Math 3336
Homework Assignment 5

Instructions

- Record your answers to the following 10 questions. Show your work when a question requires you to do so.
- Scan your work and save the file as a .pdf (make sure your work and answers are legible)
- Upload your scanned work to CASA CourseWare using the "Assignments" tab. (Click this link for instructions on how to do this).
- Homework submitted after 11:59pm on the indicated due date will be assigned a grade of 0.
- Also, **DON'T FORGET THAT** $0 \in \mathbb{N}$.
- I understand that if any of the questions from this assignment (or future ones) are shared in ways that violate our Academic Honesty Policy, then the syllabus will change. Specificlaly, Homework and Quizzes will be worth zero points.

  Name:

  Signature:

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1. Prime numbers are of particular interest to Mathematicians and Computer Scientists. As the "Mozart of Math" Dr. Terry Tao noted, prime numbers are like the atoms of mathematics, and that's one reason mathematicians are interested in **twin primes**. In particular, we are interested in knowing whether or not there are *infinitely* many twin primes. The famous **Twin Prime Conjecture** guesses that there are:

Twin Prime Conjecture: There are infinitely many twin primes.

So far no one has come up with a proof for this Conjecture (nor has anyone come up with a proof for its negation). If one wanted to prove this using a *Proof by Contradiction*, which of the following could be used as the first line?

(a) Suppose there are infinitely many twin primes.

(b) Suppose p_1 and p_2 are twin primes (we want to show $p_1 = p_2 = \infty$).

(c) Suppose there are finitely many twin primes.

(d) Suppose there are finitely many primes.

(e) Therefore there are infinitely many twin primes.

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2. There is another conjecture about prime numbers, one called "Brocard's Conjecture." To understand it, we first consider the *sequence* of prime numbers p_n (so that $p_1 = 2, p_2 = 3, \dots$).

Brocard's Conjecture : If $n > 1$, then there are at least four prime numbers between $(p_n)^2$ and $(p_{n+1})^2$.

As the word "conjecture" tells us, no one has yet proven nor dis-proven this claim. If one wanted to prove this using a *Contrapositive Proof*, which of the following could be used as the first line?

(a) Suppose $n > 1$.

(b) Suppose that there are three or fewer primes between $(p_n)^2$ and $(p_{n+1})^2$.

(c) Suppose that there are at least four prime numbers between $(p_n)^2$ and $(p_{n+1})^2$.

(d) Suppose $n > 1$ and that there are three or fewer primes between $(p_n)^2$ and $(p_{n+1})^2$.

(e) Therefore there are at least four prime numbers between $(p_n)^2$ and $(p_{n+1})^2$.

3. Use the definitions of **even** and **odd** to write a direct proof of the following proposition:

Proposition. The sum of two odd integers is an even integer.

Proof (Direct).

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4. Consider the following proposition:

Proposition. If x^2 is an even integer, then x is even.

A proof of this proposition is presented below. Identify the type of proof used.

Proof. Suppose x is an odd integer. (We will show that x^2 is an odd integer, too.)
By definition of odd this means that $x = 2m + 1$ for some $m \in \mathbb{Z}$. It follows that
$x^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$, which has the form of an odd number.
Therefore x^2 is odd. $\square$

(a) This is a direct proof.

(b) This is a contrapositive proof.

(c) This is a proof by contradiction.

(d) This is a proof of an existential claim.

(e) This isn't even a proof; it is wrong.

5. Suppose we know that integers x and y are each multiples of the integer a. Do you think that the integer $5x^2 + y$ is also a multiple of a? Complete part (a) to answer this first question.

Part (a). Test at least two examples, each time choosing three integers $x, y, a \in \mathbb{Z}$ that satisfy the hypotheses. Is the conclusion true in these cases?

Part (b). Write a direct proof that the following proposition is true:

Proposition. If $x, y \in \mathbb{Z}$ are multiples of $a \in \mathbb{Z}$, then $5x^2 + y$ is a multiple of a.

