18.369 Problem Set 3

Due Wednesday, March 6.

Problem 1: Bloch-periodic eigenproblems

Suppose that we have a periodic system with period a in the x direction, and we look for Bloch-periodic eigenfunctions $\mathbf{H}(x+a,y,z) = e^{ika}\mathbf{H}(x,y,z)$ of the $\hat{\Theta} = \nabla \times \varepsilon^{-1} \nabla \times$ operator with these boundary conditions in x, acting on a unit cell $x \in [0,a]$ (with some other boundary conditions in y and z). (That is, we *don't* rewrite in terms of the periodic Bloch envelope and use $\hat{\Theta}_k$.)

(a) Explain why $\hat{\Theta}$ is still Hermitian with these boundary conditions: when we integrated by parts, we had some boundary terms that we needed to vanish, and explain why the boundary terms from $x=0$ and $x=a$ still vanish with Bloch-periodic boundary conditions. (You can assume that the y and z boundary conditions were chosen so that those boundaries vanished.)

(b) Why do k and $k + \frac{2\pi}{a}$ give the same solutions to this Bloch-periodic eigenproblem? (Yes, we already discussed the periodicity of k from other perspectives in class, but you should be able to see it directly here without reference to any of our previous arguments.)

Problem 2: Periodic waveguide guidance proof

In class, we showed by a variational proof that any $\varepsilon(y)$, in two dimensions, gives rise to at least one guided mode whenever $\varepsilon(y)^{-1} = \varepsilon_{\text{lo}}^{-1} - \Delta(y)$ for $\int \Delta > 0$ and $\int |\Delta| < \infty$.[1] At least, we showed it for the H_z polarization ($\mathbf{H}$ in the $\hat{\mathbf{z}}$ direction). Now, you will show the same thing much more generally, but using the same basic technique.

(a) Let $\varepsilon(x,y)^{-1} = 1 - \Delta(x,y)$ be a periodic function $\Delta(x,y) = \Delta(x+a,y)$, with $\int|\Delta| < \infty$ and $\int_0^a \int_{-\infty}^{\infty} \Delta(x,y)dxdy > 0$. Prove that at least one H_z-polarized guided mode exists, by choosing an appropriate (simple!) trial function of the form $\mathbf{H}(x,y) = u(x,y)e^{ikx}\hat{\mathbf{z}}$. That is, show by the variational theorem that $\omega^2 < c^2k^2$ for the lowest-frequency eigenmode. (It is sufficient to show it for $|k| \le \pi/a$, by periodicity in k-space; for $|k| > \pi/a$, the light line is not $\omega = c|k|$.)

(b) Prove the same thing as in (a), but for the E_z polarization ($\mathbf{E}$ in the $\hat{\mathbf{z}}$ direction). Hint: you will need to pick a trial function of the form $\mathbf{H}(x,y) = [u(x,y)\hat{\mathbf{x}} + v(x,y)\hat{\mathbf{y}}]e^{ikx}$ where u and v are some (simple!) functions such that $\nabla \cdot \mathbf{H} = 0$.[2]

[1] As in class, the latter condition on Δ will allow you to swap limits and integrals for any integrand whose magnitude is bounded above by some constant times $|\Delta|$ (by Lebesgue's dominated convergence theorem).

[2] You might be tempted, for the TM polarization, to use the $\mathbf{E}$ form of the variational theorem that you derived in problem 1, since the proof in that case will be somewhat simpler: you can just choose $\mathbf{E}(x,y) = u(x,y)e^{ikx}\hat{\mathbf{z}}$ and you will have $\nabla \cdot \varepsilon\mathbf{E} = 0$ automatically. However, this will lead to an inequivalent condition $\int(\varepsilon - 1) > 0$ instead of $\int \Delta = \int \frac{\varepsilon-1}{\varepsilon} > 0$.