# Non-linear Relationships & Potential Problems

## Section 3.3

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# Outline

# Two Important Assumptions

1. The **additive** assumptions means that the effect of changes in a predictor $X_j$ on the response $Y$ is independent of the values of the other predictors.

2. The **linear** assumptions means that the change in the response $Y$ due to a one-unit change in $X_j$ is constants, regardless of the value of $X_J$.

# Non-Linear Relationships

- The linear regression model assumes a linear relationship between the response and the predictors.
- The true relationship between the response and the predictors my be non-linear.
- The *polynomial regression* is a very simple way to directly extend the linear model to accommodate non-linear relationships.

# Polynomial Regression

- **Polynomial regression** is a form of regression analysis in which the relationship between the predictor $x$ and the response $y$ is modeled as an $n^{th}$ degree polynomial in $x$.
- Model:

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \beta_3 x_i^3 + \cdots + \beta_m x_i^m + \epsilon_i$$

  for $i = 1, 2, \ldots, n$.
- We need to keep $m < n$
- In `R` we use `lm(y~poly(x,m))`.
- For more information see: `https://datascienceplus.com/fitting-polynomial-regression-r/`

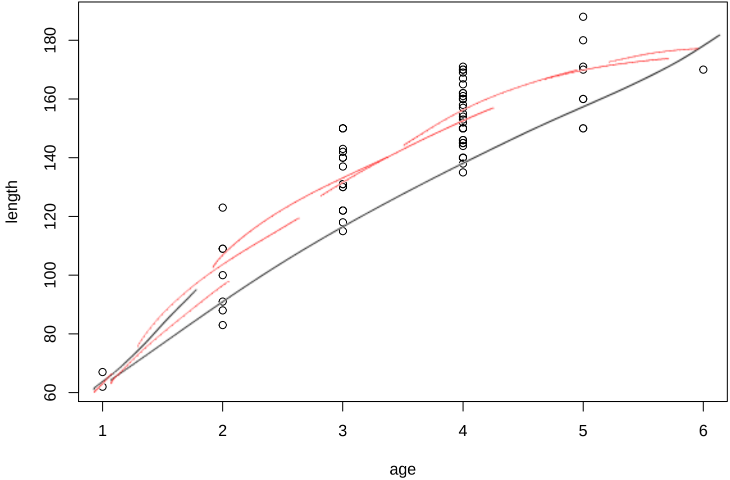
# Non-Linear Relationship Example

In 1981, n = 78 bluegills were randomly sampled from Lake Mary in Minnesota. The researchers (Cook and Weisberg, 1999) measured and recorded the following data (`https://onlinecourses.science.psu.edu/stat501/sites/onlinecourses.science.psu.edu.stat501/files/data/bluegills/index.txt`):

- Response (y): length (in mm) of the fish
- Potential predictor (x1): age (in years) of the fish

The researchers were primarily interested in learning how the length of a bluegill fish is related to it age.

# Scatterplot



# Linear Summary

```
> summary(fish.lm)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   62.649      5.755   10.89   <2e-16 ***
age           22.312      1.537   14.51   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 12.51 on 76 degrees of freedom
Multiple R-squared:  0.7349,Adjusted R-squared:  0.7314
F-statistic: 210.7 on 1 and 76 DF,  p-value: < 2.2e-16
```

