18.369 Problem Set 4 Solutions

Problem 1: (5+5+10 points)

(a) Let us write $(\hat{A}^{(0)}+\Delta\hat{A})\psi=\lambda(\hat{B}^{(0)}+\Delta\hat{B})\psi$, where $\psi=\psi^{(0)}+\psi^{(1)}+\cdots$ and $\lambda=\lambda^{(0)}+\lambda^{(1)}+\cdots$ are expansions of the new eigensolutions in powers of the perturbation Δ (since we don't have to worry about breaking degeneracies, by assumption). If we keep only the zero-th order terms, we get the unperturbed problem $\hat{A}^{(0)}\psi^{(0)}=\lambda^{(0)}\hat{B}^{(0)}\psi^{(0)}$. If we only keep terms only up to the 1st order, we get:

$$\hat{A}^{(0)}\psi^{(1)}+\Delta\hat{A}\psi^{(0)}=\lambda^{(1)}\hat{B}^{(0)}\psi^{(0)}+\lambda^{(0)}\hat{B}^{(0)}\psi^{(1)}+\lambda^{(0)}\Delta\hat{B}\psi^{(0)}+O(\Delta^2).$$

Taking the inner product with $\psi^{(0)}$ on both sides, and using the Hermitian property to operate $\hat{A}^{(0)}$ to the left and the fact that $\lambda^{(0)}$ is real, the $\lambda^{(0)}\left\langle\psi^{(0)},\hat{B}^{(0)}\psi^{(1)}\right\rangle$ terms cancel on both sides, and we obtain:

$$\lambda^{(1)}=\frac{\left\langle\psi^{(0)},\Delta\hat{A}\psi^{(0)}\right\rangle-\lambda^{(0)}\left\langle\psi^{(0)},\Delta\hat{B}\psi^{(0)}\right\rangle}{\left\langle\psi^{(0)},\hat{B}^{(0)}\psi^{(0)}\right\rangle},$$

which is the generalized version of first-order perturbation theory.

(b) For $\nabla\times\nabla\times\mathbf{E}=\frac{\omega^2}{c^2}\varepsilon\mathbf{E}$ with a small change $\Delta\varepsilon$, we have $\Delta\hat{A}=0$ and $\Delta\hat{B}=\Delta\varepsilon$. Also, to first order, $\Delta(\frac{\omega^2}{c^2})=2\frac{\omega\Delta\omega}{c^2}=\lambda^{(1)}$. Plugging this in above and dividing through by $2\omega/c^2$, we have:

$$\Delta\omega=-\frac{\omega}{2}\frac{\left\langle\mathbf{E}^{(0)},\Delta\varepsilon\mathbf{E}^{(0)}\right\rangle}{\left\langle\mathbf{E}^{(0)},\varepsilon\mathbf{E}^{(0)}\right\rangle},$$

which is the same as the expression we derived from the $\mathbf{H}$ eigenproblem in class.

Problem 3: (10+5+5 points)

See solutions notebook.

Problem 4: (5+5+5+5 points)

See solutions notebook.