Simulating the Forward Problem of Magneto-Acousto-Electric Tomography

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Magneto-Acousto-Electric-Tomography (MAET)- Basic Set-up

- Biological tissue of unknown conductivity $\sigma(x)$ is placed in a conductive medium, and subjected to a magnetic field B.
- Acoustic waves with velocity $V(x, t)$ are generated with a transducer, causing movement in the tissue.
- A pair of electrodes measures the change in potential at the boundary.
- The goal is to recover $\sigma(x)$, forming a conductivity map of the tissue.

Magneto-Acousto-Electric-Tomography (MAET)- Basic Set-up

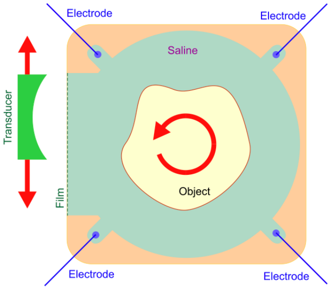

Figure 1: MAET Schematic

Physics of MAET

- **Lorentz Current:** Charges moving with velocity $V(x,t)$ generate Lorentz Currents

$$J_L(x,t) = \sigma(x)V(t,x) \times B.$$

- **Ohmic Current:** Lorentz Force generates electric potential, $u(t,x)$, which obeys Ohm's Law. Ohmic Current is given by

$$J_O(x,t) = \sigma(x)\nabla u(x,t).$$

- **Field is Divergence Free:** With total current $J = J_O + J_L$, we have

$$\nabla J = 0.$$

- Therefore,

$$\nabla \cdot \sigma(x)\nabla u(x,t) = -\nabla \cdot (\sigma(x)B \times V).$$

Relating Measurements to Desired Quantities

- We are able to measure differences of potential on the boundary, $M(t) = u(z_1, t) - u(z_2, t)$.

$$M(t) = \int_\Omega J_I(x) \cdot B \times V(x,t) dx.$$

- Since $V(x,t)$ solves the wave equation, it satisfies

$$V(x,t) = \frac{1}{\rho} \nabla \phi(x,t)$$
$$p(x,t) = \frac{\partial}{\partial t} \phi(x,t),$$

where $\phi(x,t)$ is the velocity potential, and $p(x,t)$ is pressure.

- Therefore,

$$M(t) = \frac{1}{\rho} B \cdot \int_\Omega \phi(x,t) \nabla \times J_I(x).$$

- $J_I(x)$ is "lead current". Does not actually exist. Will be discussed further.

Relating Measurements to Desired Quantities

- In practice, we can use plane waves, $\phi(x,\tau,\omega) = \delta(x \cdot \omega - \tau)$, then

$$M(\omega,\tau) = \frac{1}{\rho} B \cdot \int_{\omega} \delta(x \cdot \omega - \tau) \nabla \times J_I(x) dx = \frac{1}{\rho} B \cdot R(\nabla \times J_I(x))(\omega,\tau).$$

- Therefore, the difference in potential measured by our electrodes is proportional to the Radon Transform of the curl of Lead currents.
- The overall MAET procedure involves inverting the radon transform of the measurements to obtain curl of lead currents, from which one can obtain lead currents, and then finally the conductivity.

Basic Model of Lead Currents

- $J_I(x)$ desribes the sensitivity of the system to a dipole.
- Lead currents arise physically by running a unit charge through the system in the absence of a magnetic field, and in the absense of acoustic waves.

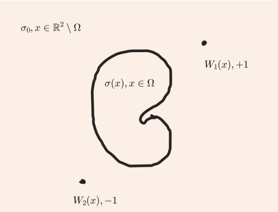

Figure 2: Basic Setup for Lead Currents

The Conductivity Equation

- The conductivity equation takes conductivity as input, and returns electric potential.
- For convenience, we can split the total electric potential into smooth and non-smooth components, $u(x)$ and $w(x)$.
- $u(x)$ is the potential in the absence of sinks/sources, and is unknown.
- $w(x)$ is the potential arising from point sources, and is known.
- Lead current is given by

$$J_l(x) = \sigma(x)\nabla(u(x) + w(x))$$

- $w(x) = -\frac{1}{\sigma_0}\sum_{j=1}^{n} W_j \Phi(x - y^{(j)})$, with $\Phi(x) = \frac{1}{2\pi}\ln(|x|)$.
- The governing equation is the Conductivity Equation

$$\nabla \cdot \sigma(x)\nabla(u(x) + w(x)) = 0, \quad \lim_{x\to\infty} u(x) = 0.$$

