1. Let $A = \begin{bmatrix} 1 & -1 & 1 & 3 \\ -1 & 1 & 0 & -2 \\ 3 & -3 & 1 & 7 \\ 2 & -2 & 1 & 5 \end{bmatrix}$.

   a) (1 points) Remind the definition of Null(A), i.e. the null space of A.

   b) (3 points) Find a basis for Null(A).

   c) (1 points) Using the rank theorem, determine the dimension of Col(A), i.e. the column space of A.

Solution.

a) A is a 4x4 matrix so we have $Null(A) = \{\vec{x} \in \mathbb{R}^4 \mid A\vec{x} = \vec{0}\}$.

b) First, we row reduce the matrix A. It reads

$$\begin{bmatrix} 1 & -1 & 1 & 3 \\ -1 & 1 & 0 & -2 \\ 3 & -3 & 1 & 7 \\ 2 & -2 & 1 & 5 \end{bmatrix} \begin{matrix} \sim \\ R_2 + R_1 \\ R_3 - 3R_1 \\ R_4 - 2R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -1 & -1 \end{bmatrix} \begin{matrix} \sim \\ R_1 - R_2 \\ R_3 + 2R_2 \\ R_4 + R2 \end{matrix} \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The vector parametric form of the solutions are given by:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 + -2x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Thus $B = \{\vec{v_1}, \vec{v_2}\}$ is a basis for Null(A) where $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

c) The basis of Null(A) from b) has two vectors, thus dim(Null(A)) = 2. Using the rank theorem, we have:

$$dim(Col(A)) + dim(Null(A)) = 4.$$

Thus $dim(Col(A)) = 4 - 2 = 2$.

2. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 0 \\ 2 & -3 & 2 \end{bmatrix}$.

   a) Find A^{-1}.

   b) Use A^{-1} to solve the matrix equation $A\vec{x} = \vec{b}$, where $\vec{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

Solution.

a)

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array}\right] \underset{\substack{R_2 - R_1 \\ R_3 - 2R_1}}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right] \underset{-R_2}{\sim}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right] \underset{R_3 + R_2}{\overset{R_1 + R_2}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -1 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \overset{\substack{R_1 - 2R_3 \\ R_2 - R_3}}{\sim}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 1 & -2 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

Therefore, $A^{-1} = \begin{bmatrix} 4 & 1 & -2 \\ 2 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix}$.

b)

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 4 & 1 & -2 \\ 2 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ 4 \end{bmatrix}.$$

3. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 4 & 0 \\ -3 & 2 & 4 \end{bmatrix}$.

a) Find $\det A$ using any method from class. If you use a cofactor expansion, clearly indicate which row/column you are using.

b) Let B be a 3×3 invertible matrix such that $\det B = 2$. Compute

$$\det\left(\frac{1}{16}B\left(\operatorname{adj}(A)\right)\right).$$

Hint: recall that $\operatorname{adj}(A) = (\det A)A^{-1}$. You do not need to compute A^{-1}, simply use the properties of determinant.

Solution.

a) We first do row operations, and then do a cofactor expansion along the third column

$$\det A = \begin{vmatrix} 1 & -1 & 2 \\ -1 & 4 & 0 \\ -3 & 2 & 4 \end{vmatrix} \underset{R_3 - 2R_1}{=} \begin{vmatrix} 1 & -1 & 2 \\ -1 & 4 & 0 \\ -5 & 4 & 0 \end{vmatrix} = 2(-1)^{1+3} \begin{vmatrix} -1 & 4 \\ -5 & 4 \end{vmatrix}$$
$$= 2((-1)(4) - 4(-5)) = 2(-4 + 20) = 32.$$

b) Using properties of determinant, we have

$$\det\left(\frac{1}{16}B\left(\operatorname{adj}(A)\right)\right) = \det\left(\frac{1}{16}B(\det A)A^{-1}\right) = \left(\frac{\det A}{16}\right)^3 \det\left(BA^{-1}\right)$$
$$= 2^3\left(\det B\right)\left(\det(A^{-1})\right) = 8\frac{\det B}{\det A}$$
$$= 8\frac{2}{32} = \frac{16}{32} = \frac{1}{2}$$

4. Let $A = \begin{bmatrix} a-3 & a \\ a-1 & 2a-3 \end{bmatrix}$.

   a) Calculate $A + 3I_2$.

   b) Find all the values of a so that $A + 3I_2$ is singular.

Solution.

a)
$$A + 3I_2 = \begin{bmatrix} a-3 & a \\ a-1 & 2a-3 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a \\ a-1 & 2a \end{bmatrix}$$

b) $A + 3I_2$ is singular if and only if its determinant is zero. We first compute the determinant:

$$\det(A+3I_2) = \begin{vmatrix} a & a \\ a-1 & 2a \end{vmatrix} = a(2a)-a(a-1) = a(2a-(a-1)) = a(2a-a+1) = a(a+1)$$

Therefore, $A + 3I_2$ is singular if and only if $a(a+1) = 0$, i.e., $a = 0$ or $a = -1$.

5. For each of the following, determine if the statement is true or false.

   a) $\det\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = 0$.

   b) For any two matrices A and B, if $AB = I_n$ for some positive integer n, then A is invertible.

   c) If A is a square invertible matrix, the columns of A^T are linearly independent.

Solution.

a) FALSE. This matrix is not square.

b) FALSE. A and B might not be square. For example, take $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $AB = I_2$, but A and B are not square so they cannot be considered invertible.

c) TRUE. If A is invertible, so is A^T, so its columns must be linearly independent (using the invertible matrix theorem).

Bonus. Let C be an $n \times n$ matrix such that $C^3 = C^T$. Find all possible values for $\det(C)$. *Hint*: apply determinant on both sides.

Solution. Following the hint, we apply determinant on both sides and use the properties of determinant:

$$\begin{aligned} \det\left(C^3\right) &= \det\left(C^T\right) \\ (\det C)^3 &= \det C \\ (\det C)^3 - \det C &= 0 \\ (\det C)((\det C)^2 - 1) &= 0 \end{aligned}$$

So, either $\det C = 0$ or $(\det C)^2 = 1$, i.e., $\det C = 0, -1$ or 1.