Simulating the Forward Problem of Magneto-Acousto-Electric Tomography

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1 Introduction

Magneto-Acousto-Electric Tomography (MAET) is a multi-physics medical imaging modality of relevance to cancer screening. The method utilizes a constant magnetic field, coupled with acoustic waves, to generate an electrical signal which can be used to find the conductivity of a tissue of interest. MAET is promising for cancer screening, as tumors are thought to be conductive.

2 Physics and Mathematics of MAET

2.1 Physical Implementation and Governing Equations

The physical implementation of MAET uses a tank filled with saline solution, in which a rotatable chamber is submerged housing the tissue of interest. Above the tank rests a magnet, and along an edge of the tank is a transducer, which can move side-to-side. The magnet creates a constant magnetic field, while the transducer generates acoustic waves to create motion in the tissue. The resulting electrical signals, which we measure with a pair of electrodes, can be understood mathematically through the Lorentz force and Ohm's law.

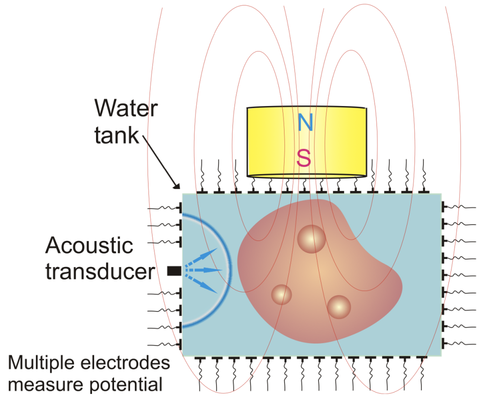

Figure 1: MAET Schematic

If a tissue of conductivity $\sigma(x)$, supported on a simply connected domain, Ω, moves with velocity $V(x,t)$ within a constant magnetic field B, the resulting Lorentz force will create separation of charges, and thus generate Lorentz currents, $J_L(x,t)$ given by

$$J_L(x,t) = \sigma(x) B \times V(x,t).$$

In addition to the Lorentz currents, there is an electrical potential $u(x,t)$, which generates Ohmic currents, satisfying Ohm's Law

$$J_O(x,t) = \sigma(x) \nabla u(x,t).$$

Since there are no sources/sinks within the conductive tissue, the total current is divergence free

$$\nabla \cdot (J_L + J_O) = 0.$$

Therefore,

$$\nabla \cdot \sigma(x) \nabla u(x,t) = -\nabla \cdot (\sigma(x) B \times V(x,t)), \tag{1}$$

where the normal component of total current vanishes:

$$\frac{\partial}{\partial n} u(z)|_{\partial\Omega} = -(B \times V(z)) \cdot n(z), \quad z \in \partial\Omega \tag{2}$$

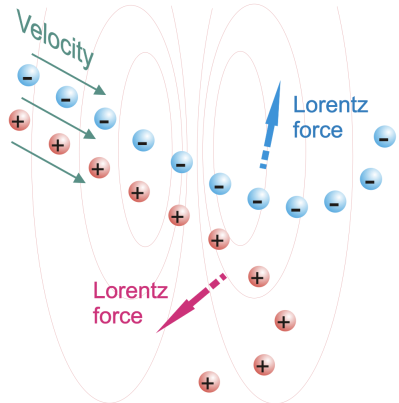

Figure 2: Lorentz Force - Separation of Charges

Equations (1) and (2) can be used to relate measurements of electric potential on the boundary to the conductivity of the tissue.

