Numerical Methods for the Conductivity Equation

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Motivation: Magneto-Acousto-Electric-Tomography (MAET)

- Based on measurements of the electrical potential arising when an acoustic wave propogates through conductive medium placed in a magnetic field.
- MAET is an example of an inverse problem for the conductivity of biological tissue
- We focused on understanding the *direct* problem this semester.

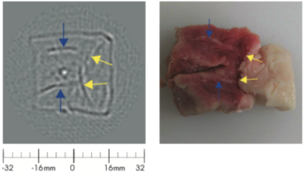

Figure: Sample Reconstruction

Basic Physics of Current

- Let $E(x)$ denote an electric field over $\mathbb{R}^2$.
- Let $\sigma(x)$ denote the conductivity over $\mathbb{R}^2$.
- Current is given by $\sigma(x)E(x)$.
- We assume there exists a potential $u(x)$, for which $E(x) = \nabla u(x)$.
- Then current is given by $\sigma(x)\nabla u(x)$.
- In the absence of sinks and sources, the divergence of current is 0.

Current Induced by Point Sources

Consider the case where conductivity is constant on $\mathbb{R}^2$. Suppose we inject currents W_j at a finite set of points $y^{(j)}$, where $\sum_{j=1}^{n} W_j = 0$. The resulting electric potential is given by

$$w(x) = -\frac{1}{\sigma_0}\sum_{j=1}^{n} W_j \phi(x - y^{(j)})$$

where $\Delta\phi(x) = \delta(x)$. $\phi(x)$ is known as the fundamental solution to Laplace's Equation. $\phi(x) = \frac{1}{2\pi}\ln|x|$.

The Problem we are Considering

- Suppose now we have a region Ω, not containing any of the injection points $y^{(j)}$, where conductivity is non-constant. ie, conductivity is given by some function $\sigma(x), x \in \Omega$.
- By superposition, we can split the resulting electric potential into the potential given by the injected currents, and the potential arising when conductivity is no longer constant. Then current is given by $\sigma(x)\nabla(u(x) + w(x))$.
- Divergence of total current is 0 away from the injection points, and $u(x)$ vanishes at ∞.

The Conductivity Equation - Setup

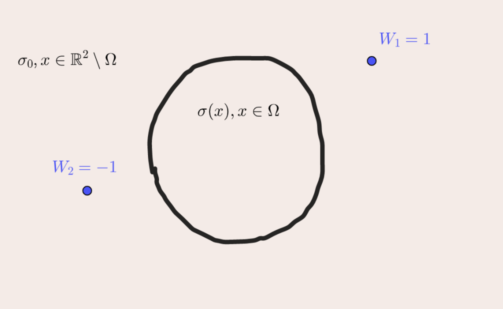

Figure: Basic Setup

$$\nabla \cdot \sigma(x)\nabla(u(x) + w(x)) = 0, \quad x \in \mathbb{R}^2 \setminus \cup_{j=1}^{n} y^{(j)}.$$

Conductivity Equation - Fredholm Reduction

- For convenience, we let $U(x) = \Delta u(x)$.
- It can be shown that the conductivity equation can be expressed as an integral equation of the second kind

$$U(x)+\nabla \ln(\sigma(x))\cdot\nabla[\int_\Omega U(y)\phi(x-y)dy] = -\nabla\sigma(x)\cdot\nabla w(x), x \in \Omega.$$

- Such equations are known to be well-posed.

Primary Problem- Convolution with a Singular Kernel

- We want to make a numerical solver to reconstruct currents for arbitrary $\sigma(x)$.
- Since we are using the Fredholm Equation of the Second kind formulation, we need to handle convolutions with a singular kernel $u(x) = \int_\Omega U(y)\frac{1}{2\pi}\ln|y-x|dy$
- Result of the convolution will satisfy the following conditions:
- (1) $\Delta u(x) = U(x), x \in \Omega$
- (2) $\Delta u(x) = 0, x \notin \Omega$.
- (3) $u(x)$ behaves like a Logarithm asymptotically.
- A function satisfying (1)-(3) will be unique.
- For simplicity, we take $\Omega = [-a/2, a/2]^2$, but the following approach should work for any Ω.

