difficult and complex than math itself; these subjects deal with richer, harder-to-exactly-specify aspects of human expression.

Interestingly enough, it is by focusing in on statements (and open sentences) that logic and mathematics accomplishes so much. By only considering such well-behaved and puny or pathetic objects, we are are able to build and apply and re-apply careful rules and procedures to produce newer and more complex statements, and all with crystal clear precision. The collection of new statements mathematicians produce is an overwhelming and ongoing enterprise, arguably as rich and beautiful as any other human achievement What's more is that many of these statements can be used in real-world applications!

In this section you read about **statements (propositions), non-statements, open sentences (predicates)**.

Definition 1.1. *A* ***statement*** *or* ***proposition*** *is a sentence that is either true or false (but not both). We say that statements are sentences that "have a truth value." We use capital letters like P, Q, R, etc. to denote these.*

Definition 1.2. *An* ***open sentence*** *or* ***predicate*** *is a non-statement that contains variables, and when those variables are replaced the sentence becomes a statement. We use expressions like $P(x), Q(s,t)$, etc., to denote open sentences and their variables.*

Statements (Propositions)	Non-Statements
sentences that are true or false (not both)	Open Sentences (variables need subst.) Commands Questions Self-refuting sentences

2. Operations on Statements

This section has a kind of "serious sounding" name, doesn't it? But, thankfully, "operating on statements" just means "creating new statements from old ones." In this section we will discuss three main ways to do this: *negation*, *conjunction (and)*, and *disjunction (or)*. A fourth way, *exclusive-or*, is also mentioned.

Negation. Given a statement P, we can form a new statement, $\neg P$, called **the negation of** P in a simple way. This new statement inserts the phrase "it is not true that" in front of the original statement, thereby negating its meaning and truth value:

$$\neg P : \text{ it is not true that } P.$$

Computer scientists tend to use the symbol $\neg$ while many mathematicians use $\sim$, but both mean the same thing, and they are both pronounced "not P."

Example 2.1. *Consider the statement* $P : 3+4 = 9$ *whose negation can be written as*

$$\neg P : \text{ it is not true that } 3 + 4 = 9.$$

There is a simpler and clearer way to write out this particular negation, namely

$$\neg P : 3 + 4 \neq 9.$$

and this is the rule-of-thumb we will follow when dealing with negated statements. Whenever possible, we will express a negation in as short and simplistic a form as we can.

Example 2.2. *Consider the statement* Q : 35 *is an odd number. Which of the following statements are correct ways of writing* $\neg Q$ *?*

(1) *35 is an even number.*

(2) *It is not true that 35 is an odd number.*

(3) *35 is not odd.*

(4) *35 is a number.*

(5) *There is no remainder when 35 is divided by 2.*

Based on your knowledge about odd and even numbers, you probably figured out that statements (1), (2), (3) and (5) are all different but equally valid ways to write $\neg Q$. Of course, sentences (3) and (1) are probably the *best* or *better* ways to express this negation.

Thinking a bit more about our examples above reveals a nice pattern: a statement and its negation have *opposite* truth values. In Example 2.1, P is false while $\neg P$ is true, whereas in Example 2.2 Q is true and $\neg Q$ is false. This is, of course, the defining property of the negation operator, and it is nicely encapsulated in the following table:

P	$\neg P$
T	F
F	T

The table above is called **a truth table**, and the statement P in it does not refer to any particular sentence. Rather, this table defines for us how negation creates a new statement by showing us the truth value for $\neg P$ given the truth value for P.

Conjunction (And). Given two statements, P and Q, we can create a new statement, $P \wedge Q$, that is pronounced "P **and** Q." Some like to think of this as "conjoining" two statements – hence the fancy-sounding-name "**conjunction**." More important, though, is the fact that this way of combining two statements is very similar to how we naturally connect two English sentences with the word "and." One can see this by examining the defining truth table:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

The statement $P \wedge Q$ only returns a value of "true" when *both* statements P and Q are individually true. In all other cases $P \wedge Q$ evaluates to "false."

Consider this example of a strange-sounding English sentence:

Mars is a planet in our solar system *and* birds don't exist.

We naturally understand this sentence to be false, and can probably imagine that the sort of person who would speak it is very interesting and is full of amazing (even if incorrect) ideas. But *why* do we regard this as a false sentence? What, exactly, makes it untrue? Once we recognize that this is, of course, a compound sentence that *conjoins* the two shorter ones

P : Mars is a planet in our solar system.

Q : Birds don't exist.

the issue becomes clear. The word "*and*" is connecting P and Q in a way where *both* statements are asserted simultaneously. We (hopefully) agree that one part, P, is true and that the other statement, Q, is false. When we form $P \wedge Q$, the entire expression is therefore false.

Example 2.3. *All of the following statements are false. Make sure you understand why this is so.*

(1) *35 is an odd number and* $\sin(\pi) = 1$.

(2) $1 + 1 = 1$ *and the derivative of tangent is co-tangent.*

(3) $2^3 = 9$ *and* $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.

Example 2.4. *Consider the statement* $P : 5^2 < 10$. *Can you come up with a statement* Q *so that* $P \wedge Q$ *is true? If you can, write one down. If not, explain why.*

Disjunction (Or). Connecting two statements P, Q with the word **or** produces a new statement, $P \vee Q$, that is pronounced "P or Q" and asserts *either* P or Q (or both). This expression is also referred to as **a disjunction**. Reusing our strange example above, the sentence

Mars is a planet in our solar system *or* birds don't exist

is understood to be true because at least one of the components is true. This is captured or defined more generally by the following truth table:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

The only time $P \vee Q$ returns a value of 'false' is when both P and Q are false statements, and this matches well the ways we use the English word "or," at least in many instances.

