18.369 Problem Set 2 Solutions

Problem 3: Cylindrical symmetry (5+10+5pts)

(a) If $f(\phi)$ is an eigenfunction of $\hat{\phi}$ for all $\phi \in \mathbb{R}$, we know from class that $f(\phi)$ must be an exponential function $e^{\kappa\phi}$ (or a multiple thereof) for some κ For the translation group, this number κ was a completely arbitrary imaginary number (disallowing diverging solutions). Here, however, we have a further constraint: rotation by 2π must be the same as doing nothing, and therefore we *must* have $f(2\pi) = f(0) = 1$. This implies $2\pi\kappa = i2\pi m$ where m is an *integer* (so that $e^{-i2\pi m} = 1$), i.e. $\kappa = im$ for $m \in \mathbb{Z}$. (You could, of course, choose the opposite sign convention.)

(b) Combining the previous part with translation-invariance in z means that ψ must be of the form $\psi = \psi_{mk}(r)e^{im\phi+ikz}$. That is, we now have a *one*-dimensional function and should obtain a *one*-dimensional Hermitian eigenproblem in r.

If we plug this ψ into $-\nabla^2\psi = \frac{\omega^2}{c^2}\psi$, we can apply the form of ∇^2 in cylindrical coordinates and obtain:

$$-\psi''_{km} - \frac{1}{r}\psi'_{km} + (\frac{1}{r^2}m^2 + k^2)\psi_{km} = \frac{\omega^2}{c^2}\psi_{km},$$

where primes denote differentiation by r, which after a slight rearrangment becomes:

$$r^2\psi''_{km} + r\psi'_{km} + (k_\perp^2 r^2 - m^2)\psi_{km} = 0,$$

where $k_\perp^2 = \frac{\omega^2}{c^2} - k^2$. Now, make a change of variables: $x = k_\perp r$, $y(x) = \psi_{km}(x/k_\perp)$, and we find:

$$x^2y'' + xy' + (x^2 - m^2)y = 0,$$

which is precisely Bessel's equation of order m. The solutions of this are $J_m(x)$ and $Y_m(x)$, the Bessel functions of the first and second kinds. However, $Y_m(x)$ diverges at $x = 0$, which we can't allow (physical wave solutions can't blow up in empty space). So, the eigensolution must by $y(x) = J_m(x)$ (choosing an arbitrary amplitude of 1), and thus

$$\psi(r,\phi,z) = J_m(k_\perp r)e^{im\phi+ikz}.$$

We're not done yet, because we haven't found $k_\perp$. This is determined by the boundary condition $\psi|_{r=R} = 0$. That means we must have $J_m(k_\perp R) = 0$, or $k_\perp^{(n)} = x_{m,n}/R$ where the $x_{m,n}$ are the zeros of J_m. Solving for ω from $k_\perp$, we finally get a discrete sequence of bands:

$$\omega_{mk}^{(n)}(k) = c\sqrt{k^2 + \left(\frac{x_{m,n}}{R}\right)^2}.$$

Note, moreover, that $J_{-m} = J_m$, and therefore the bands for $m \neq 0$ are *doubly degenerate* (this actually also follows from mirror symmetry). To sketch the bands, we can choose dimensionless units ($c = 1$, $R = 1$), and the result is shown in figure 1.

(c) The $\psi_{mk}(r)e^{im\phi+kz}$ must be orthogonal ($\int \psi_1^*\psi_2 r\,dr\,d\phi\,dz = 0$ for eigenfunctions of different eigenvalues) since they are eigenfunctions of a Hermitian operator. For two different m and k values, they are orthogonal since $\int e^{i(m_1-m_2)\phi}d\phi = 0$ for $m_1 \neq m_2$ (which follows from group theory since they correspond to different representations). For the *same* m, we must have that the r integral is zero, and hence, changing variables to $u = r/R$, we have:

$$\int_0^1 J_m(x_{m,n}u) \cdot J_m(x_{m,n'}u)\,u\,du = 0$$

for different zeros $n \neq n'$. (This is a well-known identity that is used in Fourier-Bessel series expansions.)