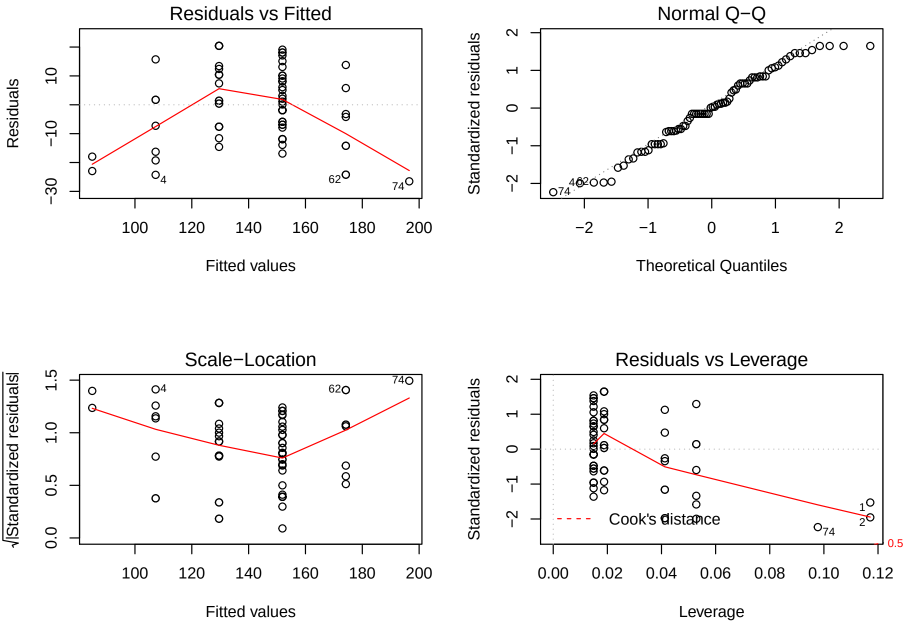
Equation: $\widehat{length} = 62.649 + 22.312 * age$

Is this a "good" equation to determine the relationship between length & age? Yes

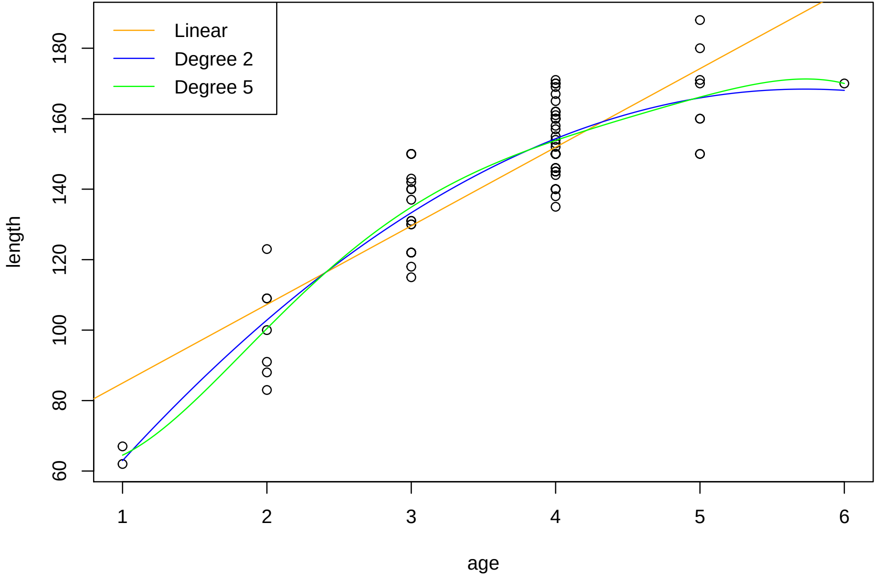
$H_0: \beta_1 = 0$ vs $H_A: \beta_1 \neq 0$ P-value $\approx 0$

$R^2 = 0.73$

Linear ?



