Proof of Sherman-Morrison formula

$(A+uv^t)x=b$ △

$\xrightarrow{\times A^{-1}} A^{-1}(A+uv^t)x = A^{-1}b = y$

$(I+A^{-1}uv^t)x = x + \underbrace{A^{-1}u}_{w}\,\underbrace{v^t x}_{\alpha \to \text{scalar}} = y$

$x + w\alpha = y$ *

$\xrightarrow{\times v^t} v^t x + v^t w \alpha = v^t y \longrightarrow \alpha + v^t A^{-1} u \alpha = v^t y$

$$\alpha = \frac{v^t y}{1+v^t A^{-1} u}$$

* $\longrightarrow x = y - w\alpha = y - \dfrac{w v^t y}{1+v^t A^{-1} u}$

$x = A^{-1}b - \dfrac{A^{-1}uv^tA^{-1}b}{1+v^tA^{-1}u} = \left(A^{-1} - \dfrac{A^{-1}uv^tA^{-1}}{1+v^tA^{-1}u}\right)b$ 1

△ $\longrightarrow x = (A+uv^t)^{-1}b$ ۲

1 , ۲ $\longrightarrow (A+uv^t)^{-1} = A^{-1} - \dfrac{A^{-1}uv^tA^{-1}}{1+v^tA^{-1}u}$

اثبات از راه ضرب کردن

$$(A+uv^t)\left(A^{-1} - \frac{A^{-1}uv^tA^{-1}}{1+v^tA^{-1}u}\right)$$

$$= AA^{-1} + uv^tA^{-1} - \frac{AA^{-1}uv^tA^{-1} + uv^tA^{-1}uv^tA^{-1}}{1+v^tA^{-1}u}$$

$$= I + uv^tA^{-1} - \frac{u(1+v^tA^{-1}u)v^tA^{-1}}{\underbrace{1+v^tA^{-1}u}_{\text{scalar}}}$$

$$= I + uv^tA^{-1} - uv^tA^{-1} = I$$

reverse direction

$$\left(A^{-1} - \frac{A^{-1}uv^tA^{-1}}{1+v^tA^{-1}u}\right)(A+uv^t) = I + A^{-1}uv^t - \frac{A^{-1}uv^t + A^{-1}uv^tA^{-1}uv^t}{1+v^tA^{-1}u}$$

$$= I + A^{-1}uv^t - \frac{A^{-1}uv^t(1+A^{-1}uv^t)}{1+v^tA^{-1}u}$$

$$= I + A^{-1}uv^t - A^{-1}uv^t = I$$