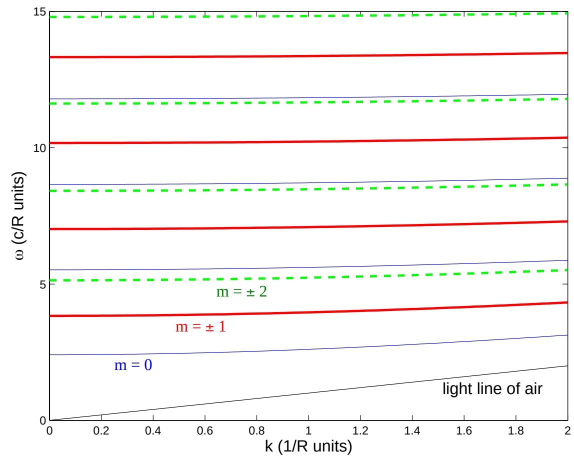

Figure 1: The first few bands of the scalar wave equation in a cylindrical waveguide. Plotted are lowest four bands for $m = 0$ (blue), $m = \pm 1$ (thick red), and $m = \pm 2$ (dashed green). Also, shown, for reference, is the light line $\omega = ck$ of air (all modes lie above this and are therefore extended in the air core). In particualr, the shapes of the curves are hyperbolas.

Problem 2: 2d Waveguide Modes (5+15 points)

(a) Maxwell's equations are (in terms of $\mathbf{H}$) given by the eigen-equation $\nabla \times \frac{1}{\varepsilon} \nabla \times \mathbf{H} = \frac{\omega^2}{c^2} \mathbf{H}$. Suppose that we replace ε by $\alpha\varepsilon$ where α is some constant. By inspection, one obtains the *same* eigensolution $\mathbf{H}$ with ω replaced by $\omega/\sqrt{\alpha}$ (we just divided both sides by α). Thus, scaling epsilon everywhere by a constant just trivially scales the eigenvalues. [We could have alternatively rescaled the geometry and fields: $\varepsilon(\mathbf{x}) \to \varepsilon(\mathbf{x}\sqrt{\alpha})$ and $\mathbf{H}(\mathbf{x}) \to \mathbf{H}(\mathbf{x}\sqrt{\alpha})$.] Therefore, we can set $\varepsilon_{lo} = 1$ (that is, $\alpha = 1/\varepsilon_{lo}$), without loss of generality.

(b) Since we are looking for "TM" solutions $E_z(x,y) = e^{ikx} E_k(y)$, i.e. with $\mathbf{E}$ in the z direction, then similar to class the eigen-equation simplifies to $-\nabla^2 E_z = \frac{\omega^2}{c^2} \varepsilon E_z$, and when we plug in the e^{ikx} form we get:

$$-\frac{d^2}{dy^2} E_y = (\omega^2 \varepsilon - k^2) E_y$$

(where I have chosen $c = 1$ units for simplicity).

(i) In any region where ε is constant, the above equation is solved simply by sines and cosines if $\omega^2\varepsilon - k^2 > 0$ and by exponentials otherwise. Since we have a $y = 0$ mirror plane, the solutions can be chosen either even or odd, and therefore in the $|y| < h/2$ region we have solutions $E_k = A\cos(k_\perp y)$ or $A\sin(k_\perp y)$, where

$$k_\perp = \sqrt{\omega^2 \varepsilon_{hi} - k^2}.$$

If $k_\perp$ is imaginary, these become cosh and sinh solutions, but we will see below that this won't happen. In the $|y| > h/2$ region, since we are looking for solutions below the light line ($\omega^2 \varepsilon_{lo} < k^2$), we must have exponentials...and requiring the solutions to be finite at infinity we must have $E_k = Be^{-\kappa y}$ for $y > h/2$ and $\pm Be^{\kappa y}$ for $y < -h/2$ (with $\pm$ depending on whether the state is even or odd, where:

$$\kappa = \sqrt{k^2 - \omega^2 \varepsilon_{lo}} = \sqrt{k^2(1-f) - k_\perp^2 f},$$

where we define $f = \varepsilon_{lo}/\varepsilon_{hi} < 1$ (the dielectric contrast), and we have used the definition of $k_\perp$ from above.

