18.369 Problem Set 2

Due Wednesday, 28 February 2024.

Problem 1: Cylindrical symmetry

Suppose that we have a *cylindrical* metallic waveguide—that is, a perfect metallic tube with radius R, which is uniform in the z direction. The interior of the tube is simply air ($\varepsilon = 1$).

(a) This structure has continuous rotational symmetry around the z axis, correspond to commuting with rotations $\hat{\phi}$ by any angle ϕ (which is sometimes called the C_∞ symmetry group[1]). If $f(\phi)$ is an eigenfunction of $\hat{\phi}$ for all $\phi \in \mathbb{R}$, we know from class that $f(\phi)$ must[2] be an exponential function $e^{\kappa\phi}$ (or a multiple thereof) for some κ since $\hat{\phi}_1\hat{\phi}_2 = \widehat{\phi_1 + \phi_2}$. What are the possible values of κ, and why?

(b) For simplicity, consider the (Hermitian) *scalar* wave equation $-\nabla^2\psi = \frac{\omega^2}{c^2}\psi$ with $\psi|_{r=R} = 0$. Show that, when we look for solutions ψ that are eigenfunctions of all $\hat{\phi}$ from above, and have z dependence e^{ikz} (from the translational symmetry), then we obtain a Bessel equation (Google it if you've forgotten Mr. Bessel, or never learned). Write the solutions in terms of Bessel functions, assuming that you are given their zeros $x_{m,n}$ (i.e. $J_m(x_{m,n}) = 0$ for $n = 1, 2, \ldots$, where J_m is the Bessel function of the first kind...if you Google for "Bessel function zeros" you can find them tabulated). Sketch the dispersion relation $\omega(k)$ for a few bands.

(c) From the general orthogonality property of Hermitian-operator eigenfunctions, derive/prove an orthogonality integral for the Bessel functions. (No, just looking one up on Wikipedia doesn't count.)

[1] It also has an infinite set of mirror planes containing the z axis, but let's ignore these for now. If they are included, the group is called $C_{\infty\mathrm{v}}$.

[2] Assuming it is anywhere continuous, to exclude crazy non-measurable non-constructive counterexamples.

Problem 2: 2d Waveguide Modes

Consider the two-dimensional dielectric waveguide of thickness h that we first introduced in class:

$$\varepsilon(y) = \begin{cases} \varepsilon_{hi} & |y| < h/2 \\ \varepsilon_{lo} & |y| \geq h/2 \end{cases},$$

where $\varepsilon_{hi} > \varepsilon_{lo}$. Look for solutions with the "TM" or "E_z" polarization $\mathbf{E} = E_z(x,y)\hat{\mathbf{z}}e^{-i\omega t}$. The boundary conditions are that E_z is continuous and $\partial E_z/\partial y$ ($\sim H_x$) is continuous, and that we require the fields to be finite at $x, y \to \pm\infty$,

(a) Prove that we can set $\varepsilon_{lo} = 1$ without loss of generality, by a change of variables in Maxwell's equations. In the subsequent sections, therefore, set $\varepsilon_{lo} = 1$ for simplicity.

(b) Find the guided-mode solutions $E_z(x,y) = e^{ikx}E_k(y)$, where the corresponding eigenvalue $\omega(k) < ck$ is below the light line.

(i) Show for the $|y| < h/2$ region the solutions are of sine or cosine form, and that for $|y| > h/2$ they are decaying exponentials. (At this point, you can't easily prove that the arguments of the sines/cosines are real, but that's okay—you will be able to rule out the possibility of imaginary arguments below.)

(ii) Match boundary conditions (E_z and H_x are continuous) at $y = \pm h/2$ to obtain an equation relating ω and k. You should get a transcendental equation that you cannot solve explicitly. However, you can "solve" it graphically and learn a lot about the solutions—in particular, you might try plotting the left and right hand sides of your equation (suitably arranged) as a function of $k_\perp = \sqrt{\frac{\omega}{c^2}^2\varepsilon_{hi} - k^2}$, so that you have two curves and the solutions are the intersections (your curves will be parameterized by k, but try plotting them for one or two typical k).

(iii) From the graphical picture, derive an exact expression for the number of guided modes as a function of k. Show that there is exactly one guided mode, with even symmetry, as $k \to 0$, as we argued in class.

Problem 3: Evanescent modes in waveguides

In class, we looked at H_z-polarized solutions in a 2d metallic waveguide, formed by an $\varepsilon = \mu = 1$ region between two PEC walls at $x = 0$ and $x = L$, and found that (in the absence of sources) it satisfied the eigen-equation $-\nabla^2 H_z = \omega^2 H_z$ (for $\varepsilon_0 = \mu_0 = 1$ units) with "Neumann" boundary conditions $H_z'(0) = H_z'(L) = 0$. Exploiting translational symmetry in y, we looked for solutions $H_z(x, y) = u_k(x)e^{iky}$ with a given propagation constant ("wavevector") $k \in \mathbb{R}$, and found

$$u_{k,n} = \sin\left(\frac{n\pi x}{L}\right) \qquad \text{for } n = 0, 1, \ldots$$

$$\omega_n(k) = \pm\sqrt{\left(\frac{n\pi}{L}\right)^2 + k^2},$$

so that the dispersion relation is a set of hyperbolas (except for the $n = 0$ solution $\omega_0 = \pm|k|$).

