

Mass-radius relation of self-gravitating Bose-Einstein condensates with a central black hole

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We determine the mass-radius relation of self-gravitating Bose-Einstein condensates with an attractive $-1/r$ external potential created by a central mass. Following our previous work [P.H. Chavanis, Phys. Rev. D **84**, 043531 (2011)], we use an analytical approach based on a Gaussian ansatz. We consider the case of noninteracting bosons as well as the case of self-interacting bosons with a repulsive or an attractive self-interaction. These results may find application in the context of dark matter halos made of self-gravitating Bose-Einstein condensates. In that case, the central mass may mimic a supermassive black hole. We apply our results to ultralight axions with an attractive self-interaction. We determine how the central black hole affects the mass-radius relation and the maximum mass of axionic halos found in our previous papers. Our approximate analytical results based on the Gaussian ansatz are compared with exact analytical results obtained in particular limits.

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I. INTRODUCTION

The nature of dark matter remains one of the most important mysteries of modern cosmology. It has been proposed that dark matter could be made of bosons in the form of Bose-Einstein condensates (BECs) and that dark matter halos could correspond to giant self-gravitating BECs [1–91] (see the introduction of [31] for a short historic of this model). To account for the mass and size of dark matter halos, the mass of the bosons must be extraordinarily small, between $10^{-3} - 10^{-22}$ eV/ c^2 (see Appendix D of [80]). The quantum nature of the bosonic particles may solve important problems that the standard cold dark matter (CDM) model encounters at small (galactic) scales such as the cusp-core problem [92], the missing satellite problem [93–96], and the too big to fail problem [97]. As a result, there is a huge activity on the BEC dark matter (BECDM) model. Apart from its astrophysical applications, this model is also interesting on a physical point of view since it combines fundamental concepts of quantum mechanics (like Bose-Einstein condensation or superfluidity) and gravity. It is fascinating to realize that quantum mechanics may manifest itself at the scale of dark matter halos and that it may stabilize them in the same manner that it stabilizes ordinary matter at atomic scales.

In Refs. [31, 32], we have determined the mass-radius relation of self-gravitating BECs in Newtonian gravity described by the Gross-Pitaevskii-Poisson (GPP) equations. We have considered the possibility that the bosons are noninteracting or self-interacting with a scattering length a_s . In Ref. [31] we have used a Gaussian ansatz to obtain an approximate analytical expression of the mass-radius relation. In Ref. [32] we have compared our approximate analytical results with the exact ones obtained by determining the ground state of the GPP equations numerically. We found a reasonable agreement between the numerical and the analytical results showing that the Gaussian ansatz can provide a useful qualitative description of self-gravitating BECs at equilibrium. Furthermore, it allows us to play easily with the parameters and to incorporate new effects into the problem.

In the noninteracting case ($a_s = 0$), there exist equilibrium states for any mass M and they are stable. The mass-radius relation is given by [3, 31, 32]:

$$R_{99}^{\text{exact}} = 9.946 \frac{\hbar^2}{GMm^2}, \quad (1)$$

where R_{99} is the radius containing 99% of the mass. The radius decreases as the mass increases.

When the self-interaction between bosons is repulsive ($a_s > 0$), we found [31, 32] that equilibrium states also exist for any mass M and that they are stable. The radius decreases with the mass but it remains always larger than the gravitational Thomas-Fermi (TF) radius [6, 13, 20, 22, 31]:

$$R_{\text{TF}}^{\text{exact}} = \pi \left(\frac{a_s \hbar^2}{Gm^3} \right)^{1/2} \quad (2)$$

obtained when $M \rightarrow +\infty$. Comparing Eqs. (1) and (2), we obtain the mass scale [31]:

$$M_s \sim \frac{\hbar}{\sqrt{Gma_s}}. \quad (3)$$

The noninteracting limit corresponds to $M \ll M_s$ and $R \gg R_{\text{TF}}$. The TF limit corresponds to $M \gg M_s$ and $R \sim R_{\text{TF}}$. In that limit, the equilibrium states have approximately the same radius R_{TF} independently of their mass M .

When the self-interaction between bosons is attractive ($a_s < 0$), we found that equilibrium states exist only below a maximum mass¹ [31, 32]:

$$M_{\text{max}}^{\text{exact}} = 1.012 \frac{\hbar}{\sqrt{Gm|a_s|}} \quad (4)$$

corresponding to a radius

$$(R_{99}^*)^{\text{exact}} = 5.5 \left(\frac{|a_s| \hbar^2}{Gm^3} \right)^{1/2}. \quad (5)$$

For $M < M_{\text{max}}$ there are two branches of solutions on the mass-radius relation $M(R)$. The equilibrium states on the decreasing branch ($R > R_*$) are stable while the equilibrium states on the increasing branch ($R < R_*$) are unstable. Therefore, R_* is the minimum radius for stable equilibrium states. The noninteracting limit corresponds to $M \ll M_{\text{max}}$ and $R \gg R_*$. The nongravitational limit corresponds to $M \ll M_{\text{max}}$ and $R \ll R_*$. In that case, the mass-radius relation is given by [32]:

$$R_{99}^{\text{exact}} = 3.64 \frac{|a_s|}{m} M \quad (6)$$

but these equilibrium states are unstable.

One of the most serious dark matter particle candidates is the axion [73]. This is a bosonic particle with an attractive self-interaction ($a_s < 0$). As a result, dilute axion stars (or more generally dilute axionic clusters) can exist only below the maximum mass given by Eq. (4). For QCD axions with $m = 10^{-4} \text{ eV}/c^2$ and $a_s = -5.8 \times 10^{-53} \text{ m}$, we find $M_{\text{max}}^{\text{exact}} = 6.46 \times 10^{-14} M_{\odot} = 1.29 \times 10^{17} \text{ kg} = 2.16 \times 10^{-8} M_{\oplus}$ and $(R_{99}^*)^{\text{exact}} = 3.26 \times 10^{-4} R_{\odot} = 227 \text{ km} = 3.56 \times 10^{-2} R_{\oplus}$ which are of the order of the asteroids size. QCD axions can form mini “axion stars” but they cannot form dark matter halos of relevant size. However, string theory predicts the existence of axions with a very small mass up to $10^{-34} \text{ eV}/c^2$ [98]. For ultralight axions (ULAs), the maximum mass given by Eq. (4) is of the order of the galactic mass ($\sim 10^8 M_{\odot}$ or larger).² Therefore, ULAs can form “axionic clusters” of the size of dark matter halos. For $M > M_{\text{max}}$, the system undergoes a gravitational collapse.³ An estimate of the collapse time has been obtained analytically in [74] from the Gaussian ansatz. However, the Gaussian ansatz is not able to describe the complex collapse dynamics of the system. A detailed study of the collapse process requires solving the GPP equations, or the Klein-Gordon-Einstein (KGE) equations, numerically. It is then found that the system first undergoes gravitational collapse (implosion) until collisions between axions stop the collapse and lead to an explosion accompanied by the emission of relativistic axions with a characteristic radiation (bosenova) [79]. There is also the possibility to form dense axion stars (or dense axionic clusters) [68]. Finally, the collapse of very massive axion stars (or axionic clusters) can lead to the formation of a black hole [77]. The phase transitions between dilute and dense axion stars have been studied in Ref. [85] with the Gaussian ansatz. This analytical study is able to reproduce the numerical results of Braaten *et al.* [68] and to display a tricritical point between dilute axion stars, dense axion stars and black holes similar to the one found by Helfer *et al.* [77].

In this paper, we complete our former study [31]. Using a Gaussian ansatz, we study how the mass-radius relation of self-gravitating BECs is modified when there is a massive object at the center of the system. In the case of BECDM halos, the central object could represent a supermassive black hole. Indeed, supermassive black holes are purported to exist at the centers of the galaxies.

The paper is organized as follows. In Sec. II we present the exact GPP equations describing self-gravitating BECs with a central black hole and the approximate equations obtained from the Gaussian ansatz. In Sec. III we consider particular cases of physical interest and identify characteristic mass and length scales. In Sec. IV we treat the general case using dimensionless variables. The Appendices provide additional results. Dimensionless variables are introduced

¹ This maximum mass can be expressed in various forms (depending on the parameter used to measure the self-interaction of the bosons) as detailed in Sec. IV of [85].

² The precise characteristics of the dark matter particle are not known. For that reason, we prefer to remain general (and therefore necessarily a bit vague) in order to cover all the possibilities. We refer to Appendix D of [80] and to Ref. [85] for numerical applications (see also [1–91]).

³ This may concern the solitonic core of large dark matter halos as suggested in [85].

in Appendix A. In Appendix B we explain how the general formalism of self-gravitating BECs developed in Ref. [82] can be generalized in the presence of a central mass (black hole). In Appendix C we derive a general expression of the gravitational (potential) energy of a self-gravitating polytropic sphere in the presence of an external potential, possibly created by a central black hole. In Appendices D-J, we derive the exact mass-radius relation of self-gravitating BECs with a central black hole in particular limits of the theory. These exact results are compared with the approximate ones obtained with the Gaussian ansatz usually giving a good qualitative agreement.

II. SELF-GRAVITATING BECS WITH A CENTRAL BLACK HOLE

A. Gross-Pitaevskii-Poisson equations

We consider a self-gravitating BEC at $T = 0$ whose complex wavefunction $\psi(\mathbf{r}, t)$ is described by the GPP equations [82]:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m(\Phi + \Phi_{\text{ext}})\psi + m \frac{dV}{d|\psi|^2} \psi, \quad (7)$$

$$\Delta \Phi = 4\pi G |\psi|^2, \quad (8)$$

where $\Phi(\mathbf{r}, t)$ is the gravitational potential produced by the system, $\Phi_{\text{ext}}(\mathbf{r})$ is a fixed external potential, and $V(|\psi|^2)$ is the self-interaction potential of the bosons. The mass density of the bosons is $\rho = |\psi|^2$. The GPP equations conserve the total mass and the total energy which can be written as

$$M = \int |\psi|^2 d\mathbf{r}, \quad (9)$$

$$E_{\text{tot}} = \frac{\hbar^2}{2m^2} \int |\nabla \psi|^2 d\mathbf{r} + \frac{1}{2} \int |\psi|^2 \Phi d\mathbf{r} + \int |\psi|^2 \Phi_{\text{ext}} d\mathbf{r} + \int V(|\psi|^2) d\mathbf{r}. \quad (10)$$

The energy includes the kinetic energy Θ , the gravitational energy W , the potential energy of the external potential W_{ext} , and the internal energy U [82].