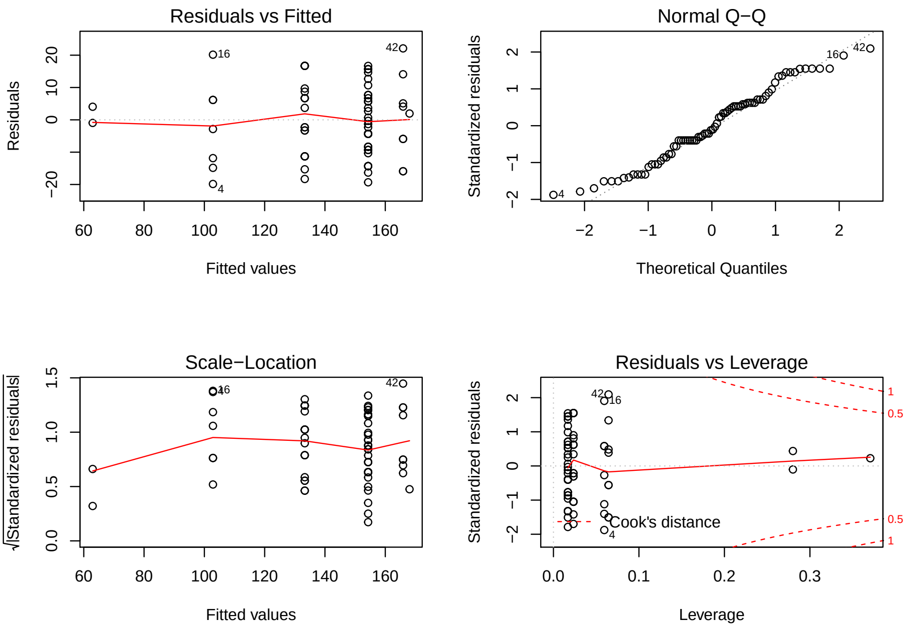
Residuals vs Fitted
Residuals
Fitted values
4
62
74
Normal Q−Q
Standardized residuals
Theoretical Quantiles
74
4
62
Scale−Location
√|Standardized residuals|
Fitted values
4
62
74
Residuals vs Leverage
Standardized residuals
Leverage
Cook's distance
1
2
74
0.5


Linear
Degree 2
Degree 5
180
160
140
120
100
80
60
length
1
2
3
4
5
6
age

# Polynomial Regression R

```
> fish.lm2 = lm(length~poly(age,2),data = index)
> summary(fish.lm2)

Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)      143.603      1.235 116.290  < 2e-16 ***
poly(age, 2)1    181.565     10.906  16.648  < 2e-16 ***
poly(age, 2)2    -54.517     10.906  -4.999 3.67e-06 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 10.91 on 75 degrees of freedom
Multiple R-squared:  0.8011,Adjusted R-squared:  0.7958
F-statistic: 151.1 on 2 and 75 DF,  p-value: < 2.2e-16
```

$$\widehat{\text{length}} = 143.603 + 181.565 * \text{age} - 54.517 * \text{age}^2$$



Residuals vs Fitted
Residuals
Fitted values
Normal Q−Q
Standardized residuals
Theoretical Quantiles
Scale−Location
√|Standardized residuals|
Fitted values
Residuals vs Leverage
Standardized residuals
Leverage
Cook's distance

# Cubic Regression

```
> #Cubic Polynomial
> fish.lm3 = lm(length~poly(age,3),data = index)
> summary(fish.lm3)

Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)      143.603      1.243 115.544  < 2e-16 ***
poly(age, 3)1    181.565     10.976  16.541  < 2e-16 ***
poly(age, 3)2    -54.517     10.976  -4.967 4.25e-06 ***
poly(age, 3)3      2.234     10.976   0.203    0.839     Not Significant
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 10.98 on 74 degrees of freedom
Multiple R-squared:  0.8012,Adjusted R-squared:  0.7932
F-statistic: 99.44 on 3 and 74 DF,  p-value: < 2.2e-16
```

$$\widehat{\text{length}} = 143.603 + 181.565 * \text{age} - 54.517 * \text{age}^2 + 2.234 * \text{age}^3$$

# Lab Questions

We will use the Boston data for these questions. Make sure you load the `MASS` library.

```
library(MASS)
```

1. Perform a linear regression on medv (response variable) onto lstat (predictor). What is the adjusted $R^2$ for this model?

   a) 0.5441
   b) 0.5432
   c) 0.0002
   d) 0.95

2. Draw a scatterplot between medv (y) and lstat (x). Does it appear linear?

   a) Yes
   b) No

In R type in

```
par(mfrow = c(2,2))
plot(fit.lm)
```

3. Do we see a pattern in the residuals?
   a) Yes
   b) No

   If there is a pattern then the linear model may not be the best model.

