18.369 Problem Set 5

Due Tuesday, May 5.

Problem 1: Group Velocity and Material Dispersion

In class, we showed (following the book) that the group velocity $d(\omega^2)/dk = \langle H_k, \frac{\partial \hat{\Theta}_k}{\partial k} H_k \rangle / \langle H_k, H_k \rangle$ was equal to Poynting flux divided by energy density (both averaged over the unit cell).

Now, go through a similar Hellman–Feynman derivation, but in this case assume that we have a lossless *dispersive* material with a real $\varepsilon(\mathbf{x}, \omega)$. In this case, when you take the k derivative, apply the chain rule to obtain a term with $\frac{\partial \varepsilon}{\partial \omega} \frac{d\omega}{dk}$ on the right-hand side. Solve for $d\omega/dk$ and show that it is Poynting flux divided by energy density, but the energy density is now the "Brillouin" energy density of a lossless dispersive medium, which we gave in the notes for Lecture 6:

$$\frac{1}{4}\left[\frac{\partial(\omega\varepsilon)}{\partial\omega}|\mathbf{E}|^2 + |\mathbf{H}|^2\right]$$

(for $\mu = 1$, where we have an additional $1/2$ factor from the time average).

Problem 2: Brillouin zones and band diagrams

The accompanying Jupyter notebook gives an example calculation of the TM (E_z) band diagram and gap for a 2d hexagonal lattice of dielectric rods (radius $0.2a$, $\varepsilon = 12$) in air. Modify this calculation so that the angle between the primitive lattice vectors is **75**° instead of 60°.

(a) Compute the new Brillouin zone irreducible Brillouin zone, and work out the coordinates of the corners of the I.B.Z.

(b) Compute the TM band diagram for $\mathbf{k}$ points at the corners of the I.B.Z. and some points interpolated along the edges in between. See the comment in the notebook about coordinate systems for the $\mathbf{k}$ points!

(c) Plot the band diagram and the ε structure (over several periods) similar to the hexagonal-lattice example.

(d) Compute the gap size and compare it to the gap for the hexagonal lattice.

Problem 3: Transmission spectra

Based on the sample code in the Jupyter notebook, compute the TM transmission spectrum for planewaves incident upon N_x layers of the hexagonal lattice of rods from problem 2.

(a) Compute the transmission spectrum for frequencies from $0.2c/a$ to $0.8c/a$ as a function of N_x, for $N_x = 1, 2, 3, 5, 6$, and plot them (on a single plot). The transmission spectrum should be normalized by dividing by the transmission for $N_x = 0$ (no holes). Relate the features of this transmission spectrum to the band diagram in the sample code.

(b) Predict analytically at what frequency ω_0 you should start to see additional diffracted orders in the reflected wave (i.e. reflected waves at angles in *addition* to the normal 0° reflection). Now, modify the simulation to use a TM *continuous-wave* (CW) source and output E_z at the end and show that there is a qualitative change in the reflected field pattern if you put in a frequency *just* below ω_0 versus a frequency *just* above ω_0. If you look *just* below ω_0, then you will have to increase the "pad" parameter in order to see an undisturbed 0° reflection pattern far from the crystal—why?