

MATH 3336 : FINAL REVIEW

INSTRUCTIONS

- This is not an assignment. Neither work nor answers are to be submitted.
 - Also, **DON'T FORGET THAT** $0 \in \mathbb{N}$.
-

1. Suppose the function $f(n)$ is $O(n^5)$, then...

- (a) $f(n)$ is also $\Theta(n^5)$
 - (b) $f(n)$ is also $O(n^k)$ for all $k \in \{0, 1, 2, 3, 4\}$
 - (c) $f(n)$ is also $O(n^k)$ for all $k \geq 5$
 - (d) $f(n) = n^5$
-

2. Suppose you have an algorithm that consists of a loop, and the input for the algorithm is a list of length n .

(A) When the length of the list increases by one, the number of times the loop is repeated doubles– at least this happens in the worst-case scenario. Based on this information, the worst-case run-time for the algorithm is...

- (a) $\Theta(\log_2 n)$
- (b) $\Theta(2^n)$
- (c) $\Theta(n^2)$
- (d) $\Theta(n)$
- (e) $\Theta(2)$

(B) When the length of the list is doubled, the number of times the loop is repeated increases by one – at least this happens in the worst-case scenario. Based on this information, the worst-case run-time for the algorithm is...

- (a) $\Theta(\log_2 n)$
- (b) $\Theta(2^n)$
- (c) $\Theta(n^2)$
- (d) $\Theta(n)$
- (e) $\Theta(2)$

(C) When the length of the list increases by one, the number of times the loop is repeated also increases by one – at least this happens in the worst-case scenario. Based on this information, the worst-case run-time for the algorithm is...

- (a) $\Theta(\log_2 n)$
- (b) $\Theta(2^n)$
- (c) $\Theta(n^2)$
- (d) $\Theta(n)$
- (e) $\Theta(2)$

3. Fill in the truth tables for each of the following logical operators:

P	Q	$P \Rightarrow Q$
T	T	
T	F	
F	T	
F	F	

P	Q	$P \wedge Q$
T	T	
T	F	
F	T	
F	F	

P	Q	$P \oplus Q$
T	T	
T	F	
F	T	
F	F	

P	Q	$P \iff Q$
T	T	
T	F	
F	T	
F	F	

P	Q	$P \vee Q$
T	T	
T	F	
F	T	
F	F	

4. An expression involving many abstract statements is a **contradiction** when ...

- (a) Its truth table contains only F
 - (b) Its truth table contains only T
 - (c) Its truth table is blank
 - (d) The first and last two rows of its truth table end with F
 - (e) None of the above
-

5. If the congruence equation $ax \equiv b \pmod{n}$ has multiple solutions from the set $\{0, 1, 2, \dots, n-1\}$ then...

- (a) $a = 0$
 - (b) $\gcd(a, n) = 1$
 - (c) $\gcd(a, n) \nmid b$
 - (d) $\gcd(a, n) > 1$
 - (e) None of the above
-

6. How is a second-order, linear, homogeneous recurrence equation related to its characteristic equation?

- (a) The recurrence relation $a_n = ba_{n-1} + ca_{n-2}$ has as its characteristic equation $x^2 + bx + c = 0$
 - (b) The recurrence relation $a_n = ba_{n-1} + ca_{n-2}$ has as its characteristic equation $x^2 - bx - c = 0$
 - (c) The recurrence relation $a_n = ba_{n-1} + ca_{n-2}$ has as its characteristic equation $x^2 + cx + b = 0$
 - (d) The recurrence relation $a_n = ba_{n-1} + ca_{n-2}$ has as its characteristic equation $x^2 - cx - b = 0$
 - (e) In general, there is no way to relationship between these two objects.
-

7. Is the function $\log x$ in $O(x)$?

- (a) Yes, of course!
- (b) No, of course not!
- (c) This is an open question that no one has yet proved is true or proved is false.

8. Carefully read the statement P below.

$$P : \exists a \in \mathbb{Z}, \forall b \in \mathbb{Z}, a + b = 2022.$$

Which, if any, of the following correctly expresses $\neg P$?

