Notes on time evolution and conservation laws in electromagnetism

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1 Maxwell's equations

The time evolution of Maxwell's equations is governed by Ampere's law $\nabla \times \mathbf{H} = \frac{\partial}{\partial t}(\mathbf{E} + \mathbf{P}) + \mathbf{J}$ and Faraday's Law $-\nabla \times \mathbf{E} = \frac{\partial}{\partial t}(\mathbf{H} + \mathbf{M}) + \mathbf{K}$, where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, $\mathbf{P}$ and $\mathbf{M}$ are the electric and magnetic polarization densities, and $\mathbf{J}$ and $\mathbf{K}$ are the electric and magnetic current densities, respectively. (Magnetic current is the flow of magnetic charge, which is experimentally zero, but it is convenient to include the possibility of $\mathbf{K} \neq 0$ theoretically.) It is nice to write this in the form $\hat{C}\psi = \frac{\partial}{\partial t}(\psi + \phi) + \xi$, or:

$$\frac{\partial}{\partial t} \underbrace{\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}}_{\psi} = \underbrace{\begin{pmatrix} & \nabla\times \\ -\nabla\times & \end{pmatrix}}_{\hat{C}} \psi - \frac{\partial}{\partial t} \underbrace{\begin{pmatrix} \mathbf{P} \\ \mathbf{M} \end{pmatrix}}_{\phi} - \underbrace{\begin{pmatrix} \mathbf{J} \\ \mathbf{K} \end{pmatrix}}_{\xi},$$

so that ψ is the six-component field state, ϕ is the six-component polarization, ξ is the six-component current, and $\hat{C}$ is the 6×6 "curl" operator. (For these notes I am using "natural" units in which $\varepsilon_0 = \mu_0 = 1$.) Our goal in these notes is to analyze some of the key algebraic properties of these equations, especially as they relate to time evolution and conservation laws. The basics of electromagnetic energy can be found in standard textbooks [1], but a more complete treatment of arbitrary dispersive materials can be found in the literature as recently reviewed and extended by Welters [2].

1.1 The anti-Hermitian curl operator

The 6×6 curl operator $\hat{C}$ is anti-Hermitian ($\hat{C}^\dagger = -\hat{C}$) under the usual inner product $\langle \psi, \psi' \rangle = \int \mathbf{E}^* \cdot \mathbf{E}' + \mathbf{H}^* \cdot \mathbf{H}'$ with most boundary conditions. The key identity [1] to derive this is, as usual,

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}).$$

Using this, we can integrate by parts over the domain Ω:

$$\begin{aligned} \langle \psi, \hat{C}\psi' \rangle &= \int_\Omega \left[\mathbf{E}^* \cdot (\nabla \times \mathbf{H}') - \mathbf{H}^* \cdot (\nabla \times \mathbf{E}') \right] dV \\ &= \int_\Omega \left[(\nabla \times \mathbf{E})^* \cdot \mathbf{H}' - (\nabla \times \mathbf{H})^* \cdot \mathbf{E}' - \nabla \cdot (\mathbf{E}^* \times \mathbf{H}' + \mathbf{E}' \times \mathbf{H}^*) \right] dV \\ &= \langle -\hat{C}\psi, \psi' \rangle - \oiint_{\partial\Omega} (\mathbf{E}^* \times \mathbf{H}' + \mathbf{E}' \times \mathbf{H}^*) \cdot d\mathbf{A}, \end{aligned}$$

where $\partial\Omega$ denotes the boundary of Ω and $d\mathbf{A}$ is the outward normal. Hence, if we have boundary conditions such that the surface integral ($\oiint_{\partial\Omega}$) vanishes, then $\hat{C}^\dagger = -\hat{C}$. This will be the case in

an infinite domain for square-integrable fields (which must decay faster than $1/|\mathbf{x}|$ asymptotically), or in a finite domain for periodic boundary conditions (where the boundary terms cancel), or in a finite domain with perfect-electric-conductor (PEC) boundary conditions (where $\mathbf{E}$ is normal to the boundary on $\partial\Omega$), or for perfect-magnetic conductor (PMC) boundary conditions (where $\mathbf{H}$ is normal to the boundary).

One consequence of this, which we use below, is that

$$2\,\mathrm{Re}\langle\psi, \hat{C}\psi\rangle = \langle\psi, \hat{C}\psi\rangle + \langle\hat{C}\psi, \psi\rangle = \langle\psi, \hat{C}\psi\rangle + \langle\psi, -\hat{C}\psi\rangle = 0,$$

i.e. $\langle\psi, \hat{C}\psi\rangle$ is purely imaginary.