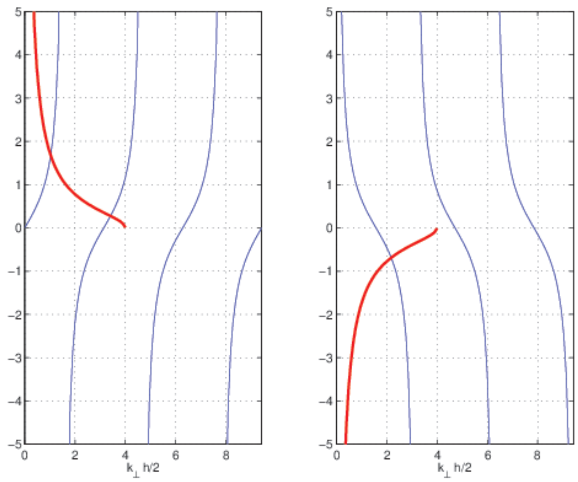

Figure 2: Plot of the two transcendental equations for even modes (left plot) and odd modes (right plot) as a function of $k_\perp h/2$. The thick lines show the right hand sides, while the thin lines show the left hand sides (tan or cot) of the equations, and the intersections correspond to guided-mode solutions. This plot is for the particular case of $f = 0.1$ and $kh/2 = 2$.

(ii) Let's consider first the *even* solutions (cosine). Continuity of E_k implies that $A\cos(k_\perp h/2) = Be^{-\kappa h/2}$, and continuity of $E_k' \sim H_x$ implies that $-k_\perp A\sin(k_\perp h/2) = -\kappa B e^{-\kappa h/2}$. Dividing these two equations, we find:

$$\tan(k_\perp h/2) = \frac{\kappa}{k_\perp} = \frac{\sqrt{k^2(1-f) - k_\perp^2 f}}{k_\perp}.$$

Similarly, for the *odd* solutions (sine), we obtain:

$$\cot(k_\perp h/2) = -\frac{\sqrt{k^2(1-f) - k_\perp^2 f}}{k_\perp}.$$

These are transcendental equations for $k_\perp$. We plot the left and right hand sides of these two equations in figure 2, where the intersections of the curves give the guided-mode solutions.

What about imaginary $k_\perp$ solutions? In this case, the left hand side (tan or cot) would be purely imaginary, while the right hand side would also be purely imaginary, so it seems like there might be some such solutions. Consider the even mode (tan) equation. The tangent of an imaginary $k_\perp$ is always imaginary with the *same* sign as the imaginary part of $k_\perp$, whereas the right hand side will be imaginary with the *opposite* sign ($1/i = -i$)—because of that, the two curves will *never* intersect for imaginary $k_\perp$ and there will be no solution. Conversely for the odd-mode case. So, there are no imaginary $k_\perp$ solutions, as promised—this means that the guided modes must always be *above* the light line for ε_{hi}, which makes physical sense (they must correspond to *propagating* modes in the ε_{hi} region and *evanescent* modes in the ε_{lo} regions).

(iii) We can see immediately that the right-hand side of the transcendental equations is a real number only when $k_\perp \le |k|\sqrt{\frac{1}{f} - 1} = k_\perp^{max}$. Furthermore, we will clearly have an intersection for *every* branch of the tangent/cotangent curve that passes through zero *before* $k_\perp^{max}$. The tangent curves pass through zero whenever $k_\perp h/2$ is an integer multiple of π, and the cotangent curves pass through zero when $k_\perp h/2 + \pi/2$ is an integer multiple of π. Therefore, the number of even modes is simply the number of zero crossings before $k_\perp^{max}$, namely:

$$\text{\# even modes} = \left\lfloor \frac{|k|h\sqrt{\frac{1}{f}-1}}{2\pi} \right\rfloor + 1,$$

where the $+1$ is for the first branch of the tangent (which has a zero crossing at $k_\perp = 0$ and therefore *always* intersects the right-hand-side at least once). Here, by $\lfloor x \rfloor$ we mean the greatest[1] integer $\leq x$. Similarly, the number of odd modes is also given by the number of zero crossings:

$$\text{\# odd modes} = \left\lfloor \frac{|k|h\sqrt{\frac{1}{f}-1}+\pi}{2\pi} \right\rfloor,$$

where in this case we see that we will not have *any* odd guided modes for $|k|h\sqrt{\frac{1}{f}-1} < \pi$. Therefore, as $k \to 0$ we get exactly one (even) guided mode.

