18.369 Problem Set 3

Problem 0: Scattering and energy

As in the course notes, we can write Maxwell's equations for the 6-component fields ψ as $\frac{\partial \psi}{\partial t} = \hat{C}\psi - \frac{\partial \phi}{\partial t} - \xi$, where the 6-component polarizations $\phi = \chi * \psi$ are a convolution in time with a (passive and causal) linear susceptibility $\chi(\mathbf{x},t)$ at each point $\mathbf{x}$ in space. Now, suppose we have the situation depicted in Fig. 1: sources ($\xi \neq 0$) lying *outside* some bounded volume Ω produce an incident wave ψ_{inc} that scatters off materials ($\chi \neq 0$) *inside* of Ω to produce a scattered wave ψ_{scat}. Outside of Ω there are no materials ($\chi = 0$), while inside of Ω there are no sources ($\xi = 0$).

(a) Write the total fields as $\psi = \psi_{\text{inc}} + \psi_{\text{scat}}$, where $\psi_{\text{inc}} = \begin{pmatrix} \mathbf{E}_{\text{inc}} \\ \mathbf{H}_{\text{inc}} \end{pmatrix}$ satisfies $\frac{\partial \psi_{\text{inc}}}{\partial t} = \hat{C}\psi_{\text{inc}} - \xi$ (Maxwell's equations with the same currents but no materials), and the initial conditions at $t = -\infty$ are $\psi = \psi_{\text{inc}} = \psi_{\text{scat}} = 0$.

  (i) What differential[1] equation must ψ_{scat} satisfy?

  (ii) Fourier transform in time to find equations for $\hat{\psi}_{\text{inc}}(x,\omega)$ and $\hat{\psi}_{\text{scat}}$ (in terms of $\hat{\xi}$ and $\hat{\chi}$); equivalently, look for time-harmonic solutions $\hat{\psi}_{\text{inc}} e^{-i\omega t}$ and $\hat{\psi}_{\text{scat}} e^{-i\omega t}$ from a time-harmonic source $\hat{\xi}(\mathbf{x})e^{-i\omega t}$.

(b) Applying Poynting's theorem and passivity to your time-harmonic fields, show that $\oiint_{\partial\Omega} \text{Re}[\hat{\mathbf{E}}^*_{\text{inc}} \times \hat{\mathbf{H}}_{\text{inc}}] \cdot d\mathbf{A} = 0$ and $P_{\text{abs}} = -\oiint_{\partial\Omega} \text{Re}[\hat{\mathbf{E}}^* \times \hat{\mathbf{H}}] \cdot d\mathbf{A} \geq 0$ (the absorbed power = inward total flux). (Note that here, Ω is a subset of the whole domain!)

(c) In order to ensure $P_{\text{scat}} = \oiint_{\partial\Omega} \text{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A} \geq 0$ (the scattered power = outward scattered flux), we need something more: an *outgoing boundary condition* (or "radiation" boundary condition): if we consider *any point* on sphere of radius R (around any point in Ω), then *in the limit* as $R \to \infty$ we require that $R^2\,\text{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}]$ must point *radially outward*. Use Poynting's theorem to show that such a radiation boundary condition implies $\oiint_{\partial\Omega} \text{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A} \geq 0$.[2]

(d) Show that the "extinguished power" $P_{\text{ext}} = P_{\text{abs}} + P_{\text{scat}}$ can be written in terms of an integral over $\partial\Omega$ involving $\hat{\mathbf{E}}^*_{\text{inc}} \times \hat{\mathbf{H}}_{\text{scat}}$ and $\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{inc}}$ *only*. This is a version of a famous result known as the **optical theorem**.

(e) Derive an alternative form of the optical theorem: show that P_{ext} can be written as an integral over Ω involving *only* $\hat{\psi}^*_{\text{inc}}\hat{\phi}$. (Hint: use Poynting's theorem to relate P_{abs} and P_{scat} to work on polarization currents, and write $\psi_{\text{scat}} = \psi - \psi_{\text{inc}}$.)

