Math 3336 : Final Review

Instructions

- This is not an assignment. Neither work nor answers are to be submitted.
- Also, **DON'T FORGET THAT** $0 \in \mathbb{N}$.

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1. Suppose the function $f(n)$ is $O\left(n^5\right)$, then...

(a) $f(n)$ is also $\Theta\left(n^5\right)$

(b) $f(n)$ is also $O\left(n^k\right)$ for all $k \in \{0, 1, 2, 3, 4\}$

(c) $f(n)$ is also $O\left(n^k\right)$ for all $k \geq 5$

(d) $f(n) = n^5$

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2. Suppose you have an algorithm that consists of a loop, and the input for the algorithm is a list of length n.

(A) When the length of the list increases by one, the number of times the loop is repeated doubles– at least this happens in the worst-case scenario. Based on this information, the worst-case run-time for the algorithm is...

(a) $\Theta\left(\log_2 n\right)$
(b) $\Theta\left(2^n\right)$
(c) $\Theta\left(n^2\right)$
(d) $\Theta(n)$
(e) $\Theta(2)$

(B) When the length of the list is doubled, the number of times the loop is repeated increases by one – at least this happens in the worst-case scenario. Based on this information, the worst-case run-time for the algorithm is...

(a) $\Theta\left(\log_2 n\right)$
(b) $\Theta\left(2^n\right)$
(c) $\Theta\left(n^2\right)$
(d) $\Theta(n)$
(e) $\Theta(2)$

(C) When the length of the list increases by one, the number of times the loop is repeated also increases by one – at least this happens in the worst-case scenario. Based on this information, the worst-case run-time for the algorithm is...

(a) $\Theta\left(\log_2 n\right)$
(b) $\Theta\left(2^n\right)$
(c) $\Theta\left(n^2\right)$
(d) $\Theta(n)$
(e) $\Theta(2)$

3. Fill in the truth tables for each of the following logical operators:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

4. An expression involving many abstract statements is a **contradiction** when ...

(a) Its truth table contains only F

(b) Its truth table contains only T

(c) Its truth table is blank

(d) The first and last two rows of its truth table end with F

(e) None of the above

5. If the congruence equation $ax \equiv b \bmod n$ has multiple solutions from the set $\{0, 1, 2, \ldots, n-1\}$ then...

(a) $a = 0$

(b) $\gcd(a, n) = 1$

(c) $\gcd(a, n) \nmid b$

(d) $\gcd(a, n) > 1$

(e) None of the above

6. How is a second-order, linear, homogeneous recurrence equation related to its characteristic equation?

(a) The recurrence relation $a_n = ba_{n-1} + ca_{n-2}$ has as its characteristic equation $x^2 + bx + c = 0$

(b) The recurrence relation $a_n = ba_{n-1} + ca_{n-2}$ has as its characteristic equation $x^2 - bx - c = 0$

(c) The recurrence relation $a_n = ba_{n-1} + ca_{n-2}$ has as its characteristic equation $x^2 + cx + b = 0$

(d) The recurrence relation $a_n = ba_{n-1} + ca_{n-2}$ has as its characteristic equation $x^2 - cx - b = 0$

(e) In general, there is no way to relationship between these two objects.

7. Is the function $\log x$ in $O(x)$?

(a) Yes, of course!

(b) No, of course not!

(c) This is an open question that no one has yet proved is true or proved is false.

8. Carefully read the statement P below.

$$P : \exists\, a \in \mathbb{Z},\, \forall\, b \in \mathbb{Z},\, a + b = 2022.$$

Which, if any, of the following correctly expresses $\neg P$?

