

Superfluid Dark Matter around Black Holes

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Abstract

Superfluid dark matter, consisting of self-interacting light particles that thermalize and condense to form a superfluid in galaxies, provides a novel theory that matches the success of the standard Λ CDM model on cosmological scales while simultaneously offering a rich phenomenology on galactic scales. Within galaxies, the dark matter density profile consists of a nearly homogeneous superfluid core surrounded by an isothermal envelope. In this work we compute the density profile of superfluid dark matter around supermassive black holes at the center of galaxies. We show that, depending on the fluid equation of state, the dark matter profile presents distinct power-law behaviors, which can be used to distinguish it from the standard results for collisionless dark matter.

1 Introduction

The standard Λ Cold Dark Matter (Λ CDM) model, in which dark matter (DM) consists of non-relativistic, purely gravitationally-interacting particles, fits extremely well the background expansion history, the shape of the cosmic microwave background and matter power spectra, as well as the abundance and mass function of galaxy clusters, with strong evidence coming from observations on the largest scales.

However, as simulations and observations of galaxies have improved, a number of challenges appeared when one focuses on galactic scale phenomena [1]. In particular, in the absence of baryons, the numerically predicted Navarro-Frank-White (NFW) density profile [2] in inner regions of galaxies and clusters seems to mismatch the one inferred from observations, giving rise to the so-called “core-cusp problem”. Once baryons are included in simulations [3], they play a crucial role in modifying the cuspy cold dark matter distribution to cored profiles, even though assumptions have to be made about the stellar evolution history of galaxies.

Furthermore, striking correlations between the gravitational acceleration of baryons and their distribution in disk galaxies are found [4, 5], which represent a generalized version of the baryonic Tully-Fisher relation connecting the fourth power of the asymptotic circular velocity to the total baryonic mass of disk galaxies [6–9]. Much progress has been made to derive such correlations in the context of CDM, though their small scatter remains puzzling [10–12]. Alternatively, the observed galactic relations may originate from interactions between baryons and dark matter [13, 14].

Straightforward extensions to the standard model consist in endowing DM with self-interactions

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that are strong enough to modify the galactic core [15, 16]. Another well-studied extension is to “fuzz out” the cuspy profiles by considering extremely light DM particles, with mass $m \lesssim 10^{-21}$ eV, called fuzzy dark matter [17, 18]. These ultra-light particles have a de Broglie wavelength of order kiloparsec, thus behaving as a Bose-Einstein condensate on astrophysical scales. (See [19] for a review.)

In this paper we instead focus on a third possibility, represented by superfluid dark matter. Over a certain mass range and scattering length, these bosonic particles can support condensation at high density and low temperature [20]. An enticing phenomenological aspect of superfluid dark matter models is the possibility of phonons mediating long-range interactions between baryons [21–25]. The simplest superfluid theory, with quartic self-interactions, was initially thought to be ruled out in Ref. [26], using observations of galactic rotation curves and Bullet Cluster constraints [27, 28]. However, the idea has been revitalized recently in Refs. [29–31], by relaxing the simplifying assumptions of global thermal equilibrium and spherical symmetry made in Ref. [26].

One of the most compelling features of most galaxies, including the Milky Way [32–34], is that they are believed to host a supermassive black hole (SMBH) at their center [35, 36]. This hypothesis may apply as well to other bound structures containing dark matter, such as galaxy clusters or satellite systems. These SMBHs with masses in the range 10^6 – $10^{10} M_\odot$ are believed to power active galactic nuclei and quasars, and may be generated from baryonic processes, such as the collapse of Pop III stars [37] and supermassive stars [38], or primordial black holes [39–41], followed by mergers and baryonic gas accretion, even though their origin is still unknown.

The presence of such black holes is expected to strongly modify the density profile of the DM around it, depending both on the properties of the DM particles and the formation history of the central black hole. For example, for collisionless DM, assuming that the SMBH grows adiabatically from a smaller seed by gas accretion [42], the BH will perturb the particle orbits, resulting into a DM spike characterized by a power-law behavior with slope $-9/4$ in its outer region [43]. On the other hand, the gravitational scattering of stars in the inner region can heat up the DM fluid and give rise to a softened profile with slope $-3/2$ [44–46] (see also Ref. [47] for a recent analysis) or even to disruption [48], while DM annihilations can result into a smoother cusp, with slope $-1/2$ [49, 50]. Other spikes can be obtained for alternative BH formation histories, such as from direct collapse of gas inside DM halos [51], from the growth of inspiralling seed [52], or in the presence of DM self-interactions [53–56]. Focusing on the latter, finally, the presence of particle collisions will wash out any initial and/or adiabatically altered particle distribution on a collisional relaxation time scale, and the cusp will follow the power law behavior with slope $-3/4$ as found in Ref. [54].

In this work we will evaluate the profile of superfluid DM around massive black holes. In Sec. 2 we will review the basic formalism of the superfluid DM model. In Section 3 we will calculate the DM density profile in different regions around SMBHs. In Sec. 4 we put these results together to discuss the entire DM density profile and corresponding mass function. Section 5 is devoted to the conclusions. In the following we use natural units $\hbar = c = 1$.

2 Brief review of superfluid dark matter

In this section we review the basic notions of the theory of superfluid DM. Following Refs. [21–24, 29, 30], DM forms a superfluid inside galaxies with a coherence length of order the size of galaxies. These DM particles should be light and characterized by sufficiently strong self-interactions, such that they thermalize and create a core at the center of galaxies.

In a theory of self-interacting bosons, a superfluid state may be obtained through Bose-Einstein condensation, which occurs if two requirements are fulfilled. The first requirement amounts to demanding that the de Broglie wavelength, $\lambda_{\text{dB}} \sim 1/mv$, of DM particles overlap in the (cold and dense enough) central region of galaxies. That is, λ_{dB} must be larger than the average interparticle separation, $d \sim (m/\rho)^{1/3}$, in terms of the DM density profile $\rho(r)$ of the galaxy and the one-dimensional DM velocity dispersion $v(r)$ before the phase transition. This condition implies an upper bound on the DM mass

$$m \lesssim \left(\frac{\rho_v}{v_v^3} \right)^{1/4}, \quad (1)$$

which is evaluated at virialization. This latter process occurs, according to standard collapse theory, when the overdensity is about 200 times the present critical density, such that [57]

$$\begin{aligned} \rho_v &= 200 \frac{3H_0^2}{8\pi G} \simeq 1.95 \times 10^{-27} \text{ g/cm}^3; \\ R_v &= \left(\frac{3M_{\text{DM}}}{4\pi\rho_v} \right)^{1/3} \simeq 203 \left(\frac{M_{\text{DM}}}{10^{12} M_\odot} \right)^{1/3} \text{ kpc}; \\ v_v &= \sqrt{\frac{1}{3} \frac{GM_{\text{DM}}}{R_v}} \simeq 85 \left(\frac{M_{\text{DM}}}{10^{12} M_\odot} \right)^{1/3} \text{ km/s}, \end{aligned} \quad (2)$$

where we have assumed $H_0 = 70 \text{ km s}^{-1} \text{Mpc}^{-1}$ for concreteness, and denoted with M_{DM} the mass of the DM halo. Substituting these expressions into Eq. (1) gives

$$m \lesssim 2.3 \left(\frac{M_{\text{DM}}}{10^{12} M_\odot} \right)^{-1/4} \text{ eV}. \quad (3)$$

This condition implies that only light enough particles might have de Broglie wavelenghts which overlap significantly at the center of galaxies, therefore forming a Bose-Einstein condensate, while heavy particles do not. The corresponding wavefunction would describe the gas of particles as a whole system rather than treating its particles individually. When, at fixed density ρ , the system temperature T drops below the critical temperature

$$T_c = \frac{2\pi}{m^{5/3}k_B} \left(\frac{\rho}{\zeta(3/2)} \right)^{2/3}, \quad (4)$$

where ζ denotes the Riemann zeta function, and k_B indicates the Boltzmann constant, a macroscopically large number of particles occupy the ground state, giving rise to a Bose-Einstein condensate. We will denote this condition as *degeneracy*.

