2. Proof by Induction

So, what *exactly* is a "proof by induction?" First and foremost: this is a proof-technique used *when you have a natural number's worth of statements, $P(n)$, that you want to argue are each true.* In other words, **a proof by induction** is used to prove an unending collection of statements that are joined by our "and"-operator, something like

$$P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge \cdots$$

One can phrase this endless string more concisely as

$$\forall n \in \mathbb{N},\ P(n).$$

Depending on what, exactly, the sentence $P(n)$ claims, you may or may not be able to prove this kind of proposition using one of our previous methods. Trying a **proof by induction** works particularly well when *there is a relationship between $P(k+1)$ and $P(k)$*, i.e. when *there is recursive structure.* Here is an outline:

Induction Outline

Proposition. $\forall n \in \mathbb{N},\ P(n)$

Proof by Induction

Base Case

Show that $P(0)$ is true.

(You may need to show $P(1), P(2), P(-1)$, etc., is true.)

Inductive Step

Show being true at one "case" implies the proposition is true at the "next case."

Show $P(k) \Rightarrow P(k+1)$

Assume $P(k)$ (this is called the **inductive hypothesis**.)

Carefully write out $P(k+1)$

Figure out how to relate $P(k+1)$ to $P(k)$

Use this recursive relationship to conclude $P(k+1)$ is true. $\square$

This outline will make more sense when you apply it to some examples (see below), but before we do this a few comments are in order.

- As mentioned in the outline, the Base Case does not necessarily need to handle the case when $n = 0$. Sometimes our Base Case starts at a higher value, sometimes at a lower value. ***As long as there is a unique, smallest value of n to use, we're good to go.***
- Carefully writing out the **inductive hypothesis** can often feel like "cheating." After all, it literally gets written as "Suppose $P(n)$ is true for some natural number $k \in \mathbb{N}$."

- However, the inductive step is not assuming $P(n)$ is true for *all* values of n. Rather, this step is establishing a kind of "causal relationship" wherein $P(k)$ being true "causes" $P(k+1)$ to be true.
- A helpful and popular analogy is to think of this proof strategy like knocking over a trail of carefully arranged dominoes. The Inductive Step shows that the dominoes are lined up, and the Base Case shows we can knock over the first one.

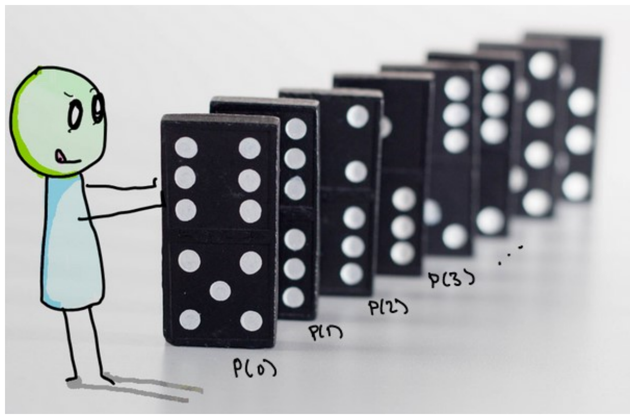

Figure 1. Proof by Induction

Example 2.1. *The notation* $n!$ *is (unfortunately) not used to denote an exciting natural number. It is pronounced "n factorial" and here is what it means:*

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$$

In particular, $5! = 120, 3! = 6, 2! = 2, 1! = 1$ *and we even declare* $0! = 1$*, too. Consider the following proposition:*

Proposition. $\forall\, n \in \mathbb{N}$, $n! \geq 1$.

Proof by Induction.
Base Case
When $n = 0$ we find $0! = 1 \geq 1$.

