$$(1)\ \underbrace{\text{a being is a man}}_{P} \ \Rightarrow\ \underbrace{\text{that being is mortal}}_{Q}.$$
$$(2)\ \underbrace{\text{Socrates is a man.}}_{P}$$
$$(3)\ \underbrace{\text{Socrates is mortal.}}_{Q}$$

In fact we can make all of this abstract by using P's and Q's :

(1) $P \Rightarrow Q$
(2) P
(3) Therefore Q

This argument is simply saying that if the conditional $P \Rightarrow Q$ is true *and* if the hypothesis P is true then it logically follows that Q is true. This style of argument has a fancy Latin name: "*Modus Ponens*" and it is contained in the truth-table that defines $\Rightarrow$ – the only row that has $P \Rightarrow Q$ marked as T *and* that has P marked as T also has Q marked as T (this is row 1).

5. Truth Tables and Logical Equivalence

You may have noticed that a statement of the form $P \wedge \neg P$ is *always* assigned a truth value of F – no matter what the truth value of P is. Something just as extreme happens for a statement of the form $P \vee \neg P$ only in the opposite direction. This is captured in the truth tables:

P	$P \wedge \neg P$
T	F
F	F

Contradiction

P	$P \vee \neg P$
T	T
F	T

Tautology

$P \wedge \neg P$ is an example of a **contradiction** which is the name we use for an abstract statement **whose truth table contains only F values**. Similarly, $P \vee \neg P$ is an example of a **tautology**, an abstract statement **whose truth table contains only T values**.

Example 5.1. *A statement of the form $(P \wedge Q) \wedge (\neg P \vee \neg Q)$ is another example of a **contradiction**, one that involves more "pieces" than the contradiction above; observe that its truth table records only F values:*

P	Q	$(P \wedge Q) \wedge (\neg P \vee \neg Q)$
T	T	F
T	F	F
F	T	F
F	F	F

Example 5.2. *Several abstract statements are provided below, and each one is labelled as a contradiction, a tautology or as* neither. *Make sure you understand* why *each label is accurate! (Using truth tables should help.)*

$$(P \vee Q) \vee (\neg P \wedge \neg Q) \text{ tautology} \tag{18}$$

$$(P \Rightarrow Q) \wedge (P \wedge \neg Q) \text{ contradiction} \tag{19}$$

$$(P \wedge Q \wedge R) \vee \Big(\neg R \Rightarrow (P \vee Q)\Big) \text{ neither} \tag{20}$$

$$\Big(P \iff Q\Big) \Rightarrow \Big((R \vee P) \iff (R \vee Q)\Big) \text{ tautology} \tag{21}$$

$$(P \wedge Q) \wedge \neg P \text{ contradiction} \tag{22}$$

$$\neg(\neg P) \iff P \text{ tautology} \tag{23}$$

The last formula in Example 5.2 *should* seem somewhat obvious or even silly once you give it some thought. We can interpret it as saying something like this

"When you negate a statement twice you're left with a sentence that has the same meaning as the original statement"

which certainly sounds very intuitive. In fact, as you may have noticed, the operation of *negation* is similar to the the algebraic act of *multiplying by* -1 in that it flips the values of our objects (either between *positive and negative* or between T *and* F), and performing it twice "cancels."

Just like we use the equal sign "=" in algebraic formulas, we'd like to use it, too, for logical formulas. In other words, we want to be able to write things like "$\neg(\neg P) = P$" much the way we write algebraic things like "$-(-2) = 2$." To do this we introduce the notion of **logical equivalence**. Two abstract statements, A and B, are said to be **logically equivalent** if the statement $A \iff B$ is a tautology; we use the equal sign to write this symbolically as $A = B$ and pronounce it "A is logically equivalent to B."

$$\boxed{A = B \text{ means } A \iff B \text{ is a tautology}}$$

There is another, just-as-good way to think about two logically equivalent statements, one that may seem easier to use when working with actual logical formulas. The bi-conditional $A \iff B$ will be a tautology precisely when the truth tables for A and B look identical.

$$\boxed{A = B \text{ means } A \text{ and } B \text{ have the same truth tables}}$$

Two abstract statements can be both logically equivalent *and* convoluted by involving far more pieces than our example $\neg(\neg P) = P$. An example should help clarify both the concept of **logical equivalence** and how complex some formulas can appear.

