# Notes on time evolution and conservation laws in electromagnetism

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## 1 Maxwell's equations

The time evolution of Maxwell's equations is governed by Ampere's law $\nabla \times \mathbf{H} = \frac{\partial}{\partial t}(\mathbf{E} + \mathbf{P}) + \mathbf{J}$ and Faraday's Law $-\nabla \times \mathbf{E} = \frac{\partial}{\partial t}(\mathbf{H} + \mathbf{M}) + \mathbf{K}$, where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, $\mathbf{P}$ and $\mathbf{M}$ are the electric and magnetic polarization densities, and $\mathbf{J}$ and $\mathbf{K}$ are the electric and magnetic current densities, respectively. (Magnetic current is the flow of magnetic charge, which is experimentally zero, but it is convenient to include the possibility of $\mathbf{K} \neq 0$ theoretically.) It is nice to write this in the form $\hat{C}\psi = \frac{\partial}{\partial t}(\psi + \phi) + \xi$, or:

$$\frac{\partial}{\partial t} \underbrace{\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}}_{\psi} = \underbrace{\begin{pmatrix} & \nabla \times \\ -\nabla \times & \end{pmatrix}}_{\hat{C}} \psi - \frac{\partial}{\partial t} \underbrace{\begin{pmatrix} \mathbf{P} \\ \mathbf{M} \end{pmatrix}}_{\phi} - \underbrace{\begin{pmatrix} \mathbf{J} \\ \mathbf{K} \end{pmatrix}}_{\xi},$$

so that $\psi$ is the six-component field state, $\phi$ is the six-component polarization, $\xi$ is the six-component current, and $\hat{C}$ is the $6 \times 6$ "curl" operator. (For these notes I am using "natural" units in which $\varepsilon_0 = \mu_0 = 1$.) Our goal in these notes is to analyze some of the key algebraic properties of these equations, especially as they relate to time evolution and conservation laws. The basics of electromagnetic energy can be found in standard textbooks [1], but a more complete treatment of arbitrary dispersive materials can be found in the literature as recently reviewed and extended by Welters [2].

### 1.1 The anti-Hermitian curl operator

The $6 \times 6$ curl operator $\hat{C}$ is anti-Hermitian ($\hat{C}^\dagger = -\hat{C}$) under the usual inner product $\langle \psi, \psi' \rangle = \int \mathbf{E}^* \cdot \mathbf{E}' + \mathbf{H}^* \cdot \mathbf{H}'$ with most boundary conditions. The key identity [1] to derive this is, as usual,

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}).$$

Using this, we can integrate by parts over the domain $\Omega$:

$$
\begin{aligned}
\langle \psi, \hat{C}\psi' \rangle &= \int_\Omega \left[ \mathbf{E}^* \cdot (\nabla \times \mathbf{H}') - \mathbf{H}^* \cdot (\nabla \times \mathbf{E}') \right] dV \\
&= \int_\Omega \left[ (\nabla \times \mathbf{E})^* \cdot \mathbf{H}' - (\nabla \times \mathbf{H})^* \cdot \mathbf{E}' - \nabla \cdot (\mathbf{E}^* \times \mathbf{H}' + \mathbf{E}' \times \mathbf{H}^*) \right] dV \\
&= \langle -\hat{C}\psi, \psi' \rangle - \oiint_{\partial\Omega} (\mathbf{E}^* \times \mathbf{H}' + \mathbf{E}' \times \mathbf{H}^*) \cdot d\mathbf{A},
\end{aligned}
$$

where $\partial\Omega$ denotes the boundary of $\Omega$ and $d\mathbf{A}$ is the outward normal. Hence, if we have boundary conditions such that the surface integral ($\oiint_{\partial\Omega}$) vanishes, then $\hat{C}^\dagger = -\hat{C}$. This will be the case in an infinite domain for square-integrable fields (which must decay faster than $1/|\mathbf{x}|$ asymptotically), or in a finite domain for periodic boundary conditions (where the boundary terms cancel), or in a finite domain with perfect-electric-conductor (PEC) boundary conditions (where $\mathbf{E}$ is normal to the boundary on $\partial\Omega$), or for perfect-magnetic-conductor (PMC) boundary conditions (where $\mathbf{H}$ is normal to the boundary).

One consequence of this, which we use below, is that

$$
2\,\mathrm{Re}\langle \psi, \hat{C}\psi \rangle = \langle \psi, \hat{C}\psi \rangle + \langle \hat{C}\psi, \psi \rangle = \langle \psi, \hat{C}\psi \rangle + \langle \psi, -\hat{C}\psi \rangle = 0,
$$

i.e. $\langle \psi, \hat{C}\psi \rangle$ is purely imaginary.