Just for fun, let's look at the TE polarization ($\mathbf{H}$ in the $\hat{\mathbf{z}}$ direction). For the $H_z = H_k e^{ikx}$ polarization, we have very similar equations except that the boundary conditions are that H_k is continuous and H_k'/ε is continuous (since $H_k' \sim D_x = D_\parallel$). Thus, for example for the $\cos(k_\perp y)$ mode (the *odd* mode, since $\mathbf{H}$ is a pseudovector), we have $-k_\perp A\sin(k_\perp h/2)/\varepsilon_{hi} = -\kappa B e^{-\kappa h/2}/\varepsilon_{lo}$. Therefore, both the tan and cot in the transcendental equations get multipled by $f = \varepsilon_{lo}/\varepsilon_{hi}$. What effect does this have on the solutions? Multiplying by $f < 1$ *decreases* the tangent curves, but does *not* change the locations of their zeros. Therefore, the *number* of modes at a given k is *unaffected*. However, the intersection point is clearly pulled towards *larger* values of $k_\perp$ when the tan/cot is shrunk, which corresponds to *smaller* values of κ, the decay rate. Therefore, the modes are *less* strongly confined for the H_z (TE) polarization. (Later in the class, we will see how this generally follows from the boundary conditions and the variational theorem.)

Problem 3: Evanescent modes in waveguides (5+10+10 points)

In class, we looked at H_z-polarized solutions in a 2d metallic waveguide, formed by an $\varepsilon = \mu = 1$ region between two PEC walls at $x = 0$ and $x = L$, and found that (in the absence of sources) it satisfied the eigen-equation $-\nabla^2 H_z = \omega^2 H_z$ (for $\varepsilon_0 = \mu_0 = 1$ units) with "Neumann" boundary conditions $H_z'(0) = H_z'(L) = 0$. Exploiting translational symmetry in y, we looked for solutions $H_z(x,y) = u_k(x)e^{iky}$ with a given propagation constant ("wavevector") $k \in \mathbb{R}$, and found

$$u_{k,n} = \cos\left(\frac{n\pi x}{L}\right) \qquad \text{for } n = 0, 1, \ldots$$

$$\omega_n(k) = \pm\sqrt{\left(\frac{n\pi}{L}\right)^2 + k^2},$$

so that the dispersion relation is a set of hyperbolas (except for the $n = 0$ solution $\omega_0 = \pm|k|$).

(a) Fixing ω and solving for $\omega_n(k_n) = \omega$, we find solutions $\pm k_n$ for $k_n = \sqrt{\omega^2 - \left(\frac{n\pi}{L}\right)^2}$. For $\omega = 1.5\pi/L$, this means that k_n is *purely imaginary* for $n > 1$: we have real **propagating** solutions $\pm|\omega|$ for $n = 0$ and $\pm\frac{\pi}{L}\sqrt{(1.5)^2 - 1}$ for $n = 1$, with the $\pm$ signs corresponding to modes propagating in the $\pm y$ directions, along with an infinite number of **evanescent** solutions $k_n = i\kappa_n$ where $\kappa_n = \frac{\pi}{L}\sqrt{n^2 - (1.5)^2}$ for $n > 1$, with the $\pm i\kappa_n$ signs corresponding to solutions that are exponentially decaying in the $\pm y$ directions, respectively.

[1] There is some ambiguity about whether to define the mode as guided when the argument of $\lfloor x \rfloor$ here is exactly an integer, because that corresponds to the case where the mode is exactly on the light line and hence has $\kappa = 0$. If we don't call that a guided mode, then we have to modify our formula by one in that case, but since this situation has measure zero in the parameter space, the question has no practical significance.

