

## 18.369 Problem Set 1 Solutions

### Problem 1: Adjoint operators (5+10+5 points)

- (a) If  $\dagger$  is conjugate-transpose of a matrix or vector, we are just using the usual linear-algebra rule that  $(AB)^\dagger = B^\dagger A^\dagger$ , hence  $\langle h, Oh' \rangle = h^\dagger(Oh') = (O^\dagger h)^\dagger h' = \langle O^\dagger h, h' \rangle$  for the Euclidean inner product.

More explicitly, if  $h$  is a column-vector and we let  $h^\dagger$  be its conjugate transpose, then  $h^\dagger$  is a row vector and  $h^\dagger h' = \sum_m h_m^* h'_m = \langle h, h' \rangle$  by the usual row-times-column multiplication rule. If  $O$  is a matrix then  $Oh' = \sum_n O_{mn} h'_n$  by the usual matrix-vector product. Then the dot product of  $h$  with  $Oh'$  is given by  $\sum_m h_m^* (\sum_n O_{mn} h'_n) = \sum_n (\sum_m O_{mn} h_m^*) h'_n$ , which is the same thing as the dot product of  $O^\dagger h$  with  $h'$  where  $O^\dagger$  is the conjugate transpose of  $O$ .

Thus, interpreting  $\dagger$  as the conjugate transpose in this finite-dimensional case is consistent with the abstract definitions given in class.

- (b) If  $\hat{O}$  is unitary and we send  $u \rightarrow \hat{O}u$  and  $v \rightarrow \hat{O}v$ , then  $\langle u, v \rangle \rightarrow \langle u, \hat{O}^\dagger \hat{O}v \rangle = \langle u, v \rangle$ , and thus inner products are preserved. Consider now two eigensolutions  $\hat{O}u_1 = \lambda_1 u_1$  and  $\hat{O}u_2 = \lambda_2 u_2$ . Then  $\langle u_1, \hat{O}^\dagger \hat{O}u_2 \rangle = \langle u_1, u_2 \rangle$  by the unitarity of  $\hat{O}$  and  $\langle u_1, \hat{O}^\dagger \hat{O}u_2 \rangle = \langle \hat{O}u_1, \hat{O}u_2 \rangle = \lambda_1^* \lambda_2 \langle u_1, u_2 \rangle$  by the eigenvector property (where we let  $\hat{O}^\dagger$  act to the left, and conjugate the eigenvalue when we factor it out, as in class). Combining these two expressions, we have  $(\lambda_1^* \lambda_2 - 1) \langle u_1, u_2 \rangle = 0$ . There are three cases, just like for Hermitian operators. If  $u_1 = u_2$ , then we must have  $\lambda_1^* \lambda_1 = 1 = |\lambda_1|^2$ , and thus the eigenvalues have unit magnitude. This also implies that  $\lambda_1^* = 1/\lambda_1$ . If  $\lambda_1 \neq \lambda_2$ , then  $(\lambda_1^* \lambda_2 - 1) = (\lambda_2/\lambda_1 - 1) \neq 0$ , and therefore  $\langle u_1, u_2 \rangle = 0$  and the eigenvectors are orthogonal. If  $\lambda_1 = \lambda_2$  but have linearly independent  $u_1 \neq u_2$  (degenerate eigenvectors, i.e. geometric multiplicity  $> 1$ ), then we can form orthogonal linear combinations (e.g. via Gram-Schmidt).
- (c) Take two vectors  $u$  and  $v$ , and consider their inner product. Then  $\langle u, \hat{O}^{-1} \hat{O}v \rangle = \langle u, v \rangle$ . By definition of the adjoint, however, if we move first  $\hat{O}^{-1}$  and then  $\hat{O}$  to act to the left, then we get  $\langle u, v \rangle = \langle \hat{O}^\dagger (\hat{O}^{-1})^\dagger u, v \rangle$ . For this to be true for all  $u$  and  $v$ , we must have  $\hat{O}^\dagger (\hat{O}^{-1})^\dagger = \mathbf{1}$  and thus  $(\hat{O}^{-1})^\dagger = (\hat{O}^\dagger)^{-1}$ . Q.E.D.