(a) $\neg P : \forall\, a \in \mathbb{Z},\, \exists\, b \in \mathbb{Z},\, a + b \neq 2022$

(b) $\neg P : \exists\, a \in \mathbb{Z},\, \forall\, b \in \mathbb{Z},\, a + b \neq 2022$

(c) $\neg P : \forall\, a \in \mathbb{Z},\, \exists\, b \in \mathbb{Z},\, a + b = 2022$

(d) $\neg P : \forall\, a \in \mathbb{Z},\, \forall\, b \in \mathbb{Z},\, a + b \neq 2022$

(e) None of the above

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9. Consider the set $F = \{\pi, \sqrt{7}, 7, 0, -1, 8\}$. Which, if any, of the following expressions represents the number of size-4 subsets of F?

(a) $\binom{8}{4}$

(b) $8!$

(c) $6!$

(d) $\binom{6}{4}$

(e) 2^6

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10. Use the division algorithm to divide 116 by 3, and then, based on your work, determine which of the following statements is true.

(a) $q = 36$ and $r = -2$

(b) $q = 35$ and $r = 1$

(c) $q = 34$ and $r = 4$

(d) $q = 34$ and $r = 0$

(e) It is impossible to use the division algorithm for this pair of integers.

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11. Melon Usk – a celebrity billionaire and self-proclaimed "genius computer scientist" – claims to have invented a way to perfectly compress files *without any loss of information*! In fact, he claims that his algorithm reduces the file size by factor of 0.5. Which of the following explains why such an algorithm cannot, in reality, work or exist?

(a) Consider the set, A, of files that are length n bit-strings as well as the set, B, of files that are length $.5 \cdot n$ bit-strings. Such an algorithm acts as a function $f : A \to B$, but since $|A| = 2^n > 2^{0.5n} = |B|$, the Pigeonhole Principle tells us two files in A must be compressed into the same file in B. As a result, the compression algorithm cannot be undone – files cannot be decompressed.

(b) Actually, such an algorithm can (and will) exist; it will be used to help run future colonies on Venus.

12. Suppose you want to prove a Proposition of the form $P \Rightarrow Q$.

(A) However, you want to do so using a direct proof. Which, if any, of the following best outlines such a proof?

(a) Begin the proof by assuming $\neg(P \Rightarrow Q) = P \wedge \neg Q$, then use relevant facts and definitions to reach a conclusion like "2 is both even and odd."

(b) Begin the proof by assuming $\neg P$, then use relevant facts and definitions to reach the conclusion $\neg Q$.

(c) Begin the proof by assuming $\neg Q$, then use relevant facts and definitions to reach the conclusion $\neg P$.

(d) Begin the proof by assuming P, then use relevant facts and definitions to reach the conclusion Q.

(e) Begin the proof by assuming Q, then use relevant facts and definitions to reach the conclusion P.

(B) However, you want to do so using a proof by contradiction. Which, if any, of the following best outlines such a proof?

(a) Begin the proof by assuming $\neg(P \Rightarrow Q) = P \wedge \neg Q$, then use relevant facts and definitions to reach a conclusion like "2 is both even and odd."

(b) Begin the proof by assuming $\neg P$, then use relevant facts and definitions to reach the conclusion $\neg Q$.

(c) Begin the proof by assuming $\neg Q$, then use relevant facts and definitions to reach the conclusion $\neg P$.

(d) Begin the proof by assuming P, then use relevant facts and definitions to reach the conclusion Q.

(e) Begin the proof by assuming Q, then use relevant facts and definitions to reach the conclusion P.

(C) However, you want to do so using a contrapositive proof. Which, if any, of the following best outlines such a proof?

(a) Begin the proof by assuming $\neg(P \Rightarrow Q) = P \wedge \neg Q$, then use relevant facts and definitions to reach a conclusion like "2 is both even and odd."

(b) Begin the proof by assuming $\neg P$, then use relevant facts and definitions to reach the conclusion $\neg Q$.

(c) Begin the proof by assuming $\neg Q$, then use relevant facts and definitions to reach the conclusion $\neg P$.

(d) Begin the proof by assuming P, then use relevant facts and definitions to reach the conclusion Q.

(e) Begin the proof by assuming Q, then use relevant facts and definitions to reach the conclusion P.

