4. Functions

When a math professor says the word "function," you probably think of familiar examples from previous classes, like $x^2, \sin(x)$ or e^x. These actually represent only one part of a function, what we might call "the rule part." Moreover, functions don't always involve formulas and explicit variables like these old friends.

A **function** requires three things: two sets and a "rule." Indeed, a **function** is just a rule between two sets, one that "associates" elements in the **domain** set with output elements in the **codomain** set. Our notation for functions goes as follows:

$$f : A \to B$$

and this expression is pronounced "f is a function from set A to set B." The set A is the **domain** of f, and the set B is the **codomain**. It is clarifying to talk of "elements in A being *sent* to elements in B by f." Indeed, the notation above tells us that *every* element $a \in A$ is "sent to" an element in B, and we usually denote this particular element as $f(a)$.

If the domain set A is inifinite, then we'll need to have some sort of abstract description of *how* $f(a)$ spits out an element in the codomain B, but if the set A is finite (and small), then we can simply state each and every input-output pairing. The familiar examples above use the first approach, relying on a formula that applies to any of the infinitely many elements in the domain $\mathbb{R}$.

Example 4.1. *Let's revisit those three familiar examples up top.*

(1) $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = x^2$ *is a function. The element* $2 \in \mathbb{R}$ *is sent to the element* $4 = 2^2 = f(2)$ *in the codomain.*

(2) $g : \mathbb{R} \to \mathbb{R}$ *given by* $g(x) = \sin(x)$ *is a function. The element* $\pi/2$ *in the domain is sent to the element* $1 = g(\pi/2) \in \mathbb{R}$.

(3) $h : \mathbb{R} \to \mathbb{R}$ *given by* $h(x) = e^x$ *is a function. The element* 0 *in the domain is sent to* $1 = e^0$ *in the codomain.*

It is worth noting that in Example 4.1 all the functions have the same domain and codomain, but for the example $g(x) = \sin(x)$ we *can* use a different codomain. Since $\sin(x)$ only outputs values between -1 and 1, we could create a technically different function $g : \mathbb{R} \to [-2, 2]$, for example. We could also use $g : \mathbb{R} \to [-1, 1]$. Officially speaking, all three of these are *different* functions even though they use the same rule and domain set.

You also likely recall that functions from Algebra, Pre-Calculus and Calculus can have restricted domains, for example

Example 4.2. *The function* $f(x) = 1/x$ *has as its largest possible domain* $\mathbb{R} - \{0\}$ *and it has* $\mathbb{R}$ *as its codomain. That is*

$$f : \Big(\mathbb{R} - \{0\}\Big) \to \mathbb{R} \text{ given by } f(x) = 1/x \text{ is a function.}$$

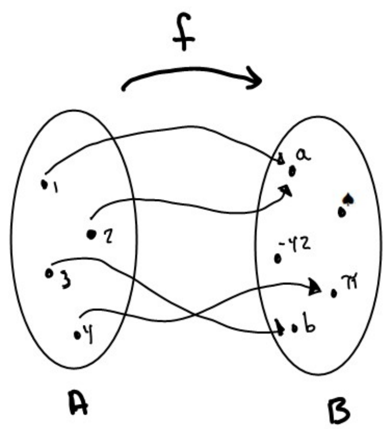
What "goes wrong" if we try to use $\mathbb{R}$ *as the domain for* $f(x) = 1/x$*?*

Of course, there are other ways to define functions, ones you may not have paid much attention to before but that are really worth investigating – mostly because they are simple *and* they clarify exactly how our definition of "function" works.

Example 4.3. *Consider the function*

$$f : \{1, 2, 3, 4\} \to \{a, \spadesuit, -42, \pi, b\}$$

given by the following diagram

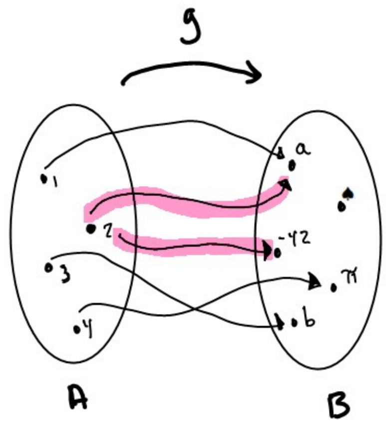


The domain of f is $A = \{1, 2, 3, 4\}$ and its codomain is $B = \{a, \spadesuit, -42, \pi, b\}$

The diagram depicts how f acts as a rule that assigns elements in A to elements in B. For instance, according to the diagram, $f(1) = a$ and $f(4) = \pi$. Note that $1, 2 \in A$ get sent to the same element in B, namely $f(1) = f(2) = a$. It is also worth noting that the element $-42 \in B$ is never "hit" or "used," and that is OK; not every element in your codomain needs to be used. However, every element in your domain needs to be used.

