Discrete Math

Lecture 18

Direct & Contrapositive Proofs

Propositions. $P \Rightarrow Q$

Direct Proofs

Direct Proof Outline

Proposition. $P \Rightarrow Q$

Assume P.

Proof. (Direct)
(First Step) Suppose P is true.
(Intermediate Steps) Use definitions related to P and Q.
(Intermediate Steps) Use previously established facts to connect P and Q.
$\vdots$
(Last Step) Conclude Q is true. $\square$

$A \subseteq B$

if $a \in A$, then $a \in B$

ex) $\mathbb{Z} \subseteq \mathbb{Q}$

Proof (direct)

Suppose $x \in \mathbb{Z}$.
(we want to show $x \in \mathbb{Q}$)
It follows that

$$x = \frac{x}{1} \in \mathbb{Q}.$$

Therefore $x \in \mathbb{Q}$. $\square$

ex) (Book of Proof, Ch. 4)

11. Suppose $a,b,c,d \in \mathbb{Z}$. If $a \mid b$ and $c \mid d$, then $ac \mid bd$.

Proof (direct)

Suppose $a|b$ and $c|d$. (we want to show $ac|bd$)

This means $\exists\, q_1 \in \mathbb{Z}$, $b = q_1 \cdot a$ and $\exists\, q_2 \in \mathbb{Z}$, $d = q_2 \cdot c$.

It follows that $b \cdot d = (q_1 \cdot a) \cdot (q_2 \cdot c)$

$= (q_1 \cdot q_2) \cdot (ac) = q \cdot (ac)$ where $q = q_1 \cdot q_2 \in \mathbb{Z}$.

Thus $bd = q \cdot (ac)$ and this

tells us $ac \mid bd$. $\square$

Contrapositive Proofs

I'm going to prove $\neg Q \Rightarrow \neg P$ instead.

Contrapositive Proof Outline

Proposition. $P \Rightarrow Q$

Proof. (Contrapositive)
(First Step) Suppose $\neg Q$ is true (i.e. Suppose Q is false).
(Intermediate Steps) Use definitions related to $\neg P$ and $\neg Q$.
(Intermediate Steps) Use previously established facts to connect $\neg Q$ and $\neg P$.

$\vdots$

(Last Step) Conclude $\neg P$ is true (i.e. conclude P is false). $\square$

ex) $\mathbb{Z} \subseteq \mathbb{Q}$ (if $x \in \mathbb{Z}$ then $x \in \mathbb{Q}$)

Proof (contra. pos.)

Suppose $x \notin \mathbb{Q}$. (we want to show $x \notin \mathbb{Z}$)

This means $x \neq \frac{a}{b}$, where $a, b \in \mathbb{Z}$.

Since $1 \in \mathbb{Z}$, we see that after using $b = 1 \in \mathbb{Z}$

$x \neq \frac{a}{1} = a$, for any $a \in \mathbb{Z}$.

In other words, $x \neq a$ for any $a \in \mathbb{Z}$, so $x \notin \mathbb{Z}$. $\square$

ex) (Book of Proof, Ch 5)

6. Suppose $x \in \mathbb{R}$. If $x^3 - x > 0$ then $x > -1$.

Proof (contrapositive)

Assume $x \leq -1$. (We want to show $x^3 - x \leq 0$.)

Note $x^3 - x = x(x^2 - 1) = x(x+1)(x-1)$.

Since $x \leq -1$, each factor is less than or equal to 0.

$x < 0$ because $x \leq -1$.

$(x+1) \leq 0$ since $x \leq -1 \Rightarrow x+1 \leq 0$.

$(x-1) < 0$ since $x \leq -1 \Rightarrow x-1 \leq -2 < 0$.

The product of three numbers each less than or equal to zero is itself ≤ 0.

Therefore $x^3 - x \leq 0$. $\square$