MATH 2318 Final Exam. Sanders Fall 2022

This exam has 10 problems, and all 10 problems will be graded. Use my supplied paper only. Return your solution sheets with the problems in order. Put your name, **last name first**, and **student id number** on each solution sheet you turn in. Each problem is worth 20 points with parts equally weighted unless otherwise indicated.

1. Consider the matrices.

$$A = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad E = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Compute the following when defined.

(a) AC (b) CD (c) $B+2E$ (d) AB^T

2. Find <u>all</u> solutions, if there are any, to the following.

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$

3. Determine if the given vector

(a) $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is in span $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$.

Please show all your work.

4. Consider the following two matrices A each which defines a linear operator $\mathcal{L}$ via matrix multiplication, $\mathcal{L}(\mathbf{x}) \equiv A\mathbf{x}$.

(a) $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 1 \end{pmatrix}$ (b) $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 4 & 9 \\ 4 & 2 & 2 \end{pmatrix}$

First determine a basis for $\mathcal{L}$'s null space, and second compute the **standard basis** for $\mathcal{L}$'s range space.

5. Each of the following matrices is invertible. Find the inverse. Please state whether you are using elimination or Cramer's rule.

(a) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

(Part (a) is worth 6 points. (b) and (c) are worth 7 points.)

6. In class we showed that $\det(I) = 1$, $\det(AB) = \det(A)\det(B)$, and we showed that A is invertible if and only if $\det(A) \neq 0$. Use these determinant properties to deduce the following.

(a) If A is invertible then $\det(A^{-1}) = 1/\det(A)$.

(b) If A and B are invertible then the product AB is invertible.

7. Compute the determinant of each matrix.

(a) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

If you use cofactors, clearly indicate which row or column you are using.

8. Calculate the eigenvalues and associated eigenvectors of the following two matrices.

(a) $\begin{pmatrix} -3 & 2 \\ -4 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$

9. You may freely use the given fact

$$A = \begin{pmatrix} -2 & 6 \\ -3 & 7 \end{pmatrix} \quad \Rightarrow \quad R^{-1}AR \equiv \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -2 & 6 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \equiv \Lambda.$$

(a) Calculate e^{At}.

(b) Find a matrix $\sqrt{A}$ such that $(\sqrt{A})^2 = A$.

10. Use Gram-Schmidt to find the requested orthogonal basis set, $\{\mathbf{e}_1, \mathbf{e}_2\}$ in part (a), and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in part (b).

(a) In $\mathbb{R}^2$ let $\mathbf{e}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

and find $\mathbf{e}_2 \in \operatorname{span}\{\mathbf{e}_1, \mathbf{x}_2\}$.

(b) In $\mathbb{R}^3$ let $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

and find $\mathbf{e}_2 \in \operatorname{span}\{\mathbf{e}_1, \mathbf{x}_2\}$, $\mathbf{e}_3 \in \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{x}_3\}$.

Your vectors need <u>not</u> have unit length.