Supplementary Materials for: [On the Problem Characteristics of Multi-objective Pseudo Boolean Functions in Runtime Analysis]

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1. Introduction

This supplementary document provides additional material for our main submission titled *"On the Problem Characteristics of Multi-objective Pseudo-Boolean Functions in Runtime Analysis"* to the Conference on Foundations of Genetic Algorithms 2025 (FOGA 25). We first present a detailed description of the mix-and-match multi-objective pseudo-Boolean functions involving the OneMax function. This is followed by formal proofs concerning the ratio of Pareto-optimal solutions for the OneJump-ZeroJump and OneJump-ZeroRoyalRoad function classes.

2. Mix-and-Match Multi-Objective Boolean Functions with OneMax

In the main paper, we only present the mixes of LeadingOnes, Jump and RoyalRoad as mixing OneMax with other functions may be trivial, as it simplifies the problem. Here, we present the mixes involving the OneMax function.

2.1. OneMax-TrailingZeroes (OMTZ)

OMTZ combines the OneMax function with the TrailingZeroes function, thus a simultaneous maximisation of the number of ones in the bit-string and the number of consecutive zeroes (from right to left). Formally, OMTZ can be formulated as follows.

Definition 2.1. (*OneMax-TrailingZeroes*). Let $x \in \{0,1\}^n$ be a bit-string of length n. The OMTZ problem is defined as:

$$f_1(x) = \sum_{i=1}^{n} x_i, \quad \text{and} \quad f_2(x) = \sum_{i=1}^{n} \prod_{j=i}^{n} (1 - x_j) \tag{1}$$

As can be seen in Table **??**, OMTZ shares some characteristics with LeadingOnes-TrailingZeroes (LOTZ), such as a linear Pareto front, no either disjoint optimal solutions or local optima, and a low ratio of Pareto optimal solutions. The primary difference between them is that OMTZ is non-symmetric with respect to its objectives, which is commonly seen in real-world problems.

The OneMax objective in OMTZ is easier to improve on than the TrailingZeroes objective since improving on OneMax involves flipping any bit from zero to one, whereas improving on TrailingZeroes involves flipping a specific one bit to zero. This difference creates an objective imbalance, a common feature in many test suites (e.g., ZDT [1]). Figure 1 illustrates the 8-bit OMTZ. As seen in Figure 1(b), more solutions concentrate in a region with fairly good value on the objective OneMax but poor value on the objective TrailingZeroes.

2.2. OneMax-ZeroJump (OMZJ)

OMZJ combines the OneMax function with the ZeroJump function, aiming to simultaneously maximise the number of ones in the bit-string while crossing a valley in the search space determined by the number of zeroes. Like OneJump-ZeroJump (OJZJ), the size of this valley is controlled by the *jump parameter* k, with larger values of k making it more difficult to reach the boundary solution for the ZeroJump objective (i.e., (0^n)). Formally, OMZJ can be formulated as follows.

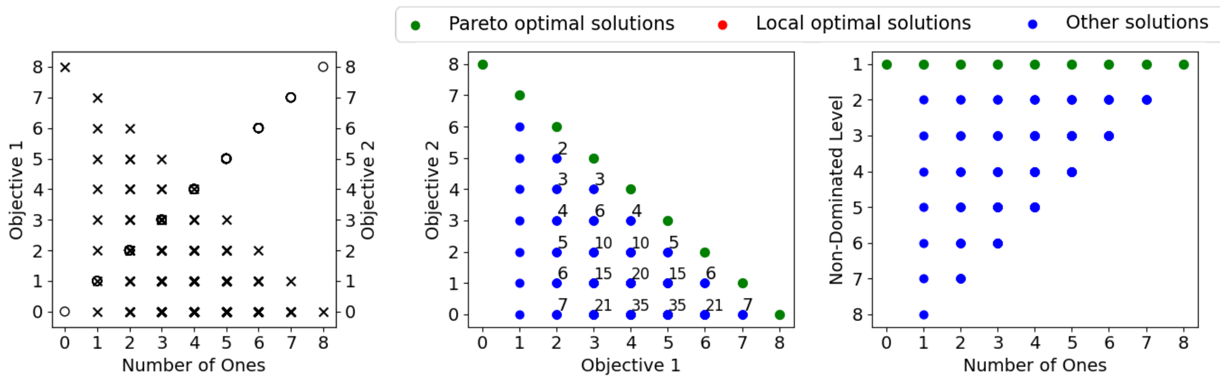
Definition 2.2. (*OneMax-ZeroJump*). Let $x \in \{0,1\}^n$ be a bit-string of length n and let k be a fixed jump parameter with $1 < k < \frac{n}{2}$. The OneMax-ZeroJump problem is defined as:

$$\begin{aligned} f_1(x) &= \sum_{i=1}^{n} x_i, \\ f_2(x) &= \begin{cases} k + |x|_0, & \text{if } |x|_0 \leq n-k \; or \; x = 0^n \\ n - |x|_0, & \text{otherwise} \end{cases} \end{aligned} \tag{2}$$

OMZJ shares some characteristics with OJZJ (see Table **??**), such as having disjoint optimal solutions. However, unlike OZJZ, OMZJ does not exhibit a low ratio of Pareto optimal solutions. The reason for this is that OMZJ has only one valley (where dominated solutions are located) from the ZeroJump objective (in contrast to the two valleys in OZJZ), thus at least half of the solutions are Pareto optimal, regardless of the value of k. Figure 2(b) plots the objective space of the 8-bit OMZJ problem, where it is clear that most of the solutions are Pareto optimal ones.

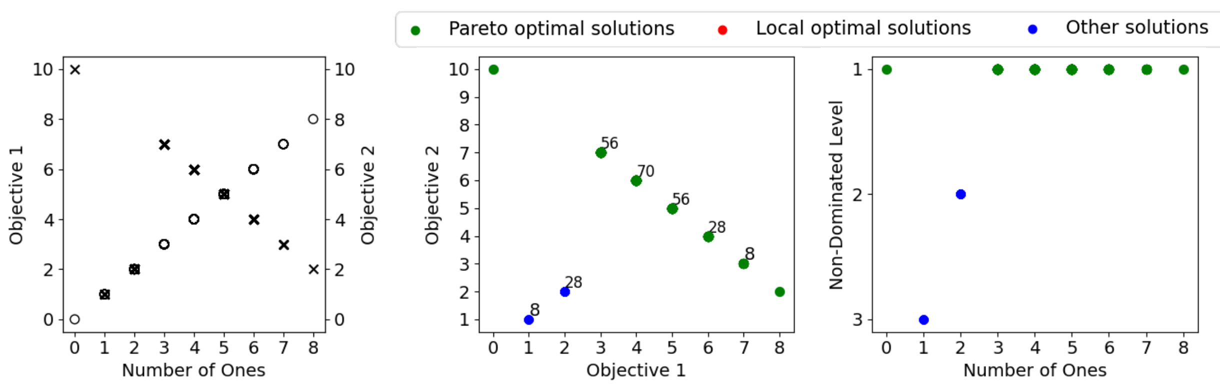
2.3. OneMax-ZeroRoyalRoad (OMZR)

OMZR combines the OneMax function with the ZeroRoyalRoad function, and it simultaneously maximises the number of ones in a bit-string and the number of blocks with all zero bits. Formally, OMZR can be formulated as follows.



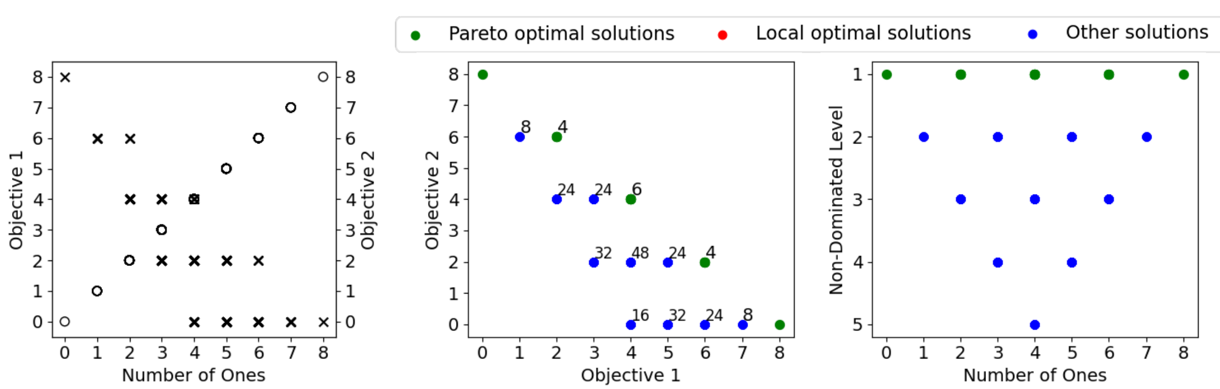
(a) Two individual objectives wrt #1 (b) Objective space (c) Nondominated level wrt #1

