A fundamental solution $\Phi(x)$ to the Laplace equation is defined by the Poisson equation with $\delta(x)$ at the right hand side:

$$\Delta\Phi(x) = \delta(x).$$

One can prove that

$$\Phi(x) = \frac{1}{2\pi}\ln|x|.$$

We want to model the electric potential $u(x) + w(x)$ and currents $I(x) = \sigma(x)\nabla(u(x) + w(x))$ flowing through a conductive medium, and induced by the currents W_j injected through points $y^{(j)}$, $j = 1, ..., n$. We request the currents to add up to 0. Then the sum of potentials $u(x)$ and $w(x)$ will decay at infinity. Potential $w(x)$ solves this problem if the conductivity $\sigma(x) \equiv \text{const} = \sigma_0$ in $\mathbb{R}^2$; it is given by the sum

$$w(x) = -\frac{1}{\sigma_0}\sum_{j=1}^{n} W_j\Phi(x - y^{(j)}), \qquad \sum_{j=1}^{n} W_j = 0.$$

Potential $u(x)$ can be viewed as a perturbation of $w(x)$ arising when $\sigma(x)$ is not constant anymore. We will assume that the support of $\sigma(x) - \sigma_0$ is contained within a simply-connected bounded region Ω not containing any of $y^{(j)}$'s. Since the divergence of total current $\sigma(x)\nabla(u(x) + w(x))$ is zero away from the injection points $y^{(j)}$'s, we obtain the following equation on $u(x)$

$$\nabla \cdot \sigma(x)\nabla u(x) = -\nabla \cdot \sigma(x)\nabla w(x), \qquad x \in \mathbb{R}^2 \backslash \bigcup_{j=1}^{n} y^{(j)}$$

with $u(x)$ vanishing at infinity (since the total potential $u + w$ should vanish, and $w(x)$ defined above also vanishes).

Due to one of the properties of harmonic functions, the singular poitns $y^{(j)}$'s can be included in the domain of definition of $u(x)$. Then, the above equations can be re-written as the system

$$\begin{cases} \nabla \cdot \sigma(x)\nabla u(x) = -\nabla\sigma(x) \cdot \nabla w(x), \qquad x \in \Omega, \\ \nabla \cdot \sigma(x)\nabla u(x) = 0, \qquad x \in \mathbb{R}^2\backslash\Omega, \\ u(x) = \mathcal{O}(1/|x|) \text{ as } x \to \infty. \end{cases} \tag{1}$$

The system above can be written as

$$\begin{cases} \sigma(x)\Delta u(x) + \nabla\sigma(x) \cdot \nabla u(x) = -\nabla\sigma(x) \cdot \nabla w(x), \qquad x \in \Omega, \\ \Delta u(x) = 0, \qquad x \in \mathbb{R}^2\backslash\Omega, \\ u(x) = \mathcal{O}(1/|x|) \text{ as } x \to \infty. \end{cases}$$

and, further,

$$\begin{cases} \Delta u(x) + \nabla\ln\sigma(x) \cdot \nabla u(x) = -\nabla\ln\sigma(x) \cdot \nabla w(x), \qquad x \in \Omega, \\ \Delta u(x) = 0, \qquad x \in \mathbb{R}^2\backslash\Omega, \\ u(x) = \mathcal{O}(1/|x|) \text{ as } x \to \infty. \end{cases}$$

Reduction to the Fredholm equation.

Given an arbitrary, smooth, compactly supported in Ω function $\psi(x)$ let us define operator Δ^{-1} as the convolution

$$v(x) = [\Delta^{-1}\psi](x) \equiv [\psi * \Phi](x) = \int_{\Omega} \psi(y)\Phi(x - y)dy, \qquad x \in \mathbb{R}^2. \tag{2}$$

Then,

$$\Delta v = \psi,$$

and, if $\int_\Omega \psi(x)dx = 0$, potential $v(x)$ decays at infinity. One can show that, if ψ does not integrate to 0, then

$$v(x) = c_0 \Phi(x) + v_1(x),$$

where $v_1(x)$ does decay at infinity, and

$$c_0 = \int\limits_\Omega \psi(x)dx.$$

Let us denote by $\varphi(x)$ the Laplacian of $u(x)$. Then our system of equations implies

$$\begin{aligned} &\varphi(x) = -\nabla \ln \sigma(x) \cdot \nabla u(x) - \nabla \ln \sigma(x) \cdot \nabla w(x), \qquad x \in \Omega, \\ &\varphi(x) = 0, \qquad x \in \mathbb{R}^2 \backslash \Omega, \end{aligned} \tag{3}$$

and, if $\varphi(x)$ is known, potential $u(x)$ can be found as a solution of the Poisson equation in the plane

$$\begin{aligned} \Delta u(x) &= \varphi(x), \qquad x \in \mathbb{R}^2, \\ u(x) &= [\Delta^{-1}\varphi](x) \end{aligned}$$

Again, the decay $u(x) = \mathcal{O}(1/|x|)$ as $x \to \infty$ is equivalent to the condition

$$\int\limits_\Omega \varphi(x)dx = 0. \tag{4}$$

Now we can re-write equation (3) as follows:

$$\varphi(x) + \nabla \ln \sigma(x) \cdot \nabla [\Delta^{-1}\varphi](x) = -\nabla \ln \sigma(x) \cdot \nabla w(x), \qquad x \in \Omega, \tag{5}$$

subject to condition (4). We show below that the condition (4) follows from (5), and therefore solving the latter equation alone yields $\varphi(x)$ satisfying both equations. Then, the unique solution $u(x)$ of system (1) is obtained as the convolution $\varphi * \Phi$.