The next example shows how "unfollowable rules" *fail* to be functions.

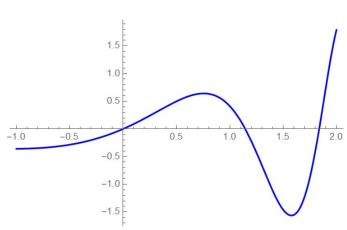
Example 4.4. *Consider the diagram below:*

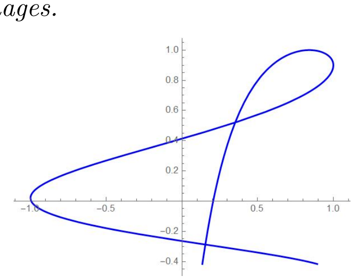


This diagram is presented and labelled in a way that suggestions $g : A \to B$ is a function, however ***this is not a function!*** *The issue is highlighted in red: g attempts to send the element $2 \in A$ to two different outputs in the codomain B. According to the diagram, $g(2) = a$ and $g(2) = -42$ – which output do we assign? We are being asked to follow to conflicting rules, and so there is no single rule.*

Way back when in previous math courses, non-examples of functions were usually presented in terms of **graphs**. Functions f with domains and codomains involving easy-to-draw sets like $\mathbb{R}$ can be visualized as ordered pairs $(x, f(x))$, and this picture is called **the graph of** f. The fact that f is a "rule" means its graph passes the familiar ***vertical line test***.

Example 4.5. *Consider the following images.*





The curve on the left is *the graph of a function $f : \mathbb{R} \to \mathbb{R}$ since this graph passes the vertical line test.*

The curve on the right, however, **is not** *the graph of a function $f : \mathbb{R} \to \mathbb{R}$ since it* **fails** *the vertical line test. If this were a graph, then the function f would produce two outputs at $x = 0$: $f(0) = 0.4$ and $f(0) = -0.25$. There are other places where multiple outputs would be produced, too (can you see them?).*

In your future math courses you will encounter functions that will take more time to understand and visualize, functions with domains and codomains like $\mathbb{R}^2$ or $\mathbb{R}^3$. Functions with impossible-to-visualize domains and codomains are also in your future, and we will use some here in Discrete-Land, too.

Example 4.6.

(1) *The function* $f : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{Z}$ *is given by the formula* $f(a,b,c) = a^2+b^2-c^2$. f *sends the following elements in the domain to elements in the codomain:*

$$f(1,2,0) = 1^2 + 2^2 - 0^2 = 5$$
$$f(3,4,5) = 3^2 + 4^2 - 5^2 = 0$$
$$f(2,4,10) = 2^2 + 4^2 - 10^2 = -80$$

Can you think of an element $(a,b,c) \in \mathbb{N}^3$ *that is sent to* -49*? Where does* f *send the element* $(5,12,13)$*?*

(2) *The function* $g : \mathbb{R} \to \mathbb{R}^2$ *is given by the formula* $g(t) = (\cos t, \sin t)$*. This function sends the following elements in the domain to elements in the codomain:*

$$0 \mapsto (\cos 0, \sin 0) = (1,0)$$
$$\pi/4 \mapsto (\cos(\pi/4), \sin(\pi/4)) = (\sqrt{2}/2, \sqrt{2}/2)$$
$$\pi \mapsto (\cos\pi, \sin\pi) = (-1,0)$$

Can you explain why all of the outputs of g*, when visualized as points in* $\mathbb{R}^2$*, lie on the standard unit circle?*

(3) *The function* $Det : \mathbb{R}^4 \to \mathbb{R}$ *is given by the formulas* $Det(a,b,c,d) = ad - bc$*. This function sends the following elements in the domain to elements in the codomain:*

$$Det(1,2,3,6) = 1\cdot 6 - 2\cdot 3 = 0$$
$$Det(1,0,0,1) = 1\cdot 1 - 0\cdot 0 = 1$$
$$Det(\sqrt{3},-2,-1,\pi) = \sqrt{3}\cdot\pi - (-2)\cdot(-1) = \pi\sqrt{3} + 2$$

For those of you who worked through a linear algebra class: can you figure out why we chose the name "Det" for this function?

