18.369 Problem Set 1 Solutions

Problem 1: Adjoints and operators (5+10+5 points)

(a) If $\dagger$ is conjugate-transpose of a matrix or vector, we are just using the usual linear-algebra rule that $(AB)^\dagger = B^\dagger A^\dagger$, hence $\langle h, Oh' \rangle = h^\dagger (Oh') = (O^\dagger h)^\dagger h' = \langle O^\dagger h, h' \rangle$ for the Euclidean inner product.

More explicitly, if h is a column-vector and we let $h^\dagger$ be its conjugate transpose, then $h^\dagger$ is a row vector and $h^\dagger h' = \sum_m h_m^* h'_m = \langle h, h' \rangle$ by the usual row-times-column multiplicaton rule. If O is a matrix then $Oh' = \sum_n O_{mn} h'_n$ by the usual matrix-vector product. Then the dot product of h with Oh' is given by $\sum_m h_m^* (\sum_n O_{mn} h'_n) = \sum_n (\sum_m O_{mn}^* h_m)^* h'_n$, which is the same thing as the dot product of $O^\dagger h$ with h' where $O^\dagger$ is the conjugate transpose of O.

Thus, as claimed in class, the abstract $\langle h, Oh' \rangle = \langle O^\dagger h, h' \rangle$ definition of $O^\dagger$ implies the usual conjugate transpose definition of $O^\dagger$ for matrices.

(b) If $\hat{O}$ is unitary and we send $u \to \hat{O}u$ and $v \to \hat{O}v$, then $\langle u, v \rangle \to \langle u, \hat{O}^\dagger \hat{O} v \rangle = \langle u, v \rangle$, and thus inner products are preserved. Consider now two eigensolutions $\hat{O}u_1 = \lambda_1 u_1$ and $\hat{O}u_2 = \lambda_2 u_2$. Then $\langle u_1, \hat{O}^\dagger \hat{O} u_2 \rangle = \langle u_1, u_2 \rangle$ by the unitarity of $\hat{O}$ and $\langle u_1, \hat{O}^\dagger \hat{O} u_2 \rangle = \langle \hat{O} u_1, \hat{O} u_2 \rangle = \lambda_1^* \lambda_2 \langle u_1, u_2 \rangle$ by the eigenvector property (where we let $\hat{O}^\dagger$ act to the left, and conjugate the eigenvalue when we factor it out, as in class). Combining these two expressions, we have $(\lambda_1^* \lambda_2 - 1) \langle u_1, u_2 \rangle = 0$. There are three cases, just like for Hermitian operators. If $u_1 = u_2$, then we must have $\lambda_1^* \lambda_1 = 1 = |\lambda_1|^2$, and thus the eigenvalues have unit magnitude. This also implies that $\lambda_1^* = 1/\lambda_1$. If $\lambda_1 \neq \lambda_2$, then $(\lambda_1^* \lambda_2 - 1) = (\lambda_2/\lambda_1 - 1) \neq 0$, and therefore $\langle u_1, u_2 \rangle = 0$ and the eigenvectors are orthogonal. If $\lambda_1 = \lambda_2$ but have linearly independent $u_1 \neq u_2$ (degenerate eigenvectors, i.e. geometric multiplicity > 1), then we can form orthogonal linear combinations (e.g. via Gram–Schmidt).

(c) Take two vectors u and v, and consider their inner product. Then $\langle u, \hat{O}^{-1} \hat{O} v \rangle = \langle u, v \rangle$. By definition of the adjoint, however, if we move first $\hat{O}^{-1}$ and then $\hat{O}$ to act to the left, then we get $\langle u, v \rangle = \langle \hat{O}^\dagger (\hat{O}^{-1})^\dagger u, v \rangle$. For this to be true for all u and v, we must have $\hat{O}^\dagger (\hat{O}^{-1})^\dagger = \mathbf{1}$ and thus $(\hat{O}^{-1})^\dagger = (\hat{O}^\dagger)^{-1}$. Q.E.D.