## 1.2 The polarizations and susceptibilities in linear time-invariant media

In *linear time-invariant* medium, the polarizations $\phi$ arise from the fields $\psi$ in a linear convolution relation

$$
\phi(t) = \chi * \psi |_t = \int_{-\infty}^{+\infty} \chi(t - t')\psi(t')dt',
$$

where $\chi(t)$ is a $6 \times 6$ *susceptibility* matrix [1, 2]. That is, the polarization is proportional to a weighted average of the fields at different times.[1] Physically, we require the material to be *causal*: polarizations must come *after* the fields that create them, not before, which leads to the condition that $\chi(t) = 0$ for $t < 0$. As we discuss below, however, in a linear medium we will also require *passivity*—the material can dissipate energy but not supply energy—and it turns out that the assumption of passivity alone is enough to guarantee causality [2].

In the frequency domain, the Fourier transform of a convolution is a multiplication. That is, if we denote the Fourier transforms by $\hat{\phi}$, $\hat{\chi}$, and $\hat{\psi}$, then

$$
\hat{\phi}(\omega) = \hat{\chi}(\omega)\hat{\psi}(\omega).
$$

The frequency dependence of $\hat{\chi}(\omega)$ is known as *material dispersion.* Passivity [2], via causality [1], turns out to guarantee that $\hat{\chi}(\omega)$ is an *analytic* function (convergent Taylor series,

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[1]Here, I have omitted the $\mathbf{x}$ dependence. Typically, we have a *local* medium, in which $\phi(t, \mathbf{x})$ depends on only on fields $\psi(t', \mathbf{x})$ at the same point $\mathbf{x}$ in space, but in some cases authors also consider *nonlocal* media in which $\phi$ can arise from fields at other points. Nonlocal models are only relevant in conductors at lengthscales of a few nanometers, however, and in practice local models are almost always accurate enough to ignore nonlocal effects.

no poles or other singularities) in the upper-half complex-$\omega$ plane (i.e. for $\mathrm{Im}\,\omega > 0$). Passivity also turns out to imply that $\mathrm{Im}[\omega\hat{\chi}(\omega)] = \frac{\omega\hat{\chi}(\omega) - \omega^*\hat{\chi}(\omega)^\dagger}{2i} \succeq 0$ (a positive-semidefinite imaginary part) in the upper-half plane (and in practice it is positive-definite for anything other than vacuum) [2]. Physically, the fields $\psi$ and polarizations $\phi$ are real quantities, and this implies that $\hat{\chi}(-\omega) = \hat{\chi}(\omega)^*$ for real $\omega$ [1]. If one adds the physical constraint that the material must cease to respond at very high frequencies, and in particular that $\hat{\chi}(\omega)$ goes to zero faster than $1/|\omega|$ for large $|\omega|$, then one obtains the famous Kramers–Kronig relations that relate integrals of the real and imaginary parts of $\hat{\chi}(\omega)$ [1].

## 1.3    Dispersionless linear media

In many important cases it is possible to neglect material dispersion, in which case matters are greatly simplified. If we have a field $\psi(t)$ that is *bandlimited*—its Fourier transform $\hat{\psi}(\omega)$ is nonzero only in a small bandwidth around some frequency $\omega_0$ (or around $\pm\omega_0$)—and in this bandwidth $\hat{\chi}$ is nearly constant, then we can approximate $\hat{\chi}(\omega) \approx \hat{\chi}(\omega_0)$. Furthermore, in most physical circumstances the matrix $\hat{\chi}$ block-diagonalizes

$$\hat{\chi}(\omega) = \begin{pmatrix} \varepsilon(\omega) - 1 & \\ & \mu(\omega) - 1 \end{pmatrix}$$

where $\varepsilon$ and $\mu$ are the electric permittivity and the magnetic permeability, respectively. (Nonzero off-diagonal blocks in $\hat{\chi}$ are known as *bi-anisotropic susceptibilities*, and are very rarely encountered—essentially unheard-of at infrared and optical frequencies.) $\varepsilon$ and $\mu$ can be $3 \times 3$ matrices in general (*anisotropic materials*), but in the common case of *isotropic* materials they are scalars. (Furthermore, at infrared and optical frequencies, we almost always have $\mu \approx 1$.)

