18.369 Problem Set 3

Due Thursday, 12 March 2020.

Problem 0: Scattering and energy

As in the course notes, we can write Maxwell's equations for the 6-component fields ψ as $\frac{\partial \psi}{\partial t} = \hat{C}\psi - \frac{\partial \phi}{\partial t} - \xi$, where the 6-component polarizations $\phi = \chi * \psi$ are a convolution in time with a (passive and causal) linear susceptibility $\chi(\mathbf{x},t)$ at each point $\mathbf{x}$ in space. Now, suppose we have the situation depicted in Fig. 1: sources ($\xi \neq 0$) lying *outside* some bounded volume Ω produce an incident wave ψ_{inc} that scatters off materials ($\chi \neq 0$) *inside* of Ω to produce a scattered wave ψ_{scat}. Outside of Ω there are no materials ($\chi = 0$), while inside of Ω there are no sources ($\xi = 0$).

(a) Write the total fields as $\psi = \psi_{\text{inc}} + \psi_{\text{scat}}$, where $\psi_{\text{inc}} = \begin{pmatrix} \mathbf{E}_{\text{inc}} \\ \mathbf{H}_{\text{inc}} \end{pmatrix}$ satisfies $\frac{\partial \psi_{\text{inc}}}{\partial t} = \hat{C}\psi_{\text{inc}} - \xi$ (Maxwell's equations with the same currents but no materials), and the initial conditions at $t = -\infty$ are $\psi = \psi_{\text{inc}} = \psi_{\text{scat}} = 0$.

   (i) What differential[1] equation must ψ_{scat} satisfy?

   (ii) Fourier transform in time to find equations for $\hat{\psi}_{\text{inc}}(x,\omega)$ and $\hat{\psi}_{\text{scat}}$ (in terms of $\hat{\xi}$ and $\hat{\chi}$); equivalently, look for time-harmonic solutions $\hat{\psi}_{\text{inc}}e^{-i\omega t}$ and $\hat{\psi}_{\text{scat}}e^{-i\omega t}$ from a time-harmonic source $\hat{\xi}(\mathbf{x})e^{-i\omega t}$.

(b) Applying Poynting's theorem and passivity to your time-harmonic fields, show that $\oiint_{\partial\Omega} \operatorname{Re}[\hat{\mathbf{E}}^*_{\text{inc}} \times \hat{\mathbf{H}}_{\text{inc}}] \cdot d\mathbf{A} = 0$ and $P_{\text{abs}} = -\oiint_{\partial\Omega} \operatorname{Re}[\hat{\mathbf{E}}^* \times \hat{\mathbf{H}}] \cdot d\mathbf{A} \geq 0$ (the absorbed power = inward total flux). (Note that here, Ω is a subset of the whole domain!)

(c) In order to ensure $P_{\text{scat}} = \oiint_{\partial\Omega} \operatorname{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A} \geq 0$ (the scattered power = outward scattered flux), we need something more: an *outgoing boundary condition* (or "radiation" boundary condition): if we consider *any point* on sphere of radius R (around any point in Ω), then *in the limit* as $R \to \infty$ we require that $R^2 \operatorname{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}]$ must point *radially outward*. Use Poynting's theorem to show that such a radiation boundary condition implies $\oiint_{\partial\Omega} \operatorname{Re}[\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{scat}}] \cdot d\mathbf{A} \geq 0$.[2]

(d) Show that the "extinguished power" $P_{\text{ext}} = P_{\text{abs}} + P_{\text{scat}}$ can be written in terms of an integral over $\partial\Omega$ involving $\hat{\mathbf{E}}^*_{\text{inc}} \times \hat{\mathbf{H}}_{\text{scat}}$ and $\hat{\mathbf{E}}^*_{\text{scat}} \times \hat{\mathbf{H}}_{\text{inc}}$ *only*. This is a version of a famous result known as the **optical theorem**.

(e) Derive an alternative form of the optical theorem: show that P_{ext} can be written as an integral over Ω involving *only* $\hat{\psi}^*_{\text{inc}}\hat{\phi}$. (Hint: use Poynting's theorem to relate P_{abs} and P_{scat} to work on polarization currents, and write $\psi_{\text{scat}} = \psi - \psi_{\text{inc}}$.)

[1] Technically, if you include a convolution $\chi *$ then you have a "pseudodifferential" operator.

