Recurrence Relations & Equations

Links: Math 3336

(from textbook chapter 5)

Recurrence Equations

Recurrence equations that relate "new" terms to "previous ones" which only involve adding, subtracting and multiplying by constants are called **linear recurrence equations**. These equations do not feature non-zero constants, these are called **homogenous** (and those that feature the constants are called **non-homogenous**)

example from textbook:

Example 4.3. *Several recursively-defined sequences are provided below (although no initial conditions are provided). Make sure you can identify which ones are linear (but non-homogeneous), which ones are linear* and *homogeneous, and which ones are not linear.*

(1) $a_n = 2a_{n-1} + 3a_{n-2}$ *(linear, homogeneous)*

(2) $b_n = 2b_{n-1} + 3b_{n-2} + \pi$ *(linear, non-homogeneous)*

(3) $d_n = -3d_{n-1} \cdot d_{n-2}$ *(non-linear)*

(4) $e_n = 2n \cdot e_{n-1}$ *(non-linear)*

(5) $g_n = g_{n-1} - g_{n-2} + 4g_{n-3}$ *(linear, homogeneous)*

(6) $h_n = 2n \cdot h_{n-1} + 3n \cdot h_{n-2} - \sqrt{2}$ *(non-linear)*

More generally, a linear, homogenous recurrence equation is expressed as:
$a_n = f(previous\ terms)$
where the function f is only allowed to multiply previous terms against constants
and then combine these expressions using addition and subtraction i.e.:
$a_n = f(previous\ terms) = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$
(where c_i are constants)
The larger the k value the more "previous terms" are used.

The **order** or **degree** of a linear, homogeneous recurrence equation is simply the number of "previous terms" on which it depends.

Solving Recurrence Equations using Iteration

When we have a recursively defined sequence $\{a_n\}$, its recurrence equation is said to be solved when an explicit formula for each term, $\{a_n\}$, has been found – and this formula is not allowed to depend on previous terms (only on the index or input variable n). These are often called closed form solution.

Iteration is when you just plug in values to the recurrence equation and you eventually notice a pattern to turn it into a closed equation (they really did not need to write three whole paragraphs for this).

Solving Recurrence Equations using Algebra

order-1, linear, homogenous, recurrence equation can always be solved iteratively, and their closed formulas have similar properties, these are called **geometric equations**, which are identified as follows:
(from textbook)

Theorem 5.1. *The linear, homogeneous, order-1 recurrence relation $a_n = c_1 a_{n-1}$ with initial condition a_0 has as the closed form solution*

$$a_n = a_0 (c_1)^n .$$

You can also solve order-2, linear, homogenous, recurrence equations with algebra by using a **characteristic equation**. You obtain this by taking the recurrence equation and solve it for zero, then turn it into a quadratic equation (i.e. turn the first term into x^2, the next one into x and the last into a constant), once you find the the roots you can plug them into a closed formula and use the roots to find unknown alphs and beta shown below.

Theorem 5.2. *The linear, homogeneous, order-2 recurrence relation*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \text{ with initial conditions } a_1 \text{ and } a_0$$

has as a closed form solution

$$a_n = \alpha\,(r_1)^n + \beta\,(r_2)^n$$
$$\text{or } a_n = \alpha\, r^n + \beta\, n\, r^n$$

where $\alpha, \beta \in \mathbb{R}$ are determined by the initial conditions a_0 and a_1, and where $r_1, r_2, r \in \mathbb{R}$ are determined by ***the charateristic equation****.*

(this gets complicated if we get comples roots, or bigger order equations, but those are not covered by this textbook so don't worry about them).

example from textbook:

Example 4.7. *Solve the recurrence equation*

$$a_n = 10a_{n-1} - 25a_{n-2}$$

with initial conditions $a_0 = 3$ and $a_1 = 35$.

The characteristic equation can be formed and solved as follows:

$$x^2 - 10x + 25 = 0$$
$$(x-5)^2 = 0$$
$$x = 5$$

We have precisely one repeated root, i.e. $r = 5$. Closed form solutions are then given by

$$a_n = \alpha 5^n + \beta\, n\, 5^n$$

Plugging in $n = 0$ and $n = 1$, respectively, yields the equations

$$3 = \alpha$$
$$35 = 5\alpha + 5\beta$$

From this we find $\alpha = 3, \beta = 4$ and

$$a_n = 3 \cdot 5^n + 4n \cdot 5^n = (3+4n)5^n.$$