18.369 Take-Home Midterm Exam: Fall 2021

Posted 3pm Tuesday November 9, due **3pm Wednesday November 10.**

Problem 0: Honor code

Copy and sign the following in your solutions:

I have not used any resources to complete this exam other than my own 18.369 notes, the course textbooks, and posted course materials.

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your signature

Problem 1: (34 points)

In problem set 2, you showed that Maxwell's equations for a time-harmonic current $\hat{\mathbf{J}}(\mathbf{x})e^{-i\omega t}$ yield the following linear equation for the time-harmonic fields $\hat{\mathbf{E}}(\mathbf{x})e^{-i\omega t}$ (with $\mu = 1$, $c = 1$):

$$\hat{A}\mathbf{E} = \left(\nabla\times\nabla\times - \omega^2\varepsilon\right)\hat{\mathbf{E}} = i\omega\hat{\mathbf{J}}.$$

(a) The scalar function $\varepsilon(\omega,\mathbf{x})$ may be complex, corresponding to (linear) gain or loss. In a passive system, the external current $\mathbf{J}$ must supply energy (do nonnegative time-averaged work), and not receive energy (have work done on it). For $\omega > 0$, show that this leads to a constraint on the operator $\Im\hat{A} = \frac{\hat{A}-\hat{A}^\dagger}{2i}$ (which is Hermitian) being positive or negative definite or semidefinite, and hence leads to a constraint on $\omega\operatorname{Im}\varepsilon$ identical to the one from class.

Note: the original posted version of this problem stated the wrong sign for the work done on $\mathbf{J}$; students received full marks if they applied this condition literally and flipped the sign of the resulting definiteness.

(b) If $\Im\hat{A} = \frac{\hat{A}-\hat{A}^\dagger}{2i}$ is positive (or negative) definite for any invertible operator $\hat{A}$, i.e. if $\langle\mathbf{F},(\Im\hat{A})\mathbf{F}\rangle > 0$ (or < 0, respectively) for any $\mathbf{F}\neq 0$, show that it follows that $\Im(\hat{A}^{-1}) = \frac{\hat{A}^{-1}-\left(\hat{A}^{-1}\right)^\dagger}{2i}$ is similarly negative (or positive) definite, respectively—note sign flip! (Note: $\Im(\hat{A}^{-1}) \neq \left(\Im\hat{A}\right)^{-1}$!)

(c) Suppose that we have solved this equation for some $\varepsilon(\mathbf{x})$, ω, and boundary conditions, and have found the resulting inverse operator $\hat{A}^{-1}$ (e.g. an a matrix inverse on the computer for a discretized approximation) such that the solution is given by

$$\hat{\mathbf{E}}(\mathbf{x}) = \hat{A}^{-1}(i\omega\hat{\mathbf{J}})$$

for any $\hat{\mathbf{J}}(\mathbf{x})$. If we perturb the materials to $\varepsilon+\Delta\varepsilon$, where $|\Delta\varepsilon(\mathbf{x})|$ is $\ll 1$ everywhere, the solution (for the same $\mathbf{J}$) changes to $\hat{\mathbf{E}}+\Delta\hat{\mathbf{E}}$ for some $\Delta\hat{\mathbf{E}}$. Expand the change in the solution $\Delta\hat{\mathbf{E}} = \Delta\hat{\mathbf{E}}^{(1)} + \Delta\hat{\mathbf{E}}^{(2)} + \cdots$ as a series of terms proportional to powers of $\Delta\varepsilon$, collect terms order-by-order, and express the *first-order* correction $\Delta\hat{\mathbf{E}}^{(1)}$ in terms of $\hat{\mathbf{E}}$, $\hat{A}^{-1}$, $\hat{\mathbf{J}}$, ω, ε, and/or $\Delta\varepsilon$.

Problem 2: (33 points)

Recall from class the dispersion relation of a dielectric waveguide in 2d (say a strip of $\varepsilon_{\text{hi}} > 1$ surrounded on either side by semi-infinite $\varepsilon = 1$) for E_z-polarized light; see, for example, figure 3 of chapter 3 of the textbook.