### Problem 2: Maxwell eigenproblems (5+5+5+5+5 points)

- (a) To eliminate  $\mathbf{H}$ , we start with Faraday's law  $\nabla \times \mathbf{E} = i\frac{\omega}{c} \mathbf{H}$  and take the curl of both sides. We obtain:

$$\nabla \times \nabla \times \mathbf{E} = \frac{\omega^2}{c^2} \varepsilon \mathbf{E}.$$

If we divide both sides by  $\varepsilon$ , we get the form of a linear eigenproblem but the operator  $\frac{1}{\varepsilon} \nabla \times \nabla \times$  is not Hermitian under the usual inner product  $\langle \mathbf{E}_1, \mathbf{E}_2 \rangle = \int \mathbf{E}_1^* \cdot \mathbf{E}_2$ —integrating by parts as in class, assuming boundary conditions such that the boundary terms vanish, we find that its adjoint is  $\nabla \times \nabla \times \frac{1}{\varepsilon}$ , which is not the same operator unless the  $\frac{1}{\varepsilon}$  commutes with the curls, which only happens if  $\varepsilon$  is a constant. However, if we leave it in the form above we have a generalized Hermitian problem with  $\hat{A} = \nabla \times \nabla \times$  and  $\hat{B} = \varepsilon$ .  $\hat{A}$  is Hermitian for the same reason that  $\hat{\Theta}$  was (it *is*  $\hat{\Theta}$  for  $\varepsilon = 1$ ), and  $\hat{B}$  is Hermitian as long as  $\varepsilon$  is real (so that  $\mathbf{H}_1^* \cdot \varepsilon \mathbf{H}_2 = (\varepsilon \mathbf{H}_1)^* \cdot \mathbf{H}_2$ ).

- (b) The proof follows the same lines as in class. [Alternatively, we could simply quote the Hermitian results from class once we prove part (c).] Consider two eigensolutions  $u_1$  and  $u_2$  (where  $\hat{A}u = \lambda \hat{B}u$ , and  $u \neq 0$ ), and take  $\langle u_2, \hat{A}u_1 \rangle$ . Since  $\hat{A}$  is Hermitian, we can operate it to the left or to the right in the inner product, and get  $\lambda_2^* \langle u_2, \hat{B}u_1 \rangle = \lambda_1 \langle u_2, \hat{B}u_1 \rangle$ , or  $(\lambda_2^* - \lambda_1) \langle u_2, \hat{B}u_1 \rangle = 0$ . There are three cases. First, if  $u_1 = u_2$  then we must have  $\lambda_1 = \lambda_1^*$  (real eigenvalues), since  $\langle u_1, \hat{B}u_1 \rangle > 0$  by definition if  $\hat{B}$  is positive definite. Second, if  $\lambda_1 \neq \lambda_2$  then we must have  $\langle u_2, \hat{B}u_1 \rangle = 0$ , which is our modified orthogonality condition. Finally, if  $\lambda_1 = \lambda_2$  but  $u_1 \neq u_2$ , then we can form a linear combination that is orthogonal (since any linear combination still is an eigenvector); e.g.

$$u_2 \rightarrow u_2 - u_1 \frac{\langle u_2, \hat{B}u_1 \rangle}{\langle u_1, \hat{B}u_1 \rangle},$$

where we have again relied on the fact that  $\hat{B}$  is positive definite (so that we can divide by  $\langle u_1, \hat{B}u_1 \rangle$ ). This is certainly true for  $\hat{B} = \varepsilon$ , since  $\langle E, \hat{B}E \rangle = \int \varepsilon |\mathbf{E}|^2 > 0$  for all  $\mathbf{E} \neq 0$  (almost everywhere) as long as we have a real  $\varepsilon > 0$  as we required in class.