Example 2.5. *The simple looking inequality $\pi \leq 4$ is an example of a **disjunction** since it asserts*

$$\pi < 4 \text{ or } \pi = 4.$$

Example 2.6. *Determine the truth values of the following statements.*

(1) $(15 \div 3 = 5) \vee (12 < 0)$

(2) *Every whole number is even or* $12 < 0$.

(3) *The derivative of a constant is zero or the sum of two odd numbers is even.*

(4) $(\sec(\pi/2) = 0) \vee (1/2 > 3/4)$

Example 2.7. *Consider the statement $P : 5^2 < 10$. Can you come up with a statement Q so that $P \vee Q$ is true? If you can, write one down. If not, explain why.*

There are some instances in the English language where the word "or" is used in a subtly different way than described above. Specifically, it can be used to combine statements so that exactly *one* is asserted, but not both. The sentence "Driver 1 or Driver 2 will win the race," for example, is usually understood to mean that only one driver will take first place. Because this use of "or" *excludes* the case where both statements P and Q are true, computer scientists call it **xor** or "**exclusive or**" and the funny-looking symbol $\oplus$ is used to denote it. This means that the statement $P \oplus Q$ means "either P or Q, but not both."

Given two statements P, Q the new statement $P \oplus Q$ can, of course, be defined by its own truth table, and it is worth your time to figure out how this table should be filled out based on what we intend $P \oplus Q$ to mean.

Example 2.8. *Complete the truth table for xor $P \oplus Q$:*

P	Q	$P \oplus Q$
T	T	
T	F	T
F	T	T
F	F	

Closing Thoughts and Section Summary

It's really interesting to emphasize, yet again, that our main ways of "operating on statements" are each entirely described by truth tables. This means that negating statements and logically connecting them with "and's" and "or's" does not have to correspond to meaningful expressions in our natural languages; these procedures can be abstractly defined using abstract statements and truth tables whose entries are filled in with certain T's and F's.

In fact, we can choose to relabel those T's and F's using other, distinct symbols. One would then be free to interpret the P's and Q's as true-or-false sentences and the entries in the different tables as codes for "true" and "false," but another person might leave all of it entirely abstract and uninterpreted. It is worth noting that computer scientists actually do this sort of thing, coding 1 as T and 0 as F. The truth-tables for $\neg$, $\wedge$, $\vee$, and $\oplus$ are filled out with 0's and 1's and, as a result, can be understood in purely computational terms. Any "meaning" that we attach to these operations is of no concern! The pure structure of the operations – the *logic* of it all – is all that matters for computers.

Of course, we motivated the definition of our logical operations using familiarity with meaningful English sentences, and this is a helpful approach to keep in mind. There are many other ways to connect statements, and we first focus on **and** and **or** precisely because they represent useful constructions in the English language. Compare our discussion of those logical connectives, for instance, with a rather randomly-defined, new logical connective.

Example 2.9. *Consider a new logical connective, $\star$, defined by the following truth table:*

P	Q	$P \star Q$
T	T	T
T	F	F
F	T	T
F	F	F

If statement P is true and statement Q is false, determine the truth value of statement $(P \star Q) \wedge P$.

Does the operation $\star$ in Example 2.9 have a meaningful interpretation the way $\wedge$ and $\vee$ do? It may or may not, but in either case we can use $\star$ as easily as we use these connectives.

Note on Open Sentences. You should also know that we *can* negate open sentences and connect them using "and," "or," and "xor," too, but doing so **results in new open sentences, not statements**. Let's say that one more time:

negating and combining open sentences creates open sentences.

These new open sentences will have truth tables once their variables have been replaced, but until then we must regard them as non-statements. The predicate $P(x) : x + 3 = 10$, for instance, negates to $\neg P(x) : x + 3 \neq 10$, but neither of these sentences are true or false until x has been replaced. Similarly, $(x+3=10) \vee (x<5)$ lacks a truth value as does $\big(x^2 > 9\big) \oplus (\sin(\theta)=1)$.

In this section you read about and worked through examples dealing with **truth tables, negation** (¬)**, and** (∧)**, or** (∨), and **xor** (⊕). The definitions of these operations are summarized in the following truth tables.

P	$\neg P$
T	F
F	T

NOT Truth Table

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

AND Truth Table

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

OR Truth Table

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

XOR Truth Table

One of your goals is to become so comfortable with these operations that you can draw up and fill out their truth tables lightning-fast, and this is best done by practicing examples and sharing your thoughts with others. The table below summarizes their properties in straightforward English terms and may assist your efforts.

Operation	Name	Properties
$\neg$	NOT	swaps truth values
$\wedge$	AND	T only when both parts are T; F otherwise
$\vee$	OR	T when one or both parts are T; F otherwise
$\oplus$	XOR	T only when one part is T; F otherwise

3. Conditional Statements

"Your 'if' is the only peacemaker; much virtue in 'if'."

– W. Shakespeare (*As You Like it*, Act 5, Scene 4)

This section is all about ***conditional statements***, which are created using two given statements P and Q and a new connective "$\Rightarrow$." The sentence $P \Rightarrow Q$ is pronounced "P implies Q," and before we discuss its possible meaning or interpretation, let's examine its truth table; in other words, let's momentarily treat $\Rightarrow$ as an arbitrary way to create new statements (much as we did with $\star$ in Example 2.9).

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth Table for $\Rightarrow$

According to this truth table the following conditional statement is true:

$$(\text{Birds do not exist}) \Rightarrow (\text{ At least ten people live in Canada })$$