The second condition requires DM *thermalization* within galaxies, that is, bosons reach their maximal entropy state via interactions. Since high densities facilitate equilibration by enhancing

interaction rates, thermalization is expected to be more efficient in the core of the environment rather than in its outskirts. Thermalization is therefore achieved when particles within a thermal radius have experienced at least one interaction over the galaxy lifetime t_g , which is of order 13 Gyr for characteristic galaxies.¹ This gives the condition

$$\Gamma t_g \gtrsim 1, \quad (5)$$

where Γ denotes the interaction rate. The latter depends on the theory at hand, and can be approximated as [58]

$$\Gamma = (1 + \mathcal{N}) \frac{\sigma}{m} \rho v, \quad (6)$$

in terms of the scattering cross section σ , and the Bose-enhancement factor

$$\mathcal{N} = \frac{\rho}{m} \left(\frac{2\pi}{mv} \right)^3, \quad (7)$$

which accounts for the fact that particles are scattering into a highly degenerate phase space. Equation (5) translates into a lower bound on the scattering cross section:

$$\frac{\sigma}{m} \gtrsim \left(\frac{m}{\text{eV}} \right)^4 \left(\frac{M_{\text{DM}}}{10^{12} M_{\odot}} \right)^{2/3} 0.1 \frac{\text{cm}^2}{\text{g}}. \quad (8)$$

For $m \lesssim \text{eV}$, this satisfies constraints [59–61] on the cross section of self-interacting DM [15].

An upper bound on DM self-interactions comes from the analysis of the Bullet Cluster event [27, 28], where in the collision of two clusters the gas component is observed to be displaced with respect to the DM component. In this case, the mean number of scatterings that a DM particle may undergo while passing through the target cluster must be negligible, giving rise to the well-known limit

$$\frac{\sigma}{m} \lesssim \frac{\text{cm}^2}{\text{g}}, \quad (9)$$

under the assumption of non-degeneracy and 2-body scattering.

The superfluid nature of DM dramatically modifies its macroscopic behavior. In particular, instead of behaving as individual collisionless particles, the condensate behaves like a homogeneous classical field configuration with finite number density, characterized by a low-energy spectrum of excitations around the homogeneous condensate, *i.e.*, phonon degrees of freedom. Such excitations can mediate an emergent long-range interaction between baryons in the superfluid phase, even though the form of these interactions is model-dependent [25].

In the following we will consider two different Lagrangians for the interacting particles, giving rise to different equations of state $P(\rho)$, that are ultimately responsible for the nature of the condensate.

¹If phase-space reshuffling due to dynamical effects is not negligible, then the dynamical time scale $t_{\text{dyn}} = r/v$ would be more appropriate.

2.1 Two-body interacting superfluid

One of the simplest interacting DM models consists of a non-relativistic massive complex scalar field ψ minimally coupled to gravity and characterized by quartic self-interaction of the form

$$\mathcal{H}_2 = \int d^3x \left[-\frac{1}{2m} \psi^\dagger(x) \nabla^2 \psi(x) + \frac{1}{2} g_2 \psi^\dagger(x) \psi^2(x) \right]. \quad (10)$$

The parameter g_2 controls the strength of the contact interactions. Upon Bose-Einstein condensation this system can exhibit superfluidity, provided that $g_2 > 0$. At zero temperature, the superfluid equation of state is given by

$$P_2 = \frac{g_2 \rho^2}{2m^2} = \frac{2\pi a}{m^3} \rho^2, \quad (11)$$

where we have introduced the $2 \rightarrow 2$ scattering length

$$a = \frac{mg_2}{4\pi}. \quad (12)$$

The scattering length can be constrained by the upper limit on the cross section from the Bullet Cluster event (Eq. (9)). Using $\sigma = 4\pi a^2$, this gives

$$a \lesssim 10^{-7} \sqrt{\frac{m}{\mu\text{eV}}} \text{ fm}. \quad (13)$$

The sound speed of phonon excitations is as usual given by

$$c_s^2 = \frac{\partial P_2}{\partial \rho} = \frac{4\pi a}{m^3} \rho. \quad (14)$$

The non-relativistic approximation inherent in Eq. (10) is valid as long as $c_s \ll 1$. Furthermore, both the equation of state and sound speed receive thermal corrections proportional to the temperature of the system, suppressed by T/T_c [24]. We ignore such corrections for simplicity.

2.2 Three-body interacting superfluid

Going beyond the theory with quartic self-interactions, one can consider superfluids with predominantly 3-body interactions, which is the relevant case for the original DM superfluidity scenario [22]. This corresponding Hamiltonian for a complex scalar field involves a hexic potential²

$$\mathcal{H}_3 = \int d^3x \left[-\frac{1}{2m} \psi^\dagger(x) \nabla^2 \psi(x) + \frac{1}{3} g_3 \psi^\dagger(x) \psi^3(x) \right], \quad (15)$$

in terms of the coupling constant g_3 controlling the strength of interactions. The corresponding zero-temperature equation of state is given by

$$P_3 = \frac{2g_3 \rho^3}{3m^3} = \frac{\rho^3}{12\Lambda^2 m^6}, \quad (16)$$

²Our primary motivation for considering $P \sim \rho^3$ is the DM superfluidity scenario of Ref. [22]. As emphasized in [22], such an equation of state likely arises from a strongly coupled system. The toy Hamiltonian of Eq. (15) is only intended to illustrate how such equation of state may arise from a microphysical model, and as such we do not consider the bound on its parameters from the Bullet Cluster constraint.

where we have introduced the cut-off scale Λ as

$$g_3 = \frac{1}{8\Lambda^2 m^3}. \quad (17)$$

Following Ref. [22] we have mind the fiducial values $m = \text{eV}$ and $\Lambda = \text{meV}$ for these parameters. In this case the sound speed is given by

$$c_s^2 = \frac{\partial P_3}{\partial \rho} = \frac{\rho^2}{4\Lambda^2 m^6}. \quad (18)$$

3 Density profile around black holes

In this section we compute the density profile of superfluid DM around a central massive black hole, with mass M_{BH} and Schwarzschild radius

$$r_{\text{BH}} = 2GM_{\text{BH}} \simeq 9.6 \cdot 10^{-7} \text{ kpc} \left(\frac{M_{\text{BH}}}{10^{10} M_{\odot}} \right). \quad (19)$$

One expects that the presence of the BH will result in enhanced interactions among DM particles, steepening the corresponding density profile from its nearly homogeneous central core with density ρ_0 . This modification will occur within the BH sphere of influence, described by a radius r_h . This can be estimated as the radius at which the DM potential is equal to the potential induced by the presence of the BH or, equivalently, where the total enclosed DM mass and the BH mass are comparable:

$$r_h = \left(\frac{3M_{\text{BH}}}{4\pi\rho_0} \right)^{1/3} \simeq 5.5 \text{ kpc} \left(\frac{M_{\text{BH}}}{10^{10} M_{\odot}} \right)^{1/3} \left(\frac{\rho_0}{10^{-24} \text{ g/cm}^3} \right)^{-1/3}. \quad (20)$$

At distances larger than r_h , the BH has negligible effect on the superfluid DM profile.

In order to simplify the nature of the problem, we assume that the distribution of DM particles is spherically-symmetric in space, isotropic in velocity, and has relaxed to a near-equilibrium state. We further assume that the BH mass M_{BH} is much smaller than the total integrated DM halo mass M_{DM} , while it dominates the total mass of all particles bound to it in the cusp.

