18.369 Midterm Solutions: Fall 2021

Problem 1: ((6+6)+11+11 points)

(a) To have no time-average work done *by* the current, we need non-negative time-average work done *on* the current, i.e.: $\frac{1}{2}\operatorname{Re}\langle\mathbf{E},\mathbf{J}\rangle \geq 0$, for the usual inner product $\langle\mathbf{F},\mathbf{G}\rangle = \int \mathbf{F}^* \cdot \mathbf{G}$. Now, we can simplify:

$$\begin{aligned}
\operatorname{Re}\langle\mathbf{E},\mathbf{J}\rangle &= \operatorname{Re}\langle\mathbf{E},\frac{\hat{A}\mathbf{E}}{i\omega}\rangle \\
&= \frac{1}{\omega}\operatorname{Im}\langle\mathbf{E},\hat{A}\mathbf{E}\rangle \\
&= \frac{1}{2i\omega}\left[\langle\mathbf{E},\hat{A}\mathbf{E}\rangle - \langle\mathbf{E},\hat{A}\mathbf{E}\rangle^*\right] \\
&= \frac{1}{2i\omega}\left[\langle\mathbf{E},\hat{A}\mathbf{E}\rangle - \langle\hat{A}\mathbf{E},\mathbf{E}\rangle\right] \\
&= \frac{1}{2i\omega}\left[\langle\mathbf{E},\hat{A}\mathbf{E}\rangle - \langle\mathbf{E},\hat{A}^\dagger\mathbf{E}\rangle\right] \\
&= \frac{1}{\omega}\langle\mathbf{E},(\Im\hat{A})\mathbf{E}\rangle.
\end{aligned}$$

Since this must be ≥ 0 for any possible $\mathbf{E}$, it follows for $\omega > 0$ that $\Im\hat{A}$ must be **positive semidefinite.**

Furthermore, since this is true for *any* $\mathbf{E}$, let's choose an $\mathbf{E}$ that vanishes on the boundary of our domain (or far away for an infinite domain), in which case (as in class) when we integrate by parts we will find that $(\nabla\times\nabla\times)^\dagger = \nabla\times\nabla\times$, since the boundary terms vanish. Hence, the curl terms will vanish in $\hat{A} = \hat{A}^\dagger$, and we are left with the $\omega^2\varepsilon$ terms. But since this is just a scalar, the adjoint (†) is simply the complex conjugate, and $\Im$ is then simply the imaginary part:

$$\frac{\Im\hat{A}}{\omega} = \frac{\operatorname{Im}(\omega^2\varepsilon)}{\omega} = \omega\operatorname{Im}(\varepsilon) \geq 0,$$

which is exactly the passivity condition from class.

(b) Say $\Im\hat{A} = \frac{\hat{A}-\hat{A}^\dagger}{2i}$ is positive definite, and we want to show that $\Im(\hat{A}^{-1})$ is negative definite for any $\mathbf{G} \neq 0$, consider

$$\begin{aligned}
\langle\mathbf{G},\Im(\hat{A}^{-1})\mathbf{G}\rangle &= \frac{1}{2i}\left[\langle\mathbf{G},\hat{A}^{-1}\mathbf{G}\rangle - \langle\mathbf{G},\left(\hat{A}^{-1}\right)^\dagger\mathbf{G}\rangle\right] \\
&= \frac{1}{2i}\left[\langle\mathbf{G},\hat{A}^{-1}\mathbf{G}\rangle - \langle\hat{A}^{-1}\mathbf{G},\mathbf{G}\rangle\right] \\
&= \frac{1}{2i}\left[\langle\hat{A}\mathbf{F},\mathbf{F}\rangle - \langle\mathbf{F},\hat{A}\mathbf{F}\rangle\right] \qquad \text{letting } \mathbf{F} = \hat{A}^{-1}\mathbf{G} \\
&= \frac{1}{2i}\left[\langle\mathbf{F},\hat{A}^\dagger\mathbf{F}\rangle - \langle\mathbf{F},\hat{A}\mathbf{F}\rangle\right] \\
&= -\langle\mathbf{F},(\Im\hat{A})\mathbf{F}\rangle < 0,
\end{aligned}$$

where we got a sign flip because we had $\hat{A}^\dagger - \hat{A}$ instead of $\hat{A} - \hat{A}^\dagger$.

Note that if we consider the case where $\hat{A}$ is just a complex number $a+ib$ (the simplest linear operator), we can see that the above result is simply a generalization of the observation that $(a+ib)^{-1} = \frac{a-ib}{a^2+b^2}$ has the opposite sign of its imaginary part.

