

## 18.369 Problem Set 2 Solutions

### Problem 1: Projection operators (5+5+5+5=20 points)

- (a) Consider the action of  $\hat{P}_i^{(\alpha)}$  on a partner function  $\phi_j^{(\alpha')}$ , as defined in the handout. We obtain  $\hat{P}_i^{(\alpha)} \phi_j^{(\alpha')} = \frac{d_\alpha}{|G|} \sum_{g \in G} D_{ii}^{(\alpha)}(g)^* \hat{g} \phi_j^{(\alpha')} = \frac{d_\alpha}{|G|} \sum_{g \in G} D_{ii}^{(\alpha)}(g)^* [\sum_{i'} \phi_{i'}^{(\alpha')} D_{i'j}^{(\alpha)}(g)]$ . If we perform the  $\sum_g$  first, then from the Great Orthogonality Theorem we get  $\sum_{i'} \delta_{ii'} \delta_{ij} \delta_{\alpha\alpha'} \phi_{i'}^{(\alpha')} = \delta_{ij} \delta_{\alpha\alpha'} \phi_i^{(\alpha')}$ . Thus, for a  $\psi = \sum_\alpha \sum_i c_i^{(\alpha)} \phi_i^{(\alpha)}$  decomposed into partner functions (as proved in the class notes),  $\hat{P}_i^{(\alpha)} \psi = c_i^{(\alpha)} \phi_i^{(\alpha)}$ . Q.E.D.
- (b)  $\sum_\alpha \hat{P}^{(\alpha)} = \sum_{g \in G} \left[ \frac{1}{|G|} \sum_\alpha \chi^{(\alpha)*}(g) d_\alpha \right] \hat{g}$ , interchanging the  $\sum_\alpha$  and the  $\sum_g$ . But  $d_\alpha$ , the dimension of the representation, is exactly the character  $\chi^{(\alpha)}(E)$ , the character of the identity element  $E$ , which is one column of the character table ( $E$  is in a conjugacy class by itself). So, the  $[\dots]$  inner sum is  $\frac{1}{|G|} \sum_\alpha \chi^{(\alpha)*}(g) \chi^{(\alpha)}(E) = \delta_{gE}$  by the orthogonality of columns: it is zero unless  $g = E$  and one otherwise. Hence, the outer sum  $\sum_g$  disappears except for the  $E$  term, and  $\sum_\alpha \hat{P}^{(\alpha)} = \hat{O}_E = 1$ . Q.E.D.
- Note that you can also prove  $\sum_\alpha \hat{P}^{(\alpha)} = 1$  by combining the result from (a) with the fact that any function is a sum of irrep partners, which immediately gives  $\sum_\alpha \hat{P}^{(\alpha)} \psi = \psi$  for any  $\psi$ . But I wanted you to see how the same fact arises from the character-table orthogonality.
- (c)  $\hat{P}_i^{(\alpha)\dagger} = \frac{d_\alpha}{|G|} \sum_{g \in G} D_{ii}^{(\alpha)}(g) \hat{g}^\dagger = \frac{d_\alpha}{|G|} \sum_{g \in G} D_{ii}^{(\alpha)}(g) \widehat{g^{-1}} = \frac{d_\alpha}{|G|} \sum_{g \in G} D_{ii}^{(\alpha)}(g^{-1}) \hat{g}$ , using unitarity  $\hat{g}^\dagger = \hat{g}^{-1} = \widehat{g^{-1}}$  and changing variables  $g \rightarrow g^{-1}$  in the summation. Note that the  $D_{ii}^*$  term in  $\hat{P}_i$  is unconjugated in  $\hat{P}_i^\dagger$ . By definition of a representation, we must have  $D^{(\alpha)}(g^{-1}) = D^{(\alpha)}(g)^{-1}$ , and if  $D$  is unitary then we must have  $D^{(\alpha)}(g)^{-1} = D^{(\alpha)}(g)^\dagger$  (the conjugate-transpose). Hence  $D_{ii}^{(\alpha)}(g^{-1}) = D_{ii}^{(\alpha)}(g)^*$ , and  $\hat{P}_i^{(\alpha)\dagger} = \hat{P}_i^{(\alpha)}$ . Q.E.D.
- (d) Inserting  $\sum_{k,\gamma} \hat{P}_k^{(\gamma)} = 1$ , we obtain  $\langle \phi_i^{(\alpha)}, \psi_j^{(\beta)} \rangle = \langle \phi_i^{(\alpha)}, \sum_{k,\gamma} \hat{P}_k^{(\gamma)} \psi_j^{(\beta)} \rangle = \langle \phi_i^{(\alpha)}, \hat{P}_j^{(\beta)} \psi_j^{(\beta)} \rangle$  by definition of a projection operator (only the  $j, \beta$  projection operator can give a nonzero result for  $\psi_j^{(\beta)}$ ). Alternatively,  $\hat{P}_j^{(\beta)} \psi_j^{(\beta)} = \psi_j^{(\beta)}$  follows directly from the defining property of a projection operator, in (a). Because  $\hat{P}_j^{(\beta)}$  is Hermitian (we always assume unitary irreps., which are guaranteed to be possible for finite groups), we then have  $\langle \phi_i^{(\alpha)}, \psi_j^{(\beta)} \rangle = \langle \phi_i^{(\alpha)}, \hat{P}_j^{(\beta)} \psi_j^{(\beta)} \rangle = \langle \hat{P}_j^{(\beta)} \phi_i^{(\alpha)}, \psi_j^{(\beta)} \rangle$ , which is zero unless  $i = j$  and  $\alpha = \beta$ . Q.E.D.