4. Type in `R` and run the summary: $\beta_0 + \beta_1 x + \beta_2 x^2$
   `fit.lm2 = lm(medv ~ poly(lstat,2),data = Boston)`.
   What is the adjusted $R^2$ for this model?

   a) 0.0002
   b) 0.055
   c) 0.6407
   d) 0.6393

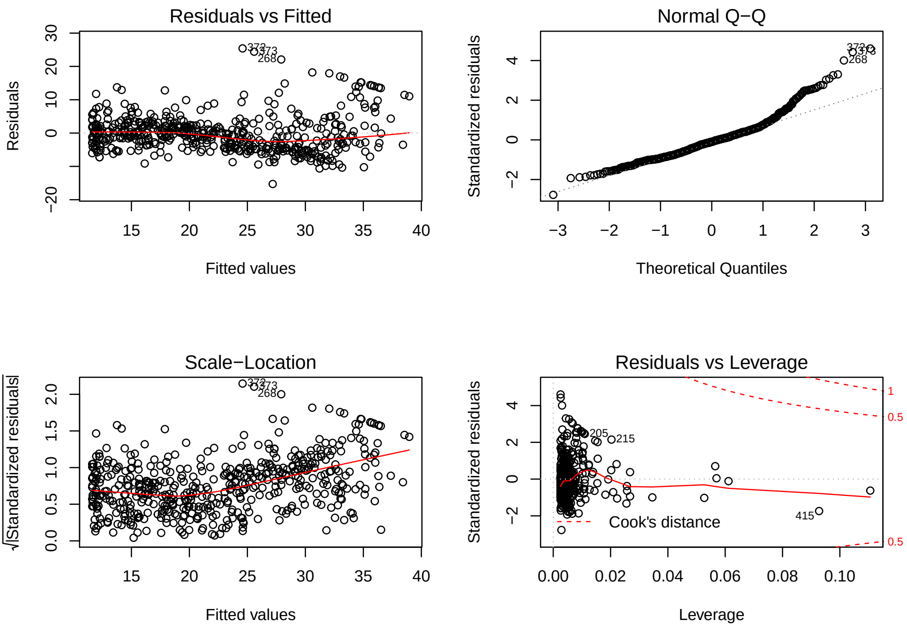
5. Run a cubic model. What is the adjusted $R^2$?

a) 0.6558
b) 0.6578
c) 0.0002
d) 0.054

lm( medv ~ poly(lstat, 3))

$\beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3$

```
Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)      22.5328     0.2399  93.937  < 2e-16 ***
poly(lstat, 3)1 -152.4595    5.3958 -28.255  < 2e-16 ***
poly(lstat, 3)2   64.2272    5.3958  11.903  < 2e-16 ***
poly(lstat, 3)3  -27.0511    5.3958  -5.013 7.43e-07 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 5.396 on 502 degrees of freedom
Multiple R-squared:  0.6578, Adjusted R-squared:  0.6558
F-statistic: 321.7 on 3 and 502 DF,  p-value: < 2.2e-16
```


Residuals vs Fitted
Residuals
Fitted values
372
373
268
Normal Q−Q
Standardized residuals
Theoretical Quantiles
Scale−Location
√|Standardized residuals|
Residuals vs Leverage
Cook's distance
Leverage
205
215
415

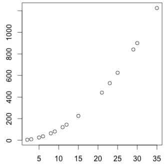
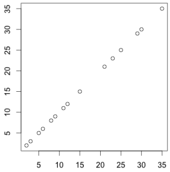
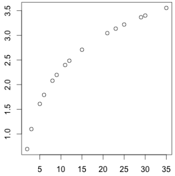
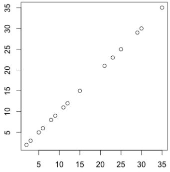
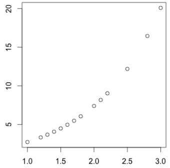
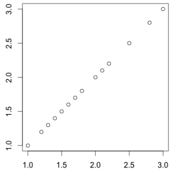
# Some Warnings About Polynomial Models

- The following list is from `https://online.stat.psu.edu/stat462/node/158/`.
- The fitted model is more reliable when it is built on a larger sample size.
- Consider how large the size of the predictors(s) will be when incorporating higher degree terms as this may cause numerical overflow for the statistical software being used.
- Do not go strictly by low $p$-values to incorporate a higher degree term, but rather just uses these to support your model only if the resulting residual plots looks reasonable.
- As a standard practice if you have an $n^{th}$ degree polynomial, always include the each $X^j$ such that $j < n$.

