Chapter 5: Eigenvalues and Eigenvectors

Goal: section 5.1 – 5.2 – 5.3

1) Eigenvectors and eigenvalues

Let $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

special real (eigenvalue)
special vector (eigenvector)

$A\vec{u} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$ and $A\vec{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\vec{v}$

⚠ A only stretches $\vec{v}$ (while it totally modify $\vec{u}$, in the sense that direction and length $A\vec{u}$ are ≠ than the ones of $\vec{u}$.)

Definition:

For $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ a scalar λ is an eigenvalue of A if there exists a nonzero $\vec{x} \in \mathbb{R}^n$ such that

$$A\vec{x} = \lambda\vec{x}$$

The vector $\vec{x}$ is then called an eigenvector of A corresponding to λ.

⚠ $\vec{x} \neq \vec{0}$ important because $A\vec{0} = \vec{0} = \lambda\vec{0}$ for any λ scalar.

Example 1:

Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

Are $\vec{u}, \vec{v}$ eigenvectors of A? If so, find the corresponding eigenvalues

Solution:

$$A\vec{u} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\vec{u}$$

$\Rightarrow$ $\vec{u}$ eigenvector of A with eigenvalue $\lambda = 2$

$$A\vec{v} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \end{pmatrix} \neq \lambda\vec{v} \text{ for any scalar } \lambda$$

$\Rightarrow$ $\vec{v}$ is not an eigenvector of A.

Example 2:

Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$, show that $\lambda = 3$ is an eigenvalue. Find a corresponding eigenvector.

Solution

$\lambda = 3$ eigenvalue of A $\Leftrightarrow$ $A\vec{x} = 3\vec{x}$ has non trivial solution

$\Leftrightarrow$ $(A - 3I_2)\vec{x} = \vec{0}$ has ______

$\Leftrightarrow$ $Null(A - 3I_2) \neq \{\vec{0}\}$ (i.e $A - I_2$ has free variable when put in REF)

$$A - 3I_2 = \begin{pmatrix} -2 & -2 \\ -1 & 1 \end{pmatrix} \underset{\frac{R_1}{-2}}{\sim} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \underset{R_2 + R_1}{\sim} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Leftrightarrow \left| \begin{array}{l} x_1 - x_2 = 0 \\ x_2 \text{ free} \end{array} \right. \Leftrightarrow \left| \begin{array}{l} x_1 = x_2 \\ x_2 \text{ free} \end{array} \right. \Leftrightarrow \lambda = 3 \text{ is an eigenvalue}$$

$$\Rightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ basis for } Null(A - 3I_2)$$

$\Rightarrow \lambda = 3$ is an eigenvalue and $x_2\begin{pmatrix}1\\1\end{pmatrix}$ $(x_2 \neq 0)$ are eigenvectors of A associated to $\lambda = 3$.

⚠ For a given eigenvalue, there are infinitely many eigenvectors.

⚠ λ eigenvalue of A if and only if $(A-\lambda I)\vec{x} = \vec{0}$ has nontrivial solution

Therefore, the null space of $A-\lambda I$ gives the set of eigenvectors corresponding to λ.

<u>Definition</u>

Let λ be an eigenvalue of $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. The set containing all of the eigenvectors of A corresponding to λ and the vector zero is called the <u>eigenspace</u> of A corresponding to λ and is denoted $\underline{E_\lambda(A)}$. We have:

$$E_\lambda(A) = Null\,(A-\lambda I)$$

(so $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$).

