5. Sequences

A **sequence** can be thought of as a (usually infinite) list, and you have already encountered these in lots of previous courses. For instance, *the sequence of (positive) perfect squares* is an unending list of natural numbers that we can present as follows

$$1, 4, 9, 16, 25, 36, 49, 64, \dots$$

In fact, the entire set of natural numbers, $\mathbb{N}$, can be presented as a sequence:

$$0, 1, 2, 3, 4, \dots$$

Note that the *set* $\mathbb{N}$ does not care about the order of these terms since elements in a set can be written down in mixed up ways, for instance $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\} = \{3, 4, 0, 1, 2, \dots\} = \{0, 2, 4, 6, 8, 3, 10, 12, 1, \dots\}$. However, as a *sequence* the order of the terms matters.

There is a more concise or compact way of writing down **sequences**, and that is to provide a generic description of the entries in the unending list. For the perfect-squares example above we can write

the first term is 1 squared
the second term is 2 squared
the third term is 3 squared
the fourth term is 4 squared

$$\vdots$$

And if you think about it just right, you'll realize that there's a function lurking behind this repetitive description! The **location** or **index** of each entry can be used as an input to describe the value of that entry. Specifically, if we let $n \in \mathbb{N}$ and use $a(n)$ to denote the sequence's n-th term, then we are left with a function that takes natural numbers as inputs and produces real numbers as outputs. In the perfect-squares case we can use $a(n) = n^2$:

$$\begin{aligned} a(n) &= n^2 \\ a(1) &= 1^2 = 1 \\ a(2) &= 2^2 = 4 \\ a(3) &= 3^2 = 9 \\ &\vdots \end{aligned}$$

As it turns out, *this is a good way to think about all sequences.* That is, a **sequence** is simply a function $a : \mathbb{N} \to \mathbb{R}$. Take a moment here. Pause and make certain you understand how the following concepts are equivalent:

$$\boxed{a : \mathbb{N} \to \mathbb{R}} = \boxed{\text{an unending list of values: } \{\, a(0), a(1), a(2), a(3), \dots \}}$$

Before we explore some examples, a few comments on notation are in order.

(1) First, you might notice that our list of perfect squares does not have a zero-th term; there is no $a(0) = 0^2$ entry, and so *technically* that sequence is a function

$a : \mathbb{N} - \{0\} \to \mathbb{R}$. This is A-OK as sequences don't have to start at $n = 0$. They can start at $n = 1$ or at other natural numbers, too.

(2) Second, there are a variety of ways that mathematicians and computer scientists use to denote sequences, including the following:

$$a(n) = a_n = \{a_n\}_{n=0}^{\infty} = \{a_n\}$$

It is actually far more common to denote the n-th entry in a sequence as a_n.

(3) Third, it is possible to have sequences of objects *other* than real numbers. One can have an endless list of sets, for example, or a sequence of matrices, etc. In this book our attention will be focused on sequences of real numbers, but the astute reader will notice that this really comes down to adjusting the codomain in our description of sequences. It is valuable to not think of sequences as functions $a : \mathbb{N} \to \mathbb{R}$ but also as functions $a : \mathbb{N} \to B$ where B is a more generic set.

Example 5.1. *Consider the sequence $a_n = 1/n$. The entry $a_2 = 1/2$ and $a_5 = 1/5$. Note that this sequence* cannot *start with $n = 0$ (why not?). What is the 10th term in this sequence? Do any terms in this sequence solve the equation $22x - 1 = 0$? Are any terms in this sequence equal to 0? 1/2?*

Example 5.2. *Consider the sequences $a_n = (-1)^n$ and $b_n = \cos(n\pi)$. Observe that $a_0 = a_2 = 1$ and that $a_{303} = -1$. Interestingly enough, the sequence b_n behaves very similarly; $b_0 = \cos(0) = 1, b_1 = \cos(\pi) = -1$, and $b_2 = \cos(2\pi) = 1$. Indeed, these sequences have the exact same terms and in the exact same order!*

Often times a sequence will be given to you but in an indirect way. This can happen, for example, when several of the terms in the sequence are provided and *you* are tasked with finding a pattern so that the remaining terms can be produced. In other words, sometimes only a few values of the function $a : \mathbb{N} \to \mathbb{R}$ are provided, and you will need to find its formula or "rule." This formula is referred to as a **closed formula** for a_n.

Example 5.3. *Find a closed formula for the sequence $\{a_n\}$ whose first six terms are the following:*

$$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \ldots$$

Hint: the denominators should look special or familiar to you.

Questions like those in Example 5.3 are, technically speaking, *bad questions*. There may be an obvious pattern that appears to describe the first six, seven or even six million terms of a sequence, but without more information we don't actually *know* that the pattern continues! Sure, we are using some pretty strong evidence, but this isn't full-proof; we can't 100%-*know* that this pattern holds for every $n \in \mathbb{N}$. As a result, we won't be dealing with questions like Example 5.3, but ***it is good to be able to spot patterns***; Example 5.3 is at least good in helping you practice this skill.

