Note on decomposing functions into partner functions

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In the representation-theory handout for 18.369, it says that any function $\psi(\vec{x})$ can be decomposed into a sum of partner functions of the different irreps of any symmetry group G. Recall that for a coordinate transformation g (a rotation or translation), I denote the corresponding transformation of functions ψ by $\hat{g}$.[1] What follows is a brief proof of that.

1. Consider the set $S = \{\hat{g}\psi \text{ for all } g \in G\}$ and the subspace $\mathcal{S}$ spanned by S. Form a basis of $\mathcal{S}$ from d elements $\psi_i = \widehat{g_i}\psi$ of S, for $i \in \{1, \ldots, d\}$ where d is the dimension of $\mathcal{S}$ (the number of linearly independent functions in S).

2. By construction, $\hat{g}\psi_j = \hat{g}\widehat{g_i}\psi = \widehat{g'}\psi \in S, \mathcal{S}$ for any $j \in \{1, \ldots, d\}$, $g \in G$. Hence $\hat{g}\psi_j = \sum_{i=1}^{d} \psi_i D_{ij}(g)$ where $D_{ij}(g)$ are some coefficients depending on i, j, and g.

3. The matrices $D(g)$ with entries $D_{ij}(g)$ form a representation of G. Proof:

$$
\begin{aligned}
\widehat{g_1}\widehat{g_2}\psi_j &= \widehat{g_1 g_2}\psi_j = \sum_{i=1}^{d} \psi_i D_{ij}(g_1 g_2) \\
&= \widehat{g_1} \sum_{k=1}^{d} \psi_k D_{kj}(g_2) = \sum_{k=1}^{d} \left[\sum_{i=1}^{d} \psi_i D_{ik}(g_1) \right] D_{kj}(g_2) \\
&= \sum_{i=1}^{d} \psi_i \left[\sum_{k=1}^{d} D_{ik}(g_1) D_{kj}(g_2) \right].
\end{aligned}
$$

   Comparing the first and last lines, which must be true for any i, j, we find $D_{ij}(g_1 g_2) = \sum_{k=1}^{d} D_{ik}(g_1) D_{kj}(g_2)$, which is exactly the formula for a matrix multiplication, so $D(g_1 g_2) = D(g_1) D(g_2)$. Hence D is a representation.

4. D must be reducible into one or more irreps $D^{(\alpha)}$ of G, i.e. we can perform a change of basis to $\tilde{D} = S^{-1} D S$ that block-diagonalize $\tilde{D}$ into irreps. Perform the same change of basis on ψ_i to obtain the corresponding basis functions $\tilde{\psi}_j = \sum_i \psi_i S_{ij}$. By construction, the $\tilde{\psi}_j$ are partners of $\tilde{D}$, and hence they are partners of the irreps that $\tilde{D}$ reduces into.

5. $\psi \in S$ since the identity $E \in G$, so ψ is in the span of the basis functions ψ_i and hence of $\tilde{\psi}_i$. Hence $\psi = \sum_i c_i \tilde{\psi}_i$ for some coefficients c_i, which from above is a sum of partner functions of one or more of various irreps of G. (Note it is easy to show that the partner functions of an irrep form a vector space: summing two partners of the same irrep or multipling them by scalars c_i yields another partner function.) Q.E.D.

[1] Some authors just use g interchangeably for rotations of the coordinate space or rotations of the Hilbert space, but for vector fields it is confusing if you don't distinguish the two. e.g. for a covariant vector field $\vec{F}(\vec{x})$, we define $\hat{g}\vec{F} = g\vec{F}(g^{-1}\vec{x})$.