In this section you read about conditional statements $P \Rightarrow Q$ and how they attempt to resemble if-then sentences. You also learned about **vacuously true** statements. Your goal is to be able to reproduce the $\Rightarrow$ truth table easily *and* use your understanding of it to form new if-then statements and determine when they are true or false.

Definition 1.3. *A* ***conditional statement*** *is a proposition of the form $P \Rightarrow Q$. The statement P is referred to as the* ***premise*** *or* ***hypothesis****. The statement Q is referred to as the* ***conclusion****.*

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth Table for $\Rightarrow$

Notation	English Phrasings
$P \Rightarrow Q$	P implies Q If P, then Q Whenever P, then also Q Q is guaranteed by P P is a sufficient condition for Q Q is a necessary condition for P Q whenever P

Definition 1.4. *A* ***vacuously true*** *statement is a conditional statement whose hypothesis is false.*

4. Bi-conditional Statements

It is also important to understand that the two conditionals $P \Rightarrow Q$ and $Q \Rightarrow P$ are *different*, often times having *different* truth values. Even though they both use the same statements P and Q, the order in which these parts are connected matters. For instance, the conditional statement

$$\text{(At least ten people live in Canada)} \Rightarrow \text{(Birds do not exist)}$$

is false while the seemingly-related statement

$$\text{(Birds do not exist)} \Rightarrow \text{(At least ten people live in Canada)}$$

is (vacuously) true.

Indeed, the conditional $Q \Rightarrow P$ *should* be different from $P \Rightarrow Q$ as it swaps around its hypothesis and conclusion. It is so different, in fact, that it gets a special name: $Q \Rightarrow P$ is called the **converse** of $P \Rightarrow Q$. The following examples should help you check your understanding of converses, especially when you make sure you can determine the truth value of each statement.

Example 4.1.

(10) *The converse of "If a whole number is a multiple of 4, then it is even" is "If a whole number is even, then it is a multiple of 4."*

(11) *The converse of* $(1+2=3) \Rightarrow \left(\sin^2(\pi/7)+\cos^2(\pi/7)=2\right)$ *is* $\left(\sin^2(\pi/7)+\cos^2(\pi/7)=2\right) \Rightarrow (1+2=3)$

(12) *The converse of "If a shape is a square, then it is a rectangle" is "If ____________________, then __________________."*

(13) *The converse of "* $2^{13}+4$ *is even* $\Rightarrow \left(9^{-1}\right)^{-1}=9$*" is*
_______________ $\Rightarrow$ __________________________

It can happen that the conditional $P \Rightarrow Q$ and its converse $Q \Rightarrow P$ are simultaneously true. If you compare the tables for these two statements you will see that this happens precisely when P and Q share the same truth value. These observations motivate the definition of our last logical connective, $\iff$.

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

The sentence $P \iff Q$ is most commonly pronounced "P if and only if Q," and is referred to as a **bi-conditional**. It asserts that statements P and Q are either both false or are both true. You can *also* interpret this statement as asserting that $P \Rightarrow Q$ and its converse are both true. There are, of course, many *other, equivalent* ways to pronounce "$P \iff Q$," which we summarize at the close of this section, but for now practice your understanding on the following examples.

Example 4.2. *Example 4.1 features four conditionals. Which ones have a* true *bi-conditional? Which ones have a* false *bi-conditional? How do you know?*

Example 4.3. *Several bi-conditional statements are provided below along with their truth values. Make sure you understand* why *their truth values are the given ones.*

(14) *A function is constant* $\iff$ *the function's derivative is constantly* 0 *(T)*

(15) $\left(\frac{3}{5}-\frac{2}{9}=\frac{17}{45}\right) \iff$ *There are no even numbers. (F)*

(16) *Birds don't exist if and only if* $2+4=6$. *(F)*

(17) *Sine is concave up if and only if* 3 *is even (T).*

Example 4.4. *Suppose P and Q are statements where $P \Rightarrow Q$ is* vacuously true*, and suppose we know that $P \iff Q$ is true. What is the truth value of statement Q?*

Closing Thoughts and Section Summary

In this section you read about and worked on **bi-conditional statements**.