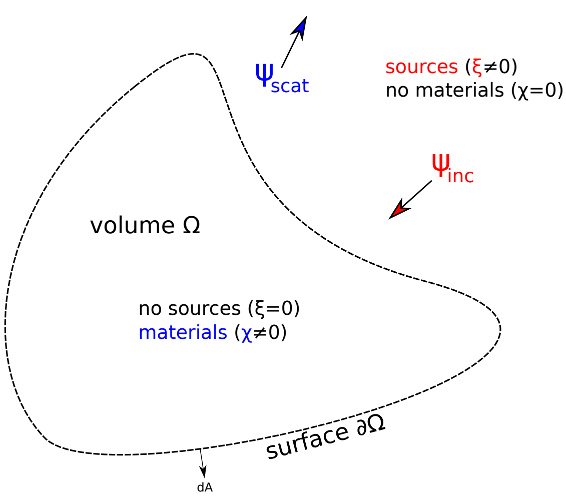

Figure 1: Schematic of an incident wave ψ_{inc} created by sources ξ that are only $\neq 0$ *outside* some volume Ω, interacting with materials $\chi \neq 0$ that lie *inside* Ω to produce an outgoing scattered wave ψ_{scat}.

Problem 1: Periodic waveguide guidance proof

In class, we showed by a variational proof that any $\varepsilon(y)$, in two dimensions, gives rise to at least one guided mode whenever $\varepsilon(y)^{-1} = \varepsilon_{\text{lo}}^{-1} - \Delta(y)$ for $\int \Delta > 0$ and $\int |\Delta| < \infty$.[3] At least, we showed it for the TE polarization ($\mathbf{H}$ in the $\hat{\mathbf{z}}$ direction). Now, you will show the same thing much more generally, but using the same basic technique.

[1] Technically, if you include a convolution $\chi *$ then you have a "pseudodifferential" operator.

[2] We also will also need some boundary condition to uniquely determine ψ_{inc}. Not necessarily an outgoing boundary condition, however—often it is convenient to have an incident wave coming in "from infinity", such as an incident planewave, perhaps with no explicit sources ($\xi = 0$) and just appropriate boundary conditions.

[3] As in class, the latter condition on Δ will allow you to swap limits and integrals for any integrand whose magnitude is bounded above by some constant times $|\Delta|$ (by Lebesgue's dominated convergence theorem).

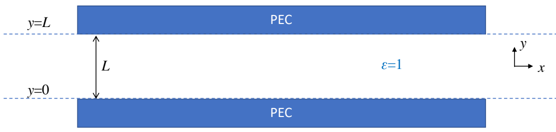

Figure 2: 2d metallic waveguide with perfect electric conductor walls, extending infinitely in the x direction.

(a) Let $\varepsilon(x,y)^{-1} = 1 - \Delta(x,y)$ be a periodic function $\Delta(x,y) = \Delta(x+a,y)$, with $\int |\Delta| < \infty$ and $\int_0^a \int_{-\infty}^{\infty} \Delta(x,y)dxdy > 0$. Prove that at least one TE guided mode exists, by choosing an appropriate (simple!) trial function of the form $\mathbf{H}(x,y) = u(x,y)e^{ikx}\hat{\mathbf{z}}$. That is, show by the variational theorem that $\omega^2 < c^2k^2$ for the lowest-frequency eigenmode. (It is sufficient to show it for $|k| \le \pi/a$, by periodicity in k-space; for $|k| > \pi/a$, the light line is not $\omega = c|k|$.)

(b) Prove the same thing as in (a), but for the TM polarization ($\mathbf{E}$ in the $\hat{\mathbf{z}}$ direction). Hint: you will need to pick a trial function of the form $\mathbf{H}(x,y) = [u(x,y)\hat{\mathbf{x}} + v(x,y)\hat{\mathbf{y}}]e^{ikx}$ where u and v are some (simple!) functions such that $\nabla \cdot \mathbf{H} = 0$.[4]

Problem 2: 2d metallic waveguide

Consider a waveguide depicted in Fig.~2: two x-invariant PEC (perfect electric conductor) walls enclosing a width-L region of $\varepsilon = 1$.

(a) Find the E_z-polarized eigenfunctions. From symmetry, this should be of the form $\mathbf{E} = E_{k,n}(y)e^{ikx-i\omega_n t}$. Hence, obtain the dispersion relation $\omega_n(k)$ for $n = 1, 2, \ldots$. Plot it for the first few bands.

