3. Strong Induction

Using so-called "Strong Induction" to prove a proposition of the form "$\forall n \in \mathbb{N}$, $P(n)$" is almost identical to Induction (or what some call "Weak Induction"), except that in the inductive step we assume *more* than just the single previous case $P(k)$, we assume *multiple* previous cases.

If we stick with our "knocking over dominoes" analogy, then Strong Induction is useful when the dominoes are larger and heavier, as in the following illustration.

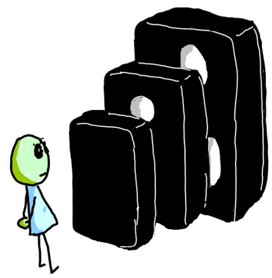

Figure 2. When Strong Induction is Needed

We'd like to show that all the dominoes can be knocked down, but it might be that $P(k+1)$ is so difficult or "heavy," that just the previous domino leaning on it, $P(k)$, is not enough. Instead, we my need *all* or *several* of the previous dominoes' combined "weight" to topple $P(k+1)$.

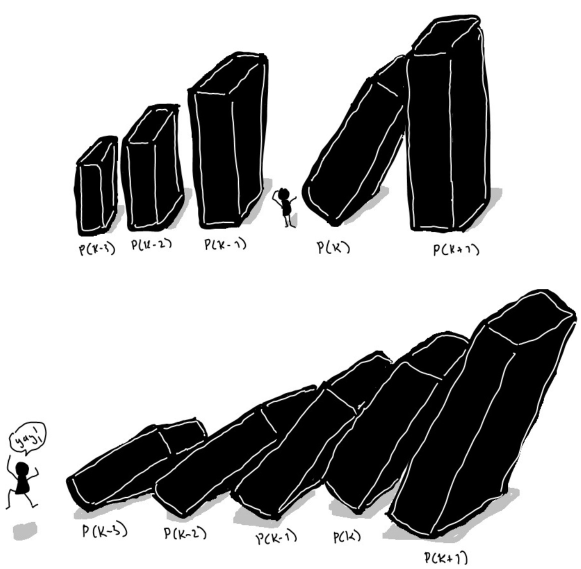

Figure 3. Proof by Strong Induction

Recursive structure is still needed or used, of course, only it takes a slightly differently looking form. Rather than relating $P(k+1)$ to $P(k)$, we will need to relate $P(k+1)$ to several previous statements like $P(k), P(k-1), P(k-2), \cdots, P(1), P(0)$. An outline for a Proof by Strong Induction is quite similar to our one for Induction, and, as usual, examples will make this technique much clearer.

Strong Induction Outline

Proposition. $\forall n \in \mathbb{N},\ P(n)$

Proof by (strong) Induction

Base Case

Show that $P(0)$ is true (or show that the first several cases are true, as needed).

Inductive Step

Show being true at several "cases" implies the proposition is true at the "next case."

Show $\Big(P(0) \wedge P(1) \wedge \cdots \wedge P(k)\Big) \Rightarrow P(k+1)$

Assume $P(0) \wedge P(1) \wedge \cdots \wedge P(k)$ (this is the **strong ind. hypothesis**.)

That is, assume $P(n)$ is true for all $0 \le n \le k$

Carefully write out $P(k+1)$

Figure out how to relate $P(k+1)$ to $P(n)$ for $0 \le n \le k$

Use this recursive relationship to conclude $P(k+1)$ is true. $\square$

The first example below is about a recursively-defined sequence, and the proof uses *two* previous cases in the inductive step. See if you can figure out why this happens (Hint: the recurrence equation involves how many "previous terms?")

The proceeding one is about natural numbers. Really, it is a result that holds for all integers, and you may have noticed it or been told about it before. Specifically, the proposition concerns the fact that many (all?) numbers can be expressed as a product of primes. For instance, $28 = 7 \cdot 2^2$ is a product of primes, as is $45 = 3^2 \cdot 5$ and $7 = 7$. The Inductive Step for this example will use *all* possible previous cases.

Figuring out how many previous cases are needed in an Induction Proof requires us to engage with the actual proposition, perhaps by testing some examples and investigating how the proposition can be made to rely on "previous" cases. So, as usual, engage these kinds of problems with an open mind and a ready-to-explore attitude.

Example 3.1. *Consider the recursively defined sequence $a_n = 3a_{n-1} + a_{n-2}$ with initial conditions $a_0 = 0$ and $a_1 = 4$.*

Proposition. a_n is even for ever $n \in \mathbb{N}$

Before reading the following proof, test some examples to make certain you think the proposition is true, then try to set up your own (strong) induction proof.

Proof by (strong) Induction.

Base Case

We need to check two base cases here, $n = 0$ and $n = 1$. The given initial conditions verify the proposition in these cases, i.e. that a_0 and a_1 are even.

Inductive Step

Suppose the proposition is true for $n = k$ and $n = k-1$, where $k \in \mathbb{N}$ and $k \geq 1$. This means a_k and a_{k-1} are even, and so $a_k = 2x$ and $a_{k-1} = 2y$ for integers x, y. (We want to show that a_{k+1} is even.) The recurrence relation allows us to write

$$a_{k+1} = 3a_k + a_{k-1} = 3 \cdot 2x + 2y = 2\left(3x + y\right).$$

Since $(3x + y) \in \mathbb{Z}$, it follows that a_{k+1} is even $\square$

Example 3.2.

Proposition. $\forall\, n \in \mathbb{N},\ n \geq 2 \Rightarrow n$ can be expressed as a product of primes.

Proof by (strong) Induction.

Base Case

We check the proposition when $n = 2$. Since 2 is already a prime number, the claim is true. (We will regard this as a "trivial product of primes.")

Inductive Step

Suppose the proposition is true for $n = k$ and for all values of $n < k$. (We want to show that $k + 1$ can be written as a product of primes.)

If $k + 1$ is prime, then it is itself a product of primes.

If $k + 1$ is not a prime number, then it is **composite** and so can be written as a product of two smaller numbers: $k + 1 = a \cdot b$. Since $a \leq k$, by our strong inductive hypothesis, it can be written as a product of primes. Similarly, $b \leq k$ can be written as a product of primes. It follows that $(k + 1) = a \cdot b$ is a product of primes. $\square$

Book of Proof has excellent discussions about Induction as well as lots of useful examples. Consult the following

- Section 10.2 (pages 187-190)
- The proposition on page 190. It states "If a tree has n vertices, then it has $n - 1$ edges." The proof that follows is discussed very well, and is probably one of the best examples of "strong" induction.
- Section 10.5 (pages 193-195) for Propositions about the Fibonacci numbers
- Exercises 25, 27 (page 196)