Inductive Step
Suppose the proposition is true for some value, $n = k \in \mathbb{N}$. This means that $k! \geq 1$. (We want to show the proposition is true when $n = k+1$; that is, we want to show that $(k+1)! \geq 1$.) It follows from the definition of factorial that

$$(k+1)! = (k+1) \cdot k \cdot (k-1) \cdot (k-2) \cdots 2 \cdot 1 = (k+1) \cdot (k!)$$

Since, by our inductive hypothesis, $k! \geq 1$, it follows that

$$(k+1)! = (k+1) \cdot (k!) \geq (k+1) \cdot 1 = k+1 \geq 1.$$

This completes the proof. $\square$

The proposition in the above example may strike you as an obvious one, but that's not really the point. Instead, make sure you understand the structure or format of the Proof by Induction used above, and make sure you understand how valuable the simple observation

$$(k+1)! = (k+1) \cdot k!$$

is! This one, single equation is why our Inductive Step works, and it is where all the "recursive structure" lies.

Example 2.2. *Consider the recursively defined sequence and initial condition*

$$a_n = -2 + 3a_{n-1} \text{ and } a_0 = 1.$$

Proposition. The sequence $\{a_n\}$ is constant; in particular $a_n = 1$ for all $n \in \mathbb{N}$.

Proof by Induction.

Base Case
The given initial condition $a_0 = 1$ verifies the Base Case.

Inductive Step
Suppose the proposition is true for some natural number $n = k \in \mathbb{N}$. This means $a_k = 1$ (we want to show $a_{k+1} = 1$, too). The given recurrence equation tells us

$$a_{k+1} = -2 + 3a_k = -2 + 3 \cdot 1 = -2 + 3 = 1$$

where we used the inductive hypothesis to substitute $a_k = 1$. This completes the proof. $\square$

Our next example concerns a formula for a sum – and formulas for sums are Mathematician's *favorite things* to prove by induction. Here's why: stopping a summation at a "next term" is naturally related to the summation being stopped at the "previous term." You'll see this recursive structure / trick used over and over in sample induction problems, and here's what it looks like it purely math symbols:

sum to $(k+1)$	$=$	(sum to k)	$+$	(last term)
$s_1 + s_2 + \cdots + s_k + s_{k+1}$	$=$	$(s_1 + s_2 + \cdots + s_k)$	$+$	s_{k+1}
$\displaystyle\sum_{i=1}^{k+1} s_i$	$=$	$\displaystyle\sum_{i=1}^{k} s_i$	$+$	s_{k+1}

The expressions s_i are just the terms of our sum, and they will change depending on what formulas we are analyzing. The important part is this: when dealing with summation formulas *you can always peel off the last term and relate your current sum to the previous one.* Pay attention to this "move" in our following example (and notice that the algebra at the end gets a little tedious to write out; you should check that it all works, though).

Example 2.3. *Consider the proposition*

$$\textbf{Proposition.}\ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

<u>Proof by Induction.</u>

<u>Base Case</u>
The smallest value of n we can use is $n = 1$. Plugging $n = 1$ into the summation gives

$$\sum_{i=1}^{1} i^2 = 1^2 = 1.$$

Plugging $n = 1$ into the proposed formula gives

$$\frac{1 \cdot (1+1)(2 \cdot 1 + 1)}{6} = \frac{2 \cdot 3}{6} = 1.$$

Since both expressions are equal, the Base Case has been verified.

<u>Inductive Step</u>
Suppose the formula is true for $n = k$. This means

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

When $n = k + 1$ our summation becomes

$$\sum_{i=1}^{k+1} i^2 = \boxed{\sum_{i=1}^{k} i^2} + (k+1)^2 = \boxed{\frac{k(k+1)(2k+1)}{6}} + (k+1)^2$$

We used our inductive hypothesis for the substitution in the equation above (the boxed terms are equal because of this hypothesis). Algebraic manipulation completes the proof:

$$\begin{aligned}
\frac{k(k+1)(2k+1)}{6} + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)\Big(k(2k+1) + 6(k+1)\Big)}{6} \\
&= \frac{(k+1)(k+2)(2(k+1)+1)}{6} \quad \square
\end{aligned}$$

Book of Proof has excellent discussions about Induction as well as lots of useful examples. Consult the following:

- Pages 180-186
- Exercises 3, 4, 5, and 8 from Chapter 10 (page 195)