(b) Ampere's law $\nabla\times\mathbf{H} = -i\omega\mathbf{E}+\mathbf{J}$ combined with Faraday's law $\nabla\times\mathbf{E} = +i\omega\mathbf{H}$ gives us $\nabla\times\nabla\times\mathbf{H} = \omega^2\mathbf{H}+\nabla\times\mathbf{J}$. Taking the z components of both sides gives $-\nabla^2 H_z = \omega^2 H_z + \frac{\partial J_y}{\partial z} - \frac{\partial J_x}{\partial y}$. For $\mathbf{J} = f(x)\delta(y)\hat{x}$, this gives:

$$(-\nabla^2-\omega^2)H_z = -f(x)\delta'(y),$$

where the right-hand side is indeed of the form $Cf(x)\delta'(y)$ for $\boxed{C=-1}$.

To get a $\delta'(y)$ on the right-hand side can only come from the $-\frac{\partial^2}{\partial^2 y}H_z$ terms on the left-hand side, so we must have a delta function in the *first* derivative:

$$\frac{\partial H_z}{\partial y} = f(x)\delta(y)+\cdots.$$

(One can also see this directly from the x component of Ampere's law $\nabla\times\mathbf{H} = -i\omega\mathbf{E}+\mathbf{J}$.) This corresponds to a **discontinuity** of

$$\boxed{H_z(x,0^+)-H_z(x,0^-) = f(x)}$$

in H_z across $y=0$.

Above, the $\cdots$ terms correspond to the other terms in $\partial H_z/\partial y$ that do *not* come from the discontinuity: the ordinary piecewise derivative. These terms must *not* have a discontinuity, as otherwise $\frac{\partial^2}{\partial^2 y}H_z$ would have an additional $\delta(y)$ term. This means that the piecewise slopes must match:

$$\boxed{\left.\frac{\partial H_z}{\partial y}\right|_{y=0^+} = \left.\frac{\partial H_z}{\partial y}\right|_{y=0^-}}.$$

So, we have *two* continuity conditions at $y=0$. This will be important below.

(c) We now write

$$H_z(x,y) = \sum_{n=0}^{\infty}\cos\left(\frac{n\pi x}{L}\right)\begin{cases} a_n^+ e^{+ik_n y} & y>0 \\ a_n^- e^{-ik_n y} & y<0 \end{cases},$$

where we have chosen the signs $\pm$ in the exponent for $y>0$ and $y<0$, respectively, to obtain *outgoing* waves (for $n=0,1$) and *decaying* waves (for $n>1$) as suggested.

Now, applying our two continuity conditions above, we have:

$$\sum_{n=0}^{\infty}\cos\left(\frac{n\pi x}{L}\right)(a_n^+-a_n^-) = f(x)$$

$$\sum_{n=0}^{\infty}\cos\left(\frac{n\pi x}{L}\right)k_n(a_n^++a_n^-) = 0.$$

A little algebra shows that these two equations are satisfied if $a_n^- = -a_n^+ = a_n$ (the H_z fields "look antisymmetric," which actually means that the $\mathbf{H}$ field is *symmetric* since it is a pseudovector, consistent with the current source being symmetric), and

$$\boxed{\sum_{n=0}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right) = f(x)/2},$$

which is a Fourier cosine series and one can solve for $a_n = \frac{1}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$, divided by 2 for a_0, by the usual cosine-series orthogonality formula. Hence, the final solution looks like

$$H_z(x,y) = \sum_{n=0}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)e^{ik_n|y|},$$

which is a superposition of two propagating terms ($n=0$ and $n=1$) and infinitely many evanescent terms.