We describe the density profile across three different regions. The outer region, at distances $r \gtrsim r_h$ from the BH, identifies the range where the BH is not effective enough in modifying the DM profile, and the standard results for superfluid DM apply. Specifically, this region describes the superfluid core with radius $R_{2,3}$ and its match to the standard NFW envelope. In the intermediate region, at radii smaller than r_h , the BH starts to modify the DM density profile due to its gravitational well, resulting in deviations from the outer profile. This intermediate region continues until relativistic effects to the DM motion become so important that they modify the properties of the superfluid and its equation of state. In particular, in the three-body interacting superfluid, the condition of degeneracy is found to break down at a certain radius r_{deg} , after which an inner region starts, further modifying the DM density profile.

3.1 Outer region

For distances larger than the BH sphere of influence r_h , the DM density profile should approach the standard results found in the literature, see Refs. [21, 22]. In particular, the DM halo is characterized by a superfluid core, where the condition of degeneracy and thermalization are satisfied, with the ambient temperature being subcritical (due to high density). Within this region, the presence of repulsive self-interactions is crucial to make the core stable. At distances larger than a critical radius, on the other hand, thermal corrections induced from the environment start to be relevant and likely result in an isothermal envelope. The dynamics on these scales is quite complex, due to possible fragmentation and tidal disruption events that can occur [29, 31].

In order to simplify the picture, we will assume that, beyond the superfluid core, the complex distribution of tidal debris can be approximated as an NFW profile [2, 62], as predicted by N -body simulations [16] ignoring the effect of baryons:

$$\rho(r) = \frac{\rho_{\text{NFW}}}{\frac{r}{r_s} \left(1 + \frac{r}{r_s}\right)^2}. \quad (21)$$

The characteristic density ρ_{NFW} and scale parameter r_s depend on the considered halo. Its size is obtained in terms of the virial radius, R_V , at which the mean DM density is about 200 times the critical density. The ratio between these scales defines the concentration parameter, $c = R_V/r_s$, which is related to the total mass by the mass-to-concentration relation [63]. Typical values of these parameters for a Milky Way-like galaxy are $\rho_{\text{NFW}} = 10^{-25} \text{ g/cm}^3$, $c = 6$ and $R_V = 200 \text{ kpc}$.

The DM density profile of the condensate halo can be computed assuming hydrostatic equilibrium. Under spherical symmetry, the pressure and acceleration are related by

$$\frac{1}{\rho(r)} \frac{dP(r)}{dr} = -\frac{d\Phi_{\text{DM}}(r)}{dr} - \frac{d\Phi_{\text{BH}}(r)}{dr} = -\frac{4\pi G}{r^2} \int_0^r dr' r'^2 \rho(r') - \frac{GM_{\text{BH}}}{r^2}, \quad (22)$$

where we have assumed the BH potential to be $\Phi_{\text{BH}} = -GM_{\text{BH}}/r$. The BH sphere of influence r_h defined above describes the distance at which the DM and BH contributions equal each other in the gravitational potential. At larger radii $r \gtrsim r_h$ one can therefore neglect the BH potential contribution on the right-hand side of Eq. (22), such that

$$\frac{1}{\rho(r)} \frac{dP(r)}{dr} \simeq -\frac{4\pi G}{r^2} \int_0^r dr' r'^2 \rho(r'); \quad r \gtrsim r_h. \quad (23)$$

The corresponding profile therefore depends on the equation of state of superfluid DM (*i.e.*, $P_2 \propto \rho^2$ or $P_3 \propto \rho^3$) and, as we will show later, will give rise to a superfluid core of size $R_{2,3}$, respectively.

In what follows, we derive the density profile of the superfluid core under different assumptions for the particle self-interactions.

a) Two-body interacting superfluid: In the case where DM particles interact primarily via two-body interactions, the zero-temperature superfluid equation of state is given by $P_2 \sim \rho^2$, as shown in Eq. (11). Substituting this into Eq. (23), one obtains

$$\frac{4\pi a}{m^3} \frac{d\rho}{dr} = -\frac{4\pi G}{r^2} \int_0^r dr' r'^2 \rho(r'). \quad (24)$$

It is convenient to rewrite this equation in terms of dimensionless variables Ξ_2 and ξ_2 , defined by

$$\Xi_2 \equiv \frac{\rho(r)}{\rho_0}; \quad \xi_2 \equiv \sqrt{\frac{Gm^3}{a}}r. \quad (25)$$

Differentiating Eq. (24) with respect to r , and expressing the result in the new variables, it is straightforward to obtain the $n = 1$ Lane-Emden equation

$$\frac{1}{\xi_2^2} \frac{d}{d\xi_2} \left(\xi_2^2 \frac{d\Xi_2}{d\xi_2} \right) = -\Xi_2. \quad (26)$$

The boundary conditions that must be imposed are: *i*) $\Xi_2(\xi_{2h}) = 1$, to ensure continuity of the density at r_h ; and *ii*) a continuity condition on the derivative

$$\begin{aligned} \Xi'_2(\xi_{2h}) &= -\frac{2}{\xi_{2h}^2} \frac{G^{3/2} M_{\text{BH}} m^{9/2}}{4\pi\rho_0 a^{3/2}} \\ &\simeq -\frac{2 \cdot 10^{-2}}{\xi_{2h}^2} \left(\frac{M_{\text{BH}}}{10^{10} M_{\odot}} \right) \left(\frac{m}{\mu\text{eV}} \right)^{9/2} \left(\frac{a}{10^{-11} \text{fm}} \right)^{-3/2} \left(\frac{\rho_0}{10^{-24} \text{g/cm}^3} \right)^{-1}, \end{aligned} \quad (27)$$

where the factor of 2 in the first line is obtained from the equality of the DM and BH potential at r_h . The corresponding solution is found to vanish at $\xi_2 = \xi_{2h} + \pi$, and approximately given by

$$\rho(r) \simeq \rho_0 \frac{\sin \left[\sqrt{\frac{Gm^3}{a}}(r - r_h) \right]}{\sqrt{\frac{Gm^3}{a}}(r - r_h)}, \quad (28)$$

giving rise to a condensate of size

$$R_2 = \sqrt{\frac{a}{Gm^3}} (\xi_2 - \xi_{2h}) \simeq 55 \text{kpc} \left(\frac{m}{\mu\text{eV}} \right)^{-3/2} \left(\frac{a}{10^{-11} \text{fm}} \right)^{1/2}. \quad (29)$$

In estimating the size of the condensate we have focused on a DM mass and scattering length compatible with the Bullet cluster constraint.

b) Three-body interacting superfluid: Adopting instead the superfluid equation of state $P_3 \sim \rho^3$, given in Eq. (16), one gets the equation

$$\frac{1}{8\Lambda^2 m^6} \frac{d\rho^2}{dr} = -\frac{4\pi G}{r^2} \int_0^r dr' r'^2 \rho(r'). \quad (30)$$

Introducing the dimensionless variables Ξ_3 and ξ_3 as

$$\Xi_3^{1/2} \equiv \frac{\rho(r)}{\rho_0}; \quad \xi_3 \equiv \sqrt{\frac{32\pi G\Lambda^2 m^6}{\rho_0}} r, \quad (31)$$

it is straightforward to obtain in this case the $n = 1/2$ Lane-Emden equation

$$\frac{1}{\xi_3^2} \frac{d}{d\xi_3} \left(\xi_3^2 \frac{d\Xi_3}{d\xi_3} \right) = -\Xi_3^{1/2}. \quad (32)$$