Figure 1. **OneMax-TrailingZeroes (OMTZ)** ($n = 8$) (a) Two individual objectives (OneMax and TrailingZeroes) of the OMTZ problem with respect to the number of ones. (b) Objective space, where the number associated with a solution means how many solutions in the decision space map to that solution. (c) The level of solutions with respect to the number of ones based on the Pareto non-dominated sorting. In (b) and (c), green, red and blue points indicate Pareto optimal solutions, local optimal solutions, and other solutions respectively.



(a) Two individual objectives wrt #1 (b) Objective space (c) Nondominated level wrt #1

Figure 2. **OneMax-ZeroJump (OMZJ)** ($n = 8$, jump parameter $k = 2$) (a) Two individual objectives (OneMax and ZeroJump) of the OMZJ problem with respect to the number of ones. (b) Objective space, where the number associated with a solution means how many solutions in the decision space map to that solution. (c) The level of solutions with respect to the number of ones based on the Pareto non-dominated sorting. In (b) and (c), green, red and blue points indicate Pareto optimal solutions, local optimal solutions, and other solutions respectively.



(a) Two individual objectives wrt #1 (b) Objective space (c) Nondominated level wrt #1

Figure 3. **OneMax-ZeroRoyalRoad (OMZR)** ($n = 8$, $b = 4$ where b denotes the number of blocks) (a) Two individual objectives (OneMax and ZeroRoyalRoad) of the OMZR problem with respect to the number of ones. (b) Objective space, where the number associated with a solution means how many solutions in the decision space map to that solution. (c) The level of solutions with respect to the number of ones based on the Pareto non-dominated sorting. In (b) and (c), green, red and blue points indicate Pareto optimal solutions, local optimal solutions and other solutions, respectively.

Definition 2.3. (*OneMax-ZeroRoyalRoad*). Let $x \in \{0,1\}^n$ be a bit-string of length n, partitioned into b disjoint blocks $S_1, S_2, \ldots, S_b$, each of length ℓ (where $n = b \times \ell, b > 1$). The problem is defined as:

$$f_1(x) = \sum_{i=1}^{n} x_i \quad \text{and} \quad f_2(x) = \sum_{j=1}^{b} \prod_{i \in S_j} (1 - x_i) \tag{3}$$

Like OneRoyalRoad-ZeroRoyalRoad (ORZR), the block design of the ZeroRoyalRoad objective in the OMZR problem leads to disjoint optimal solutions and a low ratio of Pareto optimal solutions. However, unlike ORZR, OMZR does not exhibit Pareto local optima, as any non-optimal solution can improve on its OneMax objective by flipping a zero to one. Figure 3 illustrates the 8-bit OMZR problem. As can be seen in Figure 3(b), all dominated solutions can move rightward by improving on their OneMax objective (without affecting the ZeroRoyalRoad objective) until reaching a Pareto optimal solution. This may make OMZR easier to be dealt with than the problem consisting of two RoyalRoad functions (i.e., ORZR).

3. Ratio of Pareto optimal solutions

3.1. Ratio of Pareto optimal solutions of OJZJ

For a OneJump-ZeroJump (OJZJ) problem, the ratio of Pareto optimal solutions is $R(n,k) = \frac{2^n - 2\sum_{s=n-k}^{n-1}\binom{n}{s}}{2^n}$. The following propositions prove that (1) for a large k near $\frac{n}{2}$, this ratio can be very low; (2) For a smaller $k < \frac{n}{\ln n}$, this ratio becomes no less than 0.5.

