PRINTABLE VERSION

Quiz 6

You scored 100 out of 100

Question 1

Your answer is CORRECT.

Consider the open sentence $P(n)$ below:

$$P(n):\ n! = n(n-1)!$$

What is $P(0)$?

a) ○ $P(0):\ 0! = 0.$

b) ◉ $P(n)$ does not make sense when $n = 0$.

c) ○ $P(0):\ n! = n(n-1)!$

d) ○ $P(0):\ 0! = 0 - 1$

e) ○ $P(0):\ 0! = 1! \cdot 0!$

Question 2

Your answer is CORRECT.

Consider the open sentence $P(n)$ below:

$$P(n):\ 5^{2n} - 1 \text{ is a multiple of 3}$$

What is $P(k+1)$?

a) ◉ $P(k+1):\ 5^{2k+2} - 1$ is a multiple of 3

b) ○ $P(k+1):\ 5^{2k+1} - 1$ is a multiple of 3

c) ○ $P(k+1):\ 5^{k+1} - 1$ is a multiple of 3

d) ○ None of the above.

Question 3

Your answer is CORRECT.

A mathematician wishes to prove the following proposition using a Proof by Induction:

The sum of the first n consecutive odd numbers equals n^2.

Which of the following can be used as the Base Case part of her proof?

a) ○ $\sum_{i=1}^{1} i = 1 = 1^2$

b) ○ Suppose the proposition is true for some value, say $n = k$. We now prove it is true when $n = k + 1$.

c) ○ $\sum_{i=1}^{1} (2i-2) = 0 = 0^2$

d) ◉ $\sum_{i=1}^{1} (2i-1) = 1 = 1^2$

e) ○ None of the above.

Question 4

Your answer is CORRECT.

A mathematician wishes to prove the following proposition using a Proof by Induction:

$$n^2 - 3n + 4 \quad \text{is even for all } n \geq 1 \ .$$

Which of the following can be used as part of the Inductive Step in her proof?

a) ○ $(k+1)^2 - 3(k+1) + 4$ is even.

b) ◉
$(k+1)^2 - 3(k+1) + 4 = k^2 + 2k + 1 - 3k - 3 + 4 = \underline{k^2 - 3k + 4} + 2k - 2 = \underline{\text{even integer}} + 2(k-1)$
is even.

c) ○ $\underline{(k+1)^2 - 3(k+1) + 4} = \underline{k^2 - 3k + 4}$ is even.

d) ○ Prove the proposition for $n = 1$.

Question 5

Your answer is CORRECT.

Consider $P(n)$ below:

$$P(n): \ 1 + 4 + 4^2 + \cdots + 4^{n-1} + 4^n = \frac{4^{n+1} - 1}{3}.$$

Which of the following equations is both accurate *and* helpful in showing $P(k) \Rightarrow P(k+1)$?

a) ○ $P(0): \sum_{i=0}^{0} 4^i = 4^0 = 1 = \frac{4-1}{3}$ is true.

b) ○ $\sum_{i=0}^{k+1} 4^i = 1 + 4 + 4^2 + \cdots + \underline{4^k} + 4^{k+1} = \underline{(\frac{4^{k+1} - 1}{3})} + 4^{k+1}$

c) ◉ $\sum_{i=0}^{k+1} 4^i = \underline{1 + 4 + 4^2 + \cdots + 4^k} + 4^{k+1} = \underline{\sum_{i=0}^{k} 4^i} + 4^{k+1}$

d) ○ $\sum_{i=0}^{k+1} 4^i = \underline{\sum_{i=0}^{k} 4^i} + 1 = \underline{(\frac{4^{k+2} - 1}{3})}$

e) ○ None of the above.

Question 6

Your answer is CORRECT.

A proof is presented below, but portions of it are missing.

Proposition. $F_n \leq 2^n$ for all $n \geq 1$. (Here F_n denotes the nth Fibonacci number.)

Proof. (By $\underset{1}{\underline{\hspace{6cm}}}$)

(Base Case.) We have two base cases.
When $n = 1, F_1 = 1 \leq 2^1$, and when $n = 2, F_2 = 1 \leq 2^2$.
Therefore the base case holds.

(Inductive Step). Suppose the proposition is true for some values $n \leq k$.
In particular, we will suppose it is true for $n = k$ and $n = k - 1$.
That is, we will assume $F_k \leq \underset{2}{\underline{\hspace{2cm}}}$ and $F_{k-1} \leq 2^{k-1}$.
We want to show $\underset{3}{\underline{\hspace{7cm}}}$
The recurrence relation for the Fibnoacci sequence tells us that
$F_{k+1} = F_k + \underset{4}{\underline{\hspace{1.5cm}}}$
By our inductive hypothesis this sum is $\leq 2^k + 2^{k-1} \leq 2^k + 2^k = 2 \cdot 2^k = \underset{5}{\underline{\hspace{1.5cm}}}$

This completes the proof. □

Carefully read and complete the proof. Which of the following best fills in space 3?

a) ○ (Standard) Induction

b) ○ 2^{k+1}

c) ◉ $F_{2k+1} \leq 2^{k+1}$

d) ○ Strong Induction

e) ○ F_{k+1}

f) ○ 2^k

g) ○ F_{k-1}

Question 7

Your answer is CORRECT.

