HW3 Sols:

A.) Let $f(x) = x^T M x$

Let $\hat{x}$ be s.t.

$$\hat{x}^T M \hat{x} = 0.$$

Then $f(\hat{x}) \leq f(x) \quad \forall x$

because M is PSD.

Hence, $\hat{x}$ is a minimizer of

$\min_{x \in \mathbb{R}^n} f(x)$. Hence,

$$\nabla_x f(\hat{x}) = 0 \Rightarrow (M + M^T)\hat{x} = 0.$$

B.) Let z solve $LCP(M, q)$,

d s.t.

$$z + d \geq 0$$
$$M(z+d) + q \geq 0$$
$$q^T d = 0$$
$$(M + M^T) d = 0$$

$\Rightarrow$

$z+d$ is a solution if

$$M(z+d)+q \geq 0 \qquad \underbrace{(z+d)^T\left(M(z+d)+q\right)}=0$$

$$z+d \geq 0$$

By assumption.

$$z^T(Mz+q) + d^T(Mz+q) + z^TMd + d^TMd$$

$$= 0 + d^TMz + d^Tq + z^TMd + \tfrac{1}{2}d^T(M+M^T)d$$

$$= 0 + z^TM^Td + 0 + z^TMd + 0$$

$$= z^T(M+M^T)d = 0 \quad \checkmark$$

solution

Now: Let z and $z+d$ be solutions to $LCP(M,q)$

Want to show

i) $z+d \geq 0$

ii) $M(z+d)+q \geq 0$

iii) $q^T d = 0$

iv.) $(M+M^T)d = 0$

i) and ii) true by $z+d$ solution.

Also: $(z+d)^T (M(z+d)+q) = 0$

Now: Let $I_1: \{i: z_i > 0, (Mz+q)_i = 0\}$

$I_2: \{i: z_i = 0, (Mz+q)_i = 0\}$

$I_3: \{i: z_i = 0, (Mz+q)_i > 0\}$

$$z^T(Mz+q) + d^T(Mz+q) + z^TMd + d^TMd = 0$$

$$\underbrace{z^T(Mz+q)}_{0} + \underbrace{d^T(Mz+q)}_{} + \underbrace{z^TMd}_{} + \underbrace{d^TMd}_{} = 0$$

I_1: $=0$ | I_1: ≥ 0 | ≥ 0
I_2: $=0$ | I_2: $=0$ | (M PSD)
I_3: ≥ 0 | $I_3 = 0$

$\Rightarrow$ each term $= 0$ if is possible.

	d_i	$(Mz+q)_i$	z_i	$(Md)_i$
I_1	free	$=0$	>0	≥ 0
I_2	≥ 0	$=0$	$=0$	≥ 0
I_3	≥ 0	>0	$=0$	free

else, z and $z+d$ can't both be solutions.

$d^TMd = 0 \Rightarrow (M+M^T)d = 0$ (M PSD)

$\Rightarrow$ iv) Satisfied.

$$d^T(Mz+q) + z^T M d = 0$$

$$= d^T M z + d^T q + z^T M d$$

$$= z^T M^T d + z^T M d + d^T q$$

$$= z^T (M + M^T) d + d^T q$$

$$= d^T q = 0$$

(iii) Satisfied.

c)

$$\Sigma x - 1 = 0$$

$$\min_{x} \quad x^T c$$

$$\text{s.t.} \quad \begin{bmatrix} 1 & \cdots & 1 \\ -1 & \cdots & -1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \geq -$$

$$x \geq 0$$

$$\Downarrow$$

x^* is sol iff

$$\exists \; \lambda \in \mathbb{R}^2 \qquad c - \begin{bmatrix} 1 & -1 \\ \vdots & \vdots \\ 1 & -1 \end{bmatrix} \lambda \geq 0 \perp x \geq 0$$

$$\begin{bmatrix} 1 & \cdots & 1 \\ -1 & \cdots & -1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \geq 0 \perp \lambda \geq 0$$

Let $x^* = e_j$

$$\lambda_1 = \operatorname{Max}(c_j, 0) \geq 0$$

$$\lambda_2 = -\operatorname{Min}(c_j, 0) \geq 0$$

$$\lambda_1 - \lambda_2 = c.$$

x^*, λ Satisfy $\Rightarrow$ $x^* = e_j$ is sol