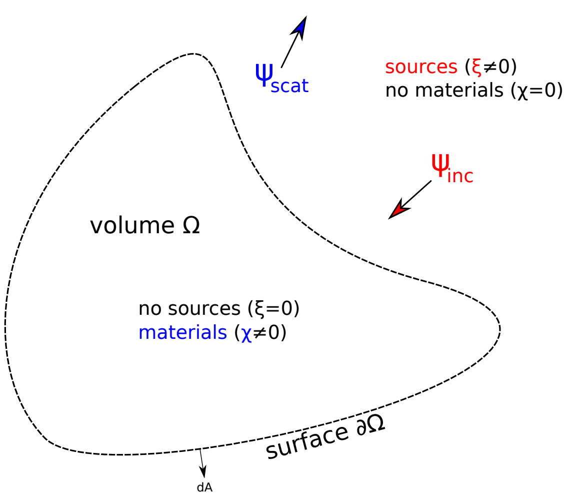

Figure 1: Schematic of an incident wave ψ_{inc} created by sources ξ that are only $\neq 0$ *outside* some volume Ω, interacting with materials $\chi \neq 0$ that lie *inside* Ω to produce an outgoing scattered wave ψ_{scat}.

Problem 1: Periodic waveguides

In class, we showed by a variational proof that any $\varepsilon(y)$, in two dimensions, gives rise to at least one guided mode whenever $\varepsilon(y)^{-1} = \varepsilon_{\text{lo}}^{-1} - \Delta(y)$ for $\int \Delta > 0$ and $\int |\Delta| < \infty$.[3] At least, we showed it for the TE polarization ($\mathbf{H}$ in the $\hat{\mathbf{z}}$ direction). Now, you

[2] We also will also need some boundary condition to uniquely determine ψ_{inc}. Not necessarily an outgoing boundary condition, however—often it is convenient to have an incident wave coming in "from infinity", such as an incident planewave, perhaps with no explicit sources ($\xi = 0$) and just appropriate boundary conditions.

[3] As in class, the latter condition on Δ will allow you to swap limits and integrals for any integrand whose magnitude is bounded above by some constant times $|\Delta|$ (by Lebesgue's dominated convergence theorem).

will show the same thing much more generally, but using the same basic technique.

(a) Let $\varepsilon(x,y)^{-1} = 1 - \Delta(x,y)$ be a periodic function $\Delta(x,y) = \Delta(x+a,y)$, with $\int |\Delta| < \infty$ and $\int_0^a \int_{-\infty}^{\infty} \Delta(x,y)dxdy > 0$. Prove that at least one TE guided mode exists, by choosing an appropriate (simple!) trial function of the form $\mathbf{H}(x,y) = u(x,y)e^{ikx}\hat{\mathbf{z}}$. That is, show by the variational theorem that $\omega^2 < c^2k^2$ for the lowest-frequency eigenmode. (It is sufficient to show it for $|k| \le \pi/a$, by periodicity in k-space; for $|k| > \pi/a$, the light line is not $\omega = c|k|$.)

(b) Prove the same thing as in (a), but for the TM polarization ($\mathbf{E}$ in the $\hat{\mathbf{z}}$ direction). Hint: you will need to pick a trial function of the form $\mathbf{H}(x,y) = [u(x,y)\hat{\mathbf{x}} + v(x,y)\hat{\mathbf{y}}]e^{ikx}$ where u and v are some (simple!) functions such that $\nabla \cdot \mathbf{H} = 0$.[4]

Problem 2: 2d Waveguide Modes

Consider the two-dimensional dielectric waveguide of thickness h that we first introduced in class:

$$\varepsilon(y) = \begin{cases} \varepsilon_{hi} & |y| < h/2 \\ \varepsilon_{lo} & |y| \ge h/2 \end{cases},$$

where $\varepsilon_{hi} > \varepsilon_{lo}$. Look for solutions with the "TM" polarization $\mathbf{E} = E_z(x,y)\hat{\mathbf{z}}e^{-i\omega t}$. The boundary conditions are that E_z is continuous and $\partial E_z/\partial y$ ($\sim H_x$) is continuous, and that we require the fields to be finite at $x, y \to \pm\infty$,

(a) Prove that we can set $\varepsilon_{lo} = 1$ without loss of generality, by a change of variables in Maxwell's equations. In the subsequent sections, therefore, set $\varepsilon_{lo} = 1$ for simplicity.

(b) Find the guided-mode solutions $E_z(x,y) = e^{ikx}E_k(y)$, where the corresponding eigenvalue $\omega(k) < ck$ is below the light line.

  (i) Show for the $|y| < h/2$ region the solutions are of sine or cosine form, and that for $|y| > h/2$ they are decaying exponentials. (At this point, you can't easily prove that the arguments of the sines/cosines are real, but that's okay—you will be able to rule out the possibility of imaginary arguments below.)

