

## 18.369 Problem Set 2

Due Wednesday, 28 February 2024.

### Problem 1: Cylindrical symmetry

Suppose that we have a *cylindrical* metallic waveguide—that is, a perfect metallic tube with radius  $R$ , which is uniform in the  $z$  direction. The interior of the tube is simply air ( $\varepsilon = 1$ ).

- (a) This structure has continuous rotational symmetry around the  $z$  axis, correspond to commuting with rotations  $\hat{\phi}$  by any angle  $\phi$  (which is sometimes called the  $C_\infty$  symmetry group<sup>1</sup>). If  $f(\phi)$  is an eigenfunction of  $\hat{\phi}$  for all  $\phi \in \mathbb{R}$ , we know from class that  $f(\phi)$  must<sup>2</sup> be an exponential function  $e^{\kappa\phi}$  (or a multiple thereof) for some  $\kappa$  since  $\hat{\phi}_1\hat{\phi}_2 = \widehat{\phi_1 + \phi_2}$ . What are the possible values of  $\kappa$ , and why?
- (b) For simplicity, consider the (Hermitian) *scalar* wave equation  $-\nabla^2\psi = \frac{\omega^2}{c^2}\psi$  with  $\psi|_{r=R} = 0$ . Show that, when we look for solutions  $\psi$  that are eigenfunctions of all  $\hat{\phi}$  from above, and have  $z$  dependence  $e^{ikz}$  (from the translational symmetry), then we obtain a Bessel equation (Google it if you've forgotten Mr. Bessel, or never learned). Write the solutions in terms of Bessel functions, assuming that you are given their zeros  $x_{m,n}$  (i.e.  $J_m(x_{m,n}) = 0$  for  $n = 1, 2, \dots$ , where  $J_m$  is the Bessel function of the first kind...if you Google for “Bessel function zeros” you can find them tabulated). Sketch the dispersion relation  $\omega(k)$  for a few bands.
- (c) From the general orthogonality property of Hermitian-operator eigenfunctions, derive/prove an orthogonality integral for the Bessel functions. (No, just looking one up on Wikipedia doesn't count.)

<sup>1</sup>It also has an infinite set of mirror planes containing the  $z$  axis, but let's ignore these for now. If they are included, the group is called  $C_{\infty v}$ .

<sup>2</sup>Assuming it is anywhere continuous, to exclude crazy non-measurable non-constructive counterexamples.

### Problem 2: 2d Waveguide Modes

Consider the two-dimensional dielectric waveguide of thickness  $h$  that we first introduced in class:

$$\varepsilon(y) = \begin{cases} \varepsilon_{hi} & |y| < h/2 \\ \varepsilon_{lo} & |y| \geq h/2 \end{cases},$$

where  $\varepsilon_{hi} > \varepsilon_{lo}$ . Look for solutions with the “TM” or “ $E_z$ ” polarization  $\mathbf{E} = E_z(x, y)\hat{\mathbf{z}}e^{-i\omega t}$ . The boundary conditions are that  $E_z$  is continuous and  $\partial E_z/\partial y$  ( $\sim H_x$ ) is continuous, and that we require the fields to be finite at  $x, y \rightarrow \pm\infty$ ,

- (a) Prove that we can set  $\varepsilon_{lo} = 1$  without loss of generality, by a change of variables in Maxwell's equations. In the subsequent sections, therefore, set  $\varepsilon_{lo} = 1$  for simplicity.
- (b) Find the guided-mode solutions  $E_z(x, y) = e^{ikx}E_k(y)$ , where the corresponding eigenvalue  $\omega(k) < ck$  is below the light line.
  - (i) Show for the  $|y| < h/2$  region the solutions are of sine or cosine form, and that for  $|y| > h/2$  they are decaying exponentials. (At this point, you can't easily prove that the arguments of the sines/cosines are real, but that's okay—you will be able to rule out the possibility of imaginary arguments below.)
  - (ii) Match boundary conditions ( $E_z$  and  $H_x$  are continuous) at  $y = \pm h/2$  to obtain an equation relating  $\omega$  and  $k$ . You should get a transcendental equation that you cannot solve explicitly. However, you can “solve” it graphically and learn a lot about the solutions—in particular, you might try plotting the left and right hand sides of your equation (suitably arranged) as a function of  $k_\perp = \sqrt{\frac{\omega^2}{c^2}\varepsilon_{hi} - k^2}$ , so that you have two curves and the solutions are the intersections (your curves will be parameterized by  $k$ , but try plotting them for one or two typical  $k$ ).
  - (iii) From the graphical picture, derive an exact expression for the number of guided modes