Problem 2: Maxwell eigenproblems (5+5+5+5+5 points)

(a) To eliminate $\mathbf{H}$, we start with Faraday's law $\nabla \times \mathbf{E} = i\frac{\omega}{c}\mathbf{H}$ and take the curl of both sides. We obtain:

$$\nabla \times \nabla \times \mathbf{E} = \frac{\omega^2}{c^2} \varepsilon \mathbf{E}.$$

If we divide both sides by ε, we get the form of a linear eigenproblem but the operator $\frac{1}{\varepsilon} \nabla \times \nabla \times$ is not Hermitian under the usual inner product $\langle \mathbf{E}_1, \mathbf{E}_2 \rangle = \int \mathbf{E}_1^* \cdot \mathbf{E}_2$—integrating by parts as in class, assuming boundary conditions such that the boundary terms vanish, we find that its adjoint is $\nabla \times \nabla \times \frac{1}{\varepsilon} \cdot$, which is not the same operator unless the $\frac{1}{\varepsilon}$ commutes with the curls, which only happens if ε is a constant. However, if we leave it in the form above we have a generalized Hermitian problem with $\hat{A} = \nabla \times \nabla \times$ and $\hat{B} = \varepsilon$. $\hat{A}$ is Hermitian for the same reason that $\hat{\Theta}$ was (it *is* $\hat{\Theta}$ for $\varepsilon = 1$), and $\hat{B}$ is Hermitian as long as ε is real (so that $\mathbf{H}_1^* \cdot \varepsilon \mathbf{H}_2 = (\varepsilon \mathbf{H}_1)^* \cdot \mathbf{H}_2$).

(b) The proof follows the same lines as in class. [Alternatively, we could simply quote the Hermitian results from class once we prove part (c).] Consider two eigensolutions u_1 and u_2 (where $\hat{A}u = \lambda\hat{B}u$, and $u \neq 0$), and take $\langle u_2, \hat{A}u_1\rangle$. Since $\hat{A}$ is Hermitian, we can operate it to the left or to the right in the inner product, and get $\lambda_2^* \langle u_2, \hat{B}u_1\rangle = \lambda_1 \langle u_2, \hat{B}u_1\rangle$, or $(\lambda_2^* - \lambda_1)\langle u_2, \hat{B}u_1\rangle = 0$. There are three cases. First, if $u_1 = u_2$ then we must have $\lambda_1 = \lambda_1^*$ (real eigenvalues), since $\langle u_1, \hat{B}u_1\rangle > 0$ by definition if $\hat{B}$ is positive definite. Second, if $\lambda_1 \neq \lambda_2$ then we must have $\langle u_2, \hat{B}u_1\rangle = 0$, which is our modified orthogonality condition. Finally, if $\lambda_1 = \lambda_2$ but $u_1 \neq u_2$, then we can form a linear combination that is orthogonal (since any linear combination still is an eigenvector); e.g.

$$u_2 \to u_2 - u_1 \frac{\langle u_2, \hat{B}u_1\rangle}{\langle u_1, \hat{B}u_1\rangle},$$

where we have again relied on the fact that $\hat{B}$ is positive definite (so that we can divide by $\langle u_1, \hat{B}u_1\rangle$). This is certainly true for $\hat{B} = \varepsilon$, since $\langle E, \hat{B}E\rangle = \int \varepsilon|\mathbf{E}|^2 > 0$ for all $\mathbf{E} \neq 0$ (almost everywhere) as long as we have a real $\varepsilon > 0$ as we required in class.