Problem 4: Poynting's theorem (15 points)

We want to compute the total radiated power P, which by Poynting's theorem should be equal to $-\frac{1}{2}\int \Re[\mathbf{E}^*\mathbf{J}]$, the total work done *by* the current $\mathbf{J}$ (note the integral over space). For $\mathbf{J}(\mathbf{x},t) = -i\omega\mathbf{p}\delta(\mathbf{x})e^{-i\omega t}$, this gives $P = \frac{\omega}{2}\Im[\mathbf{p}\cdot\mathbf{E}(\mathbf{0})]$ (evaluating at $t=0$ since the $e^{-i\omega t}$ factors cancel). However, $\mathbf{E}$ blows up at $r=0$, so we will evaluate $\mathbf{p}\cdot\mathbf{E}$ at $r\neq 0$ and take the $r\to 0$ limit (and the result had better be independent of the diretion $\mathbf{n}$ in which we go to the origin). The identity $(\mathbf{n}\times\mathbf{p})\times\mathbf{n} = \mathbf{p}-\mathbf{n}(\mathbf{n}\cdot\mathbf{p})$ will be useful, as will the Taylor expansion $e^{ikr}\approx 1+ikr-\frac{k^2r^2}{2}-i\frac{k^3r^3}{6}+O(r^4)$. Plugging this in, collecting terms by powers of r (omitting terms that immediately vanish when we take the imaginary part), and letting $\mathbf{v} = 3\mathbf{n}(\mathbf{n}\cdot p)-\mathbf{p}$, we find:

$$\begin{aligned}
\Im\left[4\pi\varepsilon_0\mathbf{E}(r\mathbf{n},0)\right] &= \Im\left(\left\{k^2\left[\mathbf{p}-\mathbf{n}(\mathbf{n}\cdot\mathbf{p})\right]\frac{1}{r}+\left[3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)\right\}\left[1+ikr-\frac{k^2r^2}{2}-i\frac{k^3r^3}{6}+\cdots\right]\right)\\
&= k^3\left[\mathbf{p}-\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\frac{\mathbf{v}}{6}+\frac{\mathbf{v}}{2}\right]+\frac{k}{r^2}(\mathbf{v}-\mathbf{v})+O(r) = k^3\left[\mathbf{p}-\mathbf{n}(\mathbf{n}\cdot\mathbf{p})+\frac{\mathbf{v}}{3}\right]+O(r)\\
&= \frac{2}{3}k^3\mathbf{p}+O(r),
\end{aligned}$$

and hence the $r\to 0$ limit of $\Im\mathbf{E}$ is well defined (independent of $\mathbf{n}$). We therefore obtain (using $\omega = ck$ and $c = 1/\sqrt{\varepsilon_0\mu_0}$):

$$P = \frac{\omega}{2}\cdot\frac{2k^3p^2}{12\pi\varepsilon_0} = \frac{ck^4p^2}{12\pi\varepsilon_0} = \frac{c^2k^4p^2}{12\pi\varepsilon_0}\sqrt{\varepsilon_0\mu_0} = \frac{c^2Z_0k^4p^2}{12\pi}$$

as desired (which matches the power computed in Jackson by integrating the outgoing Poynting flux).

Problem 5: Scattering and energy (6+6+5+5+5 points)

(a) Solutions:

(i) From $\frac{\partial\psi}{\partial t} = \hat{C}\psi-\frac{\partial\phi}{\partial t}-\xi$, $\frac{\partial\psi_{\text{inc}}}{\partial t} = \hat{C}\psi_{\text{inc}}-\xi$, and $\psi = \psi_{\text{inc}}+\psi_{\text{scat}}$, it follows that

$$\frac{\partial\psi_{\text{scat}}}{\partial t} = \hat{C}\psi_{\text{scat}}-\frac{\partial\phi}{\partial t}$$

where $\phi = \chi*\psi = \chi*(\psi_{\text{inc}}+\psi_{\text{scat}})$.

Because of the dependence on ϕ, we can't solve the ψ_{scat} equation without first solving the ψ_{inc} equation. Dividing the solutions into $\psi = \psi_{\text{inc}}+\psi_{\text{scat}}$ doesn't make Maxwell's equations easier to solve, but it does give us a new perspective on the solutions (and leads to some numerical approaches such as integral-equation formulations).

(ii) The Fourier transform merely changes $\frac{\partial}{\partial t}\to -i\omega$, giving

$$\begin{aligned}
\frac{\partial\hat{\psi}_{\text{inc}}}{\partial t} &= \hat{C}\hat{\psi}_{\text{inc}}-\hat{\xi}\\
\frac{\partial\hat{\psi}_{\text{scat}}}{\partial t} &= \hat{C}\hat{\psi}_{\text{scat}}-\hat{\phi},
\end{aligned}$$

where the convolution $\phi = \chi*\psi$ becomes a multiplication $\hat{\phi} = \hat{\chi}\hat{\psi}$.