In this paper, we consider a quartic self-interaction potential of the form

$$V(|\psi|^2) = \frac{2\pi a_s \hbar^2}{m^3} |\psi|^4. \quad (11)$$

It corresponds to the effective potential of the axions expanded at second order in $|\psi|^2$ (see, e.g., Sec. III of [85]). Since this term dominates at low densities, it describes dilute axion stars (or dilute axionic clusters). It also corresponds to the usual $|\psi|^2 \psi$ (cubic) nonlinearity present in the standard GP equation [99]. It describes short-range binary collisions between the bosons modeled by a pair contact potential $u_{\text{SR}}(\mathbf{r} - \mathbf{r}') = (4\pi a_s \hbar^2 / m^3) \delta(\mathbf{r} - \mathbf{r}')$ where a_s is the scattering length (see, e.g., Sec. II.A. of [31]). When $a_s > 0$ the self-interaction is repulsive and when $a_s < 0$ the self-interaction is attractive. When $a_s = 0$ the bosons are noninteracting. We shall consider these three possibilities. We shall also assume that there is a mass at the center of the system mimicking for example a supermassive black hole or any other massive object. Therefore, we consider an external potential of the form

$$\Phi_{\text{BH}} = -\frac{GM_{\text{BH}}}{r} \quad (12)$$

that we shall call the BH potential (the corresponding force by unit of mass created by the BH is $-\nabla \Phi_{\text{BH}} = -GM_{\text{BH}}\mathbf{r}/r^3$). As a result, the GPP equations considered in the present paper can be written as

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi - \frac{GM_{\text{BH}}m}{r} \psi + \frac{4\pi a_s \hbar^2}{m^2} |\psi|^2 \psi, \quad (13)$$

$$\Delta \Phi = 4\pi G |\psi|^2. \quad (14)$$

B. Hydrodynamic representation

Using the Madelung [100] transformation

$$\psi(\mathbf{r}, t) = \sqrt{\rho(\mathbf{r}, t)} e^{iS(\mathbf{r}, t)/\hbar}, \quad \rho = |\psi|^2, \quad \mathbf{u} = \frac{\nabla S}{m}, \quad (15)$$

the GPP equations (7) and (8) are equivalent to the hydrodynamic equations [82]:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (16)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi - \nabla \Phi_{\text{ext}} - \frac{1}{m} \nabla Q, \quad (17)$$

$$\Delta \Phi = 4\pi G \rho, \quad (18)$$

where

$$Q = -\frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} = -\frac{\hbar^2}{4m} \left[\frac{\Delta \rho}{\rho} - \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} \right] \quad (19)$$

is the quantum potential which takes into account the Heisenberg uncertainty principle and P is the pressure which is determined by the self-interaction potential from the relation [82]:

$$P(\rho) = \rho V'(\rho) - V(\rho) = \rho^2 \left[\frac{V(\rho)}{\rho} \right]'. \quad (20)$$

Inversely, the self-interaction potential is related to the pressure by

$$V(\rho) = \rho \int^\rho \frac{P(\rho')}{\rho'^2} d\rho'. \quad (21)$$

We have

$$V'(\rho) = \int^\rho \frac{P'(\rho')}{\rho'} d\rho', \quad V''(\rho) = \frac{P'(\rho)}{\rho}. \quad (22)$$

In the hydrodynamic representation, the mass (9) and the total energy (10) can be written as

$$M = \int \rho d\mathbf{r}, \quad (23)$$

$$E_{\text{tot}} = \int \rho \frac{\mathbf{u}^2}{2} d\mathbf{r} + \frac{1}{m} \int \rho Q d\mathbf{r} + \frac{1}{2} \int \rho \Phi d\mathbf{r} + \int \rho \Phi_{\text{ext}} d\mathbf{r} + \int V(\rho) d\mathbf{r}. \quad (24)$$

The total energy includes the classical kinetic energy Θ_c , the quantum kinetic energy Θ_Q , the gravitational energy W , the potential energy of the external potential W_{ext} , and the internal energy U [82].

The self-interaction potential defined by Eq. (11) can be rewritten as

$$V(\rho) = \frac{2\pi a_s \hbar^2}{m^3} \rho^2. \quad (25)$$

According to Eq. (20) it generates a pressure associated with an equation of state of the form

$$P(\rho) = \frac{2\pi a_s \hbar^2}{m^3} \rho^2. \quad (26)$$

We note that the pressure is positive when $a_s > 0$ and negative when $a_s < 0$. This equation of state can be written as

$$P(\rho) = K_2 \rho^2 \quad \text{with} \quad K_2 = \frac{2\pi a_s \hbar^2}{m^3}. \quad (27)$$

This is a polytropic equation of state of the form $P = K \rho^\gamma$ ($\gamma = 1 + 1/n$) with index $\gamma = 2$ ($n = 1$). We note that $P(\rho) = V(\rho)$.

C. Equilibrium state

In the hydrodynamic representation, an equilibrium state of the quantum Euler equations (16) and (17), obtained by taking $\partial_t = 0$ and $\mathbf{u} = \mathbf{0}$, satisfies

$$\nabla P + \rho \nabla \Phi + \rho \nabla \Phi_{\text{ext}} + \frac{\rho}{m} \nabla Q = \mathbf{0}. \quad (28)$$

This equation can be interpreted as a condition of quantum hydrostatic equilibrium. It is equivalent to the stationary solution of the GPP equations (see [82] and Appendix B5). It describes the balance between the pressure due to short-range interactions (self-interaction), the gravitational force, the external force (black hole) and the quantum force arising from the Heisenberg uncertainty principle. Combining Eq. (28) with the Poisson equation (18), we obtain the fundamental differential equation of quantum hydrostatic equilibrium

$$-\nabla \cdot \left(\frac{\nabla P}{\rho} \right) + \frac{\hbar^2}{2m^2} \Delta \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = 4\pi G \rho + \Delta \Phi_{\text{ext}}. \quad (29)$$

For the BH potential (12), we have

$$\Delta \Phi_{\text{BH}} = 4\pi G M_{\text{BH}} \delta(\mathbf{r}) \quad (30)$$

and the foregoing equation can be rewritten as

$$-\nabla \cdot \left(\frac{\nabla P}{\rho} \right) + \frac{\hbar^2}{2m^2} \Delta \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = 4\pi G \rho + 4\pi G M_{\text{BH}} \delta(\mathbf{r}). \quad (31)$$

For the quartic self-interaction potential (11), using Eq. (26), it takes the form

$$-\frac{4\pi a_s \hbar^2}{m^3} \Delta \rho + \frac{\hbar^2}{2m^2} \Delta \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = 4\pi G \rho + 4\pi G M_{\text{BH}} \delta(\mathbf{r}). \quad (32)$$

D. Exact equilibrium relations

From now on, we restrict ourselves to the case where the external potential is due to a central black hole [see Eq. (12)] and to the case of a quartic self-interaction potential [see Eq. (11)]. At equilibrium, the total energy [see Eq. (B13)] is given by

$$E_{\text{tot}} = \Theta_Q + W + W_{\text{BH}} + U, \quad (33)$$

the eigenenergy [see Eq. (B36)] is given by

$$NE = 2W + 2U + W_{\text{BH}} + \Theta_Q, \quad (34)$$

and the scalar virial theorem [see Eq. (B25)] is given by

$$2\Theta_Q + 3U + W + W_{\text{BH}} = 0, \quad (35)$$

where $U = \int P d\mathbf{r}$ with $P(\rho)$ given by Eq. (26). So far, the results are exact in the sense that they do not rely on any approximation, at least with respect to the GPP equations (7) and (8) that are our starting point. In the following sections, we shall provide approximate analytical results of the GPP equations based on a Gaussian ansatz.

E. Gaussian ansatz

Making a Gaussian ansatz for the wavefunction [82], we can write the total energy of the self-gravitating BEC as

$$E_{\text{tot}} = \frac{1}{2} \alpha M \left(\frac{dR}{dt} \right)^2 + V(R), \quad (36)$$

where $R(t)$ is the typical radius of the BEC and M is its mass.⁴ The first term corresponds to the classical kinetic energy Θ_c and the second term corresponds to the potential energy. The conservation of energy, $\dot{E}_{\text{tot}} = 0$, provides the following equation determining the temporal evolution of the radius of the BEC:

$$\alpha M \frac{d^2 R}{dt^2} = -\frac{dV}{dR}. \quad (37)$$

This is similar to the equation of motion of a fictive particle of mass αM in a potential $V(R)$. The potential associated with the equation of state (26) and with the BH potential (12) is (see [82] and Appendix B 6):

$$V(R) = \sigma \frac{\hbar^2 M}{m^2 R^2} - \nu \frac{GM^2}{R} + \zeta \frac{2\pi a_s \hbar^2 M^2}{m^3 R^3} - \lambda \frac{GM_{\text{BH}} M}{R}. \quad (38)$$

The first term is the quantum kinetic energy Θ_Q , the second term is the gravitational energy W , the third term is the internal energy U and the fourth term is the potential energy W_{BH} due to the BH. The coefficients appearing in the foregoing equations are

$$\alpha = \frac{3}{2}, \quad \sigma = \frac{3}{4}, \quad \zeta = \frac{1}{(2\pi)^{3/2}}, \quad \nu = \frac{1}{\sqrt{2\pi}}, \quad \lambda = \frac{2}{\sqrt{\pi}}. \quad (39)$$

At equilibrium, we have

$$E_{\text{tot}} = \sigma \frac{\hbar^2 M}{m^2 R^2} - \nu \frac{GM^2}{R} + \zeta \frac{2\pi a_s \hbar^2 M^2}{m^3 R^3} - \lambda \frac{GM_{\text{BH}} M}{R}. \quad (40)$$

$$NE = \sigma \frac{\hbar^2 M}{m^2 R^2} - 2\nu \frac{GM^2}{R} + 2\zeta \frac{2\pi a_s \hbar^2 M^2}{m^3 R^3} - \lambda \frac{GM_{\text{BH}} M}{R}. \quad (41)$$

F. The mass-radius relation

An equilibrium state of the self-gravitating BEC is obtained by extremizing $E_{\text{tot}}(\dot{R}, R)$ at fixed mass. From Eq. (36), we first get the condition $\dot{R} = 0$ meaning that an equilibrium state is static. We then obtain the condition $V'(R) = 0$. Computing the first derivative of $V(R)$ giving

$$V'(R) = -2\sigma \frac{\hbar^2 M}{m^2 R^3} + \nu \frac{GM^2}{R^2} - 3\zeta \frac{2\pi a_s \hbar^2 M^2}{m^3 R^4} + \lambda \frac{GM_{\text{BH}} M}{R^2}, \quad (42)$$

and writing $V'(R) = 0$, we obtain the mass-radius relation

$$-2\sigma \frac{\hbar^2 M}{m^2 R^3} + \nu \frac{GM^2}{R^2} - 6\pi\zeta \frac{a_s \hbar^2 M^2}{m^3 R^4} + \lambda \frac{GM_{\text{BH}} M}{R^2} = 0 \quad (43)$$

or, equivalently,

$$M = \frac{2\sigma \frac{\hbar^2}{m^2 R^3} - \lambda \frac{GM_{\text{BH}}}{R^2}}{\nu \frac{G}{R^2} - 6\pi\zeta \frac{a_s \hbar^2}{m^3 R^4}}. \quad (44)$$

G. The pulsation

An equilibrium state is stable if and only if it is a (local) minimum of $E_{\text{tot}}(\dot{R}, R)$, or equivalently of $V(R)$, at fixed mass. Therefore, it must satisfy the condition $V''(R) > 0$. Computing the second derivative of $V(R)$, we get

$$V''(R) = 6\sigma \frac{\hbar^2 M}{m^2 R^4} - 2\nu \frac{GM^2}{R^3} + 24\pi\zeta \frac{a_s \hbar^2 M^2}{m^3 R^5} - 2\lambda \frac{GM_{\text{BH}} M}{R^3}. \quad (45)$$

⁴ For a Gaussian density profile, the relation between the radius R and the radius R_{99} containing 99% of the mass is $R_{99} = 2.38167R$ [31]. We must keep this relation in mind when we compare the results from the Gaussian ansatz with the exact results, i.e., we must compare the exact results with the approximate ones expressed in terms of R_{99} , not in terms of R .