2.2 Measurements of MAET

For simplicity, let us assume we are using only one pair of electrodes to measure electrical potential in our apparatus. We let $M(t)$ denote our measurement at time t. If our electrodes are located at points $z_1, z_2 \in \partial\Omega$, we have

$$M(t) = u(t, z_2) - u(t, z_1).$$

Consider a virtual current, $J_{\text{Lead}}(x)$ that would flow in the object if a unit current were injected at the point z_2 and extracted at the point z_1. It can be shown using Green's Second identity that

$$M(t) = \int_{\Omega} J_{\text{Lead}}(x) \cdot B \times V(t, x) dx.$$

The appearance of the lead currents can be understood as an example of the reciprocity principle in physics, which may be described by noting that the source and measurement sites may be swapped without changing the detected signal. In practice, it is difficult to work with the velocity field $V(t, x)$. We note that $V(t, x)$ solves the wave equation, and satisfies

$$V(t, x) = \frac{1}{\rho} \nabla \phi(t, x)$$

where velocity potential $\phi(t, x)$ is the time anti-derivative of pressure $p(t, x)$:

$$p(t, x) = \frac{\partial}{\partial t} \phi(t, x).$$

Using the substitution $V(t,x) = \frac{1}{\rho}\nabla\phi(t,x)$ one obtains

$$M(t) = \frac{B}{\rho}\int_{\Omega}\phi(t,x)(\nabla\times J_{\text{Lead}}(x))dx. \tag{3}$$

Since we are able to rotate the object, as well as move the transducer from side to side, we are able to generate plane waves, given mathematically by the relation

$$\phi(\tau,\omega,x) = \delta(x\cdot\omega-\tau).$$

Then

$$M(\omega,\tau) = \frac{B}{\rho}\int_{\Omega}\delta(x\cdot\omega-\tau)(\nabla\times J_{\text{Lead}}(x))dx = \frac{B}{\rho}\cdot(R(\nabla\times J_{\text{Lead}}(x))), \tag{4}$$

where R denotes the Radon transform. Therefore, each measurement of difference of potential is proportional to the Radon transform of the curl of lead currents. The Radon transform is well-known, and can be inverted to yield the curl of lead current. Once the curl of lead current is obtained, one can obtain the lead current, then the divergence of the logarithm of conductivity, and finally the conductivity.

3 The Forward Problem

Since the MAET framework under consideration measures the Radon transform of the curl of lead currents, we present a method of simulating lead currents induced by a pair of electrodes.

3.1 The Conductivity Equation

Given an electric field $E(x)$ over $\mathbb{R}^2$, and a conductivity function $\sigma(x)$ over $\mathbb{R}^2$, current is given by the product

$$J(x) = \sigma(x)E(x).$$

We assume there exists a potential function $u(x)$, for which $E(x) = \nabla u(x)$. Then current is given by

$$J(x) = \sigma(x)\nabla u(x).$$

In the absence of sinks and sources of charges, we expect no net accumulation of charges, and so divergence of current is 0.
Consider the case where conductivity is constant over $\mathbb{R}^2$, $\sigma(x) = \sigma_0$. Suppose we inject currents W_j at a finite set of points $y^{(j)}$, where $\sum_{j=1}^{n} W_j = 0$. The resulting electric potential is given by

$$w(x) = -\frac{1}{\sigma_0}\sum_{j=1}^{n} W_j\phi(x-y^{(j)}),$$

where $\phi(x) = \frac{1}{2\pi}\ln|x|$. $\phi(x)$ is the fundamental solution to Laplace's Equation, and satisfies

$$\Delta\phi(x) = \delta(x).$$

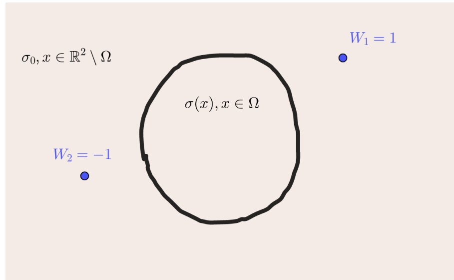

Figure 3: Conductivity Equation- Setup with Two Point Sources

Consider now a simply-connected region $\Omega \in \mathbb{R}^2$, not containing any of the injection points $y^{(j)}$, where conductivity is non-constant. That is, conductivity is given by some smooth function $\sigma(x)$ which is non-constant on Ω.
By superposition, we can split the resulting electric potential into the potential given by the point sources/injected currents and the potential arising when conductivity is no longer constaint. This is convenient, as we can split the potential into smooth and non-smooth components. Then current is given by

$$J(x) = \sigma(x)\nabla\left(u(x) + w(x)\right).$$

Since our model requires that $\sum_{j=1}^{n} W_j = 0$, we preserve the property that the divergence of total current is 0 away from the injection points. We also assume that the smooth potential $u(x)$ vanishes at ∞. This defines the *conductivity equation*,

$$\begin{cases} \nabla \cdot \sigma(x)\left(u(x) + w(x)\right) = 0, \quad x \in \mathbb{R}^2 \setminus \cup_{j=1}^{n} y^{(j)} \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{5}$$

