Math 3336
Homework Assignment 10

Instructions

- Record your answers to the following 10 questions. Show your work when a question requires you to do so.
- Scan your work and save the file as a .pdf (make sure your work and answers are legible)
- Upload your scanned work to CASA CourseWare using the "Assignments" tab. (Click this link for instructions on how to do this).
- Homework submitted after 11:59pm on the indicated due date will be assigned a grade of 0.
- Also, **DON'T FORGET THAT** $0 \in \mathbb{N}$.
- I understand that if any of the questions from this assignment (or future ones) are shared in ways that violate our Academic Honesty Policy, then the syllabus will change. Specificlaly, Homework and Quizzes will be worth zero points.

Name:

Signature:

1. (Part A) Write a proof of the following proposition (make certain to label the style of proof you are using):

$$\textbf{Proposition. } \forall\, a, b \in \mathbb{Z},\ (a+b)^3 \equiv \left(a^3 + b^3\right) \bmod 3$$

(Part B) One might wonder if the above result holds for any power, $n \in \mathbb{N}$, and not only $n = 3$. Find a counter-example to the claim

$$\forall\, a, b \in \mathbb{Z}, (a+b)^4 \equiv \left(a^4 + b^4\right) \bmod 4$$

.

2. (Part A) Complete this multliplication table for the integers-mod-5 (also notated as $\mathbb{Z}_5$):

$\times_5$	0	1	2	3	4
0					
1					
2					
3					
4					

No work or written-out computations need be included for this part of the question – just make sure all of the table-entries are expressed as integers in the set $\{0, 1, 2, 3, 4\}$.

(Part B) Use your answers from Part A to complete the following sentences:

The multiplicative inverse of 2 (mod 5) is $2^{-1} =$ _______

The multiplicative inverse of 3 (mod 5) is $3^{-1} =$ _______

The multiplicative inverse of 4 (mod 5) is $4^{-1} =$ _______

You do not need to include any explanations for your answers to Part B (but you may do so if you like).

(Part C) Use your work in Part B to solve the following congruence equations (please show your work so that it is legible):

i $2x \equiv 7 \bmod 5$

ii $3x \equiv 6 \bmod 5$

iii $4x \equiv 8 \bmod 5$

(Part D) Explain why the equation $10x \equiv 3 \bmod 5$ has no solutions.

3. (Part A) Fill in the two tables below; one is an "addition table" for the integers mod-2, and the other is a "multiplication table." (Note: "the integers mod-2" is commonly notated as $\mathbb{Z}_2$).

$+_2$	0	1
0		
1		

Addition in $\mathbb{Z}_2$

$\times_2$	0	1
0		
1		

Multiplication in $\mathbb{Z}_2$

No work or written-out computations need be included for this part of the question – just make sure all of the table entries are expressed as integers from the set $\{0, 1\}$.

(Part B) Compare the multiplication table for $\mathbb{Z}_2$ to a truth-table for our old friend the $\wedge$ operator, but let's slightly rewrite this truth table as follows:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

$\rightsquigarrow$

$\wedge$	F	T
F		
T		

Complete this new version of our old truth-table, filling in missing T's and F's where appropriate (no explanation need be included in this part of the question). Finally, rewrite this completed truth table, only everywhere you have an "F" replace it with a 0, and everywhere you have a "T" replace *it* with a 1.

What do you notice about our newly-written $\wedge$ truth table?

(Part C) Repeat the same process carried out in Part B, only this time do it for the logical operator XOR, $\oplus$.

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

$\rightsquigarrow$

$\oplus$	F	T
F		
T		

4. Find all solutions to the congruence equation

$$4x \equiv 6 \bmod 10$$

and express those solutions mod-10 (i.e. as integers from the set $\{0, 1, 2, \cdots, 9\}$). Or, if there are no solutions, explain why. (Be sure to show your work in a neat, legible way – assuming any was needed to answer this question.)

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5. It is well known that the equation $x^2 + 1 = 0$ has no solutions in the set $\mathbb{R}$.

Are there solutions if we instead use the set $\mathbb{Z}_{10}$? That is, does the analogous equation $x^2 + 1 \equiv 0 \bmod 10$ have any solutions? If there are solutions, find them and express them mod-10 (i.e. as integers from the set $\{0, 1, \cdots 9\}$) and neatly show your work.

6. How many solutions are there to the equation $x^2 - 1 = 0$ in $\mathbb{Z}_8$?

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7. When attempting to solve the equation $ax \equiv b \bmod n$, we should first...

(a) Compute $\gcd(a, b)$ and see if it divides n.

(b) Compute $\gcd(a, n)$ and see if it divides b.

(c) Compute $\gcd(b, n)$ and see if it divides a.

(d) Compute $\operatorname{lcm}(a, b)$ and see if it divides n.

(e) Apply the division algorithm to a and b.

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8. When attempting to solve the equation $ax \equiv b \bmod n$, if $\gcd(a, n) = 1$ then...

(a) There is exactly one solution in $\mathbb{Z}$.

(b) There is exactly one solution in $\mathbb{R}$.

(c) There is exactly one solution in $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$.

(d) There are multiple solutions in $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$.

(e) There are no solutions.

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9. When attempting to solve the equation $ax \equiv b \bmod n$, if $\gcd(a, n)|b$ and $gcd(a, n) > 1$ then...

(a) There is exactly one solution in $\mathbb{Z}$.

(b) There is exactly one solution in $\mathbb{R}$.

(c) There is exactly one solution in $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$.

(d) There are multiple solutions in $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$.

(e) There are no solutions.

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10. Which problems from this homework set did you find most beneficial? Most interesting? Which ones did you find least helpful? Least interesting?