In[33]:= `kvec = Grad[phase[r, alpha], {r, alpha}, "Polar"]`

Out[33]=

$$\left\{\text{phase}^{(1,0)}[r, \text{alpha}], \frac{\text{phase}^{(0,1)}[r, \text{alpha}]}{r}\right\}$$

In[34]:= `k = Sqrt[Part[kvec, 1]^2 + Part[kvec, 2]^2]`

Out[34]=

$$\sqrt{\frac{\text{phase}^{(0,1)}[r, \text{alpha}]^2}{r^2} + \text{phase}^{(1,0)}[r, \text{alpha}]^2}$$

In[35]:= `B = 1 - k^2`

Out[35]=

$$1 - \frac{\text{phase}^{(0,1)}[r, \text{alpha}]^2}{r^2} - \text{phase}^{(1,0)}[r, \text{alpha}]^2$$

In[36]:= `div = Div[kvec * B, {r, alpha}, "Polar"]`

Out[36]=

$$\frac{1}{r}\left(\frac{\text{phase}^{(0,2)}[r, \text{alpha}]\left(1 - \frac{\text{phase}^{(0,1)}[r,\text{alpha}]^2}{r^2} - \text{phase}^{(1,0)}[r, \text{alpha}]^2\right)}{r} + \right.$$

$$\text{phase}^{(1,0)}[r, \text{alpha}]\left(1 - \frac{\text{phase}^{(0,1)}[r, \text{alpha}]^2}{r^2} - \text{phase}^{(1,0)}[r, \text{alpha}]^2\right) + \frac{1}{r}\text{phase}^{(0,1)}[r, \text{alpha}]$$

$$\left.\left(-\frac{2\,\text{phase}^{(0,1)}[r, \text{alpha}]\,\text{phase}^{(0,2)}[r, \text{alpha}]}{r^2} - 2\,\text{phase}^{(1,0)}[r, \text{alpha}]\,\text{phase}^{(1,1)}[r, \text{alpha}]\right)\right) +$$

$$\left(1 - \frac{\text{phase}^{(0,1)}[r, \text{alpha}]^2}{r^2} - \text{phase}^{(1,0)}[r, \text{alpha}]^2\right)\text{phase}^{(2,0)}[r, \text{alpha}] +$$

$$\text{phase}^{(1,0)}[r, \text{alpha}]\left(\frac{2\,\text{phase}^{(0,1)}[r, \text{alpha}]^2}{r^3} - \right.$$

$$\left.\frac{2\,\text{phase}^{(0,1)}[r, \text{alpha}]\,\text{phase}^{(1,1)}[r, \text{alpha}]}{r^2} - 2\,\text{phase}^{(1,0)}[r, \text{alpha}]\,\text{phase}^{(2,0)}[r, \text{alpha}]\right)$$

In[37]:= `lap = Laplacian[phase[r, alpha], {r, alpha}, "Polar"]`

Out[37]=

$$\frac{\frac{\text{phase}^{(0,2)}[r,\text{alpha}]}{r} + \text{phase}^{(1,0)}[r, \text{alpha}]}{r} + \text{phase}^{(2,0)}[r, \text{alpha}]$$

In[38]:= `biharm = Laplacian[lap, {r, alpha}, "Polar"]`

Out[38]=

$$\frac{2\left(\frac{phase^{(0,2)}[r,alpha]}{r} + phase^{(1,0)}[r, alpha]\right)}{r^3} -$$

$$\frac{2\left(-\frac{phase^{(0,2)}[r,alpha]}{r^2} + \frac{phase^{(1,2)}[r,alpha]}{r} + phase^{(2,0)}[r, alpha]\right)}{r^2} +$$

$$\frac{\frac{2\,phase^{(0,2)}[r,alpha]}{r^3} - \frac{2\,phase^{(1,2)}[r,alpha]}{r^2} + \frac{phase^{(2,2)}[r,alpha]}{r} + phase^{(3,0)}[r, alpha]}{r} + \frac{1}{r}$$