3.2 Reduction to Fredholm Equation

Due to one of the properties of harmonic functions, the singular points $y^{(j)}$ can be included in the domain of definition of $u(x)$, and equation (5) can be rewritten as

$$\begin{cases} \nabla \cdot \sigma(x)\nabla u(x) = -\nabla\sigma(x) \cdot \nabla w(x), \quad x \in \Omega \\ \nabla \cdot \sigma(x)\nabla u(x) = 0, \quad x \in \mathbb{R}^2 \setminus \Omega \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases}$$

Further, we can express the conductivity equation as

$$\begin{cases} \Delta u(x) + \nabla \ln\sigma(x) \cdot \nabla u(x) = -\nabla \ln\sigma(x) \cdot \nabla w(x), \quad x \in \Omega \\ \Delta u(x) = 0, \quad x \in \mathbb{R}^2 \setminus \Omega \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{6}$$

For convenience, given any function $v(x)$, we define the operator Δ^{-1} as the convolution

$$c(x) = \left[\Delta^{-1}v\right](x) = \left[v * \phi\right](x) = \int_\Omega v(y)\phi(x-y)dy, \quad x \in \mathbb{R}^2.$$

Let $U(x)$ denote the laplacian of $u(x)$, $U(x) = \Delta u(x)$. Then equation (6) implies

$$U(x) = \begin{cases} -\nabla \ln \sigma(x) \cdot \nabla u(x) - \nabla \ln \sigma(x) \cdot \nabla w(x) & x \in \Omega \\ 0, & x \in \mathbb{R}^2 \backslash \Omega, \end{cases} \tag{7}$$

and if $U(x)$ is known, potential $u(x)$ can be found as a solution of the Poisson equation in the plane

$$\begin{aligned} \Delta u(x) &= U(x), x \in \mathbb{R}^2 \\ \implies u(x) &= \left[\Delta^{-1}U\right](x). \end{aligned}$$

Thus, we can rewrite equation (7) as follows:

$$U(x) + \nabla \ln \sigma(x) \cdot \nabla \left[\Delta^{-1}U\right](x) = -\nabla \ln \sigma(x) \cdot \nabla w(x), \quad x \in \Omega. \tag{8}$$

It can be shown that

$$\frac{\partial}{\partial x_k}\left[\Delta^{-1}U\right](x) = \frac{1}{2\pi}\int_\Omega U(y)\frac{x_k - y_k}{|x-y|^2}dy.$$

Thus, our equation can be written in full as

$$U(x) + \frac{1}{2\pi}\left[\frac{\partial \ln \sigma(x)}{\partial x_1}\int_\Omega U(y)\frac{x_1 - y_1}{|x-y|^2}dy + \frac{\partial \ln \sigma(x)}{\partial x_2}\int_\Omega U(y)\frac{x_2 - y_2}{|x-y|^2}dy\right] = -\nabla \ln \sigma(x) \cdot \nabla w(x).$$

Using integration by parts, and recalling the definition $\phi(x) = \frac{1}{2\pi}\ln|x|$, we can rewrite our equation as

$$U(x) + \left[\frac{\partial \ln \sigma(x)}{\partial x_1}\int_\Omega \phi(x-y)\frac{\partial}{\partial y_1}U(y)dy + \frac{\partial \ln \sigma(x)}{\partial x_2}\int_\Omega \phi(x-y)\frac{\partial}{\partial y_2}U(y)dy\right] = -\nabla \ln \sigma(x) \cdot \nabla w(x). \tag{9}$$

Equation (9) is in the form of a Fredholm integral equation of the second kind. Such equations can be solved via GMRES, provided one can compute the convolutions having singular kernels. In the following section, we outling a procedure to compute convolutions with the fundamental solution of Laplace's equation, $\phi(x)$.

