Supplementary Materials for: [On the Problem Characteristics of Multi-objective Pseudo Boolean Functions in Runtime Analysis]

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1. Introduction

This supplementary document provides additional material for our main submission titled *"On the Problem Characteristics of Multi-objective Pseudo-Boolean Functions in Runtime Analysis"* to the Conference on Foundations of Genetic Algorithms 2025 (FOGA 25). We first present a detailed description of the mix-and-match multi-objective pseudo-Boolean functions involving the OneMax function. Then, we give the formal properties (e.g., Pareto set, Pareto local optimal set) of the proposed benchmarks. This is followed by formal proofs concerning the ratio of Pareto-optimal solutions for the OneJump-ZeroJump and OneJump-ZeroRoyalRoad function classes.

2. Mix-and-Match Multi-Objective Boolean Functions with OneMax

In the main paper, we only present the mixes of LeadingOnes, Jump and RoyalRoad as mixing OneMax with other functions may be trivial, as it simplifies the problem. Here, we present the mixes involving the OneMax function.

2.1. OneMax-TrailingZeroes (OMTZ)

OMTZ combines the OneMax function with the TrailingZeroes function, thus a simultaneous maximisation of the number of ones in the bit-string and the number of consecutive zeroes (from right to left). Formally, OMTZ can be formulated as follows.

Definition 2.1. (*OneMax-TrailingZeroes*). Let $x \in \{0,1\}^n$ be a bit-string of length n. The OMTZ problem is defined as:

$$f_1(x) = \sum_{i=1}^{n} x_i, \quad \text{and} \quad f_2(x) = \sum_{i=1}^{n} \prod_{j=i}^{n} (1 - x_j) \tag{1}$$

OMTZ shares some characteristics with LeadingOnes-TrailingZeroes (LOTZ), such as a linear Pareto front, no disjoint optimal solutions, no local optima, and a low ratio of Pareto optimal solutions. The primary difference between them is that OMTZ is non-symmetric with respect to its objectives, which is commonly seen in real-world problems.

The OneMax objective in OMTZ is easier to improve on than the TrailingZeroes objective, since improving on OneMax involves flipping any bit from zero to one, whereas improving on TrailingZeroes involves flipping a specific one bit to zero. This difference creates an objective imbalance, a common feature in many test suites (e.g., ZDT [1]). Figure 1 illustrates the 8-bit OMTZ. As seen in Figure 1(b), more solutions concentrate in a region with fairly good value on the objective OneMax but poor value on the objective TrailingZeroes.

2.2. OneMax-ZeroJump (OMZJ)

OMZJ combines the OneMax function with the ZeroJump function, aiming to simultaneously maximise the number of ones in the bit-string while crossing a valley in the search space determined by the number of zeroes. Like OneJump-ZeroJump (OJZJ), the size of this valley is controlled by the *jump parameter* k, with larger values of k making it more difficult to reach the boundary solution for the ZeroJump objective (i.e., (0^n)). Formally, OMZJ can be formulated as follows.

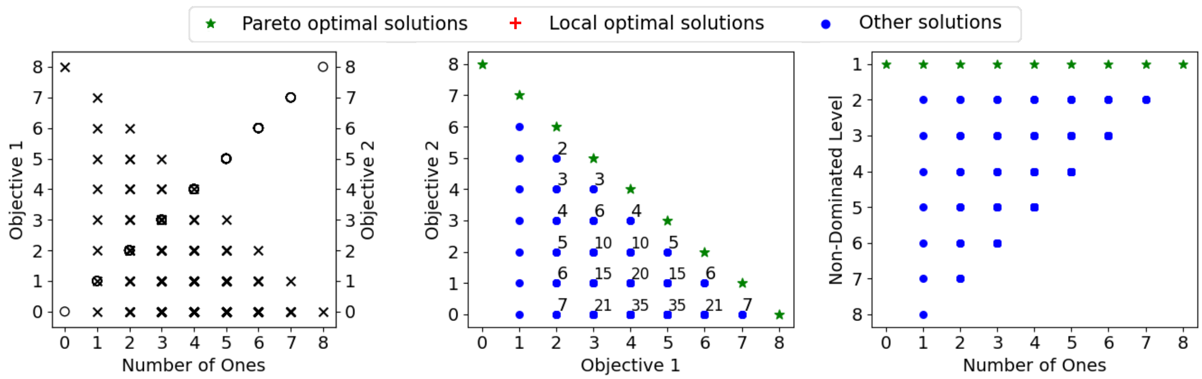
Definition 2.2. (*OneMax-ZeroJump*). Let $x \in \{0,1\}^n$ be a bit-string of length n and let k be a fixed jump parameter with $1 < k < \frac{n}{2}$. The OneMax-ZeroJump problem is defined as:

$$\begin{aligned} f_1(x) &= \sum_{i=1}^{n} x_i, \\ f_2(x) &= \begin{cases} k + |x|_0, & \text{if } |x|_0 \le n - k \textit{ or } x = 0^n \\ n - |x|_0, & \text{otherwise} \end{cases} \end{aligned} \tag{2}$$

OMZJ shares some characteristics with OJZJ, such as having disjoint optimal solutions. However, unlike OZJZ, OMZJ does not exhibit a low ratio of Pareto optimal solutions. This is because OMZJ has only one valley (where dominated solutions are located) from the ZeroJump objective (in contrast to the two valleys in OZJZ), thus at least half of the solutions are Pareto optimal, regardless of the value of k. Figure 2(b) plots the objective space of the 8-bit OMZJ problem, where it is clear that most of the solutions are Pareto optimal ones.