The square of the complex pulsation of the system about an equilibrium state is given by [82]:

$$\omega^2 = \frac{1}{\alpha M} V''(R). \quad (46)$$

Therefore

$$\omega^2 = \frac{6\sigma}{\alpha} \frac{\hbar^2}{m^2 R^4} - \frac{2\nu}{\alpha} \frac{GM}{R^3} + \frac{24\pi\zeta}{\alpha} \frac{a_s \hbar^2 M}{m^3 R^5} - \frac{2\lambda}{\alpha} \frac{GM_{\text{BH}}}{R^3}. \quad (47)$$

On the other hand, differentiating the mass-radius relation (43) with respect to R and using Eqs. (45) and (46), we obtain the identity (see Eq. (315) of [82]):

$$\omega^2 = -\frac{1}{\alpha M} \left(\frac{2\sigma \hbar^2}{m^2 R^3} - \lambda \frac{GM_{\text{BH}}}{R^2} \right) \frac{dM}{dR}. \quad (48)$$

This relation shows that the pulsation vanishes ($\omega = 0$) at a turning point of mass ($dM/dR = 0$) in agreement with the Poincaré theory of linear series of equilibria [101]. On the other hand, the term in parenthesis vanishes at a turning point of radius ($dR/dM = 0$).

III. PARTICULAR CASES

In this section, we consider particular cases of the mass-radius relation (44).

A. Nongravitational + noninteracting case

In the nongravitational + noninteracting case ($G = a_s = 0$), the equilibrium states exist for unique value of the radius

$$R_{\text{B}} = \frac{2\sigma}{\lambda} \frac{\hbar^2}{GM_{\text{BH}} m^2}, \quad (49)$$

independent of their mass M . The prefactor is equal to 1.33. This can be interpreted as a gravitational Bohr radius. The pulsation is given by

$$\omega_B^2 = \frac{\lambda}{\alpha} \frac{GM_{\text{BH}}}{R_{\text{B}}^3} = \frac{\lambda^4}{8\alpha\sigma^3} \frac{G^4 M_{\text{BH}}^4 m^6}{\hbar^6}. \quad (50)$$

The prefactors are equal to 0.752 and 0.320 consecutively. The equilibrium states are all stable ($\omega_B^2 > 0$).

Remark: The Schrödinger equation with an attractive (gravitational) $1/r$ potential can be solved analytically (see Appendix D). This corresponds to the gravitational Bohr atom. The approximate results (49) and (50) can be compared to the exact ones from Eqs. (D12) and (D19).

B. Nongravitational + TF case

In the nongravitational + TF case ($G = \hbar = 0$), the mass-radius relation is given by

$$M = \frac{\lambda}{6\pi\zeta} \frac{GM_{\text{BH}} m^3 R^2}{a_s \hbar^2} \quad (51)$$

provided that $a_s > 0$ (there is no equilibrium state when $a_s < 0$). The prefactor is equal to 0.943. The radius increases as the mass increases. The pulsation is given by

$$\omega^2 = \frac{2\lambda}{\alpha} \frac{GM_{\text{BH}}}{R^3}. \quad (52)$$

The prefactor is equal to 1.50. The equilibrium states are all stable ($\omega^2 > 0$).

Remark: When $G = \hbar = 0$ the equation of hydrostatic equilibrium (32) can be solved analytically (see Appendix E). The approximate results (51) and (52) can be compared to the exact ones from Eqs. (E12) and (E17).

C. Nongravitational + no BH case

In the nongravitational + no BH case ($G = M_{\text{BH}} = 0$), the mass-radius relation is given by

$$M = \frac{\sigma}{3\pi\zeta} \frac{m}{|a_s|} R \quad (53)$$

provided that $a_s < 0$ (there is no equilibrium state when $a_s > 0$). The prefactor is equal to 1.25. The radius increases as the mass increases. The pulsation is given by

$$\omega^2 = -\frac{2\sigma}{\alpha} \frac{\hbar^2}{m^2 R^4}. \quad (54)$$

The prefactor is equal to 1. The equilibrium states are all unstable ($\omega^2 < 0$).

Remark: When $G = M_{\text{BH}} = 0$ the wave function of the BEC is the solution of the nongravitational GP equation with an attractive self-interaction ($a_s < 0$). This equation has a stationary solution in the form of a soliton which can be obtained numerically [32]. The exact mass-radius relation is given by Eq. (6). These equilibrium states are unstable. Other exact results are given in [31, 32] and in Appendix F.

D. TF + noninteracting case

In the TF + noninteracting case ($\hbar = a_s = 0$), there is no equilibrium state.

E. Noninteracting + no BH case

In the noninteracting + no BH case ($a_s = M_{\text{BH}} = 0$), the mass-radius relation is given by

$$M = \frac{2\sigma}{\nu} \frac{\hbar^2}{Gm^2 R}. \quad (55)$$

The prefactor is equal to 3.76. The radius decreases as the mass increases. The pulsation is given by

$$\omega^2 = \frac{2\sigma}{\alpha} \frac{\hbar^2}{m^2 R^4}. \quad (56)$$

The prefactor is equal to 1. The equilibrium states are all stable ($\omega^2 > 0$).

Remark: When $a_s = M_{\text{BH}} = 0$, the equation of hydrostatic equilibrium (32) can be solved numerically [3, 31, 32] leading to the exact mass-radius relation from Eq. (1). Other exact results are given in [31, 32] and in Appendix G.

F. TF + no BH case

In the TF + no BH case ($\hbar = M_{\text{BH}} = 0$), the equilibrium states exist for unique value of the radius

$$R_{\text{TF}} = \left(\frac{6\pi\zeta}{\nu} \right)^{1/2} \left(\frac{a_s \hbar^2}{Gm^3} \right)^{1/2} \quad (57)$$

independent of their mass M , provided that $a_s > 0$ (there is no equilibrium state when $a_s < 0$). The prefactor is equal to 1.73. Their pulsation is given by

$$\omega^2 = \frac{2\nu}{\alpha} \frac{GM}{R_{\text{TF}}^3}. \quad (58)$$

The prefactor is equal to 0.532. The equilibrium states are all stable ($\omega^2 > 0$).

Remark: When $\hbar = M_{\text{BH}} = 0$, the BEC is equivalent to a classical polytrope of index $n = 1$ (see Appendix H). The equation of hydrostatic equilibrium (32) can be solved analytically [6, 13, 20, 22, 31] following standard results [102]. The exact radius is given by Eq. (2). The pulsation can be obtained from the Ledoux formula giving $\omega_{\text{Ledoux}}^2 = 0.123 GM/R_{\text{TF}}^3$ or by numerically solving the Eddington equation of pulsation giving $\omega_{\text{exact}}^2 = 0.121 GM/R_{\text{TF}}^3$ [31]. These exact formulae can be compared to the approximate results from Eqs. (57) and (58).

G. Nongravitational case

In the nongravitational case ($G = 0$), the mass-radius relation is given by

$$M = \frac{2\sigma \frac{\hbar^2}{m^2 R^3} - \lambda \frac{GM_{\text{BH}}}{R^2}}{-6\pi\zeta \frac{a_s \hbar^2}{m^3 R^4}}. \quad (59)$$

The pulsation can be written as

$$\omega^2 = \frac{6\sigma}{\alpha} \frac{\hbar^2}{m^2 R^4} + \frac{24\pi\zeta}{\alpha} \frac{a_s \hbar^2 M}{m^3 R^5} - \frac{2\lambda}{\alpha} \frac{GM_{\text{BH}}}{R^3}. \quad (60)$$

Using the mass-radius relation (59), the identity from Eq. (48) reduces to

$$\omega^2 = \frac{6\pi\zeta}{\alpha} \frac{a_s \hbar^2}{m^3 R^4} \frac{dM}{dR}. \quad (61)$$

1. Repulsive self-interaction

When $a_s > 0$, the mass-radius relation is represented in Fig. 1. The radius increases as the mass increases. There is a minimum radius R_B given by Eq. (49). According to Eq. (61) the equilibrium states are all stable (S) since $a_s > 0$ and $M'(R) > 0$ implying $\omega^2 > 0$.

For $R \rightarrow R_B^+$, the mass tends towards zero. This corresponds to the nongravitational + noninteracting limit (see Sec. III A). In that limit, the pulsation ω_B is given by Eq. (50).

For $R \rightarrow +\infty$, the mass-radius relation is given by Eq. (51). This corresponds to the nongravitational + TF limit (see Sec. III B). In that limit, the pulsation is given by Eq. (52).

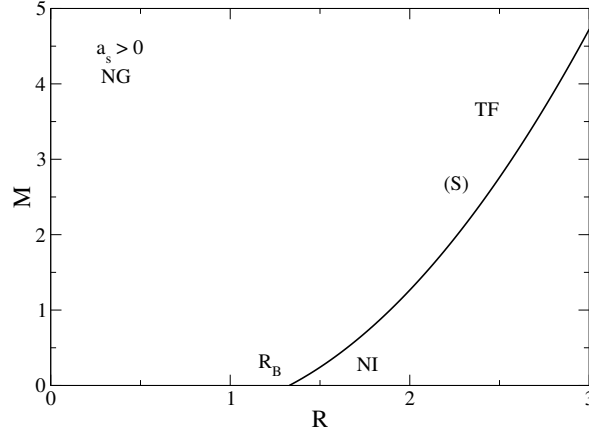


FIG. 1: Mass-radius relation of nongravitational ($G = 0$) BECs with a repulsive self-interaction ($a_s > 0$) in the presence of a central BH. We have normalized the radius by $\hbar^2/GM_{\text{BH}}m^2$ and the mass by $\hbar^2/GM_{\text{BH}}ma_s$. This amounts to taking $\hbar = G = M_{\text{BH}} = m = a_s = 1$ in the dimensional equations.

