{R_2 - 4R_1}{\sim} \begin{pmatrix} 1 & 2 & -3 \\ 0 & 0 & 2 \end{pmatrix}$$

do not need to add $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as it won't change.

$$\Rightarrow A \underset{R_2/2}{\sim} \begin{pmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \end{pmatrix} \overset{R_1+3R_2}{\sim} \begin{array}{c} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \end{array}$$

→ basic variable x_1, x_3
free variable x_2.

→ $\begin{cases} x_1 + 2x_2 = 0 \\ x_3 = 0 \end{cases}$ ⟶ $\left.\begin{cases} x_1 = -2x_2 \\ x_3 = 0 \\ (x_2 \text{ free}) \end{cases}\right\}$ parametric form of the solution set.

Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \vec{x} = \begin{pmatrix} -2x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = x_2 \vec{v}$ } Vector parametric form of the solution set.

with $\vec{v} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

⚠ solution set = span$\{\vec{v}\}$

Nonhomogeneous Linear Systems

Example: Describe the solution set of

$$\begin{cases} 2x_1 + 4x_2 - 6x_3 = 0 \\ 4x_1 + 8x_2 - 10x_3 = 4 \end{cases}$$

$$\begin{pmatrix} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 4 \end{pmatrix} \underset{R_1/2}{\sim} \begin{pmatrix} 1 & 2 & -3 & 0 \\ 4 & 8 & -10 & 4 \end{pmatrix} \underset{R_2-4R_1}{\sim} \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 2 & 4 \end{pmatrix}$$

$$\underset{R_2/2}{\sim} \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \underset{R_1+3R_2}{\sim} \begin{pmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$\Leftrightarrow \begin{cases} x_1 + 2x_2 = 6 \\ x_3 = 2 \end{cases}$

basic variable: x_1, x_3
free variable: x_2

$$\begin{cases} x_1 = 6 - 2x_2 \\ x_2 \text{ free} \\ x_3 = 2 \end{cases}$$ parametric form

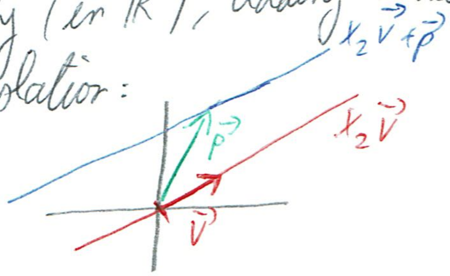
$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 - 2x_2 \\ x_2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \vec{p} + x_2 \vec{v}$$ Vector parametric form.

where $\vec{p} = \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

Notice that $\vec{p}$ is just a particular solution of the equation obtained by setting $x_2 = 0$.

Geometrically (in $\mathbb{R}^2$), adding a vector $\vec{p}$ to a solution set $x_2\vec{v}$ acts as a translation:



<u>Theorem:</u>
Suppose $A\vec{x} = \vec{b}$ is consistent for some $\vec{b}$, and let $\vec{p}$ be a solution. Then the solution set of $A\vec{x} = \vec{b}$ is the set of vectors of the form:

$$\vec{y} = \vec{p} + \vec{x}_h$$

where $\vec{x}_h$ is any solution of $A\vec{x} = \vec{0}$.

solution set nonhomogeneous = particular solution + solution set of homogeneous eq

<u>Example:</u> Let $A = \begin{pmatrix} 1 & 7 & 5 \\ 1 & 0 & 5 \\ -1 & 2 & -5 \end{pmatrix}$

(a) Solve $A\vec{x} = \vec{0}$

(b) Solve $A\vec{x} = \vec{b}$ with $\vec{b} = \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}$ without performing any EROs.

(a) $\begin{pmatrix} 1 & 7 & 5 \\ 1 & 0 & 5 \\ -1 & 2 & -5 \end{pmatrix} \underset{\substack{R_2 - R_1 \\ R_3 + R_1}}{\sim} \begin{pmatrix} 1 & 7 & 5 \\ 0 & -7 & 0 \\ 0 & 9 & 0 \end{pmatrix} \underset{R_2/-7}{\sim} \begin{pmatrix} 1 & 7 & 5 \\ 0 & 1 & 0 \\ 0 & 9 & 0 \end{pmatrix} \underset{\substack{R_1 - 7R_2 \\ R_3 - 9R_2}}{\sim} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(RHS = 0 so not needed)

Basic variable: x_1, x_2
Free variable: x_3

and $\begin{cases} x_1 + 5x_3 = 0 \\ x_2 = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -5x_3 \\ x_2 = 0 \\ x_3 \text{ free} \end{cases}$

or $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} = x_3 \vec{v}$

(b) We know solutions are of the form $\vec{p} + x_3\vec{v}$ where $\vec{p}$ is a particular solution that can be found by setting $x_3 = 0$. It reads

$$\begin{cases} x_1 + 7x_2 = 5 \\ x_1 = -2 \\ -x_1 + 2x_2 = 4 \end{cases}$$

Plug $x_1 = -2$ in first equation gives: $7x_2 = 5 - x_1 = 7 \Rightarrow x_2 = 1$.

(notice that $-x_1 + 2x_2 = 2 + 2 = 4$) $\Rightarrow \vec{p} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow \vec{x} = x_3 \vec{v} + \vec{p}$