# Potential Problems in Linear Regression

1. Non-linearity of the response-predictor relationships.
2. Correlation of error terms.
3. Non-constant variance of error terms.
4. Outliers.
5. High-leverage points.
6. Collinearity.

We have talked already about how to overcome some of these problems.

| Original Scatter-plot | Type of plot and Transformation | Modified Scatter-plot |
|---|---|---|
|   | ← looks like $y = x^2$ transform by changing $(x, y)$ into $(x, \sqrt{y})$ → |   |
|   | ← looks like $y = \log x$ transform by changing $(x, y)$ into $(x, e^y)$ → |   |
|   | ← looks like $y = e^x$ transform by changing $(x, y)$ into $(x, \log y)$ → |   |

# Correlation of Error Terms

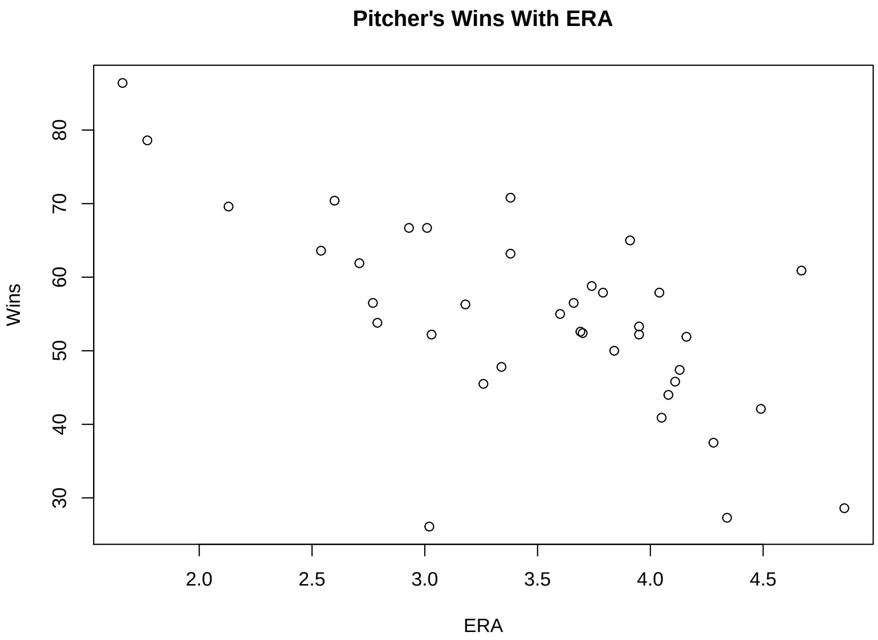
- Assumption of the linear regression model is that the error terms, $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are uncorrelated.
- For example if $\epsilon_i$ is positive provides little or no information about the sign of $\epsilon_{i+1}$.
- If there is correlation among the error terms then the estimated standard errors will tend to underestimate the true standard errors. This results in narrower confidence and prediction intervals.
- Time series data is an example of correlation among the error terms.
- Residual plot is best way to tell if there is correlation.

# Non-constant Variance of Error Terms

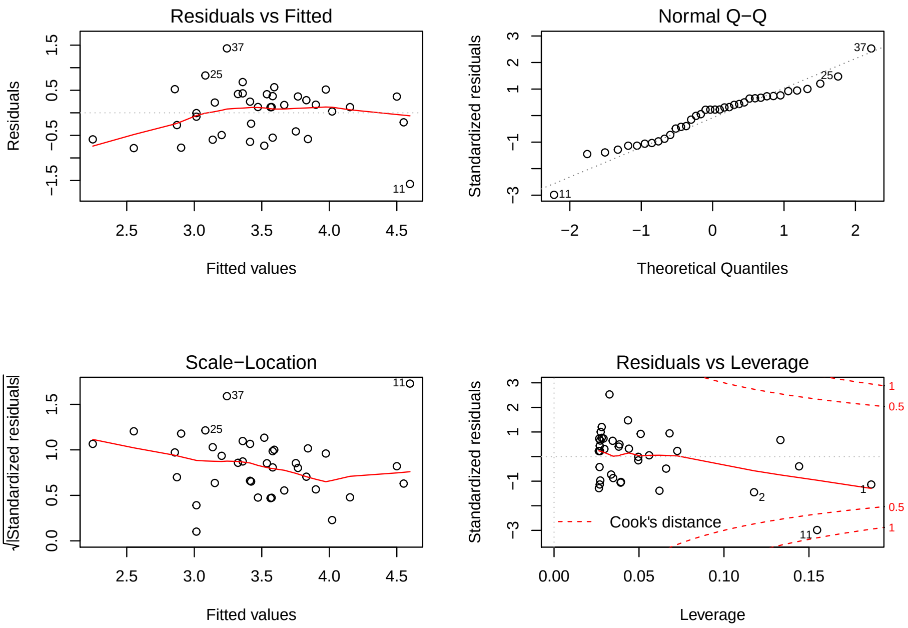
- **Heterscedasticity** is where the variances of the error terms increase with the value of the response. This will appear as a *funnel shape* in the residual plot.
- Possible solution is to transform the $Y$ using $log(Y)$ or $\sqrt{Y}$.