Neglecting dispersion, therefore, and assuming non-bianisotropic media, Maxwell's equations simplify to

$$\frac{\partial \psi}{\partial t} = \begin{pmatrix} \varepsilon^{-1} & \\ & \mu^{-1} \end{pmatrix} \left[ \hat{C}\psi - \xi \right] = (1 + \hat{\chi})^{-1} \left[ \hat{C}\psi - \xi \right].$$

# 2    Conservation laws in lossless media

Suppose we have the simple case of lossless, dispersionless time-independent media with real $\varepsilon > 0$ and $\mu > 0$.

## 2.1    Conservation of energy

In that case, define

$$\langle \psi, \psi' \rangle_{\hat{\chi}} = \langle \psi, (1 + \hat{\chi})\psi' \rangle = \langle (1 + \hat{\chi})\psi, \psi' \rangle = \int \left[ \mathbf{E}^* \cdot \varepsilon \mathbf{E}' + \mathbf{H}^* \cdot \mu \mathbf{H}' \right].$$

This turns out to be a conserved "energy" (actually *twice* the physical energy in the fields, as shown below):

$$\frac{\partial}{\partial t} \langle \psi, \psi \rangle_{\hat{\chi}} = \left\langle (1 + \hat{\chi}) \frac{\partial \psi}{\partial t}, \psi \right\rangle + \left\langle \psi, (1 + \hat{\chi}) \frac{\partial \psi}{\partial t} \right\rangle$$

$$= \left\langle \hat{C}\psi - \xi, \psi \right\rangle + \left\langle \psi, \hat{C}\psi - \xi \right\rangle$$

$$= -2 \operatorname{Re} \langle \psi, \xi \rangle,$$

where we have used the anti-Hermitian property of $\hat{C}$ from above. Thus, if there are no sources ($\xi = 0$) then $\langle \psi, \psi \rangle_{\hat{\chi}}$ is *conserved* (constant in time). More generally, as we shall see below, we can interpret $2 \operatorname{Re} \langle \psi, \xi \rangle$ as being (proportional to) the work done on the currents by the fields, so that if this quantity is positive then the energy $\langle \psi, \psi \rangle_{\hat{\chi}}$ in the fields is decreasing.

Another way of expressing this is in terms of the *time-evolution operator*

$$\hat{U}_t = e^{(1+\hat{\chi})^{-1} \hat{C} t},$$

defined such that $\psi(t) = \hat{U}_t \psi(0)$ solves the source-free ($\xi = 0$) problem. Under the $\langle \psi, \psi' \rangle_{\hat{\chi}}$ inner product, this operator is *unitary* ($\hat{U}_t^\dagger = \hat{U}_t^{-1}$) because the exponent is anti-Hermitian:

$$\langle \psi, (1 + \hat{\chi})^{-1} \hat{C} \psi' \rangle_{\hat{\chi}} = \langle \psi, \hat{C} \psi' \rangle = \langle -\hat{C} \psi, \psi' \rangle = \langle -(1 + \hat{\chi})^{-1} \hat{C} \psi, \psi' \rangle_{\hat{\chi}}.$$

Unitary time-evolution is equivalent to conservation of norm (energy): $\langle \psi(t), \psi(t) \rangle_{\hat{\chi}} = \langle \hat{U}_t \psi(0), \hat{U}_t \psi(0) \rangle_{\hat{\chi}} = \langle \psi(0), \psi(0) \rangle_{\hat{\chi}}$.

More generally, in the presence of sources $\xi \neq 0$, we can still express the solutions at time $t$ in terms of $\psi(0)$, the currents, and the time-evolution operator $\hat{U}_t$ by:

$$\psi(t) = \hat{U}_t \psi(0) - \int_0^t \hat{U}_{t-t'} (1 + \hat{\chi})^{-1} \xi(t') dt',$$

as can be verified by plugging this expression into the $\frac{\partial \psi}{\partial t}$ equation above and applying the Leibniz rule of calculus: $\frac{\partial}{\partial t} \left[ \int_0^t \hat{U}_{t-t'} (1 + \hat{\chi})^{-1} \xi(t') dt \right] = \hat{U}_0^{\nearrow 1} (1 + \hat{\chi})^{-1} \xi(t) + \int_0^t \frac{\partial \hat{U}_{t-t'}}{\partial t} (1 + \hat{\chi})^{-1} \xi(t') dt$, where the second term gives us our $(1 + \hat{\chi})^{-1} \hat{C}$ factor and the first term gives us the $-\xi$ term in the $\frac{\partial \psi}{\partial t}$ equation.