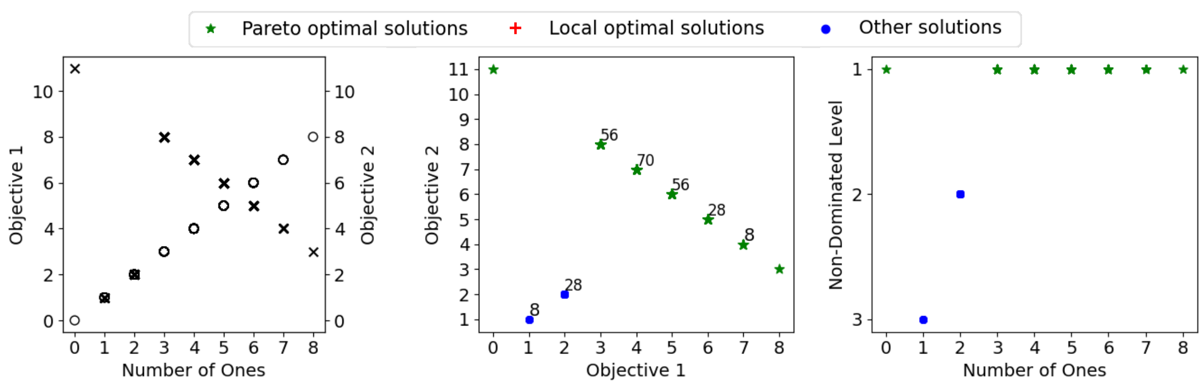
2.3. OneMax-ZeroRoyalRoad (OMZR)

OMZR combines the OneMax function with the ZeroRoyalRoad function, and it simultaneously maximises the number of ones in a bit-string and the number of blocks



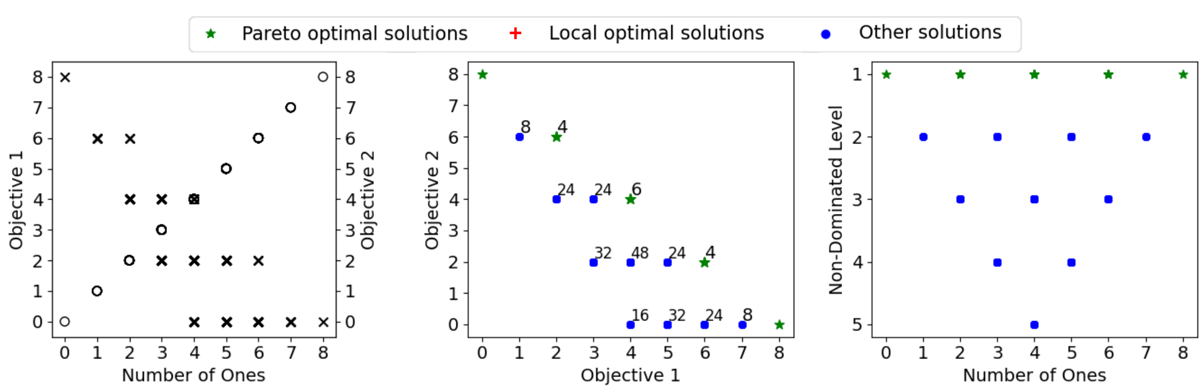
(a) Two individual objectives wrt #1 (b) Objective space (c) Nondominated level wrt #1

Figure 1. **OneMax-TrailingZeroes (OMTZ)** ($n = 8$) (a) Two individual objectives (OneMax and TrailingZeroes) of the OMTZ problem with respect to the number of ones. (b) Objective space, where the number associated with a solution means how many solutions in the decision space map to that solution. (c) The level of solutions with respect to the number of ones based on the Pareto non-dominated sorting. In (b) and (c), green, red and blue points indicate Pareto optimal solutions, local optimal solutions, and other solutions respectively.



(a) Two individual objectives wrt #1 (b) Objective space (c) Nondominated level wrt #1

Figure 2. **OneMax-ZeroJump (OMZJ)** ($n = 8$, jump parameter $k = 3$) (a) Two individual objectives (OneMax and ZeroJump) of the OMZJ problem with respect to the number of ones. (b) Objective space, where the number associated with a solution means how many solutions in the decision space map to that solution. (c) The level of solutions with respect to the number of ones based on the Pareto non-dominated sorting. In (b) and (c), green, red and blue points indicate Pareto optimal solutions, local optimal solutions, and other solutions respectively.



(a) Two individual objectives wrt #1 (b) Objective space (c) Nondominated level wrt #1

Figure 3. **OneMax-ZeroRoyalRoad (OMZR)** ($n = 8$, $b = 4$ where b denotes the number of blocks) (a) Two individual objectives (OneMax and ZeroRoyalRoad) of the OMZR problem with respect to the number of ones. (b) Objective space, where the number associated with a solution means how many solutions in the decision space map to that solution. (c) The level of solutions with respect to the number of ones based on the Pareto non-dominated sorting. In (b) and (c), green, red and blue points indicate Pareto optimal solutions, local optimal solutions and other solutions, respectively.

with all zero bits. Formally, OMZR can be formulated as follows.

Definition 2.3. (*OneMax-ZeroRoyalRoad*). Let $x \in \{0,1\}^n$ be a bit-string of length n, partitioned into b disjoint blocks $S_1, S_2, \ldots, S_b$, each of length ℓ (where $n = b \times \ell, b > 1$). The problem is defined as:

$$f_1(x) = \sum_{i=1}^{n} x_i \quad \text{and} \quad f_2(x) = \sum_{j=1}^{b} \big(\ell \prod_{i \in S_j} (1 - x_i)\big) \quad (3)$$

Like OneRoyalRoad-ZeroRoyalRoad (ORZR), the block design of the ZeroRoyalRoad objective in the OMZR problem leads to disjoint optimal solutions and a low ratio of Pareto optimal solutions. However, unlike ORZR, OMZR does not exhibit Pareto local optima, as any non-optimal solution can improve on its OneMax objective by flipping a zero to one. Figure 3 illustrates the 8-bit OMZR problem. As can be seen in Figure 3(b), all dominated solutions can move rightward by improving on their OneMax objective (without affecting the ZeroRoyalRoad objective) until reaching a Pareto optimal solution. This may make OMZR easier to be dealt with than the problem consisting of two RoyalRoad functions (i.e., ORZR).

3. Formal Properties of the New Benchmarks

In this section, we present the formal properties, such as the Pareto front and the Pareto set under general parameters, for each newly introduced benchmark. The properties are summarised in Table 1. The proofs are given below.

We start from the benchmarks presented in the main paper, namely OneRoyalRoad-ZeroRoyalRoad, LeadingOnes-ZeroJump, LeadingOnes-ZeroRoyalRoad and OneJump-ZeroRoyalRoad.

3.1. OneRoyalRoad-ZeroRoyalRoad (ORZR)

Proposition 3.1. *Let $n, b, \ell \in \mathbb{N}, n = b \cdot \ell, b > 1$, given a bit-string of length n, partitioned into b disjoint blocks $x = S_1 S_2 \ldots S_b$, each with the same length ℓ. For the benchmark ORZR, the Pareto optimal set is*

$$\mathcal{P} = \{x \mid |\{S \mid S = 1^\ell\}| + |\{S \mid S = 0^\ell\}| = b\} \quad (4)$$

Proof. For each S block, it either contributes to one objective only (by being 1^ℓ or 0^ℓ) by a constant quantity ell or contributes to nothing. Denoting 1^ℓ or 0^ℓ as a completed block. This implies that all blocks must be completed to become a Pareto optimal solutions, as otherwise a solution is dominated by another solution that completes its uncompleted block.

