18.369 Problem Set 3 Solutions

Problem 0: (6+6+5+5+5 points)

(a) Solutions:

(i) From $\frac{\partial\psi}{\partial t} = \hat{C}\psi - \frac{\partial\phi}{\partial t} - \xi$, $\frac{\partial\psi_{\text{inc}}}{\partial t} = \hat{C}\psi_{\text{inc}} - \xi$, and $\psi = \psi_{\text{inc}} + \psi_{\text{scat}}$, it follows that

$$\frac{\partial\psi_{\text{scat}}}{\partial t} = \hat{C}\psi_{\text{scat}} - \frac{\partial\phi}{\partial t}$$

where $\phi = \chi * \psi = \chi * (\psi_{\text{inc}} + \psi_{\text{scat}})$.

Because of the dependence on ϕ, we can't solve the ψ_{scat} equation without first solving the ψ_{inc} equation. Dividing the solutions into $\psi = \psi_{\text{inc}} + \psi_{\text{scat}}$ doesn't make Maxwell's equations easier to solve, but it does give us a new perspective on the solutions (and leads to some numerical approaches such as integral-equation formulations).

(ii) The Fourier transform merely changes $\frac{\partial}{\partial t} \to -i\omega$, giving

$$\begin{aligned}
-i\omega\hat{\psi}_{\text{inc}} &= \hat{C}\hat{\psi}_{\text{inc}} - \hat{\xi} \\
-i\omega\hat{\psi}_{\text{scat}} &= \hat{C}\hat{\psi}_{\text{scat}} + i\omega\hat{\phi} = \hat{C}\hat{\psi}_{\text{scat}} + i\omega\hat{\chi}\hat{\psi}_{\text{scat}} + i\omega\hat{\chi}\hat{\psi}_{\text{inc}}
\end{aligned}$$

where the convolution $\phi = \chi * \psi$ becomes a multiplication $\hat{\phi} = \hat{\chi}\hat{\psi}$.

(b) Note: when we apply Poynting's theorem (from section 3 of the notes) to a time-harmonic field $\psi(\mathbf{x},t) = \hat{\psi}(\mathbf{x})e^{-i\omega t}$, then the $\frac{\partial}{\partial t}\left[\frac{1}{2}\langle\psi,\psi\rangle\right]$ term is zero. In general, the $e^{-i\omega t}$ terms cancel from all of the products thanks to the complex conjugations.

If we apply the remaining of Poynting's theorem *for the volume* Ω to the $\hat{\psi}_{\text{inc}}$ equation, there is no ϕ term in the equation and $\xi = 0$ inside Ω so the $\langle\psi_{\text{inc}},\xi\rangle$ integral vanishes as well! So the only remaining term is $-\oiint_{\partial\Omega}\text{Re}[\hat{\mathbf{E}}^*_{\text{inc}} \times \hat{\mathbf{H}}_{\text{inc}}] \cdot d\mathbf{A}$ on the right-hand side, with the left-hand-side equal to zero, so $\oiint_{\partial\Omega}\text{Re}[\hat{\mathbf{E}}^*_{\text{inc}} \times \hat{\mathbf{H}}_{\text{inc}}] \cdot d\mathbf{A} = 0$ as desired.

Applying Poynting's theorem to the $\hat{\psi}$ equation in Ω, the $\langle\psi,\xi\rangle$ term is still $= 0$ for an integral over Ω because $\xi = 0$ there, but we *do* have a ϕ term. As shown in section 3.2 of the notes, the ϕ term in the frequency domain simplifies to $\langle\hat{\psi}, \text{Im}[\omega\hat{\chi}]\hat{\psi}\rangle$, and as we explained in the notes one always has $\text{Im}[\omega\hat{\chi}] \geq 0$ (positive semidefinite) for a passive material. Hence $\langle\hat{\psi}, \text{Im}[\omega\hat{\chi}]\hat{\psi}\rangle \geq 0$, and it follows from Poynting's theorem that

$$\langle\hat{\psi}, \text{Im}[\omega\hat{\chi}]\hat{\psi}\rangle = -\oiint_{\partial\Omega}\text{Re}[\hat{\mathbf{E}}^* \times \hat{\mathbf{H}}] \cdot d\mathbf{A} = P_{\text{abs}} \geq 0$$

as desired.