Numerical Solution - Fredholm Equation

- The conductivity equation can be written as an integral equation of the second kind:

$$U(x) + \frac{1}{2\pi}\left[\frac{\partial \ln \sigma(x)}{\partial x_1}\int_\Omega \ln|x-y|\frac{\partial}{\partial y_1}U(y)dy + \frac{\partial \ln \sigma(x)}{\partial x_2}\int_\Omega \ln|x-y|\frac{\partial}{\partial y_2}U(y)dy\right] = -\nabla \ln \sigma(x)\cdot\nabla w(x).$$

- One just needs to handle the singularities in the integrals, and apply GMRES routine.
- GMRES will output $U(x) = \Delta u(x)$. Potential $u(x)$ can be recovered by convolution with $\Phi(x)$.

Numerical Solution - Handling Singular Kernels

- Since we are using the Fredholm Equation of the Second kind formulation, we need to handle convolutions with a singular kernel $u(x) = \int_\Omega U(y) \frac{1}{2\pi} \ln |y - x| dy$
- Result of the convolution will satisfy the following conditions:
- (1) $\Delta u(x) = U(x), x \in \Omega$
- (2) $\Delta u(x) = 0, x \notin \Omega$.
- (3) $u(x)$ behaves like a Logarithm asymptotically.
- A function satisfying (1)-(3) will be unique.

Convolution Method - Descrption

Our goal is to compute $u(x) = \int_\Omega U(y) \frac{1}{2\pi} \ln |y - x| dy$.

- Step 1- Compute the solution having 0 boundary conditions on a larger square via Sine series.
- Step 2- Smoothly cutoff the corners of the resulting solution.
- Step 3- Compute the laplacian of the resulting function.
- Step 4- Set the laplacian to 0 over the small square.
- Step 5- Compute the convolution of the resulting with the fundamental solution, taking the convolution over the small square.
- Step 6- Subtract the result of the convolution from the smoothly cutoff solution.

