3336

Office Hour

10:00 am

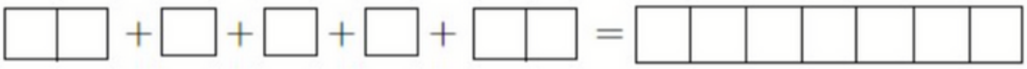
Quiz 9 deadline extended through today (before 11:59 pm)

Quiz 10, next HW uploading today!

→ the Division Algorithm

$\boxed{\text{Quiz 9}}$

A landscape architect is creating a straight path using combinations of 1×1 square tiles and 1×2 rectangular tiles. She accomplishes this by lying one tile down and then placing the subsequent tile next to the first one, moving from left to right. An example of this process is shown below where a 1×7 path is created using two rectangular and three square tiles.



Define the sequence $\{a_n\}$ to count the number of ways our landscape architecht can create a $1 \times n$ path; this means a_1 = the number of ways a 1×1 path can be constructed, a_2 = the number of ways a 1×2 path can be constructed, and so on. Make sure you understand why $a_1 = 1$ and $a_2 = 2$. Explore the sequence a_n some more until you discover a *recursive structure* or *recurrence relation* that can be used to determine the value of a_4.

a_n = # of ways a $1 \times n$ path can be made using 1×1 or 1×2 tiles

a_1 = # of ways a 1×1 path can be made

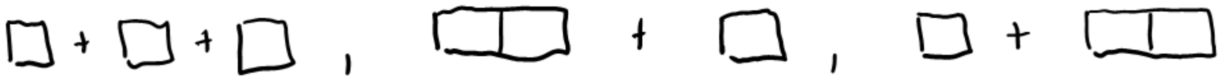
□ ← only one way

a_2 = # of ways a 1×2 path can be made

two 1×1 tiles OR one 1×2 tile

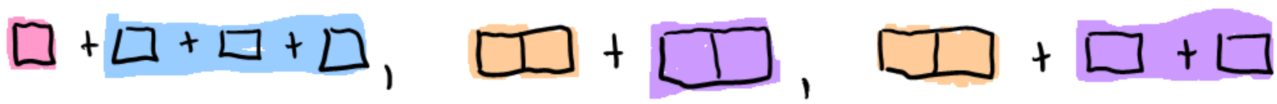
$\Big\}\ a_2 = 2$

a_3 = # of ways a 1x3 path can be made



$$\boxed{a_3 = 3}$$

a_4 = # of ways a 1x4 path can be made



$$\boxed{a_4 = 5}$$

a_4 = # of 1x3 paths + # of 1x2 paths

add a 1x1 tile

add a 1x2 tile

$a_1, a_2, a_3, a_4, \ldots$

1, 2, 3, 5, 8, ... ← looks like Fibonacci

recursion : $a_n = a_{n-1} + a_{n-2}$

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every 1x5 path = a 1x4 path + a 1x1 tile
OR
a 1x3 path + a 1x2 tile

$a_5 = a_4 + a_3$

every 1xn path = a 1x(n-1) path + a 1x1 tile
OR
a 1x(n-2) path + a 1x2 tile

$a_n = a_{n-1} + a_{n-2}$

The previous question was about searching through sorted arrays using a "linear search," and counting the number of steps involved in such a search gave us a sequence with recursive structure. There are *other* ways Computer Scientists search through sorted arrays, including one called a "binary search." This method gives rise to the more interesting sequence $\{b_n\}$ which satisfies the recurrence relation and initial conditions

$$b_n = b_{\lceil \frac{n}{2} \rceil} + 1$$
$$b_1 = 1$$

For example, this recurrence relation tells us that the second term $b_2 = b_{\lceil \frac{2}{2} \rceil} + 1 = b_{\lceil 1 \rceil} + 1 = b_1 + 1 = 2$. Determine the value of b_{20}.

a) $b_{20} = 6$
b) $b_{20} = \log_2(20)$
c) $b_{20} = 4$
d) $b_{20} = 2^{20}$
e) $b_{20} = 16$

Question 1

$$b_n = b_{\lceil \frac{n}{2} \rceil} + 1$$

$$b_2 = b_{\lceil \frac{2}{2} \rceil} + 1 = b_{\lceil 1 \rceil} + 1 = b_1 + 1 = 1 + 1 = 2$$

$$b_3 = b_{\lceil \frac{3}{2} \rceil} + 1 = b_2 + 1 = 2 + 1 = 3$$

$$b_4 = b_{\lceil \frac{4}{2} \rceil} + 1 = b_2 + 1 = 2 + 1 = 3$$

$$b_5 = b_4 + 1 = 3 + 1 = 4$$

$$\vdots$$

$$b_{20} = b_{\lceil \frac{20}{2} \rceil} + 1 = b_{\lceil 10 \rceil} + 1 = b_{10} + 1 = 6$$

$$b_{10} = b_{\lceil \frac{10}{2} \rceil} + 1 = b_{\lceil 5 \rceil} + 1 = b_5 + 1 = 4 + 1 = 5$$