Comparing Eqs. (49) and (51), we obtain the mass scale

$$M_s^{\text{NG}} \sim \frac{\hbar^2}{GM_{\text{BH}}ma_s}. \quad (62)$$

The noninteracting limit is valid for $M \ll M_s^{\text{NG}}$ and $R \sim R_B$. The TF limit is valid for $M \gg M_s^{\text{NG}}$ and $R \gg R_B$. For a given mass M , the radius of the BEC is given by

$$R = \frac{\sigma m \hbar^2 + \sqrt{\sigma^2 m^2 \hbar^4 + 6\pi\zeta \lambda G m^3 a_s \hbar^2 M_{\text{BH}} M}}{\lambda G M_{\text{BH}} m^3}. \quad (63)$$

There is no equilibrium state without BH. For a given mass M , the radius decreases as the BH mass increases. We have

$$R \sim \frac{2\sigma\hbar^2}{\lambda G m^2 M_{\text{BH}}} \quad (M_{\text{BH}} \rightarrow 0), \quad (64)$$

$$R \sim \left(\frac{6\pi\zeta}{\lambda}\right)^{1/2} \left(\frac{a_s \hbar^2 M}{G m^3 M_{\text{BH}}}\right)^{1/2} \quad (M_{\text{BH}} \rightarrow +\infty). \quad (65)$$

2. Attractive self-interaction

When $a_s < 0$, the mass-radius relation is represented in Fig. 2. There is a maximum mass $M_{\text{max}}^{\text{NG}}$ at R_*^{NG} and a maximum radius R_B given by Eq. (49). According to Eq. (61) the branch where $M(R)$ is decreasing corresponds to stable (S) equilibrium states since $a_s < 0$ and $M'(R) < 0$ implying $\omega^2 > 0$ while the branch where $M(R)$ is increasing corresponds to unstable (U) equilibrium states since $a_s < 0$ and $M'(R) > 0$ implying $\omega^2 < 0$.

For $R \rightarrow 0$, the mass-radius relation is given by Eq. (53). This corresponds to the nongravitational + no BH limit (see Sec. III C). In that limit, the pulsation is given by Eq. (54).

For $R \rightarrow R_B^-$, the mass tends towards zero. This corresponds to the nongravitational + noninteracting limit (see Sec. III A). In that limit, the pulsation ω_B is given by Eq. (50).

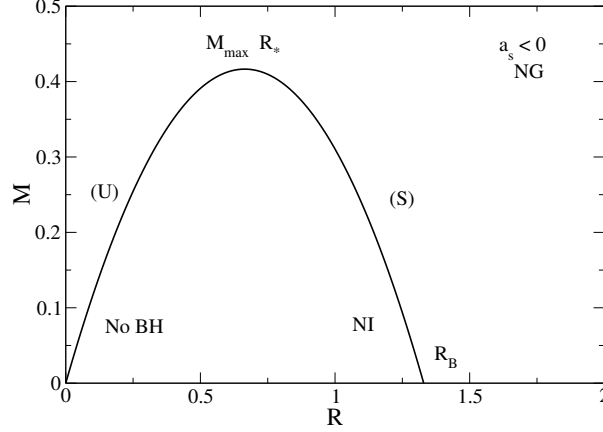


FIG. 2: Mass-radius relation of nongravitational ($G = 0$) BECs with an attractive self-interaction ($a_s < 0$) in the presence of a central BH. We have normalized the radius by $\hbar^2/GM_{\text{BH}}m^2$ and the mass by $\hbar^2/GM_{\text{BH}}m|a_s|$. This amounts to taking $\hbar = G = M_{\text{BH}} = m = |a_s| = 1$ in the dimensional equations.

The maximum mass $M_{\text{max}}^{\text{NG}}$ and the corresponding radius R_*^{NG} are given by

$$M_{\text{max}}^{\text{NG}} = \frac{\sigma^2}{6\pi\zeta\lambda} \frac{\hbar^2}{GM_{\text{BH}}m|a_s|}, \quad (66)$$

$$R_*^{\text{NG}} = \frac{\sigma}{\lambda} \frac{\hbar^2}{GM_{\text{BH}}m^2} = \frac{R_B}{2}. \quad (67)$$

We note the identity

$$M_{\text{max}}^{\text{NG}} = \frac{\sigma}{6\pi\zeta} \frac{m}{|a_s|} R_*^{\text{NG}}. \quad (68)$$

According to Eq. (61) the pulsation vanishes ($\omega^2 = 0$) at the maximum mass ($M'(R) = 0$). The turning point of mass separates stable from unstable equilibrium states in agreement with the Poincaré criterion. On the other hand, we find that there is a maximum pulsation

$$\omega_{\text{max}}^2 = 0.540 \frac{G^4 M_{\text{BH}}^4 m^6}{\hbar^6} \quad \text{at} \quad R_{\omega} = 0.886 \frac{\hbar^2}{GM_{\text{BH}}m^2}. \quad (69)$$

The no BH limit is valid for $M \ll M_{\max}^{\text{NG}}$ and $R \ll R_B$. The noninteracting limit is valid for $M \ll M_{\max}^{\text{NG}}$ and $R \sim R_B$. For a given mass M , the radius of the BEC with or without central black hole is given by

$$R = \frac{\sigma m \hbar^2 \pm \sqrt{\sigma^2 m^2 \hbar^4 - 6\pi\zeta\lambda G m^3 |a_s| \hbar^2 M_{\text{BH}} M}}{\lambda G M_{\text{BH}} m^3}, \quad R_0 = \frac{3\pi\zeta |a_s| M}{\sigma m}. \quad (70)$$

The relative deviation is

$$\frac{\Delta R}{R_0} = \frac{R - R_0}{R_0} = \frac{\sigma^2 \hbar^2}{3\pi\zeta\lambda |a_s| G M_{\text{BH}} m} \left[1 \pm \sqrt{1 - \frac{6\pi\zeta\lambda G m |a_s| M_{\text{BH}} M}{\sigma^2 \hbar^2}} \right] - 1. \quad (71)$$

For a given mass M , there is an equilibrium state only for

$$M_{\text{BH}} \leq (M_{\text{BH}})_{\max}(M) = \frac{\sigma^2 \hbar^2}{6\pi\zeta\lambda G m |a_s| M}. \quad (72)$$

There is no stable solution without BH. On the stable branch, the radius decreases as the BH mass increases. On the unstable branch, the radius increases as the BH mass increases. We have

$$R \sim \frac{2\sigma \hbar^2}{\lambda G m^2 M_{\text{BH}}} \quad (M_{\text{BH}} \rightarrow 0, \text{ stable branch}), \quad (73)$$

$$R \rightarrow \frac{3\pi\zeta |a_s| M}{\sigma m} \quad (M_{\text{BH}} \rightarrow 0, \text{ unstable branch}), \quad (74)$$

$$R \rightarrow \frac{6\pi\zeta |a_s| M}{\sigma m} \quad (M_{\text{BH}} \rightarrow (M_{\text{BH}})_{\max}(M)). \quad (75)$$

H. Noninteracting case

In the noninteracting case ($a_s = 0$), the mass-radius relation is given by

$$M = \frac{2\sigma \frac{\hbar^2}{m^2 R^3} - \lambda \frac{G M_{\text{BH}}}{R^2}}{\nu \frac{G}{R^2}} \quad (76)$$

and the pulsation by

$$\omega^2 = \frac{6\sigma}{\alpha} \frac{\hbar^2}{m^2 R^4} - \frac{2\nu}{\alpha} \frac{G M}{R^3} - \frac{2\lambda}{\alpha} \frac{G M_{\text{BH}}}{R^3}. \quad (77)$$

Using the mass-radius relation (76), the identity from Eq. (48) reduces to

$$\omega^2 = -\frac{\nu}{\alpha} \frac{G}{R^2} \frac{dM}{dR}. \quad (78)$$

The mass-radius relation is represented in Fig. 3. The radius decreases as the mass increases. There is a maximum radius R_B given by Eq. (49). According to Eq. (78) the equilibrium states are all stable (S) since $M'(R) < 0$ implying $\omega^2 > 0$.

For $R \rightarrow 0$ the mass-radius is given by Eq. (55) and the mass tends towards $+\infty$. This corresponds to the noninteracting + no BH limit (see Sec. III E). In that limit, the pulsation is given by Eq. (56).

For $R \rightarrow R_B^-$, the mass tends towards zero. This corresponds to the noninteracting + nongravitational limit (see Sec. III A). In that limit, the pulsation ω_B is given by Eq. (50).

Comparing Eqs. (49) and (55), we obtain the BH mass scale M_{BH} . The no BH limit is valid for $M \gg M_{\text{BH}}$ and $R \ll R_B$. The nongravitational limit is valid for $M \ll M_{\text{BH}}$ and $R \sim R_B$. For a given mass M , the radius of the BEC with or without central BH is given by

$$R = \frac{2\sigma}{\nu} \frac{\hbar^2}{G m^2 (M + \frac{\lambda}{\nu} M_{\text{BH}})}, \quad R_0 = \frac{2\sigma}{\nu} \frac{\hbar^2}{G m^2 M}. \quad (79)$$

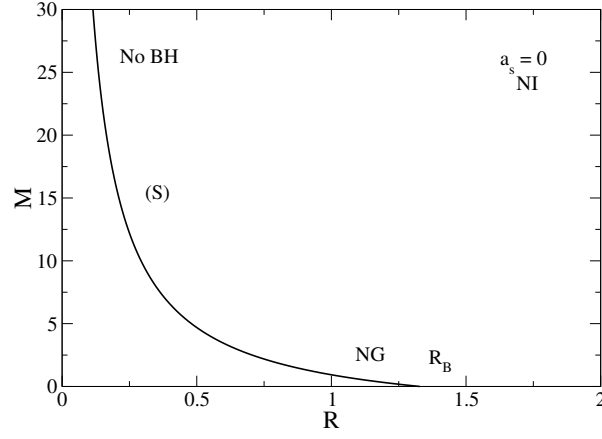


FIG. 3: Mass-radius relation of noninteracting ($a_s = 0$) self-gravitating BECs in the presence of a central BH. We have normalized the radius by $\hbar^2/GM_{\text{BH}}m^2$ and the mass by M_{BH} . This amounts to taking $\hbar = G = M_{\text{BH}} = m = 1$ in the dimensional equations.