The boundary conditions are $\Xi_3(\xi_{3h}) = 1$, and

$$\begin{aligned}\Xi'_3(\xi_{3h}) &= -\frac{2}{\xi_{3h}^2} \frac{32\sqrt{2\pi}G^{3/2}M_{\text{BH}}\Lambda^3m^9}{\rho_0^{5/2}} \\ &\simeq -\frac{3 \cdot 10^{-2}}{\xi_{3h}^2} \left(\frac{M_{\text{BH}}}{10^{10}M_\odot} \right) \left(\frac{m}{\text{eV}} \right)^9 \left(\frac{\Lambda}{\text{meV}} \right)^3 \left(\frac{\rho_0}{10^{-24}\text{g/cm}^3} \right)^{-5/2}.\end{aligned}\quad (33)$$

The numerical solution is found to vanish at $\xi_3 \simeq \xi_{3h} + 2.75$, and can be analytically fitted as

$$\rho(r) \simeq \rho_0 \cos^{1/2} \left[\frac{\pi(r - r_h)}{2R_3} \right], \quad (34)$$

in terms of the size of the central soliton

$$R_3 = \sqrt{\frac{\rho_0}{32\pi G\Lambda^2 m^6}} (\xi_3 - \xi_{3h}) \simeq 44 \text{ kpc} \left(\frac{m}{\text{eV}} \right)^{-3} \left(\frac{\Lambda}{\text{meV}} \right)^{-1} \left(\frac{\rho_0}{10^{-24}\text{g/cm}^3} \right)^{1/2}. \quad (35)$$

Compared to Eq. (29), we see that the size of the condensate for three-body interactions is comparable to the two-body case. (Notice, in particular, the different fiducial mass of DM particles in the two cases: eV for the cubic case, compared to μeV for the quadratic interactions.) Another key difference is that R_3 depends on the central density ρ_0 , whereas R_2 does not.

3.2 Intermediate region

For radii $r \lesssim r_h$, the presence of the BH modifies the density profile due to its gravitational potential. We can describe a region, dubbed “intermediate”, where the BH mainly dictates the characteristic velocity of gravitationally bound DM particles as

$$v(r) = \sqrt{\frac{GM_{\text{BH}}}{r}}, \quad (36)$$

which implies that DM particles at smaller radii move faster. Assuming energy equipartition, one can ascribe a gas temperature to DM particles via

$$k_B T(r) = \frac{1}{3} m v^2(r), \quad (37)$$

such that the temperature profile in this region is given by

$$T(r) = \frac{1}{3k_B} \frac{GmM_{\text{BH}}}{r}. \quad (38)$$

Within this region, we will assume that the BH influence is, however, not strong enough to modify the equation of state of the DM fluid. In the following we will therefore show how the profile changes assuming different equations of state for the superfluid DM.

Since r_h by definition marks the distance where the DM and BH contributions equal each other, one can make the simplifying assumption that at smaller radii the BH dominates the gravitational potential. The condition of hydrostatic equilibrium, given by Eq. (22), thus becomes

$$\frac{dP(r)}{dr} \simeq -\rho(r) \frac{GM_{\text{BH}}}{r^2}; \quad r \lesssim r_h. \quad (39)$$

However, we will find that the non-relativistic approximation breaks down at sufficiently small radii, when DM velocities approach the speed of light. Anticipating this, one must therefore consider the full general relativity treatment. Neglecting the small contribution to the stress-energy tensor of DM particles orbiting the black hole, one can adopt the Schwarzschild metric to describe the spherically-symmetric space-time as

$$ds^2 = - \left(1 - \frac{2GM_{\text{BH}}}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM_{\text{BH}}}{r}} + r^2 d\Omega^2. \quad (40)$$

The relativistic generalization of Eq. (39) is then given by [54]

$$\frac{dP(r)}{dr} = - \frac{\rho(r) + P(r)}{1 - 2GM_{\text{BH}}/r} \frac{GM_{\text{BH}}}{r^2}. \quad (41)$$

The solution to this equation provides therefore the density profile of the superfluid DM in the intermediate region.

In what follows we will solve Eq. (41) for the two-body and three-body superfluid equations of state. Such treatment is only valid, however, provided that DM is sufficiently cold to be in a superfluid state. If the DM temperature exceeds the critical temperature T_c , given by Eq. (4), then the condition of degeneracy for Bose-Einstein condensation breaks down. In other words, the assumption of Bose-Einstein degeneracy ($T < T_c$) is valid provided that³

$$v^2 < \frac{6\pi\rho^{2/3}}{m^{8/3}\zeta^{2/3}(3/2)}. \quad (42)$$

With the identification $v^2 = GM_{\text{BH}}/r$, degeneracy breaks down at a radius r_{deg} given by

$$r_{\text{deg}} \rho^{2/3}(r_{\text{deg}}) = \frac{GM_{\text{BH}} m^{8/3} \zeta^{2/3} (3/2)}{6\pi}. \quad (43)$$

For distances $r \lesssim r_{\text{deg}}$, which denote the beginning of the “inner region”, the DM component is no longer degenerate, and its equation of state is instead approximated by the ideal gas law, $P \propto \rho v^2$. As we will see below, it may happen that r_{deg} is smaller than the BH horizon, in which case the DM remains in the superfluid state all the way to the BH horizon. Let us stress that, on the other hand, thermalization never breaks down as we approach the BH.

a) Two-body interacting superfluid: Implementing the equation of state $P_2 \propto \rho^2$, given by Eq. (11), one finds that Eq. (41) becomes

$$\frac{d\rho}{dr} = - \frac{1}{1 - 2GM_{\text{BH}}/r} \frac{GM_{\text{BH}}}{r^2} \left(\frac{m^3}{4\pi a} + \frac{\rho}{2} \right). \quad (44)$$

³We use the non-relativistic expression for T_c for simplicity. As the gas becomes relativistic, Eq. (42) will receive corrections. For instance, in the ultra-relativistic regime $T \gg m$, the critical temperature instead scales as $T_c \propto \varrho^{1/3}$ [64], in terms of the charge density ϱ . However, this regime does not apply to the superfluid particles, which are relativistic but not ultra-relativistic, since at most their velocities become $v \simeq 1/2$ close to the BH accretion radius $4GM_{\text{BH}}$.

This gives the profile

$$\rho(r) = \frac{m^3}{2\pi a} \left[\left(\frac{1 - 2GM_{\text{BH}}/r_h}{1 - 2GM_{\text{BH}}/r} \right)^{1/4} - 1 \right] + \left(\frac{1 - 2GM_{\text{BH}}/r_h}{1 - 2GM_{\text{BH}}/r} \right)^{1/4} \rho_0, \quad (45)$$

where we have fixed the integration constant by requiring that $\rho(r_h) = \rho_0$. As expected, the DM density increases with decreasing distance to the BH.

The non-relativistic regime corresponds to large distances, $r, r_h \gg 2GM_{\text{BH}}$, wherein Eq. (45) simplifies to

$$\rho(r) = \frac{GM_{\text{BH}}m^3}{4\pi a} \left(\frac{1}{r} - \frac{1}{r_h} \right) + \rho_0. \quad (46)$$

It is easy to see that this is a solution to Eq. (39). In particular, at short distances the profile becomes a $1/r$ power-law:

$$\rho(r) \simeq \frac{GM_{\text{BH}}m^3}{4\pi ar}. \quad (47)$$

As discussed above, this solution is only valid provided that the DM temperature remains below critical. Substituting the relativistic density profile of Eq. (45) (with $r_h \gg 2GM_{\text{BH}}$ and neglecting the subleading contribution proportional to ρ_0 at small distances), Eq. (42) becomes

$$v^3 < \frac{3\sqrt{6\pi}}{\zeta(3/2)ma} \left[\frac{1}{(1 - 2v^2)^{1/4}} - 1 \right]. \quad (48)$$

In other words, the gas remains degenerate provided that

$$\frac{\zeta(3/2)ma}{3\sqrt{6\pi}} < \frac{1}{v^3} \left[\frac{1}{(1 - 2v^2)^{1/4}} - 1 \right]. \quad (49)$$

It is easy to see that this condition is always satisfied for the characteristic values of the DM mass and scattering length we focus on, implying that the degeneracy condition holds all the way to the radius $4GM_{\text{BH}}$ at which accretion becomes important.

