5 points

1. Let $A = \begin{bmatrix} -3 & 2 & 4 \\ 1 & -1 & 2 \\ -1 & 4 & 0 \end{bmatrix}$.

   a) Compute det$A$ by using any of the algorithms we saw in class.

   b) Let B be a 3×3 matrix such that det$B = 2$. Compute $\det(4BA^{-1})$.

Solution.

a)

$$\det A = \begin{vmatrix} -3 & 2 & 4 \\ 1 & -1 & 2 \\ -1 & 4 & 0 \end{vmatrix} \overset{R_1 \leftrightarrow R_2}{=} -\begin{vmatrix} 1 & -1 & 2 \\ -3 & 2 & 4 \\ -1 & 4 & 0 \end{vmatrix} \underset{\substack{R_2 + 3R_1 \\ R_3 + R_1}}{=} -\begin{vmatrix} 1 & -1 & 2 \\ 0 & -1 & 10 \\ 0 & 3 & 2 \end{vmatrix} \underset{R_3 + 3R_2}{=}$$

$$= -\begin{vmatrix} 1 & -1 & 2 \\ 0 & -1 & 10 \\ 0 & 0 & 32 \end{vmatrix} = -(1)(-1)(32) = 32.$$

b) $\det(4BA^{-1}) = 4^3(\det B)(\det(A^{-1})) = 4^3(2)\frac{1}{32} = \frac{4^3}{16} = 4.$

4 points

2. Let $A = \begin{bmatrix} -1 & c-1 & 1-c \\ -c-2 & 2c-3 & 4-c \\ -c-2 & c-1 & 2 \end{bmatrix}$. Use a determinant to find all the values of c for which A is not invertible. Use any algorithm from class.

Solution. A is singular (not invertible) if and only if $\det A = 0$.

$$\det A = \begin{vmatrix} -1 & c-1 & 1-c \\ -c-2 & 2c-3 & 4-c \\ -c-2 & c-1 & 2 \end{vmatrix} \overset{-R_1}{=} -\begin{vmatrix} 1 & 1-c & c-1 \\ -c-2 & 2c-3 & 4-c \\ -c-2 & c-1 & 2 \end{vmatrix} \underset{\substack{R_2 + (c+2)R_1 \\ R_3 + (c+2)R_1}}{=}$$

$$= -\begin{vmatrix} 1 & 1-c & c-1 \\ 0 & -c^2+c-1 & c^2+2 \\ 0 & 1-c^2 & c^2+c \end{vmatrix}$$

Doing a cofactor expansion along column 1:

$$\det A = -1(-1)^{1+1}\begin{vmatrix} -c^2+c-1 & c^2+2 \\ 1-c^2 & c^2+2 \end{vmatrix} = -((-c^2+c-1)(c^2+c) - (c^2+2)(1-c^2))$$

$$= c(c^2-c+1)(c+1) + (c^2+2)(1-c)(1+c) = (c+1)(c^3-c^2+c+c^2-c^3+2-2c) = (c+1)(2-c)$$

Therefore A is not invertible when $c = -1$ or $c = 2$.

3 points

3. Let A be an $n \times n$ matrix such that $A^TA = I_n$. Show that $\det(A) = \pm 1$.

Solution. Note that since A is $n \times n$, so is A^T. Therefore, we can use the property $\det(AA^T) = (\det A)(\det(A^T))$. Applying determinant to $AA^T = I_n$:

$$\begin{aligned} \det(AA^T) &= \det(I_n) \\ (\det A)(\det(A^T)) &= 1 \\ (\det A)^2 &= 1 \\ \det A &= \pm\sqrt{1} \\ \det A &= \pm 1. \end{aligned}$$

4 points

4. For each of the following, determine if the statement is true or false. Provide a short reasoning (one or two sentences).

a) If the determinant of a square matrix is zero, then the matrix has either one row or column of zeros.

b) Let A be an $n \times n$ matrix, and let $\vec{b}$ be a given vector in $\mathbb{R}^n$. If the system $A\vec{x} = \vec{b}$ is consistent, then $\det A \neq 0$.

c) If the columns of a square matrix A are linearly dependent, then $\det A = 0$.

d) If A is a square matrix whose diagonal entries are all zero, then $\det A = 0$.

Solution.

a) FALSE. For example, we have $\det\left(\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}\right) = -4 + 4 = 0$ but the matrix does not have any row or column of zeros.

b) FALSE. Consider $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. The system $A\vec{x} = \vec{b}$ is consistent, but $\det A = 0$.

c) TRUE. This is a result of the invertible matrix theorem.

d) FALSE. $\det\left(\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}\right) = 2$ and the diagonal entries of this matrix are all zero.