The relative deviation is

$$\frac{\Delta R}{R_0} = \frac{R - R_0}{R_0} = -\frac{\lambda}{\nu} \frac{M_{\text{BH}}}{M + \frac{\lambda}{\nu} M_{\text{BH}}}. \quad (80)$$

For a given mass M , the radius decreases as the BH mass increases. We have

$$R \rightarrow \frac{2\sigma}{\nu} \frac{\hbar^2}{Gm^2M} \quad (M_{\text{BH}} \rightarrow 0), \quad (81)$$

$$R \sim \frac{2\sigma}{\lambda} \frac{\hbar^2}{Gm^2M_{\text{BH}}} \quad (M_{\text{BH}} \rightarrow +\infty). \quad (82)$$

I. TF case

In the TF case ($\hbar = 0$), the mass-radius relation is given by

$$M = \frac{-\lambda \frac{GM_{\text{BH}}}{R^2}}{\nu \frac{G}{R^2} - 6\pi\zeta \frac{a_s \hbar^2}{m^3 R^4}} \quad (83)$$

provided that $a_s > 0$ (there is no equilibrium state when $a_s < 0$). The pulsation is given by

$$\omega^2 = -\frac{2\nu}{\alpha} \frac{GM}{R^3} + \frac{24\pi\zeta}{\alpha} \frac{a_s \hbar^2 M}{m^3 R^5} - \frac{2\lambda}{\alpha} \frac{GM_{\text{BH}}}{R^3}. \quad (84)$$

The identity from Eq. (48) reduces to

$$\omega^2 = \frac{\lambda}{\alpha M} \frac{GM_{\text{BH}}}{R^2} \frac{dM}{dR}. \quad (85)$$

The mass-radius relation is represented in Fig. 4. The radius increases as the mass increases. There is a maximum radius R_{TF} given by Eq. (57). According to Eq. (85) the equilibrium states are all stable since $M'(R) > 0$ implying $\omega^2 > 0$.

For $R \rightarrow 0$, the mass-radius relation is given by Eq. (51). This corresponds to the TF + nongravitational limit (see Sec. IIIB). In that limit, the pulsation is given by Eq. (52).

For $R \rightarrow R_{\text{TF}}^-$, the mass tends towards $+\infty$ as

$$M \sim \frac{\lambda}{2\nu} \frac{M_{\text{BH}} R_{\text{TF}}}{R_{\text{TF}} - R}. \quad (86)$$

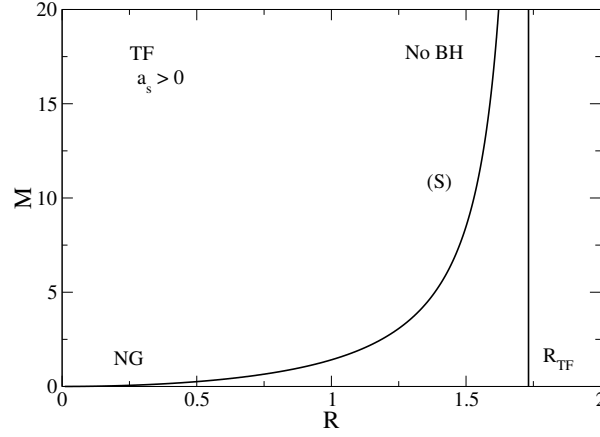


FIG. 4: Mass-radius relation of self-gravitating BECs with a repulsive self-interaction ($a_s > 0$) in the TF limit ($\hbar = 0$) in the presence of a central black hole. We have normalized the radius by $(a_s \hbar^2 / G m^3)^{1/2}$ and the mass by M_{BH} . This amounts to taking $\hbar = G = M_{\text{BH}} = m = a_s = 1$ in the dimensional equations.

This corresponds to the TF + no BH limit (see Sec. III F). The pulsation is given by Eq. (58). It behaves as

$$\omega^2 \sim \frac{\lambda}{\alpha} \frac{G M_{\text{BH}}}{R_{\text{TF}}^2} \frac{1}{R_{\text{TF}} - R}. \quad (87)$$

Comparing Eqs. (51) and (57), we obtain the BH mass scale M_{BH} . The nongravitational limit is valid for $M \ll M_{\text{BH}}$ and $R \ll R_{\text{TF}}$. The no BH limit is valid for $M \gg M_{\text{BH}}$ and $R \sim R_{\text{TF}}$. For a given mass M , the radius of the BEC with or without central BH is given by

$$R = \left(\frac{6\pi\zeta}{\nu} \right)^{1/2} \left(\frac{a_s \hbar^2}{G m^3} \right)^{1/2} \frac{1}{\sqrt{1 + \frac{\lambda}{\nu} \frac{M_{\text{BH}}}{M}}}, \quad R_0 = \left(\frac{6\pi\zeta}{\nu} \right)^{1/2} \left(\frac{a_s \hbar^2}{G m^3} \right)^{1/2} = R_{\text{TF}}. \quad (88)$$

The relative deviation is

$$\frac{\Delta R}{R_0} = \frac{R - R_0}{R_0} = \frac{1}{\sqrt{1 + \frac{\lambda}{\nu} \frac{M_{\text{BH}}}{M}}} - 1. \quad (89)$$

For a given mass M , the radius decreases as the BH mass increases. We have

$$R \rightarrow \left(\frac{6\pi\zeta}{\nu} \right)^{1/2} \left(\frac{a_s \hbar^2}{G m^3} \right)^{1/2} = R_{\text{TF}} \quad (M_{\text{BH}} \rightarrow 0), \quad (90)$$

$$R \sim \left(\frac{6\pi\zeta}{\lambda} \right)^{1/2} \left(\frac{a_s \hbar^2 M}{G m^3 M_{\text{BH}}} \right)^{1/2} \quad (M_{\text{BH}} \rightarrow +\infty). \quad (91)$$

Remark: When $\hbar = 0$, the BEC is equivalent to a classical polytrope of index $n = 1$ in the presence of a central BH. The equation of hydrostatic equilibrium (32) can be solved analytically (see Appendix I). The exact mass-radius relation is given by Eq. (I14) and the pulsation by Eq. (I22). There is a minimum pulsation ω_{min} (see Fig. 17 in Appendix I). These exact formulae can be compared to the approximate results from Eqs. (83) and (84). From the Gaussian ansatz there is a minimum pulsation

$$\omega_{\text{min}}^2 = 1.56 \left(\frac{G m^3}{a_s \hbar^2} \right)^{3/2} G M_{\text{BH}} \quad \text{at} \quad R'_\omega = 1.34 \left(\frac{a_s \hbar^2}{G m^3} \right)^{1/2}. \quad (92)$$

J. No BH case

This case has been treated in detail in Refs. [31, 32].

IV. DIMENSIONLESS STUDY IN THE GENERAL CASE

In this section, we consider the general case. We use the dimensionless variables introduced in our previous papers [74, 85]. For convenience, they are recalled in Appendix A.

A. The effective potential

In terms of the dimensionless variables, the total energy of the self-gravitating BEC, and the equation determining the temporal evolution of its typical radius, are given by

$$E_{\text{tot}} = \frac{1}{2}M \left(\frac{dR}{dt} \right)^2 + V(R) \quad (93)$$

and

$$M \frac{d^2 R}{dt^2} = -V'(R). \quad (94)$$

The effective potential is given by

$$V(R) = \frac{M}{R^2} - \frac{M^2}{R} \pm \frac{M^2}{3R^3} - \frac{\mu M}{R}, \quad (95)$$

where

$$\mu = \frac{\lambda}{\nu} M_{\text{BH}}. \quad (96)$$

We stress that the BH mass has also been normalized by the mass scale from Eq. (A1). Here and in the following, the upper sign corresponds to a repulsive self-interaction ($a_s > 0$) and the lower sign corresponds to an attractive self-interaction ($a_s < 0$).

B. The mass-radius relation

Cancelling the first derivative of the effective potential given by

$$V'(R) = -\frac{2M}{R^3} + \frac{M^2}{R^2} \mp \frac{M^2}{R^4} + \frac{\mu M}{R^2}, \quad (97)$$

we obtain the mass-radius relation

$$M = \frac{2R - \mu R^2}{R^2 \mp 1}. \quad (98)$$

C. The pulsation

The pulsation is given by

$$\omega^2 = \frac{V''(R)}{M}. \quad (99)$$

Since

$$V''(R) = \frac{6M}{R^4} - \frac{2M^2}{R^3} \pm \frac{4M^2}{R^5} - \frac{2\mu M}{R^3}, \quad (100)$$

we obtain

$$\omega^2 = \frac{6}{R^4} - \frac{2M}{R^3} \pm \frac{4M}{R^5} - \frac{2\mu}{R^3}. \quad (101)$$

We also have the identity:

$$\omega^2 = -\frac{\mu}{MR^3} \left(\frac{2}{\mu} - R \right) \frac{dM}{dR}. \quad (102)$$

D. Repulsive self-interaction

For a repulsive self-interaction ($a_s > 0$), the mass-radius relation and the pulsation are given by

$$M = \frac{2R - \mu R^2}{R^2 - 1}, \quad \omega^2 = \frac{6}{R^4} - \frac{2M}{R^3} + \frac{4M}{R^5} - \frac{2\mu}{R^3}. \quad (103)$$

The mass vanishes at the gravitational Bohr radius

$$R_B = \frac{2}{\mu} \quad (104)$$

and is infinite at the TF radius

$$R_{\text{TF}} = 1. \quad (105)$$

There is a critical BH mass corresponding to

$$\mu_* = 2 \quad (106)$$

at which $R_B = R_{\text{TF}} = 1$. In this very special case, the radius of the BEC exists at a unique value $R = 1$ whatever its mass M .

$$1. \quad \mu < \mu_*$$

When $\mu < \mu_*$ we are in the situation where $R_B > R_{\text{TF}}$. The mass-radius relation is plotted in Fig. 5. The radius decreases as the mass increases.⁵ There is a maximum radius R_B and a minimum radius R_{TF} . According to Eq. (102), the equilibrium states are all stable (S) since $R < R_B$ and $M'(R) < 0$, implying $\omega^2 > 0$.

For $R \rightarrow R_{\text{TF}}$, the mass tends towards infinity as

$$M \sim \frac{2 - \mu}{2(R - R_{\text{TF}})}. \quad (107)$$

This corresponds to the TF + no BH limit. In that limit the pulsation tends towards infinity as

$$\omega^2 \sim \frac{2 - \mu}{R - R_{\text{TF}}}. \quad (108)$$

For $R \rightarrow R_B$, the mass tends towards zero as

$$M \sim \frac{2\mu^2}{4 - \mu^2}(R_B - R). \quad (109)$$

This corresponds to the nongravitational + noninteracting limit. In that limit the pulsation tends towards

$$\omega_B = \frac{\mu^2}{2\sqrt{2}}. \quad (110)$$

Substituting R_B given by Eq. (104) into Eq. (107) or substituting R_{TF} given by Eq. (105) into Eq. (109), we find that the transition between these two regimes occurs for $M_t \sim M_{\text{BH}}$. The TF + no BH limit is valid for $M \gg M_{\text{BH}}$ and $R \sim R_{\text{TF}}$. The nongravitational + noninteracting limit is valid for $M \ll M_{\text{BH}}$ and $R \sim R_B$.

