4 points

1. Show that
$$B = \left\{ \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$
is a basis for $M_{2\times 2}$. *Hint*: Recall that $\dim(M_{2\times 2}) = 4$, and note that B contains four matrices.

Solution. Since B contains four matrices and $\dim(M_{2\times 2}) = 4$, it is enough to show that B is linearly independent. Let c_1, c_2, c_3, $c_4 \in \mathbb{R}$ and consider:
$$c_1 \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$
Working on the left hand side:
$$\begin{bmatrix} c_1 - c_2 + 2c_3 & -3c_1 + c_2 \\ 2c_1 - c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$
This leads to the following linear system:
$$\begin{array}{rcrcrcrcl} c_1 & - & c_2 & + & 2c_3 & & & = & 0 \\ -3c_1 & + & c_2 & & & & & = & 0 \\ 2c_1 & & & - & c_3 & & & = & 0 \\ & & & & & & c_4 & = & 0 \end{array}$$
To solve this homogeneous linear system, we row reduce the associated matrix:
$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ -3 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \sim \\ R_2 + 3R_1 \\ R_3 - 2R_1 \\ \end{matrix} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -2 & 6 & 0 \\ 0 & 2 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \sim \\ R_3 + R_2 \\ \end{matrix} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -2 & 6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
This is an echelon form of the matrix, from which we see that all the variables are basic, meaning that the only solution is the trivial one. Therefore, B is linearly independent. By the basis theorem, since B is a linearly independent set that contains four matrices, and since $\dim(M_{2\times 2}) = 4$, we conclude that B is a basis for $M_{2\times 2}$.

9 points

2. Let $A = \begin{bmatrix} -4 & 0 & 2 \\ 2 & 4 & -8 \\ 2 & 0 & -4 \end{bmatrix}$.

   a) Find the characteristic polynomial of A.
   b) Find the eigenvalues of A and state their algebraic multiplicity.
   c) Find a basis for each eigenspace of A.

Solution.

a)
$$\begin{aligned} C_A(\lambda) = \det(A - \lambda I) &= \begin{vmatrix} -4-\lambda & 0 & 2 \\ 2 & 4-\lambda & -8 \\ 2 & 0 & -4-\lambda \end{vmatrix} \\ &= (-1)^{2+2}(4-\lambda) \begin{vmatrix} -4-\lambda & 2 \\ 2 & -4-\lambda \end{vmatrix} \\ &= (4-\lambda)((-4-\lambda)^2 - 4) = (4-\lambda)(\lambda^2 + 8\lambda + 16 - 4) \\ &= (4-\lambda)(\lambda^2 + 8\lambda + 12) = (4-\lambda)(\lambda+6)(\lambda+2). \end{aligned}$$

b) The eigenvalues of A are the roots of $C_A(\lambda)$, i.e.,

$$\lambda_1 = 4,\ \lambda_2 = -6,\ \lambda_3 = -2$$
$$a_{\lambda_1} = 1,\ a_{\lambda_2} = 1,\ a_{\lambda_3} = 1.$$

c)
- $E_{\lambda_1}(A)$.

$$A - 4I = \begin{bmatrix} -8 & 0 & 2 \\ 2 & 0 & -8 \\ 2 & 0 & -8 \end{bmatrix} \underset{R_3 - R_2}{\overset{-1/8 R_1}{\sim}} \begin{bmatrix} 1 & 0 & -1/4 \\ 2 & 0 & -8 \\ 0 & 0 & 0 \end{bmatrix} \underset{R_2 - 2R_1}{\sim} \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 0 & -15/2 \\ 0 & 0 & 0 \end{bmatrix} \underset{-2/5 R_2}{\sim}$$

$$\sim \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1 + 1/4 R_2}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$x_1 = 0$$
$$x_2 \text{ is free}$$
$$x_3 = 0,$$

so that $\vec{x} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. A basis for $E_{\lambda_1}(A)$ is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, $g_{\lambda_1} = 1$.

- $E_{\lambda_2}(A)$.

$$A + 6I = \begin{bmatrix} 2 & 0 & 2 \\ 2 & 10 & -8 \\ 2 & 0 & 2 \end{bmatrix} \underset{\substack{R_2 - R_1 \\ R_3 - R_1}}{\sim} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 10 & -10 \\ 0 & 0 & 0 \end{bmatrix} \underset{\sim}{\overset{1/10 R_1}{1/2 R_2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$x_1 = -x_3$$
$$x_2 = x_3$$
$$x_3 \text{ is free},$$

so that $\vec{x} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. A basis for $E_{\lambda_2}(A)$ is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$, $g_{\lambda_2} = 1$.

- $E_{\lambda_3}(A)$.

$$A + 2I = \begin{bmatrix} -2 & 0 & 2 \\ 2 & 6 & -8 \\ 2 & 0 & -2 \end{bmatrix} \underset{\substack{R_2 + R_1 \\ R_3 + R_1}}{\sim} \begin{bmatrix} -2 & 0 & 2 \\ 0 & 6 & -6 \\ 0 & 0 & 0 \end{bmatrix} \underset{\sim}{\overset{-1/2 R_1}{1/6 R_2}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$x_1 = x_3$$
$$x_2 = x_3$$
$$x_3 \text{ is free},$$

so that $\vec{x} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. A basis for $E_{\lambda_3}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, $g_{\lambda_3} = 1$.

4 points

3. Let A be an $n \times n$ invertible matrix. Show that if λ is an eigenvalue of A with associated eigenvector $\vec{v}$, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with associated eigenvector $\vec{v}$. *Hint*: Consider $A\vec{v} = \lambda\vec{v}$ and multiply by A^{-1}.

   Solution. Since A is invertible, we know that $\lambda \neq 0$. Moreover, we have that

   $$A\vec{v} = \lambda\vec{v}.$$

   Multiplying by A^{-1} on both sides from the left:

   $$\begin{aligned} A^{-1}A\vec{v} &= A^{-1}(\lambda\vec{v}) \\ \vec{v} &= \lambda A^{-1}\vec{v} \\ A^{-1}\vec{v} &= \frac{1}{\lambda}\vec{v}. \end{aligned}$$

   Therefore, $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with associated eigenvector $\vec{v}$.

3 points

4. For each of the following, determine if the statement is true or false. Provide a short reasoning (one or two sentences).

   a) If $\vec{v}$ is an eigenvector of a square matrix A with corresponding eigenvalue λ, then $2\vec{v}$ is also an eigenvector of a square matrix A with corresponding eigenvalue λ.
   b) If the characteristic polynomial of a square matrix has a nonzero constant term, then the matrix is invertible.
   c) If an eigenvalue of a square matrix has algebraic multiplicity 1, then the dimension of the associated eigenspace is 1.

   Solution.

   a) TRUE. Note that
   $$A(2\vec{v}) = 2A\vec{v} = 2\lambda\vec{v} = \lambda(2\vec{v}).$$
   Therefore, $2\vec{v}$ is an eigenvector with associated eigenvalue λ.
   b) TRUE. Let $C_A(\lambda) = a_n\lambda^n + \cdots + a_1\lambda + a_0$, for some coefficients $a_n, \ldots, a_1, a_0$, and with $a_0 \neq 0$. Then, $C_A(0) = a_n0^n + \cdots + a_10 + a_0 = a_0$. Since $a_0 \neq 0$, then $C_A(0) = a_0 \neq 0$. Therefore, 0 is not a root of $C_A(\lambda)$ and thus it is not an eigenvalue of A, which implies that A is invertible.
   c) TRUE. Since $1 \leq g_\lambda \leq a_\lambda = 1$, we have $g_\lambda = 1$.