18.369 Problem Set 4 Solutions

Problem 1: (10+5 points)

(a) Let us write $(\hat{A}^{(0)} + \Delta\hat{A})\psi = \lambda(\hat{B}^{(0)} + \Delta\hat{B})\psi$, where $\psi = \psi^{(0)} + \psi^{(1)} + \cdots$ and $\lambda = \lambda^{(0)} + \lambda^{(1)} + \cdots$ are expansions of the new eigensolutions in powers of the perturbation Δ (since we don't have to worry about breaking degeneracies, by assumption). If we keep only the zero-th order terms, we get the unperturbed problem $\hat{A}^{(0)}\psi^{(0)} = \lambda^{(0)}\hat{B}^{(0)}\psi^{(0)}$. If we only keep terms only up to the 1st order, we get:

$$\hat{A}^{(0)}\psi^{(1)} + \Delta\hat{A}\psi^{(0)} = \lambda^{(1)}\hat{B}^{(0)}\psi^{(0)} + \lambda^{(0)}\hat{B}^{(0)}\psi^{(1)} + \lambda^{(0)}\Delta\hat{B}\psi^{(0)} + O(\Delta^2).$$

Taking the inner product with $\psi^{(0)}$ on both sides, and using the Hermitian property to operate $\hat{A}^{(0)}$ to the left and the fact that $\lambda^{(0)}$ is real, the $\lambda^{(0)}\left\langle \psi^{(0)}, \hat{B}^{(0)}\psi^{(1)} \right\rangle$ terms cancel on both sides, and we obtain:

$$\lambda^{(1)} = \frac{\left\langle \psi^{(0)}, \Delta\hat{A}\psi^{(0)} \right\rangle - \lambda^{(0)}\left\langle \psi^{(0)}, \Delta\hat{B}\psi^{(0)} \right\rangle}{\left\langle \psi^{(0)}, \hat{B}^{(0)}\psi^{(0)} \right\rangle},$$

which is the generalized version of first-order perturbation theory.

(b) For $\nabla\times\nabla\times\mathbf{E} = \frac{\omega^2}{c^2}\varepsilon\mathbf{E}$ with a small change $\Delta\varepsilon$, we have $\Delta\hat{A} = 0$ and $\Delta\hat{B} = \Delta\varepsilon$. Also, to first order, $\Delta(\frac{\omega^2}{c^2}) = 2\frac{\omega\Delta\omega}{c^2} = \lambda^{(1)}$. Plugging this in above and dividing through by $2\omega/c^2$, we have:

$$\Delta\omega = -\frac{\omega}{2}\frac{\left\langle \mathbf{E}^{(0)}, \Delta\varepsilon\mathbf{E}^{(0)} \right\rangle}{\left\langle \mathbf{E}^{(0)}, \varepsilon\mathbf{E}^{(0)} \right\rangle},$$

which is the same as the expression we derived from the **H** eigenproblem in class.

Problem 2: (10+5+5 points)

See Jupyter notebook.

Problem 3: (5+5+5+10 points)

See Jupyter notebook.