Proposition 3.1. *For the OJZJ problem with a sufficiently large $n \in \mathbb{Z}^+$ and a large $k = \left\lfloor \frac{n}{2} \right\rfloor - 1$, the ratio of Pareto optimal solutions of OJZJ converges to 0 as $n \to \infty$:*

$$\lim_{n\to\infty} R(n,k) = \lim_{n\to\infty} \frac{2^n - 2\sum_{s=n-k}^{n-1}\binom{n}{s}}{2^n} = 0 \tag{4}$$

Proof. We begin by simplifying the ratio $R(n,k)$.

$$\begin{aligned} R(n,k) &= \frac{2^n - 2\sum_{s=n-k}^{n-1}\binom{n}{s}}{2^n} \\ &= 1 - \frac{2\sum_{s=n-k}^{n-1}\binom{n}{s}}{2^n} = 1 - \frac{\sum_{s=1}^{k}\binom{n}{s}}{2^{n-1}} \end{aligned}$$

For simplicity, we choose $k = \left\lfloor \frac{n}{2} \right\rfloor - 1$ as a representative large value near $\frac{n}{2}$. We analyse two cases based on the parity of n.

Case 1: n is even. Let $n = 2m$, where m is a positive integer, we have $k = m - 1$ and

$$\sum_{s=1}^{k}\binom{n}{s} = \sum_{s=1}^{m-1}\binom{2m}{s} = 2^{2m-1} - \frac{1}{2}\binom{2m}{m} - 1$$

Thus, the ratio becomes:

$$R(2m, m-1) = 1 - \frac{2^{2m-1} - \frac{1}{2}\binom{2m}{m} - 1}{2^{2m-1}} = \frac{\frac{1}{2}\binom{2m}{m} + 1}{2^{2m-1}}$$

Using Stirling's approximation [?], $\binom{2m}{m} \approx \frac{4^m}{\sqrt{\pi m}}$, Therefore, for large m:

$$R(2m, m-1) \approx \frac{\frac{1}{2}\binom{2m}{m} + 1}{2^{2m-1}} \approx \frac{\frac{1}{2} \cdot \frac{4^m}{\sqrt{\pi m}}}{\frac{1}{2} \cdot 4^m} \approx \sqrt{\frac{2}{\pi n}}$$

Case 2: n is odd. Let $n = 2m+1$, where m is a positive integer. Then we have $k = m - 1$ and

$$\sum_{s=1}^{k}\binom{n}{s} = \sum_{s=1}^{m-1}\binom{2m+1}{s} = 2^{2m} - \binom{2m+1}{m} - 1$$

Thus, the ratio becomes:

$$R(2m+1, m-1) = 1 - \frac{2^{2m} - \binom{2m+1}{m} - 1}{2^{2m}} = \frac{\frac{2m+1}{m+1}\binom{2m}{m}}{2^{2m}} + \frac{1}{2^{2m}}$$

Using Stirling's approximation again, since for $\frac{2m+1}{m+1} \approx 2$ for large m, we have:

$$R(2m+1, m-1) \approx \frac{\frac{2m+1}{m+1}\binom{2m}{m}}{2^{2m}} + \frac{1}{2^{2m}} \approx \sqrt{\frac{8}{\pi n}} + \frac{1}{2^{n-1}}$$

As $m \to \infty$, the term $\frac{1}{2^{n-1}}$ becomes negligible. Therefore:

$$R(2m+1, m-1) \approx \sqrt{\frac{8}{\pi n}}$$

Finally, both cases converge towards 0 as $n \to \infty$. □

Proposition 3.2. *For the OJZJ problem with $n \in \mathbb{Z}^+$, if n is sufficiently large and the jump parameter $k < \frac{n}{\ln n}$, we have*

$$R(n,k) \geq 0.5 \tag{5}$$

Proof. $R(n,k) \geq 0.5$ implies:

$$\begin{aligned} R(n,k) = 1 - \frac{\sum_{s=1}^{k}\binom{n}{s}}{2^{n-1}} &\geq 0.5 \\ \sum_{s=1}^{k}\binom{n}{s} &\leq 2^{n-2} \end{aligned}$$