For those solutions in $\mathcal{P}$, all blocks are completed and we have $f_1 + f_2 = n$. Therefore, solutions in $\mathcal{P}$ are non-dominated to each other, as they cannot improve one objective without reducing another.

For $x \notin \mathcal{P}$, they have at least one uncompleted block. Completing that block yields a solution that dominates x. Therefore, $x \notin \mathcal{P}$ are dominated. □

The corresponding Pareto front is thereby $\{\big(i\ell, (b-i)\ell\big) \mid i \in \{0, 1, \ldots, b\}\}$.

Proposition 3.2. *With the same notation above, given the block length $\ell > 3$, the local Pareto optimal solutions of ORZR are the following.*

$$\mathcal{L} = \big\{x \mid \forall S : |S|_1 > 1 \wedge |S|_0 > 1, \quad (5)$$
$$|\{S \mid s = 1^\ell\}| + |\{S \mid s = 0^\ell\}| < b\big\} \quad (6)$$

Proof. We first show that the solutions $x \in \mathcal{L}$ are non-dominated by its 1-bit neighbours. Since solutions in $\mathcal{L}$ have the property $\forall S : |S|_1 > 1 \wedge |S|_0 > 1$, all uncompleted blocks cannot be completed (become 1^ℓ or 0^ℓ) via one bit-flip, implying flipping a bit in an uncompleted block has no effect. On the other hand, flipping any bit in completed blocks will reduce one objective without improving another. Thus, $x \in \mathcal{L}$ cannot be improved via a single bit flip, meaning that it is not dominated by any of its 1-bit neighbours.

Then, we show that the solutions $x \notin \mathcal{P} \cup \mathcal{L}$ are dominated by one of its neighbours. Since $\forall S : |S|_1 > 1 \wedge |S|_0 > 1$ is false, we have $\exists S : |S|_0 \leq 1 \vee |S|_1 \leq 1$. Since $x \notin \mathcal{P}$, there exists at least one uncompleted block (i.e., $|S|_1 \neq 0 \wedge |S|_0 \neq 0$), thus we have $\exists S : |S|_0 = 1 \vee |S|_1 = 1$. For such block, flipping that one 1 left (or one 0 left) completes this block and yields improvement on one objective, which dominates x. □

Remark. The above proof of the Pareto local optima relies on the condition $\ell > 3$. This is because, the condition $\forall S : |S|_1 > 1 \wedge |S|_0 > 1$ implies that the minimum $|S|_1$ and $|S|_0$ are 2, thus $\ell = |S| = |S|_1 + |S|_0 \geq 4$. Therefore, for any $\ell \leq 3$, the Pareto local optimal set $\mathcal{L}$ is empty.

3.2. LeadingOnes-ZeroJump (LOZJ)

Proposition 3.3. *Let $n \in \mathbb{N}, 1 < k < \frac{n}{2}$, given a bit-string of length n, for the benchmark LOZJ, the Pareto optimal set is*

$$\mathcal{P} = \{\, 0^n \,\} \cup \{\, 1^i 0^{n-i} \mid i \in \{k, k+1, \ldots, n\}\} \quad (7)$$

Proof. Let $i = f_1(x)$ be the number of leading ones, and $t = |x|_0$. Since the first i bits of x are 1s and the last t bits are 0.

$$i + t \leq n,$$

First, we show that all solutions in $\mathcal{P}$ are non-dominated. 0^n is non-dominated, as it is the optimal solution of the ZeroJump. As for $x_i = 1^i 0^{n-i}$ with $i \geq k$, we have $f_1(x_i) = i$ and $f_2(x_i) = k + n - i$. Since f_1 has i and f_2 has $-i$, it is obvious that no point dominates another in $\mathcal{P}$.

Then, we show that for any $x' \notin \mathcal{P}$, it is dominated by some $x \in \mathcal{P}$. Denote its leading ones as i' and $t' = |x'|_0$, we consider two cases.

Case 1: $i' < k$. We can compare it with $x = 1^k 0^{n-k}$. Then, we have$f_1(x') = i < k = f_1(x)$. As for f_2, by 0^{n-k}, $f_2(x)$ already has maximum ZeroJump value below 0^n, thus $f_2(x') \leq f_1(x)$.

TABLE 1. SUMMARY OF GLOBAL AND LOCAL PARETO PROPERTIES FOR THE NEW BENCHMARKS. x DENOTES A SOLUTION AND EACH SOLUTION IS A BIT-STRING OF LENGTH n; k DENOTES THE JUMP PARAMETER OF THE JUMP FUNCTION COMPONENT; b AND ℓ DENOTES THE NUMBER OF BLOCKS AND THE CORRESPONDING BLOCK LENGTH OF THE ROYALROAD FUNCTION COMPONENT.

Benchmark	Pareto set $\mathcal{P}$	Pareto front	Local optima $\mathcal{L}$
ORZR	$\{x \mid \lvert\{S = S_i = 1^\ell\}\rvert + \lvert\{S = 0^\ell\}\rvert = b\}$	$\{(i\ell, (b-i)\ell) \mid i = 0, \dots, b\}$	$\{x \mid \forall S : \lvert S\rvert_1 > 1 \ \wedge\ \lvert S\rvert_0 > 1,\ \lvert\{S = 1^\ell\}\rvert + \lvert\{S = 0^\ell\}\rvert < b\}$ ($\ell > 3$)
LOZJ	$\{0^n\} \cup \{1^i 0^{n-i} \mid i = k, \dots, n\}$	$\{(0, n+k)\} \cup \{(i, n+k-i) \mid i = k, \dots, n\}$	$\{1^i s \mid i < k,\ \lvert s\rvert_0 = n-k\}$
LOZR	$\{1^{i\ell} 0^{n-i\ell} \mid i = 0, \dots, b\}$	$\{(i\ell, (b-i)\ell) \mid i = 0, \dots, b\}$	$\{1^i s \mid i \in \{0, \ell, \dots, (b-1)\ell\},\ S_{\lceil i/\ell\rceil+1} = 0^\ell,\ \forall j > \lceil i/\ell\rceil,\ \lvert S_j\rvert_1 > 1\}$
OJZR	$\{1^n\} \cup \{x \mid \lvert x\rvert_1 \le n-k,\ \lvert\{1^\ell\}\rvert + \lvert\{0^\ell\}\rvert = b\}$ (case $(n-k) \bmod \ell = 0$)	$\{(n+k, 0)\} \cup \{(i\ell+k, n-i\ell) \mid i = 0, \dots, \lfloor k/\ell\rfloor\}$	$\{x \mid \lvert x\rvert_1 = n-k,\ \lvert\{0^\ell\}\rvert < \lfloor k/\ell\rfloor\}$
OJZR	$\{1^n\} \cup \{x \mid \lvert x\rvert_1 < n-k,\ \lvert\{1^\ell\}\rvert + \lvert\{0^\ell\}\rvert = b\} \cup \{x \mid \lvert x\rvert_1 = n-k,\ \lvert\{0^\ell\}\rvert = \lfloor k/\ell\rfloor\}$ (case $(n-k) \bmod \ell > 0$)	$\{(n+k, 0)\} \cup \{(n-k, \lfloor k/\ell\rfloor\ell)\} \cup \{(i\ell+k, n-i\ell) \mid i = 0, \dots, \lfloor k/\ell\rfloor\}$	same as above
OMTZ	$\{0^n\} \cup \{1^i 0^{n-i} \mid i = 0, \dots, n\}$	$\{(i, n-i) \mid i \in \{0, 1, \dots, n\}\}$	none
OMZJ	$\{0^n\} \cup \{x \mid \lvert x\rvert_0 \le n-k\}$	$\{(0, n+k))\} \cup \{(i, n+k-i) \mid i = k, \dots, n\}$	none
OMZR	$\{x \mid \lvert\{1^\ell\}\rvert + \lvert\{0^\ell\}\rvert = b\}$	$\{(i\ell, (b-i)\ell) \mid i = 0, \dots, b\}$	none