### Problem 2: Symmetries of a field in a square metal box (5+15 points)

- (a) For  $\mathbf{E} = E_z(x, y) \hat{\mathbf{z}}$  in air ( $\epsilon = 1$ ), we get  $\nabla \times \nabla \times \mathbf{E} = -\nabla^2 E_z \hat{\mathbf{z}} = \frac{\omega^2}{c^2} E_z \hat{\mathbf{z}}$ . The solutions of this are sines and cosines, and to satisfy the  $E_z = 0$  boundary conditions at  $x = 0, L$  and  $y = 0, L$  we must have:

$$E_z = A \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}y\right)$$

where  $n$  and  $m$  are positive integers and  $A$  is an amplitude. The corresponding frequency is  $\omega = c \frac{\pi}{L} \sqrt{n^2 + m^2}$ . This solutions are plotted in Figure 1 for the first few values of  $n$  and  $m$ .

- (b) Since  $\mathbf{E}$  is a vector, we can treat  $E_z$  as an ordinary scalar field here (no funny  $-1$  factors under reflections). Therefore, from the field plots we can see that  $n = m = 1$  (and, in fact, any  $n = m = \text{odd}$ ) transforms as  $\Gamma_1$  (the trivial representation) and  $n = m = 2$  (and any  $n = m = \text{even}$ ) transforms as  $\Gamma_4$  (even with diagonal mirror planes and odd with horizontal/vertical mirrors). Any  $n \neq m$  state is (at least) doubly degenerate [with the  $(m, n)$  state], such as the  $(n, m) = (2, 1)$  and  $(1, 2)$  states shown in Fig. 1. There are three cases [which can be understood in a variety of ways, e.g. by acting projection operators on the  $(n, m)$  solution]:

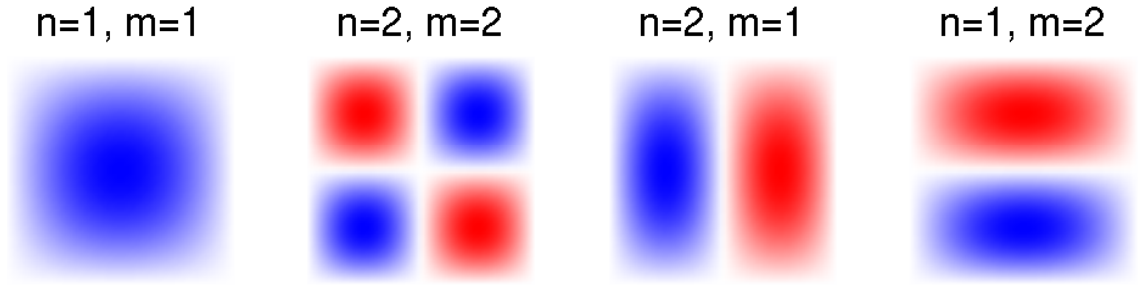


Figure 1: Electric fields ( $E_z$ ) of eigenmodes inside a 2d metal box. Blue/white/red indicate positive/zero/negative fields.

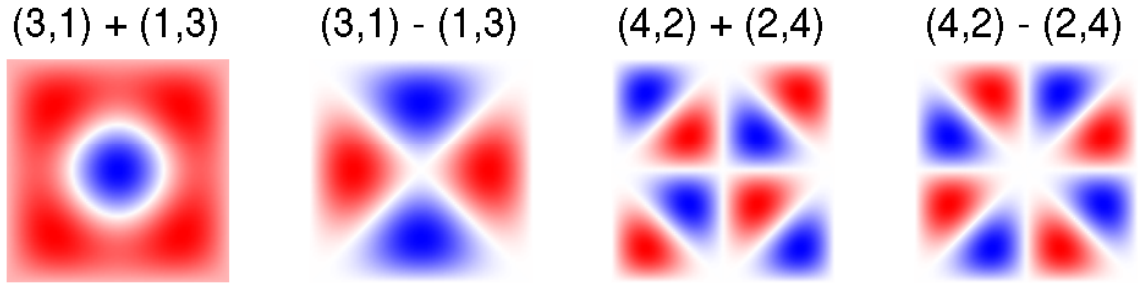


Figure 2: Electric fields ( $E_z$ ) of eigenmodes inside a 2d metal box for accidentally degenerate  $(n, m)$  states  $n \neq m$ . By forming linear combinations as shown, we can create eigenmodes which transform as, from left to right:  $\Gamma_1$ ,  $\Gamma_3$ ,  $\Gamma_4$ , and  $\Gamma_2$ . Blue/white/red indicate positive/zero/negative fields.

- (i) If  $n$  is even and  $m$  is odd or vice versa, e.g.  $(n, m) = (2, 1)$ , then it is a partner of  $\Gamma_5$ .
- (ii) If both  $n$  and  $m$  are odd, e.g.  $(n, m) = (3, 1)$ , then it forms a  $2 \times 2$  *reducible* representation with  $(m, n)$ . To decompose it into irreducible representations, we just take the sum and difference of the two eigenmodes. The  $(n, m) + (m, n)$  state is a partner of  $\Gamma_1$ , and  $(n, m) - (m, n)$  is a partner of  $\Gamma_3$ . This is shown in the left two panels of Fig. 2.
- (iii) If both  $n$  and  $m$  are even, e.g.  $(n, m) = (4, 2)$ , then it again forms a  $2 \times 2$  reducible representation with  $(m, n)$ . The  $(n, m) + (m, n)$  state is a partner of  $\Gamma_4$ , and  $(n, m) - (m, n)$  is a partner of  $\Gamma_2$ . This is shown in the right two panels of Fig. 2.