## 2.2 Symmetry and conservation laws

The simplest case of a symmetry is an operation $g$ (a rotation/reflection and/or a translation) along with its corresponding operation $\hat{g}$ on $\psi$ that **commutes** with the linear operator(s) describing our system (and preserves the boundary condition). Any rotation/reflection/translation commutes with the curl operator $\nabla \times$ and hence with $\hat{C}$, so the symmetries in Maxwell's equations are determined by the distribution of materials. $\hat{g}$ is a symmetry if it commutes with $\hat{\chi}$ (i.e., if the materials are **invariant** under that rotation/reflection/translation). Hence, a symmetry $\hat{g}$ commutes with $(1 + \hat{\chi})^{-1} \hat{C}$ and time evolution $\hat{U}_t = e^{(1+\hat{\chi})^{-1} \hat{C} t}$. It turns out that this implies a conservation law: eigenvalues of $\hat{g}$ are conserved.

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In particular, suppose that at time $t = 0$ we have fields $\psi(0)$ that are eigenfunctions of $\hat{g}$ with an eigenvalue $\lambda$, i.e. $\hat{g}\psi(0) = \lambda\psi(0)$. It then follows by commutation that $\hat{g}\psi(t) = \hat{g}\hat{U}_t\psi(0) = \hat{U}_t\hat{g}\psi(0) = \lambda\hat{U}_t\psi(0) = \lambda\psi(t)$: the fields $\psi(t)$ are *still* eigenfunctions of the symmetry operation $\hat{g}$ with the *same* eigenvalue $\lambda$ *at all times.* More generally, if $\xi \neq 0$, we can see from the general $\psi(t)$ formula above that we need *both* $\psi(0)$ and $\xi(t')$ (for all $0 \leq t' \leq t$) to be eigenfunctions of $\hat{g}$ with the *same* eigenvalue $\lambda$ in order for $\psi(t)$ to be as well.

For example, suppose $g$ is a mirror symmetry, which has eigenvalues $\pm 1$ since $\hat{g}^2 = 1$: the eigenfunctions are even/odd (symmetric/anti-symmetric) functions. If you have mirror-symmetric materials $\hat{\chi}$, and you start out with (e.g.) an *odd* $\psi(0)$ at $t = 0$ (and any currents $\xi$ are also odd), then the solutions at *all* times are odd—the "even/oddness is conserved."

### 2.2.1 General case: Conservation of "irrep"

For a more general symmetry group $G$ (of rotations, reflections, and/or translations $g \in G$), understanding the full impact is best done from the perspective of **group representation theory**. Using this tool, we can define a projection operator $\hat{P}^{(\alpha)} = \frac{d_\alpha}{|G|} \sum_{g \in G} \chi^{(\alpha)}(g)^* \hat{g}$, where $\chi^{(\alpha)}(g) = \operatorname{tr} D^{(\alpha)}(g)$ is the character of an irreducible representation (*irrep*) $D^{(\alpha)}(g)$ of $G$ with dimension $d_\alpha$ and $\hat{g}$ is the transformation (rotation/reflection/translation) acting on functions $\psi$ (or $\phi$ or $\xi$) corresponding to the operation $g$ on the coordinates.[2] $\hat{P}^{(\alpha)}\psi$ projects any function $\psi$ onto a function (possibly zero) that transforms as a partner function (basis function) of the irrep $\alpha$. It turns out that the irrep is "conserved" in the following sense: if your currents are partners of irrep $\alpha$, and your fields $\psi$ are partners of $\alpha$ at *any* time, then the fields are partners at *all* times.

If the currents $\xi$ are partners (possibly zero) of $\alpha$, that means $\hat{P}^{(\alpha)}\xi = \xi$ for all times. Furthermore, if $G$ is the symmetry group of the problem, then by definition $\hat{P}^{(\alpha)}$ commutes with $(1 + \hat{\chi})^{-1}\hat{C}$ and hence with the time-evolution operator $\hat{U}_t$. Suppose at time $t = 0$ our fields $\psi(0)$ are partners of $\alpha$, i.e $\hat{P}^{(\alpha)}\psi(0) = \psi(0)$, where the dependence $\psi(t, \mathbf{x})$ on $\mathbf{x}$ is implied. We can write $\psi(t)$ via $\hat{U}_t$ and $\xi$ as explained above. Then, since $\hat{P}^{(\alpha)}$ again commutes with $\hat{U}_t$ and $(1 + \hat{\chi})^{-1}$, it immediately follows that $\hat{P}^{(\alpha)}\psi(t) = \psi(t)$, and $\psi(t)$ is a partner of irrep $\alpha$.