$$\left(-\frac{\frac{phase^{(0,2)}[r,alpha]}{r} + phase^{(1,0)}[r, alpha]}{r^2} + \frac{-\frac{phase^{(0,2)}[r,alpha]}{r^2} + \frac{phase^{(1,2)}[r,alpha]}{r} + phase^{(2,0)}[r, alpha]}{r} +\right.$$

$$\left.\frac{\frac{\frac{phase^{(0,4)}[r,alpha]}{r} + phase^{(1,2)}[r,alpha]}{r} + phase^{(2,2)}[r, alpha]}{r} + phase^{(3,0)}[r, alpha]\right) + phase^{(4,0)}[r, alpha]$$

In[39]:= `RHS = -2 * div - biharm`

Out[39]=

$$-\frac{2\left(\frac{phase^{(0,2)}[r,alpha]}{r} + phase^{(1,0)}[r, alpha]\right)}{r^3} +$$

$$\frac{2\left(-\frac{phase^{(0,2)}[r,alpha]}{r^2} + \frac{phase^{(1,2)}[r,alpha]}{r} + phase^{(2,0)}[r, alpha]\right)}{r^2} -$$

$$2\left(\frac{1}{r}\left(\frac{phase^{(0,2)}[r, alpha]\left(1 - \frac{phase^{(0,1)}[r,alpha]^2}{r^2} - phase^{(1,0)}[r, alpha]^2\right)}{r} + \right.\right.$$

$$phase^{(1,0)}[r, alpha]\left(1 - \frac{phase^{(0,1)}[r, alpha]^2}{r^2} - phase^{(1,0)}[r, alpha]^2\right) + \frac{1}{r}phase^{(0,1)}[r, alpha]$$

$$\left.\left(-\frac{2\,phase^{(0,1)}[r, alpha]\,phase^{(0,2)}[r, alpha]}{r^2} - 2\,phase^{(1,0)}[r, alpha]\,phase^{(1,1)}[r, alpha]\right)\right) +$$

$$\left(1 - \frac{phase^{(0,1)}[r, alpha]^2}{r^2} - phase^{(1,0)}[r, alpha]^2\right)phase^{(2,0)}[r, alpha] +$$

$$phase^{(1,0)}[r, alpha]\left(\frac{2\,phase^{(0,1)}[r, alpha]^2}{r^3} - \right.$$

$$\left.\left.\frac{2\,phase^{(0,1)}[r, alpha]\,phase^{(1,1)}[r, alpha]}{r^2} - 2\,phase^{(1,0)}[r, alpha]\,phase^{(2,0)}[r, alpha]\right)\right) -$$

$$\frac{\frac{2\,phase^{(0,2)}[r,alpha]}{r^3} - \frac{2\,phase^{(1,2)}[r,alpha]}{r^2} + \frac{phase^{(2,2)}[r,alpha]}{r} + phase^{(3,0)}[r, alpha]}{r} -$$

$$\frac{1}{r}$$

$$\left(-\frac{\frac{phase^{(0,2)}[r,alpha]}{r} + phase^{(1,0)}[r, alpha]}{r^2} + \frac{-\frac{phase^{(0,2)}[r,alpha]}{r^2} + \frac{phase^{(1,2)}[r,alpha]}{r} + phase^{(2,0)}[r, alpha]}{r} + \right.$$

$$\left.\frac{\frac{\frac{phase^{(0,4)}[r,alpha]}{r} + phase^{(1,2)}[r,alpha]}{r} + phase^{(2,2)}[r, alpha]}{r} + phase^{(3,0)}[r, alpha]\right) - phase^{(4,0)}[r, alpha]$$

