18.369 Midterm Solutions: Fall 2021

Problem 1: ((6+6)+11+11 points)

Note: in the problem as originally posted, the passivity condition was mis-stated (or at least, stated very confusingly). In a passive system, the *bound* currents $\partial \mathbf{P}/\partial t$ in the material must do no net work. However, the external ("free") current $\mathbf{J}$ must *supply* (expend) non-negative energy, and not receive energy (in a closed system, the rate of work done *by* the free currents equals the rate of work done *on* the bound currents, by Poynting's theorem). This is the opposite of the sign as originally stated, so with the problem as states you should have gotten the opposite of the definite-ness conditions below.

(a) To have no time-average work done *on* the current, we need non-negative time-average work done *by* the current, i.e.: $\boxed{\frac{1}{2}\operatorname{Re}\langle \mathbf{E}, \mathbf{J}\rangle \leq 0}$,for the usual inner product $\langle \mathbf{F}, \mathbf{G}\rangle = \int \mathbf{F}^* \cdot \mathbf{G}$. Now, we can simplify:

$$
\begin{aligned}
\operatorname{Re}\langle \mathbf{E}, \mathbf{J}\rangle &= \operatorname{Re}\langle \mathbf{E}, \frac{\hat{A}\mathbf{E}}{i\omega}\rangle \\
&= \frac{1}{\omega}\operatorname{Im}\langle \mathbf{E}, \hat{A}\mathbf{E}\rangle \\
&= \frac{1}{2i\omega}\left[\langle \mathbf{E}, \hat{A}\mathbf{E}\rangle - \langle \mathbf{E}, \hat{A}\mathbf{E}\rangle^*\right] \\
&= \frac{1}{2i\omega}\left[\langle \mathbf{E}, \hat{A}\mathbf{E}\rangle - \langle \hat{A}\mathbf{E}, \mathbf{E}\rangle\right] \\
&= \frac{1}{2i\omega}\left[\langle \mathbf{E}, \hat{A}\mathbf{E}\rangle - \langle \mathbf{E}, \hat{A}^\dagger\mathbf{E}\rangle\right] \\
&= \frac{1}{\omega}\langle \mathbf{E}, (\Im\hat{A})\mathbf{E}\rangle.
\end{aligned}
$$

Since this must be ≤ 0 for any possible $\mathbf{E}$, it follows for $\omega > 0$ that $\Im\hat{A}$ must be **negative semidefinite.**

Furthermore, since this is true for *any* $\mathbf{E}$, let's choose an $\mathbf{E}$ that vanishes on the boundary of our domain (or far away for an infinite domain), in which case (as in class) when we integrate by parts we will find that $(\nabla\times\nabla\times)^\dagger = \nabla\times\nabla\times$, since the boundary terms vanish. Hence, the curl terms will vanish in $\hat{A} = \hat{A}^\dagger$, and we are left with the $\omega^2\varepsilon$ terms. But since this is just a scalar, the adjoint ($\dagger$) is simply the complex conjugate, and $\Im$ is then simply the imaginary part:

$$
\frac{\Im\hat{A}}{\omega} = \frac{\operatorname{Im}(-\omega^2\varepsilon)}{\omega} = -\omega\operatorname{Im}(\varepsilon) \leq 0,
$$

which implies exactly the passivity condition $\boxed{\omega\operatorname{Im}(\varepsilon) \geq 0}$ from class. (More explicitly, the operator $-\omega\operatorname{Im}(\varepsilon)$ is negative semidefinite, i.e. $-\langle \mathbf{E}, \omega\operatorname{Im}(\varepsilon)\mathbf{E}\rangle \leq 0$ for *all* $\mathbf{E}$, which implies $\omega\operatorname{Im}(\varepsilon) \geq 0$ almost everywhere in space.)