⁵ When $\mu < \mu_*$ the mass-radius relation has no extremum. The condition $M'(R) = 0$ yields the second degree equation $R^2 - \mu R + 1 = 0$ whose discriminant $\Delta = \mu^2 - 4$ is negative ($\Delta < 0$).

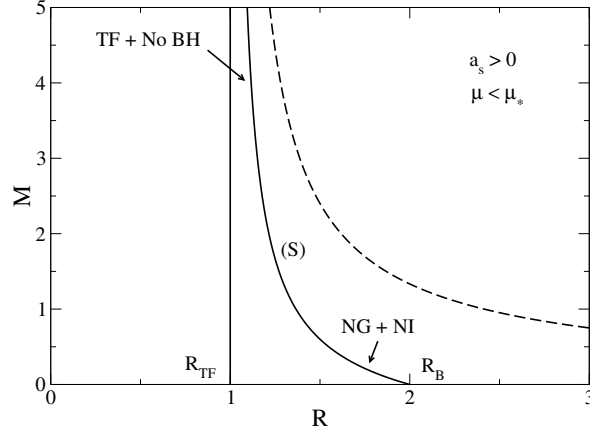


FIG. 5: Mass-radius relation of self-gravitating BECs with a repulsive self-interaction ($a_s > 0$) in the presence of a BH with a mass $\mu < \mu_*$ (specifically $\mu = 1$). The dashed line corresponds to $\mu = 0$ (no black hole) [31].

2. $\mu > \mu_*$

When $\mu > \mu_*$ we are in the situation where $R_B < R_{\text{TF}}$. The mass-radius relation is plotted in Fig. 6. The radius increases as the mass increases.⁶ There is a minimum radius R_B and a maximum radius R_{TF} . According to Eq. (102), the equilibrium states are all stable (S) since $R > R_B$ and $M'(R) > 0$ implying $\omega^2 > 0$.

For $R \rightarrow R_B$, the mass tends towards zero as

$$M \sim \frac{2\mu^2}{\mu^2 - 4}(R - R_B). \quad (111)$$

This corresponds to the nongravitational + noninteracting limit. In that limit the pulsation tends towards

$$\omega_B = \frac{\mu^2}{2\sqrt{2}}. \quad (112)$$

For $R \rightarrow R_{\text{TF}}$, the mass tends towards infinity as

$$M \sim \frac{\mu - 2}{2(R_{\text{TF}} - R)}. \quad (113)$$

This corresponds to the TF + no BH limit. In that limit the pulsation tends towards infinity as

$$\omega^2 \sim \frac{\mu - 2}{R_{\text{TF}} - R}. \quad (114)$$

The nongravitational + noninteracting limit is valid for $M \ll M_{\text{BH}}$ and $R \sim R_B$. The TF + no BH limit is valid for $M \gg M_{\text{BH}}$ and $R \sim R_{\text{TF}}$.

3. General results

For a given mass M , the radius of the BEC with or without central BH is given by

$$R = \frac{1 + \sqrt{1 + (M + \mu)M}}{M + \mu}, \quad R_0 = \frac{1 + \sqrt{1 + M^2}}{M}. \quad (115)$$

⁶ When $\mu > \mu_*$, the mass-radius relation has no extremum in the physical range $[R_B, R_{\text{TF}}]$ where the mass is positive. The condition $M'(R_e) = 0$ yields the second degree equation $R_e^2 - \mu R_e + 1 = 0$, with a positive discriminant $\Delta = \mu^2 - 4 > 0$, which determines the extrema of mass M_e . Combining the equation $R^2 - \mu R + 1 = 0$ with Eq. (103) we get $M_e = -R_e < 0$. Therefore, the extrema of mass correspond to an unphysical negative mass.

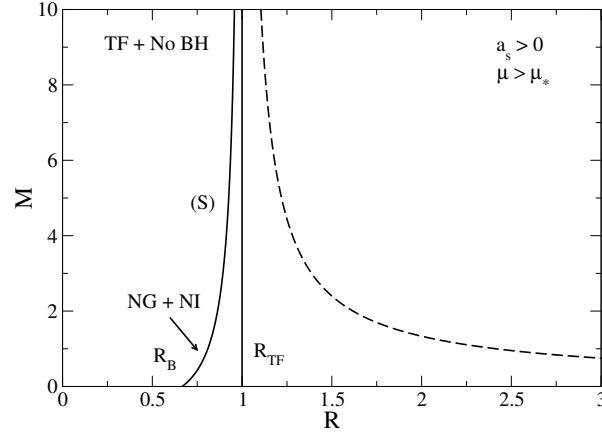


FIG. 6: Mass-radius relation of self-gravitating BECs with a repulsive self-interaction in the presence of a BH with a mass $\mu > \mu_*$ (specifically $\mu = 3$). The dashed line corresponds to $\mu = 0$ (no black hole) [31].

The relative deviation is

$$\frac{\Delta R}{R_0} = \frac{R - R_0}{R_0} = \frac{1 + \sqrt{1 + (M + \mu)M}}{1 + \sqrt{1 + M^2}} \frac{M}{M + \mu} - 1. \quad (116)$$

For a given mass M , the radius decreases as the BH mass increases (see Fig. 7). We have

$$R \rightarrow \frac{1 + \sqrt{1 + M^2}}{M} \quad (\mu \rightarrow 0), \quad (117)$$

$$R \sim \left(\frac{M}{\mu}\right)^{1/2} \quad (\mu \rightarrow +\infty). \quad (118)$$

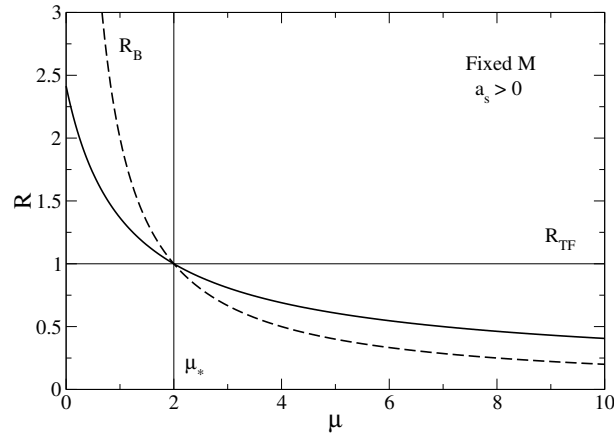


FIG. 7: Radius R of the BEC with a repulsive self-interaction as a function of the BH mass μ for a fixed value of the mass M (specifically $M = 1$). For $\mu < \mu_*$ the radius of the BEC is larger than R_{TF} and smaller than R_B . For $\mu > \mu_*$ the radius of the BEC is larger than R_B and smaller than R_{TF} .

The square complex pulsation ω^2 is plotted as a function of the radius R in Fig. 8 for $\mu > \mu_c \simeq 2.83$. It starts from ω_B^2 at $R = R_B$, decreases, reaches a minimum $\omega_{\min}^2(\mu)$, and increases towards infinity as $R \rightarrow R_{TF}$. The existence of a minimum pulsation $\omega_{\min}(\mu)$ for sufficiently large values of μ is consistent with the exact results obtained in the TF approximation (see Appendix I). We find that the minimum square pulsation increases linearly with the BH mass (see Fig. 9). Its asymptotic behavior for $\mu \rightarrow +\infty$ can be obtained from Eq. (92) yielding $\omega_{\min}^2 \sim 10.8\mu$. For $\mu_* < \mu < \mu_c$, the square pulsation increases monotonically from ω_B^2 at $R = R_B$ to infinity as $R \rightarrow R_{TF}$. For $\mu < \mu_*$, the square pulsation decreases monotonically from infinity as $R \rightarrow R_{TF}$ to ω_B^2 at $R = R_B$.

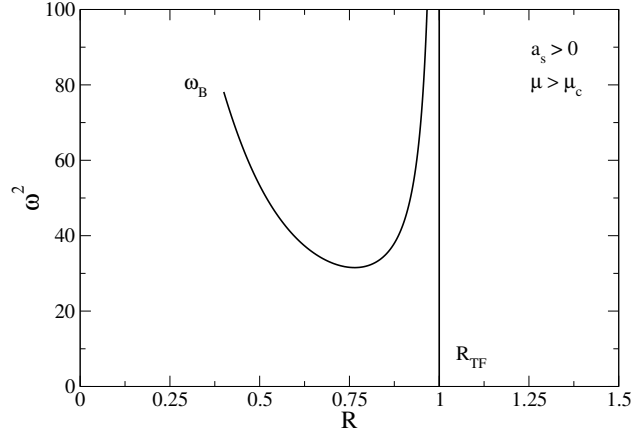


FIG. 8: Square complex pulsation ω^2 as a function of the radius R for self-gravitating BECs with a repulsive self-interaction ($a_s > 0$) in the presence of a BH with a mass $\mu > \mu_c \simeq 2.83$ (specifically $\mu = 5$).

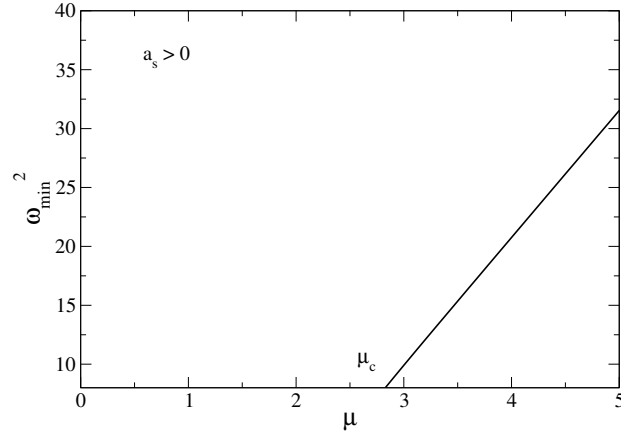


FIG. 9: Minimum square pulsation ω_{\min}^2 as a function of the BH mass μ (a linear fit gives $\omega_{\min}^2 = 10.8\mu - 22.5$).

E. Attractive self-interaction

For an attractive self-interaction ($a_s < 0$) the mass-radius relation and the pulsation are given by

$$M = \frac{2R - \mu R^2}{R^2 + 1}, \quad \omega^2 = \frac{6}{R^4} - \frac{2M}{R^3} - \frac{4M}{R^5} - \frac{2\mu}{R^3}. \quad (119)$$

The mass vanishes at $R = 0$ and at the gravitational Bohr radius R_B given by Eq. (104). The mass-radius relation is plotted in Fig. 10. There is a maximum mass $M_{\max}(\mu)$ and a maximum radius R_B . For a given mass $M < M_{\max}$ there are two branches of solution. According to Eq. (102), the equilibrium states with $R > R_*$ are stable (S) since $R < R_B$ and $M'(R) < 0$, implying $\omega^2 > 0$, while the equilibrium states with $R < R_*$ are unstable (U) since $R < R_B$ and $M'(R) > 0$, implying $\omega^2 < 0$.