### Problem 3: Symmetries of a field in a triangular metal box (5+5+5+5+5pts)

Now we consider the solutions in a *triangular* box with side  $L$ .

- (a) The different symmetry operations in the space group of a triangle are shown in Figure 3 : three mirror planes  $\sigma$ , and counter-clockwise rotation  $C_3$  by  $120^\circ$  (and also  $C_3^{-1}$ , clockwise rotation), and of course the identity  $E$ . There are three conjugacy classes:  $\{E\}$ ,  $\{\sigma_1, \sigma_2, \sigma_3\}$ , and  $\{C_3, C_3^{-1}\}$ . This is because  $\sigma_3 = C_3^{-1} \sigma_1 C_3$ ,  $\sigma_2 = C_3 \sigma_1 C_3^{-1}$ , and  $C_3^{-1} = \sigma_1 C_3 \sigma_1$ . The multiplication table of the group is:

| $\circ$    | $E$        | $C_3$      | $C_3^{-1}$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ |
|------------|------------|------------|------------|------------|------------|------------|
| $E$        | $E$        | $C_3$      | $C_3^{-1}$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ |
| $C_3$      | $C_3$      | $C_3^{-1}$ | $E$        | $\sigma_3$ | $\sigma_1$ | $\sigma_2$ |
| $C_3^{-1}$ | $C_3^{-1}$ | $E$        | $C_3$      | $\sigma_2$ | $\sigma_3$ | $\sigma_1$ |
| $\sigma_1$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $E$        | $C_3$      | $C_3^{-1}$ |
| $\sigma_2$ | $\sigma_2$ | $\sigma_3$ | $\sigma_1$ | $C_3^{-1}$ | $E$        | $C_3$      |
| $\sigma_3$ | $\sigma_3$ | $\sigma_1$ | $\sigma_2$ | $C_3$      | $C_3^{-1}$ | $E$        |

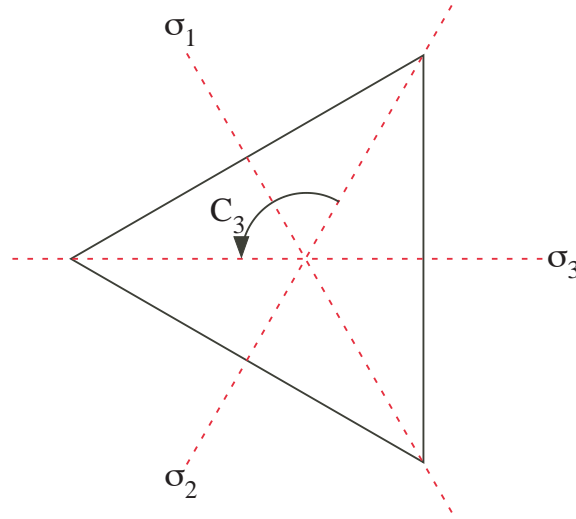


Figure 3: Symmetries of  $C_{3v}$  (triangle symmetries): three mirror planes  $\sigma$ , and rotation  $C_3$  by  $120^\circ$ .

- (b) The character table of  $C_{3v}$  must have only three representations since there are three classes, and the sum of the squares of the dimensions must equal 6 (the number of elements in the group). From this, the only possibility is for the representations to have dimensions 1, 1, and 2 (this gives the first column of the table). The first row must be the trivial representation, and by applying the orthogonality relations we get the other two rows:

|            | $E$ | $2C_3$ | $3\sigma$ |
|------------|-----|--------|-----------|
| $\Gamma_1$ | 1   | 1      | 1         |
| $\Gamma_2$ | 1   | 1      | -1        |
| $\Gamma_3$ | 2   | -1     | 0         |

where  $\Gamma_{1...3}$  are traditional names for these three representations.

- (c) For  $\Gamma_1$  and  $\Gamma_2$ , the representations are one-dimensional and are therefore simply numbers equal to the characters in the character table ( $\pm 1$ , from above). For  $\Gamma_3$ , we must first construct partner functions. Let's guess  $f(\mathbf{x}) = x$ . If we then operate the different group elements on this, recalling the coordinates rotated counter-clockwise by an angle  $\theta$  are multiplied by the  $2 \times 2$  matrix  $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ , we get:

| $E$ | $C_3$              | $C_3^{-1}$         | $\sigma_1$         | $\sigma_2$         | $\sigma_3$ |
|-----|--------------------|--------------------|--------------------|--------------------|------------|
| $x$ | $(-x+y\sqrt{3})/2$ | $(-x-y\sqrt{3})/2$ | $(-x-y\sqrt{3})/2$ | $(-x+y\sqrt{3})/2$ | $x$        |

and therefore if we operate the  $\Gamma_3$  projection operator  $\hat{P}^{(3)} = \frac{2}{6}(2\hat{\sigma}_E - \hat{\sigma}_{C_3} - \hat{\sigma}_{C_3^{-1}})$  on  $f(\mathbf{x}) = x$  we get simply  $x$ —thus, the function  $x$  must itself be a partner function for  $\Gamma_3$  and no other representation. Moreover, the operation of the space group on  $x$  is clearly spanned by the orthogonal functions  $x$  and  $y$ , and so we must have a unitary representation given simply by the  $2 \times 2$  rotation matrices that transform the functions  $\{x, y\}$ :

