18.369 Problem Set 3

Problem 0: Scattering and energy

As in the course notes, we can write Maxwell's equations for the 6-component fields ψ as $\frac{\partial \psi}{\partial t} = \hat{C}\psi - \frac{\partial \phi}{\partial t} - \xi$, where the 6-component polarizations $\phi = \chi * \psi$ are a convolution in time with a (passive and causal) linear susceptibility $\chi(\mathbf{x}, t)$ at each point $\mathbf{x}$ in space. Now, suppose we have the situation depicted in Fig. 1: sources ($\xi \neq 0$) lying *outside* some bounded volume Ω produce an incident wave ψ_{inc} that scatters off materials ($\chi \neq 0$) *inside* of Ω to produce a scattered wave ψ_{scat}. Outside of Ω there are no materials ($\chi = 0$), while inside of Ω there are no sources ($\xi = 0$).

(a) Write the total fields as $\psi = \psi_{\text{inc}} + \psi_{\text{scat}}$, where $\psi_{\text{inc}} = \begin{pmatrix} \mathbf{E}_{\text{inc}} \\ \mathbf{H}_{\text{inc}} \end{pmatrix}$ satisfies $\frac{\partial \psi_{\text{inc}}}{\partial t} = \hat{C}\psi_{\text{inc}} - \xi$ (Maxwell's equations with the same currents but no materials), and the initial conditions at $t = -\infty$ are $\psi = \psi_{\text{inc}} = \psi_{\text{scat}} = 0$.

   (i) What differential[1] equation must ψ_{scat} satisfy?

   (ii) Fourier transform in time to find equations for $\hat{\psi}_{\text{inc}}(x, \omega)$ and $\hat{\psi}_{\text{scat}}$ (in terms of $\hat{\xi}$ and $\hat{\chi}$); equivalently, look for time-harmonic solutions $\hat{\psi}_{\text{inc}} e^{-i\omega t}$ and $\hat{\psi}_{\text{scat}} e^{-i\omega t}$ from a time-harmonic source $\hat{\xi}(\mathbf{x}) e^{-i\omega t}$.

[1] Technically, if you include a convolution $\chi *$ then you have a "pseudodifferential" operator.

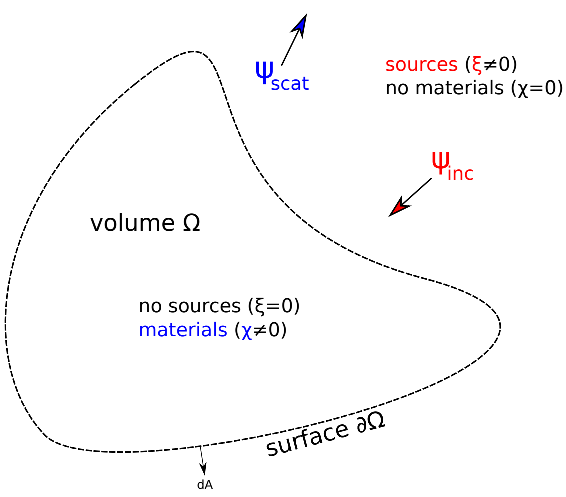

Figure 1: Schematic of an incident wave ψ_{inc} created by sources ξ that are only $\neq 0$ *outside* some volume Ω, interacting with materials $\chi \neq 0$ that lie *inside* Ω to produce an outgoing scattered wave ψ_{scat}.

(b) Applying Poynting's theorem and passivity to your time-harmonic fields, show that $\oiint_{\partial\Omega} \text{Re}[\hat{\mathbf{E}}^*_{\text{inc}} \times \hat{\mathbf{H}}_{\text{inc}}] \cdot d\mathbf{A} = 0$ and $P_{\text{abs}} = -\oiint_{\partial\Omega} \text{Re}[\hat{\mathbf{E}}^* \times \hat{\mathbf{H}}] \cdot d\mathbf{A} \geq 0$ (the absorbed power = inward total flux). (Note that here, Ω is a subset of the whole domain!)

(c) In order to ensure $P_{\text{scat}} = \oiint_{\partial\Omega} \text{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A} \geq 0$ (the scattered power = outward scattered flux), we need something more: an *outgoing boundary condition* (or "radiation" boundary condition): if we consider *any point* on sphere of radius R (around any point in Ω), then *in the limit* as $R \to \infty$ we require that $R^2\,\text{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}]$ must point *radially outward*. Use Poynting's theorem to show that such a radiation boundary condition implies $\oiint_{\partial\Omega} \text{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A} \geq 0$.[2]

(d) Show that the "extinguished power" $P_{\text{ext}} = P_{\text{abs}} + P_{\text{scat}}$ can be written in terms of an integral over $\partial\Omega$ involving $\hat{\mathbf{E}}^*_{\text{inc}} \times \hat{\mathbf{H}}_{\text{scat}}$ and $\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{inc}}$ *only*. This is a version of a famous result known as the **optical theorem**.

(e) Derive an alternative form of the optical theorem: show that P_{ext} can be written as an integral over Ω involving *only* $\hat{\psi}^*_{\text{inc}}\hat{\phi}$. (Hint: use Poynting's theorem to relate P_{abs} and P_{scat} to work on polarization currents, and write $\psi_{\text{scat}} = \psi - \psi_{\text{inc}}$.)

Problem 1: Periodic waveguide guidance proof

In class, we showed by a variational proof that any $\varepsilon(y)$, in two dimensions, gives rise to at least one guided mode whenever $\varepsilon(y)^{-1} = \varepsilon_{\text{lo}}^{-1} - \Delta(y)$ for $\int \Delta > 0$ and $\int |\Delta| < \infty$.[3] At least, we showed it for the TE polarization ($\mathbf{H}$ in the $\hat{\mathbf{z}}$ direction). Now, you will show the same thing much more generally, but using the same basic technique.

