7. Translating between English and Logic

Natural languages like English can be used to formulate statements, of course, but their precise logical meaning can sometimes be unclear or require some effort to understand. Using logical symbols avoids this, but this approach can sometimes obscure a simple concept that's more easily expressed in "everyday language." Here are two examples.

Example 7.1. *Consider the English and logical versions of statement P:*

P : *Every integer is either odd or even, but not both.*

P : $\forall\, x \in \mathbb{Z}$, $x = 2a$ *(for some $a \in \mathbb{Z}$)* $\oplus$ $x = 2b + 1$ *(for some $b \in \mathbb{Z}$).*

Example 7.2. *Consider the English and logical versions of statement Q:*

Q : *Whenever a real number is squared the result is a non-negative real numer.*

Q : $x \in \mathbb{R} \Rightarrow x^2 \geq 0$.

Statement P in Example 7.1 is probably easier to read as an English sentence, but the purely logical version is entirely precise. We should pause here to acknowledge that the second version of P uses two unstated definitions, that of **even** and **odd**. We will collect such definitions throughout this book (and formally summarize them in closing sections). As we progress through this material we will need more interesting examples of statements, and so definitions of old (and new!) concepts will appear.

The English version of statement Q uses the word "whenever" in place of the words "if" and "then," and there is a rich variety of ways to denote conditionals as mentioned in Section 4. This may be a matter of personal taste, but the logical version of Q seems shorter and clearer.

The main point of these examples, though, is to emphasize that we will often want or need to *translate between natural language statements and logical ones.* This can be especially useful when confronted with complicated looking statements like the ones below.

Example 7.3. *Consider the statement*

$$P : \forall\, x \in \mathbb{R},\ \exists\, y \in \mathbb{R}, y + x = 2.$$

*This is an example of a statement with **nested quantifiers**, and the order of these quantifiers matters (see the next example)! To see that P is true we carefully take apart the statement, step-by-step. We can also rewrite P in a natural language:*

P : *Given any real number, x, we can find another one, y, so that $y + x = 2$.*

Example 7.4. *Consider the statement*

$$Q : \exists\, y \in \mathbb{R},\ \forall\, x \in \mathbb{R},\ y + x = 2.$$

This complicated-looking sentence claims that there exists a special real number (named y) that, when added with any other real number *results in* 2. *Your familiarity with real numbers should tell you that Q is false.*

Example 7.5. ***The Saturday Night Live Example.*** *First, watch this video clip of an early episode from the show Saturday Night Live (you may want to start the video at the 45s mark and stop at around 2:38).*

The entire joke here is a misunderstanding of a nested-quantifier statement; quantifiers have been switched and so the meaning becomes absurd.

$$P_1: \ \forall \text{ 11 second periods}, \exists \text{ a person in NYC who gets mugged.}$$

$$P_2: \ \exists \text{ a person in NYC who gets mugged every 11 seconds.}$$

If a statement features a repeated quantifier, though, the order does **not** matter.

Example 7.6. *The statement*

$$P: \quad \forall\, q \in \mathbb{Q}, \forall\, a \in \mathbb{Z}, \; q \cdot a \in \mathbb{Q}$$

is true (pause a moment to make sure you understand why). The meaning and truth value of this statement do not change when we alter the order of the repeated quantifiers:

$$P: \quad \forall\, a \in \mathbb{Z}, \forall\, q \in \mathbb{Q}, \; q \cdot a \in \mathbb{Q}$$

Your work with the logical operations $\neg, \wedge, \vee$ and $\oplus$ have likely helped you become adept at translating between English sentences and logical ones that feature these symbols, but if you want some more practice, Book of Proof's Section 2.9 (pages 57-59) includes additional examples.

Closing Thoughts and Summary. In this section you read about and worked on translating between purely logical statements and English language ones. Particular focus was placed on statements containing multiple or **nested quantifiers**. These can take some time to get used to, but an important lesson is that, when the quantifiers are different, **their order matters**.

We also used the definition of **even integer** and that of **odd integer** in one of our examples. As such, we record these defitions here.

Definition 1.12. ***(Even, Odd)***

An integer $x \in \mathbb{Z}$ is called ***even*** *if there exists $a \in \mathbb{Z}$ such that $x = 2a$.*

An integer $y \in \mathbb{Z}$ is called ***odd*** *if there exists $b \in \mathbb{Z}$ such that $y = 2b + 1$.*

Lastly, you learned that 1970's New York City was a dangerous place (if only for one resident).