(c) First, let us verify that $\langle \mathbf{E}, \mathbf{E}'\rangle_B = \langle \mathbf{E}, \hat{B}\mathbf{E}'\rangle$ is indeed an inner product. Because $\hat{B}$ is self-adjoint, we have $\langle \mathbf{E}', \mathbf{E}\rangle_B = \langle \mathbf{E}', \hat{B}\mathbf{E}\rangle = \langle \hat{B}\mathbf{E}', \mathbf{E}\rangle = \langle \mathbf{E}, \hat{B}\mathbf{E}'\rangle^* = \langle \mathbf{E}, \mathbf{E}'\rangle_B^*$. Bilinearity follows from bilinearity of $\langle \cdot, \cdot\rangle$ and linearity of $\hat{B}$. Positivity $\langle \mathbf{E}, \mathbf{E}\rangle_B = \langle \mathbf{E}, \hat{B}\mathbf{E}\rangle > 0$ except for $\mathbf{E} = 0$ (almost everywhere) follows from positive-definiteness of $\hat{B}$. All good!

Now, Hermiticity of $\hat{B}^{-1}\hat{A}$ follows almost trivially from Hermiticity of $\hat{A}$ and $\hat{B}$: $\langle \mathbf{E}, \hat{B}^{-1}\hat{A}\mathbf{E}'\rangle_B = \langle \mathbf{E}, \cancel{\hat{B}\hat{B}^{-1}}\hat{A}\mathbf{E}'\rangle = \langle \hat{A}\mathbf{E}, \mathbf{E}'\rangle = \langle \hat{A}\mathbf{E}, \hat{B}^{-1}\hat{B}\mathbf{E}'\rangle = \langle \hat{B}^{-1}\hat{A}\mathbf{E}, \hat{B}\mathbf{E}'\rangle = \langle \hat{B}^{-1}\hat{A}\mathbf{E}, \mathbf{E}'\rangle_B$, where we have used the fact, from problem 1, that Hermiticity of $\hat{B}$ implies Hermiticity of $\hat{B}^{-1}$. Q.E.D.

(d) If $\mu \neq 1$ then we have $\mathbf{B} = \mu\mathbf{H} \neq \mathbf{H}$, and when we eliminate $\mathbf{E}$ or $\mathbf{H}$ from Maxwell's equations we get:

$$\nabla \times \frac{1}{\varepsilon}\nabla \times \mathbf{H} = \frac{\omega^2}{c^2}\mu\mathbf{H}$$

$$\nabla \times \frac{1}{\mu}\nabla \times \mathbf{E} = \frac{\omega^2}{c^2}\varepsilon\mathbf{E}$$

with the constraints $\nabla \cdot \varepsilon\mathbf{E} = 0$ and $\nabla \cdot \mu\mathbf{H} = 0$. These are both generalized Hermitian eigenproblems (since μ and $\nabla \times \frac{1}{\mu}\nabla\times$ are both Hermitian operators for the same reason ε and $\nabla \times \frac{1}{\varepsilon}\nabla\times$ were). Thus, the eigenvalues are real and the eigenstates are orthogonal through μ and ε, respectively, as proved above. To prove that ω is real, we consider an eigenfunction H. Then $\langle H, \hat{\Theta}H\rangle = \frac{\omega^2}{c^2}\langle H, \mu H\rangle$ and we must have $\omega^2 \geq 0$ since $\hat{\Theta}$ is positive semi-definite (from class) and μ is positive definite (for the same reason ε was, above). The $\mathbf{E}$ eigenproblem has real ω for the same reason (except that μ and ε are swapped).

Alternatively, as in part (c), we can write them as ordinary Hermitian eigenproblems with a modified inner product, e.g. $\frac{1}{\varepsilon}\nabla \times \frac{1}{\mu}\nabla \times \mathbf{E} = \frac{\omega^2}{c^2}\mathbf{E}$, where $\frac{1}{\varepsilon}\nabla \times \frac{1}{\mu}\nabla\times$ is Hermitian and positive semidefinite under the $\langle \mathbf{E}, \mathbf{E}'\rangle_B = \int \mathbf{E}^* \cdot \varepsilon\mathbf{E}\prime$ inner product as above. The results then follow.

