18.369 Take-Home Midterm Exam: Spring 2024

Posted 3pm Tuesday April 9, due **3pm Wednesday April 10.**

Problem 0: Honor code

Copy and sign the following in your solutions:

I have not used any resources to complete this exam other than my own 18.369 notes, the textbook, and posted 18.369 course materials.

your signature

Problem 1: (5+5+5 points)

Recall from class that if we have an x-periodic problem with period a, i.e. its symmetry group consists of translations $\hat{n}f(x) = f(x-na)$ for all integers n, then the eigenfunctions of $\hat{n}$ are Bloch-periodic functions $f_k(x)e^{ikx}$ where f_k is periodic, corresponding to eigenvalues $D^{(k)}(n) = e^{-ikna}$ for arbitrary real k (but $D^{(k)} = D^{(k+\frac{2\pi}{a})}$). Equivalently, these are the partner functions of the 1d irreps $D^{(k)}$ (which are, in fact, the only possible unitary irreps).

(a) Suppose that space is looped around in a torus/circle with period Na, so that translation $\hat{N}$ is equivalent to translation $\hat{0}$ (= identity). In this case, what are the irreps?

(b) For this finite N-period torus, what are the projection operators for your irreps, applied to some arbitrary function $f(x)$? The projection operator involves a sum, and (as you saw in pset 4) the sum of the projection operators over all irreps is the identity — show that in the limit $N \to \infty$ (infinite periodic systems), at least one of these two sums becomes an integral (you may need to rearrange a normalization factor).

(c) Suppose that we have Maxwell's equations in an x-periodic system $\varepsilon(x,y,z) = \varepsilon(x+a,y,z)$. You saw in pset 1 that the response to a time-harmonic current source is given by the solution to $(\nabla\times\nabla\times - \omega^2\varepsilon)\mathbf{E} = i\omega\mathbf{J}$ ($\varepsilon_0 = \mu_0 = c = 1$ units); here, the operator on the left commutes with $\hat{n}$. Now, however, suppose that $\mathbf{J}$ is *not* periodic or Bloch-periodic, and just is some arbitrary current distribution (e.g. maybe it is localized in a finite region of space).

Explain how we could still solve for $\mathbf{E}$ by solving a *set* of problems with Bloch-periodic boundaries and summing or integrating the results. (Hint: apply your projection operator.)

(This is a very practical computational tool for modeling periodic systems with non-periodic sources!)

Problem 2 (15 points)

Refer to the solutions for problem 3 of pset 2 (recently updated to fix a couple of typos), in which we put a time-harmonic current source $\mathbf{J} = f(x)\delta(y)e^{-i\omega t}\hat{x}$ into a PEC waveguide (waves traveling along the y direction) and found the (H_z-polarized) solutions as an infinite cosine series of propagating + evanescent modes.

Suppose that the cosine-series coefficients of $f(x)/2$ are $a_n = 0$ for n even and $a_n = \frac{1}{n^2}$ for n odd (corresponding to $f(x)$ proportional to $x - L/2$, but you need not go through the integrals). Compute the rate of power $P(\omega)$ expended by the current source (e.g. with the help of Poynting's theorem), and sketch a plot of $P(\omega)$ a function of frequency $\omega \geq 0$ (capturing the main qualitative features).

Problem 3 (5+5+5 points)

In class, we have mostly considered the simplified case where the electric polarization density $\mathbf{P}$ responds *instantaneously* and linearly to the electric field at the same point in space: $\mathbf{P}(\mathbf{x}) = \chi_e \mathbf{E}(\mathbf{x})$ for some constant "susceptibility" $\chi_e = \varepsilon - 1$. More generally, however, changing $\mathbf{P}$ involves physically moving electric charges, and so it cannot happen instantaneously.

Imagine a simple model in which the charges are attached to immobile atoms with linear springs, so that they can bounce back and forth with some frequency ω_0 in the absence of external forces. (This actually turns out to be a reasonable semiclassical model of an atom with an "electric dipole transition" energy difference $\hbar\omega_0$ between a ground state and an excited state.) Of course, there also needs to be some "friction" so that the atom slowly settles down (to its ground state) when it is left alone. A simple model for this is to describe $\mathbf{P}$ by a damped harmonic oscillator at each point in space:

$$\ddot{\mathbf{P}} = -\omega_0^2 \mathbf{P} - \gamma_0 \dot{\mathbf{P}} + \sigma_0 \mathbf{E}$$

for a frequency ω_0, a damping rate $\gamma_0 > 0$, and a coupling coefficient $\sigma_0 > 0$ to the electric field at that point in space.

(a) Assuming time-harmonic fields $\mathbf{E}$ and $\mathbf{P}$ proportional to $e^{-i\omega t}$, show that you obtain the same Maxwell equations as in class but with a *frequency-dependent* permittivity $\varepsilon(\omega, \mathbf{x})$ (where the $\mathbf{x}$ dependence arises if your coefficients above vary with $\mathbf{x}$).

(b) Sketch a plot of $\Re\varepsilon - 1 = \Re\chi_e$ and $\Im\varepsilon = \Im\chi_e$ versus ω/ω_0, in the regime where the friction term is fairly small (but not zero!) compared the ω_0 term (i.e. where the loss rate is small, say 5%, compared to ω_0). $\Im\varepsilon$ should have a peak—what parameters determine the width, amplitude, and location of this peak?

(c) Show that your equations are consistent with the passivity condition $\omega\Im[\varepsilon] \geq 0$ from class.