4 points 1. Let $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$. Construct a 2×2 nonzero matrix B such that AB is the zero matrix. *Hint*: Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Compute AB and make it equal to the zero matrix.

Solution. Let $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. Then $AB = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 3b_{11} - 6b_{21} & 3b_{12} - 6b_{22} \\ -b_{11} + 2b_{21} & -b_{12} + 2b_{22} \end{bmatrix}$.
Since we want $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, we have

$$3b_{11} - 6b_{21} = 0 \tag{1}$$
$$-b_{11} + 2b_{21} = 0 \tag{2}$$
$$3b_{12} - 6b_{22} = 0 \tag{3}$$
$$-b_{12} + 2b_{22} = 0 \tag{4}$$

From equation (2), we have $b_{11} = 2b_{21}$. Plug into equation (1): $6b_{21} - 6b_{21} = 0 \Rightarrow 0 = 0$. Thus, we can pick any b_{21}. For example, pick $b_{21} = 1$. Then, $b_{11} = 2$.

Similarly, from equation (4), we get $b_{12} = 2b_{22}$. Plug into equation (3): $6b_{22} - 6b_{22} = 0 \Rightarrow 0 = 0$. Thus we are free to pick b_{22}. Choose $b_{22} = -1 \Rightarrow b_{12} = -2$. Thus,

$$B = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}.$$

5 points 2. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$.

a) Find A^{-1} using the algorithm we saw in class.

b) Use A^{-1} to solve the linear system $A\vec{x} = \vec{b}$, where $\vec{b} = \begin{bmatrix} -5 \\ -3 \\ 2 \end{bmatrix}$.

Solution.

a)

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \underset{\substack{R_2 - 2R_1 \\ R_3 - 3R_1}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -3 & -2 & -3 & 0 & 1 \end{array}\right] \underset{-R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & -3 & -2 & -3 & 0 & 1 \end{array}\right]$$

$$\underset{R_3 + 3R_2}{\overset{R_1 - R_2}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3 & -3 & 1 \end{array}\right] \underset{R_2 - R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 3 & -3 & 1 \end{array}\right]$$

Thus, $A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 2 & -1 \\ 3 & -3 & 1 \end{bmatrix}$.

b) $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 2 & -1 \\ 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}$

4 points 3. Let A and B be $n \times n$ invertible matrices such that

$$(A^T + I_n)^{-1} = (BA^{-1})^T.$$

Find A^{-1}. *Note*: Your formula for A^{-1} should not depend on A or A^T. *Hint*: Try applying inverse and transpose on both sides. Then, manipulate the equation algebraically to obtain a matrix C such that $AC = I_n$.

Solution.

Solution 1:

$$\begin{aligned}
\text{Invert both sides} \quad (A^T + I_n)^{-1} &= (BA^{-1})^T \\
((A^T + I_n)^{-1})^{-1} &= ((BA^{-1})^T)^{-1} \\
A^T + I_n &= ((BA^{-1})^{-1})^T \\
A^T + I_n &= ((A^{-1})^{-1}B^{-1})^T \\
A^T + I_n &= (AB^{-1})^T \\
\text{Transpose both sides} \quad (A^T + I_n)^T &= ((AB^{-1})^T)^T \\
(A^T)^T + I_n^T &= AB^{-1} \\
A + I_n &= AB^{-1} \\
AB^{-1} - A &= I_n \\
A(B^{-1} - I_n) &= I_n \\
\Rightarrow A^{-1} &= B^{-1} - I_n.
\end{aligned}$$

Solution 2:

$$\begin{aligned}
\text{Evaluate the transpose on the RHS} \quad (A^T + I_n)^{-1} &= (BA^{-1})^T \\
(A^T + I_n)^{-1} &= (A^{-1})^T B^T \\
(A^T + I_n)^{-1} &= (A^T)^{-1} B^T \\
\text{Invert both sides} \quad ((A^T + I_n)^{-1})^{-1} &= ((A^T)^{-1} B^T)^{-1} \\
A^T + I_n &= (B^T)^{-1}((A^T)^{-1})^{-1} \\
A^T + I_n &= (B^T)^{-1} A^T \\
(B^T)^{-1} A^T - A^T &= I_n \\
((B^T)^{-1} - I_n) A^T &= I_n \\
\Rightarrow (A^T)^{-1} &= (B^T)^{-1} - I_n \\
\Rightarrow (A^{-1})^T &= (A^T)^{-1} = (B^{-1})^T - I_n \\
\Rightarrow A^{-1} &= ((A^{-1})^T)^T = ((B^{-1})^T - I_n)^T = ((B^{-1})^T)^T - I_n^T \\
\Rightarrow A^{-1} &= B^{-1} - I_n.
\end{aligned}$$

3 points

4. For each of the following, determine if the statement is true or false. Provide a short reasoning (one or two sentences).

   a) Let A be an $n \times n$ matrix, and $\vec{b}$ be a vector in $\mathbb{R}^n$. If the system $A\vec{x} = \vec{b}$ has infinitely many solutions, then A is not invertible.

   b) Let A be an invertible square matrix. If $AB = AC$, then $B = C$.

   c) If A and B are square matrices such that $AB = BA$, then $A^{-1} = B$.

Solution.

a) TRUE. A is invertible if and only any linear system $A\vec{x} = \vec{b}$ has a unique solution. Therefore, if $A\vec{x} = \vec{b}$ has infinitely many solutions, then A must not be invertible.

b) TRUE.

$$\begin{aligned} AB &= AC \\ A^{-1}(AB) &= A^{-1}(AC) \\ I_n B &= I_n C \\ B &= C \end{aligned}$$

c) FALSE. Let A be the zero matrix. Then $AB = BA = 0$, but A is not invertible.