Incidentally, at the accretion radius the fluid sound speed (Eq. (14)) is

$$c_s^2(r = 4GM_{\text{BH}}) = \frac{4\pi a}{m^3} \rho(4GM_{\text{BH}}) \simeq 0.39, \quad (50)$$

showing that we have already approached the relativistic regime at some larger radius. As discussed earlier, for simplicity we neglect corrections to the non-relativistic superfluid equation of state, $P_2 \propto \rho^2$. In this approximation, the profile (45) thus remains valid up to a radius of order the BH horizon, where it will drop off due to accretion effects. See next subsection for details.

b) Three-body interacting superfluid: Applying the same procedure to the equation of state in Eq. (16), one finds that Eq. (41) in this case becomes

$$\frac{d\rho}{dr} = -\frac{1}{1 - 2GM_{\text{BH}}/r} \frac{GM_{\text{BH}}}{r^2} \left(\frac{4\Lambda^2 m^6}{\rho} + \frac{\rho}{3} \right). \quad (51)$$

The solution is

$$\rho(r) = \sqrt{12\Lambda^2 m^6 \left[\left(\frac{1 - 2GM_{\text{BH}}/r_h}{1 - 2GM_{\text{BH}}/r} \right)^{1/3} - 1 \right] + \rho_0^2 \left(\frac{1 - 2GM_{\text{BH}}/r_h}{1 - 2GM_{\text{BH}}/r} \right)^{1/3}}, \quad (52)$$

where we have once again fixed the integration constant by requiring $\rho(r_h) = \rho_0$. In the non-relativistic regime ($r, r_h \gg 2GM_{\text{BH}}$), this reduces to

$$\rho(r) = \sqrt{8\Lambda^2 m^6 GM_{\text{BH}} \left(\frac{1}{r} - \frac{1}{r_h} \right) + \rho_0^2}, \quad (53)$$

which solves the non-relativistic condition of hydrostatic equilibrium (Eq. (39)). In particular, at short distances the profile becomes a power-law $\rho \sim 1/\sqrt{r}$:

$$\rho(r) \simeq \sqrt{\frac{8\Lambda^2 m^6 GM_{\text{BH}}}{r}}, \quad (54)$$

which is a milder growth than the $1/r$ profile (Eq. (47)) for the case $P_2 \propto \rho^2$.

As discussed above, this assumption of superfluidity is only valid for large enough distances such that DM is sub-critical. Substituting the relativistic density profile of Eq. (52), once again with $r_h \gg 2GM_{\text{BH}}$ and neglecting the subleading contribution proportional to ρ_0 , the criterion for degeneracy (Eq. (42)) gives

$$v^2 < \frac{(12)^{1/3} 6\pi}{\zeta(3/2)} \left(\frac{\Lambda}{m} \right)^{2/3} \left[\frac{1}{(1 - 2v^2)^{1/3}} - 1 \right]^{1/3}. \quad (55)$$

It is easy to see that this can only be satisfied in the regime $v^2 \ll 1$, *i.e.*, as long as

$$v < v_{\text{deg}} \simeq 0.3 \left(\frac{m}{\text{eV}} \right)^{-1/2} \left(\frac{\Lambda}{\text{meV}} \right)^{1/2}. \quad (56)$$

Thus the DM particles cease to be degenerate around the same distance as they become relativistic. The corresponding radius at which the degeneracy condition is broken is estimated to be

$$r_{\text{deg}} \simeq \frac{GM_{\text{BH}}\zeta(3/2)m}{8(3\pi)^{3/2}\Lambda} \simeq 5.4 \cdot 10^{-6} \text{kpc} \left(\frac{M_{\text{BH}}}{10^{10} M_{\odot}} \right) \left(\frac{m}{\text{eV}} \right) \left(\frac{\Lambda}{\text{meV}} \right)^{-1}. \quad (57)$$

Comparing this value with Eq. (19) shows that the degeneracy condition is violated before the BH horizon is reached. For $r < r_{\text{deg}}$, the fluid equation of state is more aptly described by the ideal gas law, $P \propto \rho v^2$. This regime will be discussed below.

Note that the fluid sound speed (Eq. (18)) at the degeneracy radius is given by

$$c_s^2(r_{\text{deg}}) = \frac{\rho^2(r_{\text{deg}})}{4\Lambda^2 m^6} \simeq 0.2 \frac{\Lambda}{\text{meV}} \frac{\text{eV}}{m}. \quad (58)$$

This implies that the non-relativistic equation of state $P \sim \rho^3$ assumed above should receive significant corrections before reaching r_{deg} . As in the two-body case, we ignore this issue for simplicity.

3.3 Inner region (three-body case)

We have seen that, while the two-body interacting superfluid remains degenerate all the way to the accretion radius $r = 4GM_{\text{BH}}$, this is not so in the three-body case. For the latter, degeneracy breaks down at r_{deg} , which is larger than $4GM_{\text{BH}}$.

Our goal is to derive the DM density and velocity profiles in this innermost region, $4GM_{\text{BH}} < r < r_{\text{deg}}$. We follow the results outlined in Ref. [54], where the interested reader can find additional details. Let us stress that, for simplicity, we neglect the role that gravitational scattering off stars plays in establishing the steady-state distribution of the dark matter around the central object, see Ref. [47] for a recent analysis.

As mentioned already, although the DM component is no longer degenerate in this region, it nevertheless remains thermal, thanks to the large number of interactions around the central BH. Therefore its equation of state can be approximated as an ideal gas,

$$P = nk_{\text{B}}T = \frac{\rho v^2}{3}. \quad (59)$$

Within full general relativity, the density and temperature profile of the dark matter particles around the central BH are determined by the condition of hydrostatic equilibrium, given by Eq. (41), together with a heat equation within the gravothermal fluid approximation [54, 65–67]:

$$q_r = -\frac{\kappa}{\sqrt{1 - \frac{2GM_{\text{BH}}}{r}}} \frac{d}{dr} \left(\sqrt{1 - \frac{2GM_{\text{BH}}}{r}} T \right) = -\kappa \left(\frac{dT}{dr} + \frac{T}{1 - 2GM_{\text{BH}}/r} \frac{GM_{\text{BH}}}{r^2} \right). \quad (60)$$

where q_r is the radial component of the heat flux. For a virialized gas at rest in a stationary, spherically-symmetric gravitational field, q_r is the only non-zero component of the heat flux four-vector [68]. Equation (60) is a relativistic Fourier's law, which states that the heat flux is proportional to the temperature gradient, with proportionality constant given by the conductivity κ .

