2. How do we prove conditional statements? (Proving $P \Rightarrow Q$)

One can use a variety of proofs to explain why certain Theorems or Propositions are true. In this chapter will will focus on three proof strategies or "formats:"

- Direct Proofs
- Proving the Contrapositive
- Proof by Contradiction

The first two options – Direct Proofs and Contrapositive Proofs – are especially useful for many if-then statements. Indeed, you should think of these methods as your "go-to options," the ones you want to try first. Every now and then these won't work very well, and in these situations you'll have good reason to try that third Contradiction option.

2.1. Direct Proofs. The idea behind a Direct Proof (for a Proposition of the form $P \Rightarrow Q$) is quite simple:

Direct Proof Outline

Proposition. $P \Rightarrow Q$

Proof. (Direct)

(First Step) Suppose P is true.

(Intermediate Steps) Use definitions related to P and Q.

(Intermediate Steps) Use previously established facts to connect P and Q.

$\vdots$

(Last Step) Conclude Q is true. $\square$

When attempting to write your own Direct Proof feelings of uncertainty are natural. For instance, you might wonder exactly which definitions help you "move from" P "towards" Q, and you will also wonder how best to word these "moves." All of this is normal, though, and reading examples and practice writing your own proofs are the best ways to help.

Sometimes a given Proposition may seem very confusing, and this makes writing a proof impossible. In this situation, as in so many others in mathematics, **your first step should be to explore examples related to the proposition**. Doing this will help you understand what it is saying, and it convince you that the Proposition is true.

Example 2.1. *Consider the following proposition:*

Proposition. The sum of two even integers is an even integer.

Before proceeding with a proof, **explore this proposition by testing actual examples**. *We'll pause right now to wait – take your time and explore as many examples as you need. Your goal is to convince yourself "Yes, I agree that this Proposition is correct."*

Having used examples to explore and understand this proposition, we now want to double check that we can view it as an if-then statement. Given your work on statements and sets, this may seem relatively simple; in fact, we can easily rewrite this proposition as

Proposition. $\forall\, x, y \in \mathbb{Z}$, ($x$ is even) $\wedge$ (y is even) $\Rightarrow$ $x + y$ is even.

A direct proof of this proposition will start by assuming the hypothesis – in this case that both x and y are even – and then it will end by concluding that $x + y$ is even.

Proof (Direct)

(1) Suppose $x, y \in \mathbb{Z}$ are even.

(2) This means $\exists\, m, n \in \mathbb{Z}$, $x = 2m$ and $y = 2n$.

(3) ???????????

(Last Step) Therefore $x + y$ is even. $\square$

Note the attempted direct proof written above is in "draft form." The steps are labelled for the reader's convenience, and if you understand why Line (2) makes sense to write down then great! Many students first drafting a proof of this Proposition will understandably struggle to fill in Line (3) (or any additional lines), but if you understand why the last step is written the way it is, then great again! You understand the structure of this proof. Before reading on, **pause here and take a few minutes to try and figure out what sentence(s) could be used in Line (3).**

Okay, so let's finish this example by competing the direct proof, only let's present it without labelled steps. Here it is, all together.

Proposition. $\forall\, x, y \in \mathbb{Z}$, ($x$ *is even*) $\wedge$ (y *is even*) $\Rightarrow x + y$ *is even* .

Proof (Direct) Suppose $x, y \in \mathbb{Z}$ are even. This means $\exists\, m, n \in \mathbb{Z}$, $x = 2m$ and $y = 2n$. It follows that $x + y = 2m + 2n = 2(m + n)$, and since $(m + n) \in \mathbb{Z}$, we see that $x + y$ is even. $\square$

Example 2.2. *How might one set up a direct proof for the following proposition?*

Proposition. $\{x+1 : x \in \mathbb{N} \text{ and } x \text{ is odd } \} \subseteq \{ \text{ even integers } \}$

As usual, make sure that you understand the objects involved in the proposition before trying to prove anything – does the set on the left make sense? Does the one on the right-side make sense? (If not, spend some time exploring examples of their elements.)