as a function of  $k$ . Show that there is exactly one guided mode, with even symmetry, as  $k \rightarrow 0$ , as we argued in class.

### Problem 3: Evanescent modes in waveguides

In class, we looked at  $H_z$ -polarized solutions in a 2d metallic waveguide, formed by an  $\varepsilon = \mu = 1$  region between two PEC walls at  $x = 0$  and  $x = L$ , and found that (in the absence of sources) it satisfied the eigen-equation  $-\nabla^2 H_z = \omega^2 H_z$  (for  $\varepsilon_0 = \mu_0 = 1$  units) with “Neumann” boundary conditions  $H'_z(0) = H'_z(L) = 0$ . Exploiting translational symmetry in  $y$ , we looked for solutions  $H_z(x, y) = u_k(x)e^{iky}$  with a given propagation constant (“wavevector”)  $k \in \mathbb{R}$ , and found

$$u_{k,n} = \cos\left(\frac{n\pi x}{L}\right) \quad \text{for } n = 0, 1, \dots$$

$$\omega_n(k) = \pm \sqrt{\left(\frac{n\pi}{L}\right)^2 + k^2},$$

so that the dispersion relation is a set of hyperbolas (except for the  $n = 0$  solution  $\omega_0 = \pm|k|$ ).

- (a) We can alternatively fix  $\omega$  and solve for the propagation constant  $k_n(\omega)$  and the corresponding  $H_z$  solutions. Let  $\omega = 1.5\pi/L$ . What do the possible solutions  $H_z$  look like for  $n = 0, 1, 2, 3, \dots$ ? The solutions that are exponentially growing/decaying are called “evanescent” modes.
- (b) Suppose we introduce an electric current source  $\mathbf{J} = f(x)\delta(y)e^{-i\omega t}\hat{x}$  into Maxwell’s equations: here, a delta function in  $y$  and some function  $f(x)$  as the  $x$  dependence, with a frequency  $\omega$ , oriented in the  $x$  direction (a “current sheet”). Show that Maxwell’s equations in this case take the form  $(-\nabla^2 - \omega^2)H_z = Cf(x)\delta'(y)$  for some constant  $C$ , where  $\delta'(y)$  is the derivative of a delta function (recall that  $\int \delta'(y)g(y) = -g'(0)$ , or equivalently  $\delta'(y)$  is the second derivative of a unit step function). Any solution  $H_z$  must have some kind of discontinuity or “kink” at  $y = 0$  — what is it?

- (c) In the presence of this current source  $f(x)$ , write the resulting magnetic field  $H_z(x, y)$  everywhere in  $x \in [0, L]$  as an infinite series  $\sum_n$  of solutions from your  $k_n(\omega)$  modes above, with a different series for  $y > 0$  and  $y < 0$ . (Hint: you should find that by imposing the correct “kink” condition on  $H_z$  across  $y = 0$  from above, you get an equation in terms of a Fourier series for  $f(x)$  that allows you determine the coefficients of the series.) You should assume boundary conditions of *outgoing and/or decaying waves* as  $y \rightarrow \pm\infty$ : that is, for  $y > 0$ , the solutions should be a superposition of terms that look like  $e^{iky}$  for either real  $k > 0$  (propagating upwards) or imaginary  $k = i\kappa$  with  $\kappa > 0$  (decaying upwards), and similarly (with opposite signs: propagating/decaying downwards) for  $y < 0$ . You *must* include the evanescent modes, as otherwise you will be unable to find a solution for an arbitrary<sup>3</sup>  $f(x)$ .