Case 2: $i' \in \{k, \dots, n\}$. Since x' is not in $\mathcal{P}$, it cannot have the form $1^{i'}0^{n-i'}$, then $i' + t' < n$. We compare it with $x = 1^{i'}0^{n-i'}$ in $\mathcal{P}$. Then, we have $f_1(x') = i' = f_1(x)$ and $f_2(x') = t' < n - i' = f_2(x)$.

In both cases, x strictly dominates x', hence no $x' \notin \mathcal{P}$ is Pareto-optimal. □

The corresponding Pareto front of LOZJ is $\{(0, n+k)\} \cup \{(k, n), (k+1, n-1), \dots, (n, k)\}$.

Notably, besides the global optima, there are Pareto local optimal solutions in LOZJ.

Proposition 3.4. *With the same notation above, the local Pareto optimal solutions of LOZJ are the following.*

$$\mathcal{L} = \{\, 1^i s \mid i < k, |s|_0 = n - k\} \tag{8}$$

where s is a sub bit-string of length $n - i$.

Proof. For any solution x, let i denotes the number of leading ones and $t = |x|_0$.

Firstly, we show that solutions in $\mathcal{L}$ are Pareto local optimal solutions. Flipping any 1 among the first i bits only reduces the LeadingOnes objective. Flipping any 1 to 0 in s only reduces the ZeroJump objective, as s has more than $n-k$ 0s and drop into the valley. Flipping any 0 to 1 in s may or may not increase the LeadingOnes objective, but it definitely reduces the ZeroJump objective as there are fewer zeroes. For all cases, flipping one bit cannot generate any solution that dominates solution in $\mathcal{L}$.

Secondly, we show that any $x \notin \mathcal{P} \cup \mathcal{L}$ is dominated by at least one of its neighbour (i.e., can be improved via a single bit flip). Similarly, we denote i as the number of leading ones and $t = |x|_0$. We consider two cases.

Case 1: $i < k$. Since it is not in $\mathcal{P}$, we have $t \neq n - k$. If $t < n - k$, then there is at least one 1 among the non-leading bits because $t < n-k < n-i$, so flipping any such 1 immediately improves the ZeroJump objective. On the other hand, if $t > n - k$, then $f_2(x) = n - t = i$. Here, flipping any 0 to 1 can improve the ZeroJump objective (and may improve the LeadingOnes objective if that 0 is at position $i+1$). For both cases, that improved neighbouring solution dominates the current solution.

Case 2: $i \geq k$. Since $x \notin \mathcal{P} \cup \mathcal{L}$, we have $i + t < n$. Here, there exists at least one 1 that is not in the first $i+1$ 1s (otherwise i will be larger). Flipping such 1 to 0 improve the ZeroJump objective without affecting the LeadingOnes objective, thus dominating the current solution x.

In all cases, x has a one-bit neighbour that dominates it, therefore not being Pareto local optimal. □

3.3. LeadingOnes-ZeroRoyalRoad (LOZR)

Proposition 3.5. *Let $n \in \mathbb{N}, 1 < k < \frac{n}{2}$, for the benchmark LOZR, partitioned into b disjoint blocks $x = S_1S_2 \dots S_b$, each with the same length ℓ. For the benchmark LOZR, the Pareto optimal set is*

$$\mathcal{P} = \{\, 1^i 0^{n-i} \mid i \in \{0, \ell, 2\ell, \dots, b\ell\}\} \tag{9}$$

Proof. Firstly, we show that solutions in $\mathcal{P}$ are non-dominated. Let $i = f_1(x)$ be the number of leading ones. Since $n = b\ell$, the format 1^i0^{n-i} where $i \in \{0, \ell, 2\ell, \dots, b\ell\}$ indicates that the trailing zeroes resembles $\frac{n-i}{\ell}$ blocks of 0^ℓ. This yields an objective of $(i, n-i)$. Clearly, i and $-i$ in the objective values indicate that solutions in this set are non-dominated to each other.

Secondly, we show that $x \notin \mathcal{P}$ are dominated. For any solutions $x \notin \mathcal{P}$, let $i = f_1(x)$, we can rewrite those solutions as 1^i0s for $i \in \{0, \ldots, n-1\}$ and arbitrary s where $|s| = n-2$. If $i \mod \ell = 0$, the rest of the bits cannot be entirely 0^{n-i} as $x \notin \mathcal{P}$, indicating that there exist blocks that are not 0^ℓ. Clearly, x is dominated by the solution 1^i0^{n-i} as 1^i0^{n-i} has a higher ZeroRoyalRoad objective. If $i \mod \ell \neq 0$, it means that flipping the 0 at the $i+1$ position (before s) will not interrupt any 0^ℓ blocks in s (if there is any) but only improve the LeadingOnes objective, making x dominated by this new solution. □

The corresponding Pareto front is thereby $\{(i\ell, (b-i)\ell) \mid i \in \{0, 1, \ldots, b\}\}$, identical to the Pareto front of ORZR.