For $R \rightarrow 0$, the mass tends towards zero as

$$M \sim 2R. \quad (120)$$

This corresponds to the nongravitational + no BH limit. In that limit the square pulsation tends towards $-\infty$ as

$$\omega^2 \sim -\frac{2}{R^4}. \quad (121)$$

As R increases, the mass $M(R)$ reaches a maximum $M_{\max}(\mu)$ at $R_*(\mu)$ then decreases and vanishes at $R = R_B$. For $R \rightarrow R_B$, the mass tends towards zero as

$$M \sim \frac{2\mu^2}{4 + \mu^2}(R_B - R). \quad (122)$$

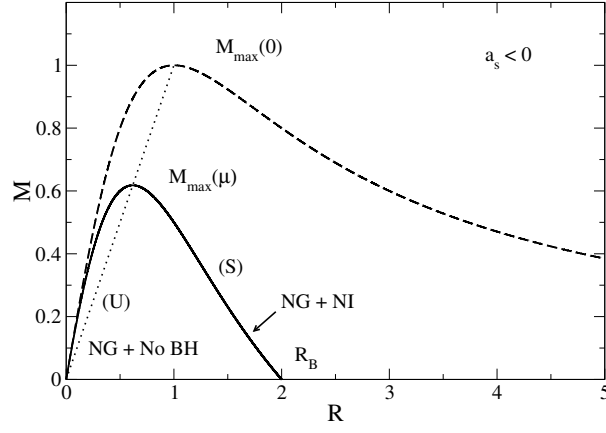


FIG. 10: Mass-radius relation of self-gravitating BECs with an attractive self-interaction ($a_s < 0$) in the presence of a BH with a mass μ (specifically $\mu = 1$). The dashed line corresponds to $\mu = 0$ (no black hole) [31]. The maximum $(R_*(\mu), M_{\max}(\mu))$ of the curve $M(R)$ lies on the straight line $M = R$ (dotted line) parametrized by μ going from zero (top) to infinity (bottom).

This corresponds to the nongravitational + noninteracting limit. In that limit the pulsation tends towards

$$\omega_B = \frac{\mu^2}{2\sqrt{2}}. \quad (123)$$

The condition $M'(R_*) = 0$ yields the second degree equation

$$R_*^2 + \mu R_* - 1 = 0. \quad (124)$$

By eliminating μ between Eqs. (119) and (124) we find that

$$M_{\max}(\mu) = R_*(\mu). \quad (125)$$

By solving Eq. (124) we get

$$M_{\max}(\mu) = R_*(\mu) = -\frac{\mu}{2} + \frac{1}{2}\sqrt{\mu^2 + 4}. \quad (126)$$

The maximum mass (or minimum radius) is plotted as a function of μ in Fig. 11. For $\mu \rightarrow 0$:

$$M_{\max}(\mu) = R_*(\mu) \simeq 1 - \frac{\mu}{2} + \frac{\mu^2}{8} - \frac{\mu^4}{128} + \dots \quad (127)$$

For $\mu \rightarrow +\infty$:

$$M_{\max}(\mu) = R_*(\mu) \sim \frac{1}{\mu}. \quad (128)$$

We note that $R_*(\mu) \sim R_B(\mu)/2$. This returns the results of Sec. III G 2 valid in the nongravitational limit. The maximum mass and the minimum radius are smaller than the values they would have in the absence of a central BH. The nongravitational + no BH limit is valid for $M \ll M_{\max}$ and $R \ll R_*$. The nongravitational + noninteracting limit is valid for $M \ll M_{\max}$ and $R \sim R_B$.

For a given mass M , the radius of the BEC with or without central BH is given by

$$R = \frac{1 \pm \sqrt{1 - (M + \mu)M}}{M + \mu}, \quad R_0 = \frac{1 \pm \sqrt{1 - M^2}}{M}. \quad (129)$$

The relative deviation is

$$\frac{\Delta R}{R_0} = \frac{R - R_0}{R_0} = \frac{1 \pm \sqrt{1 - (M + \mu)M}}{1 \pm \sqrt{1 - M^2}} \frac{M}{M + \mu} - 1. \quad (130)$$

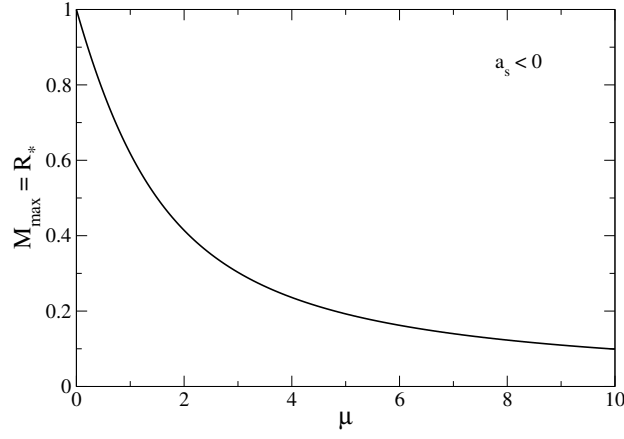


FIG. 11: Maximum mass and minimum radius of self-gravitating BECs with an attractive self-interaction ($a_s < 0$) as a function of the BH mass μ .

For a given mass $M \leq 1$, there is an equilibrium state only for

$$\mu \leq \mu_{\max}(M) = \frac{1 - M^2}{M}. \quad (131)$$

On the stable branch, the radius decreases as the BH mass increases. On the unstable branch, the radius increases as the BH mass increases (see Fig. 12). We have

$$R \rightarrow \frac{1 + \sqrt{1 - M^2}}{M} \quad (\mu \rightarrow 0, \text{ stable branch}), \quad (132)$$

$$R \rightarrow \frac{1 - \sqrt{1 - M^2}}{M} \quad (\mu \rightarrow 0, \text{ unstable branch}), \quad (133)$$

$$R \rightarrow M \quad (\mu \rightarrow \mu_{\max}(M)). \quad (134)$$

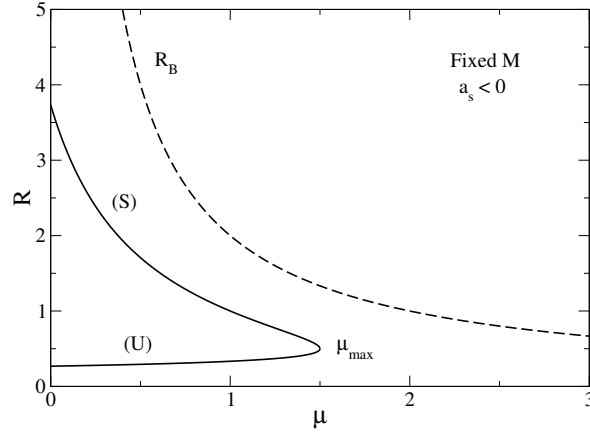


FIG. 12: Radius R of the BEC with an attractive self-interaction as a function of the BH mass μ for a fixed value of the mass $M \leq 1$ (specifically $M = 0.5$).

The square complex pulsation ω^2 is plotted as a function of the radius R in Fig. 13. It starts from $-\infty$ as $R \rightarrow 0$, vanishes at R_* (corresponding to the maximum mass), reaches a maximum $\omega_{\max}^2(\mu)$, and decreases towards ω_B^2 as $R \rightarrow R_B$. The existence of a maximum pulsation ω_{\max} was previously noted in the absence of a BH [31, 32, 74, 85]. We find that the maximum pulsation increases with the BH mass (see Fig. 14). For $\mu = 0$, we have $\omega_{\max}(0) = 0.4246$ (see Appendix I of [85]). For $\mu \rightarrow +\infty$, using Eq. (69) valid in the nongravitational limit, we get $\omega_{\max}^2 \sim 0.211\mu^4$.

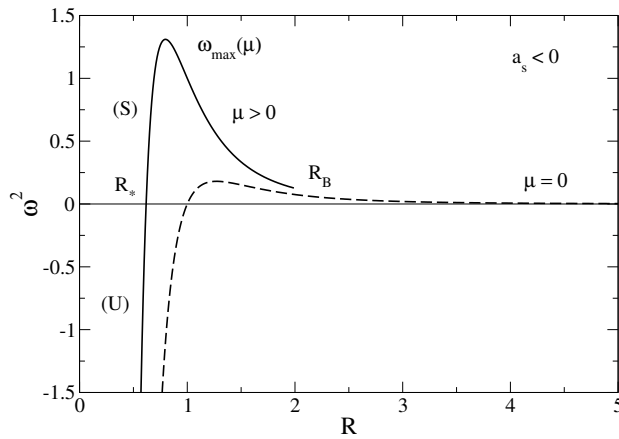


FIG. 13: Square complex pulsation ω^2 as a function of the radius R for self-gravitating BECs with an attractive self-interaction ($a_s < 0$) in the presence of a BH with a mass μ (specifically $\mu = 1$). The dashed line corresponds to $\mu = 0$ (no black hole) [31]. In that case $\omega_{\max}(0) = 0.4246$ corresponding to $R = 1.272$ and $M = 0.9717$ (see Appendix I of [85]).

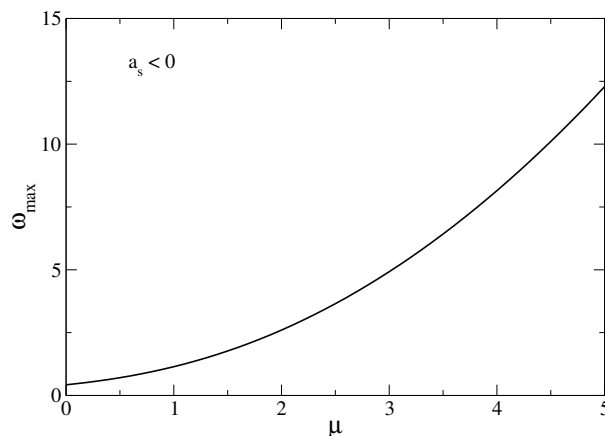


FIG. 14: Maximum pulsation ω_{\max} as a function of the BH mass μ .

V. CONCLUSION

In this paper, we have studied the influence of a black hole that would be present at the center of a self-gravitating BEC representing a dark matter halo. In particular, we have studied how the black hole modifies the mass-radius relation of self-gravitating BECs obtained in our previous papers [31, 32]. These results may find applications in the context of dark matter halos made of bosons like ultralight axions. Using a Gaussian ansatz, we have obtained general analytical results valid for noninteracting and self-interacting bosons.

In the noninteracting case ($a_s = 0$), there exists a stable equilibrium state for any mass M . The radius of the BEC decreases as the mass increases and remains always smaller than the gravitational Bohr radius R_B .

