$\boxed{\text{5 points}}$ 1. Use the adjugate to compute the inverse of A.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 4 \end{bmatrix}$$

Solution.

We compute the cofactors of A:

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix} = 8 - 0 = 8$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} = -(0 - 2) = 2$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = 0 - 4 = -4$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 2 \\ 0 & 4 \end{vmatrix} = -(-4 - 0) = 4$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -(0 + 2) = -2$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = -1 - 4 = -5$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -(1 - 0) = -1$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$$

Therefore, the cofactor matrix of A is

$$C = \begin{bmatrix} 8 & 2 & -4 \\ 4 & 0 & -2 \\ -5 & -1 & 2 \end{bmatrix}$$

The adjugate matrix of A is

$$\text{adj}A = C^T = \begin{bmatrix} 8 & 4 & -5 \\ 2 & 0 & -1 \\ -4 & -2 & 2 \end{bmatrix}$$

To compute the determinant of A, we do a cofactor expansion along the first column:

$$\det A = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = 1(8) + 0(4) + 2(-5) = 8 - 10 = -2.$$

Finally, the inverse of A is given by

$$A^{-1} = \frac{1}{\det A}\text{adj}A = -\frac{1}{2}\begin{bmatrix} 8 & 4 & -5 \\ 2 & 0 & -1 \\ -4 & -2 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -2 & 5/2 \\ -1 & 0 & 1/2 \\ 2 & 1 & -1 \end{bmatrix}.$$

3 points

2. Show that $H = \left\{ \begin{bmatrix} a & 2a \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a subspace of $M_{2\times 2}(\mathbb{R})$. *Hint*: try to write H as the span of a set of matrices.

Solution. Note that any element $A \in H$ can be written as

$$A = \begin{bmatrix} a & 2a \\ 0 & b \end{bmatrix} = a\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$H = \text{Span}\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Since H is a span of a set of 2×2 matrices, H is a subspace of $M_{2\times 2}(\mathbb{R})$.

3 points

3. For each of the following, determine if the statement is true or false. Provide a short reasoning (one or two sentences).

a) The set of invertible 2×2 matrices is a subspace of $M_{2\times 2}(\mathbb{R})$.

b) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

c) A set containing a finite number of vectors in $\mathbb{R}^n$ cannot be a subspace of $\mathbb{R}^n$.

Solution.

a) FALSE. The zero 2×2 matrix is not invertible, so this set fails the first property of subspaces.

b) FALSE. $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$, so it cannot be a subspace of $\mathbb{R}^3$.

c) FALSE. The set $\{\vec{0}\}$ is a subspace of $\mathbb{R}^n$. However, for any other set containing a finite number of vectors, we can always find a scalar such that the scalar times any vector in the set will not be in the set.