(b) Note: when we apply Poynting's theorem (from section 3 of the notes) to a time-harmonic field $\psi(\mathbf{x},t) = \hat{\psi}(\mathbf{x})e^{-i\omega t}$, then the $\frac{\partial}{\partial t}\left[\frac{1}{2}\langle\psi,\psi\rangle\right]$ term is zero. In general, the $e^{-i\omega t}$ terms cancel from all of the products thanks to the complex conjugations.

If we apply the remaining of Poynting's theorem *for the volume* Ω to the $\hat{\psi}_{\text{inc}}$ equation, there is no ϕ term in the equation and $\xi = 0$ inside Ω so the $\langle \psi_{\text{inc}}, \xi \rangle$ integral vanishes as well! So the only remaining term is $-\oiint_{\partial\Omega} \Re[\hat{\mathbf{E}}^*_{\text{inc}} \times \hat{\mathbf{H}}_{\text{inc}}] \cdot d\mathbf{A}$ on the right-hand side, with the left-hand-side equal to zero, so $\oiint_{\partial\Omega} \Re[\hat{\mathbf{E}}^*_{\text{inc}} \times \hat{\mathbf{H}}_{\text{inc}}] \cdot d\mathbf{A} = 0$ as desired.

Applying Poynting's theorem to the $\hat{\psi}$ equation in Ω, the $\langle \psi, \xi \rangle$ term is still $= 0$ for an integral over Ω because $\xi = 0$ there, but we *do* have a ϕ term. As shown in section 3.2 of the notes, the ϕ term in the frequency domain simplifies to $\langle \hat{\psi}, \Im[\omega\hat{\chi}]\hat{\psi} \rangle$, and as we explained in the notes one always has $\Im[\omega\hat{\chi}] \geq 0$ (positive semidefinite) for a passive material. Hence $\langle \hat{\psi}, \Im[\omega\hat{\chi}]\hat{\psi} \rangle \geq 0$, and it follows from Poynting's theorem that

$$\langle \hat{\psi}, \Im[\omega\hat{\chi}]\hat{\psi} \rangle = -\oiint_{\partial\Omega} \Re[\hat{\mathbf{E}}^* \times \hat{\mathbf{H}}] \cdot d\mathbf{A} = P_{\text{abs}} \geq 0$$

as desired.

(c) Consider a ball B of radius R (the interior of the sphere ∂B of radius R) centered on some point in Ω, for any R large enough so that B encloses Ω (i.e. $\Omega \subseteq B$), and apply Poynting's theorem to the volume $B \backslash \Omega$ that lies *between* $\partial\Omega$ and ∂B. Poynting's theorem immediately implies that

$$0 = -\oiint_{\partial(B\backslash\Omega)} \Re[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A} = \oiint_{\partial\Omega} \Re[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A} - \oiint_{\partial B} \Re[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A}$$

$$\implies P_{\text{scat}} = \oiint_{\partial\Omega} \Re[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A} = \oiint_{\partial B} \Re[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A},$$

where $d\mathbf{A}$ pointing *out* of Ω and B for $\partial\Omega$ and ∂B, respectively. More generally, by a similar argument one can show that P_{scat} is equal to the Poynting flux through *any* surface enclosing Ω.

Since outgoing boundary conditions require $R^2\Re[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}]$ to point radially outward for large R, it follows that the *integrand* $\Re[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot \mathbf{n}$ of the $\oiint_{\partial B}$ integral must be ≥ 0 for a sufficiently large R. If the integrand is nonnegative, then the integral $\oiint_{\partial B} \Re[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A}$ must also be ≥ 0. Since $P_{\text{scat}} = \oiint_{\partial B} \Re[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A}$ for *all* radii R that enclose Ω, it follows that $P_{\text{scat}} \geq 0$.

