6. Quantifiers

There are three objects or symbols that we call "quantifiers," and they are *incredibly* useful in expressing the pure logic behind lots of sentences, including the following every-day examples.

> "Every dog is a mammal."
>
> "There is a unique English letter that is both a consonant and a vowel."
>
> "There is a whole number whose square equals 4."

The three logical symbols used to capture key parts of these kinds of expressions are

$$\begin{aligned} \forall : &\quad \text{"for every," "for all," "all"} \\ \exists! : &\quad \text{"there exists a unique"} \\ \exists : &\quad \text{"there exists,", "some"} \end{aligned}$$

Before discussing how these "quantifiers" work, we first need to preview the topic of **Sets** – without doing this we won't have many examples to consider or work through. The next chapter is devoted entirely to **Sets**, so we'll only scratch the surface in this section (and we'll repeat everything said here again at the start of Chapter 2).

6.1. Preview of Sets. A **set** is a collection of objects. There, that's it. Honestly, that's all a set is. Its really not more complicated than that. For instance, you can think of a drawer of socks as a set – its a collection of socks. Sets can be far more abstract, too, like "the set of *all* socks," "the set of all whole numbers" or "the set of all imagined creatures." The objects contained in a set are called **elements**, so that your smelly gym sock is one element in the sock-drawer-set, and there are lots of elements in the all-imagined-creatures set (including this one, this one, and even this one).

Computer-science-interested readers would do well to think of a set as a particular **data type**. Computer scientists routinely work with a large variety of these things, including *lists, sets, arrays,* and *strings.* The popular programming language Python notates "sets" the way mathematicians often do: listing elements between squiggly braces "{" and "}."

Figure 1. Defining a set in Python

The set above consists of the three elements written in between the squiggly braces. Although that set has been named "thisset," it is more common to use capital letters for set names. We'll rename that example as S. We then use the notation "$\in$" to stand for "is an element of," and this lets us talk about S as follows:

$$\text{apple} \in S, \ \text{banana} \in S, \ \text{cherry} \in S.$$

Unlike other data types, sets are not ordered. For the example S above this means that the following sets are all the same or equal:

$$S = \{\text{apple, banana, cherry}\} = \{\text{apple, cherry, banana}\} = \{\text{banana, cherry, apple}\}.$$

Set elements can also be repeated and they still only count as one object; the set remains the same. For instance

$$S = \{\text{apple, banana, cherry}\} = \{\text{apple, banana, apple, cherry}\}.$$

Math courses tend to focus on "mathy sets," ones containing certain types of numbers or functions or matrices, but it is always worth keeping in mind that a "set" is a rather broad concept, one that can be talked about in lots of non-mathy ways, too. Anyways, you may not realize it but you've had lots of experience with some famous math sets. We'll want to use these when thinking about quantifiers *and* throughout the remainder of the class, so we might as well write the most popular ones down right now.

Example 6.1. *Important Sets to Know About*

- *The empty set:* $\emptyset = \{\ \}$
- *The natural numbers:* $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$
- *The integers:* $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, \cdots\}$
- *The rational numbers:* $\mathbb{Q} = \{$ *all fractions of integers* $\}$
- *The real numbers:* $\mathbb{R} = \{$ all *numbers along the solid number line*$\}$

Example 6.2. *Consider the two sets* $A = \{-1, 0, 5\}$ *and* $B = \{2, 4, 6, 8, 10, \cdots\}$.

(1) *How many elements are in A? How many elements are in B?*

(2) *Are there any elements $x \in A$ that satisfy $x^2 - 4x - 5 = 0$?*

(3) *Are there any elements $y \in B$ that satisfy $y - 1 = 2$?*

6.2. The Universal Quantifier $\forall$. The symbol "$\forall$" is literally pronounced "for all." Sometimes it is pronounced "all" or "for every," but it always sets up the same kind of sentence, one that talks about *all* possible objects.

