3336

Office Hour

11:00 am

Last week = induction ("weak" + "strong")
This week = structural induction
} Induction

A mathematician wishes to prove the following proposition using a Proof by Induction:

$n^2 - 3n + 4$ is even for all $n \geq 1$.

Which of the following can be used as part of the Inductive Step in her proof?

$P(n)$: $n^2 - 3n + 4$ is even

1) explore examples

$n=1$ $P(1)$: $1^2 - 3\cdot1 + 4 = -2+4 = 2$ is even ✓

$n=2$ $P(2)$: $2^2 - 3\cdot2 + 4 = 4-6+4 = 2$ is even ✓

to prove this by induction we first check a base case.

Base Case $n=1$

When $n=1$, the expression $n^2-3n+4 = 2$ which is even.

Inductive Step (prove $P(k) \Rightarrow P(k+1)$)

Suppose $P(n)$ is true when $n = k \in \mathbb{N}$.

This means $k^2 - 3k + 4$ is even.

We want to show $P(k+1)$ is true; we WTS $(k+1)^2 - 3(k+1) + 4$ is even.

It follows that

$$(k+1)^2 - 3(k+1) + 4 = k^2 + 2k + 1 - 3k - 3 + 4$$
$$= k^2 - 3k + 4 \quad + 2k + 1 - 3$$
$$= 2a \quad + \quad 2k - 2$$
$$= 2(a + k - 1)$$

which is an even number and where $k^2 - 3k + 4 = 2a$ follows from our ind. hyp. $\square$

Strong induction

- more base cases
- inductive step: $P(k-1) \wedge P(k) \Rightarrow P(k+1)$
  $P(1) \wedge \ldots$

Consider the recursively defined sequence $\{a_n\}$ with recurrence relation

$$a_n = 6a_{n-1} + 9a_{n-2}$$

Suppose we want to use Induction to prove a statement $P(n)$ about this sequence. Our Inductive Step would have us proving " $(\text{previous cases}) \Rightarrow P(k+1)$." How many "previous cases" will be needed in this situation?

1) explore sequence

$a_0 = 2$
$a_1 = 3$
} initial conditions

$a_2 = 6a_1 + 9a_0$
$= 6 \cdot 3 + 9 \cdot 2$
$= 36$

$a_3 = 6 \cdot a_2 + 9 \cdot a_1$
$= 6 \cdot 36 + 9 \cdot 3 = 243$

$\forall n \geq 1, \; 3 \mid a_n$ = "3 divides a_n"

$P(n)$: $\underline{3 \mid a_n}$

two base cases + strong induction (using two previous cases)

Ind. Step $(P(k-1) \wedge P(k) \Rightarrow P(k+1))$

ind. hyp. → $\boxed{\text{Suppose } P(k) \wedge P(k-1) \text{ are true.}}$ This means $a_k + a_{k-1}$ are divisible by 3.

(we want to show a_{k+1} is divisible by 3)

It follows that $a_{k+1} = 6a_k + 9a_{k-1}$

$$= 6 \cdot (3a) + 9 \cdot (3b)$$

where $a, b \in \mathbb{N}$ and we used our (strong) ind. hyp.

$a_{k+1} = 3(6a + 9b)$ is divisible by 3. $\square$

Structural Induction

more general

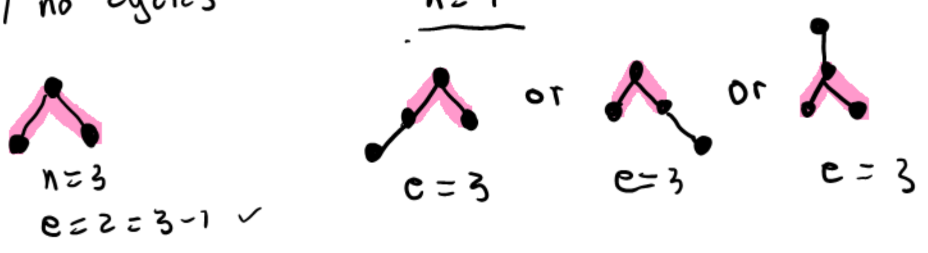
- "weak" + "strong" induction relies on the basic, recursive structure of $\mathbb{N}$

  i.e. "there's a next case"

- struct. ind. is used when there is no "next" but there is some recursive structure

ex) Every tree w/ n nodes has n−1 edges.

tree is a graph w/ no cycles



every tree w/ k+1 nodes is a tree w/ k nodes that has one additional node and one additional edge.

Recursive Structure!!

Base Case $(n=2)$

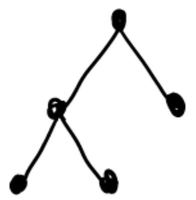
has $2-1=1$ edge ✓

Recursive Step

Suppose every tree w/ k nodes has $k-1$ edges.

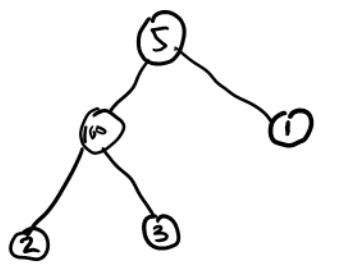
(we want to show every tree w/ $k+1$ nodes has $(k+1)-1=k$ edges)

Let T be a tree with $k+1$ nodes. Then $T = X \cup \{\text{edge, node}\}$ where X is a tree with k nodes. By our hyp., X has $k-1$ edges. It now follows that T has k edges. $\square$



binary tree

(no numbers/values in nodes!)



binary tree

binary tree = a tree where every node has 0, 1, or 2 edges

full binary tree = a tree where every node has 0 or 2 edges