| $E$  | $C_3$   | $C_3^{-1}$  | $\sigma_1$  | $\sigma_2$  | $\sigma_3$                                      |
|--|---|---|---|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$ | $\frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$ | $\frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$ | $\frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ |

where to get  $\sigma_1$  and  $\sigma_2$  we used  $\sigma_1 = \sigma_3 C_3$  and  $\sigma_2 = \sigma_3 C_3^{-1}$  from the multiplication table above. The unitarity of these matrices follows immediately from the fact that they come from the  $2 \times 2$  rotation matrices, and can be easily verified in any case. Their traces clearly match those in the character table.

Note that the  $2 \times 2$  (in 2d) or  $3 \times 3$  (in 3d) rotation matrices always form a representation, but in

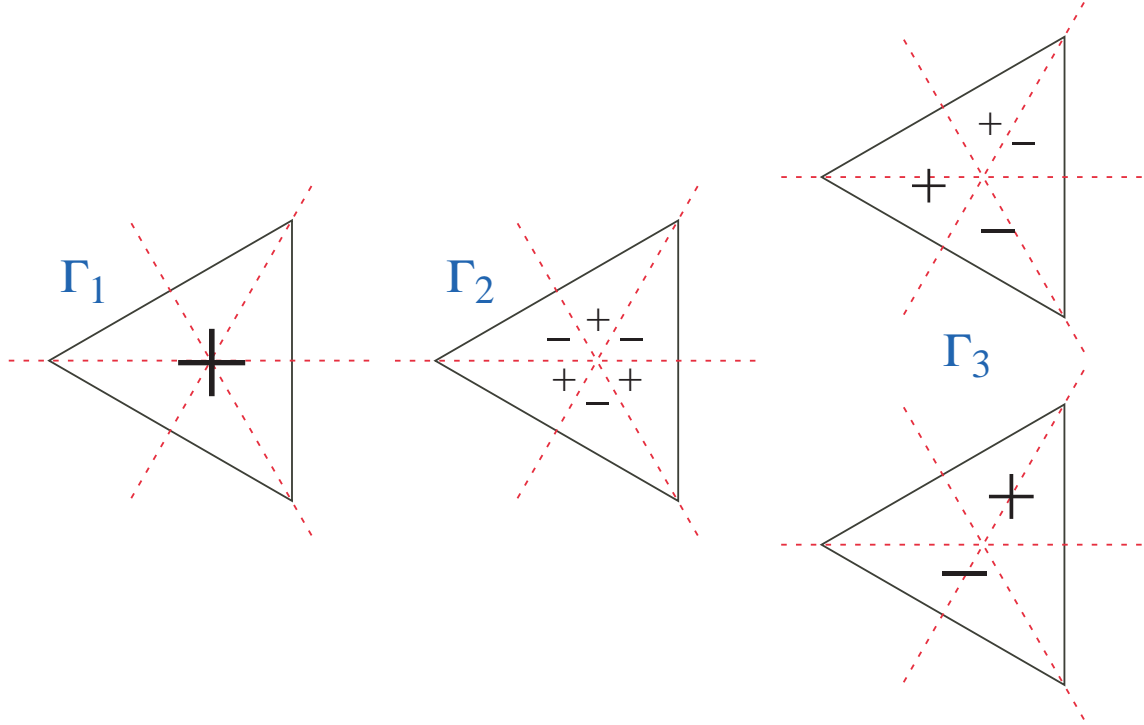


Figure 4: Sketch of possible  $E_z$  field patterns (for the lowest- $\omega$  modes of each symmetry) in the triangular cavity corresponding to the three representations. Note that a  $\Gamma_3$  mode must be doubly degenerate with two field patterns roughly as shown.

some groups these matrices are reducible, and in other groups there are multiple 2d irreps and the rotation matrices only give you one of these.

- (d)  $E_z$  field patterns that transform as these representations are very crudely sketched in Figure 4, where  $+$  and  $-$  denote maxima and minima of the field. Note that because  $\mathbf{E}$  is a vector, the component  $E_z$  transforms as an ordinary scalar in the  $xy$  plane, and “even” and “odd” fields are what we expect; note also that the boundary conditions require  $E_z$  to go to zero at the edges of the triangle, so all extrema must lie in the interior. The  $\Gamma_3$  mode must be doubly degenerate, of course, and can be chosen so that one mode is even with respect to a *single* one of the mirror planes and the other mode is odd with respect to that mirror plane, as shown. (For example, in our  $\{x, y\}$  representation matrices above, this even/odd plane was  $\sigma_3$ , although of course the modes could be rotated to be even/odd around any of the three  $\sigma$ ’s.) From the representation matrices, we can see that one degenerate partner may be found by operating  $(\hat{O}_{C_3} - \hat{O}_{C_3^{-1}})/\sqrt{3}$  on the other—i.e., by the difference of its  $120^\circ$  and  $-120^\circ$  rotations (this will give an *orthogonal* mode because it will have opposite even-odd symmetry under one of the mirror planes). The lowest-order  $E_z$  mode should be  $\Gamma_1$  (fewest nodes  $\rightarrow$  smallest  $\nabla^2$  term in eigenproblem). A less crude sketch would be to show contours of the field as in class, but in lieu of that I opted to show you an exact numerical calculation of the first few eigenmodes of this cavity, in Figure. 5, which illustrates all three representations (note that the lowest  $\omega$  mode of each representation looks much like our “guess”). The lowest frequency mode of  $E_z$  transforms like  $\Gamma_1$  because that has the fewest oscillations (only a single extremum).

