Chapter 1

Logic

"Logic is the beginning of wisdom... not the end of it."
– Ancient Vulcan Proverb

You've likely taken a few math courses before this one, including a class called "Algebra" where you were asked to solve lots of problems. With this experience in mind, the following exercise might seem like a simple task:

$$\text{Solve } x - 4 = 9 \text{ for } x.$$

The solution $x = 13$ takes little time to figure out, but much more time is needed to carefully explain *how* you arrived at this answer. Consider, for instance, how you would explain your thinking to someone who has far less experience manipulating these kinds equations. A younger child, say, would likely ask "Why?" every time you described one part of your process.

The steps taken to solve our equation ultimately come down to ***logic***: a careful and methodical way of thinking that allows us to make new conclusions from given information. In short, we use *logic* to reason towards conclusions. For our equation above, we were given information (that the unknown variable x satisfies the condition $x - 4 = 9$), and then we were asked to make a conclusion (one about the value of x itself). The tools we used to make that conclusion were, in this case, straightforward rules of algebra (adding 4 to both sides). Such rules of algebra are themselves based on rules of *logic*.

All said and done, this exercise is nothing more than an exercise in logic. To make this point clear, consider this rewording:

$$\text{Given that } x - 4 = 9 \text{ is true, what conclusions can we make about } x\text{?}$$

The meaning of the problem has not changed, and, indeed, lurking behind every exercise in mathematics is a question about drawing conclusions, one that uses *logic*. Logic does not only apply to equations and rules of Algebra, though. As you will see in this chapter, logic applies to a larger collection of objects called "*statements*."

So what, exactly, are we going to *do* in this chapter on logic? First we'll explore this world of "*statements*" and make sure we understand what they are and are not, some of the different ways in which they can be combined and created, and how to use special tables to share that understanding. The rules that tell us how this all works are the so-called rules of "propositional logic." Next we'll explore the so-called rules of "predicate logic," and these rules will allow us to handle more interesting kinds of statements. Lastly, our chapter will conclude with some tips for translating between official "statements of logic" and less formal English sentences.

1. Statements

A **statement (or proposition)** is a sentence that is either true or false (but not both). There are lots of examples of statements, of course, and you've come across many in your own life (you are reading one right now, in fact). Mathematics provides us with lots of examples, too, since we take the word 'sentence' to include mathy sentences like equations and inequalities.

Example 1.1. *The sentence "Houston is a city in Texas." is an example of a statement because it is* true.

The sentence "Every dog has brown eyes." is an example of a statement because it is false.

The sentence "$3 + 2 = 32$" is an example of a statement because it is ________.

Another way to say that the first sentence in Example 1.1 is "a true statement" is to say that the sentence "has a truth value of true." The second sentence of that example has a "truth value" of false, and, in general, we say that statements are sentences that have (or can be assigned) a single "truth value" either T or F.

Whenever a new term is introduced (in this book or, really, in any other math textbook you use in the future!) it is useful to consider *non-examples*, too. In this case, we should also take a moment to reflect on sentences *that are **not** statements*.

Example 1.2. *The sentence "Shut the door." is not a statement. This sentence is neither true nor false; it is a command.*

The sentence "Is the window open?" is not a statement. This sentence is neither true nor false; it is a question.

The sentence "$x - 4 = 9$" is not a statement. This sentence is true only if we substitute $x = 13$ and it is false if we substitute other values for x (such as $x = 11$). Until x has been substituted it is neither true nor false.

The third sentence in Example 1.2 is of interest to us because it has the possibility of being true or false (once its variable, x, has been assigned a value), while the other two sentences in Example 1.2 lack this possibility! The non-statement "$x - 4 = 9$" is an example of an ***open sentence***: a sentence that contains one or more variables, and will become an *actual* statement once those variables have been replaced. Some logicians use the term ***predicate*** instead of ***open sentence***, and, if you like, you can think of these as "pre-statements," sentences *waiting* to be turned in to statements once appropriate substitutions have been made for their variables.

Example 1.3. *The sentence "*$\cos(\theta) = 1$*" is an* ***open sentence****.*

The sentence "$t + 3 > 7$ *is an* ***open sentence****.*

The sentence "She is 6 feet tall." is an ***open sentence****. (What is the "variable" in this open sentence?)*

It is common to label statements using capital letters such as P, Q, R, etc, which is what we will do throughout our book. This lets us to refer to specific statements more briefly as well as refer to abstract statements (as in the next section). An open sentence with variables x, y will be similarly labelled as $P(x, y)$ or $Q(x, y)$, etc.