1.2 Polarizations and susceptibilities in linear time-invariant media

In *linear time-invariant* medium, the polarizations ϕ arise from the fields ψ in a linear convolution relation

$$\phi(t) = \chi * \psi|_t = \int_{-\infty}^{+\infty} \chi(t-t')\psi(t')dt',$$

where $\chi(t)$ is a 6×6 *susceptibility* matrix [1, 2]. That is, the polarization is proportional to a weighted average of the fields at different times.[1] Physically, we require the material to be *causal*: polarizations must come *after* the fields that create them, not before, which leads to the condition that $\chi(t) = 0$ for $t < 0$. As we discuss below, however, in a linear medium we will also require *passivity*—the material can dissipate energy but not supply energy—and it turns out that the assumption of passivity alone is enough to guarantee causality [2].

In the frequency domain, the Fourier transform of a convolution is a multiplication. That is, if we denote the Fourier transforms by $\hat{\phi}$, $\hat{\chi}$, and $\hat{\psi}$, then

$$\hat{\phi}(\omega) = \hat{\chi}(\omega)\hat{\psi}(\omega).$$

The frequency dependence of $\hat{\chi}(\omega)$ is known as *material dispersion.* Passivity [2], via causality [1], turns out to guarantee that $\hat{\chi}(\omega)$ is an *analytic* function (convergent Taylor series, no poles or other singularities) in the upper-half complex-ω plane (i.e. for $\mathrm{Im}\,\omega > 0$). Passivity also turns out to imply that $\mathrm{Im}[\omega\hat{\chi}(\omega)] = \frac{\omega\hat{\chi}(\omega) - \omega^*\hat{\chi}(\omega)^\dagger}{2i} \succeq 0$ (a positive-semidefinite imaginary part) in the upper-half plane (and in practice it is positive-definite for anything other than vacuum) [2]. Physically, the fields ψ and polarizations ϕ are real quantities, and this implies that $\hat{\chi}(-\omega) = \hat{\chi}(\omega)^*$ for real ω [1]. If one adds the physical constraint that the material must cease to respond at very high frequencies, and in particular that $\hat{\chi}(\omega)$ goes to zero faster than $1/|\omega|$ for large $|\omega|$, then one obtains the famous Kramers–Kronig relations that relate integrals of the real and imaginary parts of $\hat{\chi}(\omega)$ [1].

1.3 Dispersionless linear media

In many important cases it is possible to neglect material dispersion, in which case matters are greatly simplified. If we have a field $\psi(t)$ that is *bandlimited*—its Fourier transform $\hat{\psi}(\omega)$ is nonzero only in a small bandwidth around some frequency ω_0 (or around $\pm\omega_0$)—and in this bandwidth $\hat{\chi}$ is nearly constant, then we can approximate $\hat{\chi}(\omega) \approx \hat{\chi}(\omega_0)$. Furthermore, in most physical circumstances the matrix $\hat{\chi}$ block-diagonalizes

$$\hat{\chi}(\omega) = \begin{pmatrix} \varepsilon(\omega) - 1 & \\ & \mu(\omega) - 1 \end{pmatrix}$$

[1] Here, I have omitted the $\mathbf{x}$ dependence. Typically, we have a *local* medium, in which $\phi(t, \mathbf{x})$ depends on only on fields $\psi(t', \mathbf{x})$ at the same point $\mathbf{x}$ in space, but in some cases authors also consider *nonlocal* media in which ϕ can arise from fields at other points. Nonlocal models are only relevant in conductors at lengthscales of a few nanometers, however, and in practice local models are almost always accurate enough to ignore nonlocal effects.

where ε and μ are the electric permittivity and the magnetic permeability, respectively. (Nonzero off-diagonal blocks in $\hat{\chi}$ are known as *bi-anisotropic susceptibilities*, and are very rarely encountered—essentially unheard-of at infrared and optical frequencies.) ε and μ can be 3×3 matrices in general (*anisotropic materials*), but in the common case of *isotropic* materials they are scalars. (Furthermore, at infrared and optical frequencies, we almost always have $\mu \approx 1$.)