(a) We can alternatively fix ω and solve for the propagation constant $k_n(\omega)$ and the corresponding H_z solutions. Let $\omega = 1.5\pi/L$. What do the possible solutions H_z look like for $n = 0, 1, 2, 3, \ldots$? The solutions that are exponentially growing/decaying are called "evanescent" modes.

(b) Suppose we introduce an electric current source $\mathbf{J} = f(x)\delta(y)e^{-i\omega t}\hat{x}$ into Maxwell's equations: here, a delta function in y and some function $f(x)$ as the x dependence, with a frequency ω, oriented in the x direction (a "current sheet"). Show that Maxwell's equations in this case take the form $(-\nabla^2 - \omega^2)H_z = Cf(x)\delta'(y)$ for some constant C, where $\delta'(y)$ is the derivative of a delta function (recall that $\int \delta'(y)g(y) = -g'(0)$, or equivalently $\delta'(y)$ is the second derivative of a unit step function). Any solutionH_z must have some kind of discontinuity or "kink" at $y = 0$ — what is it?

(c) In the presence of this current source $f(x)$, write the resulting magnetic field $H_z(x, y)$ everywhere in $x \in [0, L]$ as an infinite series $\sum_n$ of solutions from your $k_n(\omega)$ modes above, with a different series for $y > 0$ and $y < 0$. (Hint: you should find that by imposing the correct "kink" condition on H_z across $y = 0$ from above, you get an equation in terms of a Fourier series for $f(x)$ that allows you determine the coefficients of the series.) You should assume boundary conditions of *outgoing and/or decaying waves* as $y \to \pm\infty$: that is, for $y > 0$, the solutions should be a superposition of terms that look like e^{iky} for either real $k > 0$ (propagating upwards) or imaginary $k = i\kappa$ with $\kappa > 0$ (decaying upwards), and similarly (with opposite signs: propagating/decaying downwards) for $y < 0$. You *must* include the evanescent modes, as otherwise you will be unable to find a solution for an arbitrary[3] $f(x)$.

Problem 4: Poynting's theorem

In Jackson's *Classical Electrodynamics* textbook (or in many similar books), the electric field of a radiating dipole $\mathbf{J}(\mathbf{x}, t) = -i\omega\mathbf{p}\delta(x)\delta(y)\delta(z)e^{-i\omega t}$ at the origin in vacuum is given in spherical coordinates (radius r, radial direction $\mathbf{n}$), with $k = \omega/c$, as

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\frac{1}{r} + [3\mathbf{n}(\mathbf{n}\cdot\mathbf{p}) - \mathbf{p}]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)\right\}e^{i(kr-\omega t)}.$$

Note the divergence as $r \to 0$. The total power radiated by this dipole is computed by Jackson from the Poynting flux through an enclosing sphere to be $P = \frac{c^2 Z_0 k^4}{12\pi}|\mathbf{p}|^2$, where $Z_0 = \sqrt{\mu_0/\varepsilon_0}$. Show that you obtain the *same* P by taking the $r \to 0$ limit of of the work $-\frac{1}{2}\operatorname{Re}[\mathbf{E}^*\mathbf{J}]$ expended by $\mathbf{J}$ (regardless of the direction $\mathbf{n}$ from which you approach the origin). (Taylor expanding e^{ikr} may be helpful.)

[3] Technically, we must restrict ourselves to functions $f(x)$ that have a Fourier series; this includes all remotely realistic possibilities.

Problem 5: Scattering and energy

As in the course notes, we can write Maxwell's equations for the 6-component fields ψ as $\frac{\partial\psi}{\partial t} = \hat{C}\psi - \frac{\partial\phi}{\partial t} - \xi$, where the 6-component polarizations $\phi = \chi * \psi$ are a convolution in time with a (passive and causal) linear susceptibility $\chi(\mathbf{x}, t)$ at each point $\mathbf{x}$ in space. Now, suppose we have the situation depicted in Fig. 1: sources ($\xi \neq 0$) lying *outside* some bounded volume Ω produce an incident wave ψ_{inc} that scatters off materials ($\chi \neq 0$) *inside* of Ω to produce a scattered wave ψ_{scat}. Outside of Ω there are no materials ($\chi = 0$), while inside of Ω there are no sources ($\xi = 0$).

(a) Write the total fields as $\psi = \psi_{\text{inc}} + \psi_{\text{scat}}$, where $\psi_{\text{inc}} = \begin{pmatrix} \mathbf{E}_{\text{inc}} \\ \mathbf{H}_{\text{inc}} \end{pmatrix}$ satisfies $\frac{\partial\psi_{\text{inc}}}{\partial t} = \hat{C}\psi_{\text{inc}} - \xi$ (Maxwell's equations with the same currents but no materials), and the initial conditions at $t = -\infty$ are $\psi = \psi_{\text{inc}} = \psi_{\text{scat}} = 0$.