Assuming a steady-state solution [69], one can show that the DM cluster is virialized and at rest on a dynamical time scale, such that the mean fluid velocity is everywhere negligible, resulting into a constant total radiated heat:

$$r^2 q_r = \frac{D}{\sqrt{1 - \frac{2GM_{\text{BH}}}{r}}}, \quad (61)$$

where D is a constant. Using this, Eq. (60) implies

$$\frac{dT}{dr} = \frac{D}{\kappa r^2 (1 - 2GM_{\text{BH}}/r)^{3/2}} - \frac{T}{1 - 2GM_{\text{BH}}/r} \frac{GM_{\text{BH}}}{r^2}. \quad (62)$$

The only quantity left to specify is the thermal conductivity κ . In the non-relativistic limit, using equipartition $k_{\text{B}}T = \frac{1}{2}mv^2$, one can show that it is given in terms of the scattering cross section σ and DM particle mass m by [60, 66, 67]⁴

$$\kappa = \frac{\sqrt{3}}{2} k_{\text{B}} v \left(\frac{\sigma}{B} + \frac{m^2}{AC\sigma\rho^2 H^2} \right)^{-1} \quad (63)$$

⁴The expression for the thermal conductivity is obtained by matching the relativistic heat flux equation to the Newtonian result of kinetic theory, as done in Ref. [54].

where $C \approx \frac{290}{385}$ is a constant determined by N-body simulations [67], $B = \frac{25\sqrt{\pi}}{32}$ is found perturbatively in Chapman-Enskog theory [70], and $A = \sqrt{\frac{16}{\pi}}$ for hard-sphere interactions. For gravitationally bound particles, the gravitational scale height H is the minimum of the radius r and the Jeans scale:

$$H = \min \left(r, \sqrt{\frac{v^2}{4\pi G\rho}} \right). \quad (64)$$

Which of the two contributions in Eq. (63) dominates the thermal conductivity depends on the ratio $\frac{\ell}{H}$, where $\ell = \frac{m}{\rho\sigma}$ is the mean free path (MFP). The first contribution dominates in the short MFP regime, $\ell \ll H$, while the second dominates in the long MFP regime, $\ell \gg H$.

In our case, it turns out that $H \simeq r$. To see this, we can compare r and $\sqrt{v^2/4\pi G\rho}$ at the degeneracy radius. Using Eqs. (54), (56) and (57), we obtain

$$\sqrt{\frac{v_{\text{deg}}^2}{4\pi G\rho(r_{\text{deg}})}} \simeq 4 \times 10^7 r_{\text{deg}} \left(\frac{M_{\text{BH}}}{10^{10} M_{\odot}} \right)^{-1} \left(\frac{m}{\text{eV}} \right)^{-11/4} \left(\frac{\Lambda}{\text{meV}} \right)^{3/4} \gg r_{\text{deg}}. \quad (65)$$

As we will see, this hierarchy is maintained when one considers the radial evolution of the velocity and density profile for distances $r \lesssim r_{\text{deg}}$. Therefore we henceforth set

$$H = r. \quad (66)$$

Furthermore, it can be similarly argued that we are in the long MFP regime. Indeed, at the degeneracy radius,

$$\frac{\ell}{r_{\text{deg}}} = \frac{m}{\sigma\rho(r_{\text{deg}})r_{\text{deg}}} \simeq 3 \times 10^5 \left(\frac{\sigma/m}{\text{cm}^2/\text{g}} \right)^{-1} \left(\frac{M_{\text{BH}}}{10^{10} M_{\odot}} \right)^{-1} \left(\frac{m}{\text{eV}} \right)^{-7/2} \left(\frac{\Lambda}{\text{meV}} \right)^{-1/2}. \quad (67)$$

We will see that this ratio increases as r decreases. It follows that

$$\kappa \simeq \frac{\sqrt{3}}{2} A C k_B \frac{\sigma\rho^2 r^2 v}{m^2}. \quad (68)$$

In the relativistic regime, temperature and velocity dispersion are related by $k_B T = \frac{1}{3}\gamma mv^2$, with $\gamma = \frac{1}{\sqrt{1-v^2}}$. Using this, together with Eq. (59), the condition of hydrostatic equilibrium (Eq. (41)) and the heat equation (62) can be cast as equations for density and velocity:

$$\begin{aligned} \frac{1}{\rho} \frac{d(\rho v^2)}{dr} &= -\frac{3+v^2}{1-2GM_{\text{BH}}/r} \frac{GM_{\text{BH}}}{r^2}; \\ \frac{d(\gamma v^2)}{dr} &= \frac{3k_B D}{m\kappa r^2(1-2GM_{\text{BH}}/r)^{3/2}} - \frac{\gamma v^2}{1-2GM_{\text{BH}}/r} \frac{GM_{\text{BH}}}{r^2}. \end{aligned} \quad (69)$$

Following Ref. [54], it is convenient to define dimensionless Newtonian quantities

$$\rho_N \equiv \frac{\rho}{\gamma\rho(r_{\text{deg}})} = \frac{mn}{\rho(r_{\text{deg}})}; \quad v_N^2 \equiv \frac{\gamma v^2}{v_{\text{deg}}^2}, \quad (70)$$

normalized at the degeneracy radius. Defining also a dimensionless radius $\tilde{r} = r/r_{\text{deg}}$, Eqs. (69) become

$$\begin{aligned}\frac{d\rho_N}{d\tilde{r}} &= -\frac{3\gamma}{v_N^2(1-2v_{\text{deg}}^2/\tilde{r})}\frac{\rho_N}{\tilde{r}^2} - \frac{2}{v_N^3(1-2v_{\text{deg}}^2/\tilde{r})^{3/2}}\frac{\tilde{D}}{\rho_N\gamma^{3/2}\tilde{r}^4}; \\ \frac{dv_N}{d\tilde{r}} &= \frac{1}{v_N^2(1-2v_{\text{deg}}^2/\tilde{r})^{3/2}}\frac{\tilde{D}}{\rho_N^2\gamma^{3/2}\tilde{r}^4} - \frac{v_N}{2(1-2v_{\text{deg}}^2/\tilde{r})}\frac{v_{\text{deg}}^2}{\tilde{r}^2},\end{aligned}\quad (71)$$

where

$$\gamma = \frac{v_N^2 v_{\text{deg}}^2}{2} + \sqrt{1 + \frac{v_N^4 v_{\text{deg}}^4}{4}}; \quad \tilde{D} = \frac{\sqrt{3}mD}{AC\sigma r_{\text{deg}}^3 \rho(r_{\text{deg}})^2 v_{\text{deg}}^3}. \quad (72)$$

The constant \tilde{D} can be estimated by requiring that particles deep inside the gravitational potential well of the black hole, but outside its horizon, plunge directly into the black hole as accretion processes. This occurs at the radius [54, 71]

$$r_{\text{mb}} = 4GM_{\text{BH}}. \quad (73)$$

Such a radius corresponds to marginally bound circular orbits in a Schwarzschild geometry, with energy M_{BH} and angular momentum per unit mass $4GM_{\text{BH}}$. A particle with larger angular momentum and smaller energy has an inner turning point at $r \gtrsim 4GM_{\text{BH}}$. Thus any particle that reaches it, would be necessarily captured by the BH.⁵ The net result is that we can set an inner boundary at $r = r_{\text{mb}}$ where the DM density plummets and can be approximated to be equal to zero. The value of \tilde{D} is determined numerically by finding the point at which the energy density reaches its maximum ($\rho' = 0$ and $\rho'' < 0$) and then drops off, giving $\tilde{D} \simeq -1.37$ for the values of the model parameters chosen above.

In the next Section we will show the result of numerically integrating Eqs. (71). It turns out that the density profile in the inner region ($r_{\text{mb}} < r < r_{\text{deg}}$) is well-approximated by a power-law [54]

$$\rho(r) \simeq \rho(r_{\text{deg}}) \left(\frac{r}{r_{\text{deg}}}\right)^{-3/4}, \quad r \lesssim r_{\text{deg}}, \quad (74)$$

until it reaches the inner boundary where $\rho(r_{\text{mb}}) \rightarrow 0$. This power-law is the same as in self-interacting DM [54]. Using these results, it is easy to confirm that the approximations $H \simeq r$ and $\ell \gg r$ are justified over the entire inner region.

4 Dark matter profile and mass function

In this Section we can finally review the full shape of the superfluid DM density profile and the corresponding mass function. Following the structure of the previous sections, we discuss the results for both equations of state of interest. For this purpose, it will be instructive to compare

⁵We notice that this radius corresponds to the Bondi radius $r_B = GM_{\text{BH}}/v_{\text{rel}}^2$ of a BH accreting particles with relativistic velocities $v_{\text{rel}} \approx 1/2$ [72]. This quantity describes the region of space around the BH where accretion is relevant, and we therefore expect that already at distances $r \gtrsim r_{\text{mb}}$ the assumption of static equilibrium may be violated since accretion would be relevant. We leave this refinement to future work.