(b) Say $\Im\hat{A} = \frac{\hat{A}-\hat{A}^\dagger}{2i}$ is positive definite, and we want to show that $\Im(\hat{A}^{-1})$ is negative definite for any $\mathbf{G} \neq 0$, consider

$$
\begin{aligned}
\langle \mathbf{G}, \Im(\hat{A}^{-1})\mathbf{G}\rangle &= \frac{1}{2i}\left[\langle \mathbf{G}, \hat{A}^{-1}\mathbf{G}\rangle - \langle \mathbf{G}, \left(\hat{A}^{-1}\right)^\dagger \mathbf{G}\rangle\right] \\
&= \frac{1}{2i}\left[\langle \mathbf{G}, \hat{A}^{-1}\mathbf{G}\rangle - \langle \hat{A}^{-1}\mathbf{G}, \mathbf{G}\rangle\right] \\
&= \frac{1}{2i}\left[\langle \hat{A}\mathbf{F}, \mathbf{F}\rangle - \langle \mathbf{F}, \hat{A}\mathbf{F}\rangle\right] \qquad \text{letting } \mathbf{F} = \hat{A}^{-1}\mathbf{G} \\
&= \frac{1}{2i}\left[\langle \mathbf{F}, \hat{A}^\dagger\mathbf{F}\rangle - \langle \mathbf{F}, \hat{A}\mathbf{F}\rangle\right] \\
&= -\langle \mathbf{F}, (\Im\hat{A})\mathbf{F}\rangle < 0,
\end{aligned}
$$

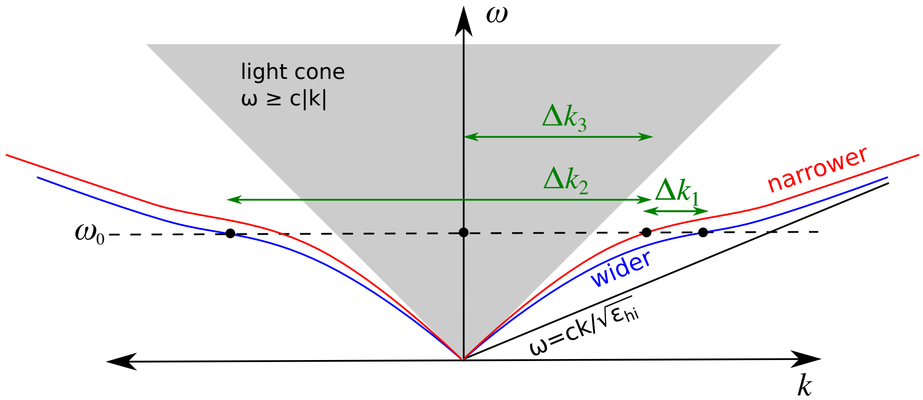

Figure 1: Sketch of dispersion relations for fundamental modes of two ε_{hi} waveguides in air ($\varepsilon = 1$), one waveguide wider than the other.

where we got a sign flip because we had $\hat{A}^\dagger - \hat{A}$ instead of $\hat{A} - \hat{A}^\dagger$.

(As shown above, $\Im\hat{A}$ for the Maxwell case is actually negative semidefinite, in which case we obtain that $\Im(\hat{A}^{-1})$, related to the Green's function of the problem, is positive semidefinite.)

Note that if we consider the case where $\hat{A}$ is just a complex number $a+ib$ (the simplest linear operator), we can see that the above result is simply a generalization of the observation that $(a+ib)^{-1} = \frac{a-ib}{a^2+b^2}$ has the opposite sign of its imaginary part.

