18.369 Problem Set 1

Due Wednesday, 14 February 2024.

Problem 1: Adjoints and operators

(a) We defined the adjoint $\dagger$ of operators $\hat{O}$ by: $\langle H_1, \hat{O}H_2\rangle = \langle \hat{O}^\dagger H_1, H_2\rangle$ for all H_1 and H_2 in the vector space. Show that for a *finite-dimensional* Hilbert space, where H is a column vector h_n ($n = 1, \cdots, d$), $\hat{O}$ is a square $d \times d$ matrix, and $\langle H^{(1)}, H^{(2)}\rangle$ is the ordinary conjugated dot product $\sum_n h_n^{(1)*} h_n^{(2)}$, the above adjoint definition corresponds to the conjugate-transpose for matrices. (Thus, as claimed in class, "swapping rows and columns" is the *consequence* of the "real" definition of transposition/adjoints, not the source.)

Note: In the **subsequent** parts of this problem, you may *not* assume that $\hat{O}$ is finite-dimensional (nor may you assume any specific formula for the inner product). Use only the abstract definitions of adjoint and linear operators on Hilbert spaces, along with the key properties of inner products: $\langle u, v\rangle = \langle v, u\rangle^*$, $\langle u, \alpha v + \beta w\rangle = \alpha\langle u, v\rangle + \beta\langle u, w\rangle$ (for arbitrary complex scalars α, β), and $\|u\|^2 = \langle u, u\rangle \geq 0$ ($= 0$ if and only if[1] $u = 0$).

(b) If a linear operator $\hat{O}$ satisfies $\hat{O}^\dagger = \hat{O}^{-1}$, then the operator is called **unitary**. Show that a unitary operator preserves inner products (that is, if we apply $\hat{O}$ to every element of a Hilbert space, then their inner products with one another are unchanged). Show that the eigenvalues u of a unitary operator have unit magnitude ($|u| = 1$) and that its eigenvectors can be chosen to be orthogonal to one another.

(c) For a non-singular operator $\hat{O}$ (i.e. $\hat{O}^{-1}$ exists), show that $(\hat{O}^{-1})^\dagger = (\hat{O}^\dagger)^{-1}$. (Thus, if $\hat{O}$ is Hermitian then $\hat{O}^{-1}$ is also Hermitian.)

[1] Technically, we mean $u = 0$ "almost everywhere" (e.g. excluding isolated points).

Problem 2: Maxwell eigenproblems

(a) As in class, assume $\varepsilon(\mathbf{x})$ real and positive (and that all function spaces are chosen so that the integrals you need exist etc.). In class, we eliminated $\mathbf{E}$ from Maxwell's equations to get an eigenproblem in $\mathbf{H}$ alone, of the form $\hat{\Theta}\mathbf{H}(\mathbf{x}) = \frac{\omega^2}{c^2}\mathbf{H}(\mathbf{x})$. Show that if you instead eliminate $\mathbf{H}$, you *cannot* get a Hermitian eigenproblem in $\mathbf{E}$ for the usual inner product $\langle \mathbf{E}_1, \mathbf{E}_2\rangle = \int \mathbf{E}_1^* \cdot \mathbf{E}_2$ except for the trivial case $\varepsilon =$ constant. Instead, show that you get a *generalized Hermitian eigenproblem*: an equation of the form $\hat{A}\mathbf{E}(\mathbf{x}) = \frac{\omega^2}{c^2}\hat{B}\mathbf{E}(\mathbf{x})$, where *both* $\hat{A}$ and $\hat{B}$ are Hermitian operators.

(b) For *any* generalized Hermitian eigenproblem where $\hat{B}$ is positive definite (i.e. $\langle \mathbf{E}, \hat{B}\mathbf{E}\rangle > 0$ for all $\mathbf{E}(\mathbf{x}) \neq 0$), show that the eigenvalues (i.e., the solutions of $\hat{A}\mathbf{E} = \lambda\hat{B}\mathbf{E}$) are real and that different eigenfunctions $\mathbf{E}_1$ and $\mathbf{E}_2$ satisfy a modified kind of orthogonality. Show that $\hat{B}$ for the $\mathbf{E}$ eigenproblem above was indeed positive definite.