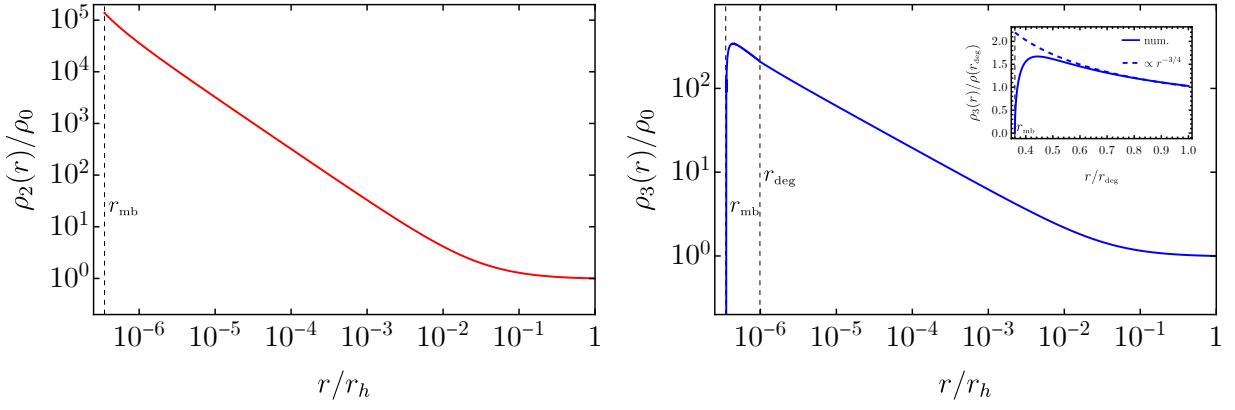


Figure 1: Density profiles of the superfluid DM around a supermassive black hole with mass $M_{\text{BH}} = 10^{10} M_{\odot}$, for the equations of state $P \propto \rho^2$ (Left, with $m = \mu\text{eV}$ and $a = 10^{-11} \text{ fm}$) and $P \propto \rho^3$ (Right, with $m = \text{eV}$ and $\Lambda = \text{meV}$). The vertical dashed lines denote the position of the accretion radius r_{mb} , and the degeneracy radius r_{deg} . The inset in the Right Panel shows the comparison between the density profile in the inner region found numerically (solid line) and the power-law fit $\rho \propto r^{-3/4}$ (dashed line).

the superfluid DM results with those for collisionless (CDM) and self-interacting DM (SIDM). Their density profiles are approximately power-law,

$$\rho(r) \simeq \rho_0 \left(\frac{r}{r_h} \right)^{-\beta}, \quad (r_{\text{mb}} \lesssim r \lesssim r_h), \quad (75)$$

with $\beta = 9/4$ (CDM) and $3/4$ (SIDM, for velocity-independent cross-section). The corresponding mass function is

$$M(r) = 4\pi \int_{r_{\text{mb}}}^r \rho(r') r'^2 dr' = \frac{4\pi}{3-\beta} \rho_0 r_h^3 \left[\left(\frac{r}{r_h} \right)^{3-\beta} - \left(\frac{r_{\text{mb}}}{r_h} \right)^{3-\beta} \right]. \quad (76)$$

4.1 Two-body interacting superfluid

For the two-body equation of state $P \propto \rho^2$, the superfluid DM density profile found in the previous Sections (Eqs. (28) and (45)) is given by

$$\rho_2(r) = \begin{cases} \frac{m^3}{2\pi a} \left[\left(\frac{1-2GM_{\text{BH}}/r_h}{1-2GM_{\text{BH}}/r} \right)^{1/4} - 1 \right] + \left(\frac{1-2GM_{\text{BH}}/r_h}{1-2GM_{\text{BH}}/r} \right)^{1/4} \rho_0 & r_{\text{mb}} \lesssim r \lesssim r_h; \\ \rho_0 \frac{\sin \left[\sqrt{\frac{Gm^3}{a}} (r-r_h) \right]}{\sqrt{\frac{Gm^3}{a}} (r-r_h)} & r_h \lesssim r \lesssim r_h + R_2. \end{cases} \quad (77)$$

As discussed in Sec. 3.2, for the values of the scattering length a we are interested in, the DM remains degenerate all the way to the accretion radius r_{mb} . Therefore, at the level of our approximations, we cannot resolve the behavior of the density profile in the vicinity of r_{mb} , where $\rho(r)$ should turn over and plummet.

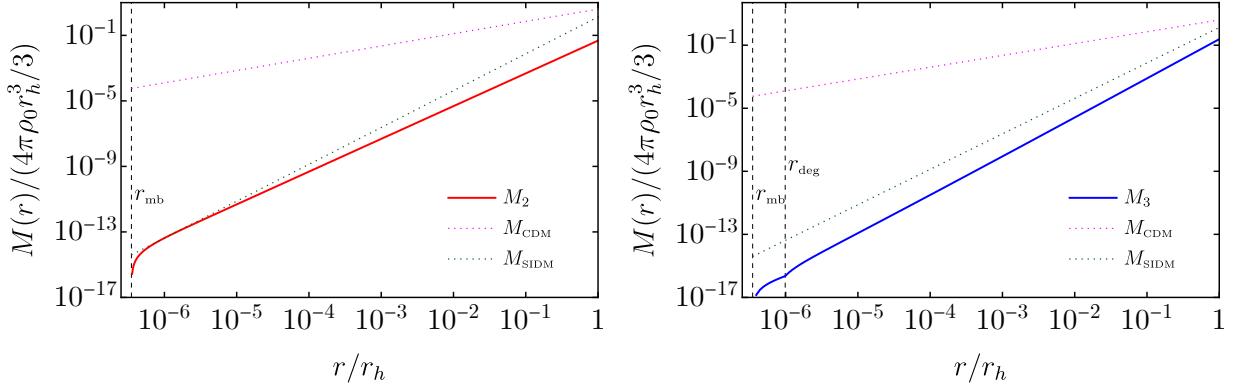


Figure 2: *Superfluid DM halo mass profile around a supermassive black hole with mass $M_{\text{BH}} = 10^{10} M_{\odot}$, assuming the equations of state $P \propto \rho^2$ (left) and $P \propto \rho^3$ (right).* The dotted magenta and green lines indicate the mass function for the models of collisionless and self-interacting dark matter. We stress that we have fixed the central density ρ_0 to compute the mass function.

Figure 1 (Left Panel) shows the corresponding density profile around a $10^{10} M_{\odot}$ supermassive black hole, with fiducial parameter values $m = \mu\text{eV}$ and $a = 10^{-11} \text{ fm}$. As one can appreciate, the profile becomes increasingly steep as we approach the BH, until the accretion radius where $\rho(r)$ is expected to drop. Our results show that the density can increase by orders of magnitude within the BH sphere of influence ($r < r_h$) with respect to the case where the BH is absent.

Let us now evaluate the DM mass function, $M_2(r) = 4\pi \int_{r_{\text{mb}}}^r \rho_2(r') r'^2 dr'$. This gives

$$M_2(r) = \begin{cases} \frac{m^3}{2a} GM_{\text{BH}} (r^2 - r_{\text{mb}}^2) & r_{\text{mb}} \lesssim r \lesssim r_h ; \\ \frac{m^3}{2a} GM_{\text{BH}} (r_h^2 - r_{\text{mb}}^2) + \frac{4}{3}\pi\rho_0 (r^3 - r_h^3) & r_h \lesssim r \lesssim r_h + R_2 . \end{cases} \quad (78)$$

To obtain a simpler analytical expression, we have used the approximate power-law profile $\rho_2(r) \sim 1/r$ of Eq. (47) in the region $r_{\text{mb}} \lesssim r \lesssim r_h$, and assumed a constant profile $\rho_2(r) \simeq \rho_0$ in the outer region $r_h \lesssim r \lesssim r_h + R_2$.