(c) Plugging $\Delta\hat{\mathbf{E}} = \Delta\hat{\mathbf{E}}^{(1)} + \Delta\hat{\mathbf{E}}^{(2)} + \cdots$ into our equation for $\mathbf{E}$, and realizing that $\hat{A}$ is just perturbed to $\hat{A} - \omega^2\Delta\varepsilon$, we obtain:

$$(\hat{A} - \omega^2\Delta\varepsilon)(\mathbf{E} + \Delta\hat{\mathbf{E}}^{(1)} + \Delta\hat{\mathbf{E}}^{(2)} + \cdots) = i\omega\mathbf{J}.$$

Collecting terms order-by-order, we find

$$\begin{aligned} \hat{A}\mathbf{E} &= i\omega\mathbf{J} \qquad \text{(zero-th order)} \\ \hat{A}\Delta\hat{\mathbf{E}}^{(1)} - \omega^2\Delta\varepsilon\mathbf{E} &= 0 \qquad \text{(first order)} \\ &\vdots \end{aligned}$$

and hence the first-order correction is

$$\Delta\hat{\mathbf{E}}^{(1)} = \hat{A}^{-1}\omega^2\Delta\varepsilon\mathbf{E} = \hat{A}^{-1}\omega^2\Delta\varepsilon(\hat{A}^{-1}i\omega\mathbf{J}).$$

Essentially, $\omega^2\Delta\varepsilon\mathbf{E}$ acts like an additional "current" source $-i\omega\Delta\varepsilon\mathbf{E}$ in the unperturbed problem. This is also known as a **first Born approximation**.

Problem 2: (12+7+7+7 points)

(a) The sketch is shown in Fig. 1. Some key features:

(i) For each waveguide, the $\omega(k)$ of the fundamental mode must lie beneath the light cone $\omega \geq c|k|$ for all $k \neq 0$, as proved in class.

(ii) The wider waveguide's dispersion curve must lie *below* the narrower waveguide at every k. Intuitively, as discussed in class, increasing ε by increasing the width of the ε_{hi} region should *decrease* the frequency. (More formally, we could show from perturbation theory that $\omega(k)$ at each k decreases monotonically with the waveguide width, but this is not required here.)

(iii) As $|k| \to 0$, both dispersion curves should approach the light line of air (from below).

(iv) For large $|k|$, both dispersion curves should approach the light line $\omega = c|k|/\sqrt{\varepsilon_{\text{hi}}}$ of the high-ε material.

(b) The key point is that we want to break conservation of k to make the two different modes at the same ω interact: a period a allows modes at a multiple of $2\pi/a$ to couple to one another, because k is only conserved modulo $2\pi/a$. If $|\Delta\varepsilon|$ is small, the dispersion curves $\omega(k)$ should be almost unchanged, so we can read off of the original dispersion relation which Δk we want to allow. To couple the two forward-propagating modes, we want $2\pi/a = \Delta k_1$ as labeled in Fig. 1, so $\boxed{a = 2\pi/\Delta k_1}$. This sort of device is also known as a **grating coupler**.

Technically, we could have any integer multiple of $2\pi/a$ equal to Δk_1, so if we divide $2\pi/k_1$ by any integer it should also work. However, if we think of the Fourier-series expansion of $\Delta\varepsilon$, the first Fourier coefficient is usually the largest—and this is certainly the case for the square-wave modulation indicated in the problem—and so the coupling will generally be strongest for the Δk_1 equal to the smallest multiple of $2\pi/a$ (corresponding to the smallest nonzero frequency in the Fourier series of $\Delta\varepsilon$).

(c) All that changes is Δk, since we now want to couple to the backward-propagating modes: we should now use $\boxed{a = 2\pi/\Delta k_2}$ for Δk_2 as labeled in Fig. 1.

(d) All that changes is Δk, since we now want to couple to the **light cone** (radiation) at $k = 0$ (corresponding to planewaves **perpendicular** to the waveguide): we should now use $\boxed{a = 2\pi/\Delta k_3}$ for Δk_3 as labeled in Fig. 1.