6. Consider the following proposition:

Proposition. Let A and B be two sets. If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ then $A \subseteq B$.

A proof of this proposition is presented below. Identify the type of proof used.

Proof. Suppose $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. We want to show that $A \subseteq B$, and to do this let $a \in A$. (We will show that $a \in B$.) It follows that $\{a\} \subseteq A$, and by definition of Power Set we have $\{a\} \in \mathcal{P}(A)$. By assumption $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, and so $\{a\} \in \mathcal{P}(B)$. By definition of Power Set it follows that $\{a\} \subseteq B$ and so $a \in B$. $\square$

(a) This is a direct proof.

(b) This is a contrapositive proof.

(c) This is a proof by contradiction.

(d) This is a proof of an existential claim.

(e) This isn't even a proof; it is wrong.

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7. A proposition and parts of a proof by contradiction are shown below.

Proposition. Let $a, b \in \mathbb{R}$. If a is rational and ab is irrational, then b is irrational.

Proof. (By Contradiction) .
Suppose a is rational, ab is irrational but that b is ____________________.
By definition of rational, this means that there exist integers $m, n \in \mathbb{Z}$ (with $n \neq 0$) where $a = m/n$.
Similarly, there exist integers $u, v \in \mathbb{Z}$ (with $v \neq 0$) where ___ $= u/v$.
It now follows

$$ab \text{ is irrational (by assumption)}$$
$$\text{and } ab = \left(\frac{m}{n}\right)\left(\frac{u}{v}\right) = \frac{mu}{nv} \in \underline{\quad\quad}.$$

Therefore ab is both __________ and rational, and this is a _________________ $\Rightarrow\Leftarrow$

Fill in the empty spaces with appropriate symbols, words or phrases. (No work need be included for this part.)

Also: test this Proposition on at least one example and include this work below.

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8. Consider the following proposition:

Proposition. $\exists x, y \in \mathbb{N},\ x \neq y\ \wedge\ x^y = y^x$

Of the following options provided, select the one that could be used as the main part in a proof of this proposition.

(a) Consider the natural number $x = 1$.

(b) Consider the natural numbers $x = 1$ and $y = 1$.

(c) Consider the natural numbers $x = 3$ and $y = 9$.

(d) Consider the natural numbers $x = 2$ and $y = 4$.

(e) Consider the natural numbers $x = 0$ and $y = 1$.

In the space below provide some light computations that explain why your choice above works.

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9. Consider the following proposition and the proof that follows:

Proposition. If the recursively defined sequence $a_n = -2 + 3a_{n-1}$ has initial condition $a_0 = 1$, then the sequence is *constant*.

Proof. Let $\{a_n\}$ be defined as in the proposition and assume $a_0 = 1$, but also assume a_n is not a constant sequence. This means there must exist a natural number, $m \in \mathbb{N}$, with $a_m \neq 1$.

If $a_m > 1$ then, using the recurrence relation, we find that $1 < a_m = -2 + 3a_{m-1}$. Adding 2 to this inequality and then dividing by 3 yields the inequality $1 < a_{m-1}$. We can repeat this process a total of m times to find that $1 < a_0$, contradicting our initial condition.

If $a_m < 1$ then using the recurrence relation, we find that $1 > a_m = -2 + 3a_{m-1}$ which can be rewritten as $1 > a_{m-1}$. Repeating this process a total of m times we find that $1 > a_0$, contradicting our initial condition. $\Rightarrow\Leftarrow$

What type of proof was used?

(a) This is a direct proof.

(b) This is a contrapositive proof.

(c) This is a proof by contradiction.

(d) This is a proof of an existential claim.

(e) This isn't even a proof; it is wrong.

10. What did you learn (or re-learn) by working through this assignment? Which questions, if any, were particularly helpful? Which ones, if any, were unhelpful?