The  $H_z$  field sketches, in Figure 6, are somewhat different, for two reasons. First,  $\mathbf{H}$  is a pseudovector, so  $H_z$  transforms as a pseudo-scalar and our normal conceptions of “even” and “odd” are reversed. Second, because of the boundary conditions the extrema of  $H_z$  tend to occur on the boundaries of the

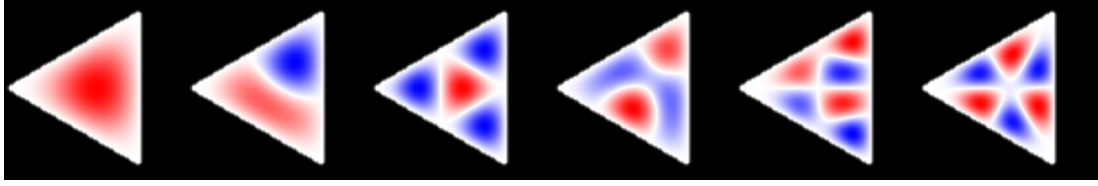


Figure 5:  $E_z$  field plots of first few lowest- $\omega$  modes in a triangular air cavity surrounded by metal (black) with side  $L$ , from a numerical calculation. The corresponding frequencies  $\omega L/2\pi c$  (and representations) are, from left to right: 1.16 ( $\Gamma_1$ ), 1.78 ( $\Gamma_3$ ), 2.32 ( $\Gamma_1$ ), 2.60 ( $\Gamma_3$ ), 2.93 ( $\Gamma_3$ ), and 3.08 ( $\Gamma_2$ ). Note that the second, fourth, and fifth modes (from left) are doubly degenerate (their degenerate partner is not shown, but can be found by subtracting  $120^\circ$  and  $-120^\circ$  rotations of the field). (Slight asymmetries in the  $\Gamma_2$  state are due to the finite computational grid resolution.)

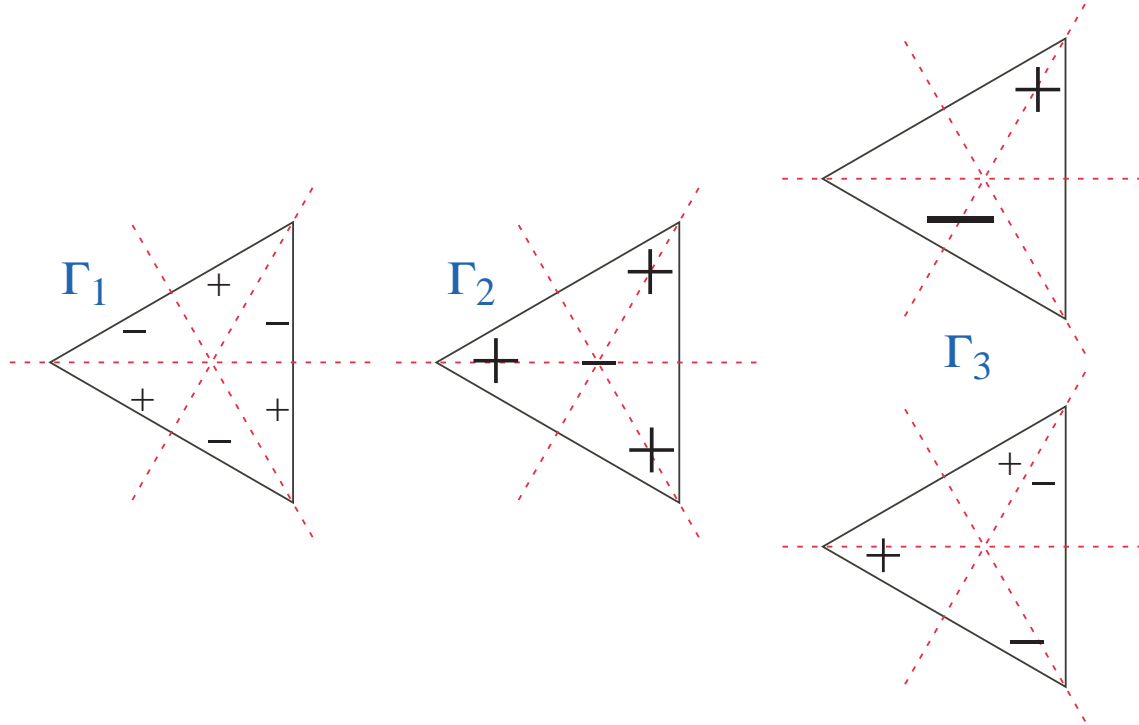


Figure 6: Sketch of possible  $H_z$  field patterns (for low- $\omega$  modes) in the triangular cavity corresponding to the three representations. Note that a  $\Gamma_3$  mode must be doubly degenerate with two field patterns roughly as shown.