<u>Example 1</u>: Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{pmatrix}$ and let $\lambda = -2$ be an eigenvalue of A. Find a basis for $E_{-2}(A)$ $(= E_\lambda(A))$

<u>Solution</u>: $E_\lambda(A) = Null(A-\lambda I_3) = Null\,(A+2I_3)$

$$A+2I_3 = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -1 \\ 1 & -1 & 0 \\ 4 & -13 & 3 \end{pmatrix}$$

$$A+2I_3 \underset{R_1\leftrightarrow R_2}{\sim} \begin{pmatrix} 1 & -1 & 0 \\ 3 & 0 & -1 \\ 4 & -13 & 3 \end{pmatrix} \underset{\substack{R_2-3R_1 \\ R_3-4R_1}}{\sim} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & -9 & 3 \end{pmatrix} \underset{\substack{\frac{R_2}{3} \\ \frac{R_3}{3}}}{\sim} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & -3 & 1 \end{pmatrix}$$

$$\underset{\substack{R_1+R_2 \\ R_3+3R_2}}{\sim} \begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow \left| \begin{array}{l} x_1 - \frac{x_3}{3} = 0 \\ x_2 - \frac{x_3}{3} = 0 \\ x_3 \text{ free} \end{array} \right. \Rightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix}$$

A basis for $E_{-2}(A)$ is given by $\begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix}$ $(=$ basis for $Null(A+2I_3))$

Example 2.

Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and let $\lambda = -1$ be an eigenvalue of A.
Find a basis for $E_{-1}(A)$ $(= Null(A+I_3))$

Solution

$$A+I_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \underset{\substack{R_2-R_1 \\ R_3-R_1}}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow \left| \begin{array}{l} x_1 = -x_2 - x_3 \\ x_2, x_3 \text{ free} \end{array} \right. \Rightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

A basis for $E_{-1}(A)$ is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Theorem:

The eigenvalues of a triangular matrix are the entries on its diagonal

⚠ How to find eigenvalue of $A \in M_{nn}(\mathbb{R})$? → see next section

proof: (for $A \in M_{22}(\mathbb{R})$)

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a-\lambda & b \\ 0 & c-\lambda \end{pmatrix}$$

Non trivial solution if and only if $\lambda = a$ or $\lambda = c$. $\left(\begin{pmatrix} 0 & b \\ 0 & a \end{pmatrix} \text{ and } \begin{pmatrix} a-c & b \\ 0 & 0 \end{pmatrix} \right)$

⚠ When $\lambda = 0$ is an eigenvalue of a matrix A, it means that

$$A\vec{x} = 0\vec{x} = \vec{0}$$

has non trivial solution ($\Rightarrow$ A not invertible).

Theorem (The invertible Matrix theorem)

Let A be a $n \times n$ matrix

$$A \text{ invertible} \iff 0 \text{ is not an eigenvalue of } A$$

2) The Characteristic Equation

Example:

Find all the eigenvalues of $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$

Solution:

We want to find λ such that $(A-\lambda I)\vec{x} = \vec{0}$ has non trivial solution $\iff$ $A-\lambda I$ is not invertible

$\iff \det(A-\lambda I) = 0$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - (2)(-1) = 4 - 5\lambda + \lambda^2 + 2$$

$$= \lambda^2 - 5\lambda + 6$$

$$= (\lambda-2)(\lambda-3)$$

$ax^2+bx+c=0$

$x = \dfrac{-b \pm \sqrt{b^2-4a}}{2a}$

$\Rightarrow$ eigenvalues of A are 2 and 3

Theorem

* Let $A \in M_{n\times n}(\mathbb{R})$. A scalar λ is an eigenvalue of A if and only if λ satisfies

$$\det(A-\lambda I) = 0$$

* If λ is an eigenvalue of A, then all nonzero solutions of the homogeneous system $(A-\lambda I)\vec{x} = \vec{0}$ are all of the eigenvectors of A corresponding to λ (set $= E_\lambda(A) \setminus \{\vec{0}\}$)

E remove $\vec{0}$.

Definition:

Let $A \in M_{n \times n}(\mathbb{R})$. The characteristic polynomial of A is

$$C_A(\lambda) = \det(A - \lambda I)$$

$\triangleleft$ $C_\lambda(A)$ is a polynomial of degree n. (i.e. it has a term in λ^n)

Remarks:

(*) The eigenvalues of A are the roots of $C_\lambda(A)$.

(*) $C_A(\lambda)$ has real coefficients but might have complex roots

Example:

Find the $C_A(\lambda)$ and eigenvalues of $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Find a basis for $E_\lambda(A)$ (for each eigenvalue λ of A).