Example 6.3. *Consider the statement*

$$P: \text{ Every real number produces a non-negative value when it is squared.}$$

Based on your familiarity with real numbers, you likely accept P as a true statement (however awkwardly worded), and we would like to rewrite such statements in purely logical terms. The term "every" tells us that we will need the $\forall$ symbol to accomplish this, and it can be done as follows:

$$P: \ \forall\, x \in \mathbb{R},\, x^2 \geq 0.$$

This version of P may look *like an open sentence, but because the variable x has been "quantified," this is now an actual, true-or-false statement.*

We need to use a variable-name to refer to "any real number," and our Universal Quantifier makes it clear that we are not waiting *for x to be substituted. Instead we are saying that* whenever ANY real value *is plugged in, the result holds.*

Example 6.4. *Determine the truth values of the following universally-quantified statements.*

(1) $\forall\, x \in \{1, \pi, 1/2\},\, x > 1/4.$

(2) $\forall\, \theta \in \mathbb{R},\, \sin^2\theta + \cos^2\theta = 1.$

(3) $\forall\, a, b \in \mathbb{Z},\, a^2 + b \leq 0.$

(4) $\forall a \in \mathbb{Z}$, $a^2 + a$ *is even.*

(5) $\forall\, x \in \mathbb{R},\, 1/x^2 > 0.$

Statements (3) and (5) in Example 6.4 are both false, and it is worth pointing out *why*. For (3), the assertion is that *every single* choice of integers a and b result in $a^2 + b$ being less than zero, but we *can select* $a = -3$ and $b = 5$. When plugged into the formula we get 14, and so not *every* choice works! The "for every" claim is not true!

Something similar happens with statement (5), but it may strike you as more subtle. There *is* a real number value we can substitute for x that makes $1/x^2 > 0$ false, but it may have escaped your attention. If we decide to use $0 \in \mathbb{R}$ and "plug it in" to the formula we obtain the output $1/0^2$. This expression is neither larger nor smaller than zero, and that's because it isn't even a number – this expression is undefined.

Notice that *all* of the universally-quantified statements in Example 6.4 use *sets*, and they are quite important. Generally speaking, universally-quantified statements have the form

$$\boxed{\forall\, x \in U, \text{ some condition holds.}} = \boxed{\forall\, x \in U,\, P(x).}$$

The set U is so important, in fact, that it gets its own special name; these are called **universal sets** (we usually use the letter U to name them), and if you change the universal set you can change the statement's truth value. Worse yet, if the universal set isn't mentioned, then confusion can result and the sentence may fail to be either true or false. The following example attempts to demonstrate this.

Example 6.5. *Consider thesentence*

$$P: \quad \textit{Every number becomes a larger number when it is squared.}$$

That word "every" tells us we can and should use $\forall$ *when rewriting* P*. We can do so as*

$$P: \quad \forall\, x,\, x^2 \geq x.$$

No ***universal set*** U *is mentioned, though. This is because our original English sentence mentions "number" without specifying the* kind or type*. Consider the more specific statements* P_1 *and* P_2*, each interpreting "number" with a different universal set.*

$$P_1: \ \forall\, x \in \mathbb{N},\, x^2 \geq x.$$

$$P_2: \ \forall\, x \in \mathbb{R},\, x^2 \geq x.$$

Your experience with natural numbers should convince you that P_1 *is true, but* P_2 *is false. It is not true that* any real number *squares to a larger one; consider as a counter-example the real number* $x = 1/2$*. In this instance* $x^2 = (1/2)^2 = 1/4$ *and* $1/4 \not\geq 1/2$*.*

Many mathematical statements will *not* mention a universal set U – but this is only done *when* U *is understood from context.* Example 6.5 above is one of the (somewhat rare) incidents in which there is not enough context to figure out U. Generally speaking, if a universal set is not specified, then we are to use the *largest* one possible. In fact, not only do mathematicians leave U unmentioned, *with conditional statements we'll leave off the quantifier* $\forall$! The following sub-section clarifies this tradition.

6.3. "For all" and "Implication". Take two open sentences, say $P(x)$ and $Q(x)$, and connect them with our implication operator to form the conditional

$$P(x) \Rightarrow Q(x).$$

Mathematicians (and most Computer Scientists) read this as a **universally-quantified** if-then statement, *even though* $\forall$ *and* U *are not mentioned.* Uptight Logicians and Philosophers get mad at us and insist that because key notation was omitted, the conditional is open. In other words

$P(x) \Rightarrow Q(x)$	
Weird Philosophers read as "Duh, what? This open. Duh."	Mathematicians (correctly) read as $\forall\, x \in U,\, P(x) \Rightarrow Q(x)$.

Statements that omit the symbol $\forall$ are referred to as ***implicitly quantified***. Some standard examples are included in the following example.