[2] We also will also need some boundary condition to uniquely determine ψ_{inc}. Not necessarily an outgoing boundary condition, however—often it is convenient to have an incident wave coming in "from infinity", such as an incident planewave, perhaps with no explicit sources ($\xi = 0$) and just appropriate boundary conditions.

[3] As in class, the latter condition on Δ will allow you to swap limits and integrals for any integrand whose magnitude is bounded above by some constant times $|\Delta|$ (by Lebesgue's dominated convergence theorem).

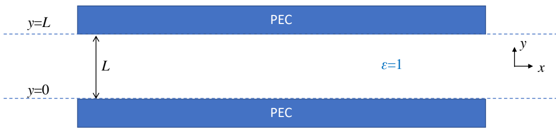

Figure 2: 2d metallic waveguide with perfect electric conductor walls, extending infinitely in the x direction.

(a) Let $\varepsilon(x,y)^{-1} = 1 - \Delta(x,y)$ be a periodic function $\Delta(x,y) = \Delta(x+a,y)$, with $\int |\Delta| < \infty$ and $\int_0^a \int_{-\infty}^{\infty} \Delta(x,y)dxdy > 0$. Prove that at least one TE guided mode exists, by choosing an appropriate (simple!) trial function of the form $\mathbf{H}(x,y) = u(x,y)e^{ikx}\hat{\mathbf{z}}$. That is, show by the variational theorem that $\omega^2 < c^2k^2$ for the lowest-frequency eigenmode. (It is sufficient to show it for $|k| \le \pi/a$, by periodicity in k-space; for $|k| > \pi/a$, the light line is not $\omega = c|k|$.)

(b) Prove the same thing as in (a), but for the TM polarization ($\mathbf{E}$ in the $\hat{\mathbf{z}}$ direction). Hint: you will need to pick a trial function of the form $\mathbf{H}(x,y) = [u(x,y)\hat{\mathbf{x}} + v(x,y)\hat{\mathbf{y}}]e^{ikx}$ where u and v are some (simple!) functions such that $\nabla \cdot \mathbf{H} = 0$.[4]

Problem 2: 2d metallic waveguide

Consider a waveguide depicted in Fig.~2: two x-invariant PEC (perfect electric conductor) walls enclosing a width-L region of $\varepsilon = 1$.

(a) Find the E_z-polarized eigenfunctions. From symmetry, this should be of the form $\mathbf{E} = E_{k,n}(y)e^{ikx-i\omega_n t}$. Hence, obtain the dispersion relation $\omega_n(k)$. Plot it for the first few bands.

(b) Now, suppose we put in a "point-dipole" current source $\mathbf{J}(x,y) = \delta(x)\delta(y-L/2)\hat{z}e^{-i\omega t}$ in the middle of the waveguide. This is mirror-symmetric (even), so you should be able to find a mirror-symmetric solution that consists of right-traveling ($k > 0$) waves for $x > 0$ and left-traveling ($k < 0$) waves for $x < 0$, where in each region ($x \gtrless 0$) you expand the solution *in the basis of the eigenfunctions*:

$$E_z = e^{-i\omega t} \begin{cases} \sum_{n=0}^{\infty} c_n E_n(y) e^{+ik_n x} & x > 0 \\ \sum_{n=0}^{\infty} c_n E_n(y) e^{-ik_n x} & x < 0 \end{cases},$$

where c_n are some coefficients to be determined, k_n is the positive solution of $\omega_n(k_n) = \omega$ (i.e. the $k > 0$ matching $\omega$for mode n), and $E_n = E_{k_n,n}$ is the corresponding eigenfunction. Solve for the coefficients c_n by plugging this E_z into Maxwell's equations with that current source.

Hint: To get $\delta(x)$, recall that the derivative of a discontinuous function is a delta function multiplied by the amplitude of the discontinuity. Maxwell's equations contain a $\nabla^2 E_z$ term, so to match the δ function source term, you must have a discontinuity in the *slope* $\partial E_z/\partial x$ at $x = 0$, while E_z itself must be continuous (otherwise you would get a discontinuity in the *first* derivative and a "delta derivative" from $\partial^2 E_z/\partial x^2$). From this discontinuity, you will $\partial^2 E_z/\partial x^2 = \delta(x)\sum c_n \cdot$ (some Fourier basis), and then you can use the usual Fourier orthogonality relations to match this to $\delta(y - L/2)$.

(c) Using your answer from the previous part, give a formula for the power P *expended* by the current source ($P = -\mathrm{Re}\int \mathbf{J}^* \cdot \mathbf{E}dxdy$, from class). (This power is also called the "local density of states" (LDOS) at that position and frequency, a subject we will return to later in class.) Sketch this power $P(\omega)$ as a function of frequency ω—notice anything strange?

(d) Using Meep (either installed on your own computer or via mybinder.org), compute and polot $P(\omega)$ (in arbitrary units) for exactly the situation in the previous part, with $L = 1$ (= a choice of distance units). It turns out that Meep has a built-in feature to compute the power expended by a dipole source (the "LDOS"): google "Meep LDOS tutorial". Compare to your analytical solution from the previous part.

[4] You might be tempted, for the TM polarization, to use the $\mathbf{E}$ form of the variational theorem that you derived in problem 1, since the proof in that case will be somewhat simpler: you can just choose $\mathbf{E}(x,y) = u(x,y)e^{ikx}\hat{\mathbf{z}}$ and you will have $\nabla \cdot \varepsilon\mathbf{E} = 0$ automatically. However, this will lead to an inequivalent condition $\int(\varepsilon - 1) > 0$ instead of $\int \Delta = \int \frac{\varepsilon-1}{\varepsilon} > 0$.