(c) Alternatively, show that $\hat{B}^{-1}\hat{A}$ is Hermitian under a modified inner product $\langle \mathbf{E}, \mathbf{E}'\rangle_B = \langle \mathbf{E}, \hat{B}\mathbf{E}'\rangle$ for Hermitian $\hat{A}$ and $\hat{B}$ and positive-definite $\hat{B}$ with respect to the original $\langle \mathbf{E}, \mathbf{E}'\rangle$ inner product; the results from the previous part then follow.

(d) Show that *both* the $\mathbf{E}$ and $\mathbf{H}$ formulations lead to generalized Hermitian eigenproblems (or. equivalently, Hermitian with a modified inner product) with real ω if we allow magnetic materials $\mu(\mathbf{x}) \neq 1$ (but require μ real, positive, and independent of $\mathbf{H}$ or ω).

(e) μ and ε are only ordinary numbers for *isotropic* media. More generally, they are 3×3 matrices (technically, rank 2 tensors)—thus, in an *anisotropic medium,* by putting an applied field in one direction, you can get dipole moment in different direction in the material. What conditions on these 3×3 matrices still give a generalized Hermitian eigenproblem in $\mathbf{E}$ (or $\mathbf{H}$) with real eigen-frequencies ω?

Problem 3: Linear responses and symmetry

Suppose that we introduce a nonzero current $\mathbf{J}(\mathbf{x})e^{-i\omega t}$ into Maxwell's equations at a given frequency ω for linear time-invariant materials described (at that ω) by some scalar $\varepsilon(\mathbf{x})$ and $\mu(\mathbf{x}) = 1$, and we want to find the resulting time-harmonic electric field $\mathbf{E}(\mathbf{x})e^{-i\omega t}$.

(a) Show that this results in a linear equation of the form $\hat{A}\mathbf{E} = \mathbf{b}(\mathbf{x})$, where $\hat{A}$ is some linear operator and $\mathbf{b}$ is some known right-hand side proportional to the current density $\mathbf{J}$.

(b) Suppose that we have a symmetry g of the problem (e.g. a mirror flip), so that $\hat{g}$ (the corresponding operation on $\mathbf{E}$ commutes with $\hat{A}$, and preserves the boundary conditions (i.e. if $\mathbf{E}$ satisfies the boundary conditions, then $\hat{g}\mathbf{E}$ does too). Show that if $\mathbf{J}$ is an eigenfunction of $\hat{g}$ with some eigenvalue α, then the solution $\mathbf{E}$ is also an eigenfunction of $\hat{g}$ with the same eigenvalue. (e.g. if g is a mirror symmetry, then an anti-symmetric current $\hat{g}\mathbf{J} = -\mathbf{J}$ produces an anti-symmetric field).

(c) Look up a picture online of the magnetic field $\mathbf{B} = \mathbf{H}$ (for $\mu = 1$) of a current loop, e.g. `https://en.m.wikipedia.org/wiki/File:Magnetic_field_due_to_current.svg` is a nice image on Wikipedia. (Technically, this image is probably for a magneto-static current loop $\omega = 0$, but you'll get a very similar image for a sufficiently small ω, so that the wavelength $2\pi c/\omega$ is much larger than the diameter of the current loop.) If the current loop lies in the xy plane, then this has mirror symmetry in z, i.e. if $g = \sigma_z$ is the $z \to -z$ mirror-flip operator, then $\hat{\sigma}_z\mathbf{J} = \mathbf{J}$. But from the picture, it may not look like the magnetic field has mirror symmetry. Does this contradict your result in the previous part? Explain. (Hint: how are $\mathbf{E}$ and $\mathbf{H}$ related?)

(d) Formally, $\mathbf{E} = \hat{A}^{-1}\mathbf{b}$, where $\hat{A}^{-1}$ is related to the *Green's function* of the system. What happens if ω is one of the eigenfrequencies of our Θ operator? (No rigorous solution required, just a few words about what you expect to happen physically in such a case; you can suppose for simplicity that you have a problem in a finite domain with impenetrable walls, e.g. PEC boundary conditions.)