Discrete Math

Lecture 16

Sequences

a sequence (of real numbers) is a function

$$a : \mathbb{N} \longrightarrow \mathbb{R}$$

normally : an unending list of numbers

ex) perfect squares

1, 4, 9, 16, 25, 36, ...

1st term, 2nd term, 5th term, 6th term

$a(1) = a_1 = 1$

$a(2) = a_2 = 4$

$a(3) = a_3 = 9$

$\vdots$

$$a : \mathbb{N} - \{0\} \longrightarrow \mathbb{R}$$

$$a(n) = a_n = n^2$$

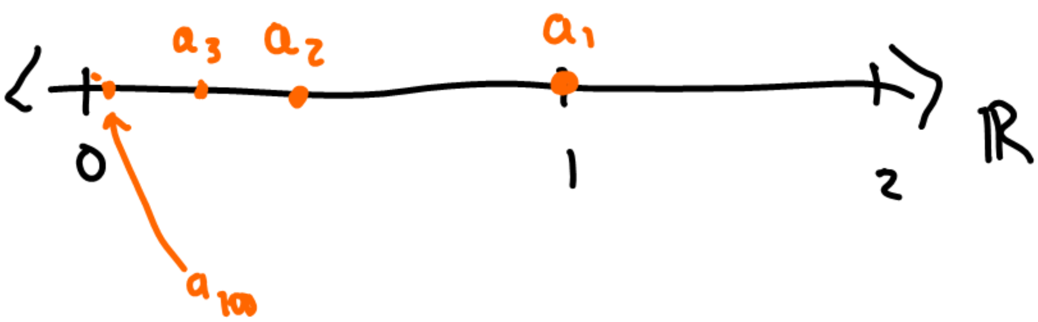
ex) $a_n = \frac{1}{n}$

$\left(a_0 = \frac{1}{0} \text{ undefined} \right)$

$a_1 = \frac{1}{1} = 1 \qquad a_2 = \frac{1}{2} \qquad a_3 = \frac{1}{3} \qquad \ldots$

$a_{100} = \frac{1}{100}$

Calculus view



Example 5.3. *Find a closed formula for the sequence $\{a_n\}$ whose first six terms are the following:*

$$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \ldots$$

Hint: the denominators should look special or familiar to you.

$a_7 = \frac{1}{49}$

$a_6 = \frac{1}{36}$

$a_5 = \frac{1}{25}$

$a_4 = \frac{1}{16}$

$a_3 = \frac{1}{9}$

$a_2 = 1/4$

pattern

$a_n = \frac{1}{n^2}$

Example 5.2. *Consider the sequences* $a_n = (-1)^n$ *and* $b_n = \cos(n\pi)$. *Observe that* $a_0 = a_2 = 1$ *and that* $a_{303} = -1$. *Interestingly enough, the sequence* b_n *behaves very similarly;* $b_0 = \cos(0) = 1, b_1 = \cos(\pi) = -1$, *and* $b_2 = \cos(2\pi) = 1$. *Indeed, these sequences have the exact same terms and in the exact same order!*

$a_n = (-1)^n$

$a_0 = (-1)^0 = 1$

$a_1 = (-1)^1 = -1$

$a_2 = (-1)^2 = 1$

$a_3 = (-1)^3 = -1$

$a_4 = (-1)^4 = 1$

$\{1, -1, 1, -1, 1, -1, \ldots\}$

$b_n = \cos(\pi \cdot n)$

$b_0 = \cos(0) = 1$

$b_1 = \cos(\pi) = -1$

$b_2 = \cos(2\pi) = 1$

$b_3 = \cos(3\pi) = -1$

$b_4 = \cos(4\pi) = 1$

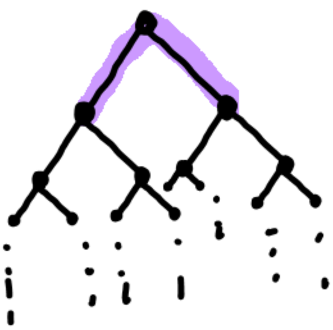
$\{1, -1, 1, -1, 1, -1, \ldots\}$

Recursively Defined Sequences /

Sequences w/ Recursive Structure

"Recursion" ← important, big theme in our class smaller / previous parts explain bigger / future ones

Visual ex "full binary tree"



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ex) $\underbrace{a_n = 2 + a_{n-1}}$ (recursively defined)

a recurrence relation

a recurrence equation

$a_0 = 4$ initial condition

$a_1 = 2 + a_0 = 2+4 = 6 = 2 + 4$

$a_2 = 2 + \boxed{a_1} = 2 + \boxed{2 + a_0} = 2+2+4$

$= 2+6 = 8$

$$a_3 = 2 + \boxed{a_2} = 2 + \boxed{2 + \boxed{a_1}} = 2 + 2 + \boxed{2 + 4}$$

$$= 2 + 2 + 2 + 4$$

$$= 10$$

a pattern: $a_n = (\text{add } n \text{ } 2\text{'s}) + 4$

$$a_n = \underbrace{(2 + 2 + \cdots + 2)}_{n\text{-times}} + 4$$

"closed formula" $\longrightarrow$ $\boxed{a_n = 2n + 4}$

ex) follow up example check that

$$a_n = 2n + 4$$

satisfies or solves the recurrence eqn & i.c.

$$a_n = 2 + a_{n-1}$$

$$a_0 = 4$$

check initial cond.

$n=0 \quad a(0) = a_0 = 2\cdot 0 + 4 = 4$ ✓

check recursive structure

We want $\boxed{a_n = a_{n-1} + 2}$

we have $a_n = 2n+4$

$LHS = a_n = \underline{2n+4}$

$RHS = \boxed{a_{n-1}} + 2 = \boxed{2(n-1)+4} + 2$

$= 2n - 2 + 4 + 2$

$= \underline{2n+4}$

Since LHS = RHS, we have $a_n = a_{n-1} + 2$

ex| Fibonacci Numbers

$F_0 = 1$, $F_1 = 1$ $\longleftarrow$ two initial conditions

$F_n = F_{n-1} + F_{n-2}$

$1, 1, \underset{F_2}{2}, \underset{F_3}{3}, \underset{F_4}{5}, \ldots$

ex| $a_n = 2a_{n-1} + 1$

$a_0 = 1$

$a_1 = 2 \cdot a_0 + 1 = 2 \cdot 1 + 1 = 3$

$a_2 = 2 \cdot \boxed{a_1} + 1 = \underbrace{2 \cdot (2 \cdot 1 + 1)}_{\text{pattern}}$

$= 2 \cdot 3 + 1$

$= 7$

$a_3 =$