(d) We just plug in the definitions in terms of Poynting-flux integrals:

$$P_{\text{ext}} = \underbrace{-\oiint_{\partial\Omega} \Re[\hat{\mathbf{E}}^* \times \hat{\mathbf{H}}] \cdot d\mathbf{A}}_{P_{\text{abs}}} + \underbrace{\oiint_{\partial\Omega} \Re[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A}}_{P_{\text{scat}}}.$$

Then, we substitute the identities $\mathbf{E} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{scat}}$ and $\mathbf{H} = \mathbf{H}_{\text{inc}} + \mathbf{H}_{\text{scat}}$, and note that the minus sign in the P_{abs} term leads to several cancellations. Another term vanishes due to the $\oiint_{\partial\Omega} \Re[\hat{\mathbf{E}}^*_{\text{inc}} \times \hat{\mathbf{H}}_{\text{inc}}] \cdot d\mathbf{A} = 0$ property that we showed above. The only remaining terms are the scat–inc cross terms:

$$\boxed{P_{\text{ext}} = -\oiint_{\partial\Omega} \Re[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{inc}} + \hat{\mathbf{E}}^*_{\text{inc}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A}}.$$

(e) From the first part above (or section 3.2 of the notes), we have $P_{\text{abs}} = \Re\langle \hat{\psi}, -i\omega\hat{\phi} \rangle = \Im\langle \hat{\psi}, \omega\hat{\phi} \rangle$. (The absorbed power is the work done **on** the polarization currents $-i\omega\hat{\phi}$ **by** the total field $\hat{\psi}$.) In the same way, applying Poynting's theorem to ψ_{scat} yields

$$\Re\langle \hat{\psi}_{\text{scat}}, -i\omega\hat{\phi} \rangle = -\oiint_{\partial\Omega} \Re[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A} = -P_{\text{scat}}.$$

(The scattered power is the work done **by** the polarization currents $-i\omega\hat{\phi}$ **on** the scattered field $\hat{\psi}_{\text{scat}}$.) Combining these two, we obtain

$$P_{\text{ext}} = \underbrace{\Re\langle\hat{\psi}, -i\omega\hat{\phi}\rangle}_{P_{\text{abs}}} \underbrace{-\Re\langle\hat{\psi}_{\text{scat}}, -i\omega\hat{\phi}\rangle}_{P_{\text{abs}}} = \boxed{\Re\langle\hat{\psi}_{\text{inc}}, -i\omega\hat{\phi}\rangle}.$$

Comment: The optical theorem is often phrased in terms of a "forward-scattering amplitude." Recall that $\hat{\psi}_{\text{scat}}$ is produced by the polarization currents $\frac{\partial\phi}{\partial t} = -i\omega\hat{\phi}$. If $\hat{\psi}_{\text{inc}}$ is a planewave $\sim e^{i\mathbf{k}\cdot\mathbf{x}}$, then $\langle\hat{\psi}_{\text{inc}}, -i\omega\hat{\phi}\rangle$ is a Fourier transform: the $e^{i\mathbf{k}\cdot\mathbf{x}}$ *Fourier component* of the polarization current. From class, a current $\sim e^{i\mathbf{k}\cdot\mathbf{x}}$ in vacuum (like the $\hat{\psi}_{\text{scat}}$ equations given $\hat{\phi}$) generates fields $\sim e^{i\mathbf{k}\cdot\mathbf{x}}$, which are the "forward-scattered" fields. So $\langle\hat{\psi}_{\text{inc}}, -i\omega\hat{\phi}\rangle$ is proportional to the amplitude of the forward-scattered fields.

Comment: It is also worth commenting on the physical intuition behind the optical theorem. The basic idea is that any scattered or absorbed light leaves a shadow, so one can determine P_{ext} by looking at the darkness of the shadow. The shadow, a region on the "forward" side where $\hat{\psi}$ is small, corresponds to a forward-scattered $\hat{\psi}_{\text{scat}}$ that partially *cancels* $\hat{\psi}_{\text{inc}}$. Hence, P_{ext} is determined by the $\hat{\psi}_{\text{scat}}$ that is generated in the forward direction (the same direction as $\hat{\psi}_{\text{inc}}$), i.e. the forward-scattering amplitude.