  (ii) Match boundary conditions (E_z and H_x are continuous) at $y = \pm h/2$ to obtain an equation relating ω and k. You should get a transcendental equation that you cannot solve explicitly. However, you can "solve" it graphically and learn a lot about the solutions—in particular, you might try plotting the left and right hand sides of your equation (suitably arranged) as a function of $k_\perp = \sqrt{\frac{\omega^2}{c^2}\varepsilon_{hi} - k^2}$, so that you have two curves and the solutions are the intersections (your curves will be parameterized by k, but try plotting them for one or two typical k).

  (iii) From the graphical picture, derive an exact expression for the number of guided modes as a function of k. Show that there is exactly one guided mode, with even symmetry, as $k \to 0$, as we argued in class.

[4] You might be tempted, for the TM polarization, to use the $\mathbf{E}$ form of the variational theorem that you derived in problem 1, since the proof in that case will be somewhat simpler: you can just choose $\mathbf{E}(x,y) = u(x,y)e^{ikx}\hat{\mathbf{z}}$ and you will have $\nabla \cdot \varepsilon\mathbf{E} = 0$ automatically. However, this will lead to an inequivalent condition $\int(\varepsilon - 1) > 0$ instead of $\int \Delta = \int \frac{\varepsilon - 1}{\varepsilon} > 0$.

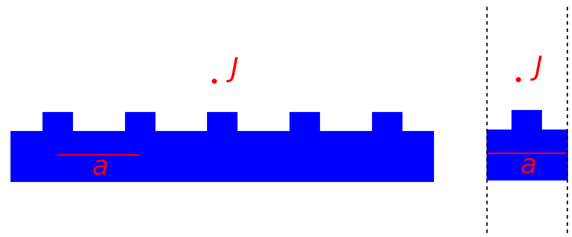

Figure 2: Schematic for problem 1. *Left*: a time-harmonic point source $\mathbf{J}$ above a periodic surface. *Right:* the problem can be reduced to solving a set of problems with point sources in a single unit cell, with periodic boundary conditions on the fields.

Problem 3: Point sources & periodicity

Suppose we are in 2d (xy plane), working with the TM polarization ($\mathbf{E}$ out of plane), and have a periodic (period a) surface shown in Fig 1(left). Above the surface is a time-harmonic point source $\mathbf{J} = \delta(x)\delta(y)e^{-i\omega t}\hat{z}$ (choosing the origin to be the location of the point source, for convenience). As you saw in pset 2, you can define a frequency-domain problem $(\nabla\times\nabla\times - \omega^2\varepsilon)\mathbf{E} = i\omega\mathbf{J}$ (setting $\mu_0 = \varepsilon_0 = 1$ for convenience) for the time-harmonic fields in response to this current.

In this problem, you will explain how to take advantage of the fact that the structure (but not the source or fields!) is periodic, by reducing it to a set of problems of the form shown in Fig. 1(right): solving for the fields of the *same* point source $\mathbf{J}$, but in a *single unit cell* of the structure with *Bloch-periodic boundary conditions* on the fields.

(a) Show that the total resulting electric field $\mathbf{E}$can be written as a superposition of solutions $\mathbf{E}_k$ to $(\nabla\times\nabla\times - \omega^2\varepsilon)\mathbf{E}_k = i\omega\mathbf{J}$ in a unit-cell domain with Bloch-periodic boundary conditions. Hint, the following identity is useful:

$$\delta(x) = \frac{a}{2\pi}\int_0^{2\pi/a}\left[\sum_{n=-\infty}^{\infty}\delta(x-na)e^{ikna}\right]dk$$

and recall conservation of irrep.

(b) Instead of considering an infinite domain, it is often useful to consider a "supercell:" suppose that we have N periods, with periodic boundary conditions $\mathbf{E}(x+Na) = \mathbf{E}(x)$ and $\mathbf{J}(x+Na) = \mathbf{J}(x)$. Show that we can *still* write this $\mathbf{E}$ as a superposition of Bloch-periodic $\mathbf{E}_k$ solutions in individual unit cells, exactly as in the previous part, but with some restriction on k.

(c) Suppose that we want to compute the radiated power P (per unit z) from $\mathbf{J}$ by integrating the Poynting flux through a plane above the current ($y = y_0 > 0$):

$$P = \frac{1}{2}\int_{-\infty}^{\infty}\hat{y}\cdot\mathrm{Re}\left[\mathbf{E}^*(x,y_0)\times\mathbf{H}(x,y_0)\right]dx.$$

Show that $P = \frac{a}{2\pi}\int_0^{2\pi/a}P_k dk$, a simple integral of powers P_k computed *separately* for each periodic subproblem above.

(Hint: orthogonality of partner functions. These infinite integrals can be somewhat awkward to work with, so it might be easier to consider P for a supercell with N periods, and then take the limit as $N\to\infty$ of your final expression.)