A practical guide for MAE 511

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1 Frames

Define a frame by a point and three orthogonal unit vectors. It is assumed there is an inertial frame $\bar{O}$ and a relative frame $\bar{B}$:

$$\vec{O} = \{O, \vec{i}_{\bar{O}}, \vec{j}_{\bar{O}}, \vec{k}_{\bar{O}}\} \tag{1}$$

$$\vec{B} = \{B, \vec{i}_{\bar{B}}, \vec{j}_{\bar{B}}, \vec{k}_{\bar{B}}\} \tag{2}$$

2 Rotation matrices

The rotation matrix ${}^{\bar{B}}[C]^{\bar{O}}$, which rotates vectors from the $\bar{O}$ to the $\bar{B}$ frame, is termed the Direction Cosine Matrix:

$${}^{\bar{B}}[C]^{\bar{O}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \tag{3}$$

Using three Euler angles θ, ϕ, ψ, each rotation about a single axis is defined as follows:

$$[R_X(\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & sin(\theta) \\ 0 & -sin(\theta) & cos(\theta) \end{bmatrix} \tag{4}$$

$$[R_Y(\phi)] = \begin{bmatrix} cos(\phi) & 0 & -sin(\phi) \\ 0 & 1 & 0 \\ sin(\phi) & 0 & cos(\phi) \end{bmatrix} \tag{5}$$

$$[R_Z(\psi)] = \begin{bmatrix} cos(\psi) & sin(\psi) & 0 \\ -sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

The Direction Cosine Matrix is calculated by choosing a sequence of the three angles above. An example of several common sequences are defined as follows:

2.1 XYZ sequence

$$\begin{aligned} {}^{\bar{B}}[C]^{\bar{O}} &= [R_X(\theta)][R_Y(\phi)][R_Z(\psi)] \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & sin(\theta) \\ 0 & -sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} cos(\phi) & 0 & -sin(\phi) \\ 0 & 1 & 0 \\ sin(\phi) & 0 & cos(\phi) \end{bmatrix} \begin{bmatrix} cos(\psi) & sin(\psi) & 0 \\ -sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{7}$$