# 18.369 Problem Set 3 Solutions

## Problem 1: Bloch-periodic eigenproblems (10+5 points)

Suppose that we have a periodic system with period $a$ in the $x$ direction, and we look for Bloch-periodic eigenfunctions $\mathbf{H}(x+a,y,z) = e^{ika}\mathbf{H}(x,y,z)$ of the $\hat{\Theta} = \nabla\times\varepsilon^{-1}\nabla\times$ operator with these boundary conditions in $x$, acting on a unit cell $x \in [0,a]$ (with some other boundary conditions in $y$ and $z$). (That is, we *don't* rewrite in terms of the periodic Bloch envelope and use $\hat{\Theta}_k$.)

(a) From class, integrating by parts with $\hat{\Theta}$ over a domain $\Omega$ yielded:

$$\langle \mathbf{H}, \hat{\Theta}\mathbf{H}' \rangle = \langle \hat{\Theta}\mathbf{H}, \mathbf{H}' \rangle + \oiint_{\partial\Omega} [\mathbf{E}' \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H}'] \cdot d\mathbf{A}$$

for any fields $\mathbf{H}$ and $\mathbf{H}'$ (with corresponding electric fields $\mathbf{E} = \frac{i}{\omega\varepsilon}\nabla\times\mathbf{H}$) in our Hilbert space (i.e., satisfying our boundary conditions). Now, if we have a period-$a$ unit cell $x \in [0,a]$, two of these boundary terms are at $x = 0$ (with $d\mathbf{A} = -\hat{x}dydz$) and at $x = a$ (with $d\mathbf{A} = +\hat{x}dydz$). The $[\cdots]$ integrand terms are $E_y' H_z^* - E_z' H_y^* + E_y^* H_z' - E_z^* H_y'$ . However, this integrand is *exactly the same* at $x = 0$ and $x = a$, because *both* the $E$ and $H$ terms differ by $e^{ika}$ at $x = a$ (compared to $x = 0$) for Bloch-periodic boundary conditions, but the complex conjugations cause these phase factors to *cancel*. Due to the sign flip in $d\mathbf{A} = \pm\hat{x}dydz$ at the two boundaries, that means that the $x = 0$ and $x = a$ boundary integrals *sum to zero*.

Therefore, $\hat{\Theta}$ is still Hermitian with Bloch-periodic boundaries in $x$, assuming that the $y$ and $z$ boundary terms also vanish (e.g. they are periodic too, or PEC, or ...).

(b) $k$ and $k + \frac{2\pi}{a}$ give the same solutions, because they are *the same boundary conditions* $\mathbf{H}(x+a,y,z) = e^{ika}\mathbf{H}(x,y,z)$: $e^{ika}$ is the same if you shift $k$ by $2\pi/a$ .

## Problem 2: Periodic waveguide guidance proof (10+15 points)

In both parts of this problem, similar to class, we need to prove that the Rayleigh quotient satisfies $\langle H, \hat{\Theta}_{\mathbf{k}} H \rangle / \langle H, H \rangle < k^2$ for some trial function $H$, or equivalently that

$$\int_0^a \int_{-\infty}^{\infty} (1-\Delta) \left|(\nabla + i\mathbf{k}) \times \mathbf{H_k}\right|^2 dx\,dy - k^2 \int_0^a \int_{-\infty}^{\infty} |\mathbf{H_k}|^2 \,dx\,dy < 0$$

for the trial Bloch envelope $\mathbf{H_k} = \mathbf{H}e^{-ikx}$, $\mathbf{k} = k\hat{\mathbf{x}}$, and $\varepsilon^{-1} = 1 - \Delta$.