Proposition 3.6. *With the same notation above, the local Pareto optimal solutions of LOZR are the following.*

$$\begin{aligned}\mathcal{L} = \{ 1^i s \mid\ & \forall S_j, j > \lceil \frac{i}{\ell} \rceil : |S|_1 > 1, \\ & S_{\lceil \frac{i}{\ell} \rceil + 1} = 0^\ell \\ & i \in \{0, \ell, 2\ell, \ldots, (b-1)\ell\}\}\end{aligned} \tag{10}$$

where s is an arbitrary sub bit-string of length $n - i$.

Proof. We first show that the solutions $x \in \mathcal{L}$ are non-dominated by its 1-bit neighbours. The first condition $\forall S_j, j > \lfloor \frac{i}{\ell} \rfloor : |S|_1 > 1$ only considers the blocks that do not contain any leading ones. This condition implies that the rest of the blocks cannot be turned into 0^ℓ via flipping an 1 to 0, thus the ZeroRoyalRoad objective cannot be improved. The second condition implies that the first block that does not contain leading ones is an all-zeroes block, and the third condition implies that extending the leading ones will destroy this all-zeroes block, thus the LeadingOnes objective cannot be improved without reducing the ZeroRoyalRoad objective. In both cases, flipping one bit cannot yield any solution that dominates the current one.

Then, we show that the solutions $x \notin \mathcal{P} \cup \mathcal{L}$ are dominated by one of its neighbours. Consider the case $i \notin \{0, \ell, 2\ell, \ldots, (b-1)n\}$, the block containing position i is not completed and extending the leading ones to $i+1$ can improve the LeadingOnes objective without reducing the ZeroRoyalRoad objective, as it has not reached the next block yet. As for the case $i \notin \{0, \ell, 2\ell, \ldots, (b-1)n$, since $x \notin \mathcal{L}$, the conditions $\forall S_j, j > \lceil \frac{i}{\ell} \rceil : |S|_1 > 1$ and $S_{\lceil \frac{i}{\ell} \rceil + 1} = 0^\ell$ cannot hold at the same time. If the former does not hold, then there exists a block that does not contain any leading ones, but there is only one 1. Flipping this 1 to 0 yields a 0^ℓ block and improves the ZeroRoyalRoad objective. If the latter does not hold, extending the leading ones does not break any 0^ℓ and thus it only improves the LeadingOnes objective. For all cases, the solution x can be improved by a single bit flip and this new solution dominates x. □

3.4. OneJump-ZeroRoyalRoad (OJZR)

As illustrated in the main paper, the Pareto set of OJZR can be linear or non-linear, depending on the condition $(n-k) \mod \ell = 0$ where n is the length of the bit-string, k is the jump parameter of OneJump and ell is the block length of ZeroRoyalRoad. These two cases have different Pareto set, and thus we treat them as two propositions as the following.

Proposition 3.7. *Let $n \in \mathbb{N}, 1 < k < \frac{n}{2}$, given a bit-string of length n, partitioned into b disjoint blocks $x = S_1S_2 \ldots S_b$, each with the same length $\ell < k$. For the benchmark OJZR, when $(n-k) \mod \ell = 0$, the Pareto optimal set is*

$$\begin{aligned}&\mathcal{P} = \{ 1^n \} \cup \\ &\{ x \mid |x|_1 \leq n-k,\ |\{S \mid S = 1^\ell\}| + |\{S \mid S = 0^\ell\}| = b\}\end{aligned} \tag{11}$$

Proof. One may notice that this Pareto set is similar to the Pareto set of ORZR (Prop. 3.1). The key difference is that the OneJump function makes those Pareto solutions with $n-k < |x|_1 < n$ no longer Pareto optimal, leaving the rest Pareto optimal solutions in $\mathcal{P}$.

For any solution in $x \in \mathcal{P}$, breaking 0^ℓ blocks decreases the ZeroRoyalRoad objective, completing more 0^ℓ blocks reduces the OneJump objective. Therefore, solutions in $\mathcal{P}$ are non-dominated to each other.

For $x \notin \mathcal{P}$, we show that they are dominated by considering three cases.

Case 1: $n-k < |x|_1 < n$. Here, the OneJump objective drops into the valley. Any solution that flips one 1 to 0 has a strictly larger OneJump objective and possibly larger ZeroRoyalRoad objective (if it completes a 0^ℓ block).

Case 2: $|x|_1 < n-k$. Since this solution x is not in $\mathcal{P}$, we have $|\{S \mid S = 1^\ell\}| + |\{S \mid S = 0^\ell\}| < b$, implying that there exists at least one block that mixes 1s and 0s. Here, turning a 0 1 yields a solution that dominates x.

Case 3: $|x|_1 = n-k$. In this case, only the global optima 1^n is better than x on the first objective. However, like in case 2, at least one block is a mix between 0s and 1s. Since $n \mod \ell = 0$ by definition and $(n-k) \mod \ell = 0$ by assumption, there must be more than one mix block, as otherwise this assumption cannot hold. By swapping the 1s and 0s between different blocks, one can complete at least one 0^ℓ block (increasing the ZeroRoyalRoad) without changing the number of 1s (keeping the OneJump unchanged). This yields a solution that dominates x.

For all cases, $x \notin \mathcal{P}$ are dominated. □

In the case $(n-k) \mod \ell = 0$, the corresponding Pareto front is $\{(n+k, 0)\} \cup \{(i\ell+k, n-i\ell) \mid i \in \{0, 1, \ldots, \lfloor \frac{k}{\ell} \rfloor\}\}$.

Proposition 3.8. *Let $n \in \mathbb{N}, 1 < k < \frac{n}{2}$, given a bit-string of length n, partitioned into b disjoint blocks $x = S_1S_2 \ldots S_b$, each with the same length $\ell < k$. For the benchmark OJZR, when $(n-k) \mod \ell > 0$, the Pareto optimal set is*

$$\begin{aligned}&\mathcal{P} = \{ 1^n \} \cup \\ &\{x \mid |x|_1 < n-k,\ |\{S \mid S = 1^\ell\}| + |\{S \mid S = 0^\ell\}| = b\} \cup \\ &\{x \mid |x|_1 = n-k,\ |\{S \mid S = 0^\ell\}| = \lfloor \frac{k}{\ell} \rfloor\}\end{aligned} \tag{12}$$

Proof. Similar to Prop.3.7, solutions in the first and the second components $\{1^n\} \cup \{x \mid |x|_1 < n-k,\ |\{S \mid S = 1^\ell\}| + |\{S \mid S = 0^\ell\}| = b\}$ are non-dominated to each other. The third component $\{x \mid |x|_1 = n-k, |\{S \mid S = 0^\ell\}| = \lfloor \frac{k}{\ell} \rfloor\}$ corresponds to the objective $(n-k, \lfloor \frac{k}{\ell} \rfloor \ell)$ in the objective space. Solutions in the third component have exactly $\lfloor \frac{k}{\ell} \rfloor$ because there are only $n - |x|_1 = k$ zeroes available. They are not dominated by 1^n as they have a higher ZeroRoyalRoad objective. They are also not dominated by any solution in the third component (those with $|x|_1 < n-k$) as they have higher OneJump objective with $|x|_1 = n-k$. The reason for their occurrence is explained in the main text.

