18.369 Problem Set 5 Solutions

Problem 1: (20 points)

Our derivation is similar to before, except that we have an additional term $-\varepsilon^{-2}\frac{\partial\varepsilon}{\partial\omega}\frac{d\omega}{dk}$ from the derivative $\frac{d}{dk}$ acting on the frequency argument of $\varepsilon^{-1}(\mathbf{x},\omega)$ inside $\hat{\Theta}_k$. This gives us:

$$\frac{d(\omega^2)}{dk} = 2\omega\frac{d\omega}{dk} = \frac{\langle H_k, \frac{\partial\hat{\Theta}_k}{\partial k}H_k\rangle - \frac{d\omega}{dk}\langle H_k, (\nabla + i\mathbf{k})\times\varepsilon^{-2}\frac{\partial\varepsilon}{\partial\omega}(\nabla+i\mathbf{k})\times H_k\rangle}{\langle H_k, H_k\rangle}.$$

If we solve for $\frac{d\omega}{dk}$, we obtain:

$$\frac{d\omega}{dk} = \frac{\langle H_k, \frac{\partial\hat{\Theta}_k}{\partial k}H_k\rangle}{2\omega\langle H_k, H_k\rangle + \langle H_k, (\nabla+i\mathbf{k})\times\varepsilon^{-2}\frac{\partial\varepsilon}{\partial\omega}(\nabla+i\mathbf{k})\times H_k\rangle},$$

where the only new term compared to class is

$$\begin{aligned}\langle H_k, (\nabla+i\mathbf{k})\times\varepsilon^{-2}\frac{\partial\varepsilon}{\partial\omega}(\nabla+i\mathbf{k})\times H_k\rangle &= \langle(\nabla+i\mathbf{k})\times H_k, \varepsilon^{-2}\frac{\partial\varepsilon}{\partial\omega}(\nabla+i\mathbf{k})\times H_k\rangle \\ &= \langle -i\omega\varepsilon E_k, \varepsilon^{-2}\frac{\partial\varepsilon}{\partial\omega}(-i\omega\varepsilon E_k)\rangle \\ &= \omega^2\int\frac{\partial\varepsilon}{\partial\omega}|E|^2,\end{aligned}$$

using Ampere's law and the fact that $E_k = Ee^{-ikx}$. From class, $2\langle H_k, H_k\rangle = 2\int|H|^2 = \int(|H|^2 + \varepsilon|E|^2)$ for time-harmonic fields. So, putting it all together, and using $\frac{\partial(\omega\varepsilon)}{\partial\omega} = \varepsilon + \omega\frac{\partial\varepsilon}{\partial\omega}$, we have

$$\frac{d\omega}{dk} = \frac{\langle H_k, \frac{\partial\hat{\Theta}_k}{\partial k}H_k\rangle}{\omega\int\left[\frac{\partial(\omega\varepsilon)}{\partial\omega}|\mathbf{E}|^2 + |\mathbf{H}|^2\right]} = \frac{\frac{1}{4\omega}\langle H_k, \frac{\partial\hat{\Theta}_k}{\partial k}H_k\rangle}{\frac{1}{4}\int\left[\frac{\partial(\omega\varepsilon)}{\partial\omega}|\mathbf{E}|^2 + |\mathbf{H}|^2\right]},$$

and the numerator was shown in class (and in the book) to be the time-average Poynting flux. Thus the energy density is modified to the Brillouin formula $\frac{1}{4}\left[\frac{\partial(\omega\varepsilon)}{\partial\omega}|\mathbf{E}|^2 + |\mathbf{H}|^2\right]$ as desired.

Problem 2: (10+10 points)

(a) See the solution notebook for the calculatoins and plots.

In the band diagram, there was a TM gap from a frequency of 0.28 to $0.42c/a$. In the transmission spectrum, we clearly see a dip in the transmission—decreasing *exponentially* with N_x as expected (since fields are evanescent in the gap)—around these frequencies. However, if we look closely, we see that the transmission dip is **wider** than the overall band gap, from about 0.24 to 0.42. What's happening is that our structure is periodic in y, so k_y is conserved, and a normal-incident planewave **only couples to** $k_y = 0$ **bands**, which are those along the Γ–X portion of the band diagram. If we just look at Γ–X, then the gap is indeed wider, from 0.24 to 0.42 c/a!

Also as expected from class, the transmission dip is largest near the middle of the gap, since we showed that the evanescent decay rates increase away from the band edges.

Another feature that you might notice is that, as we increase N_x, we see that a transmission dip is appearing round 0.71–0.74 c/a. If you look closely at the Γ–X band diagram you will see that there is indeed **another gap** (just for Γ–X, not for the whole Brillouin zone) at those frequencies. Since that gap is much smaller, the evanescent decay rate is much slower, and so the transmission dip is weaker for the same N_x.

(b) From the analysis in class (and chapter 10), k_y is conserved up to integer multiples of $2\pi/a$. Since the incident wave is $k_y = 0$, the first diffracted order corresponds to $k_y = \pm 2\pi/a$. However, the corresponding $k_x = \sqrt{(\omega/c)^2 - k_y^2}$ is imaginary (evanescent) until $\omega \geq 2\pi c/a$, or a frequency $f \geq c/a$. Since $c/a = 1$ in our Meep units, this means we expect to see an additional diffracted order actually propagating away in the reflected (and transmitted) wave for $\boxed{f \geq 1}$ in Meep units.

The resulting fields are shown for $f = 0.95c/a$ and $f = 1.05c/a$ in the solution notebook, and clearly exhibit the transition to additional diffracted orders for $f > 1\,c/a$.

The solution notebook also shows the results for $f = 0.995\,c/a$, much closer to the diffraction threshold. In this case, we needed to increase the "pad" parameter to increase the width of the air region. The reason for this is that, as $f \to 1^-$ (that is, approaching 1 from *below*), the first reflected diffracted order's evanescent decay rate gets slower and slower ($k_x \to i0^-$ from above), so we see its transverse oscillations in the field farther and farther from the crystal.