Subsets & Power Sets

We say a set B <u>is a subset</u> of another set A ("$B \subseteq A$") if every $b \in B$ satisfies $b \in A$

notation: $B \subseteq A$ allows for $B = A$!

$B \subset A$ there are elements in A that are <u>not</u> in B!

ex 1

$A = \{a, b, c, \ldots, y, z\}$

$B = \{a, e, i, o, u\}$

$B \subseteq A$ ✓

$B \subset A$ ✓

ex 2

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0\}$

$\mathbb{R} = \{\text{all real numbers}\}$ ← this set has its own class called "real analysis"

$\mathbb{N} \subseteq \mathbb{Z} \quad (\mathbb{N} \subset \mathbb{Z})$

$z \in \mathbb{Z} \quad = \frac{z}{1} \in \mathbb{Q} \rightarrow \mathbb{Z} \subseteq \mathbb{Q}$

$(\mathbb{Z} \subset \mathbb{Q})$

$\mathbb{Q} \subseteq \mathbb{R} \quad (\mathbb{Q} \subset \mathbb{R})$

all together: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

fascinating extension!

$\sqrt{2} \approx 1.41\ldots \notin \mathbb{Q}$

$\pi \notin \mathbb{Q}$

$\mathbb{R}$ fills in the many, interesting gaps in $\mathbb{Q}$

(it "completes $\mathbb{Q}$")

we can (and do!) extend further:

complex numbers

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

<u>Fact</u> the empty set, $\emptyset = \{ \}$, is a subset of <u>all</u> sets!

Given any set S, $\emptyset \subseteq S$.

$\boxed{\text{Why?}}$ weird: there are no elements in $\emptyset$.

to say $A \subseteq B$ is to carry out a test: if $x \in A$, then $x \in B$.

$A \subseteq B$ means this test always works!

↳ Apply this to $\emptyset$. $\emptyset \subseteq \mathbb{N}$

take any $x \in \emptyset$, is $x \in \mathbb{N}$?

how can this fail? $\underbrace{x \in \emptyset \text{ and } x \notin \mathbb{N}}$

never happens!

there is no $x \in \emptyset$!

$$\boxed{\emptyset \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}}$$

<u>ex 2</u> $S = \{ ☺, 1, w \}$

What are <u>all</u> the subsets of S?

$\emptyset = \{ \}$, $\{☺\}$, $\{1\}$, $\{w\}$

$\{☺, 1\}, \ \{☺, w\}, \ \{1, w\}$

$\{☺, 1, w\} = S$

S has eight subsets!

If S satisfies $|S| = 4$, how many subsets does it have?

<u>ex 3</u> $\quad A = \{⌂\}$

what are the subsets of A?

$\emptyset = \{\ \}, \quad A = \{⌂\}$ $\qquad$ two subsets

<u>Def.</u> (Power Set) The <u>Power Set</u> of a given set S is

$$\mathcal{P}(S) = \{\text{all subsets of } S\}$$

$$= \{X : X \subseteq S\}$$

note

- $\mathcal{P}(S)$ is a set of sets (can feel a bit weird)
- $\emptyset \in \mathcal{P}(S)$ (its still true that $\emptyset \subseteq \mathcal{P}(S)$)

$S = \{\text{☺}, 1, w\}$ in this example

$$\mathcal{P}(S) = \left\{ \begin{matrix} \emptyset, \{\text{☺}\}, \{1\}, \{w\} \\ \{\text{☺}, 1\}, \{\text{☺}, w\}, \{w, 1\}, S \end{matrix} \right\}$$

- $S \in \mathcal{P}(S)$

Summary

$$X \in \mathcal{P}(S) \longleftrightarrow X \subseteq S$$

Forming a power set results in a more complicated set. In particular,

$$\text{if } |S| = n, \quad \text{then } |\mathcal{P}(S)| = 2^n$$

power sets of infinite sets are even cooler / weirder!

