18.369 Problem Set 1

Problem 1: Adjoints and operators

(a) We defined the adjoint $\dagger$ of operators $\hat{O}$ by: $\langle H_1, \hat{O}H_2\rangle = \langle \hat{O}^\dagger H_1, H_2\rangle$ for all H_1 and H_2 in the vector space. Show that for a *finite-dimensional* Hilbert space, where H is a column vector h_n ($n = 1, \cdots, d$), $\hat{O}$ is a square $d \times d$ matrix, and $\langle H^{(1)}, H^{(2)}\rangle$ is the ordinary conjugated dot product $\sum_n h_n^{(1)*} h_n^{(2)}$, the above adjoint definition corresponds to the conjugate-transpose for matrices. (Thus, as claimed in class, "swapping rows and columns" is the *consequence* of the "real" definition of transposition/adjoints, not the source.)

Note: In the **subsequent** parts of this problem, you may *not* assume that $\hat{O}$ is finite-dimensional (nor may you assume any specific formula for the inner product). Use only the abstract definitions of adjoint and linear operators on Hilbert spaces, along with the key properties of inner products: $\langle u,v\rangle = \langle v,u\rangle^*$, $\langle u, \alpha v + \beta w\rangle = \alpha\langle u,v\rangle + \beta\langle u,w\rangle$ (for arbitrary complex scalars α, β), and $\|u\|^2 = \langle u,u\rangle \geq 0$ ($= 0$ if and only if[1] $u = 0$).

(b) If a linear operator $\hat{O}$ satisfies $\hat{O}^\dagger = \hat{O}^{-1}$, then the operator is called **unitary**. Show that a unitary operator preserves inner products (that is, if we apply $\hat{O}$ to every element of a Hilbert space, then their inner products with one another are unchanged). Show that the eigenvalues u of a unitary operator have unit magnitude ($|u| = 1$) and that its eigenvectors can be chosen to be orthogonal to one another.

(c) For a non-singular operator $\hat{O}$ (i.e. $\hat{O}^{-1}$ exists), show that $(\hat{O}^{-1})^\dagger = (\hat{O}^\dagger)^{-1}$. (Thus, if $\hat{O}$ is Hermitian then $\hat{O}^{-1}$ is also Hermitian.)

[1] Technically, we mean $u = 0$ "almost everywhere" (e.g. excluding isolated points).

Problem 2: Maxwell eigenproblems

(a) As in class, assume $\varepsilon(\mathbf{x})$ real and positive (and that all function spaces are chosen so that the integrals you need exist etc.). In class, we eliminated $\mathbf{E}$ from Maxwell's equations to get an eigenproblem in $\mathbf{H}$ alone, of the form $\hat{\Theta}\mathbf{H}(\mathbf{x}) = \frac{\omega^2}{c^2}\mathbf{H}(\mathbf{x})$. Show that if you instead eliminate $\mathbf{H}$, you *cannot* get a Hermitian eigenproblem in $\mathbf{E}$ for the usual inner product $\langle \mathbf{E}_1, \mathbf{E}_2\rangle = \int \mathbf{E}_1^* \cdot \mathbf{E}_2$ except for the trivial case $\varepsilon =$ constant. Instead, show that you get a *generalized Hermitian eigenproblem*: an equation of the form $\hat{A}\mathbf{E}(\mathbf{x}) = \frac{\omega^2}{c^2}\hat{B}\mathbf{E}(\mathbf{x})$, where *both* $\hat{A}$ and $\hat{B}$ are Hermitian operators.

(b) For *any* generalized Hermitian eigenproblem where $\hat{B}$ is positive definite (i.e. $\langle \mathbf{E}, \hat{B}\mathbf{E}\rangle > 0$ for all $\mathbf{E}(\mathbf{x}) \neq 0$), show that the eigenvalues (i.e., the solutions of $\hat{A}\mathbf{E} = \lambda\hat{B}\mathbf{E}$) are real and that different eigenfunctions $\mathbf{E}_1$ and $\mathbf{E}_2$ satisfy a modified kind of orthogonality. Show that $\hat{B}$ for the $\mathbf{E}$ eigenproblem above was indeed positive definite.