(a) We will choose $u(x,y) = e^{-|y|/L}$ for some $L > 0$, exactly as in class—that is, it is the simplest conceivable periodic function of $x$, a constant. Thus,

$\int |u|^2 = 2a \int_0^\infty e^{-2y/L} dy = aL$ over the unit cell. In this case, the variational criterion above becomes, exactly as in class except for the factor of $a$:

$$\begin{aligned}
\int_0^a \int_{-\infty}^{\infty} (1-\Delta)\left(k^2 + L^{-2}\right) e^{-2|y|/L} dx dy - k^2 a L &< 0 \\
&= \frac{a}{L} - \int_0^a \int_{-\infty}^{\infty} \Delta \cdot \left(k^2 + L^{-2}\right) e^{-2|y|/L} dx\, dy,
\end{aligned}$$

which becomes negative in the limit $L \to \infty$ thanks to our assumption that $\int_0^a \int_{-\infty}^{\infty} \Delta(x,y)\, dx\, dy > 0$. Note that the fact that $\int |\Delta| < \infty$ ensures that we can interchange the limits and integration, via the dominated convergence theorem discussed in class.

(b) Let us guess that we can choose $u(y)$ and $v(y)$ to be functions of $y$ only (i.e., again the trivial constant-function periodicity in $x$). The fact that $\nabla \cdot \mathbf{H} = 0$ implies that $(\nabla + i\mathbf{k}) \cdot [u(y)\hat{\mathbf{x}} + v(y)\hat{\mathbf{y}}] = 0 = iku + v'$, and therefore $u = iv'/k$. Therefore, it is convenient to choose $v(y)$ to be a smooth function so that $u$ is differentiable. Let us choose

$$v(y) = e^{-y^2/2L^2}$$

in which case $u(y) = -\frac{iy}{kL^2} e^{-y^2/2L^2}$. Recall the Gaussian integrals $\int_{-\infty}^{\infty} e^{-y^2/L^2} dy = L\sqrt{\pi}$ and $\int_{-\infty}^{\infty} y^2 e^{-y^2/L^2} dy = L^3\sqrt{\pi}/2$. So, $\int |\mathbf{H}|^2 = a \int |u|^2 + |v|^2 = aL\sqrt{\pi}[1 + \frac{1}{k^2L^2}]$. Also, $(\nabla + i\mathbf{k}) \times [u(y)\hat{\mathbf{x}} + v(y)\hat{\mathbf{y}}] = (ikv - u')\hat{\mathbf{z}}$. So,

$$|\nabla \times \mathbf{H}|^2 = |(\nabla + i\mathbf{k}) \times \mathbf{H_k}|^2 = |u'|^2 + k^2|v|^2 = k^2 \left[1 + \frac{1}{k^4L^4}\left(1 - \frac{y^2}{L^2}\right)\right] e^{-y^2/L^2}.$$

Then, if we look at our variational criterion, we have two terms: $\int |\nabla \times \mathbf{H}|^2$ and $-\int \Delta \cdot |\nabla \times \mathbf{H}|^2$. Again, we can swap limits with integration in the latter by the dominated convergence theorem. Combining the former with the $-k^2 \int |\mathbf{H}|^2$ term in the variational criterion, we get:

$$\begin{aligned}
\int |\nabla \times \mathbf{H}|^2 - k^2 \int |\mathbf{H}|^2 &= a \int_{-\infty}^{\infty} k^2 \left[1 + \frac{1}{k^4L^4}\left(1 - \frac{y^2}{L^2}\right)\right] e^{-y^2/L^2} dy - k^2 aL\sqrt{\pi}\left[1 + \frac{1}{k^2L^2}\right] \\
&= a \int_{-\infty}^{\infty} \frac{k^2}{k^4L^4}\left(1 - \frac{y^2}{L^2}\right) e^{-y^2/L^2} dy - \frac{k^2 aL\sqrt{\pi}}{k^2L^2} \\
&= \frac{a}{k^2L^4} L\sqrt{\pi}\left(1 - \frac{L^2}{2L^2}\right) - \frac{a\sqrt{\pi}}{L},
\end{aligned}$$

which goes to zero as $L \to \infty$. Thus:

$$\int (1-\Delta)\, |(\nabla + i\mathbf{k}) \times \mathbf{H_k}|^2 - k^2 \int |\mathbf{H_k}|^2 \to -k^2 \int_0^a \int_{-\infty}^{\infty} \Delta(x,y)\, dx\, dy < 0.$$

as $L \to \infty$. Q.E.D.