- (a) $\neg P : \forall a \in \mathbb{Z}, \exists b \in \mathbb{Z}, a + b \neq 2022$
 - (b) $\neg P : \exists a \in \mathbb{Z}, \forall b \in \mathbb{Z}, a + b \neq 2022$
 - (c) $\neg P : \forall a \in \mathbb{Z}, \exists b \in \mathbb{Z}, a + b = 2022$
 - (d) $\neg P : \forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}, a + b \neq 2022$
 - (e) None of the above
-

9. Consider the set $F = \{\pi, \sqrt{7}, 7, 0, -1, 8\}$. Which, if any, of the following expressions represents the number of size-4 subsets of F ?

- (a) $\binom{8}{4}$
 - (b) $8!$
 - (c) $6!$
 - (d) $\binom{6}{4}$
 - (e) 2^6
-

10. Use the division algorithm to divide 116 by 3, and then, based on your work, determine which of the following statements is true.

- (a) $q = 36$ and $r = -2$
 - (b) $q = 35$ and $r = 1$
 - (c) $q = 34$ and $r = 4$
 - (d) $q = 34$ and $r = 0$
 - (e) It is impossible to use the division algorithm for this pair of integers.
-

11. Melon Usk – a celebrity billionaire and self-proclaimed “genius computer scientist” – claims to have invented a way to perfectly compress files *without any loss of information!* In fact, he claims that his algorithm reduces the file size by factor of 0.5. Which of the following explains why such an algorithm cannot, in reality, work or exist?

- (a) Consider the set, A , of files that are length n bit-strings as well as the set, B , of files that are length $.5 \cdot n$ bit-strings. Such an algorithm acts as a function $f : A \rightarrow B$, but since $|A| = 2^n > 2^{0.5n} = |B|$, the Pigeonhole Principle tells us two files in A must be compressed into the same file in B . As a result, the compression algorithm cannot be undone – files cannot be decompressed.
- (b) Actually, such an algorithm can (and will) exist; it will be used to help run future colonies on Venus.

12. Suppose you want to prove a Proposition of the form $P \Rightarrow Q$.

- (A) However, you want to do so using a direct proof. Which, if any, of the following best outlines such a proof?
- (a) Begin the proof by assuming $\neg(P \Rightarrow Q) = P \wedge \neg Q$, then use relevant facts and definitions to reach a conclusion like “2 is both even and odd.”
 - (b) Begin the proof by assuming $\neg P$, then use relevant facts and definitions to reach the conclusion $\neg Q$.
 - (c) Begin the proof by assuming $\neg Q$, then use relevant facts and definitions to reach the conclusion $\neg P$.
 - (d) Begin the proof by assuming P , then use relevant facts and definitions to reach the conclusion Q .
 - (e) Begin the proof by assuming Q , then use relevant facts and definitions to reach the conclusion P .
- (B) However, you want to do so using a proof by contradiction. Which, if any, of the following best outlines such a proof?
- (a) Begin the proof by assuming $\neg(P \Rightarrow Q) = P \wedge \neg Q$, then use relevant facts and definitions to reach a conclusion like “2 is both even and odd.”
 - (b) Begin the proof by assuming $\neg P$, then use relevant facts and definitions to reach the conclusion $\neg Q$.
 - (c) Begin the proof by assuming $\neg Q$, then use relevant facts and definitions to reach the conclusion $\neg P$.
 - (d) Begin the proof by assuming P , then use relevant facts and definitions to reach the conclusion Q .
 - (e) Begin the proof by assuming Q , then use relevant facts and definitions to reach the conclusion P .
- (C) However, you want to do so using a contrapositive proof. Which, if any, of the following best outlines such a proof?
- (a) Begin the proof by assuming $\neg(P \Rightarrow Q) = P \wedge \neg Q$, then use relevant facts and definitions to reach a conclusion like “2 is both even and odd.”
 - (b) Begin the proof by assuming $\neg P$, then use relevant facts and definitions to reach the conclusion $\neg Q$.
 - (c) Begin the proof by assuming $\neg Q$, then use relevant facts and definitions to reach the conclusion $\neg P$.
 - (d) Begin the proof by assuming P , then use relevant facts and definitions to reach the conclusion Q .
 - (e) Begin the proof by assuming Q , then use relevant facts and definitions to reach the conclusion P .
-

13. Consider the proposition about natural numbers $n \in \mathbb{N}$

Proposition. $\forall n \geq 1, 3^n - 1$, is even .