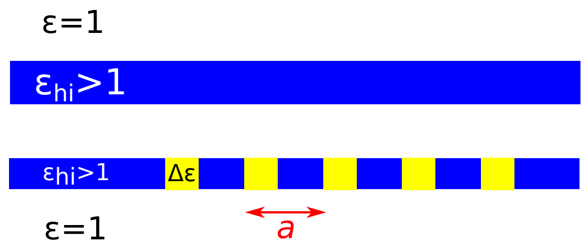

Figure 1: Schematic of two dielectric waveguides, one (above) slightly wider than the other, far enough apart so that the guided modes only slightly "see" the other waveguide. In the narrower waveguide, we add a slight ($|\Delta\varepsilon| \ll 1$) periodic (period a) modulation to help the two waveguides couple to one another.

(a) Sketch (qualitatively) the dispersion relation $\omega(k)$ for the fundamental (lowest-ω) mode, along with the light cone. On the same plot, sketch (in a different color) the dispersion relation for the fundamental mode if you *increased* the *width* of the waveguide (the width of the ε_{hi} region) slightly.

(b) Now, suppose that you put the two waveguides side by side, but far enough apart that they only slightly interact (are slightly perturbed) via the evanescent tails of the guided modes, as shown in Fig. 1. Suppose you want to *input* light in the fundamental mode of the *first* (narrower) waveguide at some frequency ω_0, but you want power to couple/scatter into the fundamental mode of the *second* waveguide propagating in the *same* direction. To improve this coupling, you periodically (period a) modulate the dielectric constant of the waveguide by a slight ($|\Delta\varepsilon| \ll 1$) amount, as shown in Fig. 1.

What period a should you choose for your grating in order to increase the power transfer between these two modes? (In particular, indicate how you would calculate a from your dispersion relations above.)

(c) How does your answer from the previous part change if you want to couple input from the first waveguide into light traveling in the *opposite* direction in the second waveguide?

(d) How does your answer change if there is *no* second waveguide, and you instead want to couple input from the first waveguide into planewaves radiating away perpendicularly from the waveguide?

Problem 3: (33 points)

Suppose that you have a 2d circular PEC cavity, where the interior of the cavity is filled with $\varepsilon(\mathbf{x}) > 0$ that has N-fold rotational symmetry but *not* mirror symmetry. This is called C_N symmetry, as depicted in Fig. 2 for $N = 12$. That is, its symmetry group consists of C_N^n for $n = 0, 1, \ldots, N-1$. (Note that $C_N^{n+N} = C_N^n$.)

(a) What are the irreps $D^{(k)}$ of this symmetry group? For simplicity, denote by $D^{(k)}(n)$ the representation matrix for C_N^n.

(b) What is the form of a partner function $\mathbf{H}(\mathbf{x})$ of $D^{(k)}$, and hence what do the eigenfunctions look like? (Hint: should be reminiscent of a Bloch wave.)

(c) You should have found only 1d irreps, which means that we don't expect degeneracies *from geometric symmetry alone*. However, show that if you *also* apply "time-reversal symmetry" of $\hat{\Theta}$ (complex-conjugate the eigen-equation), you *do* predict 2-fold degeneracies.

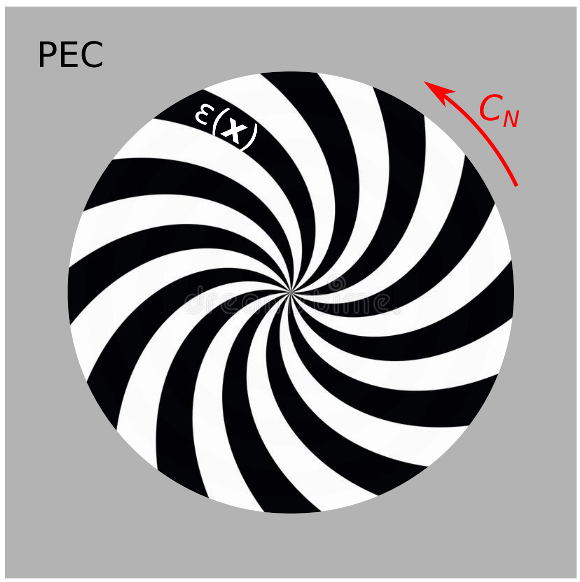

Figure 2: Schematic of a circular metallic cavity whose interior $\varepsilon > 0$ has N-fold rotational symmetry (C_N), but no mirror symmetry. (In this illustration, $N = 12$.)