Solution

(*) $C_A(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \overset{R_1 + \lambda R_2}{\underset{R_3 - R_2}{=}} \begin{vmatrix} 0 & 1-\lambda^2 & 1+\lambda \\ 1 & -\lambda & 1 \\ 0 & 1+\lambda & -\lambda-1 \end{vmatrix}$

expansion 1st column.

$$= (1)(-1)^{2+1} \begin{vmatrix} 1-\lambda^2 & 1+\lambda \\ 1+\lambda & -\lambda-1 \end{vmatrix}$$

$\Rightarrow C_A(\lambda) = -\left((1-\lambda)^2(-\lambda-1) - (1+\lambda)^2\right) = -\left(-(1-\lambda)(1+\lambda)(1+\lambda) - (1+\lambda)^2\right)$

$$= (1-\lambda)(1+\lambda)^2 + (1+\lambda)^2 = (1+\lambda)^2(1-\lambda+1) = (1+\lambda)^2(2-\lambda)$$

The eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = 2$.

(*) Basis for $E_{\lambda_1}(A) \to \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ see page 4 of lecture note (Chapter 5)

(*) Basis for $E_{\lambda_2}(A)$ = basis for $Null(A - 2I_3)$

$$A - 2I_3 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \overset{R_1 \leftrightarrow R_2}{\sim} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix} \underset{\substack{R_2+2R_1 \\ R_3-R_1}}{\sim} \begin{pmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix}$$

$$\underset{\substack{R_3/-3 \\ R_2/3}}{\sim} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \underset{R_3-R_2}{\sim} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \underset{R_1+2R_2}{\sim} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad \underline{\underline{REF}}$$

__substitution:__ $\begin{cases} x_1 = x_3 \\ x_2 = x_3 \\ x_3 \text{ free} \end{cases} \Rightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ basis for $E_{\lambda_2}(A)$

__Remark:__ A is a 3×3 matrix with two distincts eigenvalue:

$\lambda_1 = -1$ (double root)

$\lambda_2 = 2$ (single root)

Notice that $\dim E_{\lambda_1} = 2$ and $\dim E_{\lambda_2} = 1$

(⚠ λ double root $\not\Rightarrow$ $\dim E_\lambda = 2$ in general)

Definition

Let $A \in M_{n\times n}(\mathbb{R})$ and let λ be an eigenvalue of A.

⊛ The algebraic multiplicity of λ, denoted a_λ, is the # of times λ appears as a root of $C_A(\lambda)$

⊛ The geometric multiplicity of λ, denoted g_λ, is $\dim(E_\lambda(A))$

Theorem:

Let $A \in M_{n\times n}(\mathbb{R})$. For any eigenvalue λ of A, we have:

$$1 \leq g_\lambda \leq a_\lambda \leq n$$

Example:

Find the eigenvalues of A, their multiplicities and a basis for $E_\lambda(A)$.

where $A = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$

Solution:

$$C_A(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & 0 \\ 5 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - (0)(5) = (1-\lambda)^2$$

$\Rightarrow \lambda_1 = 1$ is the only eigenvalue of A with algebraic multiplicity $a_{\lambda_1} = 2$

$$A - I_2 = \begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix} \underset{R_1 \leftrightarrow R_2}{\sim} \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix} \underset{\frac{R_1}{5}}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \underline{\text{REF}}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 \text{ free} \end{cases} \Rightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

A basis for E_{λ_1} is $\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$

So $\dim(E_{\lambda_1})=1 \Rightarrow g_{\lambda_1}=1 < a_{\lambda_1}=2$

Definition

For $n\times n$ matrices A and B, we say that A and B are similar if there is an invertible matrix P such that:

$$P^{-1}AP=B \quad (\text{or equivalently } A=PBP^{-1}$$

Theorem:

If A and B are similar then they have the same characteristic polynomial. Hence, the same eigenvalues.

⚠ If A and B have same eigenvalue it does NOT imply that A and B are similar.