(c) Consider a ball B of radius R (the interior of the sphere ∂B of radius R) centered on some point in Ω, for any R large enough so that B encloses Ω (i.e. $\Omega \subseteq B$), and apply Poynting's theorem to the volume $B\backslash\Omega$ that lies *between* $\partial\Omega$ and ∂B. Poynting's theorem immediately implies that

$$0 = -\oiint_{\partial(B\backslash\Omega)}\text{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A} = \oiint_{\partial\Omega}\text{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A} - \oiint_{\partial B}\text{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A}$$

$$\Longrightarrow P_{\text{scat}} = \oiint_{\partial\Omega}\text{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A} = \oiint_{\partial B}\text{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A},$$

where $d\mathbf{A}$ pointing *out* of Ω and B for $\partial\Omega$ and ∂B, respectively. More generally, by a similar argument one can show that P_{scat} is equal to the Poynting flux through *any* surface enclosing Ω.

Since outgoing boundary conditions require $R^2\,\mathrm{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}]$ to point radially outward for large R, it follows that the *integrand* $\mathrm{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot \mathbf{n}$ of the $\oiint_{\partial B}$ integral must be ≥ 0 for a sufficiently large R. If the integrand is nonnegative, then the integral $\oiint_{\partial B} \mathrm{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A}$ must also be ≥ 0. Since $P_{\text{scat}} = \oiint_{\partial B} \mathrm{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A}$ for *all* radii R that enclose Ω, it follows that $P_{\text{scat}} \geq 0$.

(d) We just plug in the definitions in terms of Poynting-flux integrals:

$$P_{\text{ext}} = \underbrace{-\oiint_{\partial\Omega} \mathrm{Re}[\hat{\mathbf{E}}^* \times \hat{\mathbf{H}}] \cdot d\mathbf{A}}_{P_{\text{abs}}} + \underbrace{\oiint_{\partial\Omega} \mathrm{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A}}_{P_{\text{scat}}}.$$

Then, we substitute the identities $\mathbf{E} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{scat}}$ and $\mathbf{H} = \mathbf{H}_{\text{inc}} + \mathbf{H}_{\text{scat}}$, and note that the minus sign in the P_{abs} term leads to several cancellations. Another term vanishes due to the $\oiint_{\partial\Omega} \mathrm{Re}[\hat{\mathbf{E}}^*_{\text{inc}} \times \hat{\mathbf{H}}_{\text{inc}}] \cdot d\mathbf{A} = 0$ property that we showed above. The only remaining terms are the scat–inc cross terms:

$$\boxed{P_{\text{ext}} = -\oiint_{\partial\Omega} \mathrm{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{inc}} + \hat{\mathbf{E}}^*_{\text{inc}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A}}.$$

(e) From the first part above (or section 3.2 of the notes), we have $P_{\text{abs}} = \mathrm{Re}\langle \hat{\psi}, -i\omega\hat{\phi} \rangle = \mathrm{Im}\langle \hat{\psi}, \omega\hat{\phi} \rangle$. (The absorbed power is the work done **on** the polarization currents $-i\omega\hat{\phi}$ **by** the total field $\hat{\psi}$.) In the same way, applying Poynting's theorem to ψ_{scat} yields

$$\mathrm{Re}\langle \hat{\psi}_{\text{scat}}, -i\omega\hat{\phi} \rangle = -\oiint_{\partial\Omega} \mathrm{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A} = -P_{\text{scat}}.$$

(The scattered power is the work done **by** the polarization currents $-i\omega\hat{\phi}$ **on** the scattered field $\hat{\psi}_{\text{scat}}$.) Combining these two, we obtain

$$P_{\text{ext}} = \underbrace{\mathrm{Re}\langle \hat{\psi}, -i\omega\hat{\phi} \rangle}_{P_{\text{abs}}} \underbrace{- \mathrm{Re}\langle \hat{\psi}_{\text{scat}}, -i\omega\hat{\phi} \rangle}_{P_{\text{abs}}} = \boxed{\mathrm{Re}\langle \hat{\psi}_{\text{inc}}, -i\omega\hat{\phi} \rangle}.$$

