MATH 3336

HOMEWORK ASSIGNMENT 7

INSTRUCTIONS

- Record your answers to the following 10 questions. Show your work when a question requires you to do so.
- Scan your work and save the file as a .pdf (make sure your work and answers are legible)
- Upload your scanned work to CASA CourseWare using the "Assignments" tab. (Click this link for instructions on how to do this).
- Homework submitted after 11:59pm on the indicated due date will be assigned a grade of 0.
- Also, **DON'T FORGET THAT** $0 \in \mathbb{N}$.
- I understand that if any of the questions from this assignment (or future ones) are shared in ways that violate our Academic Honesty Policy, then the syllabus will change. Specificlaly, Homework and Quizzes will be worth zero points.

  Name:

  Signature:

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1. (10 points) Consider the set $S \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ defined by the following rules:

- $(1, 0, 0), (0, 1, 0), (0, 0, 3) \in S$
- If $(x, y, z), (a, b, c) \in S$, then $(xa, yb, zc) \in S$.
- If $(x, y, z) \in S$ then $(y, x, z) \in S$

Consider the proposition

Proposition. $\forall (x, y, z) \in S,\ 3|(xyz)$

Why would Structural Induction be an appropriate method to use when attempting to prove this proposition?

(a) Structural Induction is a good idea because S is a set.

(b) Structural Induction is a good idea because S is an infinite set.

(c) Structural Induction is a good idea because S has recursive structure (but it is not clear or obvious how to "order" the elements of S).

(d) Structural Induction is a good idea because S has recursive structure and its elements involve numbers.

(e) Structural Induction is NOT a good idea; "regular" induction can be used in a simple way.

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2. (10 points) Consider the same set $S \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ from Problem 1 and the following proposition about it:

Proposition. $\forall\, (x, y, z) \in S,\ 3|(xyz)$

Which of the following accurately describes the "Base Case" portion of a proof-by-Structural-Induction one would use for this proposition?

(a) Show that the proposition is true for the elements $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 3)$.

(b) Show that the proposition is true for the element $(0, 0, 0)$.

(c) Show that the proposition is true when $n = 0$.

(d) Suppose the proposition is true for all elements in $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$.

(e) None of the above.

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3. (10 points) Consider the same set $S \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ from Problem 1 and the following proposition about it:

Proposition. $\forall\, (x, y, z) \in S,\ 3|(xyz)$

Which of the following could be used as the "Recursive Step" portion of a proof-by-Structraul-Induction one would use for this proposition?

(a) Suppose the proposition is true for $(x, y, z), (a, b, c)$. It follows that $(u, v, w) = (xa, yb, zc)$ satisfies

$$uvw = (xa)(yb)(zc) = (xyz)(abc) = (3m)(3k) \text{ for some } m, k \in \mathbb{Z}.$$

Our inductive hypothesis was used for the last equation, and it now follows that $3|(uvw)$.

(b) Suppose the proposition is true for (x, y, z). This means $3|(xyz)$. It follows that the proposition is true for (y, x, z) since $(yxz) = (xyz)$ and so $3|(yxz)$, too.

(c) One would need to include both answers (a) and (b) for the "recursive step."

(d) One does not need to include either answer (a) or (b) for the "recursive step" since the proposition only needs to be established when $n = k + 1$ (based on it being true at $n = k$).

(e) None of the above.

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4. (10 points) Consider the claimed formula

$$\forall\, n \geq 1,\ \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

Consider the following piece of advice for proving this formula using induction: "For the inductive step it will be useful to keep in mind that a sum of $k + 1$ terms can be expressed as a sum of k terms (plus one last term)."

(a) This is great advice!

(b) This is terrible advice!

5. (10 points) One special type of graph is called **a cycle graph**. It is common to notate a Cycle Graph with n vertices as C_n. Which, if any, of the facts about Cycle Graphs would be *most* useful or relevant in the "inductive" portion of an induction proof?

(a) C_n has n edges.

(b) Every vertex of C_n is joined to exactly two other vertices in C_n.

(c) C_n can be drawn as a regular n-gon.

(d) $C_n = C_{n-1}$ plus one additional vertex and one additional edge.

(e) $C_n = C_{n-1}$ plus one additional vertex.

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6. (10 points) One special type of graph is called **a complete graph**. It is common to notate a Complete Graph with n vertices as K_n, and this type of graph is defined via the following property: every pair of distinct vertices in K_n are joined by one edge. Which, if any, of the following claims about Complete Graphs is both true and informative about "recursive structure?"

(a) $K_n = K_{n-1}$ plus one additional vertex and $n-1$ additional edges.

(b) $K_3 = C_3$

(c) $K_n = K_{n-1}$ plus one additional edge and n additioinal vertices

(d) $K_n = C_n$

(e) None of the above

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7. (10 points) Recall that an $n \times n$ matrix is called a **diagonal matrix** if all of its entries are 0, with the possible exception of entries along its the diagonal. Here are a few examples:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\pi & 0 \\ 0 & 0 & 22 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

The third example in our collection above is supposed to represent an arbitrary, diagonal $n \times n$ matrix with diagonal entries d_i. There is a lovely fact about the determinant of diagonal matrices which we present as the following

Proposition. $\det \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = d_1 d_2 \cdots d_n$

If we want to prove this for *all* $n \times n$ matrices using induction, which of the following facts would be most helpful?

(a) A recursive definition of the determinant, one that relates the determinant of $n \times n$ matrices to $(n-1)\times(n-1)$ matrices.

(b) The fact that the determinant changes sign when two rows are interchanged.

(c) A geometric definition of the determinant, one that involves the n-dimensional volume of a higher-dimensional parallelepiped.

(d) The fact that the determinant equals zero if and only if the rows (and/or columns) of the matrix are linearly dependent.

(e) The fact that the determinant of a matrix equals the determinant of its transpose.

8. (10 points) Consider the recursive function a_n as defined by the recurrence equation and initial conditions $a_n = a_{n-1} + 4a_{n-2}$ and $a_0 = 3, a_1 = -6$. If we wanted to prove that $3|a_n$ (for every $n \in \mathbb{N}$), then ...

(a) we would need to show $3|a_0$ for our Base Case.

(b) we would need to show $3|a_1$ for our Base Case.

(c) we would need to show $3|a_0$ and $3|a_1$ for our Base Case.

(d) we would need to show $a_0|3$ for our Base Case.

(e) we would need to show $a_1|3$ for our Base Case.

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9. (10 points) Which of the following equations / inequalities shows the "recursive structure" of the factorial formula?

(a) $k! = 2! + k! - 2$

(b) $(k+1)! = k! + 1!$

(c) $0! = 1$

(d) $(k+1)! = (k+1) \cdot k!$

(e) $k! \geq k$

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10. 10 points Imagine you are tutoring a student who is taking our course. How would you describe the idea of "Proofs by Induction" to them? What would be most important for them to know?