Neglecting dispersion, therefore, and assuming non-bianisotropic media, Maxwell's equations simplify to

$$\frac{\partial \psi}{\partial t} = \begin{pmatrix} \varepsilon^{-1} & \\ & \mu^{-1} \end{pmatrix} \left[\hat{C}\psi - \xi\right] = (1+\hat{\chi})^{-1}\left[\hat{C}\psi - \xi\right].$$

2 Conservation laws

2.1 Conservation of energy in lossless media

Suppose we have the simple case of lossless, dispersionless time-independent media with real $\varepsilon > 0$ and $\mu > 0$. In that case, define

$$\langle \psi, \psi' \rangle_{\hat{\chi}} = \langle \psi, (1+\hat{\chi})\psi' \rangle = \langle (1+\hat{\chi})\psi, \psi' \rangle = \int \left[\mathbf{E}^* \cdot \varepsilon \mathbf{E}' + \mathbf{H}^* \cdot \mu \mathbf{H}'\right].$$

This turns out to be a conserved "energy" (actually *twice* the physical energy in the fields, as shown below):

$$\begin{aligned} \frac{\partial}{\partial t}\langle \psi, \psi \rangle_{\hat{\chi}} &= \left\langle (1+\hat{\chi})\frac{\partial \psi}{\partial t}, \psi \right\rangle + \left\langle \psi, (1+\hat{\chi})\frac{\partial \psi}{\partial t} \right\rangle \\ &= \left\langle \hat{C}\psi - \xi, \psi \right\rangle + \left\langle \psi, \hat{C}\psi - \xi \right\rangle \\ &= -2\,\mathrm{Re}\langle \psi, \xi \rangle, \end{aligned}$$

where we have used the anti-Hermitian property of $\hat{C}$ from above. Thus, if there are no sources ($\xi = 0$) then $\langle \psi, \psi \rangle_{\hat{\chi}}$ is *conserved* (constant in time). More generally, as we shall see below, we can interpret $2\,\mathrm{Re}\langle \psi, \xi \rangle$ as being (proportional to) the work done on the currents by the fields, so that if this quantity is positive then the energy $\langle \psi, \psi \rangle_{\hat{\chi}}$ in the fields is decreasing.

Another way of expressing this is in terms of the *time-evolution operator*

$$\hat{U}_t = e^{(1+\hat{\chi})^{-1}\hat{C}t},$$

defined such that $\psi(t) = \hat{U}_t\psi(0)$ solves the source-free ($\xi = 0$) problem. Under the $\langle \psi, \psi' \rangle_{\hat{\chi}}$ inner product, this operator is *unitary* ($\hat{U}_t^\dagger = \hat{U}_t^{-1}$) because the exponent is anti-Hermitian:

$$\langle \psi, (1+\hat{\chi})^{-1}\hat{C}\psi' \rangle_{\hat{\chi}} = \langle \psi, \hat{C}\psi' \rangle = \langle -\hat{C}\psi, \psi' \rangle = \langle -(1+\hat{\chi})^{-1}\hat{C}\psi, \psi' \rangle_{\hat{\chi}}.$$

Unitary time-evolution is equivalent to conservation of norm (energy): $\langle \psi(t), \psi(t) \rangle_{\hat{\chi}} = \langle \hat{U}_t\psi(0), \hat{U}_t\psi(0) \rangle_{\hat{\chi}} = \langle \psi(0), \psi(0) \rangle_{\hat{\chi}}$.

2.2 Conservation of irrep in lossless media

As is reviewed by many authors [3], the key tool for describing the consequences of symmetry for linear systems is *group representation theory,* and it turns out that this yields a kind of conservation law from irreducible representations (*irreps*) of the symmetry group. If we have a symmetry group G (of rotations, reflections, and/or translations $g \in G$), recall that we can define the projection operator $\hat{P}^{(\alpha)} = \frac{d_\alpha}{|G|}\sum_{g\in G}\chi^{(\alpha)}(g)^*\hat{g}$, where $\chi^{(\alpha)}(g) = \mathrm{tr}\, D^{(\alpha)}(g)$ is the character of an

irreducible representation (*irrep*) $D^{(\alpha)}(g)$ of G with dimension d_α and $\hat{g}$ is the transformation (rotation/reflection/translation) acting on functions ψ (or ϕ or ξ) corresponding to the operation g on the coordinates.[2] $\hat{P}^{(\alpha)}\psi$ projects any function ψ onto a function (possibly zero) that transforms as a partner function (basis function) of the irrep α. It turns out that the irrep is "conserved" in the following sense: if your currents are partners of irrep α, and your fields ψ are partners of α at *any* time, then the fields are partners at *all* times.

If the currents ξ are partners (possibly zero) of α, that means $\hat{P}^{(\alpha)}\xi = \xi$ for all times. Furthermore, if G is the symmetry group of the problem, then by definition $\hat{P}^{(\alpha)}$ commutes with $(1+\hat{\chi})^{-1}\hat{C}$ and hence with the time-evolution operator $\hat{U}_t$. Suppose at time $t = 0$ our fields $\psi(0)$ are partners of α, i.e $\hat{P}^{(\alpha)}\psi(0) = \psi(0)$, where the dependence $\psi(t, \mathbf{x})$ on $\mathbf{x}$ is implied. We can write $\psi(t)$ via $\hat{U}_t$ as:

$$\psi(t) = \hat{U}_t\psi(0) - \int_0^t \hat{U}_{t-t'}\left(1+\hat{\chi}\right)^{-1}\xi(t')dt'.$$

(It is easily verified that this satisfies our PDE: just plug it in.) Therefore, since $\hat{P}^{(\alpha)}$ commutes with $\hat{U}_t$ and $(1+\hat{\chi})^{-1}$, it immediately follows that $\hat{P}^{(\alpha)}\psi(t) = \psi(t)$, and $\psi(t)$ is a partner of irrep α.