3.3 Handling convolutions with Singular Kernels

Consider the result of the convolution

$$u(x) = \int_\Omega U(y)\frac{1}{2\pi}\ln|y-x|dy.$$

We consider smooth functions $U(x) = \Delta u(x)$ that are 0 outside of Ω. $u(x)$ should satisfy the following conditions

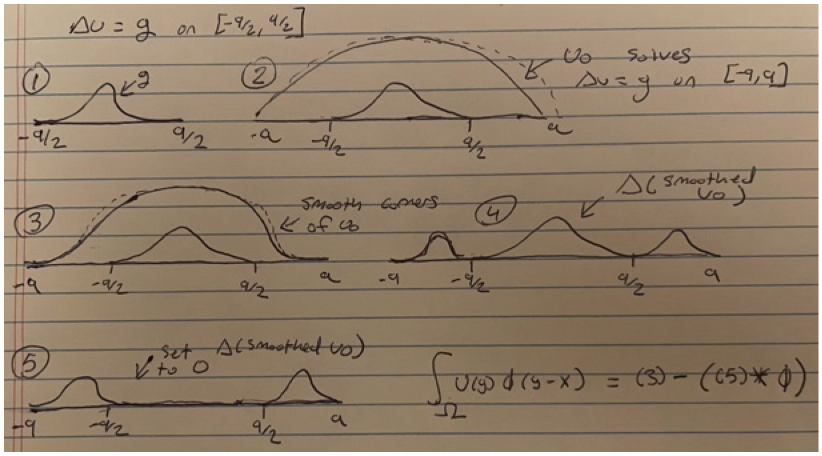

Figure 4: Convolution Method

(i) $\Delta u(x) = U(x), x \in \Omega$

(ii) $\Delta u(x) = 0, x \notin \Omega$

(iii) $u(x)$ behaves like a logarithm asymptotically.

Properties (i)-(iii) are enough to uniquely determine a function. For simplicity, consider a square region, $\Omega = [-a/2, a/2]^2$. The following procedure will generate a function satisfying properties (i)-(iii) above.

- (1) Compute the solution having 0 boundary conditions on a larger square, $[-a, a]^2$, via Sine series.
- (2) Smoothly cutoff the corners of the resulting solution, without perturbing any part of the solution lying within $[-a/2, a/2]^2$.
- (3) Compute the Laplacian of the resulting function.
- (4) Set the Laplacian to 0 over the small square.
- (5) Compute the convolution of the result with the fundamental solution. Note that this function is 0 over $\Omega = [-a/2, a/2]^2$, so there are no singularities in this convolution.
- (6) Subtract the result of the convolution from the result obtained in step 2.

Figure (4) provides a visual depiction of the method, on a generic function that is supported on $[-a/2, a/2]$.

We test the algorithm numerically on the function $(1 - \widehat{\sin(x)}) \ln(|x|)$. This function has a Laplacian which we can exactly compute, and behaves like a logarithm asymptotically. The sequence of transformations is presented below.