# Outliers

- An **outlier** is a point for which $y_i$ is far from the value predicted by the model.
- Outliers can have an effect on the estimated regression parameters, RSE and $R^2$.
- Scatterplot and residual plot would be the best to detect outliers.

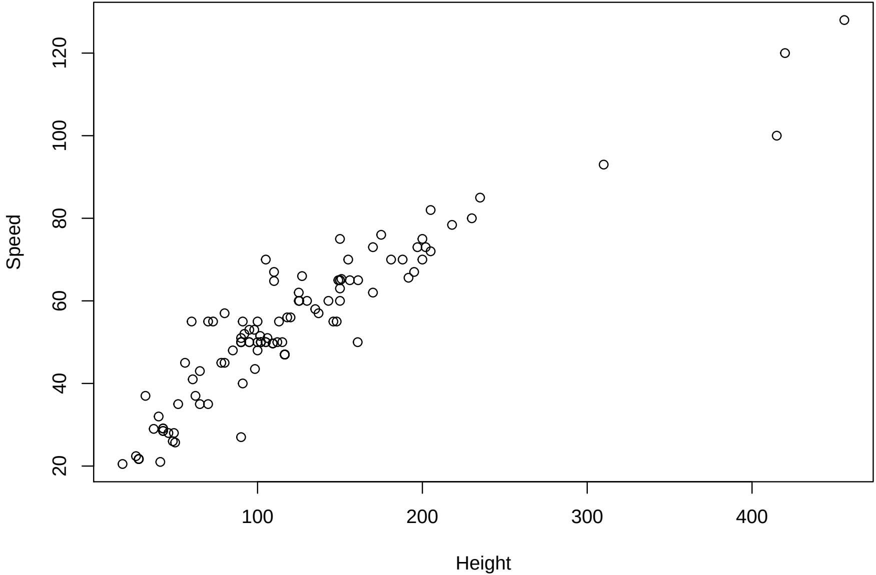

Pitcher's Wins With ERA
Wins
80
70
60
50
40
30
2.0
2.5
3.0
3.5
4.0
4.5
ERA

```
> era.lm = lm(ERA ~ Wins,data = Era)
> summary(era.lm)

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  5.614231   0.405423  13.848 5.43e-16 ***
Wins        -0.038957   0.007229  -5.389 4.56e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.5744 on 36 degrees of freedom
Multiple R-squared:  0.4465,Adjusted R-squared:  0.4311
F-statistic: 29.04 on 1 and 36 DF,  p-value: 4.557e-06
```