The simplest example is that of a problem with a mirror symmetry σ, in which case $G = \{E, \sigma\}$ where E is the identity. The irreps are $\chi^{(1)} = (1, 1)$ and $\chi^{(2)} = (1, -1)$, with the corresponding partners being even and odd functions. If you have even current sources, and even fields at *any* time, then fields are even at *all* times; similarly for odd currents and fields.

2.3 Poynting's theorem in lossless media

To make explicit the connection with the classical notion of energy, we can briefly derive Poynting's theorem to relate electromagnetic energy to mechanical energy [1]. The key to deriving Poynting's theorem is to compute the rate of *mechanical* work done by the fields on the currents. Recall that the electric field $\mathbf{E}$ is the force per unit charge, and $\mathbf{J}$ is current (charge/time) per area. If we think of $\mathbf{J}$ as a charge density ρ (charge/volume) moving at a velocity $\mathbf{v}$, then we can write $\mathbf{J} = \rho\mathbf{v}$. But since $\mathbf{E}\rho$ is force per unit volume, and force $\cdot\, \mathbf{v}$ is work per time, it follows that $\mathbf{E}\cdot\mathbf{J}$ is the work per time per volume done *by* the field *on* the charges. Similar for $\mathbf{H}\cdot\mathbf{K}$ for the (apparently fictitious) case of magnetic charges. Hence, we can interpret

$$\langle\psi, \xi\rangle$$

as the rate of change of mechanical energy (the power *expended* by the fields ψ on the currents ξ). From above,

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\langle\psi, \psi\rangle_{\hat{\chi}}\right] + \langle\psi, \xi\rangle = 0,$$

so if we interpret $\frac{1}{2}\langle\psi, \psi\rangle_{\hat{\chi}}$ as the *electromagnetic energy* (in a lossless, dispersionless system), then total energy is conserved.

Above, we assumed that the surface term from the integration-by-parts of $\hat{C}$ vanished, which is the case if we are considering a closed system from which no energy escapes (or enters). More generally, if we imagine that the integration domain Ω is some arbitrary volume—not necessarily the entire domain of the PDE—and put back in the surface terms from the integration by parts,

[2] If g is a 3×3 rotation matrix $g^{-1} = g^T$ acting on the coordinates $\mathbf{x}$, for example, then $\hat{g}\psi|_{\mathbf{x}} = \begin{pmatrix} g\mathbf{E}(g^{-1}\mathbf{x}) \\ g\mathbf{H}(g^{-1}\mathbf{x}) \det g \end{pmatrix}$, where the extra $\det g$ factor arises because the magnetic field is a pseudo-vector.

we obtain:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\langle\psi,\psi\rangle_{\hat{\chi}}\right] + \langle\psi,\xi\rangle = \frac{1}{2}\left[\langle\psi,\hat{C}\psi\rangle + \langle\hat{C}\psi,\psi\rangle\right]$$
$$= -\oiint_{\partial\Omega}[\mathbf{E}^* \times \mathbf{H}]\cdot d\mathbf{A},$$

where the *Poynting*[3] *flux* $\mathbf{E}^* \times \mathbf{H}$ is interpreted as the *electromagnetic energy flux*: the flow of energy (per time, per area) out through the boundary $\partial\Omega$; for physical, real fields the complex conjugation in $\mathbf{E}^*$ is irrelevant, but it becomes important in the Fourier domain (i.e. for time-harmonic fields) in Sec. 3. The left-hand side is the time derivative of the *total* (electromagnetic + mechanical) energy in Ω and the right-hand side is the rate at which energy is flowing *into* Ω. That is, since we defined $d\mathbf{A}$ as the *outward* normal, when the Poynting vector is *inwards* (into Ω) we obtain positive right-hand side and hence *increasing* total energy.