For those $x \notin \mathcal{P}$, similar to Prop. 3.7, we may consider the cases $n-k < |x|_1 < n$, $|x|_1 < n-k$ and $|x|_1 = n-k$. The proof of the first two cases is the same as Prop.3.7. As for the third case, the solutions left from $\mathcal{P}$ are the $\{x \mid |x|_1 = n-k, |x|_1 = n-k, |\{S \mid S = 0^\ell\}| < \lfloor \frac{k}{\ell} \rfloor\}$. It is easy to see that they have the same OneJump objective but a smaller ZeroRoyalRoad objective compared to the third component of $\mathcal{P}$. Therefore, $x \notin \mathcal{P}$ are all dominated. ☐

In the case $(n-k) \mod \ell > 0$, the corresponding Pareto front is $\{(n+k, 0)\} \cup \{(n-k, \lfloor \frac{k}{\ell} \rfloor \ell)\} \cup \{(i\ell + k, n - i\ell) \mid i \in \{0, 1, \dots, \lfloor \frac{k}{\ell} \rfloor\}\}$.

Unlike the Pareto set, the Pareto local optimal set for OJZR has only one case, as it is mainly attributed to the OneJump objective. They are the solutions right below the Pareto optimal solution at $f_1(x) = n-k$, as shown in the following.

Proposition 3.9. *With the same notation above, the local Pareto optimal solutions of OJZR are the following.*

$$\mathcal{L} = \{x \mid |x|_1 = n-k, |\{S \mid S = 0^\ell\}| < \lfloor \frac{k}{\ell} \rfloor\} \tag{13}$$

Proof. Firstly, we show every solution $x \in \mathcal{L}$ is not dominated by any of its neighbours. If we flip any 1 to 0, it simply reduces the OneJump objective. If we flip any 0 to 1, it reduces the OneJump objective (as $|x|_1 > n-k$ now lies in the "valley"). Since no neighbour can yield improvement, $x \in \mathcal{L}$ are not dominated by any of its neighbours.

Then, we show that $x \notin \mathcal{P} \cup \mathcal{L}$ are dominated by at least one of its neighbours. For solutions with $|x|_1 < n-k$, since it is not in $\mathcal{P}$, there exists at least one mix block, meaning that there are mixes of 1s and 0s. Flipping 0 to 1 in such block can improve the OneJump objective without changing the ZeroRoyalRoad objective. For solutions with $n-k < |x|_1 < n$, flipping any 1 to 0 increase the OneJump objective (escaping from the "valley") and possibly increases the second objective. For all cases, $x \notin \mathcal{P} \cup \mathcal{L}$ have at least one neighbour that dominates itself. ☐

3.5. OneMax-TrailingZeroes (OMTZ)

Now we move to the benchmarks involving OneMax.

Proposition 3.10. *Let $n \in \mathbb{N}$, given a bit-string of length n, for the benchmark OMTZ, the Pareto optimal set is*

$$\mathcal{P} = \{0^n\} \cup \{1^i 0^{n-i} \mid i \in \{0, 1, \dots, n\}\} \tag{14}$$

Proof. The TrailingZeroes objective enforces the structure of the Pareto set to be $1^i 0^{n-i}$. If the "leading ones" component does not follow this structure (i.e., having extra 0s on the left), they are dominated by the solution who follow this structure. Therefore, the Pareto set of OMTZ is the same as the LeadingOnes-TrailingZeroes benchmark. ☐

The corresponding Pareto front is the same as the LeadingOnes-TrailingZeroes as well.

3.6. OneMax-ZeroJump (OMZJ)

Proposition 3.11. *Let $n \in \mathbb{N}, 1 < k < \frac{n}{2}$, given a bit-string of length n, for the benchmark OMZJ, the Pareto optimal set is*

$$\mathcal{P} = \{0^n\} \cup \{x \mid |x|_0 \le n-k\} \tag{15}$$

Proof. Since the Jump function resembles an OneMax function with a valley, the OneJump-ZeroJump function resembles an OneMinMax with some solutions no longer being Pareto optimal as they are in one of the two valleys. Similarly, for OMZJ, it resembles an OneMinMax with only one valley $n-k < |x|_0 < n$. The rest of the solutions, denoted by $\mathcal{P}$ here, are Pareto optimal solutions of OMZJ. ☐

The corresponding Pareto front of OMZJ is $\{0^n\} \cup \{(i, n+k-i) \mid i \in \{k, k+1, \dots, n\}\}$.

3.7. OneMax-ZeroRoyalRoad (OMZR)

Proposition 3.12. *Let $n, b, \ell \in \mathbb{N}, n = b \cdot \ell, b > 1$, given a bit-string of length n, partitioned into b disjoint blocks $x = S_1 S_2 \dots S_b$, each with the same length ℓ. For the benchmark OMZR, the Pareto optimal set is*

$$\mathcal{P} = \{x \mid |\{S \mid S = 1^\ell\}| + |\{S \mid S = 0^\ell\}| = b\} \tag{16}$$

Proof. The ZeroRoyalRoad objective enforces the block structure for the OneMax objective (block of 1s) for the Pareto optimal solution. If the 1s in the bit-string does not follow the block structure (i.e., break 0^ℓ blocks or having extra 0s), they are dominated by the solutions who follow the structure. Therefore, the Pareto set of OMZR is the same as the OneRoyalRoad-ZeroRoyalRoad. ☐

The corresponding Pareto front is the same as the OneRoyalRoad-ZeroRoyalRoad as well.

4. Ratio of Pareto optimal solutions

4.1. Ratio of Pareto optimal solutions of OJZJ

For a OneJump-ZeroJump (OJZJ) problem, the ratio of Pareto optimal solutions is $R(n,k) = \frac{2^n - 2\sum_{s=n-k+1}^{n-1}\binom{n}{s}}{2^n}$. The following propositions prove that (1) for a large k near

$\frac{n}{2}$, this ratio can be very low; (2) For a smaller $k < \frac{n}{\ln n}$, this ratio becomes no less than 0.5.

