Factorials & Permutations

Question: how many length n non-repeating lists / strings are there that are made from n symbols

ex) $n=5$ $X=\{a,b,c,d,e\}$

$$\underset{5}{\_}\ \underset{4}{\_}\ \underset{3}{\_}\ \underset{2}{\_}\ \underset{1}{\_}$$

mult. principle

$\Rightarrow$ # of such lists / strings

$= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ = "5 factorial"

$= 120$ $= 5!$

more generally, when counting the # of length-n non-repeating lists / strings made using n symbols, we find

$$\underbrace{\underset{n}{\_}\ \underset{n-1}{\_}\ \underset{n-2}{\_}\ \cdots\ \underset{1}{\_}}_{n \text{ slots}}$$

mult. princ. $\Rightarrow$ there are $\boxed{n(n-1)\cdot(n-2)\cdots 1}$ such lists

$\boxed{n!}$

Definition 3.1 If n is a non-negative integer, then $n!$ is the number of lists of length n that can be made from n symbols, without repetition. Thus $0! = 1$ and $1! = 1$. If $n > 1$, then $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$.

<u>Comment</u> $0! = $ # of length-0 lists (non-repeating) made from 0 symbols

$= 1$ (the empty list ( ))

<u>another, non-listy way to see why $0! = 1$</u>

note: $6! = 6 \cdot \boxed{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$= 6 \cdot 5!$

$5! = 5 \cdot 4!$

$\vdots$

$2! = 2 \cdot 1!$

$\boxed{1} = 1! = \boxed{1} \cdot \boxed{0!}$

$$n! = n \cdot (n-1)!$$

$\uparrow$ must $= 1$

<u>ex)</u> how many non-repeating, length-5 strings can be made using symbols $X = \{v, w, x, y, z\}$?

$\underset{5}{\_}\ \underset{4}{\_}\ \underset{3}{\_}\ \underset{2}{\_}\ \underset{1}{\_}$ $5! = 120$ such strings exist

one example is: $y\,x\,w\,z\,v$ is one such string.

each such string is called a "<u>permutation of X</u>"

another permutation : v x z y w , v x w z y , ...

a permutation of X (where $|X| = n$)

is a non-repeating, length-n string that uses elements of X as its symbols.

a k-permutation of X is a non-repeating, length-k string that uses elements of X as its symbols.

ex | $X = \{v, w, x, y, z\}$

1-permutations of X : v w x y z $\quad 5$

5-permutations = permutations of X : $\quad 5! = 120$

2-permutations of X : vw vx vy vz ... $\quad 5 \cdot 4 = 20$

$\underset{5}{\_}\ \underset{4}{\_}$ $\quad \boxed{5 \cdot 4 \text{ of these!}}$

3-permutations of X : $\boxed{5 \cdot 4 \cdot 3 = 60 \text{ of these}}$

$\underset{5}{\_}\ \underset{4}{\_}\ \underset{3}{\_}$

4-permutations : $\boxed{\begin{array}{c} 5\cdot 4\cdot 3\cdot 2 = 120 \\ \text{of these} \end{array}}$

now try this : if $|X| = 100$, how many 6-permutations will there be?

$\underline{\ \ }_{100}\ \underline{\ \ }_{99}\ \underline{\ \ }_{98}\ \underline{\ \ }_{97}\ \underline{\ \ }_{96}\ \underline{\ \ }_{95}$

$100\cdot 99\cdot 98\cdot 97\cdot 96\cdot 95$

what about 70-permutations?

$\underbrace{\underline{\ \ }_{100}\ \underline{\ \ }_{99}\ \underline{\ \ }_{98}\ \underline{\ \ }_{97}\ \cdots\ \underline{\ \ }}_{70 \text{ slots}}$ $100-70+1 = 31$

$100\cdot 99\cdot \cdots \cdot 70$

How many k-permutations on a set $|X| = n$ are there? $(k \le n)$

$\underbrace{\underline{\ \ }_{n}\ \underline{\ \ }_{n-1}\ \underline{\ \ }_{n-2}\ \cdots\ \underline{\ \ }_{n-k+1}}_{k \text{ slots}}$

$$n\cdot(n-1)(n-2)\cdots(n-k+1)$$

$$= \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)\cdot(n-k)\cdots 1}{(n-k)(n-k-1)\cdots 1}$$

$$= \frac{n!}{(n-k)!}$$

What happens if $k > n$?

How many k-permutations on n elements are there?

ex) $X = \{v, w, x, y, z\}$ $n = 5$

$k = 7$

7-permutation = a length-7 list w/ no repeats that is made using symbols from X

$\underline{w}\ \underline{z}\ \underline{x}\ \underline{v}\ \underline{y}\ \underline{?!}\ \underline{?!}$ None of these!!

ex) what about 0-permutations of n elements?

exactly one length-0 string : the empty list!

notation $P(n,k)$ = # of k-permutations of n elements

$$= \begin{cases} \dfrac{n!}{(n-k)!} & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k > n \end{cases}$$

example

length 7

$n=7$

9. How many permutations of the letters ~~A~~, ~~B~~, ~~C~~, *D*, *E*, *F*, *G* are there in which the three letters ABC appear consecutively, in alphabetical order?

4 3 2 1 A B C

type 1

$4 \cdot 3 \cdot 2 \cdot 1 = \boxed{24}$

4 3 2 A B C 1

type 2

$\boxed{24}$

4 3 A B C 2 1

type 3

$\boxed{24}$

4 A B C 3 2 1

type 4

$\boxed{24}$

A B C 4 3 2 1

type 5

$\boxed{24}$

a total of $5 \cdot 24 = 120$ such strings.

14. Five of ten books are arranged on a shelf. In how many ways can this be done?

order matters

i.e. a "list" of books

asking about 5-permutations of 10 elements

$$P(10,5) = \frac{10!}{(10-5)!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot \cancel{5\cdot 4\cdot 3\cdot 2\cdot 1}}{\cancel{5\cdot 4\cdot 3\cdot 2\cdot 1}}$$

$$= 30{,}240$$