2.4 Poynting's theorem in dispersive/lossy media

Let us now return to the case of an arbitrary 6×6 dispersive medium whose polarization is a convolution $\phi = \chi * \psi$ with the susceptibility $\chi(t, \mathbf{x})$. In this case, we obtain:

$$\frac{\partial}{\partial t}\langle\psi,\psi\rangle = \langle\psi,\hat{C}\psi - \frac{\partial\phi}{\partial t} - \xi\rangle + \langle\hat{C}\psi - \frac{\partial\phi}{\partial t} - \xi,\psi\rangle$$

and hence

$$\boxed{\frac{\partial}{\partial t}\left[\frac{1}{2}\langle\psi,\psi\rangle\right] + \langle\psi,\frac{\partial\phi}{\partial t}\rangle + \langle\psi,\xi\rangle = -\oiint_{\partial\Omega}[\mathbf{E}^* \times \mathbf{H}]\cdot d\mathbf{A}}.$$

On the right-hand side is the Poynting flux into the domain, from the integration-by-parts of $\hat{C}$ as before. On the left hand side, the first term is the rate of change of the electromagnetic energy $\frac{1}{2}\langle\psi,\psi\rangle$ in vacuum ($\varepsilon = \mu = 1$): this can be thought of as the energy stored "purely" in the fields. The third term $\langle\psi,\xi\rangle$ is the rate of work done on the currents ξ as before, i.e. the rate of change of the mechanical energy in these "free" or "external" charges. The second term $\langle\psi,\partial\phi/\partial t\rangle$ is similar, where $\partial\phi/\partial t$ is the "bound" current [1]: this is a *physical* current corresponding of the motion of the polarizations (the little charges "bound" to the material that are polarizing in response to the fields). That is, the second term is the rate of work done *on the material* by the fields.

2.5 Conservation of irrep

The time-evolution operator $\hat{U}_t$ is more complicated to write down explicitly in a general dispersive medium, so I omit it here. However, it is still the case that it commutes with a projection operator $\hat{P}^{(\alpha)}$, because the time evolution must be invariant under the symmetry operations $\hat{g}$ (again, this is essentially the definition of the symmetry group of the problem). Hence it will still be the case (proof as in the lossless case) that if your currents are partners of irrep α, and the fields at some time t are partners of α, that the fields ψ will be partners of α at all times: the "irrep is conserved."

[3] Named after John Henry Poynting, who developed the first version of this equation in 1884 ("On the transfer of energy in the electromagnetic field," *Philosophical Transactions* vol. 175, pp. 343–361). The pre-vector notation, following Maxwell from 1865, is horrifying.

3 Complex time-harmonic fields: Instantaneous, real, and reactive power

In linear time-invariant systems, we often write sinusoidally oscillating "time-harmonic" quantities as the real parts of complex exponentials, simply because complex exponentials are much easier to work with than sines and cosines.[4] In the case of Maxwell's equations in linear time-invariant media, we can work with time-harmonic complex currents $\xi = \hat{\xi}(\mathbf{x})e^{-i\omega t}$, fields $\psi = \hat{\psi}(\mathbf{x})e^{-i\omega t}$, and polarizations $\phi = \hat{\phi}(\mathbf{x})e^{-i\omega t}$ (with $\hat{\phi} = \hat{\chi}\hat{\psi}$) and eventually take the real parts to get the physical solutions $\mathrm{Re}[\psi]$ etcetera. That is, linearity means that it doesn't matter if you take the real parts first and then solve Maxwell's equations, or solve Maxwell's equations and take the real parts at the end... and the latter is a lot easier! What does this mean for Poynting's theorem?

Since *both* the complex time-harmonic fields *and* their real parts solve Maxwell's equations, we can apply Poynting's theorem from above to *either* the "unphysical" complex quantities *or* to the physical real parts. If we apply it to the real parts, we obtain an equation involving the **instantaneous power**, which describes what the energy is actually doing in the real system at a microscopic timescale:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\langle \mathrm{Re}\,\psi, \mathrm{Re}\,\psi\rangle\right] + \langle \mathrm{Re}\,\psi, \frac{\partial(\mathrm{Re}\,\phi)}{\partial t}\rangle + \langle \mathrm{Re}\,\psi, \mathrm{Re}\,\xi\rangle = -\oiint_{\partial\Omega} [(\mathrm{Re}\,\mathbf{E}) \times (\mathrm{Re}\,\mathbf{H})] \cdot d\mathbf{A}.$$