Example 1.4. *Determine whether the statements P, Q, R and S (given below) are true or false.*

$$P : \tan(\pi/4) = 1$$
$$Q : \textit{ the derivative of a polynomial is a polynomial.}$$
$$R : \pi \leq e$$
$$S : \textit{ Every even whole number is a multiple of 4.}$$

Example 1.5. *The open sentences below are non-statements, each containing one variable. Find a value you can assign this variable that makes the open sentence a true statement.*

$$P(\theta) : \tan(\theta) = 1$$
$$Q(x) : x^2 = 0$$
$$R(t) : \int_0^t 2x\,dx = 9$$

It is useful to point out a couple of curious facts about certain sentences here. First, some sentences *appear* to have an unsubstituted variable when, in reality, the variable is removable or actually replaced by the sentence itself. Consider the sentence

P : The function cosine outputs numbers between -1 and 1.

Take a moment to recall some of your knowledge about cosine to make certain you understand that this is a true statement. Now consider this rewording:

$$P : \text{Whenever a real number } \theta \text{ is inputted into cosine}$$
$$\text{the output satisfies } -1 \le \cos\theta \le 1.$$

Both of these sentences express the same idea, but the second version uses a variable and so *appears to be* an open sentence.

The term "whenever" is the key ingredient in the second version of P, and we will focus on these kinds of statements in Section 1.6. For now, though, it is enough to understand that the second sentence "removes" the variable θ by talking about the sorts of things (real numbers) we can plug in for θ. Compare this to the open sentence $Q(x)$ in Example 1.5 above where the variable x is not "removed" or "plugged in" by any of the language used in the actual sentence.

Second, while statements can be true or false and open sentences are neither true nor false (at least not until their variables are replaced), there are *other*, interesting examples of sentences that are *both* true *and* false! The most famous example is the so-called "Liar's Paradox" which can stated as follows:

$$LP : \text{This sentence is false.}$$

What are we to make of this sentence? Should we regard it as something that is true or false (i.e. should we count it as a "statement")? Or should we count this as a non-statement, like an open sentence or a command or a question are regarded as non-statements?

To answer these questions we need to conduct a thought experiment, one that is better done by you on your own and not simply read about in a book. Hopefully the following questions will guide you through this fascinating puzzle.

Step 1. Suppose LP is true. Does LP truthfully talk about itself?

Step 2. Suppose LP is false. Does LP truthfully talk about itself?

If you carefully reason through both Steps 1 and 2, you will come to the astounding conclusion that *if* LP has a truth value, then it must be both true and false! This makes LP a non-statement (since statements must be either true or false, but not both).

Closing Thoughts and Section Summary

Once you've fully digested this section you may come to the suspicion that statements (and even open sentences) are rather puny or pathetic kinds of things. They can only express somewhat simple ideas, ones that are (or upon substitution will be) true or false. By definition they exclude a whole lot of interesting and useful things we want to be able to say as human beings!

Asking questions and making demands are essential parts of our everyday discussions, for instance, and these are non-statements. Self-referential sentences like LP are similarly common, useful and interesting, but many of them aren't statements either. Metaphors and poetic language and song lyrics? Forget about 'em! All of these things are simply too complicated to be statements. This is one reason why many prominent mathematicians regard subjects like English and Art as more difficult and complex than math itself; these subjects deal with richer, harder-to-exactly-specify aspects of human expression.

Interestingly enough, it is by focusing in on statements (and open sentences) that logic and mathematics accomplishes so much. By only considering such well-behaved and puny or pathetic objects, we are are able to build and apply and re-apply careful rules and procedures to produce newer and more complex statements, and all with crystal clear precision. The collection of new statements mathematicians produce is an overwhelming and ongoing enterprise, arguably as rich and beautiful as any other human achievement What's more is that many of these statements can be used in real-world applications!

In this section you read about **statements (propositions), non-statements, open sentences (predicates)**.

Definition 1.1. *A* ***statement*** *or* ***proposition*** *is a sentence that is either true or false (but not both). We say that statements are sentences that "have a truth value." We use capital letters like P, Q, R, etc. to denote these.*

Definition 1.2. *An* ***open sentence*** *or* ***predicate*** *is a non-statement that contains variables, and when those variables are replaced the sentence becomes a statement. We use expressions like $P(x), Q(s, t)$, etc., to denote open sentences and their variables.*

Statements (Propositions)	Non-Statements
sentences that are true or false (not both)	Open Sentences (variables need subst.) Commands Questions Self-refuting sentences

2. Operations on Statements

This section has a kind of "serious sounding" name, doesn't it? But, thankfully, "operating on statements" just means "creating new statements from old ones." In this section we will discuss three main ways to do this: *negation*, *conjunction (and)*, and *disjunction (or)*. A fourth way, *exclusive-or*, is also mentioned.

Negation. Given a statement P, we can form a new statement, $\neg P$, called **the negation of** P in a simple way. This new statement inserts the phrase "it is not true that" in front of the original statement, thereby negating its meaning and truth value:

$$\neg P: \text{ it is not true that } P.$$

Computer scientists tend to use the symbol $\neg$ while many mathematicians use $\sim$, but both mean the same thing, and they are both pronounced "not P."