Math 3336
Homework Assignment 9

Instructions

- Record your answers to the following 6 questions. Show your work when a question requires you to do so.
- Scan your work and save the file as a .pdf (make sure your work and answers are legible)
- Upload your scanned work to CASA CourseWare using the "Assignments" tab. (Click this link for instructions on how to do this).
- Homework submitted after 11:59pm on the indicated due date will be assigned a grade of 0.
- Also, **DON'T FORGET THAT** $0 \in \mathbb{N}$.
- I understand that if any of the questions from this assignment (or future ones) are shared in ways that violate our Academic Honesty Policy, then the syllabus will change. Specificlaly, Homework and Quizzes will be worth zero points.

  Name:

  Signature:

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1. Write a proof of the following proposition (make certain to label the style of proof you are using):

Proposition. Let $a, b, d \in \mathbb{Z}$. If $d|a \wedge d|b$, then $d^2 | (ab)$

2. (Part A) Use the Pigeonhole Principle to explain the following fact: when six integers are selected at random, at least two of them will have the same remainder when divided by the number 7. (Thinking about Question 3 from Quiz 10 may be helpful here. Also, you may write up your explanation as a formal proof or you may explain it in less formal language.)

(Part B) what can one conclude when six integers are selected at random and each one is divided by the number 5? Specifically, what does the "extended Pigeonhole Principle" (aka "the Division Principle") allow us to conclude about the values of their remainders?

3. Use induction to prove the following proposition:

Proposition. Let $a, p, n \in \mathbb{N}$. If p is prime and $p | (a^n)$ for all $n \geq 1$, then $p|a$

Proof (by induction)

Base Case ($n = 1$)

Inductive Step

4. (Part A) As it turns out, the greatest common divisor of 537 and 2022 is 3, i.e. $\gcd(537, 2022) = 3$. Carefully and accurately fill in the missing blanks below to show how the Euclidean Division Algorithm can be used to calculate this value (no work need be included for this part).

$$\begin{aligned}
2022 &= ___ \cdot 537 + 411 \\
537 &= ___ \cdot 411 + ___ \\
411 &= ___ \cdot ___ + ___ \\
___ &= ___ \cdot ___ + ___ \\
___ &= ___ \cdot ___ + ___ \\
___ &= ___ \cdot ___ + ___ \\
___ &= ___ \cdot ___ + ___
\end{aligned}$$

(Part B) Work backwards through the equations above to find integer coefficients $x, y \in \mathbb{Z}$ that satisfy Bezout's Identity (i.e. that solve $2022x + 537y = 3$).

5. In elementary school you may have learned about "factor trees." These are used to visualize and find the prime factors of a given integer. As it turns out, these objects are always **full binary trees** and, of course, the Fundamental Theorem of Arithmetic tells us that they necessarily "terminate" at the (unique) prime factors of a given integer.

Here is a brief summary of how these trees work, one focused on factoring the integer 12. For our first step, we choose any two factors of 12 (and if there are none, then the process ends as the number must already be prime). We store the value of 12 at a root node, and then create two edges joining it to two children nodes, each stored with one of the factors.

We continue this process until we arrive at prime numbers, thereby completing our "Factor Tree." The image below shows two different Factor Trees for the integer 12.

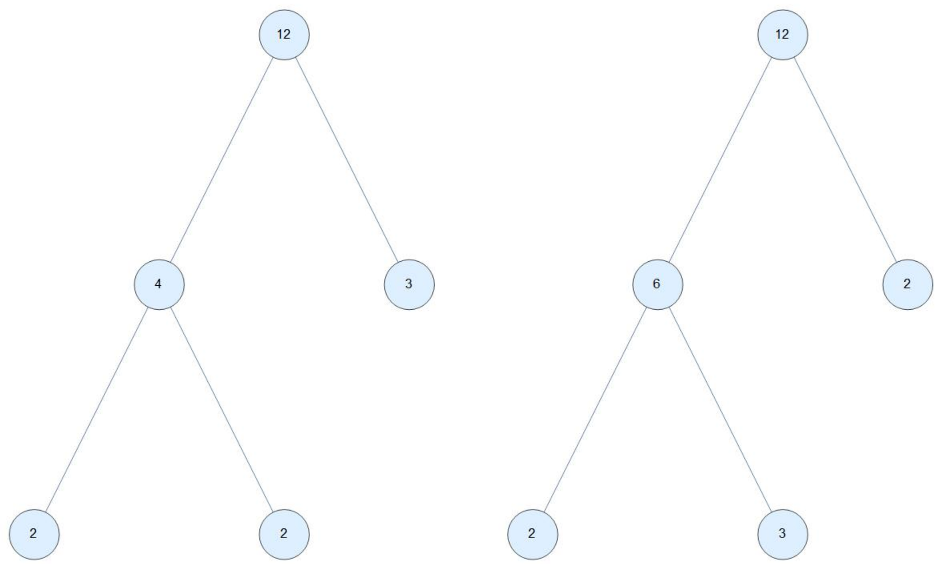

Figure 1: Factor Trees for 12

Recall the **height of a full binary tree**, $h(T)$ (an idea we discussed in our own notes and office hours). Notice that in the example above, the height of both factor trees is two, and this equals the number of distinct primes in 12's decomposition, i.e. $12 = 3 \cdot 2^4$. Doe this always happen?

(Part A) Provide an example of a factor tree T where $h(T)$ is strictly larger than the number of distinct primes. (Or briefly explain why this is impossible.)

(Part B) Provide an example of a factor tree, T, where $h(T)$ is strictly smaller than the number of distinct primes. (Or briefly explain why this is impossible.)

6. Everyone agrees that your favorite natural number is the amazing integer 4, and so let's use it for a little experiment. We will agree to use the following language: two integers $x, y \in \mathbb{Z}$ are **amazingly equivalent to one another** precisely if they have the same remainder when divided by 4. (Explore. Some. Examples.)

(Part A) Which integers are **amazingly equivalent** to -3? If you think there are infinitely many such integers, describe the set of all of them. (You do not need to compose a formal proof when explaining your answer, but you are welcome to do so.)

(Part B) Write a direct proof that explains the following: if integer x is **amazingly equivalent to 0** and if y is **amazingly equivalent to 0** then $x + y$ is also **amazingly equivalent to 0**.

7. Let $a, b \in \mathbb{Z}$. If we also know that a and b are relatively prime, then which, if any, of the following statements do we also know to be true?

(I) $\gcd(a, b) = 1$
(II) There exist integers $x, y \in \mathbb{Z}$ satisfying $ax + by = 1$.
(III) $\Big(x|a \wedge x|b\Big) \Rightarrow \Big(x = a \vee x = b\Big)$
(IV) Both a and b are prime numbers.

(a) All four statements are true.

(b) Only statement (I) is true.

(c) Only statements (I) and (II) are true.

(d) Only statements (II), (III) and (IV) are true.

(e) None of the statements are true.

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8. Let $a \in \mathbb{Z}$, and suppose we know that $5|\,(33a)$. What does Euclid's Lemma allow us to conclude?

(a) 5 must divide a.

(b) a is prime.

(c) 5 must divide 33.

(d) 33 is a multiple of a.

(e) $5a = 30$ and so $a = 6$.

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9. How many prime numbers are there?

(a) There are 10 prime numbers.

(b) There are 17 prime numbers.

(c) The are infinitely many prime numbers.

(d) There are finitely many prime numbers, but the exact amount is not currently known.

(e) There are zero prime numbers.

10. Which problems from this homework set did you find most beneficial? Most interesting? Which ones did you find least helpful?