3) Diagonalization

In many applications (e.g. dynamical systems), high power of matrices must be computed, which is an expensive calculation. The goal of this section is to factorize a $n\times n$ matrix A as:

$A=PDP^{-1}$ with D a diagonal matrix (zeros outside diagonal)

$$\Rightarrow A^n=P\underbrace{D^n}_{\text{easy to compute}}P^{-1} \quad (\text{only 2 matrix product to do})$$

Example 1: Let $D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$. Find D^2, D^3 and D^k for any positive integer k.

Solution:

$$D^2 = DD = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 3^2 & 0 \\ 0 & (-2)^2 \end{pmatrix}$$

$$D^3 = D^2 D = \begin{pmatrix} 3^2 & 0 \\ 0 & (-2)^2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 3^3 & 0 \\ 0 & (-2)^3 \end{pmatrix}$$

$$D^k = \begin{pmatrix} 3^k & 0 \\ 0 & (-2)^k \end{pmatrix} //$$

Example 2: $A = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}$. Find formula for A^k given that

$A = PDP^{-1}$ with $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

Solution

$$A^2 = AA = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)(DP^{-1}) = (PD)(DP^{-1}) = PD^2P^{-1}$$

$\Rightarrow A^k = PD^kP^{-1}$ for positive integer k.

So $$A^k = P\begin{pmatrix} 5^k & 0 \\ 0 & 4^k \end{pmatrix}P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 5^k & 0 \\ 0 & 4^k \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2(5)^k - 4^k & -5^k + 4^k \\ 2(5^k) - 2(4^k) & -5^k + 2(4^k) \end{pmatrix} //$$

Definition:

A $n\times n$ matrix A is diagonalizable if there exists an $n\times n$ invertible matrix P and a $n\times n$ diagonal matrix D such that

$$A = PDP^{-1}$$

⚠ It does not imply that $A = D$ ⚠

Question: When A is diagonalizable?

From previous example, we have:

$$A = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}$$

$$\qquad\qquad\qquad\qquad \vec{p_1} \ \ \vec{p_2}$$

Notice that $A\vec{p_1} = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 5\vec{p_1}$

$$A\vec{p_2} = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 4\vec{p_2}$$

$\Rightarrow$ $\vec{p_1}, \vec{p_2}$ are eigenvectors of A with associated eigenvalues $\lambda_1 = 5$, $\lambda_2 = 4$

Also, we have:

$$AP = (A\vec{p_1} \ A\vec{p_2}) = \begin{pmatrix} 5 & 4 \\ 5 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix} = PP$$

$\Rightarrow$ $A = PDP^{-1}$

⚠ The columns of P are eigenvectors of A and the diagonal entries of D are the associated eigenvalues.

In general, let $\vec{v}_1, \dots, \vec{v}_r$ be eigenvectors of A with associated eigenvalues $\lambda_1, \dots, \lambda_r$. Then we have:

$$A \underbrace{(\vec{v}_1 \cdots \vec{v}_r)}_{P} = \underbrace{(\vec{v}_1 \cdots \vec{v}_r)}_{P} \underbrace{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_r \end{pmatrix}}_{D}$$

If P is invertible (i.e eigenvectors are lin. indep) then $A = PDP^{-1}$

Theorem

A $n\times n$ matrix A is diagonalizable if and only if A has n lin. indep. eigenvectors. In that case, the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are the corresponding eigenvalues

Theorem:

If $\vec{v}_1, \dots, \vec{v}_p$ are eigenvectors that correspond to different eigenvalues, then $\{\vec{v}_1, \dots, \vec{v}_p\}$ is lin. indep.