5.1. Recursive Sequences. There is another way to indirectly present a sequence, one where formulas can (many times) be figured out and with 100% confidence. This happens when a sequence is defined or described **recursively**. The term **recursive** is one you're going to be seeing a lot of – both in this course and future ones – and it is used in situations where "smaller" or "previous" situations help us understand "larger" or "subsequent" ones. Said differently, any phenomenon or object that features "recursive structure" can be broken down into smaller pieces or parts, ones that help us understand the *whole* object. An example should help!

Example 5.4. *Consider the sequence defined by the formula $a_n = 2^n$. The first several terms are*

$$1, 2, 4, 8, 16, \ldots$$

The next term equals the previous one, 16, multiplied by 2; that is, the next term is $a_5 = 2 \cdot 16 = 32$. Indeed, this recursive structure or pattern appears to be true for all *terms (except the first one).*

Not only does this pattern seem to be true, we can use the closed formula for a_n to know with certainty that it is true. If the n-th term is $a_n = 2^n$, then the previous term is $a_{n-1} = 2^{n-1}$ and our familiarity with algebra allows us to compute

$$a_n = 2^n = 2^{1+n-1} = 2 \cdot 2^{n-1} = 2 \cdot a_{n-1}.$$

*The equation $a_n = 2a_{n-1}$ is an example of **recursive structure**; the equation itself is an example of what is called a **recurrence relation**, and it literally tells us that "future" terms in the sequence can be understood using "previous ones."*

Example 5.4 provided a sequence with a closed formula and then concluded that the sequence enjoys recursive properties. However, we will eventually want to turn this around and start with a **recursively defined sequence** and then (try to) find a closed formula.

Example 5.5. *Consider the recursively defined sequence*

$$a_n = 5 + a_{n-1}.$$

Check that the closed formula $a_n = 5n + 7$ satisfies this recursive property.

To check this, one needs to "test out" the proposed recurrence equation, and this will mean plugging in $n-1$ into our formula. Everywhere we see an "n" we will carefully replace it with "$n-1$" :

$$a_{n-1} = 5(n-1) + 7 = 5n - 5 + 7 = 5n + 2.$$

We now compare this to the proposed formula for a_n:

$$a_n = 5n + 7 = 5n + 2 + 5 = \underbrace{(5n+2)}_{a_{n-1}} + 5 = a_{n-1} + 5 = 5 + a_{n-1}.$$

This shows that the closed formula $a_n = 5n+7$ solves the given recurrence relation $a_n = 5 + a_{n-1}$.

Example 5.6. *There are multiple sequences that satisfy*

$$a_n = 5 + a_{n-1}$$

including the following:

$$\begin{aligned} a_n &= 5n + \pi \\ a_n &= 5n \\ a_n &= 5n - 3 \\ a_n &= 5n + 12 \end{aligned}$$

In fact, any sequence of the form

$$a_n = 5n + a_0,$$

where a_0 is any *real number, will satisfy this recurrence equation. Whatever choice one makes for a_0, its value must equal the* 0*th term in the sequence (plug in $n = 0$ into both sides to see why). The choice of a_0 is referred to as an* ***initial condition****.*

Example 5.7. *Consider the recursively defined sequence with initial condition*

$$\begin{aligned} a_n &= 2a_{n-1} + 1 \\ a_0 &= 3 \end{aligned}$$

One can use the recurrence relation to determine other entries in this sequence. For instance

$$a_1 = 2a_0 + 1 = 2 \cdot 3 + 1 = 7.$$

Indeed, the first several terms are

$$3, 7, 15, 31, 63, \ldots$$

Check that the sequence $a_n = 4 \cdot 2^n - 1$ satisfies this recurrence relation and initial condition.

Recurrence equations can involve more than just *one* previous term; the n-th term in a recursively-defined sequence can depend on *any number of previous terms*, and the more previous terms involved the more initial conditions are needed. We close with an especially famous example, ***The Fibonacci Numbers***.

Example 5.8. The Fibonacci Numbers. *Consider the recursively defined sequence*

$$F_n = F_{n-1} + F_{n-2}$$

and initial conditions

$$F_0 = F_1 = 1.$$

The first few terms in the Fibnoacci Sequence are

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

and Mathematicians and Math-Enthusiasts alike continue to find lots and lots of appearances of this sequence (here is but one video by the incredible Vi Hart on Fibonnaci numbers). A closed formula for F_n is possible, but we will revisit that topic later in this text.

Closing Thoughts and Summary. In this section you read and learned about **sequences**, which are special examples of functions.

Definition 2.4. *A* ***sequence (of numbers)*** *is a function*

$$a : \mathbb{N} \to \mathbb{R}.$$

The co-domain can be different, resulting in sequences of different objects, not just numbers. The domain can be slightly different in that we can "have the sequence begin at a natural number other than 0" by simply removing a set of consecutive naturals from $\mathbb{N}$*, starting at 0.*

Function-notation is not commonly used for sequences, so that instead of writing $a(0)$ *or* $a(7)$ *we write* a_0 *or* a_7*.*

You also read about and worked on **recursively defined sequences**, ones that often come with **initial conditions**. A **closed formula** for these sequences is sometimes available, and we will develop some techniques for finding these formulas later.