Again, the simplest example is that of a problem with a mirror symmetry $\sigma$, in which case $G = \{E, \sigma\}$ where $E$ is the identity. The irreps are $\chi^{(1)} = (1, 1)$ and $\chi^{(2)} = (1, -1)$, with the corresponding partners being even and odd functions. If you have even current sources, and even fields at *any* time, then fields are even at *all* times; similarly for odd currents and fields.

## 2.3 Poynting's theorem

To make explicit the connection with the classical notion of energy, we can briefly derive Poynting's theorem to relate electromagnetic energy to mechanical energy [1]. The key to deriving Poynting's theorem is to compute the rate of *mechanical* work done by the fields on the currents. Recall that the electric field $\mathbf{E}$ is the force per unit charge, and $\mathbf{J}$ is current

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[2]If $g$ is a $3 \times 3$ rotation matrix $g^{-1} = g^T$ acting on the coordinates $\mathbf{x}$, for example, then $\hat{g}\psi|_{\mathbf{x}} = \begin{pmatrix} g\mathbf{E}(g^{-1}\mathbf{x}) \\ g\mathbf{H}(g^{-1}\mathbf{x}) \det g \end{pmatrix}$, where the extra $\det g$ factor arises because the magnetic field is a pseudo-vector.

(charge/time) per area. If we think of $\mathbf{J}$ as a charge density $\rho$ (charge/volume) moving at a velocity $\mathbf{v}$, then we can write $\mathbf{J} = \rho\mathbf{v}$. But since $\mathbf{E}\rho$ is force per unit volume, and force $\cdot\, \mathbf{v}$ is work per time, it follows that $\mathbf{E} \cdot \mathbf{J}$ is the work per time per volume done *by* the field *on* the charges. Similar for $\mathbf{H} \cdot \mathbf{K}$ for the (apparently fictitious) case of magnetic charges. Hence, we can interpret

$$\mathrm{Re}\langle\psi, \xi\rangle$$

as the rate of change of mechanical energy, where the Re and the complex conjugations inside $\langle\cdot, \cdot\rangle$ are irrelevant for the physical, real-valued fields. From above, we have

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\langle\psi, \psi\rangle_{\hat{\chi}}\right] + \mathrm{Re}\langle\psi, \xi\rangle = 0,$$

so if we interpret $\frac{1}{2}\langle\psi, \psi\rangle_{\hat{\chi}}$ as the *electromagnetic energy* (in a lossless, dispersionless system), then total energy is conserved.

The above was if we assumed the surface term from the integration-by-parts of $\hat{C}$ vanished, which is the case if we are considering a closed system from which no energy escapes (or enters). More generally, if we imagine that the integration domain $\Omega$ is some arbitrary volume—not necessarily the entire domain of the PDE—put back in the surface terms from the integration by parts, we obtain:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\langle\psi, \psi\rangle_{\hat{\chi}}\right] + \mathrm{Re}\langle\psi, \xi\rangle = \frac{1}{2}\left[\langle\psi, \hat{C}\psi\rangle + \langle\hat{C}\psi, \psi\rangle\right]$$

$$= -\oiint_{\partial\Omega} \mathrm{Re}\left[\mathbf{E}^* \times \mathbf{H}\right] \cdot d\mathbf{A},$$

where the *Poynting*[3] *flux* $\mathrm{Re}[\mathbf{E}^* \times \mathbf{H}]$ is interpreted as the *electromagnetic energy flux*: the flow of energy (per time, per area) out through the boundary $\partial\Omega$. For physical, real-valued fields, this is simply $\mathbf{E} \times \mathbf{H}$. The left-hand side is the time derivative of the *total* (electromagnetic + mechanical) energy in $\Omega$ and the right-hand side is the rate at which energy is flowing *into* $\Omega$. That is, since we defined $d\mathbf{A}$ as the *outward* normal, when the Poynting vector is *inwards* (into $\Omega$) we obtain positive right-hand side and hence *increasing* total energy.

## 2.4  A note on complex numbers and time averages

The physical fields and polarizations are always real, so one might be tempted to simply omit the complex conjugations and Re factors in the expressions above. However, it is very convenient mathematically to allow complex solutions, and it turns out that these complex conjugations have a simple physical interpretation in the case of sinusoidally oscillating quantities.