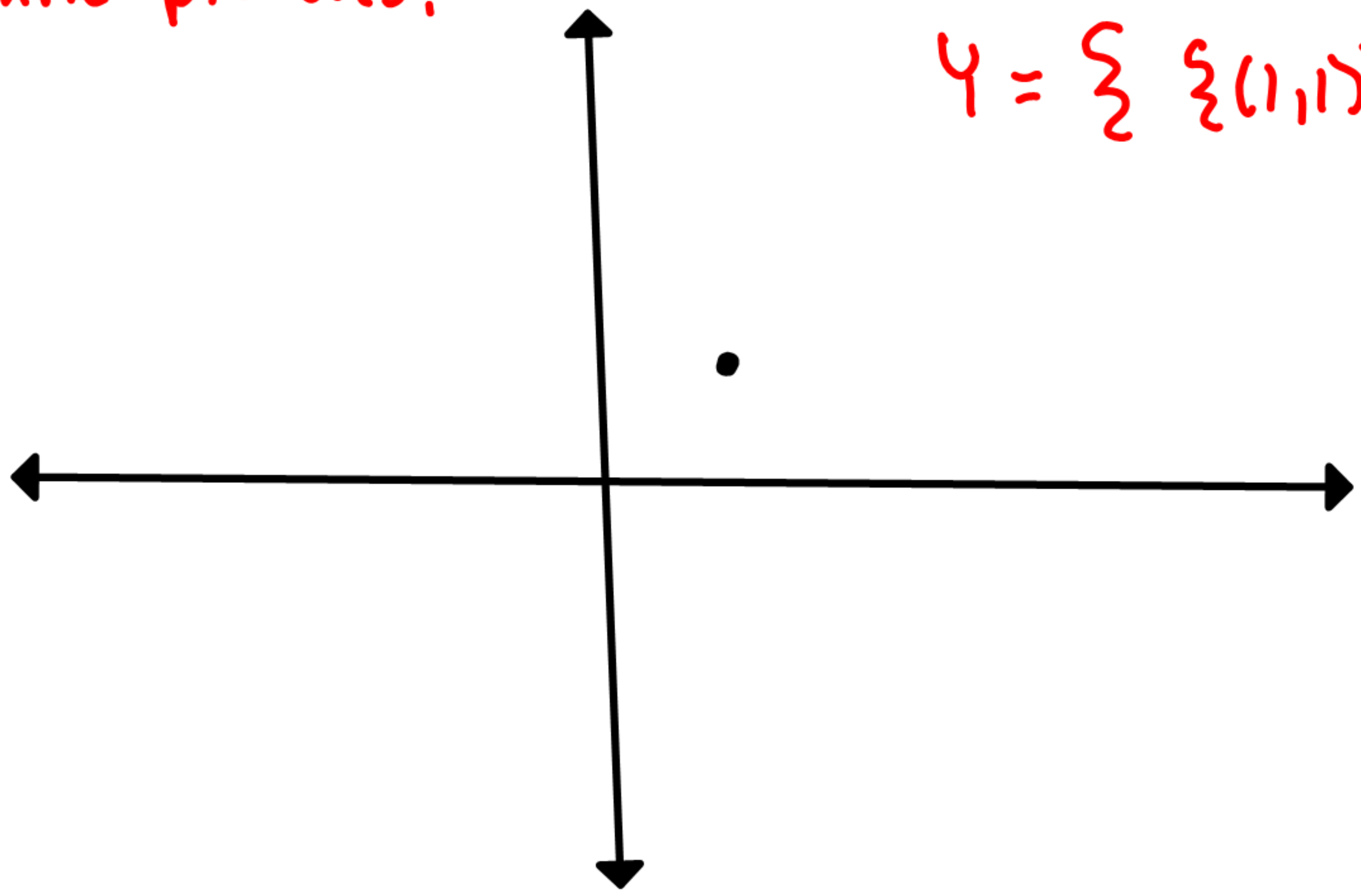
ex) $\mathcal{P}(\mathbb{N}) = \left\{ \begin{matrix} \emptyset, \mathbb{N}, \{1\}, \{2\}, \{3\}, \ldots \\ \{1,2\}, \{\ldots\} \\ \text{SO MANY!!} \end{matrix} \right\}$

fun ex $\mathcal{P}(\mathbb{R}^2)$

we can think of elements $X \in \mathcal{P}(\mathbb{R}^2)$ as

black + white pictures!

$Y = \{ \{(1,1)\} \} \in \mathcal{P}(\mathbb{R}^2)$



$\mathcal{P}(\mathbb{R}^2)$ contains every possible B+W photo... ever!

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you are familiar w/ lots of subsets of $\mathbb{R}^2$

ex $S = \{(t, t^3) \, ; \, t \in \mathbb{R}\} \subseteq \mathbb{R}^2$

$S \in \mathcal{P}(\mathbb{R}^2)$