Figure 2 (Left Panel) shows the DM halo mass function for the same parameter values as above. For comparison, we also plot the approximate halo mass functions for CDM and SIDM, given by Eq. (76). Our results show that a two-body interacting superfluid is characterized by a different profile slope compared to CDM and SIDM. In particular, the presence of strong interactions within the superfluid model give rise to a shallower growth with respect to the other two cases, resulting in a less steep profile. Physically this is due to the classical pressure in the superfluid state.

4.2 Three-body interacting superfluid

For the three-body equation of state $P \propto \rho^3$, the superfluid DM density profile is given analytically in the outer ($r_h \lesssim r \lesssim r_h + R_3$) and intermediate ($r_{\text{deg}} \lesssim r \lesssim r_h$) regions respectively by Eqs. (34)

and (52):

$$\rho_3(r) = \begin{cases} \sqrt{12\Lambda^2 m^6 \left[\left(\frac{1-2GM_{\text{BH}}/r_h}{1-2GM_{\text{BH}}/r} \right)^{1/3} - 1 \right] + \rho_0^2 \left(\frac{1-2GM_{\text{BH}}/r_h}{1-2GM_{\text{BH}}/r} \right)^{1/3}} & r_{\text{deg}} \lesssim r \lesssim r_h ; \\ \rho_0 \cos^{1/2} \left[\frac{\pi(r-r_h)}{2R_3} \right] & r_h \lesssim r \lesssim r_h + R_3 . \end{cases} \quad (79)$$

The density profile in the inner region ($r_{\text{mb}} < r < r_{\text{deg}}$) is obtained by numerically solving Eqs. (71) as described in Sec. 3.3.

In Fig. 1 (Right Panel) we show the resulting density profile for fiducial parameter values $m = \text{eV}$ and $\Lambda = \text{meV}$. As shown in the inset, the profile in the inner region is well-approximated by the power-law $\rho(r) \sim r^{-3/4}$, given by Eq. (74), which is found to coincide with the SIDM behavior. Comparing the two panels, one can appreciate that the profile for $P \propto \rho^3$ increases slightly less than in the $P \propto \rho^2$ case. This can be understood as follows. For the fiducial parameter values assumed in the two cases, the ratio of pressures is

$$\frac{P_3}{P_2} = \frac{\rho m_2^3}{24\pi a \Lambda^2 m_3^6} \simeq \left(\frac{\rho}{\rho_0} \right) \left(\frac{\rho_0}{10^{-24} \text{ g/cm}^3} \right) \left(\frac{m_2}{\mu\text{eV}} \right)^3 \left(\frac{a}{10^{-7} \text{ fm}} \right)^{-1} \left(\frac{\Lambda}{\text{meV}} \right)^{-2} \left(\frac{m_3}{\text{eV}} \right)^{-6} . \quad (80)$$

In other words, as we go to smaller radii and the density increases, the pressure in the three-body case is significantly larger, thereby allowing an overall growth in density which is milder in the three-body than in the two-body case.

Let us briefly comment on the velocity profile $v(r)$. In the inner region ($r_{\text{mb}} < r < r_{\text{deg}}$), the velocity profile is determined by numerically integrating Eqs. (71). The result is that $v(r)$ grows steeply as we approach the accretion radius r_{mb} . The increase is much sharper than the $1/\sqrt{r}$ scaling in the intermediate region. See Fig. 2 of Ref. [54].

The DM mass function in this case is given by

$$M_3(r) = \begin{cases} \frac{32\sqrt{2}\pi}{9} \Lambda m^3 \sqrt{GM_{\text{BH}}} r_{\text{deg}}^{1/4} \left(r^{9/4} - r_{\text{mb}}^{9/4} \right) & r_{\text{mb}} \lesssim r \lesssim r_{\text{deg}} ; \\ 16\sqrt{2}\pi \Lambda m^3 \sqrt{GM_{\text{BH}}} \left\{ \frac{2}{9} r_{\text{deg}}^{1/4} \left(r_{\text{deg}}^{9/4} - r_{\text{mb}}^{9/4} \right) + \frac{1}{5} \left(r^{5/2} - r_{\text{deg}}^{5/2} \right) \right\} & r_{\text{deg}} \lesssim r \lesssim r_h ; \\ \frac{16\sqrt{2}\pi}{5} \Lambda m^3 \sqrt{GM_{\text{BH}}} \left(r_h^{5/2} - r_{\text{deg}}^{5/2} \right) + \frac{4}{3}\pi \rho_0 (r^3 - r_h^3) & r_h \lesssim r \lesssim r_h + R_3 . \end{cases} \quad (81)$$

Once again, to simplify the analytical expression, we have used the approximate power-law profile $\rho_3(r) \sim 1/\sqrt{r}$ of Eq. (54) in the region $r_{\text{deg}} \lesssim r \lesssim r_h$, and assumed a constant profile $\rho_3(r) \simeq \rho_0$ in the outer region $r_h \lesssim r \lesssim r_h + R_3$. Figure 2 (Right Panel) shows the DM halo mass function for the same parameter values as before, as well as CDM and SIDM mass functions (Eq. (76)) for comparison. Comparing the two panels, we notice that the halo mass for $P \propto \rho^3$ assumes comparable values to the $P \propto \rho^2$ case.

5 Conclusions

The novel theory of DM superfluidity is able to reconcile the triumph of the Λ CDM model on cosmological scales while giving rise to a rich phenomenology on galactic scales. Within this model,

the DM is represented by self-interacting particles which are generated out-of-equilibrium and remain decoupled from baryons throughout the history of the universe. These particles are able to thermalize and form a superfluid in galaxies, with critical temperature of order \sim mK, if their mass is sufficiently light and their self-interactions sufficiently strong.

At the center of these galaxies, large and massive black holes are expected to be present [73, 74], and they should currently contribute to about 10^{-5} of the DM in the universe [75, 76]. We therefore expect that the presence of these massive objects will modify the density profile of the surrounding DM fluid, with different predictions depending on the nature of the dark matter one considers.

In this work we computed the density profile of superfluid DM around central SMBHs, for different superfluid equations of state, $P \propto \rho^2$ and $P \propto \rho^3$, corresponding to predominantly two-body and three-body interactions, respectively. We found that, depending on the distance from the central object, the DM density is characterized by different functional behavior, providing striking distinguishing features from standard predictions for collisionless DM [77].

One possible way to reveal the underlying properties of DM, and eventually its superfluid nature, could be through gravitational lensing observations [78, 79]. In particular, it is expected that the gravitational lensing of central black holes is influenced by the surrounding DM, which would modify for example the photon sphere and black hole shadow, thus providing a novel way to probe DM properties. Moreover, continuous gravitational waves emitted by isolated neutron stars and lensed by SMBHs like Sgr A* might be observable with future gravitational wave detectors. As such they can provide a new probe of the matter distribution in the galactic center [80]. Furthermore, it might be possible to probe this kind of DM profiles if we can assess that a binary merged near a SMBH, for instance via the detection of the gravitational spin Hall effect, see Ref. [81] for further details.

To improve on our analysis, it would be important to include the contribution of realistic baryon distributions, which are expected to modify the DM profile at distances larger than the BH sphere of influence, especially in the matching to the NFW envelope, as well as the role that gravitational scattering off stars plays in establishing the steady-state distribution of DM around the central object. Finally, the inclusion of an AGN disk around the central BH and the corresponding emission of jets would probably modify the density profile in the region close to the BH. We leave these further improvements to future work.

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