Theorem. Suppose function $\varphi(x)$ supported in Ω satisfies the equation (5). Then $\varphi(x)$ satisfies condition (4).

Proof. Equation (5) implies:

$$\sigma(x)\varphi(x) + \nabla\sigma(x) \cdot \nabla[\Delta^{-1}\varphi](x) = -\nabla\sigma(x) \cdot \nabla w(x), \qquad x \in \Omega.$$

Call $v(x) \equiv [\Delta^{-1}\varphi](x)$ (as defined by (2)) and observe $\varphi = \Delta v$. One then obtains:

$$\sigma(x)\Delta v(x) + \nabla\sigma(x) \cdot \nabla v(x) = -\nabla\sigma(x) \cdot \nabla w(x), \qquad x \in \Omega,$$

and further

$$\nabla \cdot \sigma(x)\nabla v(x) = -\nabla\sigma(x) \cdot \nabla w(x), \qquad x \in \Omega.$$

Now, compute the integral of φ over Ω utilizing the divergence theorem and using $\sigma(x) = \sigma_0$ on the boundary $\partial\Omega$ of Ω:

$$\begin{aligned}
\int_{\Omega} \varphi(x)dx &= \int_{\Omega} \nabla \cdot \nabla v(x)dx = \int_{\partial\Omega} n(z) \cdot \nabla v(z)dz = \frac{1}{\sigma_0} \int_{\partial\Omega} n(z) \cdot \sigma_0 \nabla v(z)dz \\
&= \frac{1}{\sigma_0} \int_{\partial\Omega} n(z) \cdot \sigma(z) \nabla v(z)dz = \frac{1}{\sigma_0} \int_{\Omega} \nabla \cdot \sigma(x) \nabla v(x)dx \\
&= -\frac{1}{\sigma_0} \int_{\Omega} \nabla \cdot \sigma(x) \nabla w(x) = -\frac{1}{\sigma_0} \int_{\partial\Omega} n(z) \cdot \sigma_0 \nabla w(z)dz \\
&= -\int_{\partial\Omega} n(z) \cdot \nabla w(z)dz = -\int_{\Omega} \Delta w(x)dx = 0,
\end{aligned}$$

where $n(z)$ is the exterior normal to $\partial\Omega$ at $z \in \partial\Omega$ and dz is the area element on $\partial\Omega$. **End_proof.**

Let us simplify $\nabla[\Delta^{-1}\varphi]$. Observe

$$\begin{aligned}
\frac{\partial}{\partial x_k}[\Delta^{-1}\varphi](x) &= \frac{\partial}{\partial x_k} \int_{\Omega} \varphi(y)\Phi(x-y)dy = \int_{\Omega} \varphi(y) \frac{\partial}{\partial x_k} \Phi(x-y)dy \\
&= \frac{1}{2\pi} \int_{\Omega} \varphi(y) \frac{\partial}{\partial x_k} \ln|x-y|dy = \frac{1}{2\pi} \int_{\Omega} \varphi(y) \frac{x_k - y_k}{|x-y|^2} dy.
\end{aligned}$$

Our equation now becomes

$$\varphi(x) + \frac{1}{2\pi} \left[\frac{\partial \ln\sigma(x)}{\partial x_1} \int_{\Omega} \varphi(y) \frac{x_1 - y_1}{|x-y|^2} dy + \frac{\partial \ln\sigma(x)}{\partial x_2} \int_{\Omega} \varphi(y) \frac{x_2 - y_2}{|x-y|^2} dy \right] = -\nabla \ln\sigma(x) \cdot \nabla w(x),$$

at each $x \in \Omega$. The integrals in the above equation are convolutions with $\frac{x_1}{|x|^2}$ and $\frac{x_2}{|x|^2}$, or $\frac{\cos\theta}{r}$ and $\frac{\sin\theta}{r}$ if written in polar coordinates. The expression in the brackets is an integral operator with weakly singular kernels, and the above equation is the Fredholm integral equation of the second kind, ideally suited for such solvers as GMRES. One only needs to write routines for computing convolutions with the above kernels.

$$\begin{aligned}
\frac{\partial}{\partial x_1} \int_{\Omega} \varphi(y) \ln|x-y|dy &= \int_{\Omega} \varphi(y) \frac{\partial}{\partial x_1} \ln\sqrt{(x-y)\cdot(x-y)}dy \\
&= \int_{\Omega} \varphi(y) \frac{x_1 - y_1}{|x-y|^2} dy
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial x_1} \int_{\Omega} \varphi(y) \ln|x-y|dy &= \int_{\Omega} \varphi(y) \frac{\partial}{\partial x_1} \ln|x-y|dy \\
&= -\int_{\Omega} \varphi(y) \frac{\partial}{\partial y_1} \ln|x-y|dy \\
&= \int_{\Omega} \ln|x-y| \frac{\partial}{\partial y_1} \varphi(y)dy
\end{aligned}$$

So

$$\varphi(x)+\frac{1}{2\pi}\left[\frac{\partial \ln\sigma(x)}{\partial x_1}\int\limits_{\Omega}\ln|x-y|\frac{\partial}{\partial y_1}\varphi(y)dy+\frac{\partial \ln\sigma(x)}{\partial x_2}\int\limits_{\Omega}\ln|x-y|\frac{\partial}{\partial y_2}\varphi(y)dy\right]=-\nabla\ln\sigma(x)\cdot\nabla w(x),$$