sentences are true or false until x has been replaced. Similarly, $(x+3=10) \vee (x<5)$ lacks a truth value as does $\left(x^2>9\right) \oplus (\sin(\theta)=1)$.

In this section you read about and worked through examples dealing with **truth tables, negation** $(\neg)$**, and** $(\wedge)$**, or** $(\vee)$, and **xor** $(\oplus)$. The definitions of these operations are summarized in the following truth tables.

P	$\neg P$
T	F
F	T

NOT Truth Table

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

AND Truth Table

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

OR Truth Table

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

XOR Truth Table

One of your goals is to become so comfortable with these operations that you can draw up and fill out their truth tables lightning-fast, and this is best done by practicing examples and sharing your thoughts with others. The table below summarizes their properties in straightforward English terms and may assist your efforts.

Operation	Name	Properties
$\neg$	NOT	swaps truth values
$\wedge$	AND	T only when both parts are T; F otherwise
$\vee$	OR	T when one or both parts are T; F otherwise
$\oplus$	XOR	T only when one part is T; F otherwise

3. Conditional Statements

"Your 'if' is the only peacemaker; much virtue in 'if'."

– W. Shakespeare (*As You Like it*, Act 5, Scene 4)

This section is all about ***conditional statements***, which are created using two given statements P and Q and a new connective "$\Rightarrow$." The sentence $P \Rightarrow Q$ is pronounced "P implies Q," and before we discuss its possible meaning or interpretation, let's examine its truth table; in other words, let's momentarily treat $\Rightarrow$ as an arbitrary way to create new statements (much as we did with $\star$ in Example 2.9).

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth Table for $\Rightarrow$

According to this truth table the following conditional statement is true:

$$(\text{Birds do not exist}) \Rightarrow (\text{ At least ten people live in Canada })$$

Make sure it is crystal clear why the weird-sounding statement above is true (examine the third row of the truth table).

Example 3.1. *Use the truth table above to determine the truth value of* $P \Rightarrow Q$ *for each of the given statements.*

(1) $P : 4 \cdot (1/2) = 2,\ Q : 3 + 5 = 8$

(2) $P : \pi^2 < 10,\ Q :$ *some triangles have only two sides*

(3) $P : \pi^2 > 10,\ Q :$ *some triangles have only two sides*

(4) $P : 3 = 2,\ Q : 8^2 = 64$

In a conditional statement $P \Rightarrow Q$, the first statement P is referred to as **the premise** or **the hypothesis** and the second one, Q, is **the conclusion**. In other words, every conditional statement is of the form

$$\text{Hypothesis} \Rightarrow \text{Conclusion}.$$

These terms match the intended meaning of a conditional statement as the sentence $P \Rightarrow Q$ is *supposed* to mean "Q is necessarily true when statement P is true." To phrase it somewhat differently $P \Rightarrow Q$ can be read as saying "the conclusion Q follows from the hypothesis P being true." In fact, there is a wide variety of ways to express this in the English language (see our section summary at the end), but they all intend to convey the same concept.

This is why conditional statements are associated with **if-then** sentences. Consider as an example the following sentence R:

$$R: \text{ If Liam eats his vegetables, then Liam gets ice cream.}$$

We can rewrite this as a conditional $P \Rightarrow Q$ in the following way:

$$R: \underbrace{\text{Liam eats his vegetables}}_{P} \Rightarrow \underbrace{\text{Liam gets ice cream}}_{Q}.$$

We understand that the conclusion, "Liam gets ice cream," follows from the hypothesis, "Liam eats his vegetables," being true. This is how, more generally, $P \Rightarrow Q$ models an English sentence "if P then Q."

It is helpful to think of a conditional or if-then as expressing a *promise* or *contractual obligation*. The sentences $P \Rightarrow Q$ and "If P then Q" assert a promise or guarantee that *if* P is true, *then* Q is also true. In our example Liam hears us *promising* that once he eats his vegetables he will be given ice cream. This leaves us with an interesting question: when could Liam accuse us of lying or breaking our promise?

In the event that Liam eats his vegetables and we then serve him ice cream, the promise is kept. No one was lied to. But in the event that Liam eats his vegetables and we do *not* give him ice cream, the promise was broken; we lied. This second secnario corresponds to the second row of our $P \Rightarrow Q$ truth table. In this row the hypothesis P is true, but the promised conclusion, Q, is false, and so the entire if-then is declared false.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth Table for $\Rightarrow$

Consider a situation where Liam does not eat his vegetables. If we decided to give him ice cream anyways would our promise be broken? If we decided to withhold ice cream would that mean we lied to Liam? The answer to both questions is "no!" Our guarantee that

$$\text{eating vegetables} \;\Rightarrow\; \text{getting ice cream}$$

has not been violated in either of these cases. That is, the promise is *not false*, and since we only ever have two options for statements, that means the promise is *true*. This is why the last two rows in our truth-table end with $P \Rightarrow Q$ being marked as true.

When a conditional $P \Rightarrow Q$ has a false hypothesis, the entire statement is referred to as **a vacuously true** sentence. The sentence is true, but only because we weren't able to show it was false; the promise "passed the test" because we couldn't even test it. Q might be true when P is false – in the vegetable and ice cream example, Liam might get ice cream for some *other* reason, one we don't know about. Q might be false when P is false. If the hypothesis of our if-then isn't true, then we are under no obligation to deliver a true or a false conclusion; the conclusion simply doesn't matter. An if-then with a false premise tells us nothing, really, and so deserves to be called "vacuous."