(e) Consider the $\mathbf{H}$ eigenproblem. (To even get this linear eigenproblem, we must immediately require ε to be an invertible matrix, and of course require ε and μ to be independent of ω or the field strength.) For the right-hand operator μ to be Hermitian, we require $\int \mathbf{H}_1^* \cdot \mu\mathbf{H}_2 = \int (\mu\mathbf{H}_1)^* \cdot \mathbf{H}_2$

for all $\mathbf{H}_1$ and $\mathbf{H}_2$, which implies that $\mathbf{H}_1^* \cdot \mu \mathbf{H}_2 = (\mu \mathbf{H}_1)^* \cdot \mathbf{H}_2$. Thus, we require the 3×3 $\mu(\mathbf{x})$ matrix to be itself Hermitian at every $\mathbf{x}$ (that is, equal to its conjugate transpose, from problem 1). (Technically, these requirements hold "almost everywhere" rather than at every point, but as usual I will gloss over this distinction as well as other technical restrictions on the functions to exclude crazy functions.) Similarly, for $\hat{\Theta}$ to be Hermitian we require $\int \mathbf{F}_1^* \cdot \varepsilon^{-1} \mathbf{F}_2 = \int (\varepsilon^{-1} \mathbf{F}_1)^* \cdot \mathbf{F}_2$ where $\mathbf{F} = \nabla \times \mathbf{H}$, so that we can move the ε^{-1} over to the left side of the inner product, and thus $\varepsilon^{-1}(\mathbf{x})$ must be Hermitian at every $\mathbf{x}$. From problem 1, this implies that $\varepsilon(\mathbf{x})$ is also Hermitian. Finally, to get real eigenvalues we saw from above that we must have μ positive definite ($\int \mathbf{H}^* \cdot \mu \mathbf{H} > 0$ for $\mathbf{H} \neq 0$); since this must be true for all $\mathbf{H}$ then $\mu(\mathbf{x})$ at each point must be a positive-definite 3×3 matrix (positive eigenvalues). Similarly, $\hat{\Theta}$ must be positive semi-definite, which implies that $\varepsilon^{-1}(\mathbf{x})$ is positive semi-definite (non-negative eigenvalues), but since it has to be invertible we must have $\varepsilon(\mathbf{x})$ positive definite (zero eigenvalues would make it singular). To sum up, we must have $\varepsilon(\mathbf{x})$ and $\mu(\mathbf{x})$ being positive-definite Hermitian matrices at (almost) every $\mathbf{x}$. (The analysis for the $\mathbf{E}$ eigenproblem is identical.)

Optional: Technically, there are a few other possibilities. In part (b), we showed that if $\hat{B}$ is positive-definite it leads to real eigenvalues etc. The same properties, however, hold if $\hat{B}$ is *negative*-definite, and if *both* $\hat{A}$ and $\hat{B}$ are negative-definite we still get real, *positive* eigenvalues. Thus, another possibility is for ε and μ to be Hermitian *negative*-definite matrices. (For a scalar $\varepsilon < 0$ and $\mu < 0$, this leads to so-called "left-handed materials" with a *negative* real index of refraction $n = -\sqrt{\varepsilon\mu}$!) Furthermore, ε and μ could both be *anti*-Hermitian instead of Hermitian (i.e., $\varepsilon^\dagger = -\varepsilon$ and $\mu^\dagger = -\mu$). More generally, for any complex number z, if we replace ε and μ by $z\varepsilon$ and μ/z, then ω is unchanged (e.g. making $z = i$ gives anti-Hermitian matrices).