- (c) First, let us verify that  $\langle \mathbf{E}, \mathbf{E}' \rangle_B = \langle \mathbf{E}, \hat{B}\mathbf{E}' \rangle$  is indeed an inner product. Because  $\hat{B}$  is self-adjoint, we have  $\langle \mathbf{E}', \mathbf{E} \rangle_B = \langle \mathbf{E}', \hat{B}\mathbf{E} \rangle = \langle \hat{B}\mathbf{E}', \mathbf{E} \rangle = \langle \mathbf{E}, \hat{B}\mathbf{E}' \rangle^* = \langle \mathbf{E}, \mathbf{E}' \rangle_B^*$ . Bilinearity follows from bilinearity of  $\langle \cdot, \cdot \rangle$  and linearity of  $\hat{B}$ . Positivity  $\langle \mathbf{E}, \mathbf{E} \rangle_B = \langle \mathbf{E}, \hat{B}\mathbf{E} \rangle > 0$  except for  $\mathbf{E} = 0$  (almost everywhere) follows from positive-definiteness of  $\hat{B}$ . All good!

Now, Hermiticity of  $\hat{B}^{-1}\hat{A}$  follows almost trivially from Hermiticity of  $\hat{A}$  and  $\hat{B}$ :  $\langle \mathbf{E}, \hat{B}^{-1}\hat{A}\mathbf{E}' \rangle_B = \langle \mathbf{E}, \hat{B}\hat{B}^{-1}\hat{A}\mathbf{E}' \rangle = \langle \hat{A}\mathbf{E}, \mathbf{E}' \rangle = \langle \hat{A}\mathbf{E}, \hat{B}^{-1}\hat{B}\mathbf{E}' \rangle = \langle \hat{B}^{-1}\hat{A}\mathbf{E}, \hat{B}\mathbf{E}' \rangle = \langle \hat{B}^{-1}\hat{A}\mathbf{E}, \mathbf{E}' \rangle_B$ , where we have used the fact, from problem 1, that Hermiticity of  $\hat{B}$  implies Hermiticity of  $\hat{B}^{-1}$ . Q.E.D.

- (d) If  $\mu \neq 1$  then we have  $\mathbf{B} = \mu \mathbf{H} \neq \mathbf{H}$ , and when we eliminate  $\mathbf{E}$  or  $\mathbf{H}$  from Maxwell's equations we get:

$$\nabla \times \frac{1}{\varepsilon} \nabla \times \mathbf{H} = \frac{\omega^2}{c^2} \mu \mathbf{H}$$

$$\nabla \times \frac{1}{\mu} \nabla \times \mathbf{E} = \frac{\omega^2}{c^2} \varepsilon \mathbf{E}$$

with the constraints  $\nabla \cdot \varepsilon \mathbf{E} = 0$  and  $\nabla \cdot \mu \mathbf{H} = 0$ . These are both generalized Hermitian eigenproblems (since  $\mu$  and  $\nabla \times \frac{1}{\mu} \nabla \times$  are both Hermitian operators for the same reason  $\varepsilon$  and  $\nabla \times \frac{1}{\varepsilon} \nabla \times$  were). Thus, the eigenvalues are real and the eigenstates are orthogonal through  $\mu$  and  $\varepsilon$ , respectively, as proved above. To prove that  $\omega$  is real, we consider an eigenfunction  $H$ . Then  $\langle H, \hat{\Theta} H \rangle = \frac{\omega^2}{c^2} \langle H, \mu H \rangle$  and we must have  $\omega^2 \geq 0$  since  $\hat{\Theta}$  is positive semi-definite (from class) and  $\mu$  is positive definite (for the same reason  $\varepsilon$  was, above). The  $\mathbf{E}$  eigenproblem has real  $\omega$  for the same reason (except that  $\mu$  and  $\varepsilon$  are swapped).

Alternatively, as in part (c), we can write them as ordinary Hermitian eigenproblems with a modified inner product, e.g.  $\frac{1}{\varepsilon} \nabla \times \frac{1}{\mu} \nabla \times \mathbf{E} = \frac{\omega^2}{c^2} \mathbf{E}$ , where  $\frac{1}{\varepsilon} \nabla \times \frac{1}{\mu} \nabla \times$  is Hermitian and positive semidefinite under the  $\langle \mathbf{E}, \mathbf{E}' \rangle_B = \int \mathbf{E}^* \cdot \varepsilon \mathbf{E}'$  inner product as above. The results then follow.

