Discrete Math

Lectures 24-27

Counting Techniques

we'll focus on counting <u>lists</u>

a <u>list</u> a finite sequence

ex| length-3 lists of letters (cap. or lower case) from the English alphabet

___ ___ ___

$2 \cdot 26 = 52$ 52 52

total $52 \cdot 52 \cdot 52 = \boxed{52^3}$

how many lists!

"multiplication principle"

\# lists = product of option entries

why?

a _ _ _

aa _ 52

ab _ 52

⋮

az _ 52

aA _ 52

aB _ 52

⋮

aZ _ 52

$52 \cdot 52$

b _ _ → 52^2

c _ _ → 52^2

⋮

z _ _ → 52^2

A _ _ → 52^2

⋮

Z _ _ → 52^2

$$\underbrace{52 \cdot 52^2}_{\boxed{52^3}}$$

Example 3.1 A standard license plate consists of three letters followed by four digits. For example, *JRB-4412* and *MMX-8901* are two standard license plates. How many different standard license plates are possible?

digits : 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

_ _ _ _ _ _ _

26 26 26 10 10 10 10

there are $26^3 \cdot 10^4$ diff. stand. license plates.

we cannot repeat symbols!

Example 3.4 A non-repetitive list of length 5 is to be made from the symbols A, B, C, D, E, F, G. The first entry must be either a B, C or D, and the last entry must be a vowel. How many such lists are possible?

7 symbols

no repetitions

_ _ _ _ _

7 options 6 options 5 options 4 options 3 options

$$\boxed{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}$$

<u>Set up</u> we're using n distinct symbols

How many length-n non-repeating lists are there?

<u>ex</u> using 5 distinct symbols (A, B, C, D, E)

How many non-repeating length-5 lists are there?

$\underline{}\ \underline{}\ \underline{}\ \underline{}\ \underline{}$
$5\ \ 4\ \ 3\ \ 2\ \ 1$

$$\boxed{\begin{aligned}&5\cdot 4\cdot 3\cdot 2\cdot 1\\&= 5!\end{aligned}}$$

$\underline{}\ \underline{}\ \underline{}\ \underline{}\ \cdots\ \underline{}$
$n\ \ n-1\ \ n-2\ \ n-3\ \cdots\ 1$

$$n\cdot(n-1)(n-2)\cdots 2\cdot 1 = n!$$

$n!$ can be defined as the # of non-repeating length-n lists using n symbols

$0! = \#$ of length-0, non-rep lists

use 0 symbols

$=$ "the empty list" $= 1$

Binary Strings = Bit Strings

- lists of 0's and 1's

ex) How many length-7 bit strings are there?

_ _ _ _ _ _ _

2 2 2

$\boxed{2^7}$

ex) How many length-7 bit strings are there that contain <u>at least</u> one 0?

$2^7 - 1$

throw away the one bit string that has NO 0's throw away $\underline{1}\ \underline{1}\ \underline{1}\ \underline{1}\ \underline{1}\ \underline{1}\ \underline{1}$

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

How many bit strings fail our condition?

$Y = \{$ b. strings that fail $\}$

How many bit strings are there?

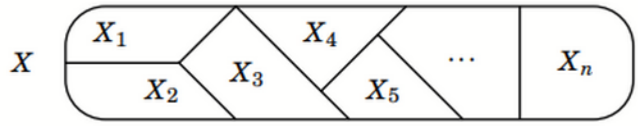
$\mathcal{U} = \{$ all b. strings $\}$

$X = \{$ b. strings that have our property $\}$

$$|X| = |\mathcal{U} - Y|$$

Fact 3.3 (Subtraction Principle)
If X is a subset of a finite set U, then $|\overline{X}| = |U| - |X|$.
In other words, if $X \subseteq U$ then $|U - X| = |U| - |X|$.

Fact 3.2 **(Addition Principle)**
Suppose a finite set X can be decomposed as a union $X = X_1 \cup X_2 \cup \cdots \cup X_n$, where $X_i \cap X_j = \emptyset$ whenever $i \neq j$. Then $|X| = |X_1| + |X_2| + \cdots + |X_n|$.



X is partitioned into subsets X_i

Example 3.6 How many **even** 5-digit numbers are there for which no digit is 0, and the digit 6 appears exactly once? For instance, 55634 and 16118 are such numbers, but not 63304 (has a 0), nor 63364 (too many 6's), nor 55637 (not even).

$X = \{$ all such lists $\}$

9 9 9 9 4

$X_1 = \{$ 6 is in first place $\}$

$X_2 = \{$ 6 is in 2nd place $\}$

$X_3 =$

$X_4 =$

$X_5 = \{$ 6 is in 5th place $\}$

6 8 8 8 3 $\quad 8^3 \cdot 3$

8 6 8 8 3 $\quad 8^3 \cdot 3$

8 8 8 8 6 $\quad 8^4$

total: $|X_1| + |X_2| + |X_3| + |X_4| + |X_5|$

$= \boxed{4 \cdot 8^3 \cdot 3 + 8^4}$

permutations

a permutation on a set X is a (non-repeating) arrangement of its elements into a list

ex) $X = \{1, 2, 3\}$

$$\left.\begin{matrix} 123 & 213 & 312 \\ 132 & 231 & 321 \end{matrix}\right\} \; 3! = 6$$

$\#$ permutations on $X = |X|!$

a k-permutation on a set X is an arrangement of k symbols / elements from X into a non-repeating list

$X = \{1, 2, 3\}$

How many 2-permutations are there?

12   21   31
13   23   32 } 6

How many 1-perms are there? } 3

1   2   3

How many 3-perms are there? } 6

$3! = 6$

How many 4-perms are there? } 0

Zero!

$|X| = n$

How many k-permutations are there?

$$\underbrace{\overline{n}\ \overline{n-1}\ \overline{n-2}\ \cdots\ \overline{n-k}}_{\text{length } k}$$

$$n \cdot (n-1)(n-2) \cdots (n-k)$$

$$= \frac{n(n-1)(n-2)\cdots(n-k)\cdots 2\cdot 1}{(n-k)\cdots 2\cdot 1}$$

$$= \frac{n!}{(n-k)!}$$

$$\# \text{ of } k\text{-permutations on an } n\text{-element set} = \begin{cases} \dfrac{n!}{(n-k)!} & \text{if } 0 \le k \le n \\ 0 & \text{if } k > n \end{cases}$$

$$P(n,k)$$

Counting # of subsets of an n-element set

$$|X| = n$$

$$|\mathcal{P}(X)| = 2^n$$

<u>ex</u> $X = \{a, b, c, d\}$

how many 2-element subsets are there?

(how many ways can we choose 2 objects from X?)

$\{a,b\}$ $\{b,c\}$ $\{c,d\}$

$\{a,c\}$ $\{b,d\}$

$\{a,d\}$

$$3 + 2 + 1 = \boxed{6}$$

If $|X| = n$, how many k-element subsets are there?

if $k > n$, answer $= 0$.

$$P(n,k) = k! \cdot (\text{answer})$$

$$\boxed{\frac{P(n,k)}{k!} = \text{answer}}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

↑

"n choose k"

Why do we care so much about counting?

- a lot of parts of problems require us to count these things!

- count # of operations an algorithm uses!

  → many of these counting procedures involve recursive structure!