Problem 3: Linear responses and symmetry (5+5+5+5 points)

(a) A current $\mathbf{J}$ modifies Ampere's law: $\nabla \times \mathbf{H} = \mathbf{J} - i\omega\varepsilon_0\varepsilon\mathbf{E}$ (where we have cancelled the $e^{-i\omega t}$ time-dependence in every term). Therefore, when we take Faraday's law $\mu^{-1}\nabla \times \mathbf{E} = i\omega\mu_0\mathbf{H}$ and operate $\nabla\times$ on both sides, we get

$$(\nabla \times \mu^{-1} \nabla \times - \frac{\omega^2}{c^2}\varepsilon)\mathbf{E} = i\omega\mu_0\mathbf{J}.$$

This is indeed of the form $\hat{A}\mathbf{E} = \mathbf{b}$, where $\hat{A} = \nabla \times \mu^{-1} \nabla \times -\omega^2\varepsilon/c^2$ is a linear operator and $\mathbf{b} = i\omega\mu_0\mathbf{J}$ is a given right-hand side. In the problem, you were told to assume $\mu = 1$, in which case the μ^{-1} term disappears; you can also employ "natural" units with $\mu_0 = \varepsilon_0 = 1$.

(b) By assumption, we have $\hat{A}$ commuting with $\hat{g}$ for any g in the space group (this also follows explicitly if $\hat{g}\varepsilon = \varepsilon$ and $\hat{g}\mu = \mu$, since $\nabla \times$ is invariant under rotations and translations), and we are also given that g leavse the domain Ω and the boundary conditions unchanged. Suppose that $\hat{A}\mathbf{E} = \mathbf{b}$, and we are given that $\hat{g}\mathbf{J} = \alpha\mathbf{J} \implies \hat{g}\mathbf{b} = \alpha\mathbf{b}$. Then $\hat{g}(\hat{A}\mathbf{E}) = \hat{g}\mathbf{b} = \alpha\mathbf{b} = \hat{A}(\hat{g}\mathbf{E})$ by commutation, so $\mathbf{b} = \hat{A}(\hat{g}\mathbf{E}/\alpha)$ by linearity. Assuming that $\mathbf{E}$ is a *unique* solution to $\hat{A}\mathbf{E} = \mathbf{b}$ (which was implied by the problem calling it "the" solution), i.e. that it is fully the specified by the equation boundary conditions, then since $\hat{g}\mathbf{E}/\alpha$ satisfies the same boundary conditions (preserved by $\hat{g}$ **assuming the boundary conditions are preserved by scaling** α — i.e. that the boundary conditions as well as the PDE are **linear**), we must have $\hat{g}\mathbf{E}/\alpha = \mathbf{E} \implies \hat{g}\mathbf{E} = \alpha\mathbf{E}$ as desired.

Supplementary comment: It is interesting to consider a case of a Dirichlet boundary condition that is preserved by $\hat{g}$ but is *not* preserved by scaling — in this case, the equations are no longer linear, e.g. doubling the current does not produce double the fields. For example, suppose that we have a square $L \times L$ box domain filled with $\varepsilon = \mu = 1$) in 2d, with the E_z polarization, with boundary conditions $E_z = 0$ on three walls (PEC) but $E_z = x(L - x)$ (a nonzero mirror-symmetric function that goes to zero at the corners) on the fourth wall parallel to x. This problem is mirror-symmetric $g = \sigma_x$, but not invariant under scaling by $\alpha \neq 1$. So, if we have an *odd* current source $\hat{\sigma_x} J_z = -J_z$, ($\alpha = -1$) the nonzero boundary prevents us from having an odd solution E_z; the resulting E_z won't have any mirror symmetry at all. Good for you if you noticed this implicit assumption of linearlity in the problem!

(c) If you look at a picture of the magnetic field, it *appears* to be *odd* under mirror flips σ_z. But actually this is okay. An even current $\hat{\sigma_z}\mathbf{J} = \mathbf{J}$ should produce an even *electric* field $\hat{\sigma_z}\mathbf{E} = \mathbf{E}$, but what does this mean for the magnetic field $\mathbf{H} = \frac{1}{i\omega}\nabla \times \mathbf{E}$?

