Discrete Math

Lecture 22

"Strong" Induction

Strong **Induction Outline**

Proposition. $\forall n \in \mathbb{N}$, $P(n)$

Proof by Induction

Base Case

more base cases

Show that $P(0)$ is true.

(You may need to show $P(1), P(2), P(-1)$, etc., is true.)

Inductive Step

Show being true at one "case" implies the proposition is true at the "next case."

Show $P(k) \Rightarrow P(k+1)$

Assume $P(k)$ (this is called the **inductive hypothesis**.)

Carefully write out $P(k+1)$

Figure out how to relate $P(k+1)$ to $P(k)$

Use this recursive relationship to conclude $P(k+1)$ is true. $\square$

$P(0) \wedge P(1) \wedge \dots \wedge P(k)$

$\boxed{\text{Previous / smaller cases} \Rightarrow \text{future / bigger ones}}$

Example 3.1. *Consider the recursively defined sequence $a_n = 3a_{n-1} + a_{n-2}$ with initial conditions $a_0 = 0$ and $a_1 = 4$.*

Proposition. a_n is even for every $n \in \mathbb{N}$

Proof by induction

Inductive Step

Suppose it's true when $n = k, k-1 \in \mathbb{N}$. That means a_k is even.

(WTS a_{k+1} is also even)

Our given recurrence eqn tells us that

$$a_{k+1} = 3 \cdot a_k + a_{k-1}$$

$$= 3 \cdot (2m) + 2n$$

$$= 2(3m + n)$$

which shows a_{k+1} is even. We used our (strong) ind. hyp, namely that $\exists m,n$, $a_k = 2m$ and $a_{k-1} = 2n$.

Base Case

Check $n=0$ + $n=1$.

These base cases are true according to the given i.c.'s $a_0 = 0$ + $a_1 = 4$ are both even. $\square$

Example 3.2.

Proposition. $\forall n \in \mathbb{N}$, $n \geq 2 \Rightarrow n$ can be expressed as a product of primes.

Proof by (strong) Induction.

Base Case

We check the proposition when $n = 2$. Since 2 is already a prime number, the claim is true. (We will regard this as a "trivial product of primes.")

Inductive Step

Suppose the proposition is true for $n = k$ and for all values of $n < k$. (We want to show that $k + 1$ can be written as a product of primes.)

Suppose $P(n)$ for $2 \leq n \leq K$

If $k + 1$ is prime, then it is itself a product of primes.

If $k + 1$ is not a prime number, then it is **composite** and so can be written as a product of two smaller numbers: $k + 1 = a \cdot b$. Since $a \leq k$, by our strong inductive hypothesis, it can be written as a product of primes. Similarly, $b \leq k$ can be written as a product of primes. It follows that $(k + 1) = a \cdot b$ is a product of primes. $\square$

$$K+1 = a \cdot b$$

$$\underbrace{a < K+1}_{a \leq K} \qquad \underbrace{b < K+1}_{b \leq K}$$

our (strong) ind hyp. assumed that if $2 \leq a \leq K$ then $\underbrace{a = \text{prod of primes}}$

$a = (p_1 p_2 \cdots p_s)$ each p_i is prime

$\underbrace{b = \text{prod of primes}}$

$b = (q_1 q_2 \cdots q_t)$ each q_i is prime

$$k+1 = a \cdot b$$

$$= (p_1 \cdots p_s) \cdot (q_1 q_2 \cdots q_t)$$

$$= \text{a product of primes}$$

Here F_n is the nth Fibonacci number. Prove that

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

an inductive proof of this does require

two base cases

& two previous "steps"

$$P(k-1) \wedge P(k) \Rightarrow P(k+1)$$

why? answer: defining recurrence equation for F_n is

$$F_n = F_{n-1} + F_{n-2}$$