18.369 Problem Set 4

Due Wed, 20 March 2024.

Problem 1: Projection operators

(You can use the Great Orthogonality Theorem [GOT] and/or the various orthogonality results for the rows/columns of the character table.)

(a) The representation-theory handout gives a formula for the projection operator from a function onto its component that transforms as a particular representation. Using the GOT, pove the correctness of this formula: in particular, for any function ψ, show that $\hat{P}_i^{(\alpha)}\psi$ transforms as the i-th partner function representation α. (You can use the fact, from class, that we showed that any function can be decomposed into a sum of partner functions of irreducible representations. Consider what $\hat{P}_i^{(\alpha)}$ does to one of thes partner functions.)

(b) Prove that the sum of the projection operators $\hat{P}^{(\alpha)}$ over all the irreducible representations α gives $\sum_\alpha \hat{P}^{(\alpha)} = \sum_{\alpha,i} \hat{P}_i^{(\alpha)} = 1$ (the identity operator), by using the column-orthogonality property of the character table.

(c) Prove that the projection operators $\hat{P}_i^{(\alpha)}$ are Hermitian for any unitary representation $D^{(\alpha)}$, assuming that $\hat{g}$ is unitary for all $g \in G$ (unitarity of $\hat{g}$ is trivial to prove for symmetry groups where g is a rotation and/or translation).

(d) Using the previous parts, prove that $\langle \phi_i^{(\alpha)}, \psi_j^{(\beta)} \rangle = 0$ if $\phi_i^{(\alpha)}$ and $\psi_j^{(\beta)}$ are partner functions of different irreducible representations $\alpha \neq \beta$, or if they correspond to different components $i \neq j$ of the same (unitary) representation $\alpha = \beta$. (Hint: insert *1* into the inner product.)

Problem 2: A square metal box

In class, we considered a two-dimensional (xy) problem of light in an $L \times L$ square of air ($\varepsilon = 1$) surrounded by perfectly conducting walls (in which $\mathbf{E} = 0$). We solved the case of $\mathbf{H} = H_z(x,y)\hat{\mathbf{z}}$ and saw solutions corresponding to five different representations of the symmetry group ($C_{4\text{v}}$).

(a) Solve for the eigenfunctions of the other polarization: $\mathbf{E} = E_z(x,y)\hat{\mathbf{z}}$ (you will need the $\mathbf{E}$ eigenproblem from problem set 1), with the boundary condition that $E_z = 0$ at the metal walls. (These are separable, i.e. $X(x)Y(y)$ for some functions X and Y.)

(b) Sketch (or plot on a computer) and classify these solutions according to the representations $\Gamma_{1\ldots5}$ of $C_{4\text{v}}$ enumerated in class. (Like in class, you will get some reducible accidental degeneracies.) Look at enough solutions to find *all five* irreps, and to illustrate the general pattern (you should find that the irreps appear in repeating patterns).

Problem 3: Group Velocity and Material Dispersion

In class, we showed (following the book) that the group velocity $d\omega/dk = \langle H_k, \frac{\partial \hat{\Theta}_k}{\partial k} H_k \rangle / \langle H_k, H_k \rangle$ was equal to Poynting flux divided by energy density (both averaged over the unit cell).

Now, go through the same derivation, but in this case assume that we have a lossless *dispersive* material with a real $\varepsilon(\mathbf{x},\omega)$, still assuming $\mu = 1$ for simplicity. In this case, when you take the k derivative (assuming $\mathbf{k} = k\hat{x}$, i.e. $k = k_x$), apply the chain rule to obtain a term with $\frac{\partial \varepsilon}{\partial \omega}\frac{d\omega}{dk}$ on the right-hand side. Solve for $d\omega/dk$ and show that it is Poynting flux divided by energy density, but the energy density is now the "Brillouin" energy density of a lossless dispersive medium, which we gave in the notes for Lecture 6:

$$\frac{1}{4}\left[\frac{\partial(\omega\varepsilon)}{\partial\omega}|\mathbf{E}|^2 + |\mathbf{H}|^2\right]$$

(for $\mu = 1$, where we have an additional 1/2 factor from the time-average).