Comment: The optical theorem is often phrased in terms of a "forward-scattering amplitude." Recall that $\hat{\psi}_{\text{scat}}$ is produced by the polarization currents $\frac{\partial \phi}{\partial t} = -i\omega\hat{\phi}$. If $\hat{\psi}_{\text{inc}}$ is a planewave $\sim e^{i\mathbf{k}\cdot\mathbf{x}}$, then $\langle \hat{\psi}_{\text{inc}}, -i\omega\hat{\phi} \rangle$ is a Fourier transform: the $e^{i\mathbf{k}\cdot\mathbf{x}}$ *Fourier component* of the polarization current. From class, a current $\sim e^{i\mathbf{k}\cdot\mathbf{x}}$ in vacuum (like the $\hat{\psi}_{\text{scat}}$ equations given $\hat{\phi}$) generates fields $\sim e^{i\mathbf{k}\cdot\mathbf{x}}$, which are the "forward-scattered" fields. So $\langle \hat{\psi}_{\text{inc}}, -i\omega\hat{\phi} \rangle$ is proportional to the amplitude of the forward-scattered fields.

Comment: It is also worth commenting on the physical intuition behind the optical theorem. The basic idea is that any scattered or absorbed light leaves a shadow, so one can determine P_{ext} by looking at the darkness of the shadow. The shadow, a region on the "forward" side where $\hat{\psi}$ is small, corresponds to a forward-scattered $\hat{\psi}_{\text{scat}}$ that partially *cancels* $\hat{\psi}_{\text{inc}}$. Hence, P_{ext} is determined by the $\hat{\psi}_{\text{scat}}$ that is generated in the forward direction (the same direction as $\hat{\psi}_{\text{inc}}$), i.e. the forward-scattering amplitude.

Problem 1: (10+15 points)

In both parts of this problem, similar to class, we need to prove that the Rayleigh quotient satisfies $\langle H, \hat{\Theta}_{\mathbf{k}} H \rangle / \langle H, H \rangle < k^2$ for some trial function H, or equivalently that

$$\int_0^a \int_{-\infty}^{\infty} (1-\Delta)\, |(\nabla + i\mathbf{k}) \times \mathbf{H}_{\mathbf{k}}|^2 \, dx\, dy - k^2 \int_0^a \int_{-\infty}^{\infty} |\mathbf{H}_{\mathbf{k}}|^2 \, dx\, dy < 0$$

for the trial Bloch envelope $\mathbf{H}_{\mathbf{k}} = \mathbf{H} e^{-ikx}$, $\mathbf{k} = k\hat{\mathbf{x}}$, and $\varepsilon^{-1} = 1 - \Delta$.

(a) We will choose $u(x,y) = e^{-|y|/L}$ for some $L > 0$, exactly as in class—that is, it is the simplest conceivable periodic function of x, a constant. Thus, $\int |u|^2 = 2a\int_0^\infty e^{-2y/L}dy = aL$ over the unit cell. In this case, the variational criterion above becomes, exactly as in class except for the factor of a:

$$\begin{aligned}\int_0^a \int_{-\infty}^{\infty} (1-\Delta)\left(k^2+L^{-2}\right)e^{-2|y|/L}dxdy - k^2 aL &< 0 \\ &= \frac{a}{L} - \int_0^a \int_{-\infty}^{\infty} \Delta\cdot\left(k^2+L^{-2}\right)e^{-2|y|/L}dx\,dy,\end{aligned}$$

which becomes negative in the limit $L \to \infty$ thanks to our assumption that $\int_0^a \int_{-\infty}^{\infty} \Delta(x,y)\,dx\,dy > 0$. Note that the fact that $\int |\Delta| < \infty$ ensures that we can interchange the limits and integration, via the dominated convergence theorem discussed in class.