This Proposition claims that one set, call it A, is a subset of another one, call it B. How might one directly prove that $A \subseteq B$ in general? Thankfully there is an easy strategy since this statement about sets is a familiar if-then:

$$\boxed{A \subseteq B \text{ means if } a \in A \text{ then } a \in B.}$$

This observation leads to a general principle:

$$\boxed{\textbf{To prove } A \subseteq B : \text{Let } a \in A \text{ be any element, and then show } a \in B.}$$

Returning to the specific Proposition above, a Direct Proof can be written as follows:

Proof. (Direct) Let $a \in \{x+1 : x \in \mathbb{N} \text{ and } x \text{ is odd } \}$.
This means $a = n+1$ for some odd $n \in \mathbb{N}$. According to the definition of odd it follows that $n = 2m+1$ for some $m \in \mathbb{N}$. We now have $a = n+1$ $= (2m+1)+1 = 2m+1+1 = 2m+2 = 2(m+1)$. This shows that a is even, and so $a \in \{\text{even integers}\}$. □

"Book of Proof" features excellent writing on and examples about this topic. See the following pages and examples:

- Chapter 4, pages 113-115, 118-123 contain excellent summaries and examples.
- Page 120 works out a proof by revising it more than once; this is an excellent example.
- The first proposition on page 122 is a useful example.
- Pages 124-126 discuss in detail how to set up cases for proofs – this is sometimes necessary or useful, as the examples show.
- Exercises 3, 7 and 9 (page 126) are good ones to try (solutions are included at the end of the book)
- Chapter 8, pages 157-161 contain excellent discussions about and examples of proving propositions about sets

2.2. Proving the Contrapositive. As we are well aware, a conditional $P \Rightarrow Q$ is logically equivalent to its contrapositive. That is

$$(P \Rightarrow Q) = (\neg Q \;\Rightarrow\; \neg P)\,.$$

Aside from being an interesting fact about truth-tables, *this tells us that whenever we need to prove $P \Rightarrow Q$ we can choose to prove $\neg Q \;\Rightarrow\; \neg P$ instead!*

Not getting anywhere directly proving a conditional? Then try directly proving its contrapositive!

The boxed slogan above is the entire strategy behind a **Proof Using the Contrapositive**.

Contrapositive Proof Outline

Proposition. $P \Rightarrow Q$

Proof. (Contrapositive)

(First Step) Suppose $\neg Q$ is true (i.e. Suppose Q is false).

(Intermediate Steps) Use definitions related to $\neg P$ and $\neg Q$.

(Intermediate Steps) Use previously established facts to connect $\neg Q$ and $\neg P$.

$\vdots$

(Last Step) Conclude $\neg P$ is true (i.e. conclude P is false). $\square$

After a little thought one realizes that there's very little difference between this approach and a "direct proof" strategy. One need only write down (and understand) a contrapositive and *then* start a direct proof.

Example 2.3. *Consider the proposition and the Contrapositive proof that follows.*

Proposition. If x is not a multiple of 3, then $6 \nmid x$.

Proof. (Contrapositive).

Let $x \in \mathbb{Z}$ and suppose $\neg\,(6 \nmid x)$. (We want to show x is a multiple of 3).
This means that $6|x$ and so $\exists\, q \in \mathbb{Z}$, $x = 6 \cdot q = 3 \cdot 2 \cdot q = 3 \cdot (2q)$
Since $2q$ is an integer, it follows that x is a multiple of 3. $\square$

Example 2.4. *Consider the proposition and the contrapositive proof that follows.*

Proposition. $\left\{x^2 : x \in \mathbb{R}\right\} \subseteq [0, \infty)$

Proof. (Contrapositive).

Suppose $y \notin [0, \infty)$. This means $y < 0$. (We want to show $y \notin \left\{x^2 : x \in \mathbb{R}\right\}$.)
We know that the square of a real number can never be negative, and so there cannot exist any real number $x \in \mathbb{R}$ that satisfies $x^2 = y$.
Therefore $y \notin \left\{x^2 : x \in \mathbb{R}\right\}$ $\square$

The example above could have easily been proven using a Direct Proof, too, but we included this contrapositive proof to help show how such proofs work. Many if-then's will be nicely provable using *either* approach, and so you will be able to choose the one that you prefer.

“Book of Proof” features excellent writing on and examples about Contrapositive proofs. See the following pages and examples:

- Chapter 5, pages 128-131 (before “Congruence of Integers”) contains an excellent summary and examples of Contrapositive Proofs.
- Chapter 5, pages 133-136 (the section labeled “5.3 Mathematical Writing”) contains great advice and tips on using words and symbols to write your own direct and contrapositive proofs.
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- Exercises 3, 5 and 7 (page 136) are good ones to try (solutions are included at the end of the book)
- Chapter 8, pages 157-161 contain excellent discussions about and examples of proving propositions about sets