Discrete Math

Lecture 5

Logical Equivalence

two new words

Contradiction : truth value is always F

tautology : truth value is always T

most common examples : $P \wedge \neg P$

natural ex : "it is raining and it is not raining"

$P \vee \neg P$

natural ex : "it is raining or it is not raining"

Recall: two statements, P and Q, are <u>logically equivalent</u> if $P \Leftrightarrow Q$ is a tautology when is $P \Leftrightarrow Q$ a tautology?

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

in order to be T, they need to have the same truth values

i.e. P & Q are logically equivalent when their truth tables are identical!

ex) $P, \neg(\neg(\neg P))$

$$\neg(\neg(\neg P)) \underset{\text{log. equiv.}}{\equiv} \neg P$$

P	$\neg P$	$\neg(\neg P)$	$\neg(\neg(\neg P))$
T	F	T	F
F	T	F	T

Example 5.2. *Several abstract statements are provided below, and each one is labelled as a contradiction, a tautology or as* neither. *Make sure you understand* why *each label is accurate! (Using truth tables should help.)*

(18) $$(P \vee Q) \vee (\neg P \wedge \neg Q) \text{ tautology}$$

(19) $$(P \Rightarrow Q) \wedge (P \wedge \neg Q) \text{ contradiction}$$

(20) $$(P \wedge Q \wedge R) \vee \Big(\neg R \Rightarrow (P \vee Q)\Big) \text{ neither}$$

(21) $$\Big(P \iff Q\Big) \Rightarrow \Big((R \vee P) \iff (R \vee Q)\Big) \text{ tautology}$$

(22) $$(P \wedge Q) \wedge \neg P \text{ contradiction}$$

(23) $$\neg(\neg P) \iff P \text{ tautology}$$

P	Q	P ∨ Q	¬P ∧ ¬Q	(P ∨ Q) ∨ (¬P ∧ ¬Q)
T	T	T	F	T
T	F	T	F	T
F	T	T	F	T
F	F	F	T	T

tautology

P	¬P	¬(¬P)	¬(¬P) ⇔ P
T	F	T	T
F	T	F	T

tautology

Example 5.4. *The statements $\neg P \vee Q$ and $\neg(P \wedge \neg Q)$ are logically equivalent. Check that this is, indeed, the case by completing their truth tables. (We have included some additional columns in these tables to help you sort out the truth values for these somewhat-complicated statements.)*

P	Q	$\neg P$	$\neg P \vee Q$
T	T	F	T
T	F	F	F
F	F	T	T
F	F	T	T

P	Q	$\neg Q$	$P \wedge \neg Q$	$\neg(P \wedge \neg Q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

$$\neg P \vee Q \equiv \neg(P \wedge \neg Q)$$

Example 5.5

Is $P \vee Q \equiv \big((P \wedge R) \vee (P \wedge \neg R)\big) \vee \big((Q \wedge R) \vee (Q \wedge \neg R)\big)$?

first "problem" these have diff. sized truth tables!

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	R	$((P \wedge R) \vee (P \wedge \neg R)) \vee ((Q \wedge R) \vee (Q \wedge \neg R))$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

this is easily fixed by adding rows to "the smaller" table

P	Q	R	$P \vee Q$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

this can be compared!

P	Q	R	$P \vee Q$	$((P \wedge R) \vee (P \wedge \neg R)) \vee ((Q \wedge R) \vee (Q \wedge \neg R))$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	F
F	F	F	F	F

Therefore, $P \vee Q = ((P \wedge R) \vee (P \wedge \neg R)) \vee ((Q \wedge R) \vee (Q \wedge \neg R))$

Important fact: all contradictions are logically equivalent.

$$\mathbb{F} = P \wedge \neg P$$

all tautologies are logically equivalent!

$$\mathbb{T} = P \vee \neg P$$

Note there is a kind of "algebra" used in establishing logical equivalences.

ex) $\neg(\neg(\neg P)) \equiv \neg P$

$\neg(\neg(\neg P)) \equiv \neg(P) \equiv \neg P$

(established $\neg(\neg P) \equiv P$)

Famous ex. of some "algebra-like" laws are

De Morgan's Laws

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

a truth table shows these are accurate!

"it's not the case that P is T or Q is T"

= "neither P nor Q is T"

= "both are false" = "P is false AND Q is false"

ex| $\neg((P \wedge Q) \vee R)$

$\underset{\text{De M}}{=} \neg(P \wedge Q) \wedge \neg R$

$\underset{\text{De M}}{=} (\neg P \vee \neg Q) \wedge \neg R$