Using the bound for binomial coefficients: $\binom{n}{s} \leq \left(\frac{en}{s}\right)^s$, we approximate the sum by the largest term multiplied by k:

$$\sum_{s=1}^{k}\binom{n}{s} \leq k \cdot \binom{n}{k} \leq k \cdot \left(\frac{en}{k}\right)^k$$

Then, we have:

$$\begin{aligned} k \cdot \left(\frac{en}{k}\right)^k &\leq 2^{n-2} \\ \ln k + k \cdot \ln\left(\frac{en}{k}\right) &\leq (n-2)\ln 2 \end{aligned}$$

For a large n, the -2 is negligible, so:

$$\ln k + k \cdot \ln\left(\frac{en}{k}\right) \leq n \ln 2$$

Assume k grows with n such that $k = \frac{n}{\ln n}$. Substitute $k = \frac{n}{\ln n}$ into the inequality:

$$\ln\left(\frac{n}{\ln n}\right) + \frac{n}{\ln n} \cdot \ln\left(\frac{en}{\frac{n}{\ln n}}\right) \leq n \ln 2$$
$$\ln\left(\frac{n}{\ln n}\right) + \frac{n}{\ln n} \cdot (1 + \ln\ln n) \leq n \ln 2$$

Rearranging into the ratio:

$$\begin{aligned}\frac{\text{LHS}}{\text{RHS}} &= \frac{\ln\left(\frac{n}{\ln n}\right) + \frac{n}{\ln n} \cdot (1 + \ln\ln n)}{n \ln 2} \\ &= \frac{\ln\left(\frac{n}{\ln n}\right) + \frac{n}{\ln n} + \frac{n \ln\ln n}{\ln n}}{n \ln 2} \\ &= \frac{\ln\left(\frac{n}{\ln n}\right)}{n \ln 2} + \frac{\frac{n}{\ln n}}{n \ln 2} + \frac{\frac{n \ln\ln n}{\ln n}}{n \ln 2}\end{aligned}$$

Since all numerator terms grow slower than $n \ln 2$, we have $\frac{\text{LHS}}{\text{RHS}} = 0$ as $n \to \infty$, implying LHS $\leq$ RHS. Therefore, the inequality

$$\ln\left(\frac{n}{\ln n}\right) + \frac{n}{\ln n} \cdot (1 + \ln\ln n) \leq n \ln 2$$

holds for a sufficiently large n.

□

Note that $k < \frac{n}{\ln n}$ is conservative in this proposition. One may transform the inequality $\ln k + k \cdot \ln\left(\frac{en}{k}\right) \leq n \ln 2$ to the format of the Lambert W function to obtain a tighter upper bound of k for $R(n, k) \geq 0.5$.

3.2. Ratio of Pareto Optimal Solutions of OJZR

When $(n-k-1) \mod \ell \neq 0$, the Pareto front shape of OJZR is concave and one Pareto optimal solution has much more corresponding bit-strings (preimages) in the decision space than the others. In this case, we show that the ratio of Pareto optimal solutions is still low in the following proposition.

Proposition 3.3. *For OJZR with problem size $n \in \mathbb{Z}^+$, jump parameter $1 \leq k < \frac{n}{2}$, and block length ℓ with $2 \leq \ell < n$, $n \mod \ell = 0$ and $(n-k-1) \mod \ell \neq 0$, let $m = \frac{n}{\ell}$, the ratio of Pareto optimal solutions is given by*

$$R(n, k) = \frac{\binom{m}{m} + \sum_{i=\lceil \frac{k+1}{\ell} \rceil}^{m} \binom{m}{i} + \binom{m}{\lfloor \frac{k+1}{\ell} \rfloor} \times \binom{n - \lfloor \frac{k+1}{\ell} \rfloor \ell}{n-k-1}}{2^n} \tag{6}$$

In what follows, we show that the ratio converges to 0 as $n \to \infty$.

