Discrete Math

Lecture 21

Proof by Induction

goal: $P(n)\ \forall\ n \in \mathbb{N}$

$P(0) \wedge P(1) \wedge P(2) \wedge \ldots$

clue: $P(k)$ is related to $P(k+1)$

Induction Outline

Proposition. $\forall n \in \mathbb{N},\ P(n)$

Proof by Induction

Base Case

Show that $P(0)$ is true.

(You may need to show $P(1), P(2), P(-1)$, etc., is true.)

usually not difficult

we can knock down the first domino!

Inductive Step

Show being true at one "case" implies the proposition is true at the "next case."

Show $P(k) \Rightarrow P(k+1)$

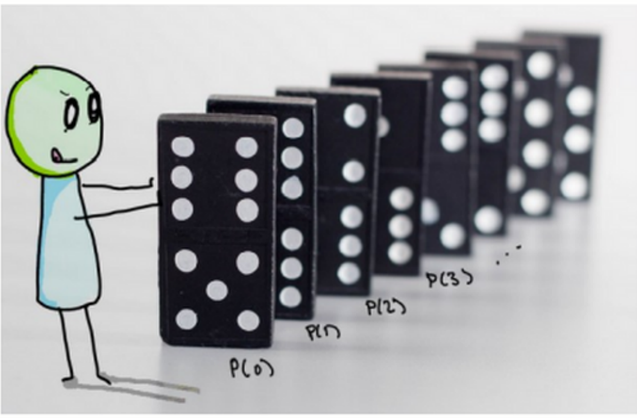
dominoes are lined up!

Assume $P(k)$ (this is called the **inductive hypothesis**.)

Carefully write out $P(k+1)$

Figure out how to relate $P(k+1)$ to $P(k)$

Use this recursive relationship to conclude $P(k+1)$ is true. □



ex] $\forall n \in \mathbb{N}, \quad n! \geq 1$

has recursive structure!

side

recall $\underline{n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1}$

& $0! = 1$

Proof (by induction)

Base Case ($P(0)$ is true)

It follows that $0! = 1 \geq 1$. This shows the base case is true

Inductive Step ($P(k) \Rightarrow P(k+1)$)

Suppose the prop. is true when $n = k \in \mathbb{N}$.

This means $\boxed{k! \geq 1}$. (we want $(k+1)! \geq 1$)

$$(k+1)! = (k+1)\underbrace{k \cdot (k-1) \cdot (k-2) \cdots 2 \cdot 1}_{k!}$$

$$(k+1)! = (k+1) \cdot \boxed{k!} \geq (k+1) \cdot \boxed{1} = k+1 \geq 1$$

where we used our inductive hypothesis in the first inequality. □

4. If $n \in \mathbb{N}$, then $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.

(Book of Proof pg. 195)

clues for induction: formula involving $n \in \mathbb{N}$

its a sum!!

$$S_{k+1} = \sum_{i=1}^{k+1} s_i = s_1 + s_2 + \cdots + s_k + s_{k+1}$$

$$= \sum_{i=1}^{k} s_i + s_{k+1}$$

$$\boxed{S_{k+1} = S_k + s_{k+1}}$$

If $n \in \mathbb{N}$, then $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.

Proof (by induction)

Base Case $n=1$

The sum on the LHS becomes $1 \cdot 2 = 2$.

The expression on the RHS becomes $\frac{1 \cdot (1+1) \cdot (1+2)}{3}$

$= \frac{1 \cdot 2 \cdot 3}{3} = 1 \cdot 2 = 2$. The Base Case is true.

Inductive Step

Suppose the prop. is true for $n = k \in \mathbb{N}$.

This means $\boxed{1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + k\cdot(k+1) = \frac{k(k+1)(k+2)}{3}}$.

(we want to show

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$
)

Our inductive hypothesis implies that

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + k(k+1) + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3}$$

$$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$$

$$= \frac{(k+1)(k+2)\left[\ k+3\ \right]}{3}$$

$$= \frac{(k+1)(k+2)(k+3)}{3}. \quad \square$$

Example 2.2. *Consider the recursively defined sequence and initial condition*

$$a_n = -2 + 3a_{n-1} \quad and \ a_0 = 1.$$

Proposition. The sequence $\{a_n\}$ is constant; in particular $a_n = 1$ for all $n \in \mathbb{N}$.

Proof by induction

Base Case $(n=0)$

$a_0 = 1$ according to the given initial conditions.

The Base Case is true.

Inductive Step

Suppose the prop. is true when $n = k \in \mathbb{N}$.

This means $a_k = 1$. (We want to show $a_{k+1} = 1$)

By the given recurrence equation,

① $a_{k+1} = -2 + 3 \cdot a_k$

② $= -2 + 3 \cdot 1$

③ $= -2 + 3 = 1$

where we used our ind. hyp. in line ② . $\square$

Cautionary words

- write out $P(k+1)$ carefully

  Substitute $n = k+1$

- use some scratch paper to explore how $P(k)$ & $P(k+1)$ are related.