To mirror flip an electric field, we do $\hat{\sigma_z}\mathbf{E}(\mathbf{x}) = \sigma_z \mathbf{E}(\sigma_z \mathbf{x})$, i.e. we flip the coordinates *and* we flip the field vector. Now, let's plug this into $\nabla \times \mathbf{E}$, by computing $\nabla \times (\hat{\sigma_z}\mathbf{E})$. We can write these out explicitly:

$$\nabla \times \mathbf{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{z},$$

$$\nabla \times (\hat{\sigma_z}\mathbf{E}) = \left(\frac{\partial(-E_z)}{\partial y} - \frac{\partial E_y}{\partial(-z)}\right)\hat{x} + \left(\frac{\partial E_x}{\partial(-z)} - \frac{\partial(-E_z)}{\partial x}\right)\hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{z},$$

where by inspection the first two terms have flipped signs, either from $E_z \to -E_z$ or $z \to -z$. Therefore, it follows that

$$\nabla \times (\hat{\sigma_z}\mathbf{E}) = -\sigma_z \left(\nabla \times \mathbf{E}|_{\sigma_z \mathbf{x}}\right),$$

with an extra minus sign compared to a naive application of the mirror-flip operation: we flip the coordinates, flip the "vector", and then multiply by -1. We can therefore write the corresponding mirror-flip operation on the magnetic field as follows:

$$\hat{\sigma_z}\mathbf{H}(\mathbf{x}) = -\sigma_z \mathbf{H}(\sigma_z \mathbf{x}).$$

Compared to the electric field, the magnetic field gets an extra -1 factor under mirror flips! (As a result, we will call the magnetic field a **pseudo-vector** rather than a true "vector.")

So, there is no contradiction: an even electric current produces an even electric field, but because the magnetic field gets an extra sign change under mirror flips, it *looks "odd"*.

(d) If ω is one of the eigenfrequencies, then $\hat{A}$ is singular—$\hat{A}$ operating on an eigenfunction $\mathbf{E}_0$ will give zero by definition (since $\nabla \times \mu^{-1}\nabla \times \mathbf{E} = \omega^2 \varepsilon \mathbf{E}/c^2$ is the generalized eigenproblem as you derived in problem set 1, equivalent to the magnetic-field eigenproblem). There are two possibilities. In a finite system (i.e. compact domain), then in the time-domain you can get a divergent (non-harmonic, linearly growing amplitude) solution, exactly like the case where you drive a harmonic oscillator at the resonant frequency. Alternatively, if $\mathbf{J}$ is orthogonal to the eigenfunction(s) $\mathbf{E}_0$ (for example, if they are partner functions of different representations), then there will be a solution, but it won't be unique because you can add any multiple of $\mathbf{E}_0$ while satistfying the equation (although one could impose some additional constraint to obtain a unique solution).

Supplementary comment: In an infinite system, the question is more subtle because the (generalized) eigenfunction in question (if it is an extended mode) can have infinitesimal overlap with $\mathbf{J}$ (if $\mathbf{J}$ is localized). In this case, it turns out you can get a finite response, but to make it unique you have to impose some additional boundary conditions. The classic example of this is a localized antenna source (e.g. a dipole) in vacuum—vacuum has (generalized) eigenfunctions (planewaves) at *every* ω ($\omega = c|\mathbf{k}|$), but the resulting field is a finite-amplitude spherical wave(s) emanating from the antenna, not a divergence (except at the antenna itself, if $\mathbf{J}$ itself diverges as for a point source). To get a unique solution, we have to impose a "radiation" boundary condition that there are no incoming waves from infinity (which would satisfy Maxwell's equations, but wouldn't be very physical). Sometimes, people express such a restriction by replacing ε with $\varepsilon + i0^+$ (for $\omega > 0$) in the operator: taking the limit of a system with infinitesimal absorption (imaginary $i0^+$ part of ε) eliminates incoming waves from infinity, and corresponds to letting the poles of $\hat{A}$ approach the real-ω axis from below; this approach to outgoing boundary conditions is sometimes called the "limiting absorption principle." We will talk more about this later in the course.