(b) Let us guess that we can choose $u(y)$ and $v(y)$ to be functions of y only (i.e., again the trivial constant-function periodicity in x). The fact that $\nabla\cdot\mathbf{H} = 0$ implies that $(\nabla + i\mathbf{k})\cdot[u(y)\hat{\mathbf{x}} + v(y)\hat{\mathbf{y}}] = 0 = iku + v'$, and therefore $u = iv'/k$. Therefore, it is convenient to choose $v(y)$ to be a smooth function so that u is differentiable. Let us choose

$$v(y) = e^{-y^2/2L^2}$$

in which case $u(y) = -\frac{iy}{kL^2}e^{-y^2/2L^2}$. Recall the Gaussian integrals $\int_{-\infty}^{\infty} e^{-y^2/L^2}dy = L\sqrt{\pi}$ and $\int_{-\infty}^{\infty} y^2 e^{-y^2/L^2}dy = L^3\sqrt{\pi}/2$. So, $\int |\mathbf{H}|^2 = a\int |u|^2 + |v|^2 = aL\sqrt{\pi}[1 + \frac{1}{k^2L^2}]$. Also, $(\nabla + i\mathbf{k})\times[u(y)\hat{\mathbf{x}} + v(y)\hat{\mathbf{y}}] = (ikv - u')\hat{\mathbf{z}}$. So,

$$|\nabla\times\mathbf{H}|^2 = |(\nabla + i\mathbf{k})\times\mathbf{H_k}|^2 = |u'|^2 + k^2|v|^2 = k^2\left[1 + \frac{1}{k^4L^4}\left(1 - \frac{y^2}{L^2}\right)\right]e^{-y^2/L^2}.$$

Then, if we look at our variational criterion, we have two terms: $\int |\nabla\times\mathbf{H}|^2$ and $-\int \Delta\cdot|\nabla\times\mathbf{H}|^2$. Again, we can swap limits with integration in the latter by the dominated convergence theorem. Combining the former with the $-k^2\int|\mathbf{H}|^2$ term in the variational criterion, we get:

$$\begin{aligned}\int |\nabla\times\mathbf{H}|^2 - k^2\int|\mathbf{H}|^2 &= a\int_{-\infty}^{\infty} k^2\left[1 + \frac{1}{k^4L^4}\left(1 - \frac{y^2}{L^2}\right)\right]e^{-y^2/L^2}dy - k^2 aL\sqrt{\pi}\left[1 + \frac{1}{k^2L^2}\right] \\ &= a\int_{-\infty}^{\infty} \frac{k^2}{k^4L^4}\left(1 - \frac{y^2}{L^2}\right)e^{-y^2/L^2}dy - \frac{k^2 aL\sqrt{\pi}}{k^2L^2} \\ &= \frac{a}{k^2L^4}L\sqrt{\pi}\left(1 - \frac{L^2}{2L^2}\right) - \frac{a\sqrt{\pi}}{L},\end{aligned}$$

which goes to zero as $L \to \infty$. Thus:

$$\int (1-\Delta)\,|(\nabla + i\mathbf{k})\times\mathbf{H_k}|^2 - k^2\int|\mathbf{H_k}|^2 \to -k^2\int_0^a \int_{-\infty}^{\infty} \Delta(x,y)\,dx\,dy < 0.$$

as $L \to \infty$. Q.E.D.

Problem 2: (5+10+10+20 points)

(a) From pset 1, the $\mathbf{E}$ eigenproblem is $\nabla\times\nabla\times\mathbf{E} = \omega^2\varepsilon\mathbf{E}$, and from class translational symmetry gives solutions $\mathbf{E} = e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{E_k}$ which satisfy a "reduced" eigenproblem in $\mathbf{E_k}(y)$ with ∇ replaced by $\nabla + i\mathbf{k}$. Also from class, the $(\nabla + i\mathbf{k})\times(\nabla + i\mathbf{k})\times\mathbf{E_k}$ operator simplifies for $E_z(y)$ to $\left(-\frac{\partial^2}{\partial y^2} + k^2\right)E_k$. Hence, our solutions in our $\varepsilon = 1$ waveguide satisfy

