Discrete Math

Lecture 6

Quantifiers

ex) Every dog is a mammal.

(This is true, so it should be a statement.)

⇝ For every creature that is a dog, it is true that that creature is a mammal.

$$\forall x \in \{\text{all creatures}\},\ x \in \{\text{all mammals}\}$$

new symbol

sets

Another preview of sets

a <u>set</u> is a collection of objects.

(think of it as a box that contains "<u>elements</u>")

ex) $\emptyset$ = "the empty set" $= \{\ \}$

ex) $S = \{☺, 1, \pi\}$

notation: ☺ $\in S$, $1 \in S$, $\pi \in S$

"is in"

(back to $\emptyset$: $1 \in \emptyset$ is false
$1 \notin \emptyset$ is true)

Famous Sets

$\emptyset$

$\mathbb{N}$ = "natural numbers" $= \{0, 1, 2, 3, 4, \ldots\}$

$\mathbb{Z}$ = "integers" $= \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$

$\mathbb{Q}$ = "rationals" = { all fractions of integers }

$$-2 = \frac{-2}{1} \in \mathbb{Q}$$

$\mathbb{R}$ = "real numbers" = {all numbers on the number line}

$5 \in \mathbb{N}$ true ✓ $5 \in \mathbb{Z}$ ✓

$-7 \notin \mathbb{N}$ true ✓ $-7 \in \mathbb{Z}$ ✓

$-7 \in \mathbb{Q}$, $-7 \in \mathbb{R}$ ✓

$\pi \in \mathbb{Z}$ is false $e \in \mathbb{R}$ ✓

$\pi \in \mathbb{R}$ ✓

Universal Quantifier $\forall$

"for all, for every, every, all"

$$\boxed{\forall x \in \mathcal{U},\quad P(x)}$$

$\mathcal{U}$ = "universal set"

ex] $\forall x \in \mathbb{N}, \quad x+1 > x$

this is true!

ex] $\forall x \in \{1, 2\}, \quad x^2 - x > 0$

when $x=2$, $2^2 - 2 = 4 - 2 = 2 > 0$ is T

when $x = 1$, $1^2 - 1 = 1 - 1 = 0 > 0$ is F

"counter-example"

the entire statement is false!

<u>note</u> our conditional statements used a "hidden" or "implied" quantifier

$$P(x) \Rightarrow Q(x)$$

math folks read this as universally quantified!

$$\boxed{\forall x \in \mathcal{U},\ P(x) \Rightarrow Q(x)}$$

ex) x is even $\Rightarrow$ x+1 is odd

$\forall x \in \mathbb{Z}$, x is even $\Rightarrow$ x+1 is odd

ex) differentiability implies continuity

($\forall f \in \{\text{functions}\}$, f being diff'able $\Rightarrow$ f is contin.)

Example 6.4. *Determine the truth values of the following universally-quantified statements.*

(1) $\forall x \in \{1, \pi, 1/2\}$, $x > 1/4$.

(2) $\forall \theta \in \mathbb{R}$, $\sin^2\theta + \cos^2\theta = 1$. this is true

(3) $\forall a, b \in \mathbb{Z}$, $a^2 + b \leq 0$. this is false $a=0, b=1$

(4) $\forall a \in \mathbb{Z}$, $a^2 + a$ *is even.* this is true

(5) $\forall x \in \mathbb{R}$, $1/x^2 > 0$.

The Existential Quantifier $\exists$

allows us to make statements about some elements in a set

ex) Some dogs have three legs.

$\exists x \in \{\text{all dogs}\}$, x has 3 legs.

"such that"

$$\boxed{\exists x \in \mathcal{U}, \quad P(x)}$$

$\exists$ is pronounced "there exists, there exist, some"

ex) $\exists\, y \in \{1,2\}, \quad y^2 - y > 0$

this is true because $y=2$ satisfies $y^2 - y > 0$ ✓

Example 6.8. *Several existentially-quantified statements are written below along with their truth values. Make sure you understand what each statement is claiming and why each truth value is assigned.*

(1) $\exists x \in \mathbb{R}, x^2 = 2$ *is true.*

(2) $\exists x \in \mathbb{N}, x^2 = 2$ *is false.*

(3) $\exists t \in \mathbb{Z}, 1/t = t$ *is true.*

(4) *"Some real numbers are negative" is true.*

(5) *"There exist students who hate math" is false.*

(6) $\exists a, b \in \mathbb{Z}, a + b = 0$ *is true.*

(1) is true. $x = \sqrt{2}$ and $x = -\sqrt{2}$ make this true.

(2) is false! notice: $x = 0$, $0^2 = 2$ is false

$x = 1$, $1^2 = 2$ is false

$x = 2$, $2^2 = 2$ is false

and all other choices square > 2

they all make this false!

(6) is true. choose $a = 5$, $b = -5$ $a + b = 5 + (-5) = 0$ ✓

$\exists!$ "there exists a unique"

ex) $\exists!\, t \in \{-1, 0, 1\}, \; t^2 = 1$

this is false; more than one element makes this true!

ex) $\exists!\, n \in \mathbb{N},\quad n^2 = 1$

this is true. only $n=1$ works.

ex) $\exists!\, x \in \{0, 2, 4, 6, 8, \ldots\},\quad x^2 = 1$

this is false because only $x = \pm 1$ work,

and neither are in this set!