Convolution Method - In Pictures

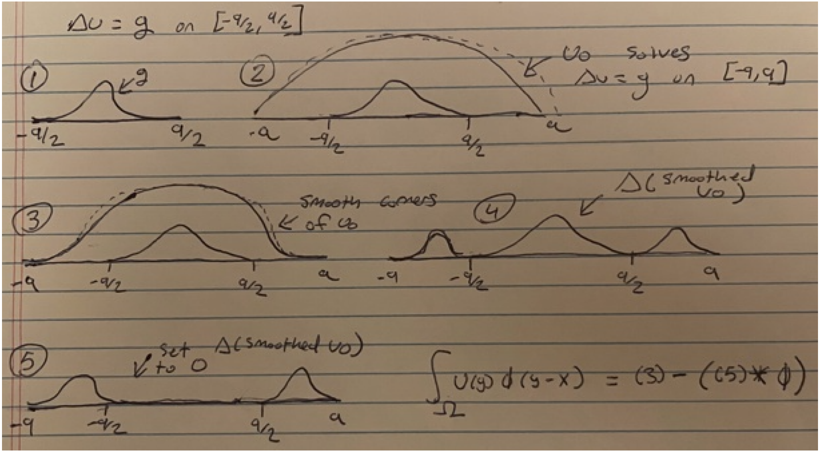

Figure 3: Convolution Method

Numerical Results- Initial Conductivity

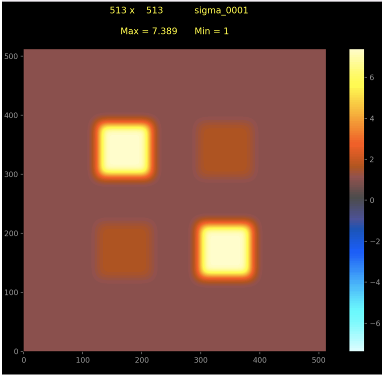

Figure 4: Conductivity $\sigma(x)$

Numerical Results- Point Source Potential

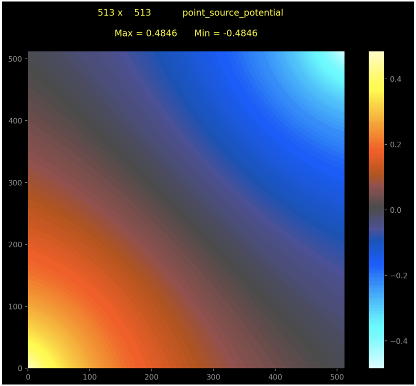

Figure 5: Point Source Potential $w(x)$

Numerical Results- Laplacian of Smooth Potential (GMRES output)

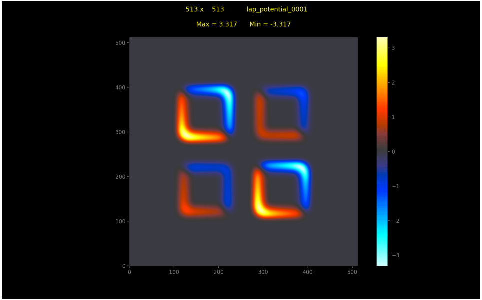

Figure 6: Laplacian of Smooth Potential $\Delta u(x)$

Numerical Results- Smooth Potential

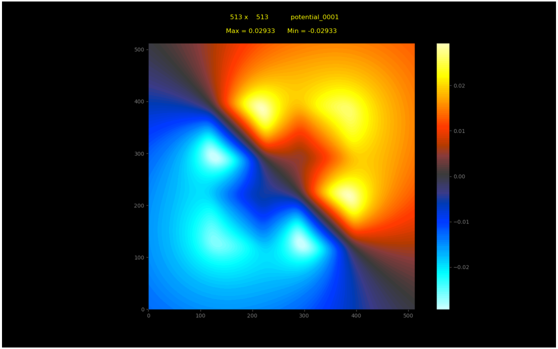

Figure 7: Smooth Potential $u(x)$

Numerical Results- GMRES Error

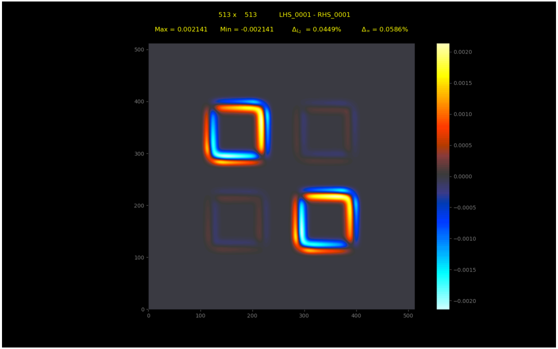

Figure 8: LHS v RHS Error

Numerical Results- Curl of Lead Currents

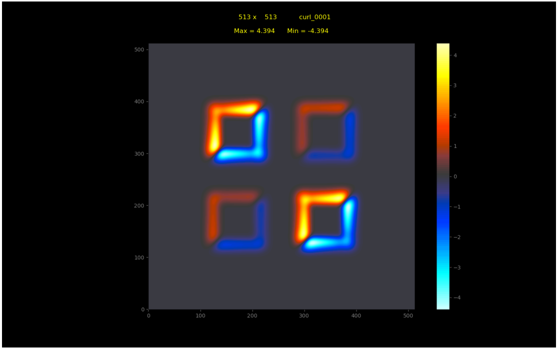

Figure 9: Curl of Lead Current $\nabla \times J_l(x)$

Numerical Results- Lead Currents

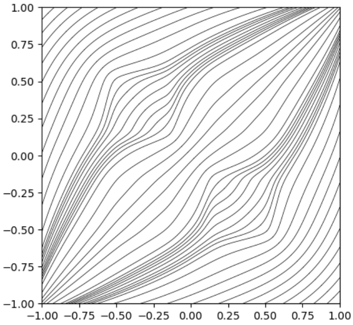

Figure 10: Depiction of Current Lines