Proof. The numerator of $R(n, k)$ comprises three terms. The first term $\binom{m}{m}$ is always 1. The second term $\sum_{i=\lceil \frac{k+1}{\ell} \rceil}^{m} \binom{m}{i}$ is maximised when the lower limit $\lceil \frac{k+1}{\ell} \rceil$ is as small as possible (i.e., 1). Thus:

$$\sum_{i=\lceil \frac{k+1}{\ell} \rceil}^{m} \binom{m}{i} \leq \sum_{i=1}^{m} \binom{m}{i} = 2^m - 1$$

The third term is a product of binomial coefficients:

$$\binom{m}{\lfloor \frac{k+1}{\ell} \rfloor} \times \binom{n - \lfloor \frac{k+1}{\ell} \rfloor \ell}{n-k-1}$$

For $\binom{m}{\lfloor \frac{k+1}{\ell} \rfloor}$, to establish a general upper bound, we use:

$$\binom{m}{\lfloor \frac{k+1}{\ell} \rfloor} \leq 2^m$$

For $\binom{n - \lfloor \frac{k+1}{\ell} \rfloor \ell}{n-k-1}$, the maximum occurs when $n - \lfloor \frac{k+1}{\ell} \rfloor \ell$ is the largest. Given $\lfloor \frac{k+1}{\ell} \rfloor \ell \geq k$:

$$\binom{n - \lfloor \frac{k+1}{\ell} \rfloor \ell}{n-k-1} \leq 2^{n - \lfloor \frac{k+1}{\ell} \rfloor \ell}$$

Thus:

$$\binom{m}{\lfloor \frac{k+1}{\ell} \rfloor} \times \binom{n - \lfloor \frac{k+1}{\ell} \rfloor \ell}{n-k-1} \leq 2^m \times 2^{n - \lfloor \frac{k+1}{\ell} \rfloor \ell} = 2^{m + n - \lfloor \frac{k+1}{\ell} \rfloor \ell}$$

Combining the terms, the numerator is bounded by:

$$\text{Numerator} \leq 1 + 2^m + 2^{m + n - \lfloor \frac{k+1}{\ell} \rfloor \ell}$$

Putting back to the ratio and substituting $m = \frac{n}{\ell}$:

$$R(n, k) \leq 2^{-n} + 2^{-n(1 - \frac{1}{\ell})} + 2^{\frac{n}{\ell} - \lfloor \frac{k+1}{\ell} \rfloor \ell}.$$

Since $k < \frac{n}{2}$, the power of the third term

$$\frac{n}{\ell} - \lfloor \frac{k+1}{\ell} \rfloor \ell < \frac{n}{\ell} - (\frac{n}{2} + 1) = -1 + n(\frac{1}{\ell} - \frac{1}{2})$$

The first two terms are negligible as $n \to \infty$. Therefore, the bound is

$$R(n, k) \leq 2^{-1 + n(\frac{1}{\ell} - \frac{1}{2})} \tag{7}$$

For $\ell > 2$, that converges towards 0 as $n \to \infty$.

However, for the edge case $\ell = 2$, this becomes 0.5.

Revisiting the ratio for case $\ell = 2$. Since $k < n/2$ and $(n-k-1) \mod 2 \neq 0$ forces k to be even, we let $k = 2p$. Then, the third term of the ratio:

$$\begin{aligned}&\binom{m}{\lfloor \frac{k+1}{\ell} \rfloor} \times \binom{n - \lfloor \frac{k+1}{\ell} \rfloor \ell}{n-k-1} \\ &= \binom{n/2}{p} \binom{n-2p}{n-2p-1} \\ &= \binom{n/2}{p} \cdot (n - 2p) = (n-k) \binom{n/2}{k/2}\end{aligned}$$

Since $\binom{n/2}{k/2} < \sum_{i=0}^{n/2} \binom{n/2}{i} = 2^{n/2}$, and $(n-k)2^{n/2}$ still grows slower than the denominator 2^n, thus the third term also converges towards 0 as $n \to \infty$.

Therefore, for all the cases, the ratio converges to 0 as $n \to \infty$.

□

References

[1] E. Zitzler, K. Deb, and L. Thiele, "Comparison of multiobjective evolutionary algorithms: Empirical results," *Evolutionary Computation*, vol. 8, no. 2, pp. 173–195, 2000.