Question 9

Consider the sequence $\{h_n\}$ that solves the recurrence relation and initial conditions

$$h_n = 4h_{n-1} + 3$$
$$h_1 = 2$$

If one uses the method of iteration to solve this recurrence relation, which closed-form expression for h_n will they find?

a) ○ $h_n = 4(n-1) + 2$
b) ○ $h_n = 4^{n-1} - 1$
c) ○ $h_n = 3 \cdot 4^{n-1} - 1$
d) ○ $h_n = 3 \cdot 4^{n-1} + 1$
e) ○ $h_n = 4^{n-1} + 1$

$h_2 = 4 \cdot h_1 + 3 = 4 \cdot 2 + 3$ $(n=2)$

$\boxed{h_3} = 4 \cdot h_2 + 3 = \boxed{4(4 \cdot h_1 + 3) + 3}$

$= 4^2 \cdot h_1 + 4 \cdot 3 + 3$

$= 4^2 \cdot 2 + 4 \cdot 3 + 3$ $(n=3)$

$h_4 = 4 \cdot h_3 + 3$

$= 4\left(4(4h_1 + 3) + 3\right) + 3$

Question 11

Consider the sequence $\{a_n\}$ that solves the recurrence relation and initial conditions

$$a_n = 20a_{n-1} - 99a_{n-2}$$
$$\boxed{a_0 = 11, \quad a_1 = 24}$$

We know that a closed-form expression for this sequence is given by $a_n = \alpha(r_1)^n + \beta(r_2)^n$ where r_1 and r_2 are the characteristic roots. Determine the values of the constants α and β.

a) ○ $\alpha = 20, \ \beta = 99$
b) ○ $\alpha = \frac{2}{97}, \ \beta = \frac{2}{-75}$
c) ○ $\alpha = \frac{97}{2}, \ \beta = \frac{-75}{2}$
d) ○ $\alpha = 11, \ \beta = 24$
e) ○ $\alpha = \frac{-97}{75}, \ \beta = 1$

$$a_n = 20a_{n-1} - 99a_{n-2}$$

$$\underbrace{a_n - 20a_{n-1} + 99a_{n-2} = 0}$$

$$x^2 - 20x + 99 = 0$$

← solve this for roots r_1 & r_2

$$(x-9)(x-11) = 0$$

$x - 9 = 0$ $\quad$ $x - 11 = 0$

$\boxed{x=9}$ $\quad$ $\boxed{x=11}$

one root $r_1 = 9$ $\quad$ other root $r_2 = 11$

our sequence $a_n = \alpha \cdot r_1^n + \beta \cdot r_2^n$

$$\boxed{a_n = \alpha \cdot 9^n + \beta \cdot 11^n}$$

how do we find α & β?

A: use initial conditions!!

$a_0 = 11 \rightarrow$ $11 = \alpha \cdot 9^0 + \beta \cdot 11^0$

$a_1 = 24 \rightarrow$ $24 = \alpha \cdot 9^1 + \beta \cdot 11^1$

$$\boxed{\begin{aligned} 11 &= \alpha \cdot 9^0 + \beta \cdot 11^0 \\ 24 &= \alpha \cdot 9^1 + \beta \cdot 11^1 \end{aligned}} \longrightarrow \begin{aligned} 11 &= \alpha + \beta \\ 24 &= 9\alpha + 11\beta \end{aligned}$$

Solve for α & β

you can use linear algebra to solve this!

$$\boxed{\alpha + \beta = 11}$$
$$9\alpha + 11\beta = 24$$

$$\longrightarrow \begin{bmatrix} 1 & 1 \\ 9 & 11 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 11 \\ 24 \end{bmatrix}$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 9 & 11 \end{bmatrix}^{-1} \begin{bmatrix} 11 \\ 24 \end{bmatrix}$$

$$\alpha = 11 - \beta$$

$$\underbrace{9(11-\beta) + 11\beta = 24}_{\text{solve for } \beta \ldots}$$

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this week Division "Algorithm"

$$\forall\, a, b \in \mathbb{Z} \ (b \neq 0),\ \exists!\, q, r \in \mathbb{Z}$$

$$a = b \cdot q + r$$

where $0 \leq r < |b|$

ex) $a = 10,\ b = 7$ $\qquad 10 = 1 \cdot 7 + 3$