For example, the instantaneous Poynting flux $(\mathrm{Re}\,\mathbf{E}^*) \times (\mathrm{Re}\,\mathbf{H})$ is an *oscillating* quantity, asd similarly for the instantaneous power $\langle \mathrm{Re}\,\psi, \mathrm{Re}\,\xi\rangle$ exerted on the currents. More explicitly, since $\mathrm{Re}\,z = \frac{z+z^*}{2}$, and recalling that the inner product $\langle\cdot,\cdot\rangle$ conjugates the first argument, we can see that

$$\begin{aligned}\langle \mathrm{Re}\,\psi, \mathrm{Re}\,\xi\rangle &= \frac{1}{4}\langle \psi + \psi^*, \xi + \xi^*\rangle = \frac{1}{4}\left(\langle\hat{\psi}^*, \hat{\xi}\rangle e^{-2i\omega t} + \langle\hat{\psi}, \hat{\xi}^*\rangle e^{+2i\omega t} + \langle\hat{\psi}, \hat{\xi}\rangle + \langle\hat{\psi}^*, \hat{\xi}^*\rangle\right) \\ &= \frac{1}{2}\left(\mathrm{Re}\left[\langle\hat{\psi}^*, \hat{\xi}\rangle e^{-2i\omega t}\right] + \mathrm{Re}\left[\langle\hat{\psi}, \hat{\xi}\rangle\right]\right),\end{aligned}$$

which is a sum of a rapidly oscillating term (as 2ω) and a constant term. Similarly for all of the other terms in Poynting's theorem.

In much of electromagnetism, however, the frequency ω is so large that the main quantities of interest are *time-averaged* powers. For example, visible light consists of frequencies $\omega/2\pi > 10^{14}$ Hz, so unless you have a rare detector that responds on femtosecond timescales all you will *ever* measure will be time-averaged quantities. The oscillating term in the instantaneous power above averages to zero, so that means that you will measure only $\frac{1}{2}\mathrm{Re}\langle\hat{\psi}, \hat{\xi}\rangle$ (and similarly for the other powers). This fact results in a neat algebraic trick (reviewed in Sec. 3.1): if we consider Poynting's theorem for the "artificial" *complex* fields (which solve Maxwell's equations!), take the real part of both sides, and divide by 2, we obtain the (so-called) **real power** Poynting's theorem

$$\frac{\partial}{\partial t}\left[\frac{1}{4}\langle\psi, \psi\rangle\right] + \frac{1}{2}\mathrm{Re}\langle\psi, \frac{\partial\phi}{\partial t}\rangle + \frac{1}{2}\mathrm{Re}\langle\psi, \xi\rangle = -\oiint_{\partial\Omega} \frac{1}{2}\mathrm{Re}\,[\mathbf{E}^* \times \mathbf{H}] \cdot d\mathbf{A},$$

which we can now interpret as the **time-averaged powers**. That is, $\frac{1}{4}\langle\psi, \psi\rangle$ is a time-averaged energy in the fields, $\frac{1}{2}\mathrm{Re}\langle\psi, \frac{\partial\phi}{\partial t}\rangle$ is a time-averaged power expended on the materials, $\frac{1}{2}\mathrm{Re}\langle\psi, \xi\rangle$ is a time-averaged ("real") power expended on the currents, and $\frac{1}{2}\mathrm{Re}\,[\mathbf{E}^* \times \mathbf{H}]$ is a time-averaged "real" Poynting flux (what you would measure on a camera, for example).

Looking at this "real power" is not the whole story, however, because the *imaginary part* of Poynting's theorem (for the complex fields) also yields a true equation

[4] This is called a "phasor" representation in electrical engineering, especially in circuit theory, where the duality of "real" and "reactive" power is more commonplace than in the full-Maxwell context.

$$\frac{1}{2}\operatorname{Im}\langle\psi, \frac{\partial\phi}{\partial t}\rangle + \frac{1}{2}\operatorname{Im}\langle\psi,\xi\rangle = -\oiint_{\partial\Omega}\frac{1}{2}\operatorname{Im}\left[\mathbf{E}^*\times\mathbf{H}\right]\cdot d\mathbf{A},$$

and in electrical engineering these terms are called **reactive powers**. It's more subtle to contrive a physical intuition about reactive power, but it is a useful constraint on the complex Maxwell solutions [4, 5, 6].

3.1 Complex numbers and time averages

For *any* time-harmonic quantities $a(t) = \hat{a}e^{-i\omega t}$ and $b(t) = \hat{b}e^{-i\omega t}$, the following relationships are extremely useful:

$$\text{time average of } \operatorname{Re}[a]\cdot\operatorname{Re}[b] = \frac{1}{2}\operatorname{Re}[a^*b] = \frac{1}{2}\operatorname{Re}[\hat{a}^*\hat{b}],$$

$$\text{time average of } \operatorname{Re}[a]^2 = \frac{1}{2}|a|^2 = \frac{1}{2}|\hat{a}|^2.$$

This can be derived by simply writing out the terms $(a+a^*)(b+b^*) = 2\operatorname{Re}(\hat{a}\hat{b}e^{-2i\omega t}) + 2\operatorname{Re}(\hat{a}^*\hat{b})$ as we did for $\langle\operatorname{Re}\psi, \operatorname{Re}\xi\rangle$ above, and noting that the first (oscillating) term time-averages to zero and the second term is a constant whose time average is itself. The second identity is merely a special case of the first.