(c) Plugging $\Delta\hat{\mathbf{E}} = \Delta\hat{\mathbf{E}}^{(1)} + \Delta\hat{\mathbf{E}}^{(2)} + \cdots$ into our equation for $\mathbf{E}$, and realizing that $\hat{A}$ is just perturbed to $\hat{A} - \omega^2\Delta\varepsilon$, we obtain:

$$(\hat{A} - \omega^2\Delta\varepsilon)(\mathbf{E} + \Delta\hat{\mathbf{E}}^{(1)} + \Delta\hat{\mathbf{E}}^{(2)} + \cdots) = i\omega\mathbf{J}.$$

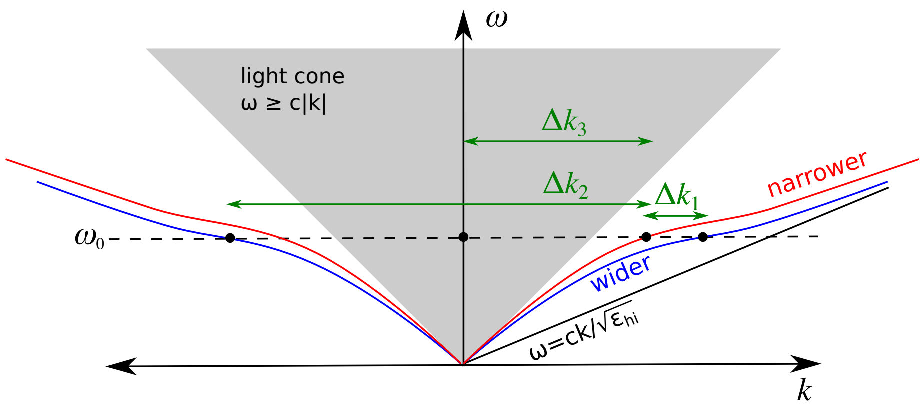

Figure 1: Sketch of dispersion relations for fundamental modes of two ε_{hi} waveguides in air ($\varepsilon = 1$), one waveguide wider than the other.

Collecting terms order-by-order, we find

$$\hat{A}\mathbf{E} = i\omega\mathbf{J} \qquad \text{(zero-th order)}$$
$$\hat{A}\Delta\hat{\mathbf{E}}^{(1)} - \omega^2\Delta\varepsilon\mathbf{E} = 0 \qquad \text{(first order)}$$
$$\vdots$$

and hence the first-order correction is

$$\Delta\hat{\mathbf{E}}^{(1)} = \hat{A}^{-1}\omega^2\Delta\varepsilon\mathbf{E} = \hat{A}^{-1}\omega^2\Delta\varepsilon(\hat{A}^{-1}i\omega\mathbf{J}).$$

Essentially, $\omega^2\Delta\varepsilon\mathbf{E}$ acts like an additional "current" source $-i\omega\Delta\varepsilon\mathbf{E}$ in the unperturbed problem. This is also known as a **first Born approximation**.

Problem 2: (12+7+7+7 points)

(a) Sketch (qualitatively) the dispersion relation $\omega(k)$ for the fundamental (lowest-ω) mode, along with the light cone. On the same plot, sketch (in a different color) the dispersion relation for the fundamental mode if you *increased* the *width* of the waveguide (the width of the ε_{hi} region) slightly.

(b) The sketch is shown in Fig. 1. Some key features:

  (i) For each waveguide, the $\omega(k)$ of the fundamental mode must lie beneath the light cone $\omega \geq c|k|$ for all $k \neq 0$, as proved in class.

  (ii) The wider waveguide's dispersion curve must lie *below* the narrower waveguide at every k. Intuitively, as discussed in class, increasing ε by increasing the width of the ε_{hi} region should *decrease* the frequency. (More formally, we could show from perturbation theory that $\omega(k)$ at each k decreases monotonically with the waveguide width, but this is not required here.)

  (iii) As $|k| \to 0$, both dispersion curves should approach the light line of air (from below).

  (iv) For large $|k|$, both dispersion curves should approach the light line $\omega = c|k|/\sqrt{\varepsilon_{\text{hi}}}$ of the high-ε material.

(c) The key point is that we want to break conservation of k to make the two different modes at the same ω interact: a period a allows modes at a multiple of $2\pi/a$ to couple to one another, because k is only conserved modulo $2\pi/a$. If $|\Delta\varepsilon|$ is small, the dispersion curves $\omega(k)$ should be almost unchanged, so we can read off of the original dispersion relation which Δk we want to allow. To couple the two forward-propagating modes, we want $2\pi/a = \Delta k_1$ as labeled in Fig. 1, so $\boxed{a = 2\pi/\Delta k_1}$. This sort of device is also known as a **grating coupler**.

Technically, we could have any integer multiple of $2\pi/a$ equal to Δk_1, so if we divide $2\pi/k_1$ by any integer it should also work. However, if we think of the Fourier-series expansion of $\Delta\varepsilon$, the first Fourier coefficient is usually the largest—and this is certainly the case for the square-wave modulation indicated in the problem—and so the coupling will generally be strongest for the Δk_1 equal to the smallest multiple of $2\pi/a$ (corresponding to the smallest nonzero frequency in the Fourier series of $\Delta\varepsilon$).