In linear systems, we often write sinusoidally oscillating quantities as the real parts of complex exponentials, simply because complex exponentials are much easier to work with

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[3]Named after John Henry Poynting, who developed the first version of this equation in 1884 ("On the transfer of energy in the electromagnetic field," *Philosophical Transactions* vol. 175, pp. 343–361). The pre-vector notation, following Maxwell from 1865, is horrifying.

than sines and cosines.[4] That is, we might work with complex quantities $a(t) = Ae^{-i\omega t}$ and $b(t) = Be^{-i\omega t}$, where $A$ and $B$ are complex amplitudes, and then at the end of our calculation obtain the physical solution from $\mathrm{Re}[a]$ and $\mathrm{Re}[b]$. In this case, the following relationships are extremely useful:

$$\text{time average of } \mathrm{Re}[a] \cdot \mathrm{Re}[b] = \frac{1}{2}\mathrm{Re}[a^*b] = \frac{1}{2}\mathrm{Re}[A^*B],$$

$$\text{time average of } \mathrm{Re}[a]^2 = \frac{1}{2}|a|^2 = \frac{1}{2}\mathrm{Re}\,|A|^2.$$

This can be derived by simply performing the time average $\frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt$ and evaluating the integral; the second identity is merely a special case of the first.

Hence, if $\frac{1}{2}[\varepsilon \mathbf{E}\cdot\mathbf{E} + \mu\mathbf{H}\cdot\mathbf{H}]$ is the physical energy density in the field, then $\frac{1}{4}\left[\varepsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2\right]$ is the time-average energy density for time-harmonic fields. Similarly, $\frac{1}{2}\mathrm{Re}[\mathbf{E}^* \times \mathbf{H}]$ is the time-average Poynting flux for time-harmonic fields, and $\frac{1}{2}\mathrm{Re}\langle\psi,\xi\rangle$ is the time-average rate of work done on the currents. This is *extremely useful*! (But remember, only for time-harmonic fields!) At infrared and optical frequencies, the fields are oscillating at $> 10^{14}$ Hz, so any physical detector will typically only measure these time-average quantities.

## 3   Conservation laws in passive linear media

Let us now return to the case of an arbitrary $6\times 6$ dispersive medium whose polarization is a convolution $\phi = \chi * \psi$ with the susceptibility $\chi(t,\mathbf{x})$. In this case, we obtain:

$$\frac{\partial}{\partial t}\langle\psi,\psi\rangle = \langle\psi, \hat{C}\psi - \frac{\partial\phi}{\partial t} - \xi\rangle + \langle\hat{C}\psi - \frac{\partial\phi}{\partial t} - \xi, \psi\rangle$$

and hence

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\langle\psi,\psi\rangle\right] + \mathrm{Re}\langle\psi,\frac{\partial\phi}{\partial t}\rangle + \mathrm{Re}\langle\psi,\xi\rangle = -\oiint_{\partial\Omega}\mathrm{Re}\left[\mathbf{E}^* \times \mathbf{H}\right]\cdot d\mathbf{A}.$$

On the right-hand side is the Poynting flux into the domain, from the integration-by-parts of $\hat{C}$ as before. On the left hand side, the first term is the rate of change of the electromagnetic energy $\frac{1}{2}\langle\psi,\psi\rangle$ in vacuum ($\varepsilon = \mu = 1$): this can be thought of as the energy stored "purely" in the fields. The third term $\mathrm{Re}\langle\psi,\xi\rangle$ is the rate of work done on the currents $\xi$ as before, i.e. the rate of change of the mechanical energy in these "free" or "external" charges. The second term $\mathrm{Re}\langle\psi,\partial\phi/\partial t\rangle$ is similar, where $\partial\phi/\partial t$ is the "bound" current [1]: this is a *physical* current corresponding of the motion of the polarizations (the little charges "bound" to the material that are polarizing in response to the fields). That is, the second term is the rate of work done *on the material* by the fields.

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[4]This is called a "phasor" representation in electrical engineering, where the $\frac{1}{2}\mathrm{Re}[A^*B]$ identity is commonplace for computing time-averaged "real" power and similar quantities. The imaginary part $\frac{1}{2}\mathrm{Im}[A^*B]$ can also be useful, and leads to relations involving the so-called "reactive" power/energy.

