HW3 Sols:

A.) Let $f(x) = x^T M x$

Let $\hat{x}$ be s.t.

$$\hat{x}^T M \hat{x} = 0.$$

Then $f(\hat{x}) \leq f(x) \quad \forall x$

because M is PSD.

Hence, $\hat{x}$ is a minimizer of

$\min_{x \in \mathbb{R}^n} f(x)$. Hence,

$$\nabla_x f(\hat{x}) = 0 \Rightarrow (M + M^T)\hat{x} = 0.$$

B.) Let z solve $LCP(M, q)$,

d s.t.
$$z + d \geq 0$$
$$M(z+d) + q \geq 0$$
$$q^T d = 0$$
$$(M + M^T) d = 0$$

$\Rightarrow$

$z+d$ is a solution if

$M(z+d)+q \geq 0$ $\qquad$ $\underbrace{(z+d)^T\left(M(z+d)+q\right)}=0$

$z+d \geq 0$

By assumption.

$$z^T(Mz+q) + d^T(Mz+q) + z^TMd + d^TMd$$

$$= 0 + d^TMz + d^Tq + z^TMd + \tfrac{1}{2}d^T(M+M^T)d$$

$$= 0 + z^TM^Td + 0 + z^TMd + 0$$

$$= z^T(M+M^T)d = 0 \checkmark$$

solution

Now: Let z and $z+d$ be solutions to $LCP(M,q)$

Want to show

i) $z+d \geq 0$

ii) $M(z+d)+q \geq 0$

iii) $q^T d = 0$

iv.) $(M+M^T)d = 0$

i) and ii) true by $z+d$ solution.

Also: $(z+d)^T\left(M(z+d)+q\right) = 0$

Now: Let $I_1 : \{i : z_i > 0,\ (Mz+q)_i = 0\}$

$I_2 : \{i : z_i = 0,\ (Mz+q)_i = 0\}$

$I_3 : \{i : z_i = 0,\ (Mz+q)_i > 0\}$

$z^T(Mz+q) + d^T(Mz+q) + z^T M d + d^T M d = 0$

$$\underbrace{z^T(Mz+q)}_{0} + \underbrace{d^T(Mz+q)}_{\substack{I_1:\ =0 \\ I_2:\ =0 \\ I_3:\ \geq 0}} + \underbrace{z^TMd}_{\substack{I_1:\ \geq 0 \\ I_2:\ =0 \\ I_3\ =0}} + \underbrace{d^TMd}_{\substack{\geq 0 \\ (M\ \text{PSD})}} = 0$$

$\Rightarrow$ each term $= 0$ if

is possible.

	d_i	$(Mz+q)_i$	z_i	$(Md)_i$
I_1	free	$=0$	>0	≥ 0
I_2	≥ 0	$=0$	$=0$	≥ 0
I_3	≥ 0	>0	$=0$	free

else, z and $z+d$ can't both be solutions.

$\hookrightarrow$ $d^TMd = 0 \Rightarrow (M+M^T)d = 0$ (M PSD)

$\Rightarrow$ iv) Satisfied.

$$d^T(Mz+q) + z^T M d = 0$$

$$= d^T M z + d^T q + z^T M d$$

$$= z^T M^T d + z^T M d + d^T q$$

$$= z^T (M + M^T) d + d^T q$$

$$= d^T q = 0$$

(iii) Satisfied.