8. Negating Statements

Although you have already read about the negation operator, some statements require more care or practice when attempting to write down useful negations. The best examples of this are statements that feature multiple quantifiers.

Example 8.1. *Let us reconsider the statement*

$$P: \quad \forall\, x \in \mathbb{R},\, \exists\, y \in \mathbb{R},\, y + x = 2$$

from Example 7.3. This features nested-quantifiers, and, as we've previously discussed, is a true sentence. Its negation must be false and can be written as

$$\neg P: \text{ It is not true that } \forall\, x \in \mathbb{R},\, \exists\, y \in \mathbb{R},\, y + x = 2.$$

We can and should rewrite this so that its meaning is clearer. This is done piece-by-piece. First, apply the negation to the "$\forall\, x \in \mathbb{R}$" clause. By negating this piece we are saying it is not true that every *real number satisfies a property; in other words, we negate this piece by claiming* there is a real number *that violates the claimed property. This first step gets us to the following:*

$$\neg P: \ \exists\, x \in \mathbb{R}, \neg\left(\exists\, y \in \mathbb{R},\, y + x = 2\right).$$

Next we negate the clause "$\exists\, y \in \mathbb{R}$," and doing this requires us to say "there is no such real number" or "every real number y violates the claimed property." This gets us to the negated statement

$$\neg P: \ \exists\, x \in \mathbb{R},\, \forall\, y \in \mathbb{R},\, \neg\left(y + x = 2\right).$$

Finally, the last part of this statement – the open sentence that expresses the condition $y + x = 2$ – can be negated quite easily, yielding

$$\neg P: \ \exists\, x \in \mathbb{R},\, \forall\, y \in \mathbb{R},\, y + x \neq 2.$$

One can explain why $\neg P$ is false by noting the following: if any real number $x \in \mathbb{R}$ is selected, then one can always choose $y = 2 - x$ so that $y + x = 2$.

Two main lessons from the example above apply more broadly: negating $\forall$ produces $\exists$, and negating $\exists$ produces $\forall$. That is

$$\boxed{\neg\left(\forall\, x \in U,\, P(x)\right) = \exists\, x \in U,\, \neg P(x) \qquad \neg\left(\exists\, x \in U,\, P(x)\right) = \forall\, x \in U,\, \neg P(x)}$$

Several examples are provided below, but before embarking on them its worth thinking about *why* we want to be able to clearly write negated statements. Indeed, there are lots of reasons, but here is an important and easy-to-overlook one: *when attempting and failing to understand a statement, it is often beneficial to instead try and understand its negation.*

Example 8.2. *The statement*

$$P: \ \forall\, x \in \mathbb{R},\, (x > 1) \ \Rightarrow\ (\ln(x) > 0)$$

is true. The false statement $\neg P$ *can be written as follows:*

$$\begin{aligned}\neg P: \ \neg\Big(\forall\, x \in \mathbb{R},\, (x > 1) \ \Rightarrow\ (\ln(x) > 0)\ \Big) \\ = \exists\, x \in \mathbb{R},\, \neg\Big(\,(x > 1) \ \Rightarrow\ (\ln(x) > 0)\Big) \\ = \exists\, x \in \mathbb{R},\, \Big(\,(x > 1) \ \wedge\ \neg\,(\ln(x) > 0)\ \Big) \\ = \exists\, x \in \mathbb{R},\, (x > 1) \ \wedge\ (\ln(x) \le 0)\end{aligned}$$

In English terms $\neg P$ *asserts the following: "There is a real number larger than* 1 *that when plugged into the natural-log function produces a non-positive output."*

Example 8.3.

$$\begin{aligned}&\neg\,(\exists\, a \in \mathbb{Z},\ a \textit{ is not even} \wedge a \textit{ is not odd}) \\ =&\forall\, x \in \mathbb{Z},\, \neg\,(a \textit{ is not even} \wedge a \textit{ is not odd}) \\ =&\forall\, x \in \mathbb{Z},\, a \textit{ is even } \ \vee\ a \textit{ is odd}\ .\end{aligned}$$

Example 8.4.

$$\begin{aligned}&\neg\,(\forall n \in \mathbb{N}, \exists\, m \in \mathbb{N},\, 2^m = n) \\ =&\exists\, n \in \mathbb{N},\, \neg\,(\exists\, m \in \mathbb{N},\, 2^m = n) \\ =&\exists\, n \in \mathbb{N},\, \forall\, m \in \mathbb{N},\, \neg\,(2^m = n) \\ =&\exists\, n \in \mathbb{N},\, \forall\, m \in \mathbb{N},\, 2^m \neq n.\end{aligned}$$

Example 8.5.

$$\neg\,(\ \textit{Everyone loves math}\ ) = \textit{Some people do not love math.}$$

Example 8.6.

$$\neg\Big(\textit{The graph of } y = e^x \textit{ lies above the } x\textit{-axis}\Big) = \exists\, a \in \mathbb{R},\, e^a \le 0.$$

More examples of negating statements can be found in Section 2.10 of Book of Proof.

8.1. Closing Thoughts and Summary. In this section you read about and worked on negating (rather complicated-looking) statements. The following table summarizes the rules that we commonly use when writing such statements (and you may have noticed that Example 8.3 used De Morgan's Laws while 8.2 used the fact that $\neg(P \Rightarrow Q) = P\ \wedge\ \neg Q$).

Statement	Negation	Commentary
$P \wedge Q$	$\neg P \vee \neg Q$	one of De Morgan's Laws
$P \vee Q$	$\neg P \wedge \neg Q$	one of De Morgan's Laws
$P \Rightarrow Q$	$P \wedge \neg Q$	breaking a promise
$\forall\, x \in U\, P(x)$	$\exists\, x \in U, \neg P(x)$	there exists a counterexample
$\exists\, x \in U,\, P(x)$	$\forall\, x \in U,\, \neg P(x)$	no element satisfies the condition

While these are the most commonly used rules when negating statements, other statement forms can be negated in useful ways, too. For example, negating a biconditional, $\neg\,(P \iff Q)$, turns out to be easily expressed, but we will leave that example (and others) for you to puzzle out with your friends and instructor.