## 3.1 Passivity

A "passive" medium is one which does *not supply energy*. In fact, all linear materials are necessarily passive, because a linear "active" (or "gain") medium would lead to runaway exponential growth of the fields—all physical gain must be nonlinear, because any energy source has to run out at some point. Mathematically, passivity means that the fields can do net work on the medium but not vice versa. From above, the net work done *on* the medium *by* the fields is precisely the work done on the polarization currents, and passivity means that this must be nonnegative:

$$\int_{-\infty}^{t} \mathrm{Re}\langle \psi, \frac{\partial}{\partial t}(\chi * \psi)\rangle dt' \geq 0$$

for all times $t$ and all fields $\psi$ (in a suitable function space) [2]. This constrains $\chi *$ to be a "passive convolution operator," and it follows (surprisingly!) that $\chi$ is causal, its Fourier transform $\hat{\chi}(\omega)$ is analytic for $\mathrm{Im}\,\omega > 0$, and $\mathrm{Im}[\omega\hat{\chi}(\omega)] = \omega\frac{\hat{\chi}-\hat{\chi}^{\dagger}}{2i}$ is positive semidefinite for $\mathrm{Im}\,\omega > 0$ as mentioned above [2].

An elegant proof that passivity implies causality is as follows [3, appendix B]. Let $\psi_+(\mathbf{x}, t)$ be any function that $= 0$ for $t < 0$, and let $\phi_+ = \chi * \psi_+$ be the corresponding polarization. Now, take any other function $\psi(\mathbf{x}, t)$ and the corresponding $\phi = \chi * \psi$. Because convolution is linear, $\psi + \lambda\psi_+$ (which $= \psi$ for $t < 0$), for any real $\lambda$, produces a polarization $\phi + \lambda\phi_+$. If we plug this into the passivity condition above for $t < 0$, we obtain

$$\int_{-\infty}^{t} \mathrm{Re}\langle \psi, \frac{\partial\phi}{\partial t}\rangle dt' + \lambda\int_{-\infty}^{t} \mathrm{Re}\langle \psi, \frac{\partial\phi_+}{\partial t}\rangle dt' \geq 0 \qquad (t < 0).$$

Now, since this must be true for *any* value of $\lambda$, even for $|\lambda| \to \infty$, it follows that the second term must be zero. Moreover, this must be true for any $\psi$, so it follows that we must have $\frac{\partial\phi_+}{\partial t} = 0$ for all $t < 0$. If the system starts at rest, so that $\phi_+ = 0$ for $t \to -\infty$, it follows that $\phi_+ = 0$ for $t < 0$. Since $\phi_+(\mathbf{x}, t) = \int_0^{\infty} \chi(t - t')\psi_+(\mathbf{x}, t')dt' = 0$ for all $t < 0$ and for all $\psi_+$, it follows that $\chi(t) = 0$ for all $t < 0$, i.e. that the susceptibility $\chi$ is causal.

Note that it is tempting to say that passivity means that $\mathrm{Re}\langle\psi, \frac{\partial\phi}{\partial t}\rangle \geq 0$, i.e. that the *rate* of work done on the material is nonnegative, but this is not actually true. Energy can be temporarily stored in the polarization of matter and then extracted by the fields, as long as you don't get out more energy than you put in. (Physically, imagine a photon that is absorbed by an atom to pump the atom into an excited state, but the atom subsequently undergoes spontaneous emission to re-emit the photon. Semiclassically, such a resonant absorption/re-emission is described by a *resonance* in $\hat{\chi}$: a pole in $\hat{\chi}$ at a complex frequency $\omega_p$ in the lower-half plane. The imaginary part $\mathrm{Im}\,\omega_p < 0$ is due to dissipation losses: physically, it represents the chance that the atom drops back down to its ground state by a non-radiative process, e.g. a vibrational collision.) Below, we will see that this $\mathrm{Re}\langle\psi, \frac{\partial\phi}{\partial t}\rangle$ term actually gives rise to the $(\varepsilon-1)|\mathbf{E}|^2 + (\mu-1)|\mathbf{H}|^2$ term that appears in the energy density of a lossless, dispersionless medium, and which represents the energy that is temporarily stored in the medium, only to be re-extracted into the fields.

## 3.2 Dissipation in the frequency domain

The rate of work done by the fields on the material, which is the rate of *energy dissipation* in the material, is given above by $\mathrm{Re}\langle\psi, \frac{\partial\phi}{\partial t}\rangle$. Suppose we Fourier transform to the frequency

domain, so that we are looking at time-harmonic fields with amplitudes $\hat{\psi}(\omega)$ and polarizations $\hat{\phi}(\omega) = \hat{\chi}(\omega)\hat{\psi}(\omega)$. The polarization current $\partial\phi/\partial t$ Fourier transforms to $-i\omega\hat{\phi}(\omega)$, and so we get a dissipation rate (at $\omega$) of:

$$\mathrm{Re}\langle\hat{\psi}, -i\omega\hat{\chi}(\omega)\hat{\psi}(\omega)\rangle = \int_\Omega \hat{\psi}^* \cdot \mathrm{Re}[-i\omega\hat{\chi}]\,\hat{\psi} = \int_\Omega \hat{\psi}^* \cdot \mathrm{Im}[\omega\hat{\chi}]\,\hat{\psi},$$

where $\mathrm{Im}[\omega\hat{\chi}] = \omega\frac{\hat{\chi}-\hat{\chi}^\dagger}{2i}$ is a direct measure of *how dissipative* the material is at $\omega$. Transparent materials are ones for which $\mathrm{Im}[\hat{\chi}] \approx 0$ (i.e., $\hat{\chi}$ is Hermitian) in the desired frequency bandwidth.