Problem 3: (11+11+11 points)

(a) Let $D(n)$ be a representation for C_N^n. Then we must have $D(n_1+n_2) = D(n_1)D(n_2)$, corresponding to $C_N^{n_1} C_N^{n_2} = C_N^{n_1+n_2}$. As in class and in homework, this immediately leads to irreps that are complex exponentials $D(n) = e^{-i\alpha n}$ (the sign in the exponent is an arbitrary convention; you could have also made it $+$). Furthermore, since $C_N^N = C_N^0 = E$, we must have $D(N) = e^{-i\alpha N} = 1$, which implies that $\alpha N = 2\pi i k$ for some integer k. So, the irreps are $\boxed{D^{(k)}(n) = e^{-\frac{2\pi i}{N} nk}}$. Furthermore, it's clear that $D^{(k+N)}(n) = D^{(k)}(n)$ for all n, k, so k and $k+N$ are the same irrep (much like a reciprocal lattice vector in periodic systems). Therefore, the unique irreps are $\boxed{k = 0, 1, \ldots, N-1}$ (much like a Brillouin zone). Indeed, since we have a finite group with 1d irreps, our character-table rules imply that we must have only N irreps.

(In fact, the character table, viewed as a matrix, is a discrete Fourier transform!)

(b) If $\mathbf{H}(\mathbf{x})$ is a partner function of $D^{(k)}$, then $\widehat{C_N}\mathbf{H}(\mathbf{x}) = C_N \mathbf{H}(C_N^{-1}\mathbf{x})$ must equal $e^{-\frac{2\pi i}{N}k}\mathbf{H}(\mathbf{x})$. It follows that $\widehat{C_N}\left[e^{-i\theta k}\mathbf{H}(\mathbf{x})\right] = e^{-i\theta k}\mathbf{H}(\mathbf{x})$—i.e., $e^{-i\theta k}\mathbf{H}(\mathbf{x})$ is N-fold rotation invariant—since C_N rotates the polar coordinate θ to $\theta - \frac{2\pi}{N}$, cancelling the $e^{-\frac{2\pi i}{N}k}$ factor. Hence, a partner $\mathbf{H}(\mathbf{x})$ of $D^{(k)}$ must be of the form

$$\mathbf{H}(\mathbf{x}) = e^{i\theta k} \underbrace{[N\text{-fold rotation-invariant function}]}_{\mathbf{H}_k(\mathbf{x})},$$

analogous to Bloch waves except that the envelope is rotationally periodic instead of translationally periodic, and k is an integer (equivalent in irrep to $k+N$).

Hence, the eigenfunctions of $\hat{\Theta}$ must be of this form as well (and must also satisfy the PEC boundary conditions of vanishing tangential electric field, i.e. vanishing tangential $\nabla \times \mathbf{H}$ at the boundary, and must be divergence-less).

(c) As for time-reverseal symmetry, since $\hat{\Theta}$ is purely real (not just Hermitian), it follows from conjugating the eigen-equation $\hat{\Theta}\mathbf{H} = \omega^2\mathbf{H} \implies \hat{\Theta}\mathbf{H}^* = \omega^2\mathbf{H}^*$, i.e. for any eigenfunction $\mathbf{H}$, the complex conjugate $\mathbf{H}^*$ is *also* an eigenfunction with the same eigenvalue. Moreover, for $\mathbf{H} = e^{i\theta k}\mathbf{H}_k(\mathbf{x})$ a partner of $D^{(k)}$, we have $\mathbf{H}^* = e^{i\theta(-k)}\mathbf{H}_k(\mathbf{x})^*$, which is a partner of $D^{(-k)} = D^{(N-k)}$, since $\mathbf{H}_k(\mathbf{x})^*$ still has N-fold rotational symmetry. This must be linearly independent of $\mathbf{H}$, and in fact orthogonal to $\mathbf{H}$, if $-k \neq k \mod N$, i.e. if $-k$ is a distinct irrep, which is true for $k \neq 0$ and, if N is even, $k \neq N/2$ (since $N/2 = N - N/2$). **So any $k \neq 0$ or $N/2$ (for even N) eigenfunction *must* be doubly degenerate!**

Note that, for even N, $D^{(N/2)} = e^{-\pi i n} = (-1)^n$. In consequence, for both $k = 0$ and $k = N/2$ (for even N), the "Bloch wave" eigenfunction $\mathbf{H}(\mathbf{x})$ can be chosen purely real.