(c) Alternatively, show that $\hat{B}^{-1}\hat{A}$ is Hermitian under a modified inner product $\langle \mathbf{E}, \mathbf{E}'\rangle_B = \langle \mathbf{E}, \hat{B}\mathbf{E}'\rangle$ for Hermitian $\hat{A}$ and $\hat{B}$ and positive-definite $\hat{B}$ with respect to the original $\langle \mathbf{E}, \mathbf{E}'\rangle$ inner product; the results from the previous part then follow.

(d) Show that *both* the $\mathbf{E}$ and $\mathbf{H}$ formulations lead to generalized Hermitian eigenproblems (or. equivalently, Hermitian with a modified inner product) with real ω if we allow magnetic materials $\mu(\mathbf{x}) \neq 1$ (but require μ real, positive, and independent of $\mathbf{H}$ or ω).

(e) μ and ε are only ordinary numbers for *isotropic* media. More generally, they are 3×3 matrices (technically, rank 2 tensors)—thus, in an *anisotropic medium,* by putting an applied field in one direction, you can get dipole moment in different direction in the material. What conditions on these 3×3 matrices still give a generalized Hermitian eigenproblem in $\mathbf{E}$ (or $\mathbf{H}$) with real eigen-frequencies ω?

Problem 3: Dispersion

In class, we considered the simplified case where the electric polarization density $\mathbf{P}$ responds *instantaneously* and linearly to the electric field at the same point in space: $\mathbf{P}(\mathbf{x}) = \chi_e \mathbf{E}(\mathbf{x})$ for some constant "susceptibility" $\chi_e = \varepsilon - 1$. More generally, however, changing $\mathbf{P}$ involves physically moving electric charges, and so it cannot happen instantaneously.

In particular, let's imagine a simple model in which the charges are attached to immobile atoms with linear springs, so that they can bounce back and forth with some frequency ω_0 in the absence of external forces. (This actually turns out to be a reasonable semiclassical model of an atom with an "electric dipole transition" energy difference $\hbar\omega_0$ between a ground state and an excited state.) Of course, there also needs to be some "friction" so that the atom slowly settles down (to its ground state) when it is left alone. Your task is to turn this model into equations, and see how it plugs into Maxwell's equations:

(a) At a given point $\mathbf{x}$ in space, $\ddot{\mathbf{P}}$ is proportional to the acceleration of charges. Write down Newton's law ($F = ma$) of motion for the charges, assuming that there are three forces acting on the charge: a "spring" restoring force proportional to $-\mathbf{P}$, a "friction" (dissipation) force proportional to $-\dot{\mathbf{P}}$, and a "driving" force proportional to $\mathbf{E}$ (all terms at $\mathbf{x}$). You should get an ODE with two unknown coefficients (but with known signs: make all of your unknown constants *positive*), along with one term proportional to ω_0^2 (make sure the oscillation frequency in the *absence* of the friction and driving terms is ω_0).

(b) Assuming time-harmonic fields $\mathbf{E}$ and $\mathbf{P}$ proportional to $e^{-i\omega t}$, show that you obtain the same Maxwell equations as in class but with a *frequency-dependent* permittivity $\varepsilon(\omega, \mathbf{x})$ (where the $\mathbf{x}$ dependence arises if your unknown coefficients above vary with $\mathbf{x}$).

(c) Sketch a plot of $\operatorname{Re}\varepsilon$ and $\operatorname{Im}\varepsilon$ versus ω. $\operatorname{Im}\varepsilon$ should have a peak—what parameters determine the width, amplitude, and location of this peak?

(d) Check that your equations imply $\omega \operatorname{Im}[\varepsilon] \geq 0\ldots$ we will see later in class that this is a necessary condition for the materials to be *passive* (dissipating energy, not supplying energy).