### Problem 4: Poynting’s theorem

In Jackson’s *Classical Electrodynamics* textbook (or in many similar books), the electric field of a radiating dipole  $\mathbf{J}(\mathbf{x}, t) = -i\omega\mathbf{p}\delta(x)\delta(y)\delta(z)e^{-i\omega t}$  at the origin in vacuum is given in spherical coordinates (radius  $r$ , radial direction  $\mathbf{n}$ ), with  $k = \omega/c$ , as

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \left\{ k^2(\mathbf{n} \times \mathbf{p}) \times \mathbf{n} \frac{1}{r} + [3\mathbf{n}(\mathbf{n} \cdot \mathbf{p}) - \mathbf{p}] \left( \frac{1}{r^3} - \frac{ik}{r^2} \right) \right\} e^{i(kr - \omega t)}.$$

Note the divergence as  $r \rightarrow 0$ . The total power radiated by this dipole is computed by Jackson from the Poynting flux through an enclosing sphere to be  $P = \frac{c^2 Z_0 k^4}{12\pi} |\mathbf{p}|^2$ , where  $Z_0 = \sqrt{\mu_0/\epsilon_0}$ . Show that you obtain the *same*  $P$  by taking the  $r \rightarrow 0$  limit of of the work  $-\frac{1}{2} \text{Re}[\mathbf{E}^* \mathbf{J}]$  expended by  $\mathbf{J}$  (regardless of the direction  $\mathbf{n}$  from which you approach the origin). (Taylor expanding  $e^{ikr}$  may be helpful.)

<sup>3</sup>Technically, we must restrict ourselves to functions  $f(x)$  that have a Fourier series; this includes all remotely realistic possibilities.

## Problem 5: Scattering and energy

As in the course notes, we can write Maxwell's equations for the 6-component fields  $\psi$  as  $\frac{\partial \psi}{\partial t} = \hat{C}\psi - \frac{\partial \phi}{\partial t} - \xi$ , where the 6-component polarizations  $\phi = \chi * \psi$  are a convolution in time with a (passive and causal) linear susceptibility  $\chi(\mathbf{x}, t)$  at each point  $\mathbf{x}$  in space. Now, suppose we have the situation depicted in Fig. 1: sources ( $\xi \neq 0$ ) lying *outside* some bounded volume  $\Omega$  produce an incident wave  $\psi_{\text{inc}}$  that scatters off materials ( $\chi \neq 0$ ) *inside* of  $\Omega$  to produce a scattered wave  $\psi_{\text{scat}}$ . Outside of  $\Omega$  there are no materials ( $\chi = 0$ ), while inside of  $\Omega$  there are no sources ( $\xi = 0$ ).

- (a) Write the total fields as  $\psi = \psi_{\text{inc}} + \psi_{\text{scat}}$ , where  $\psi_{\text{inc}} = \begin{pmatrix} \mathbf{E}_{\text{inc}} \\ \mathbf{H}_{\text{inc}} \end{pmatrix}$  satisfies  $\frac{\partial \psi_{\text{inc}}}{\partial t} = \hat{C}\psi_{\text{inc}} - \xi$  (Maxwell's equations with the same currents but no materials), and the initial conditions at  $t = -\infty$  are  $\psi = \psi_{\text{inc}} = \psi_{\text{scat}} = 0$ .