Carefully read the induction proof below (note: there is some algebra left out of line (3), but you can fill in missing parts by multiplying out terms).

Proposition. $n^3 - n$ is a multiple of 3 for all $n \geq 1$.

Proof.
(1) The proposition is true when $n = 1$ since $1^3 - 1 = 0$ (which is a multiple of 3 since $0 = 3 \cdot 0$).
(2) Suppose the proposition is true when $n = k$, i.e. that $k^3 - k = 3m$ for some integer m.
(We will show the proposition is true when $n = k + 1$).
(3) $(k+1)^3 - (k+1) = \left(k^3 - k\right) + 3\left(k^2 + k\right) = 3m + 3\left(k^2 + k\right) = 3\left(m + k^2 + k\right)$.
(4) The expression above is a multiple of 3, completing the proof. □

Which of the following best describes Line (1)?

a) ○ This line summarizes the fact that $P(k+1)$ is true (and thereby ends the proof).

b) ○ This line sets up the Inductive Step by establishing the Inductive Hypothesis (and describing the "next case" that will be shown true).

c) ◉ This line is establishing the Base Case.

d) ○ This line uses the formula's recursive structure to establish a connection between $P(k)$ and $P(k+1)$.

e) ○ This line sets up a contradiction.

Question 8

Your answer is CORRECT.

Why might it be a bad idea to prove the following proposition by induction?

$$\forall x \in \mathrm{Z},\ \sin(\pi x) = 0.$$

a) ○ The set Z is infinite.

b) ○ The variable is named x , and induction only works for n .

c) ◉ Induction is only intended for N

d) ○ The claim is false when $x = 2$

e) ○ None of the above.

Question 9

Your answer is CORRECT.

Consider the recursively defined sequence $\{a_n\}$ with recurrence relation

$$a_n = 6a_{n-1}$$

Suppose we want to use Induction to prove a statement $P(n)$ about this sequence. Our Inductive Step would have us proving "(previous cases) $\Rightarrow P(k+1)$." How many "previous cases" will be needed in this situation?

a) ○ 3 previous cases are needed; we would prove $(P(k-2) \wedge P(k-1) \wedge P(k)) \Rightarrow P(k+1)$.

b) ○ No base cases are needed.

c) ○ 2 previous cases are needed; we would prove $(P(k-1) \wedge P(k)) \Rightarrow P(k+1)$.

d) ◉ Only 1 previous case is needed; we would prove $P(k) \Rightarrow P(k+1)$.

Question 10

Your answer is CORRECT.

This question is about Induction and matrices. Specifically, it involves multiplying 2×2 matrices. If you haven't worked with multiplying these sorts of matrices together, don't worry! Ask your instructor for more details! With that out of the way, consider the fact that we can multiply a 2×2 matrix against itself, and we notate this using "exponent" notation like we do for numbers. Here is an example:

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 4 \cdot 0 & 1 \cdot 4 + 1 \cdot 4 \\ 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 4 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}$$

In fact, we can keep on multiplying this matrix against itself, as many times as we like! If you multiply this matrix against itself more and more times you might notice this pattern:

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \cdot 4 \\ 0 & 1 \end{bmatrix}$$

We can prove this pattern holds for all $n \geq 1$ by using induction. For the inductive step we assume the pattern is true for $n = k$ and compute

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & k \cdot 4 \\ 0 & 1 \end{bmatrix} = \cdots$$

Which of the following expressions accurately continues this computation and completes the Inductive Step?

a) ○ $\begin{bmatrix} 1 \cdot 1 & 4 \cdot 4k \\ 0 \cdot 0 & 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 4^2 \cdot k \\ 0 & 1 \end{bmatrix}$

b) ○ $\begin{bmatrix} 0 \cdot 4k + 1 \cdot 1 & 1 \cdot 4k + 4 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 0 & 1 \cdot 1 + 4 \cdot 0 \end{bmatrix} = \begin{bmatrix} 1 & (k+1)4 \\ 0 & 1 \end{bmatrix}$

c) ◉ $\begin{bmatrix} 1 \cdot 1 + 4 \cdot 0 & 1 \cdot 4k + 4 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 4k + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & (k+1)4 \\ 0 & 1 \end{bmatrix}$

d) ○ $\begin{bmatrix} 1 + 1 & 4 + 4k \\ 0 + 0 & 1 + 1 \end{bmatrix} = \begin{bmatrix} 2 & (k+1)4 \\ 0 & 2 \end{bmatrix}$

e) ○ $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 4 \cdot 0 & 1 \cdot 4 + 4 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 0 & 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 24 \\ 0 & 1 \end{bmatrix}$