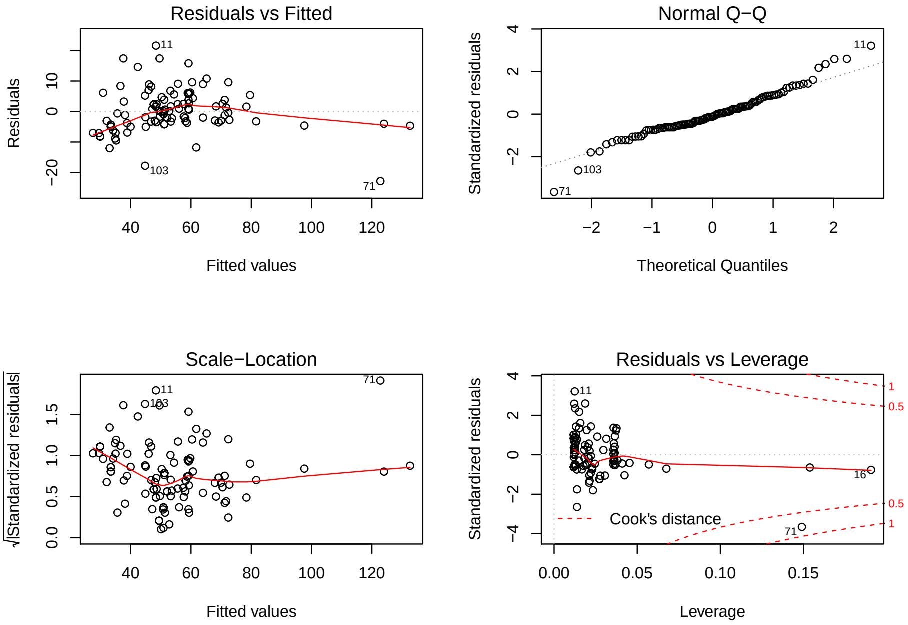

Residuals vs Fitted
Residuals
Fitted values
37
25
11
Normal Q−Q
Standardized residuals
Theoretical Quantiles
37
25
11
Scale−Location
√|Standardized residuals|
Fitted values
11
37
25
Residuals vs Leverage
Standardized residuals
Leverage
Cook's distance
1
0.5
0.5
1
2
1
11

# High Leverage Points

- **High leverage points** have an unusual value for $X$.
- High leverage observations tend to have a sizable impact on the estimated regression line.
- Can be determined by scatterplots for a simple linear regression.
- In order to quantify an observation's leverage we compute the **leverage statistic**.
- In `R` we can use the Residuals vs Leverage to see if there are high leverage observations.



Speed
120
100
80
60
40
20
100
200
300
400
Height


Residuals vs Fitted
Residuals
Fitted values
11
103
71
Normal Q−Q
Standardized residuals
Theoretical Quantiles
Scale−Location
√|Standardized residuals|
Residuals vs Leverage
Leverage
Cook's distance
16
0.5
1

# Collinearity

- **Collinearity** refers to the situation in which two or more predictor variable are closely related to one another.
- In regression this will cause difficulty to separate out the individual effects of collinear variables on the response.
- This reduces the accuracy of the estimates of the regression coefficients, it causes the standard error for $\hat{\beta}_j$ to grow.
- The **power** of the hypothesis test for $H_0 : \beta_i = 0$ - probability of correctly detecting a nonzero coefficient - is reduced by collinearity.

# Detecting Multiconllinearity

- Check the correlation matrix. In R: *cor*().
- The variance inflation factor (VIF). The VIF is the ratio of the variance of $\hat{\beta}_j$ when fitting the full model divided by the variance of $\hat{\beta}_j$ if fit on its own.
  - The smallest possible value for VIF is 1, this means there is no correlation.
  - If a VIF exceeds 4, further investigation is needed.
  - If VIF is more than 10, then there is a sign of serious multicolinearity and requires correcting.
  - The VIF for each variable can be computed using the formula:

$$VIF(\hat{\beta}_j) = \frac{1}{1 - R^2_{X_j|X_{-j}}},$$

Where $R^2_{X_j|X_{-j}}$ is the $R^2$ from a regression of $X_j$ onto all of the other predictors.

# Example of VIF

```
> library(car)
> stock3.lm = lm(Stock_Index_Price ~ Interest_Rate+Unemployment_Rate+Year,
                 data = stock_price)
> vif(stock3.lm)
Interest_Rate Unemployment_Rate              Year
     8.258442          7.890112          5.110528

> summary(lm(Interest_Rate ~ Unemployment_Rate + Year, data = stock_price))$r.squared
[1] 0.8789118
```

$$VIF(\text{Interest\_Rate}) = \frac{1}{1 - 0.8789118} = 8.258442$$