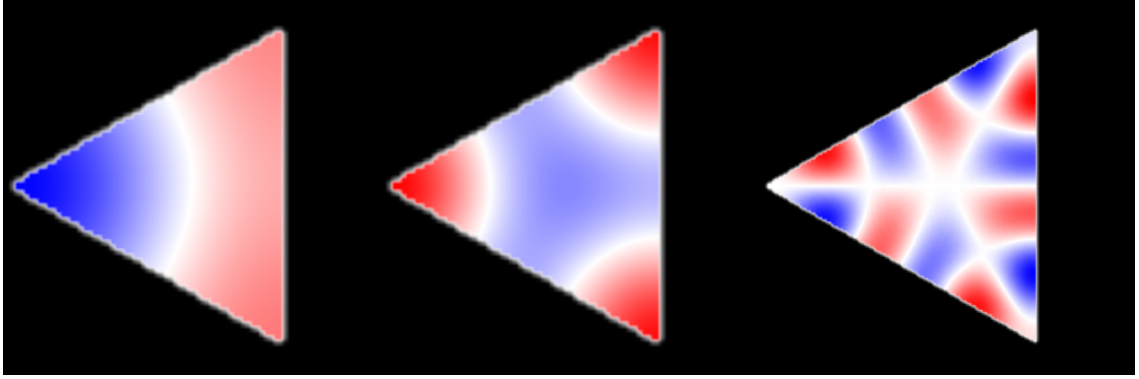


Figure 7:  $H_z$  field plots of the lowest- $\omega$  mode for each irreducible representation of  $C_{3v}$  in a triangular air cavity surrounded by metal (black) with side  $L$ , from a numerical calculation. The corresponding frequencies  $\omega L/2\pi c$  (and representations) are, from left to right: 0.66 ( $\Gamma_3$ ), 1.14 ( $\Gamma_2$ ), and 3.01 ( $\Gamma_1$ ). Note that the leftmost mode is doubly degenerate (its degenerate partner is not shown, but can be found by subtracting  $120^\circ$  and  $-120^\circ$  rotations of the field). (Slight asymmetries in the  $\Gamma_1$  state are due to the finite computational grid resolution.) There are 8 modes (not shown) with frequencies between 0.66 and 1.74.

triangle, at least for the low- $\omega$  modes. So, for example, now the  $\Gamma_1$  mode looks *odd* with respect to all of the mirror planes, and will only appear for higher- $\omega$  modes. The  $\Gamma_2$  mode is the one that looks most symmetric, but because the extrema lie on the boundaries there will tend to be a minima in the center of the triangle for the lowest- $\omega$  mode. The lowest-order  $H_z$  mode should be  $\Gamma_3$ , since it has the fewest nodal planes. The corresponding numerical calculations are shown in Figure 7. Here, the least-oscillatory (lowest- $\omega$ ) mode corresponds to  $\Gamma_3$ , with only two extrema. Note that the numerical  $\Gamma_1$  mode is higher-order (each of our sketched extrema is split into two) than our cartoon (I couldn't find any lower-order  $\Gamma_1$  modes, and I'm not quite sure why...).

- (e) We want an operation on one partner function that gives us the other (with some sign/coefficient), i.e. which corresponds to a matrix of the form  $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$ . By inspection of the  $\Gamma_3$  representation matrices from (c), there are several ways to get a matrix of this form. For example,  $(E + 2C_3)/\sqrt{3}$  or (more symmetrically)  $(C_3 - C_3^{-1})/\sqrt{3}$ , or  $(\sigma_1 - \sigma_2)/\sqrt{3} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . That means, if we have a solution  $\psi$  corresponding to one of the degenerate partner functions of  $\Gamma_3$ , then, for example,  $(\hat{\sigma}_1 - \hat{\sigma}_2)\psi/\sqrt{3}$  gives us the other orthogonal partner function.

### Problem 3: Cylindrical symmetry (5+10+5pts)

- (a) If  $D$  is a representation where  $D(\phi)$  is the matrix corresponding to rotation by an angle  $\phi$  around the  $z$  axis, then we must have  $D(\phi + \Delta\phi) = D(\phi)D(\Delta\phi)$  by definition of representations (rotating by  $\Delta\phi$  followed by  $\phi$  is exactly the same as rotating by  $\phi + \Delta\phi$ ). Therefore, as we derived in class for the translation group (which had the same property), the representations must all be of the form  $D^{(m)}(\phi) = e^{-im\phi}$  where  $m$  is some number characterizing the representation ( $k$  in class). For the translation group, this number was a completely arbitrary real number (it could even be complex, if we allowed divergent representations). Here, however, we have a further constraint: rotation by  $2\pi$  must be the same as doing nothing, and therefore we *must* have  $D(2\pi) = 1$  for  $D$  to represent the group. This implies that  $m$  is an *integer* (so that  $e^{-i2\pi m} = 1$ ).

Note, by the way, that you might be tempted to form  $2 \times 2$  representation matrices via the ordinary

$(x, y)$  rotation  $R = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}$ , but this is reducible:  $R = \exp \begin{pmatrix} 0 & i\phi \\ -i\phi & 0 \end{pmatrix}$ , which can be diagonalized into  $1 \times 1$  representations  $e^{\pm i\phi}$ .

- (b) If  $\psi(r, \phi, z)$  transforms like  $D^{(m)}$ , then rotating by  $\theta$  (sending  $\psi(r, \phi, z) \rightarrow \psi(r, \phi - \theta, z)$ ) must be just multiplication by  $e^{-im\theta}$ . This means that  $\psi$  must be of the form  $\psi_m(r, z)e^{im\phi}$ . Since we further have translational symmetry in  $z$ , we must have  $\psi = \psi_{mk}(r)e^{im\phi + ikz}$ . That is, we now have a *one*-dimensional function and should obtain a *one*-dimensional Hermitian eigenproblem in  $r$ .