Proposition 4.1. *For the OJZJ problem with a sufficiently large $n \in \mathbb{Z}^+$ and a large $k = \lfloor \frac{n}{2} \rfloor - 1$, the ratio of Pareto optimal solutions of OJZJ converges to 0 as $n \to \infty$:*

$$\lim_{n\to\infty} R(n,k) = \lim_{n\to\infty} \frac{2^n - 2\sum_{s=n-k+1}^{n-1}\binom{n}{s}}{2^n} = 0 \quad (17)$$

Proof. We begin by simplifying the ratio $R(n,k)$.

$$\begin{aligned} R(n,k) &= \frac{2^n - 2\sum_{s=n-k+1}^{n-1}\binom{n}{s}}{2^n} \\ &= 1 - \frac{2\sum_{s=n-k+1}^{n-1}\binom{n}{s}}{2^n} = 1 - \frac{\sum_{s=1}^{k-1}\binom{n}{s}}{2^{n-1}} \end{aligned}$$

For simplicity, we choose $k = \lfloor \frac{n}{2} \rfloor - 1$ as a representative large value near $\frac{n}{2}$. We analyse two cases based on the parity of n.

Case 1: n is even. Let $n = 2m$, where m is a positive integer, we have $k = m - 1$ and

$$\sum_{s=1}^{k-1}\binom{n}{s} = \sum_{s=1}^{m-2}\binom{2m}{s} = 2^{2m-1} - \frac{1}{2}\binom{2m}{m} - \binom{2m}{m-1} - 1$$

Thus, the ratio becomes:

$$\begin{aligned} R(2m, m-1) &= 1 - \frac{2^{2m-1} - \frac{1}{2}\binom{2m}{m} - \binom{2m}{m-1} - 1}{2^{n-1}} \\ &= \frac{\frac{1}{2}\binom{2m}{m} + \binom{2m}{m-1} + 1}{2^{2m-1}} \end{aligned}$$

Using Stirling's approximation [2], $\binom{2m}{m} \approx \frac{4^m}{\sqrt{\pi m}}$, Therefore, for large m, $\frac{m}{m+1} \approx 1$, thus we have:

$$R(2m, m-1) \approx \frac{(\frac{1}{2} + \frac{m}{m+1})\frac{4^m}{\sqrt{\pi m}}}{\frac{1}{2}\cdot 4^m} \approx \frac{3\sqrt{2}}{\sqrt{\pi n}}$$

Case 2: n is odd. Let $n = 2m+1$, where m is a positive integer. Then we have $k = m - 1$ and

$$\begin{aligned} \sum_{s=1}^{k-1}\binom{n}{s} &= \sum_{s=1}^{m-2}\binom{2m+1}{s} \\ &= 2^{2m} - \binom{2m+1}{m} - \binom{2m+1}{m-1} - 1 \end{aligned}$$

Thus, the ratio becomes:

$$\begin{aligned} R(2m+1, m-1) &= 1 - \frac{2^{2m} - \binom{2m+1}{m} - \binom{2m+1}{m-1} - 1}{2^{2m}} \\ &= \frac{\binom{2m+1}{m} + \binom{2m+1}{m-1} + 1}{2^{2m}} \end{aligned}$$

Using Stirling's approximation again, we have:

$$R(2m+1, m-1) \approx \frac{(\frac{2m+1}{m+1} + \frac{m}{m+2}\cdot\frac{2m+1}{m+1})\frac{4^m}{\sqrt{\pi m}}}{\frac{1}{2}\cdot 4^m} \approx \frac{4\sqrt{2}}{\sqrt{\pi n}}$$

Finally, both cases converge towards 0 as $n \to \infty$. □

Proposition 4.2. *For the OJZJ problem with $n \in \mathbb{Z}^+$, if n is sufficiently large and the jump parameter $k \leq \frac{n}{2} - \sqrt{\frac{n\ln 4}{2}}$, we have*

$$R(n,k) \geq 0.5 \quad (18)$$

Proof. $R(n,k) \geq 0.5$ implies:

$$R(n,k) = 1 - \frac{\sum_{s=1}^{k-1}\binom{n}{s}}{2^{n-1}} = 1 - 2\sum_{s=0}^{k-1}\frac{\binom{n}{s}}{2^n} \geq 0.5$$

The factor 2^n is the total number of the bitstrings. Treat this as a binomial distribution:

$$X \sim \text{Bin}(n, \frac{1}{2}), \qquad \Pr[X = s] = \frac{\binom{n}{s}}{2^n},$$

We have:

$$\begin{aligned} R(n,k) &= 1 - 2\sum_{s=0}^{k-1}\frac{\binom{n}{s}}{2^n} \\ &= 1 - 2\Pr[X \leq k-1] \geq 0.5 \\ \Pr[X \leq k-1] &\leq \frac{1}{4} \end{aligned}$$

Then, with the Chernoff bound [3] (Theorem 4.2)

$$\Pr[X \leq k-1] \leq \exp(-\frac{\delta^2\mu}{2})$$

Where $\mu = \frac{n}{2}$. Rewrite $k = \frac{n}{2} - t$ and $k \leq (1-\delta)\mu$, we have $\delta = \frac{2t}{n}$ and then:

$$\begin{aligned} \Pr[X \leq k-1] \leq \exp(-\frac{(\frac{2t}{n})^2\frac{n}{2}}{2}) = \exp(-\frac{2t^2}{n}) &= \frac{1}{4} \\ \frac{2t^2}{n} &= \ln 4 \\ t &= \sqrt{\frac{n\ln 4}{2}}. \end{aligned}$$

And finally, we obtained the bound for k:

$$k \leq \frac{n}{2} - t = \frac{n}{2} - \sqrt{\frac{n\ln 4}{2}} \quad (19)$$

Therefore, choosing any $k \leq \frac{n}{2} - \sqrt{\frac{n\ln 4}{2}}$ makes the $\Pr[X \leq k-1]$ at most $\frac{1}{4}$, and therefore $R(n,k) \geq \frac{1}{2}$ □

4.2. Ratio of Pareto Optimal Solutions of OJZR

When $(n-k) \mod \ell \neq 0$, the Pareto front shape of OJZR is concave and there is one Pareto optimal solution that has much more corresponding bit-strings in the decision space than the others. In this case, we show that the ratio of Pareto optimal solutions is still low in the following proposition.