$$-E_k'' + k^2E_k = \omega^2E_k$$

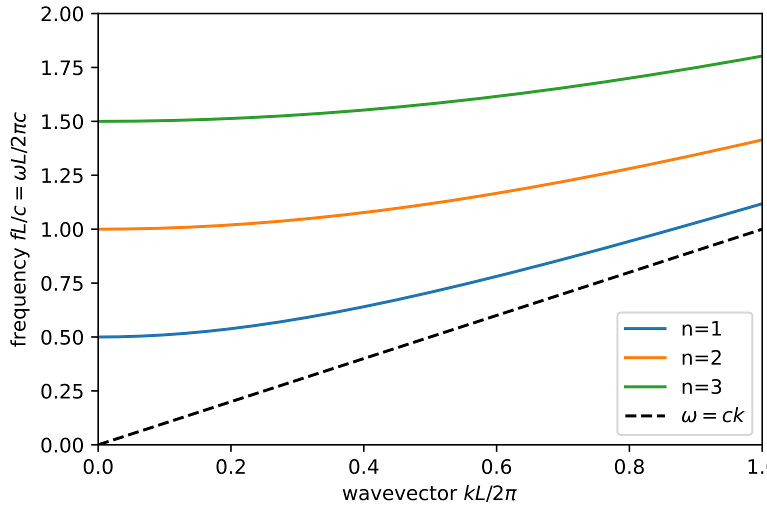

Figure 1: Problem 2(a): dispersion relations $\omega_n(k)$ of a width-L PEC waveguide. For convenience, we plot ω/c vs k in units of $2\pi/L$.

with boundary conditions $E_k(0) = E_k(L) = 0$ fromthe PEC. This is very similar to problems we've solved previously, and has solutions

$$\boxed{E_{k,n}(y) = \sin(n\pi y/L)}$$

for $n = 1, 2, \ldots$, with eigenfrequencies

$$\boxed{\omega_n(k) = c\sqrt{k^2 + (n\pi/L)^2}},$$

which is the equation of a hyperbola (I've re-inserted c, which we usually set to 1). This dispersion relation is shown in Fig. 1.

(b) We are expanding the field as

$$\boxed{E_z = e^{-i\omega t} \sum_{n=1}^{\infty} c_n \sin(n\pi y/L) e^{ik_n|x|}},$$

where k_n is obtained by solving $\omega_n(k_n) = \omega$:

$$\boxed{k_n = \sqrt{(\omega/c)^2 - (n\pi/L)^2}}.$$

Note that k_n is purely imaginary for $\omega < n\pi c/L = \omega_n(0)$, yielding *exponential decay* from $e^{ik_n|x|}$! For any given ω, *most* of the terms in our series are such decaying "evanescent" solutions — there are only a finite number of "propagating" modes with real k_n at a given ω.

Note: technically, we are assuming $\omega > 0$ in this expansion. Otherwise, we need to flip the sign of our real-k_n solutions in order for the waves to be propagating outwards. These are also called "outgoing" or "radiation" boundary conditions: we require any waves at $x \to \pm\infty$ to be propagating *outwards*.

From pset 1, Maxwell's equations including a time-harmonic current source $\mathbf{J}$ are $\nabla \times \nabla \times \mathbf{E} = \omega^2 \varepsilon \mathbf{E} + i\omega \mathbf{J}$, which for the E_z polarization in $\varepsilon = 1$ simplifies to

$$-\nabla^2 E_z = \omega^2 E_z + i\omega J_z$$

By construction, our series solution E_z solves Maxwell's equations for $x \neq 0$, where $J_z = 0$, since for $x > 0$ and $x < 0$ it consists of eigenfunctions (source-free solutions) with eigenvalue ω. So, all that remains is to handle $x = 0$. As explained in the hint, $\frac{\partial^2}{\partial x^2} e^{ik_n|x|}$ gives us two terms: the "ordinary" derivative $-k_n^2 e^{ik_n|x|}$ obtained from the $x \to 0$ limit of the second derivative (from either side), and a delta function $2ik_n\delta(x)$ from the discontinuity in the first derivative $\pm ik_n e^{ik_n|x|}$ as $x \to 0^{\pm}$. The k_n^2 term cancels the $\frac{\partial^2}{\partial y^2} + \omega^2$ terms, thanks to the eigen-equation, so all that remains is the delta function term, which helpfully matches the $\delta(x)$ in J_z. This gives the equation:

$$-2i\delta(x) \sum_{n=1}^{\infty} k_n c_n \sin(n\pi y/L) = i\omega\delta(x)\delta(y - L/2)$$

which means

$$\sum_{n=1}^{\infty} k_n c_n \sin(n\pi y/L) = -\frac{\omega}{2}\delta(y - L/2).$$

Now, we can use the general formula for the coefficients of a Fourier sine series: if $\sum b_n \sin(n\pi y/L) = f(y)$, then by orthogonality we have $b_n = \frac{2}{L}\int_0^L f(y)\sin(n\pi y/L)dy$. Plugging in the $\delta(y - L/2)$ for $f(y)$ and $k_n c_n$ for b_n, we obtain

$$\boxed{c_n = -\frac{\omega}{k_n L}\sin(n\pi/2) = \begin{cases} 0 & n \text{ even} \\ \frac{\omega}{k_n L}(-1)^{(n+1)/2} & n \text{ odd} \end{cases}}.$$

It shouldn't be surprising that $c_n = 0$ for odd n. Our current source has even symmetry with respect to the $y = L/2$ mid-plane, so by conservation of irrep it should only couple to even-y eigenfunctions, which correspond to *odd n*.

(c) Now, let us compute the time-averaged power $P = -\frac{1}{2}\operatorname{Re}\int \mathbf{J}^* \cdot \mathbf{E}dxdy$. Since $\mathbf{J}$ is a delta function, this integral is simply

$$\begin{aligned} P &= -\frac{1}{2}\operatorname{Re}[E_z(0, L/2)] \\ &= \boxed{\operatorname{Re}\left[\sum_{\text{odd } n} \frac{\omega}{2k_n L}\right] = \sum_{\text{odd } n \leq N(\omega)} \frac{\omega}{2k_n L}}. \end{aligned}$$

where $N(\omega)$ is the largest n for which k_n is **real**, i.e. for which $n \leq \omega L/\pi c$. In particular,

$$N(\omega) = \left\lfloor \frac{\omega L}{\pi c} \right\rfloor$$

where $\lfloor \cdots \rfloor$ denotes the "floor" function (the greatest integer $\leq$ the argument).

There are some interesting things to note about this solution:

(i) If you were careful with the signs, you should have gotten $\boxed{P \geq 0}$: the dipole source must be *expending* power!

(ii) For $\omega < \pi c/L = \omega_1(0)$ (the "cutoff" of the first mode), we have $\boxed{P = 0}$: the dipole expends *no* time-averaged time-averaged power at low frequencies because *there are no modes to radiate into*. Physically, such a source generates only exponentially decaying "evanescent" fields which are "trapped" near the source. If you look at the instantaneous (not time-averaged) power you would find that the dipole expends power for half of its cycle, but gets it back (received power) from the trapped fields on the other half of the cycle, so that on net it produces steady-state oscillating fields that neither decay, nor grow, nor escape to $x \to \pm\infty$.

(iii) As we approach a cutoff from above, i.e. as $\omega \to \omega_n(0)$ from above, we have $k_n \to 0$, so $P \to \infty$! This divergence in the power is also known as a **van Hove singularity** in solid-state physics, and it is a real effect—it is harder and harder to excite a mode as you approach cutoff. (Antenna engineers might say that the "radiation resistance" diverges.) Fortunately, this is an "integrable" singularity $\sim 1/\sqrt{\omega - \omega_n(0)}$, which has finite integral $\int P(\omega)d\omega$, so if you put in a pulse source you can show that the total power expended is finite.

(d) See attached Jupyter notebook file for the Meep simulations, which display precisely the $P(\omega)$ behaviors predicted in the previous part.