Example 3.2. *Which of the following statements are true? Which are vacuously true? (You may need to reword some to match the "If P then Q" form exactly.)*

(5) *You will be given a million dollars if pigs fly.*

(6) $(10 = 0) \;\Rightarrow$ *There are no birds.*

(7) $(10 > 0) \;\Rightarrow$ *There are no birds.*

(8) *If a circle has radius $r = 5$ then it encloses an area of 25π*

(9) $\sqrt{9} - 8 = -5 \;\Rightarrow\; \cot(\pi/4) = 1.$

Closing Thoughts and Section Summary

The sorts of conditional statements we have focused on in this section *can* resemble or capture the meaning of if-then English sentences, but many or most of them won't seem quite right when you think more carefully about them. Reading them as implications or promises will seem *off* or *weird*. Take, for example, statement (9) from Example 3.2:

$$\left(\sqrt{9} - 8 = -5\right) \;\Rightarrow\; \cot(\pi/4) = 1.$$

Because the premise and conclusion are both T, the logical definition of "$\Rightarrow$" tells us that this entire sentence is true. But when we read this using the words "if-then" and attach our usual understanding of those terms it seems odd:

$$\text{If } \left(\sqrt{9} - 8 = -5\right), \text{ then } \cot(\pi/4) = 1.$$

English if-then sentences usually rely on some *connection* between their hypotheses and conclusions, as in our Liam-vegetable-ice cream example. Or as in an example like "If the plants are not watered, then they will not grow." The conditional above from Ex. 3.2 is very strange by comparison, as it seems to claim that taking the difference between two particular numbers *causes* a fact about cotangent to be true, but what does the subtraction $\sqrt{9} - 8$ have to do with cotangent?

From a purely formal or logical point of view, we can take *any two, completely unrelated* statements A and B and create the new statement "if A, then B." It is OK that this new sentence is sometimes weird or even meaningless from an English-language point of view; in those instances we are simply using the purely logical, truth-table interpretation.

Open Sentences and $\Rightarrow$. Another point to preview here is this: *we actually need to use* ***open sentences*** *to capture the full meaning of most if-then sentences in English.* The true statement

If a number is a multiple of 4, then it is even

expresses a fact we all (hopefully) understand and accept, but this sentence contains *variables.* Can you spot them? In this case there is only one, and its contained in the expression "a number." No numerical value is specified, and so this part of the sentence seems like its open or waiting for some substitution to take place. All of this can be made more explicit by rewriting the sentence in various ways:

$$\text{If } x \text{ is a multiple of 4, then } x \text{ is even.}$$

$$(x \text{ is a multiple of 4}) \Rightarrow (x \text{ is even})$$

Our (true) conditional statement appears to connect two opens sentences in the following way:

$$P(x) \Rightarrow Q(x)$$

and we might be a bit suspicious or nervous of this format. In the previous section we mentioned how connecting and negating open sentences results in new *open* sentences, ones that are still neither true nor false. However, it seems that connecting open sentences with our implication-connective results in statements, ones with truth values! Indeed, most mathematicians use $\Rightarrow$ in precisely this way

$$\boxed{P(x) \Rightarrow Q(x) \text{ is not an open sentence; its a statement!}}$$

Mathematicians read the above as saying this: "whenever you plug in a value for x that makes $P(x)$ true, that some value will make $Q(x)$ true."

Kind of weird, huh? Well, don't worry too much about this right now, as we will come back to it soon in Section 6. For the moment you'll want to focus on understanding how conditional statements are built using $\Rightarrow$ according to its truth table, and you'll also want to understand when basic if-then sentences in English are true.

In this section you read about conditional statements $P \Rightarrow Q$ and how they attempt to resemble if-then sentences. You also learned about **vacuously true** statements. Your goal is to be able to reproduce the $\Rightarrow$ truth table easily *and* use your understanding of it to form new if-then statements and determine when they are true or false.

Definition 1.3. *A **conditional statement** is a proposition of the form $P \Rightarrow Q$. The statement P is referred to as the **premise** or **hypothesis**. The statement Q is referred to as the **conclusion**.*

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth Table for $\Rightarrow$

Notation	English Phrasings
$P \Rightarrow Q$	P implies Q If P, then Q Whenever P, then also Q Q is guaranteed by P P is a sufficient condition for Q Q is a necessary condition for P Q whenever P

Definition 1.4. *A **vacuously true** statement is a conditional statement whose hypothesis is false.*

4. Bi-conditional Statements

It is also important to understand that the two conditionals $P \Rightarrow Q$ and $Q \Rightarrow P$ are *different*, often times having *different* truth values. Even though they both use the same statements P and Q, the order in which these parts are connected matters. For instance, the conditional statement

$$\text{(At least ten people live in Canada)} \Rightarrow \text{(Birds do not exist)}$$

is false while the seemingly-related statement

$$\text{(Birds do not exist)} \Rightarrow \text{(At least ten people live in Canada)}$$

is (vacuously) true.

Indeed, the conditional $Q \Rightarrow P$ *should* be different from $P \Rightarrow Q$ as it swaps around its hypothesis and conclusion. It is so different, in fact, that it gets a special name: $Q \Rightarrow P$ is called the **converse** of $P \Rightarrow Q$. The following examples should help you check your understanding of converses, especially when you make sure you can determine the truth value of each statement.