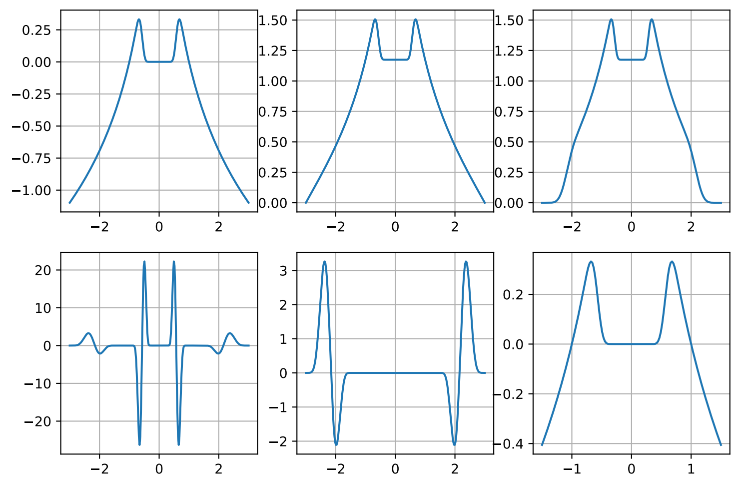

Figure 5: Convolution Method- Numerical Example

The top left cell displays the exactly computed test function, u_{exact}- we want our method to take its Laplacian as input, and return this same function on $[-1,1]$ via convolution with the fundamental solution.
The top middle cell is the result of solving poisson's equation with the Laplacian of the test function, Δu_{exact} as a right hand side, subject to 0 boundary conditions. This is computed by double application of the sine transform (since the sine transform is its own inverse) subject to some normalization. Let us call this result u_0. Note that u_0 does not smoothly decay to 0. In its current form, we cannot compute the Laplacian of this function on $[-2,2]$
The top right cell is the result of smoothing the corner of the Sine series solution, u_0, via multiplication with a smoothing function ζ. Let us call this result $u_0\zeta$. The idea here is that the profile is completely unperturbed on $[-1,1]$, but is smooth on all of $[-2,2]$.
The bottom left cell displays the Laplacian of $u_0\zeta$, $\Delta(u_0\zeta)$, and the bottom middle cell is the result of multiplying $\Delta(u_0\zeta)$ by an indicator function which is 0 on $[-1,1]$. Let us denote this indicator function by $1_{[-\bar{1},1]^2}$. Then the profile in the bottom left cell is given by $\Delta(u_0\zeta)1_{[-\bar{1},1]^2}$.
The bottom right cell is obtained in two steps. First we compute the convolution

$$c = \left[\Delta(u_0\zeta)1_{[-\bar{1},1]^2} * \phi\right],$$

over the large square $[-2,2]^2$. Then the final profile, u_{approx} is obtained by

$$u_{\text{approx}} = u_0\zeta - c = u_0\zeta - \left[\Delta(u_0\zeta)1_{[-\bar{1},1]^2} * \phi\right].$$

Note that

$$\begin{aligned}\Delta u_{\text{approx}} &= \Delta(u_0\zeta) - \Delta(\left[\Delta(u_0\zeta)1_{[-\bar{1},1]^2} * \phi\right]) \\ &= \begin{cases}\Delta u_{\text{exact}}, & x \in [-1,1]^2 \\ 0, & x \notin [-1,1]^2\end{cases}.\end{aligned}$$

Thus, u_{approx} exhibits the properties required of our convolution. The full numerical error comparing u_{exact} to u_{approx} on a 129×129 grid is displayed below.

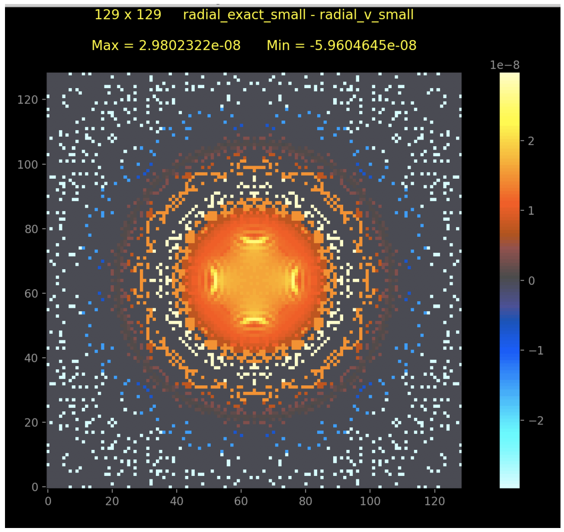

Figure 6: Convolution Method- Numerical Error

3.4 Numerical Results

Our underlying equation is equation (9),

$$U(x)+\left[\frac{\partial \ln \sigma(x)}{\partial x_1}\int_\Omega \phi(x-y)\frac{\partial}{\partial y_1}U(y)dy+\frac{\partial \ln \sigma(x)}{\partial x_2}\int_\Omega \phi(x-y)\frac{\partial}{\partial y_2}U(y)dy\right]=-\nabla \ln \sigma(x)\cdot\nabla w(x).$$