(b) Now, suppose we put in a "point-dipole" current source $\mathbf{J}(x,y) = \delta(x)\delta(y-L/2)\hat{z}e^{-i\omega t}$ in the middle of the waveguide. This is mirror-symmetric (even), so you should be able to find a mirror-symmetric solution that consists of right-traveling ($k > 0$) waves for $x > 0$ and left-traveling ($k < 0$) waves for $x < 0$, where in each region ($x \gtrless 0$) you expand the solution *in the basis of the eigenfunctions* (the solutions of the source-free Maxwell equations):

$$E_z = e^{-i\omega t} \begin{cases} \sum_{n=1}^{\infty} c_n E_n(y) e^{+ik_n x} & x \ge 0 \\ \sum_{n=1}^{\infty} c_n E_n(y) e^{-ik_n x} & x < 0 \end{cases},$$

where c_n are some coefficients to be determined, k_n is the positive solution of $\omega_n(k_n) = \omega$ (i.e. the $k > 0$ matching $\omega$for mode n), and $E_n = E_{k_n,n}$ is the corresponding eigenfunction. (Note that this E_z is continuous through $x = 0$.)

When you solve $\omega_n(k_n) = \omega$ for k_n, you should find that k_n becomes *purely* imaginary for $\omega < \omega_n(0)$ (the "cutoff" for mode n), corresponding to exponentially decaying/growing "evanescent" solutions. In an empty waveguide, we never include such solutions because they are growing exponentially towards either $x \to +\infty$ or $-\infty$. However, now that our current source has broken the translational symmetry, you *must include* these evanescent solutions in your infinite series above: choose the signs so the imaginary-k_n solutions *decay* (not blow up!) as $|x| \to \infty$. (Without these solutions, you won't have the infinite series you need to solve Maxwell's equations with the current source.)

Solve for the coefficients c_n by plugging this E_z into Maxwell's equations with that current source.

Hint: By construction, E_z is continuous with a discontinuous derivative $\frac{\partial}{\partial x}$ at $x = 0$, and it follows that the *second* derivative $\frac{\partial^2 E_z}{\partial x^2}$ yields a delta function $\delta(x)$ multiplied by the discontinuity in $\frac{\partial E_z}{\partial x}$, in addition to a $-k_n^2$ term from the "ordinary" second derivative for $x \neq 0$. (Recall that the derivative of a discontinuous function is a delta function multiplied by the amplitude of the discontinuity.[5]) Maxwell's equations contain a $\nabla^2 E_z$ term, so this $\delta(x)$ in $\frac{\partial^2 E_z}{\partial x^2}$ is exactly what you need to match the $\delta(x)$ in the source-current term. That is, you will get a term $\delta(x)\sum c_n \cdot$(some Fourier basis), and then you can use the usual Fourier orthogonality relations to match this to $\delta(x)\delta(y-L/2)$.

[4] You might be tempted, for the TM polarization, to use the $\mathbf{E}$ form of the variational theorem that you derived in problem 1, since the proof in that case will be somewhat simpler: you can just choose $\mathbf{E}(x,y) = u(x,y)e^{ikx}\hat{\mathbf{z}}$ and you will have $\nabla \cdot \varepsilon\mathbf{E} = 0$ automatically. However, this will lead to an inequivalent condition $\int(\varepsilon - 1) > 0$ instead of $\int \Delta = \int \frac{\varepsilon-1}{\varepsilon} > 0$.

[5] See `https://math.mit.edu/~stevenj/18.303/delta-notes.pdf` for some introductory notes on how to think "properly" about delta functions and derivatives. For this problem set, however, whatever handwavy rule of thumb you were taught about delta functions should suffice.

(c) Using your answer from the previous part, give a formula for the time-average power P *expended* by the current source ($P = -\frac{1}{2}\operatorname{Re}\int \mathbf{J}^* \cdot \mathbf{E}\,dxdy$, from class). (This power is also called the "local density of states" (LDOS) at that position and frequency, a subject we will return to later in class.) Sketch this power $P(\omega)$ as a function of frequency ω—notice anything interesting? (Something dramatic should happen near each "mode cutoff" where k_n changes from real to imaginary. And what happens for $\omega < \omega_1(0)$, where there are no real-k eigenmodes at all?)

(d) Using Meep (either installed on your own computer or via mybinder.org), compute and polot $P(\omega)$ (in arbitrary units) for exactly the situation in the previous part, with $L = 1$ (= a choice of distance units). It turns out that Meep has a built-in feature to compute the power expended by a dipole source (the "LDOS"): google "Meep LDOS tutorial". Compare to your analytical solution from the previous part.