(d) All that changes is Δk, since we now want to couple to the backward-propagating modes: we should now use $\boxed{a = 2\pi/\Delta k_2}$ for Δk_2 as labeled in Fig. 1.

(e) All that changes is Δk, since we now want to couple to the **light cone** (radiation) at $k = 0$ (corresponding to planewaves **perpendicular** to the waveguide): we should now use $\boxed{a = 2\pi/\Delta k_3}$ for Δk_3 as labeled in Fig. 1.

Problem 3: (11+11+11 points)

(a) Let $D(n)$ be a representation for C_N^n. Then we must have $D(n_1+n_2) = D(n_1)D(n_2)$, corresponding to $C_N^{n_1}C_N^{n_2} = C_N^{n_1+n_2}$. As in class and in homework, this immediately leads to irreps that are complex exponentials $D(n) = e^{-i\alpha n}$ (the sign in the exponent is an arbitrary convention; you could have also made it +). Furthermore, since $C_N^N = C_N^0 = E$, we must have $D(N) = e^{-i\alpha N} = 1$, which implies that $\alpha N = 2\pi i k$ for some integer k. So, the irreps are $\boxed{D^{(k)}(n) = e^{-\frac{2\pi i}{N}nk}}$. Furthermore, it's clear that $D^{(k+N)}(n) = D^{(k)}(n)$ for all n,k, so k and $k+N$ are the same irrep (much like a reciprocal lattice vector in periodic systems). Therefore, the unique irreps are $\boxed{k = 0,1,\ldots,N-1}$ (much like a Brillouin zone). Indeed, since we have a finite group with 1d irreps, our character-table rules imply that we must have only N irreps.

(In fact, the character table, viewed as a matrix, is a discrete Fourier transform!)

(b) If $\mathbf{H}(\mathbf{x})$ is a partner function of $D^{(k)}$, then $\widehat{C_N}\mathbf{H}(\mathbf{x}) = C_N\mathbf{H}(C_N^{-1}\mathbf{x})$ must equal $e^{-\frac{2\pi i}{N}k}\mathbf{H}(\mathbf{x})$. It follows that $\widehat{C_N}\left[e^{-i\theta k}\mathbf{H}(\mathbf{x})\right] = e^{-i\theta k}\mathbf{H}(\mathbf{x})$—i.e., $e^{-i\theta k}\mathbf{H}(\mathbf{x})$ is N-fold rotation invariant—since C_N rotates the polar coordinate θ to $\theta - \frac{2\pi}{N}$, cancelling the $e^{-\frac{2\pi i}{N}k}$ factor. Hence, a partner $\mathbf{H}(\mathbf{x})$ of $D^{(k)}$ must be of the form

$$\mathbf{H}(\mathbf{x}) = e^{i\theta k}\underbrace{[N\text{-fold rotation-invariant function}]}_{\mathbf{H}_k(\mathbf{x})},$$

analogous to Bloch waves except that the envelope is rotationally periodic instead of translationally periodic, and k is an integer (equivalent in irrep to $k+N$).

Hence, the eigenfunctions of $\hat{\Theta}$ must be of this form as well (and must also satisfy the PEC boundary conditions of vanishing tangential electric field, i.e. vanishing tangential $\nabla\times\mathbf{H}$ at the boundary, and must be divergence-less).

(c) As for time-reverseal symmetry, since $\hat{\Theta}$ is purely real (not just Hermitian), it follows from conjugating the eigen-equation $\hat{\Theta}\mathbf{H} = \omega^2\mathbf{H} \implies \hat{\Theta}\mathbf{H}^* = \omega^2\mathbf{H}^*$, i.e. for any eigenfunction $\mathbf{H}$, the complex conjugate $\mathbf{H}^*$ is *also* an eigenfunction with the same eigenvalue. Moreover, for $\mathbf{H} = e^{i\theta k}\mathbf{H}_k(\mathbf{x})$ a partner of $D^{(k)}$, we have $\mathbf{H}^* = e^{i\theta(-k)}\mathbf{H}_k(\mathbf{x})^*$, which is a partner of $D^{(-k)}$, since $\mathbf{H}_k(\mathbf{x})^*$ still has N-fold rotational symmetry. This must be linearly independent of $\mathbf{H}$, and in fact orthogonal to $\mathbf{H}$, if $-k \neq k \mod N$, i.e. if $-k$ is a distinct irrep, which is true for $0 < k < N$ (i.e. $k \neq 0 \mod N$). **So any $k \neq 0$ (mod N) eigenfunction *must* be doubly degenerate!**