Example 1: Diagonalize, if possible, the following matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$

Solution:

Step 1: Find eigenvalues

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (2-\lambda)^2(1-\lambda)$$

$\det(A-\lambda I) = 0 \iff \lambda = \lambda_1 = 2$ or $\lambda = \lambda_2 = 1$

Step 2 Find eigenvectors

* $\lambda_1 = 2$

$$A - \lambda_1 I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix} \overset{R_2 \leftrightarrow R_1}{\sim} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \underset{R_3 + R_1}{\sim} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left| \begin{array}{l} x_1 = -x_3 \\ x_2, x_3 \text{ free} \end{array} \right. \iff \vec{x} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = x_2 \vec{v}_1 + x_3 \vec{v}_2$$

$E_{\lambda_1} = \text{span}\{\vec{v}_1, \vec{v}_2\}$ ($\vec{v}_1, \vec{v}_2$ are eigenvectors corresponding to $\lambda_1 = 2$)

* $\lambda_2 = 1$

$$A - \lambda_2 I = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \underset{\substack{R_2 - R_1 \\ R_3 + R_1}}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \left| \begin{array}{l} x_1 = 0 \\ x_2 = -x_3 \\ x_3 \text{ free} \end{array} \right.$$

$$\Rightarrow \vec{x} = x_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = x_3 \vec{v}_3$$

$\Rightarrow \vec{v}_3$ eigenvector.

⚠ $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Step 3: construct P and D.

$$P = (\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \longleftrightarrow \lambda_1 \ \lambda_2 \ \lambda_3$

Same order.

⚠ P and D are not unique (we can switch order vectors and corresponding λ)

⚠ Notice that $a_{\lambda_1} = 2$, $a_{\lambda_2} = 1$ and $g_{\lambda_1} = 2$, $g_{\lambda_2} = 1$

In exercise, verify that $A = PDP^{-1}$.

<u>Example 2:</u> Diagonalize, if possible, the following matrix $A = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$

<u>Solution:</u>

(*) $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 4 & 6 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (2-\lambda)^2(4-\lambda) \Rightarrow \lambda_1 = 2, \lambda_2 = 4$

(*) $\underline{\lambda_1 = 2}$

$$A - 2I = \begin{pmatrix} 0 & 4 & 6 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \underset{R_3 - R_2}{\overset{R_1 - 3R_2}{\sim}} \begin{pmatrix} 0 & 4 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \underset{R_2/2}{\overset{R_1/4}{\sim}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\left| \begin{array}{l} x_1 \text{ free} \\ x_2 = x_3 = 0 \end{array} \right. \Rightarrow \vec{x} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ eigenvector for $\lambda_1 = 2$. $\left(E_{\lambda_1} = \operatorname{span}(\vec{v}_1)\right)$

(*) $\underline{\lambda_2 = 4}$

$$A - 4I = \begin{pmatrix} -2 & 4 & 6 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \underset{R_2/-2}{\overset{R_1/-2}{\sim}} \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \underset{R_1 + 2R_2}{\sim} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\left\{ \begin{array}{l} x_1 = 5x_3 \\ x_2 = x_3 \\ x_3 \text{ free} \end{array} \right. \Rightarrow \vec{x} = x_3 \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$ eigenvector for $\lambda_2 = 4$.

⚠ A not diagonalizable because we are missing an eigenvector due to $a_{\lambda_1} = 2$, $a_{\lambda_2} = 1$ but $\underline{\underline{g_{\lambda_1} = 1}}$ and $g_{\lambda_2} = 1$.

Theorem:

A $n \times n$ matrix A is diagonalizable if and only if $a_\lambda = g_\lambda$ for every eigenvalue λ of A.

Example. Is $A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ diagonalizable?

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 2 & 6-\lambda & 0 \\ 3 & 2 & 1-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda)(1-\lambda) \Rightarrow \lambda_1 = 2, \lambda_2 = 6, \lambda_3 = 1.$$

$$\Rightarrow a_{\lambda_1} = 1, a_{\lambda_2} = 1, a_{\lambda_3} = 1.$$

$1 \leq g_\lambda \leq a_\lambda = 1$

$$\Rightarrow g_{\lambda_1} = 1, g_{\lambda_2} = 1, g_{\lambda_3} = 1$$

So A is diagonalizable.

Theorem:

A $n \times n$ matrix with n distincts eigenvalues is diagonalizable.