1. Solve the following linear system

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & + & x_3 & + & 2x_4 & = & -2 \\ 3x_1 & - & 2x_2 & & & + & 4x_4 & = & 1 \\ x_1 & + & x_2 & - & 2x_3 & + & x_4 & = & 3 \\ 6x_1 & - & 4x_2 & + & 2x_3 & + & 2x_4 & = & 2 \end{array}$$

Solution.

We row reduce the augmented matrix:

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & 2 & -2 \\ 3 & -2 & 0 & 4 & 1 \\ 1 & 1 & -2 & 1 & 3 \\ 6 & -4 & 2 & 2 & 2 \end{array}\right] \begin{array}{c} \sim \\ R_2 - 3R_1 \\ R_3 - R_1 \\ R_4 - 6R_1 \end{array} \left[\begin{array}{rrrr|r} 1 & -1 & 1 & 2 & -2 \\ 0 & 1 & -3 & -2 & 7 \\ 0 & 2 & -3 & -1 & 5 \\ 0 & 2 & -4 & -10 & 14 \end{array}\right] \begin{array}{c} \sim \\ R_1 + R_2 \\ R_3 - 2R_2 \\ R_4 - 2R_2 \end{array} \left[\begin{array}{rrrr|r} 1 & 0 & -2 & 0 & 5 \\ 0 & 1 & -3 & -2 & 7 \\ 0 & 0 & 3 & 3 & -9 \\ 0 & 0 & 2 & -6 & 0 \end{array}\right]$$

$$\begin{array}{c} \sim \\ R_3/3 \\ R_4/2 \\ \end{array} \left[\begin{array}{rrrr|r} 1 & 0 & -2 & 0 & 5 \\ 0 & 1 & -3 & -2 & 7 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & -3 & 0 \end{array}\right] \begin{array}{c} \sim \\ R_1 + 2R_3 \\ R_2 + 3R_3 \\ R_4 - R_3 \end{array} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & -4 & 3 \end{array}\right] \begin{array}{c} \sim \\ \\ \\ -\frac{1}{4}R_4 \end{array} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 1 & -3/4 \end{array}\right]$$

$$\begin{array}{c} \sim \\ R_1 - 2R_4 \\ R_2 - R_4 \\ R_3 - R_4 \end{array} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & -5/4 \\ 0 & 0 & 1 & 0 & -9/4 \\ 0 & 0 & 0 & 1 & -3/4 \end{array}\right]$$

Therefore, the solution is $x_1 = 1/2$, $x_2 = -5/4$, $x_3 = -9/4$ and $x_4 = -3/4$.

2. Let

$$\left[\begin{array}{rrrrr|r} 1 & 0 & -1 & 0 & 0 & -2 \\ 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

be the reduced echelon form of the augmented matrix of a linear system $A\vec{x} = \vec{b}$.

a) (4 points) Circle the pivots in the matrix above, identify the basic variables, free variables (if any) and write the solution of the linear system in vector parametric form.

b) (1 point) What is the solution to the homogeneous system $A\vec{x} = \vec{0}$? Give your answer in vector parametric form.

Solution.

a) The pivots are circled below:

$$\left[\begin{array}{rrrrr|r} 1 & 0 & -1 & 0 & 0 & -2 \\ 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Basic variables: x_1, x_2, x_5.
Free variables: x_3, x_4.
The vector parametric form of the solution is

$$\vec{x} = \begin{bmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

b) The vector parametric form of the solution is

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

3. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1 \\ 7 \\ h \end{bmatrix}$.

a) (4 points) Find the value of h so that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

b) (3 points) For the value of h that you found in part a), find a linear dependence relation.