(i) What differential[4] equation must ψ_{scat} satisfy?

(ii) Fourier transform in time to find equations for $\hat{\psi}_{\text{inc}}(x, \omega)$ and $\hat{\psi}_{\text{scat}}$ (in terms of $\hat{\xi}$ and $\hat{\chi}$); equivalently, look for time-harmonic solutions $\hat{\psi}_{\text{inc}} e^{-i\omega t}$ and $\hat{\psi}_{\text{scat}} e^{-i\omega t}$ from a time-harmonic source $\hat{\xi}(\mathbf{x}) e^{-i\omega t}$.

(b) Applying Poynting's theorem and passivity to your time-harmonic fields, show that $\oiint_{\partial\Omega} \mathrm{Re}[\hat{\mathbf{E}}_{\text{inc}}^* \times \hat{\mathbf{H}}_{\text{inc}}] \cdot d\mathbf{A} = 0$ and $P_{\text{abs}} = -\oiint_{\partial\Omega} \mathrm{Re}[\hat{\mathbf{E}}^* \times \hat{\mathbf{H}}] \cdot d\mathbf{A} \geq 0$ (the absorbed power = inward total flux). (Note that here, Ω is a subset of the whole domain!)

(c) In order to ensure $P_{\text{scat}} = \oiint_{\partial\Omega} \mathrm{Re}[\hat{\mathbf{E}}_{\text{scat}}^* \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A} \geq 0$ (the scattered power = outward scattered flux), we need something more: an *outgoing boundary condition* (or "radiation" boundary condition): if we consider *any point* on sphere of radius R (around any point in Ω), then *in the limit* as $R \to \infty$ we require that $R^2\, \mathrm{Re}[\hat{\mathbf{E}}_{\text{scat}}^* \times \hat{\mathbf{H}}_{\text{scat}}]$ must point *radially outward.* Use Poynting's theorem to show that such a radiation boundary condition implies $\oiint_{\partial\Omega} \mathrm{Re}[\hat{\mathbf{E}}_{\text{scat}}^* \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A} \geq 0$.[5]

(d) Show that the "extinguished power" $P_{\text{ext}} = P_{\text{abs}} + P_{\text{scat}}$ can be written in terms of an integral over $\partial\Omega$ involving $\hat{\mathbf{E}}_{\text{inc}}^* \times \hat{\mathbf{H}}_{\text{scat}}$ and $\hat{\mathbf{E}}_{\text{scat}}^* \times \hat{\mathbf{H}}_{\text{inc}}$ *only.* This is a version of a famous result known as the **optical theorem**.

(e) Derive an alternative form of the optical theorem: show that P_{ext} can be written as an integral over Ω involving *only* $\hat{\psi}_{\text{inc}}^* \hat{\phi}$. (Hint: use Poynting's theorem to relate P_{abs} and P_{scat} to work on polarization currents, and write $\psi_{\text{scat}} = \psi - \psi_{\text{inc}}$.)

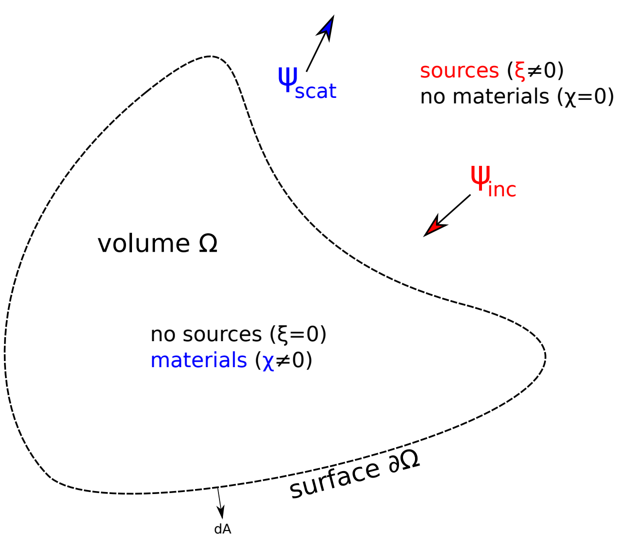

Figure 1: Schematic of an incident wave ψ_{inc} created by sources ξ that are only $\neq 0$ *outside* some volume Ω, interacting with materials $\chi \neq 0$ that lie *inside* Ω to produce an outgoing scattered wave ψ_{scat}.

[4] Technically, if you include a convolution $\chi *$ then you have a "pseudodifferential" operator.

[5] One will also choose some such boundary condition when constructing ψ_{inc}. Not necessarily an outgoing boundary condition, however—often it is convenient to have an incident wave coming in "from infinity", such as an incident planewave, perhaps with no explicit sources ($\xi = 0$) and just appropriate boundary conditions.