Example 6.6. *The following have implied quantifiers and unspecified universal sets.*

(1) x *even* $\Rightarrow$ $x+1$ *is odd.*

(2) *The function* f *being differentiable at a point* $\Rightarrow$ f *is continuous at that point.*

The first sentence can (and should) be understood to use the universal set $U_1 = \mathbb{Z}$, *while the second one uses* $U_2 = \{$ *all functions* $\}$.

6.4. The Existential Quantifier ($\exists$). Our Universal Quantifier allows us to make claims about *all elements* in a universal set – it also helps us understand the precise meaning of $P(x) \Rightarrow Q(x)$. The **existential quantifier**, notated as $\exists$ and often pronounced "there exist, there exists, there are," or "some," enables us to make a vastly different kind of sentence. Like its "for every" brother, this quantifier uses a universal set, but it only talks about *particular* elements in U. Consider the following example.

Example 6.7. *The sentence*

P : *There exist natural numbers that, when squared, equal themselves.*

makes a claim about a particular element (or elements) of $\mathbb{N}$, *and so it is rewritten using* $\exists$. *In the version written below we use a comma after the universal set,* ***and we pronounce this comma as "such that"****:*

$$P : \exists\, x \in \mathbb{N},\, x^2 = x.$$

This statement is literally read as "There exists a natural number x *such that* $x^2 = x$.*" To determine the truth value of* P *one simply needs to either find one (single!) element* $x \in \mathbb{N}$ *that makes the claimed equation true* or *one needs to explain why* no element $x \in \mathbb{N}$ *will work. (Which do you think is the case for this example? Hint: algebra.)*

Existential statements are often written as

$$\boxed{\exists\, x \in U, \text{ some condition holds.}} = \boxed{\exists\, x \in U,\, P(x).}$$

and that comma continues to be read as "such that."

Example 6.8. *Several existentially-quantified statements are written below along with their truth values. Make sure you understand what each statement is claiming and why each truth value is assigned.*

(1) $\exists\, x \in \mathbb{R},\, x^2 = 2$ *is true.*

(2) $\exists\, x \in \mathbb{N},\, x^2 = 2$ *is false.*

(3) $\exists\, t \in \mathbb{Z},\, 1/t = t$ *is true.*

(4) *"Some real numbers are negative" is true.*

(5) *"There exist students who hate math" is false.*

(6) $\exists\, a, b \in \mathbb{Z}, a + b = 0$ *is true.*

As we saw with universally-quantified statements, the universal set can be omitted, left for readers to infer. Statement (5) above, for instance, does not specify a universal set (but you can use $U = \{\text{students at your University}\}$ if you like).

6.5. Unique Existence ($\exists!$)**.** This quantifier is quite close in spirit to $\exists$, only it includes an overly-enthusiastic exclamation mark that we read as "unique." In other words, whenever the symbols $\exists!$ appear they are claiming "there exists a *unique* element" or "there is exactly one element..." The following examples demonstrate this quite well.

Example 6.9. (1) $\exists!\, x \in \mathbb{Z},\ x^2 = 0$ *is true.*

(2) $\exists!\, x \in \mathbb{Z},\ x^2 = x$ *is false.*

(3) $\exists!\, x \in \{1, 2, 3\},\ x^2 - 3x + 2 = 0$ *is false.*

(4) $\exists!\, x \in \{1, 3\},\ x^2 - 3x + 2 = 0$ *is true.*

(5) $\exists!\, x \in \{0, 3, 5\},\ x^2 - 3x + 2 = 0$ *is false.*

Note: Because the symbol "!" is commonly used to denote negation (at least in many Computer Science circles), confusion can arise. For this math text and for future math courses keep in mind the different usages (and forgive us our sins).

Closing Thoughts and Summary. In this section you read about and worked on different types of quantifiers and sets.

Definition 1.10. *A* ***set*** *is a collection of objects, and the objects in the collection are called* ***elements****. It is common to use capital letters for names of sets and the symbol $\in$ to denote set membership.*

Several important sets with special and fixed notation were mentioned in example 6.1, and readers should feel comfortable using them.

Definition 1.11. *A* ***universal set****, U, is used or assumed whenever a statement features quantifiers.*

The following table summarizes the meanings and usages of the quantifiers you learned about.

Quantifier	Pronunciation	Meaning & Use
$\forall$	for every, every, all	statements about *all* elements in a set
$\exists$	there exists, there exist, some	statements about *some* elements in a set
$\exists!$	there exists a unique	one, single element in a set