- (e) Consider the  $\mathbf{H}$  eigenproblem. (To even get this linear eigenproblem, we must immediately require  $\varepsilon$  to be an invertible matrix, and of course require  $\varepsilon$  and  $\mu$  to be independent of  $\omega$  or the field strength.) For the right-hand operator  $\mu$  to be Hermitian, we require  $\int \mathbf{H}_1^* \cdot \mu \mathbf{H}_2 = \int (\mu \mathbf{H}_1)^* \cdot \mathbf{H}_2$  for all  $\mathbf{H}_1$  and  $\mathbf{H}_2$ , which implies that  $\mathbf{H}_1^* \cdot \mu \mathbf{H}_2 = (\mu \mathbf{H}_1)^* \cdot \mathbf{H}_2$ . Thus, we require the  $3 \times 3$   $\mu(\mathbf{x})$  matrix to be itself Hermitian at every  $\mathbf{x}$  (that is, equal to its conjugate transpose, from problem 1). (Technically, these requirements hold “almost everywhere” rather than at every point, but as usual I will gloss over this distinction as well as other technical restrictions on the functions to exclude crazy functions.) Similarly, for  $\hat{\Theta}$  to be Hermitian we require  $\int \mathbf{F}_1^* \cdot \varepsilon^{-1} \mathbf{F}_2 = \int (\varepsilon^{-1} \mathbf{F}_1)^* \cdot \mathbf{F}_2$  where  $\mathbf{F} = \nabla \times \mathbf{H}$ , so that we can move the  $\varepsilon^{-1}$  over to the left side of the inner product, and thus  $\varepsilon^{-1}(\mathbf{x})$  must be Hermitian at every  $\mathbf{x}$ . From problem 1, this implies that  $\varepsilon(\mathbf{x})$  is also Hermitian. Finally, to get real eigenvalues we saw from above that we must have  $\mu$  positive definite ( $\int \mathbf{H}^* \cdot \mu \mathbf{H} > 0$  for  $\mathbf{H} \neq 0$ ); since this must be true for all  $\mathbf{H}$  then  $\mu(\mathbf{x})$  at each point must be a positive-definite  $3 \times 3$  matrix (positive eigenvalues). Similarly,  $\hat{\Theta}$  must be positive semi-definite, which implies that  $\varepsilon^{-1}(\mathbf{x})$  is positive semi-definite (non-negative eigenvalues), but since it has to be invertible we must have  $\varepsilon(\mathbf{x})$  positive definite (zero eigenvalues would make it singular). To sum up, we must have  $\varepsilon(\mathbf{x})$  and  $\mu(\mathbf{x})$  being positive-definite Hermitian matrices at (almost) every  $\mathbf{x}$ . (The analysis for the  $\mathbf{E}$  eigenproblem is identical.)

Technically, there are a couple other possibilities. In part (b), we showed that if  $\hat{B}$  is positive-definite it leads to real eigenvalues etc. The same properties, however, hold if  $\hat{B}$  is *negative*-definite, and if *both*  $\hat{A}$  and  $\hat{B}$  are negative-definite we still get real, *positive* eigenvalues. Thus, another possibility is for  $\varepsilon$  and  $\mu$  to be Hermitian *negative*-definite matrices. (For

a scalar  $\varepsilon < 0$  and  $\mu < 0$ , this leads to materials with a *negative* real index of refraction  $n = -\sqrt{\varepsilon\mu}$ !) Furthermore,  $\varepsilon$  and  $\mu$  could both be *anti*-Hermitian instead of Hermitian (i.e.,  $\varepsilon^\dagger = -\varepsilon$  and  $\mu^\dagger = -\mu$ ), although I'm not aware of any physical examples of this. More generally, for any complex number  $z$ , if we replace  $\varepsilon$  and  $\mu$  by  $z\varepsilon$  and  $\mu/z$ , then  $\omega$  is unchanged (e.g. making  $z = i$  gives anti-Hermitian matrices).