If we plug this  $\psi$  into  $-\nabla^2 \psi = \frac{\omega^2}{c^2} \psi$ , we can apply the form of  $\nabla^2$  in cylindrical coordinates and obtain:

$$-\psi_{km}'' - \frac{1}{r} \psi_{km}' + \left( \frac{1}{r^2} m^2 + k^2 \right) \psi_{km} = \frac{\omega^2}{c^2} \psi_{km},$$

where primes denote differentiation by  $r$ , which after a slight rearrangement becomes:

$$r^2 \psi_{km}'' + r \psi_{km}' + (k_{\perp}^2 r^2 - m^2) \psi_{km} = 0,$$

where  $k_{\perp}^2 = \frac{\omega^2}{c^2} - k^2$ . Now, make a change of variables:  $x = k_{\perp} r$ ,  $y(x) = \psi_{km}(x/k_{\perp})$ , and we find:

$$x^2 y'' + xy' + (x^2 - m^2)y = 0,$$

which is precisely Bessel's equation of order  $m$ . The solutions of this are  $J_m(x)$  and  $Y_m(x)$ , the Bessel functions of the first and second kinds. However,  $Y_m(x)$  diverges at  $x = 0$ , which we can't allow (physical wave solutions can't blow up in empty space). So, the eigensolution must be  $y(x) = J_m(x)$  (choosing an arbitrary amplitude of 1), and thus

$$\psi(r, \phi, z) = J_m(k_{\perp} r) e^{im\phi + ikz}.$$

We're not done yet, because we haven't found  $k_{\perp}$ . This is determined by the boundary condition  $\psi|_{r=R} = 0$ . That means we must have  $J_m(k_{\perp} R) = 0$ , or  $k_{\perp} = x_{m,n}/R$  where the  $x_{m,n}$  are the zeros of  $J_m$ . Solving for  $\omega$  from  $k_{\perp}$ , we finally get a discrete sequence of bands:

$$\omega_{mk}^{(n)}(k) = c \sqrt{k^2 + \left( \frac{x_{m,n}}{R} \right)^2}.$$

Note, moreover, that  $J_{-m} = J_m$ , and therefore the bands for  $m \neq 0$  are *doubly degenerate* (this actually also follows from mirror symmetry). To sketch the bands, we can choose dimensionless units ( $c = 1$ ,  $R = 1$ ), and the result is shown in figure 8.

- (c) The  $\psi_{mk}(r)e^{im\phi + kz}$  must be orthogonal ( $\int \psi_1^* \psi_2 r dr d\phi dz = 0$  for eigenfunctions of different eigenvalues) since they are eigenfunctions of a Hermitian operator. For two different  $m$  and  $k$  values, they are orthogonal since  $\int e^{i(m_1 - m_2)\phi} d\phi = 0$  for  $m_1 \neq m_2$  (which follows from group theory since they correspond to different representations). For the *same*  $m$ , we must have that the  $r$  integral is zero, and hence, changing variables to  $u = r/R$ , we have:

$$\int_0^1 J_m(x_{m,n}u) \cdot J_m(x_{m,n'}u) u du = 0$$

for different zeros  $n \neq n'$ . (This is a well-known identity that is used in Fourier-Bessel series expansions.)

## Problem 5: Conservation Laws (10+10+5pts)

(Clarification: the problem statement was unclear about what level of generality should be assumed about the materials. Obviously, since the problem is talking about linear operators in the frequency domain, you

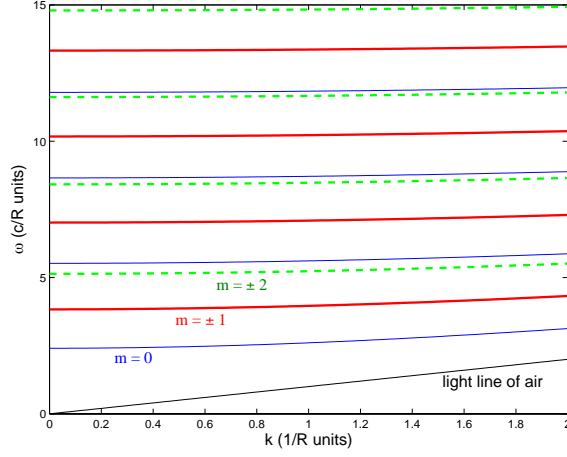


Figure 8: The first few bands of the scalar wave equation in a cylindrical waveguide. Plotted are lowest four bands for  $m = 0$  (blue),  $m = \pm 1$  (thick red), and  $m = \pm 2$  (dashed green). Also, shown, for reference, is the light line  $\omega = ck$  of air (all modes lie above this and are therefore extended in the air core). In particular, the shapes of the curves are hyperbolas.

should assume linear time-invariant materials as we have throughout the course so far. In most of the course we have then used a susceptibility  $\hat{\chi} = \begin{pmatrix} \epsilon - 1 & \\ & \mu - 1 \end{pmatrix}$ , and that is what I have done in the solutions. If you assume a completely general  $6 \times 6$  susceptibility, including bianisotropic materials, the answers are similar but it is more convenient to write them in terms of  $\psi = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$  as in the course notes, rather than just  $\mathbf{E}$ . Either approach is acceptable. It is also acceptable to set  $\mu = 1$  as we do for most of the course.)