Hence, if $\frac{1}{2}\left[\varepsilon\mathbf{E}\cdot\mathbf{E} + \mu\mathbf{H}\cdot\mathbf{H}\right]$ is the physical energy density in the field, then $\frac{1}{4}\left[\varepsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2\right]$ is the time-average energy density for time-harmonic fields. Similarly, $\frac{1}{2}\operatorname{Re}[\mathbf{E}^*\times\mathbf{H}]$ is the time-average Poynting flux for time-harmonic fields, and $\frac{1}{2}\operatorname{Re}\langle\psi,\xi\rangle$ is the time-average rate of work done on the currents. This is *extremely useful*! (But remember, only for time-harmonic fields!)

4 Passive linear media

A "passive" medium is one which does *not supply energy.* In fact, all linear materials are necessarily passive, because a linear "active" (or "gain") medium would lead to runaway exponential growth of the fields—all physical gain must be nonlinear, because any energy source has to run out at some point. Mathematically, passivity means that the fields can do net work on the medium but not vice versa. From above, the net work done *on* the medium *by* the fields is precisely the work done on the polarization currents, and passivity means that this must be nonnegative:

$$\int_{-\infty}^{t}\langle\psi, \frac{\partial}{\partial t}(\chi * \psi)\rangle dt' \geq 0$$

for all times t and all real-valued fields ψ (in a suitable function space) [2]. This constrains $\chi *$ to be a (real) "passive convolution operator," and it follows (surprisingly!) that χ is causal, its Fourier transform $\hat{\chi}(\omega)$ is analytic for $\operatorname{Im}\omega > 0$, and $\operatorname{Im}[\omega\hat{\chi}(\omega)] = \omega\frac{\hat{\chi}-\hat{\chi}^\dagger}{2i}$ is positive semidefinite for $\operatorname{Im}\omega > 0$ as mentioned above [2].

An elegant proof that passivity implies causality is as follows [7, appendix B]. Let $\psi_+(\mathbf{x},t)$ be any function that $= 0$ for $t < 0$, and let $\phi_+ = \chi * \psi_+$ be the corresponding polarization. Now, take any other function $\psi(\mathbf{x},t)$ and the corresponding $\phi = \chi * \psi$. Because convolution is linear, $\psi + \lambda\psi_+$ (which $= \psi$ for $t < 0$), for any real λ, produces a polarization $\phi + \lambda\phi_+$. If we plug this into the passivity condition above for $t < 0$, we obtain

$$\int_{-\infty}^{t}\langle\psi, \frac{\partial\phi}{\partial t}\rangle dt' + \lambda\int_{-\infty}^{t}\langle\psi, \frac{\partial\phi_+}{\partial t}\rangle dt' \geq 0 \qquad (t<0).$$

Now, since this must be true for *any* value of λ, even for $|\lambda|\to\infty$, it follows that the second term must be zero. Moreover, this must be true for any ψ, so it follows that we must have $\frac{\partial\phi_+}{\partial t} = 0$ for

all $t < 0$. If the system starts at rest, so that $\phi_+ = 0$ for $t \to -\infty$, it follows that $\phi_+ = 0$ for $t < 0$. Since $\phi_+(\mathbf{x}, t) = \int_0^\infty \chi(t - t')\psi_+(\mathbf{x}, t')dt' = 0$ for all $t < 0$ and for all ψ_+, it follows that $\chi(t) = 0$ for all $t < 0$, i.e. that the susceptibility χ is causal.

Note that it is tempting to say that passivity means that $\langle \psi, \frac{\partial \phi}{\partial t} \rangle \geq 0$, i.e. that the *rate* of work done on the material is nonnegative, but this is not actually true. Energy can be temporarily stored in the polarization of matter and then extracted by the fields, as long as you don't get out more energy than you put in. (Physically, imagine a photon that is absorbed by an atom to pump the atom into an excited state, but the atom subsequently undergoes spontaneous emission to re-emit the photon. Semiclassically, such a resonant absorption/re-emission is described by a *resonance* in $\hat{\chi}$: a pole in $\hat{\chi}$ at a complex frequency ω_p in the lower-half plane. The imaginary part $\operatorname{Im} \omega_p < 0$ is due to dissipation losses: physically, it represents the chance that the atom drops back down to its ground state by a non-radiative process, e.g. a vibrational collision.) Below, we will see that this $\operatorname{Re}\langle \psi, \frac{\partial \phi}{\partial t} \rangle$ term actually gives rise to the $(\varepsilon - 1)|\mathbf{E}|^2 + (\mu - 1)|\mathbf{H}|^2$ term that appears in the energy density of a lossless, dispersionless medium, and which represents the energy that is temporarily stored in the medium, only to be re-extracted into the fields.