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13. Consider the proposition about natural numbers $n \in \mathbb{N}$

Proposition. $\forall n \geq 1,\ 3^n - 1,$ is even .

If we wish to prove this using Induction, then our Base Case...

(a) is checked by proving the proposition is true when $n = 3$

(b) is checked by proving the proposition is true when $n \in \mathbb{N}$.

(c) is checked by proving the proposition is true when n is even.

(d) is checked by proving the proposition is true when $n = 1$

(e) is checked by proving the proposition is true when $n = 0$

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14. How many length-n bit-strings contain at least one 0?

(a) 2^{n-1}

(b) $2^n - 1$

(c) 1

(d) n

(e) $2^{n-1} + 1$

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15. If we use the Euclidean Algorithm to compute $\gcd(2022, 11)$, then how many times will the Division "Algorithm" be used?

(a) 1 time

(b) 2 times

(c) 3 times

(d) 4 times

(e) 5 times

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16. 58 integers are selected at random, and they are each divided by 5. Which of the following statements is true?

(a) At least 11 of the integers *must* have the same remainder (but we cannot know for certain that more than 11 will).

(b) At least 5 of the integers *must* have the same remainder (but we cannot know for certain that more than 5 will).

(c) At least 12 of the integers *must* have the same remainder (but we cannot know for certain that more than 12 will).

(d) At least 3 of the integers *must* have the same remainder (but we cannot know for certain that more than 3 will).

(e) All 58 integers *must* have the same remainder.

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17. Which of the following statements is true?

(a) $\emptyset \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

(b) $\emptyset \subseteq \mathbb{Z} \subseteq \mathbb{N} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

(c) $\emptyset \subseteq \mathbb{N} \subseteq \mathbb{Q} \subseteq \mathbb{Z} \subseteq \mathbb{R}$

(d) $\emptyset \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R} \subseteq \mathbb{Q}$

(e) $\emptyset \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{N} \subseteq \mathbb{Z}$

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18. Consider the set $S \subseteq \mathbb{Z} \times \mathbb{Z}$ defined by the rules

$$\begin{aligned} (1,2) &\in S \\ (x,y),(a,b) \in S \;\Rightarrow\; (x+a, y+b) &\in S \\ (x,y) \in S \;\Rightarrow\; (-x,-y) &\in S \end{aligned}$$

If we wanted to prove that every $(u,v) \in S$ satisfies $v = 2u$ using Structrual Induction, which, if any, of the following most accurately outlines our "Recursive Step?"

(a) Show that the claim is true for the base element $(1,2)$

(b) Let $(u,v) \in S$ be an arbitrary element that satisfies the claimed equation. Then show $(u+1, v+1)$ also satisfies the claimed equation.

(c) Let $(u,v),(s,t) \in S$ be a arbitrary elements that satisfy the claimed equation. Then show $(-u,-v)$ and $(u+s, v+t)$ also satisfy the claimed equation.

(d) Let $(u,v),(s,t) \in S$ be a arbitrary elements that satisfy the claimed equation. Then show $(u+1, v+1)$ and $(s+1, t+1)$ also satisfy the claimed equation.

(e) Structural Induction cannot be used here because the set S is not well-ordered.

19. Which of the following sentences is a true statement?

(a) Shut the door.

(b) 16 is a multiple of 6.

(c) $\forall x \in \mathbb{R},\, x^2 - 1 \geq 0$

(d) $\exists y \in \mathbb{Z},\, y^3 - y = 0$

(e) $|\,\mathcal{P}\,(\,\{1, -9, 5\}\,)\,| = 8$

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20. For this question consider the set $U = \{0, 1, 2, \ldots, 9\}$. Count the number of subsets whose complement has 3 elements. (Knowing the formula for counting subsets is useful. Yay! Or... Uhh... Ahhh... Let me think... Let's say "okay" instead)

(a) There are 10 such subsets.

(b) There are 120 such subsets.

(c) There are 84 such subsets.

(d) There are 36 such subsets.

(e) There are no such subsets.