Example 5.3. $\neg(P \Rightarrow Q) = (P \wedge \not Q)$. *This formula can be verified by comparing truth tables.*

P	Q	$\neg(P \Rightarrow Q)$
T	T	F
T	F	T
F	T	F
F	F	F

P	Q	$P \wedge \neg Q$
T	T	F
T	F	T
F	T	F
F	F	F

The fact that these truth-tables match up – line-for-line – tells us that the statements are logically equivalent. One can save some page space by compressing these tables into a single one:

P	Q	$\neg(P \Rightarrow Q)$	$P \wedge \neg Q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	F	F

Because the two abstract statements $\neg(P \Rightarrow Q)$ *and* $\neg P \wedge Q$ *have the same truth values, when we connect them with the bi-conditional* $\iff$ *the resulting truth table will have only* T*'s – after all, the bi-conditional returns a value of* T *precisely when each piece has the same truth-value.*

P	Q	$\neg(P \Rightarrow Q) \iff (P \wedge \neg Q)$
T	T	T
T	F	T
F	T	T
F	F	T

Another key example of logical equivalence concerns a conditional and its so-called **contrapositive**. The contrapositive of $P \Rightarrow Q$ is the new conditional $\neg Q \Rightarrow \neg P$, and these are *always* logically equivalent.

Example 5.4. ***Conditionals are logically equivalent to their contrapositives.*** *This can be seen by completing the following truth table.*

P	Q	$P \Rightarrow Q$	$\neg Q \Rightarrow \neg P$
T	T	T	
T	F		F
F	T		
F	F		

The fact that $(P \Rightarrow Q) = (\neg Q \Rightarrow \neg P)$ can also be understood using the intended *meanings* of each generic statement. We understand $P \Rightarrow Q$ to mean that a kind of promise is taking place, one that claims "whenever P is true, so is

Q." Its contrapositive is expressing that same promise, only it does so by clarifying *when that promise is violated*. It says that "If Q is false, then P was false, too," or in other, looser terms "The promised result, Q, will fail to happen only when the conditions for the promise, P, are not met." Note, though, that *both* abstract statements are talking about the *same promise*.

Example 5.5. *The statements $\neg P \vee Q$ and $\neg(P \wedge \neg Q)$ are logically equivalent. Check that this is, indeed, the case by completing their truth tables. (We have included some additional columns in these tables to help you sort out the truth values for these somewhat-complicated statements.)*

P	Q	$\neg P$	$\neg P \vee Q$
T	T		T
T	F		
F	F	T	
F	F	T	

P	Q	$\neg Q$	$P \wedge \neg Q$	$\neg(P \wedge \neg Q)$
T	T	F		
T	F	T		
F	T		F	
F	F			

Example 5.6. *A statement of the form* $P \vee Q$ *is logically equivalent to*

$$P \vee Q = \Big((P \wedge R) \vee (P \wedge \neg R) \Big) \vee \Big((Q \wedge R) \vee (Q \wedge \neg R) \Big).$$

However, when we first compare truth tables this formula appears to be wrong as the rows of one table can't *be matched up with the rows of the other! Look for yourself:*

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	R	$((P \wedge R) \vee (P \wedge \neg R)) \vee ((Q \wedge R) \vee (Q \wedge \neg R))$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

There is an easy "fix" here that allows us to both compare these truth tables and *conclude these statements should be regarded as logically equivalent. While* $P \vee Q$ *does not depend on the abstract statement* R*, there is no harm in including* R *as part of its truth-table; that is, we can extend the truth table for* $P \vee Q$ *as follows*

P	Q	R	$P \vee Q$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

We can now compare these truth tables and conclude the logical equivalence; we do this by simply combining the tables and notcing the truth values line up "row for row."

