3336

Office Hour

1:30 pm

start time

(meeting closes at 1:40 if no one shows)

HW 6 #4

4. Consider the recursively-defined sequence and initial condition

$$a_n = a_{n-1} + a_{n-2} + \cdots + a_1 + a_0$$
$$a_0 = 1$$

Write an inducion proof of the following proposition (and consider the hint that you may need to use the Proposition from problem 3 as part of your work):

Proposition. $\forall n \geq 1,\ a_n = 2^{n-1}$

Proof (by induction)

Base Case When $n=1$ it follows that $a_1 = a_0 = 1 = 2^0 = 2^{1-1}$.

Inductive Step Suppose the prop. is true when $n = k \in \mathbb{N}$. This means $a_k = 2^{k-1}$ (we want to show $a_{k+1} = 2^{(k+1)-1} = 2^k$)

$$a_{k+1} = a_k + a_{k-1} + \cdots + a_1 + a_0 = \boxed{2^k} + \underbrace{a_{k-1} + \cdots + a_1 + a_0}$$

we need "strong" induction!

we need a stronger inductive hyp. in order to know something about all of these!

First explore this proposition

$a_0 = 1$ $\qquad$ a_n = sum of the previous ones

$a_1 = a_0 = 1 = 2^0$ ✓ $n=1$ $\qquad$ $a_4 = a_3 + a_2 + a_1 + a_0 = 4 + 2 + 1 + 1 = 8 = 2^{4-1}$ ✓ $n=4$

$a_2 = a_1 + a_0 = 2 = 2^{2-1}$ $n=2$ ✓

$a_3 = a_2 + a_1 + a_0 = 2 + 1 + 1 = 4$

Inductive Step (Strong)

Suppose the proposition is true for all n, $1 \le n \le k$.

This means $a_1 = 2^0 = 1$, $a_2 = 2^1 = 2$, $a_3 = 2^2 = 4$, $\ldots$, $a_{k-1} = 2^{k-2}$, $a_k = 2^{k-1}$.

We want to show $a_{k+1} = 2^k$.

This follows from our given recursive formula;

$$a_{k+1} = a_k + a_{k-1} + \cdots + a_1 + a_0$$

$$= 2^{k-1} + 2^{k-2} + \cdots + 1 + 1 \quad = \quad 2^{k-1} + 2^{k-2} + \cdots + 2 = 2(2^{k-2} + 2^{k-3} + \cdots + 1)$$

use problem 3 ("geometric series")

$$= 2^k \text{ (using prev. problem)} \quad \square$$

Note: test your understanding by changing this problem

$$a_n = a_{n-1} + a_{n-2} + \cdots + a_1 + a_0$$

what happens if we change a_0 to π? or $3/2$? or -7?

what formula exists for a_n?

can you prove it by induction?

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2. Recall (from our notes) the definition of a "path graph with n vertices," PG(n). Use induction to prove the following proposition:

Proposition. The graph PG(n) has $n - 1$ edges.

Proof (By Induction).

Base Case. When $n = 1$, PG(1) has 1 vertex and 0 edges.

Inductive Step. (see below)

First : explore examples

$n=0$

$PG(0)$: a "path graph w/ zero vertices" an empty graph

$n=1$

$PG(1)$:

$n=1$

$\#$ edges $= 0 = n-1$ ✓

$n=2$

$PG(2)$:

$\#$ edges $= 1 = n-1$ ✓

$n=3$

$PG(3)$:

$n=3$

$\#$ edges $= 2 = n-1$ ✓

$PG(3) =$  $+$ $PG(2)$

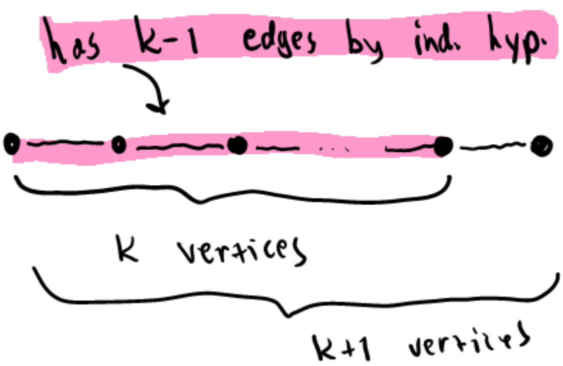
Inductive Step

Suppose $PG(n)$ has $n-1$ edges when $n = k \in \mathbb{N}$.

(we want to show $\underline{PG(k+1) \text{ has } k \text{ edges}}$)

We need to relate $PG(k+1)$ to $PG(k)$, and this can be done by observing that $PG(k+1) = PG(k)$ with one extra edge and one extra node.

By our inductive hyp., $PG(k)$ has $k-1$ edges and so $PG(k+1)$ has one more edge, giving us k edges. $\square$



This week : "structural induction"

0, 1, 2, 3, 4, 5

(weak) induction : Base Case

$P(k) \Rightarrow P(k+1)$

(strong) induction: Base Case(s)

$P(1) \land P(2) \land \cdots \land P(k-1) \land P(k) \Rightarrow P(k+1)$

small pieces

combo's of these pieces

Structural Induction! Base Case(s)

"there is no obvious notion of "next" "

good ex : full binary trees

every tree is "built" from a basic tree!

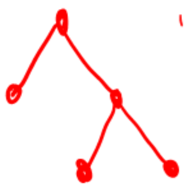
n = 1

"next full binary tree"?

n = 3

n = 5

or



n = 7

⋮