Definition 1.5. *A **bi-conditional statement** is a proposition of the form $P \iff Q$. It is true whenever P and Q have the same truth value, and it is false otherwise.*

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth Table for $\iff$

Notation	English Phrasings
$P \iff Q$	P if and only if Q
	P is necessary and sufficient for Q
	If P then Q, and conversely.
	P is logically equivalent to Q
	Q whenever P, and conversely

You also learned about **the converse** of a conditional statement and probably have noticed how $P \iff Q$ involves this converse; specifically, that the bi-conditional means the same thing as $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. Formulas like

$$\Big(P \iff Q\Big) = \Big((P \Rightarrow Q) \wedge (Q \Rightarrow P) \Big)$$

will be discussed in the next section.

Definition 1.6. *The **converse** of the conditional $P \Rightarrow Q$ is the new conditional $Q \Rightarrow P$.*

Modus Ponens and Some Basic Philosophy Stuff

The concept of "logic," at least as it was first formalized by ancient Greeks, originally focused not on generic statements or propositions as we have presented here but on common, everyday sentences. One of the most influential Western philosophers, Aristotle, presented *syllogisms* or specially-formatted arguments that exemplify exactly how all of this is supposed to work. Here is, perhaps, his most famous example:

(1) All men are mortal.

(2) Socrates is a man.

(3) Therefore Socrates is mortal.

Philosophers refer to sentences (1) and (2) as "premises" in the syllogism and sentence (3) as a "logical conclusion." These kinds of syllogisms can be understood in terms of conditional statements; sentence (1) can be rephrased as "If a being is a man, then that being is mortal," and sentence (2) can be understood as asserting the hypothesis of this conditional while sentence (3) is the conclusion. In more symbolic terms:

$$(1)\ \underbrace{\text{a being is a man}}_{P} \Rightarrow \underbrace{\text{that being is mortal}}_{Q}.$$
$$(2)\ \underbrace{\text{Socrates is a man.}}_{P}$$
$$(3)\ \underbrace{\text{Socrates is mortal.}}_{Q}$$

In fact we can make all of this abstract by using P's and Q's :

(1) $P \Rightarrow Q$
(2) P
(3) Therefore Q

This argument is simply saying that if the conditional $P \Rightarrow Q$ is true *and* if the hypothesis P is true then it logically follows that Q is true. This style of argument has a fancy Latin name: "*Modus Ponens*" and it is contained in the truth-table that defines $\Rightarrow$ – the only row that has $P \Rightarrow Q$ marked as T *and* that has P marked as T also has Q marked as T (this is row 1).

5. Truth Tables and Logical Equivalence

You may have noticed that a statement of the form $P \wedge \neg P$ is *always* assigned a truth value of F – no matter what the truth value of P is. Something just as extreme happens for a statement of the form $P \vee \neg P$ only in the opposite direction. This is captured in the truth tables:

P	$P \wedge \neg P$
T	F
F	F

Contradiction

P	$P \vee \neg P$
T	T
F	T

Tautology

$P \wedge \neg P$ is an example of a **contradiction** which is the name we use for an abstract statement **whose truth table contains only F values**. Similarly, $P \vee \neg P$ is an example of a **tautology**, an abstract statement **whose truth table contains only T values**.

Example 5.1. *A statement of the form $(P \wedge Q) \wedge (\neg P \vee \neg Q)$ is another example of a* ***contradiction****, one that involves more "pieces" than the contradiction above; observe that its truth table records only F values:*

P	Q	$(P \wedge Q) \wedge (\neg P \vee \neg Q)$
T	T	F
T	F	F
F	T	F
F	F	F