P	Q	R	$P \vee Q$	$((P \wedge R) \vee (P \wedge \neg R)) \vee ((Q \wedge R) \vee (Q \wedge \neg R))$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	F
F	F	F	F	F

One of the most famous and important examples of logical equivalence are"De Morgan's Laws." These laws tell us how to simplify negated "*or*"s and negated "*and*"s.

De Morgan's Laws
$\neg(P \wedge Q) = \neg P \vee \neg Q$
$\neg(P \vee Q) = \neg P \wedge \neg Q$

One can (and should!) verify these formulas by comparing truth tables. Example 5.6 invites you to do just that.

Example 5.7. *Verify De Morgan's Laws by comparing truth tables.*

P	Q	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	F	
T	F		T
F	T		
F	F		

P	Q	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
T	T		F
T	F		
F	T	F	
F	F		

Note: you may want or need to include additional columns in these tables to keep track of things, perhaps ones for $P \wedge Q$, $P \vee Q$, $\neg P$, and $\neg Q$.

However, it is just as important or *more* important to understand why these formulas are true using our interpretations of the connectives $\wedge$ and $\vee$. The first law, for instance, is negating an "and" statement, which we can read as claiming "both P and Q are true." What should it mean to *negate* this? Unsurprisingly, the answer is "at least one, P or Q, is false," and this is *precisely* what the statement $\neg P \vee \neg Q$ can be interpreted as saying. Can you reason similarly about the natural meaning of the second law? The following example may help!

Example 5.8. *De Morgan's Laws are used in natural-language sentences, too, The following examples might help reveal this.*

(1) *The sentence "David feels like going to a Thai restaurant or a French one" can be negated as "David wants neither Thai food nor French food." The meaning of this sentence, though, can be made clearer by rewriting it as "David does not want to go to a Thai restaurant AND he does not want to go to a French one."*

(2) *Use De Morgan's Laws to write an English-sentence negation of the following:* P : *"Casey likes both video games and Lego."*

$\neg P =$

Closing Thoughts and Section Summary

In this section you read about and worked on **contradictions** and **tautologies**

Definition 1.7. *An abstract statement is called a* ***contradiction*** *if its truth table produces only values of F.*

Definition 1.8. *An abstract statement is called a* ***tautology*** *if its truth table produces only values of T.*

You also read about and worked on the notion of **logically equivalent** statements. There are two definitions one can use here, but probably the second one is the easiest to understand.

Definition 1.9. *The notation $P = Q$ is used to say that two abstract statements, P and Q, are* ***logically equivalent****. This means*

$$\begin{aligned}
&\text{(Def. 1)}\ \ P \iff Q \text{ is a tautology.}\\[1em]
&\text{(Def. 2)}\ \ P \,\text{and}\, Q\ \text{ have the same truth tables}\\
&\qquad\qquad\qquad\qquad\text{(perhaps after adjusting their sizes)}
\end{aligned}$$

Many Logicians, Computer Scientists, and Mathematicians think about logically equivalent statements the way Algebra students think about formulas and variables. For instance, if $\mathbf{T}$ is a tautology, $\mathbf{F}$ is a contradiction, and P, Q and R are abstract statements, then the following "equations" hold:

$$P \vee \mathbf{T} = \mathbf{T} \tag{24}$$

$$P \wedge \mathbf{T} = P \tag{25}$$

$$P \wedge \mathbf{F} = \mathbf{F} \tag{26}$$

$$\neg(\neg P \wedge Q) = P \vee \neg Q \tag{27}$$

$$\neg(\neg P \Rightarrow \neg Q) = P \wedge Q \tag{28}$$

$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R) \tag{29}$$

Once can verify all of the formulas above by comparing truth tables, but once a few logical equivalences have been established, it is often "easier" to use them and check the formulas the way algebra formulas are checked. For instance, formula (27) above can be checked by appealing to De Morgan's Laws and the established equivalence $\neg(\neg P) = P$:

$$\neg(\neg P \wedge Q) \underset{\text{De Morgan}}{=} \neg(\neg P) \vee \neg Q \underset{\text{double neg.}}{=} P \vee \neg Q.$$