Solution.

a)

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 7 \\ -1 & -3 & h \end{bmatrix} \underset{\substack{R_2 - 2R_1 \\ R_3 + R_1}}{\sim} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 9 \\ 0 & -1 & h-1 \end{bmatrix} \underset{-1/3R_2}{\sim} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & -1 & h-1 \end{bmatrix} \underset{R_3 + R_2}{\sim}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & h-4 \end{bmatrix}$$

This is an echelon form. For the set to be linearly dependent, we require at least one free variable. Therefore, we need $h - 4 = 0 \Rightarrow h = 4$.

b) To obtain a linear dependence relation, we finish solving the system with $h = 4$. Note that we do not have to start from the beginning; we just have to continue with the row operations with $h = 4$:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1 - 2R_2}{\sim} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

From here we see that

$$\begin{aligned} x_1 &= -5x_3 \\ x_2 &= 3x_3 \\ x_3 &\ \text{is free.} \end{aligned}$$

Pick $x_3 = 1 \Rightarrow x_1 = -5$, $x_2 = 3$. A linear dependence relation is:

$$-5\vec{v}_1 + 3\vec{v}_2 + \vec{v}_3 = \vec{0}.$$

4. Let $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 2 \\ 2 & 0 & -2 \end{bmatrix}$, and define $T : \mathbb{R}^3 \to \mathbb{R}^3$ as $T(\vec{x}) = A\vec{x}$, for all $\vec{x} \in \mathbb{R}^3$. Find an $\vec{x}$ in $\mathbb{R}^3$ whose image under T is $\vec{b} = \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix}$. Is there more than one $\vec{x}$ in $\mathbb{R}^3$ whose image under T is $\vec{b}$? Explain.

Solution.

We must find $\vec{x}$ in $\mathbb{R}^3$ such that $T(\vec{x}) = A\vec{x} = \vec{b}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ -1 & 1 & 2 & -1 \\ 2 & 0 & -2 & 6 \end{array}\right] \quad \begin{array}{c} \sim \\ R_2 + R_1 \\ R_3 - 2R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is the REF. From here we see that

$$\begin{aligned} x_1 &= 3 + x_3 \\ x_2 &= 2 - x_3 \\ x_3 &\text{ is free.} \end{aligned}$$

Pick $x_3 = 0 \Rightarrow x_1 = 3$, $x_2 = 2$. Therefore, an $\vec{x}$ in $\mathbb{R}^3$ whose image under T is $\vec{b}$ is $\vec{x} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$. Since there is a free variable, there are infinitely many vectors in $\mathbb{R}^3$ whose image under T is $\vec{b}$.

5. For each of the following, determine if the statement is true or false. Provide a short reasoning (one or two sentences).

   a) The columns of any 4×5 matrix are linearly dependent.
   b) Let T be a linear transformation. If $T(\vec{x}) = T(\vec{y})$, then $\vec{x} = \vec{y}$.
   c) If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent, then $\{\vec{v}_1, \vec{v}_2\}$ is also linearly independent.

Solution.

   a) TRUE. At best, there are four pivots (one in every row), in which case there is one column without a pivot which leads to a free variable.
   b) FALSE. Take $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 \\ 2x_1 \end{bmatrix}$. We see that T is a linear transformation. Take $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then $T(\vec{x}) = T(\vec{y}) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, but $\vec{x} \neq \vec{y}$.
   c) TRUE. If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent, then none of the vectors can be written as a linear combination of the other ones. This implies that $\vec{v}_1$ and $\vec{v}_2$ are not a scalar multiple of each other. Therefore, $\{\vec{v}_1, \vec{v}_2\}$ is also linearly independent.

6. (BONUS) Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a linearly independent set of vectors in $\mathbb{R}^n$. Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_1 + \vec{v}_3\}$ is linearly independent.

Solution.

Let c_1, c_2, c_3 be scalars and consider

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(\vec{v}_1 + \vec{v}_3) = \vec{0}.$$

This is equivalent to

$$(c_1 + c_3)\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}.$$

Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent, the only possible solution to this equation is the trivial solution, i.e.,

$$c_1 + c_3 = 0, \quad c_2 = 0, \quad c_3 = 0.$$

Since $c_3 = 0$, we then have $c_1 = -c_3 = 0$. Therefore, $\{\vec{v}_1, \vec{v}_2, \vec{v}_1 + \vec{v}_3\}$ is linearly independent.