Firstly, we define this ratio for OJZR. As stated in Prop. 3.8, the Pareto set of OJZR is $\mathcal{P} = \{1^n\} \cup \{\, x \mid |x|_1 < n-k,\ |\{S \mid S = 1^\ell\}| + |\{S \mid S = 0^\ell\}| = b\} \cup \{\, x \mid |x|_1 = n-k, |\{S \mid S = 0^\ell\}| = \lfloor \frac{k}{\ell} \rfloor\}$. The first component has only one corresponding solution. The second component corresponds to selecting i 0^ℓ blocks out of b blocks for each $i \in \{\lceil \frac{k}{\ell} \rceil, 1, \ldots, m\}$. We start counting blocks from $lceil \frac{k}{\ell} \rceil$ as a lower number of 0^ℓ implies more 1s which already dropped into the "valley" of the OneJump function. The third component corresponds to the special solutions with the objective $(n-k, \lfloor \frac{k}{\ell} \rfloor \ell)$ that create the concave region in the Pareto front. It has $\lfloor \frac{k}{\ell} \rfloor$ completed 0^ℓ blocks out of b blocks. In the rest of the $n - \lfloor \frac{k}{\ell} \rfloor \ell$ bits, it has exactly $n-k$ 1s to be placed, yielding a total of $\binom{b}{\lfloor \frac{k}{\ell} \rfloor} \times \binom{n - \lfloor \frac{k}{\ell} \rfloor \ell}{n-k}$ possibilities. Therefore, the ratio is given by

$$R(n,k) = \frac{1 + \sum_{i=\lceil \frac{k}{\ell} \rceil}^{b} \binom{b}{i} + \binom{b}{\lfloor \frac{k}{\ell} \rfloor} \times \binom{n - \lfloor \frac{k}{\ell} \rfloor \ell}{n-k}}{2^n}. \quad (20)$$

Proposition 4.3. *For OJZR with problem size $n \in \mathbb{Z}^+$, jump parameter $1 \le k < \frac{n}{2}$, and block length ℓ with $2 \le \ell < n$, $n \mod \ell = 0$ and $(n-k) \mod \ell \neq 0$, let $b = \frac{n}{\ell}$, the ratio of Pareto optimal solutions $R(n,k)$ converges to 0 as $n \to \infty$.*

Proof. The numerator of $R(n,k)$ comprises three terms. The first term is the constant 1. The second term $\sum_{i=\lceil \frac{k}{\ell} \rceil}^{b} \binom{b}{i}$ is maximised when the lower limit $\lceil \frac{k}{\ell} \rceil$ is as small as possible (i.e., 1). Thus:

$$\sum_{i=\lceil \frac{k}{\ell} \rceil}^{m} \binom{b}{i} \le \sum_{i=1}^{b} \binom{b}{i} = 2^b - 1$$

The third term is a product of binomial coefficients:

$$\binom{b}{\lfloor \frac{k}{\ell} \rfloor} \times \binom{n - \lfloor \frac{k}{\ell} \rfloor \ell}{n-k}$$

For $\binom{b}{\lfloor \frac{k}{\ell} \rfloor}$, to establish a general upper bound, we use:

$$\binom{b}{\lfloor \frac{k}{\ell} \rfloor} \le 2^b$$

For $\binom{n - \lfloor \frac{k}{\ell} \rfloor \ell}{n-k}$, the maximum occurs when $n - \lfloor \frac{k}{\ell} \rfloor \ell$ is the largest, given:

$$\binom{n - \lfloor \frac{k}{\ell} \rfloor \ell}{n-k} \le 2^{n - \lfloor \frac{k}{\ell} \rfloor \ell},$$

We have:

$$\binom{b}{\lfloor \frac{k}{\ell} \rfloor} \times \binom{n - \lfloor \frac{k}{\ell} \rfloor \ell}{n-k} \le 2^b \times 2^{n - \lfloor \frac{k}{\ell} \rfloor \ell} = 2^{b + n - \lfloor \frac{k}{\ell} \rfloor \ell}.$$

Combining the terms, the numerator is bounded by:

$$\text{Numerator} \le 1 + 2^b + 2^{b + n - \lfloor \frac{k}{\ell} \rfloor \ell}$$

Putting it back to the ratio and substituting $b = \frac{n}{\ell}$:

$$R(n,k) \le 2^{-n} + 2^{-n(1 - \frac{1}{\ell})} + 2^{\frac{n}{\ell} - \lfloor \frac{k}{\ell} \rfloor \ell}.$$

Since $k < \frac{n}{2}$, the power of the third term

$$\frac{n}{\ell} - \lfloor \frac{k}{\ell} \rfloor \ell < \frac{n}{\ell} - (\frac{n}{2} + 1) = -1 + n(\frac{1}{\ell} - \frac{1}{2}).$$

The first two terms are negligible as $n \to \infty$. Therefore, the bound is

$$R(n,k) \le 2^{-1 + n(\frac{1}{\ell} - \frac{1}{2})} \quad (21)$$

For $\ell > 2$, that converges towards 0 as $n \to \infty$.

However, for the edge case $\ell = 2$, this becomes 0.5.

Revisiting the ratio for case $\ell = 2$. Since $k < n/2$ and $(n-k) \mod 2 \neq 0$ forces k to be odd, we let $k = 2p+1$. Then, the third term of the ratio:

$$\begin{aligned} &\binom{b}{\lfloor \frac{k}{\ell} \rfloor} \times \binom{n - \lfloor \frac{k}{\ell} \rfloor \ell}{n-k} \\ &= \binom{n/2}{p} \binom{n-2p}{n-2p-1} \\ &= \binom{n/2}{p} \cdot (n-2p) = (n-k) \binom{n/2}{k/2} \end{aligned}$$

Since $\binom{n/2}{k/2} < \sum_{i=0}^{n/2} \binom{n/2}{i} = 2^{n/2}$, and $(n-k)2^{n/2}$ still grows slower than the denominator 2^n, thus the third term also converges towards 0 as $n \to \infty$.

Therefore, for all the cases, the ratio converges to 0 as $n \to \infty$.

□

References

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[2] W. Feller, *An Introduction to Probability Theory and Its Applications, Volume 1*, 3rd ed. New York, NY, USA: John Wiley & Sons, 1971, see Section 2.4 for an introduction to Stirling's approximation and its applications to binomial coefficients.

[3] M. Mitzenmacher and E. Upfal, *Probability and Computing: Randomized Algorithms and Probabilistic Analysis*. Cambridge: Cambridge University Press, 2005.