As mentioned above, one of the consequences of passivity is that $\mathrm{Im}[\omega\hat{\chi}] \geq 0$ (in the matrix sense: positive semidefinite) for $\mathrm{Im}\,\omega > 0$, by taking the limit as $\mathrm{Im}\,\omega \to 0^+$ one can obtain the same result on the real-$\omega$ axis (albeit with some technicalities to account for the possibility of singularities on the real axis [2]), so that $\mathrm{Im}[\omega\hat{\chi}] \geq 0$ for real frequencies $\omega$ in a passive medium, and the magnitude of this imaginary part is a direct measure of how lossy the medium is at that frequency.

## 3.3   Dynamical and dispersive energy densities

From our Poynting theorem above, if we want to define an "electromagnetic energy density" such that the sum of electromagnetic + mechanical energy of free charges is conserved in the absence of incoming/outgoing Poynting flux, we should define the electromagnetic energy density as

$$\frac{1}{2}\left(\psi^* \cdot \psi\right) + \int_{-\infty}^{t} \mathrm{Re}\left[\psi^* \cdot \frac{\partial}{\partial t}(\chi * \psi)\right] dt'.$$

The first term can be interpreted as the energy stored "purely" in the fields, while the second term is the energy dumped into the material. This was dubbed the "dynamical" energy density [4, 2], and seems to be a relatively recent innovation. It is the energy density that you need to make Poynting's theorem hold in an arbitrary medium, and turns out to be a useful device for studying energy velocities.

This energy density simplifies in many cases. Suppose you have time-harmonic fields $\sim e^{-i\omega t}$, at a frequency where your materials have *negligible loss* ($\mathrm{Im}\,\hat{\chi} \approx 0$ from above), it turns out that the dynamical energy density reduces to the "dispersive" energy density [2]:

$$\frac{1}{2}\psi^* \cdot \left[1 + \frac{\partial(\omega\hat{\chi})}{\partial\omega}\right]\psi,$$

where $\frac{\partial(\omega\hat{\chi})}{\partial\omega} \geq 0$ is guaranteed from passivity [2]. In the usual case where $\hat{\chi}$ diagonalizes into isotropic $\varepsilon(\omega) - 1$ and $\mu(\omega) - 1$, this expression takes on the well-known form

$$\frac{1}{2}\left[\frac{\partial(\omega\varepsilon)}{\partial\omega}|\mathbf{E}|^2 + \frac{\partial(\omega\mu)}{\partial\omega}|\mathbf{H}|^2\right],$$

which is derived in many textbooks [1]. (Note that this includes dispersion, but neglects loss! It is *not valid* in a lossy medium.)

Finally, if we take the "dispersive" energy density for negligible loss, and *additionally* we neglect dispersion (i.e. we approximate $\varepsilon$ and $\mu$ as frequency-independent), then we obtain the formula

$$\frac{1}{2}\psi^* \cdot (1 + \hat{\chi})\psi = \frac{1}{2}\left[\varepsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2\right]$$

from above for the energy density in a dispersionless, lossless, isotropic medium.

## 3.4   Conservation and symmetry

The time-evolution operator $\hat{U}_t$ is more complicated to write down explicitly in a general dispersive medium, so I omit it here. However, it is still the case that it commutes with symmetry operations $\hat{g}$ and projection operators $\hat{P}^{(\alpha)}$, because the time evolution must be invariant under the symmetry operations $\hat{g}$ (again, this is essentially the definition of the symmetry group of the problem). Hence it will still be the case (proof as in the lossless case) that if your currents and initial conditions are eigenfunctions of $\hat{g}$, then the solutions at later times are as well with the same eigenvalue. Or, more generally if your currents are partners of irrep $\alpha$, and the fields at some time $t$ are partners of $\alpha$, that the fields $\psi$ will be partners of $\alpha$ at all times: the "irrep is conserved."

# References

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