- (a) A current  $\mathbf{J}$  modifies Ampere's law:  $\nabla \times \mathbf{H} = \mathbf{J} - i\omega\epsilon_0\epsilon\mathbf{E}$  (where we have cancelled the  $e^{-i\omega t}$  time-dependence in every term). Therefore, when we take Faraday's law  $\mu^{-1}\nabla \times \mathbf{E} = i\omega\mu_0\mathbf{H}$  and operate  $\nabla \times$  on both sides, we get

$$(\nabla \times \mu^{-1}\nabla \times - \frac{\omega^2}{c^2}\epsilon)\mathbf{E} = i\omega\mu_0\mathbf{J}.$$

This is indeed of the form  $\hat{A}\mathbf{E} = \mathbf{b}$ , where  $\hat{A} = \nabla \times \mu^{-1}\nabla \times - \omega^2\epsilon/c^2$  is a linear operator and  $\mathbf{b} = i\omega\mu_0\mathbf{J}$  is a given right-hand side.

- (b) Suppose that  $\mathbf{E}$  solves  $\hat{A}\mathbf{E} = \mathbf{b}$ , and that  $\mathbf{b}$  is a partner of some representation  $\alpha$  of the space group. Therefore,  $\hat{P}^{(\alpha)}\mathbf{b} = \mathbf{b}$ , where  $\hat{P}^{(\alpha)} = \frac{d\alpha}{|G|} \sum_g \chi^{(\alpha)}(g)^* \hat{g}$  is the projection operator for that representation. Moreover, by definition of the space group we must have  $\hat{A}$  commuting with  $\hat{g}$  for any  $g$  in the space group (this also follows explicitly if  $\hat{g}\epsilon = \epsilon$  and  $\hat{g}\mu = \mu$ , since  $\nabla \times$  is invariant under rotations and translations), and the definition of the space group also requires that  $g$  leave the domain  $\Omega$  and the boundary conditions unchanged. Since  $\hat{A}$  is linear, and  $\hat{P}^{(\alpha)}$  is merely a linear combination of the  $\hat{g}$ 's, then  $\hat{A}$  also commutes with  $\hat{P}^{(\alpha)}$ . Therefore  $\hat{A}\mathbf{E} = \mathbf{b} = \hat{P}^{(\alpha)}\mathbf{b} = \hat{P}^{(\alpha)}\hat{A}\mathbf{E} = \hat{A}\hat{P}^{(\alpha)}\mathbf{E}$ , and thus  $\hat{A}(1 - \hat{P}^{(\alpha)})\mathbf{E} = 0$ . Therefore, either  $\mathbf{E}$  is a partner of  $\alpha$  ( $\hat{P}^{(\alpha)}\mathbf{E} = \mathbf{E}$ ), or the component of  $\mathbf{E}$  that isn't a partner of  $\alpha$  lies in the null space of  $\hat{A}$ —that is,  $\hat{A}$  is singular and we didn't have a unique solution  $\mathbf{E}$  in the first place.
- (c) If  $\omega$  is one of the eigenfrequencies, then  $\hat{A}$  is singular— $\hat{A}$  operating on an eigenfunction  $\mathbf{E}_0$  will give zero by definition (since  $\nabla \times \mu^{-1}\nabla \times \mathbf{E} = \omega^2\epsilon\mathbf{E}/c^2$  is the generalized eigenproblem as you derived in problem set 1). There are two possibilities. In a finite system (i.e. compact domain), then physically you can get a divergent (non-harmonic, linearly growing amplitude) solution, exactly like the case where you drive a harmonic oscillator at the resonant frequency. Alternatively, if  $\mathbf{J}$  is orthogonal to



the eigenfunction(s)  $\mathbf{E}_0$  (for example, if they are partner functions of different representations), then there will be a solution, but it won't be unique because you can add any multiple of  $\mathbf{E}_0$  while satisfying the equation (although one could impose some additional constraint to obtain a unique solution).

**Supplementary comment:** In an infinite system, the question is more subtle because the (generalized) eigenfunction in question (if it is an extended mode) can have infinitesimal overlap with  $\mathbf{J}$  (if  $\mathbf{J}$  is localized). In this case, it turns out you can get a finite response, but to make it unique you have to impose some additional boundary conditions. The classic example of this is a localized antenna source (e.g. a dipole) in vacuum—vacuum has (generalized) eigenfunctions (planewaves) at *every*  $\omega$  ( $\omega = c|\mathbf{k}|$ ), but the resulting field is a finite-amplitude spherical wave(s) emanating from the antenna, not a divergence (except at the antenna itself, if  $\mathbf{J}$  itself diverges as for a point source). To get a unique solution, we have to impose a “radiation” boundary condition that there are no incoming waves from infinity (which would satisfy Maxwell’s equations, but wouldn’t be very physical). Sometimes, people express such a restriction by replacing  $\epsilon$  with  $\epsilon + i0^+$  (for  $\omega > 0$ ) in the operator: taking the limit of a system with infinitesimal absorption (imaginary  $i0^+$  part of  $\epsilon$ ) eliminates incoming waves from infinity, and corresponds to letting the poles of  $\hat{A}$  approach the real- $\omega$  axis from below; this approach to outgoing boundary conditions is sometimes called the “limiting absorption principle.” We will talk more about this later in the course.