Since $w(x)$ and $\ln \sigma(x)$ are known, our task is to find $U(x) = \Delta u(x)$. Once we find $U(x)$, we can recover $u(x)$ via convolution with the fundamental solution, $\phi(x)$. We use GMRES to solve for $U(x)$, where the initial condition is simply the right hand side, $-\nabla \ln \sigma(x) \cdot \nabla w(x)$. We consider the following conductivity function, supported on the square $[-1, 1]^2$. Our point sources are located at the coordinates $(1.1, 1.1)$ and $(-1.1, 1.1)$. The point source potential has the following profile on $[-1, 1]^2$. The output of GMRES is the Laplacian of the smooth potential, displayed below. We convolve the Laplacian of smooth potential and recover the smooth potential, displayed below. Having obtained the smooth electric potential $u(x)$, we can use finite differences to differentiate $u(x)$, and in combination with our conductivity $\sigma(x)$, obtain the curl of lead current, as well as draw current lines.

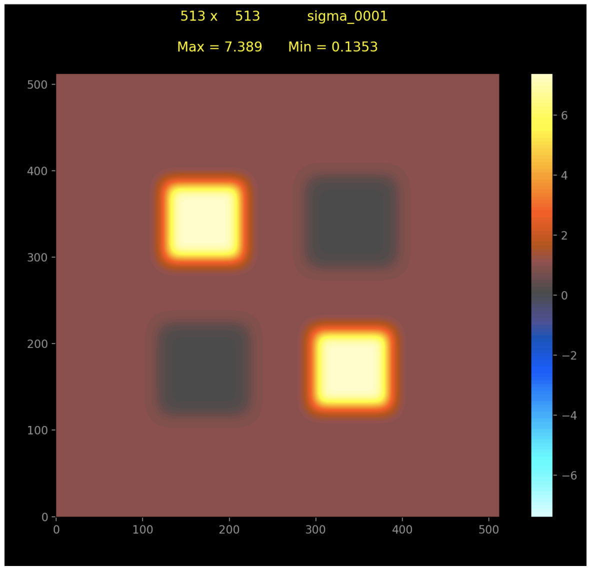

Figure 7: Conductivity

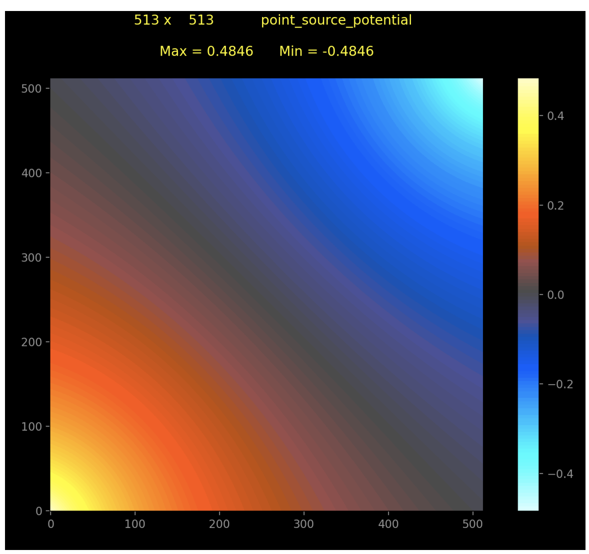

Figure 8: Point Source Potential

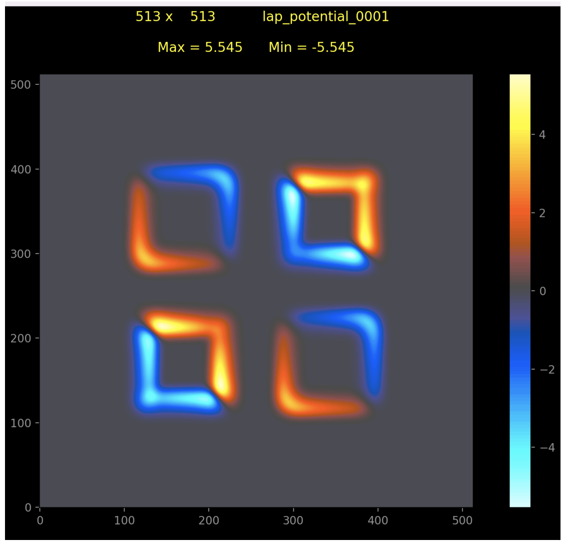

Figure 9: Laplacian of Smooth Potential

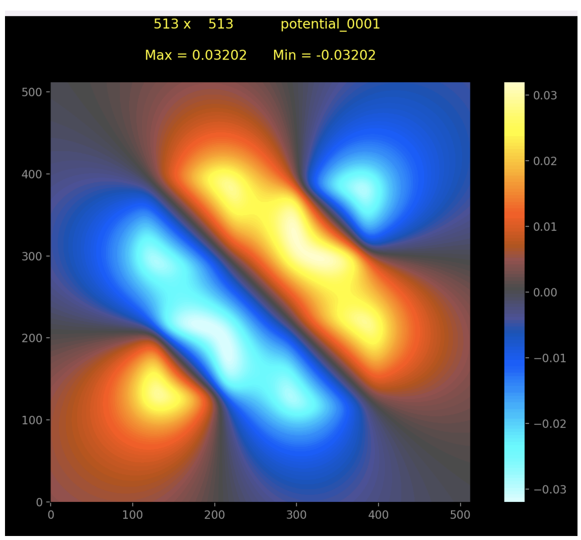

Figure 10: Smooth Potential

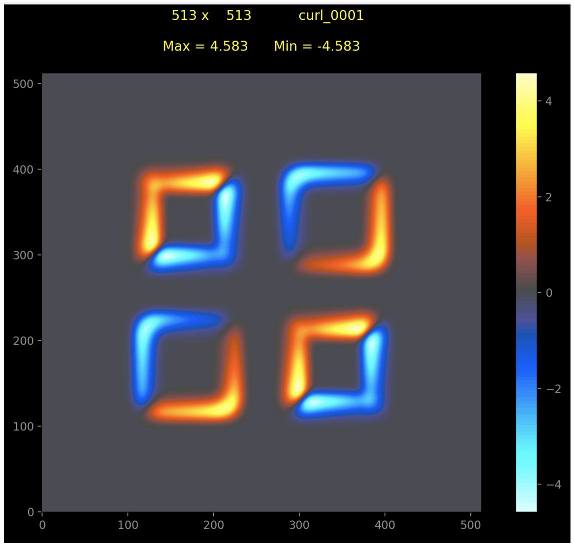

Figure 11: Curl of Lead Current

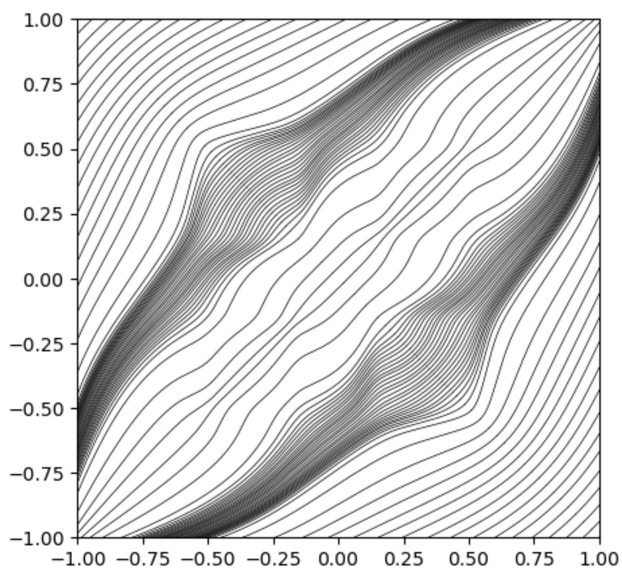

Figure 12: Current Lines

4 Codes

For completeness, we provide the main codes used to generate our figures in the previous section. All codes can be found here: conductivity project
Script for generating current lines and related outputs: currents.
Module housing supporting functions: my_conductivity.