4.1 Dissipation in the frequency domain

The rate of work done by the fields on the material, which is the rate of *energy dissipation* in the material, is given above by $\operatorname{Re}\langle \psi, \frac{\partial \phi}{\partial t} \rangle$. Suppose we Fourier transform to the frequency domain, so that we are looking at time-harmonic fields with amplitudes $\hat{\psi}(\omega)$ and polarizations $\hat{\phi}(\omega) = \hat{\chi}(\omega)\hat{\psi}(\omega)$. The polarization current $\partial \phi / \partial t$ Fourier transforms to $-i\omega\hat{\phi}(\omega)$, and so we get a time-average dissipation rate (at ω) of:

$$\operatorname{Re}\langle \hat{\psi}, -i\omega\hat{\chi}(\omega)\hat{\psi}(\omega) \rangle = \int_\Omega \hat{\psi}^* \cdot \operatorname{Re}[-i\omega\hat{\chi}]\, \hat{\psi} = \int_\Omega \hat{\psi}^* \cdot \operatorname{Im}[\omega\hat{\chi}]\, \hat{\psi},$$

where $\operatorname{Im}[\omega\hat{\chi}] = \omega \frac{\hat{\chi} - \hat{\chi}^\dagger}{2i}$ is a direct measure of *how dissipative* the material is at ω. Transparent materials are ones for which $\operatorname{Im}[\hat{\chi}] \approx 0$ (i.e., $\hat{\chi}$ is Hermitian) in the desired frequency bandwidth.

As mentioned above, one of the consequences of passivity is that $\operatorname{Im}[\omega\hat{\chi}] \geq 0$ (in the matrix sense: positive semidefinite) for $\operatorname{Im} \omega > 0$, by taking the limit as $\operatorname{Im} \omega \to 0^+$ one can obtain the same result on the real-ω axis (albeit with some technicalities to account for the possibility of singularities on the real axis [2]), so that $\operatorname{Im}[\omega\hat{\chi}] \geq 0$ for real frequencies ω in a passive medium, and the magnitude of this imaginary part is a direct measure of how lossy the medium is at that frequency.

4.2 Dynamical and dispersive energy densities

From our Poynting theorem above, if we want to define an "electromagnetic energy density" such that the sum of electromagnetic + mechanical energy of free charges is conserved in the absence of incoming/outgoing Poynting flux, we should define the electromagnetic energy density as

$$\frac{1}{2}(\psi^* \cdot \psi) + \int_{-\infty}^{t} \left[\psi^* \cdot \frac{\partial}{\partial t}(\chi * \psi) \right] dt'.$$

The first term can be interpreted as the energy stored "purely" in the fields, while the second term is the energy dumped into the material. This was dubbed the "dynamical" energy density [8, 2], and seems to be a relatively recent innovation. It is the energy density that you need to make Poynting's theorem hold in an arbitrary medium, and turns out to be a useful device for studying energy velocities.

This energy density simplifies in many cases. Suppose you have time-harmonic fields $\sim e^{-i\omega t}$, at a frequency where your materials have *negligible loss* ($\operatorname{Im} \hat{\chi} \approx 0$ from above), andd we consider

the time-averaged "real power" as in Sec. 3. It then turns out that the dynamical energy density reduces to the "dispersive" energy density [2]:

$$\frac{1}{4}\psi^* \cdot \left[1 + \frac{\partial(\omega\hat{\chi})}{\partial\omega}\right]\psi,$$

where $\frac{\partial(\omega\hat{\chi})}{\partial\omega} \geq 0$ is guaranteed from passivity [2]. In the usual case where $\hat{\chi}$ diagonalizes into isotropic $\varepsilon(\omega) - 1$ and $\mu(\omega) - 1$, this expression takes on the well-known form

$$\frac{1}{4}\left[\frac{\partial(\omega\varepsilon)}{\partial\omega}|\mathbf{E}|^2 + \frac{\partial(\omega\mu)}{\partial\omega}|\mathbf{H}|^2\right],$$

which is derived in many textbooks [1]. (Note that this includes dispersion, but neglects loss! It is *not valid* in a lossy medium.)

Finally, if we take the "dispersive" energy density for negligible loss, and *additionally* we neglect dispersion (i.e. we approximate ε and μ as frequency-independent), then we obtain the formula

$$\frac{1}{4}\psi^* \cdot (1 + \hat{\chi})\psi = \frac{1}{4}\left[\varepsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2\right]$$

from above for the time-averaged energy density of time-harmonic fields in a dispersionless, lossless, isotropic medium.

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