Convolution Method - Descrption

Our goal is to compute $u(x) = \int_\Omega U(y) \frac{1}{2\pi} \ln|y - x| dy$.

- Step 1- Compute the solution having 0 boundary conditions on a larger square via Sine series.
- Step 2- Smoothly cutoff the corners of the resulting solution.
- Step 3- Compute the laplacian of the resulting function.
- Step 4- Set the laplacian to 0 over the small square.
- Step 5- Compute the convolution of the resulting with the fundamental solution, taking the convolution over the small square.
- Step 6- Subtract the result of the convolution from the smoothly cutoff solution.

Convolution Method - In Pictures

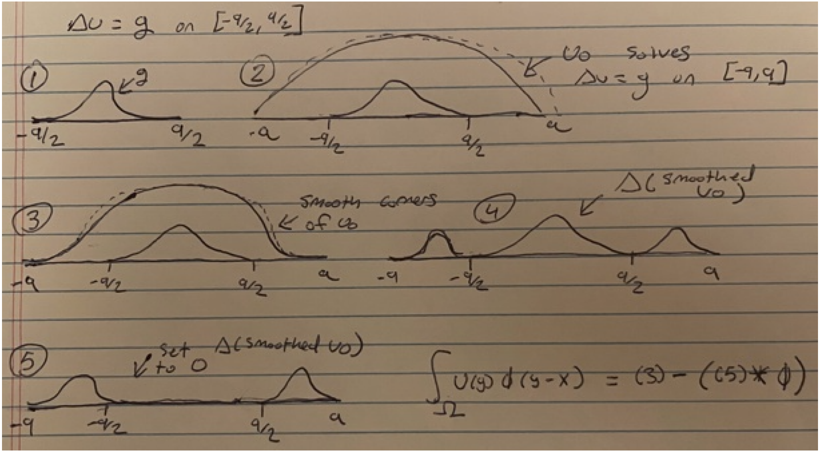

Figure: Convolution Method

Why it Works + Beneftis

- $\Delta u_{\text{final}} = U(x)$ inside of Ω.
- $\Delta u_{\text{final}} = 0$ outside of Ω.
- u_{final} behaves like a logarithm asymptotically.
- Therefore, $u_{\text{final}} = \int_\omega U(y)\phi|y - x|$.
- We completely avoid the singularity in the kernel $\phi(x)$.
- All non-trivial computations - computing the laplacian, inverting the laplcian, taking the convolution, computing partial derivatives- can be done via FFT methods.

Convolution Method- Numerical Example

We test a numerical scheme using the function $(1 - \hat{\sin}(x))\ln(|x|)$. This function has a laplacian which we can exactly compute, and behaves like a logarithm asymptotically.

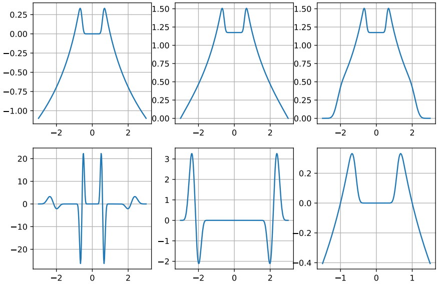

Figure: Numerical Example

Computing the Convolution- Error

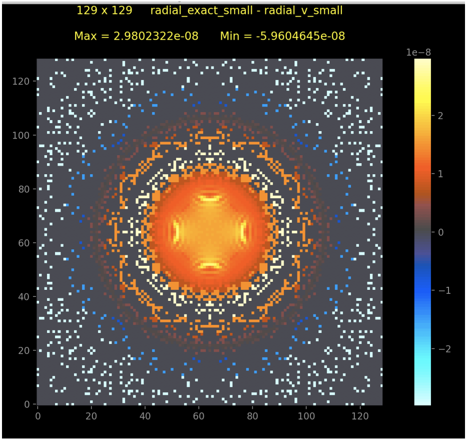

Figure: Difference between u_{final} and exact solution