In[40]:= `Simplify[RHS]`

Out[40]=

$$\frac{1}{r^4}\left(-\text{phase}^{(0,4)}[r, \text{alpha}] + \right.$$

$$2\, \text{phase}^{(0,2)}[r, \text{alpha}]\left(-2 - r^2 + 3\, \text{phase}^{(0,1)}[r, \text{alpha}]^2 + r^2\, \text{phase}^{(1,0)}[r, \text{alpha}]^2\right) +$$

$$r\left(2\, r^2\, \text{phase}^{(1,0)}[r, \text{alpha}]^3 - \text{phase}^{(1,0)}[r, \text{alpha}]\right.$$

$$\left(1 + 2\, r^2 + 2\, \text{phase}^{(0,1)}[r, \text{alpha}]^2 - 8\, r\, \text{phase}^{(0,1)}[r, \text{alpha}]\, \text{phase}^{(1,1)}[r, \text{alpha}]\right) +$$

$$2\, \text{phase}^{(1,2)}[r, \text{alpha}] + 6\, r^3\, \text{phase}^{(1,0)}[r, \text{alpha}]^2\, \text{phase}^{(2,0)}[r, \text{alpha}] +$$

$$r\left(\left(1 - 2\, r^2 + 2\, \text{phase}^{(0,1)}[r, \text{alpha}]^2\right)\text{phase}^{(2,0)}[r, \text{alpha}] - \right.$$

$$\left.\left.\left.2\, \text{phase}^{(2,2)}[r, \text{alpha}] - r\left(2\, \text{phase}^{(3,0)}[r, \text{alpha}] + r\, \text{phase}^{(4,0)}[r, \text{alpha}]\right)\right)\right)\right)$$

In[41]:= `Expand[%40]`

Out[41]=

$$-\frac{4\, \text{phase}^{(0,2)}[r, \text{alpha}]}{r^4} - \frac{2\, \text{phase}^{(0,2)}[r, \text{alpha}]}{r^2} + \frac{6\, \text{phase}^{(0,1)}[r, \text{alpha}]^2\, \text{phase}^{(0,2)}[r, \text{alpha}]}{r^4} -$$

$$\frac{\text{phase}^{(0,4)}[r, \text{alpha}]}{r^4} - \frac{\text{phase}^{(1,0)}[r, \text{alpha}]}{r^3} - \frac{2\, \text{phase}^{(1,0)}[r, \text{alpha}]}{r} -$$

$$\frac{2\, \text{phase}^{(0,1)}[r, \text{alpha}]^2\, \text{phase}^{(1,0)}[r, \text{alpha}]}{r^3} + \frac{2\, \text{phase}^{(0,2)}[r, \text{alpha}]\, \text{phase}^{(1,0)}[r, \text{alpha}]^2}{r^2} +$$

$$\frac{2\, \text{phase}^{(1,0)}[r, \text{alpha}]^3}{r} + \frac{8\, \text{phase}^{(0,1)}[r, \text{alpha}]\, \text{phase}^{(1,0)}[r, \text{alpha}]\, \text{phase}^{(1,1)}[r, \text{alpha}]}{r^2} +$$

$$\frac{2\, \text{phase}^{(1,2)}[r, \text{alpha}]}{r^3} - 2\, \text{phase}^{(2,0)}[r, \text{alpha}] + \frac{\text{phase}^{(2,0)}[r, \text{alpha}]}{r^2} +$$

$$\frac{2\, \text{phase}^{(0,1)}[r, \text{alpha}]^2\, \text{phase}^{(2,0)}[r, \text{alpha}]}{r^2} + 6\, \text{phase}^{(1,0)}[r, \text{alpha}]^2\, \text{phase}^{(2,0)}[r, \text{alpha}] -$$

$$\frac{2\, \text{phase}^{(2,2)}[r, \text{alpha}]}{r^2} - \frac{2\, \text{phase}^{(3,0)}[r, \text{alpha}]}{r} - \text{phase}^{(4,0)}[r, \text{alpha}]$$