4.1. Floor and Ceiling ($\lfloor\ \rfloor$ and $\lceil\ \rceil$). Before closing this section with a helpful summary, it'll be a good idea to emphasize two important functions that mathematicians and computer scientists use a lot – like ***a lot, a lot***. They have familiar domains and codomains, but they are named with special symbols rather than English-alphabet letters:

Example 4.7. *The* ***floor function****, $\lfloor\ \rfloor : \mathbb{R} \to \mathbb{Z}$, takes a real number as an input and rounds it down to the nearest integer. For example:*

$$\begin{aligned} \lfloor \pi \rfloor &= 3 \\ \lfloor -5.000000003 \rfloor &= -6 \\ \lfloor 1/2 \rfloor &= 0 \\ \lfloor 42 \rfloor &= 42 \end{aligned}$$

The ***ceiling function****, $\lceil\ \rceil : \mathbb{R} \to \mathbb{Z}$, takes a real number as an input and rounds it up to the nearest integer. For example:*

$$\begin{aligned} \lceil \pi \rceil &= 4 \\ \lceil -14.000007 \rceil &= -14 \\ \lceil 3/4 \rceil &= 1 \\ \lceil 1619 \rceil &= 1619 \end{aligned}$$

These are, indeed, functions so their graphs (which are interesting to draw) will pass "the vertical line test." However, these functions produce the same outputs for multiple inputs; make sure you understand why $\lceil 7 \rceil = \lceil 6.3 \rceil = \lceil 2\pi \rceil$, for instance.

Which inputs $x \in \mathbb{R}$ will result in $\lfloor x \rfloor = 3$? Are there any inputs $x \in \mathbb{R}$ that will result in $\lceil x \rceil = 5/2$? What about $\lfloor x \rfloor = \sqrt{2}$?

Closing Thoughts and Summaries. In this section you read and learned about *functions*. This idea is likely more general and easier than you were first taught.

Definition 2.1. *A* ***function*** *$f : A \to B$ is a rule that assigns to every element $a \in A$ an element $f(a) \in B$. The set A is called the* ***domain*** *of f, and the set B is called the* ***codomain****.*

The table below summarizes the main parts and finer points of functions.

Notation	Pronunciation	Commentary
$f : A \to B$	"f is a function from A to B"	all elements $a \in A$ are used not all elements $b \in B$ need to be "hit" f(single element) can't $=$ mult. elements
A	"domain"	consists of all inputs
B	"codomain"	contains all outputs (possibly more)
$f(a) = b$	"f sends a to b"	
$a \mapsto b$	"a is sent to b"	

Lastly, you also read about the floor and ceiling functions.

Definition 2.2. *The **floor function**, $\lfloor\ \rfloor : \mathbb{R} \to \mathbb{Z}$ is defined by the rule that takes a real number input and* **rounds it down** *to the nearest integer. The **ceiling function**, $\lceil\ \rceil : \mathbb{R} \to \mathbb{Z}$ is defined by the rule that takes a real number input and* **rounds it up** *to the nearest integer.*

As a closing comment, most mathematcians and "fancy math books" define functions in a slightly different way than we have, one that is equivalent to our version but that is phrased entirely in terms of sets. The notation changes, too, so that instead of saying $f(a) = b$ one would write $(a, b) \in f$ − looks weird, huh? If you'd like to challenge or further develop your set-and-logic-decoding skills, give the following alternative definition a try! Just keep in mind that the version of ***function*** you have read about in this section is the better one to use.

Definition 2.3. (Alternative Definition) *A **function**, $f : A \to B$, is a subset $f \subseteq A \times B$ that satisfies the following properties*

(1) $\forall\, a \in A,\ \exists,\ b \in B,\ (a, b) \in f$

(2) $((a, b) \in f\ \wedge\ (a, c) \in f)\ \Rightarrow\ b = c$