If we wish to prove this using Induction, then our Base Case...

- (a) is checked by proving the proposition is true when $n = 3$
 - (b) is checked by proving the proposition is true when $n \in \mathbb{N}$.
 - (c) is checked by proving the proposition is true when n is even.
 - (d) is checked by proving the proposition is true when $n = 1$
 - (e) is checked by proving the proposition is true when $n = 0$
-

14. How many length- n bit-strings contain at least one 0?

- (a) 2^{n-1}
- (b) $2^n - 1$
- (c) 1
- (d) n
- (e) $2^{n-1} + 1$

15. If we use the Euclidean Algorithm to compute $\gcd(2022, 11)$, then how many times will the Division “Algorithm” be used?

- (a) 1 time
 - (b) 2 times
 - (c) 3 times
 - (d) 4 times
 - (e) 5 times
-

16. 58 integers are selected at random, and they are each divided by 5. Which of the following statements is true?

- (a) At least 11 of the integers *must* have the same remainder (but we cannot know for certain that more than 11 will).
 - (b) At least 5 of the integers *must* have the same remainder (but we cannot know for certain that more than 5 will).
 - (c) At least 12 of the integers *must* have the same remainder (but we cannot know for certain that more than 12 will).
 - (d) At least 3 of the integers *must* have the same remainder (but we cannot know for certain that more than 3 will).
 - (e) All 58 integers *must* have the same remainder.
-

17. Which of the following statements is true?

- (a) $\emptyset \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$
 - (b) $\emptyset \subseteq \mathbb{Z} \subseteq \mathbb{N} \subseteq \mathbb{Q} \subseteq \mathbb{R}$
 - (c) $\emptyset \subseteq \mathbb{N} \subseteq \mathbb{Q} \subseteq \mathbb{Z} \subseteq \mathbb{R}$
 - (d) $\emptyset \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R} \subseteq \mathbb{Q}$
 - (e) $\emptyset \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{N} \subseteq \mathbb{Z}$
-

18. Consider the set $S \subseteq \mathbb{Z} \times \mathbb{Z}$ defined by the rules

$$\begin{aligned}(1, 2) &\in S \\ (x, y), (a, b) \in S &\Rightarrow (x + a, y + b) \in S \\ (x, y) \in S &\Rightarrow (-x, -y) \in S\end{aligned}$$

If we wanted to prove that every $(u, v) \in S$ satisfies $v = 2u$ using Structural Induction, which, if any, of the following most accurately outlines our “Recursive Step?”

- (a) Show that the claim is true for the base element $(1, 2)$
- (b) Let $(u, v) \in S$ be an arbitrary element that satisfies the claimed equation. Then show $(u + 1, v + 1)$ also satisfies the claimed equation.
- (c) Let $(u, v), (s, t) \in S$ be arbitrary elements that satisfy the claimed equation. Then show $(-u, -v)$ and $(u + s, v + t)$ also satisfy the claimed equation.
- (d) Let $(u, v), (s, t) \in S$ be arbitrary elements that satisfy the claimed equation. Then show $(u + 1, v + 1)$ and $(s + 1, t + 1)$ also satisfy the claimed equation.
- (e) Structural Induction cannot be used here because the set S is not well-ordered.

19. Which of the following sentences is a true statement?

- (a) Shut the door.
- (b) 16 is a multiple of 6.
- (c) $\forall x \in \mathbb{R}, x^2 - 1 \geq 0$
- (d) $\exists y \in \mathbb{Z}, y^3 - y = 0$
- (e) $|\mathcal{P}(\{1, -9, 5\})| = 8$

20. For this question consider the set $U = \{0, 1, 2, \dots, 9\}$. Count the number of subsets whose complement has 3 elements. (Knowing the formula for counting subsets is useful. Yay! Or... Uhh... Ahhh... Let me think... Let's say "okay" instead)

- (a) There are 10 such subsets.
- (b) There are 120 such subsets.
- (c) There are 84 such subsets.
- (d) There are 36 such subsets.
- (e) There are no such subsets.