- (i) What differential<sup>4</sup> equation must  $\psi_{\text{scat}}$  sat-

<sup>4</sup>Technically, if you include a convolution  $\chi *$  then you have

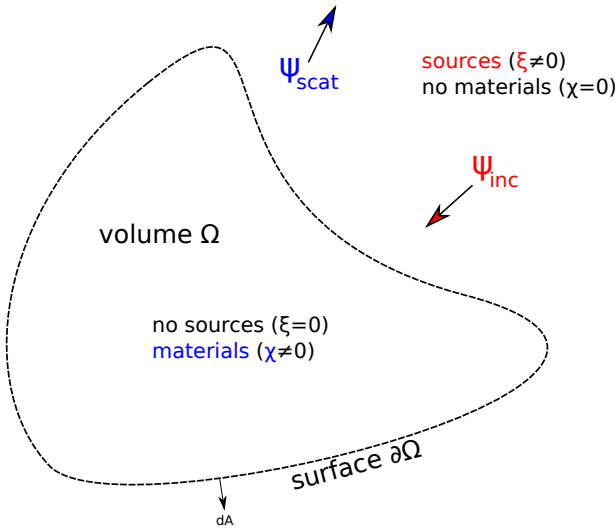


Figure 1: Schematic of an incident wave  $\psi_{\text{inc}}$  created by sources  $\xi$  that are only  $\neq 0$  *outside* some volume  $\Omega$ , interacting with materials  $\chi \neq 0$  that lie *inside*  $\Omega$  to produce an outgoing scattered wave  $\psi_{\text{scat}}$ .

isfy?

- (ii) Fourier transform in time to find equations for  $\hat{\psi}_{\text{inc}}(x, \omega)$  and  $\hat{\psi}_{\text{scat}}$  (in terms of  $\hat{\xi}$  and  $\hat{\chi}$ ); equivalently, look for time-harmonic solutions  $\hat{\psi}_{\text{inc}}e^{-i\omega t}$  and  $\hat{\psi}_{\text{scat}}e^{-i\omega t}$  from a time-harmonic source  $\hat{\xi}(\mathbf{x})e^{-i\omega t}$ .
- (b) Applying Poynting's theorem and passivity to your time-harmonic fields, show that  $\oint_{\partial\Omega} \text{Re}[\hat{\mathbf{E}}_{\text{inc}}^* \times \hat{\mathbf{H}}_{\text{inc}}] \cdot d\mathbf{A} = 0$  and  $P_{\text{abs}} = -\oint_{\partial\Omega} \text{Re}[\hat{\mathbf{E}}^* \times \hat{\mathbf{H}}] \cdot d\mathbf{A} \geq 0$  (the absorbed power = inward total flux). (Note that here,  $\Omega$  is a subset of the whole domain!)
- (c) In order to ensure  $P_{\text{scat}} = \oint_{\partial\Omega} \text{Re}[\hat{\mathbf{E}}_{\text{scat}}^* \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A} \geq 0$  (the scattered power = outward scattered flux), we need something more: an *outgoing boundary condition* (or “radiation” boundary condition): if we consider *any point* on sphere of radius  $R$  (around any point in  $\Omega$ ), then *in the limit* as  $R \rightarrow \infty$  we require that  $R^2 \text{Re}[\hat{\mathbf{E}}_{\text{scat}}^* \times \hat{\mathbf{H}}_{\text{scat}}]$  must point *radially outward*. Use Poynting's theorem to show that such a radiation boundary condition implies  $\oint_{\partial\Omega} \text{Re}[\hat{\mathbf{E}}_{\text{scat}}^* \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A} \geq 0$ .<sup>5</sup>
- (d) Show that the “extinguished power”  $P_{\text{ext}} = P_{\text{abs}} + P_{\text{scat}}$  can be written in terms of an integral over  $\partial\Omega$  involving  $\hat{\mathbf{E}}_{\text{inc}}^* \times \hat{\mathbf{H}}_{\text{scat}}$  and  $\hat{\mathbf{E}}_{\text{scat}}^* \times \hat{\mathbf{H}}_{\text{inc}}$  *only*. This is a version of a famous result known as the **optical theorem**.
- (e) Derive an alternative form of the optical theorem: show that  $P_{\text{ext}}$  can be written as an integral over  $\Omega$  involving *only*  $\hat{\psi}_{\text{inc}}^* \hat{\phi}$ . (Hint: use Poynting's theorem to relate  $P_{\text{abs}}$  and  $P_{\text{scat}}$  to work on polarization currents, and write  $\psi_{\text{scat}} = \psi - \psi_{\text{inc}}$ .)

a “pseudodifferential” operator.

<sup>5</sup>One will also choose some such boundary condition when constructing  $\psi_{\text{inc}}$ . Not necessarily an outgoing boundary condition, however—often it is convenient to have an incident wave coming in “from infinity”, such as an incident planewave, perhaps with no explicit sources ( $\xi = 0$ ) and just appropriate boundary conditions.