For a repulsive self-interaction ($a_s > 0$), there exists a stable equilibrium state for any mass M . When $\mu < 2$, the radius of the BEC decreases as the mass increases and remains between R_{TF} and R_B . When $\mu > 2$ the radius of the BEC increases as the mass increases and remains between R_B and R_{TF} .

For an attractive self-interaction ($a_s < 0$), there exists a stable equilibrium state only below a maximum mass $M_{\max}(M_{BH})$ generalizing the maximum mass found in [31, 32]. The radius of the BEC decreases as the mass increases and remains between $R_*(M_{BH})$ and R_B . On the other hand, the maximum mass $M_{\max}(M_{BH})$ decreases as the black hole mass increases.

We have resisted the temptation to make numerical applications because they would be too speculative at this stage (see footnote 2). Indeed, we do not know the characteristics (m, a_s) of the dark matter particle, nor the mass of the solitonic core M_c as a function of the halo mass M_h for self-interacting bosons in the presence of a central black hole (work is in progress in this direction). Therefore, we have provided general analytical formulae that can be used easily to make numerical applications once these quantities will be determined. This will be the object of future research.

Appendix A: Dimensionless variables

In this Appendix, we recall the expression of the dimensionless variables introduced in our previous papers [74, 85] and used in Sec. IV. Let us consider a self-gravitating BEC in the absence of a central BH. When the self-interaction of the bosons is attractive ($a_s < 0$), the maximum mass and the corresponding radius of the system are given, within the Gaussian ansatz, by [31]:

$$M_{\max} = \left(\frac{\sigma^2}{6\pi\zeta\nu} \right)^{1/2} \frac{\hbar}{\sqrt{Gm|a_s|}}, \quad (A1)$$

$$R_* = \left(\frac{6\pi\zeta}{\nu} \right)^{1/2} \left(\frac{|a_s|\hbar^2}{Gm^3} \right)^{1/2}. \quad (A2)$$

When the self-interaction of the bosons is repulsive ($a_s > 0$), these scales typically determine the transition between the noninteracting regime and the TF regime (see [31] and the Introduction).

We define a density scale, a pressure scale, an energy scale and a dynamical time scale by

$$\rho_0 = \frac{\sigma\nu}{(6\pi\zeta)^2} \frac{Gm^4}{a_s^2\hbar^2}, \quad P_0 = \frac{2\pi\sigma^2\nu^2}{(6\pi\zeta)^4} \frac{G^2m^5}{|a_s|^3\hbar^2}, \quad (A3)$$

$$V_0 = \frac{\sigma^2\nu^{1/2}}{(6\pi\zeta)^{3/2}} \frac{\hbar m^{1/2}G^{1/2}}{|a_s|^{3/2}}, \quad t_D = \frac{6\pi\zeta}{\nu} \left(\frac{\alpha}{\sigma} \right)^{1/2} \frac{|a_s|\hbar}{Gm^2}. \quad (A4)$$

We note the identities

$$M_{\max} = \frac{\sigma}{\nu} \frac{\hbar^2}{Gm^2R_*}, \quad \rho_0 = \frac{M_{\max}}{R_*^3}, \quad (A5)$$

$$V_0 = \nu \frac{GM_{\max}^2}{R_*}, \quad t_D = \left(\frac{\alpha}{\nu} \right)^{1/2} \frac{1}{\sqrt{G\rho_0}}, \quad (A6)$$

$$\frac{M_{\max}}{R_*} = \frac{\sigma}{6\pi\zeta} \frac{m}{|a_s|}. \quad (A7)$$

Using the scales from Eqs. (A1)-(A4), we introduce the dimensionless variables

$$\hat{M} = \frac{M}{M_{\max}}, \quad \hat{R} = \frac{R}{R_*}, \quad \hat{\rho} = \frac{\rho}{\rho_0}, \quad (A8)$$

$$\hat{P} = \frac{P}{P_0}, \quad \hat{V} = \frac{V}{V_0}, \quad \hat{t} = \frac{t}{t_D}, \quad \hat{\omega} = \omega t_D. \quad (A9)$$

In Sec. IV we work with these dimensionless variables but, in order to simplify the notations, we do not write the “hats”.

Appendix B: Generalized GPP equations with an algebraic and a logarithmic external potential

In Ref. [82] we have developed a general formalism applying to dissipative self-gravitating BECs in d dimensions described by the generalized GPP equations

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \left[\Phi + \frac{dV}{d|\psi|^2} + \Phi_{\text{ext}} \right] \psi - i\frac{\hbar}{2} \xi \left[\ln \left(\frac{\psi}{\psi^*} \right) - \left\langle \ln \left(\frac{\psi}{\psi^*} \right) \right\rangle \right] \psi, \quad (B1)$$

$$\Delta \Phi = S_d G |\psi|^2, \quad (B2)$$

where ξ is the friction coefficient and Φ_{ext} an arbitrary external potential. In this Appendix, we make some of our results more explicit in the case where the external potential Φ_{ext} is algebraic or logarithmic. We give the results without derivation and refer to our paper [82] for technical details.

1. Algebraic and logarithmic external potentials

We consider an algebraic external potential of the form

$$\Phi_A = -\frac{A}{r^s} \quad (\text{B3})$$

and a logarithmic external potential of the form

$$\Phi_L = B \ln r. \quad (\text{B4})$$

The harmonic potential

$$\Phi_H = \frac{1}{2}\omega_0^2 r^2 \quad (\text{B5})$$

is a particular case of Eq. (B3) corresponding to $s = -2$ and $A = -\omega_0^2/2$. The BH (central point mass) potential is given by

$$\Phi_{\text{BH}} = -\frac{1}{d-2} \frac{GM_{\text{BH}}}{r^{d-2}} \quad (d \neq 2), \quad (\text{B6})$$

$$\Phi_{\text{BH}} = GM_{\text{BH}} \ln r \quad (d = 2). \quad (\text{B7})$$

The BH potential in $d \neq 2$ dimensions is a particular case of Eq. (B3) corresponding to $s = d-2$ and $A = GM_{\text{BH}}/(d-2)$ (for $d = 3$, we obtain $s = 1$ and $A = GM_{\text{BH}}$). The BH potential in $d = 2$ dimensions is a particular case of Eq. (B4) corresponding to $B = GM_{\text{BH}}$. In the following, to make the formulae as general and useful as possible, we assume that the system is submitted to an arbitrary algebraic potential (B3), a logarithmic potential (B4), and a harmonic potential (B5).

2. Free energy

The free energy associated with the generalized GPP equations (B1) and (B2) is

$$F = \int \rho \frac{\mathbf{u}^2}{2} d\mathbf{r} + \frac{1}{m} \int \rho Q d\mathbf{r} + \frac{1}{2} \int \rho \Phi d\mathbf{r} + \int \rho \Phi_{\text{ext}} d\mathbf{r} + \int V(\rho) d\mathbf{r}. \quad (\text{B8})$$

This is the sum of the classical kinetic energy Θ_c , the quantum kinetic energy Θ_Q , the gravitational potential energy W , the external potential energy W_{ext} , and the internal energy U . The potential energies associated with the algebraic potential (B3) and with the logarithmic potential (B4) are

$$W_A = \int \rho \Phi_A d\mathbf{r} \quad \text{and} \quad W_L = \int \rho \Phi_L d\mathbf{r}. \quad (\text{B9})$$

The potential energy associated with the harmonic potential (B5) can be written as

$$W_H = \frac{1}{2}\omega_0^2 I, \quad \text{where} \quad I = \int \rho r^2 d\mathbf{r} \quad (\text{B10})$$

is the moment of inertia. The potential energy associated with the BH potential (B6) or (B7) can be written as

$$W_{\text{BH}} = \int \rho \Phi_{\text{BH}} d\mathbf{r}. \quad (\text{B11})$$

The free energy is therefore given by

$$F = \Theta_c + \Theta_Q + W + W_H + W_A + W_L + U. \quad (\text{B12})$$

At equilibrium, it reduces to

$$F = \Theta_Q + W + W_H + W_A + W_L + U. \quad (\text{B13})$$

Remark: In the case of a power-law potential $V(\rho) = K\rho^\gamma/(\gamma - 1)$, leading to a polytropic equation of state $P = K\rho^\gamma$, the internal energy is given by

$$U = \frac{1}{\gamma - 1} \int P d\mathbf{r}. \quad (\text{B14})$$

3. H -theorem and equilibrium states

The generalized GPP equations (B1) and (B2) satisfy an H -theorem for the free energy (B8) (see [82] for details). An equilibrium state extremizes F at fixed mass M . Writing the variational principle as $\delta F - (\mu/m)\delta M = 0$, where μ (chemical potential) is a Lagrange multiplier taking into the conservation of mass, we obtain

$$m\Phi + mV'(\rho) + m\Phi_{\text{ext}} + Q = \mu. \quad (\text{B15})$$

Using Eq. (22), the foregoing equation can be rewritten as

$$\Phi + \int^\rho \frac{P'(\rho')}{\rho'} d\rho' + \Phi_{\text{ext}} + \frac{Q}{m} = \frac{\mu}{m}. \quad (\text{B16})$$

Taking the gradient of this relation, one recovers the condition of quantum hydrostatic equilibrium from Eq. (28). An equilibrium state is (linearly) stable if, and only if, it is a (local) minimum of F at fixed mass M .

4. Virial theorem

The time-dependent scalar virial theorem can be written as

$$\frac{1}{2}\ddot{I} + \frac{1}{2}\xi\dot{I} = 2(\Theta_c + \Theta_Q) + d \int P d\mathbf{r} + W_{ii} + W_{ii}^{\text{ext}}. \quad (\text{B17})$$

At equilibrium ($\ddot{I} = \dot{I} = \Theta_c = 0$), the scalar virial theorem becomes

$$2\Theta_Q + d \int P d\mathbf{r} + W_{ii} + W_{ii}^{\text{ext}} = 0. \quad (\text{B18})$$

The virial of the external force is defined by

$$W_{ii}^{\text{ext}} = - \int \rho \mathbf{r} \cdot \nabla \Phi_{\text{ext}} d\mathbf{r}. \quad (\text{B19})$$

For the algebraic potential (B3) and for the logarithmic potential (B4), we obtain

$$W_{ii}^A = sW_A \quad \text{and} \quad W_{ii}^L = -BM. \quad (\text{B20})$$

For the harmonic potential (B5), we get

$$W_{ii}^H = -2W_H = -\omega_0^2 I. \quad (\text{B21})$$

For the BH potential (B6) and (B7), we get

$$W_{ii}^{\text{BH}} = (d-2)W_{\text{BH}} \quad (d \neq 2), \quad (\text{B22})$$

$$W_{ii}^{\text{BH}} = -GM_{\text{BH}}M \quad (d = 2). \quad (\text{B23})$$

In particular, in $d = 3$ dimensions, we have $W_{ii}^{\text{BH}} = W_{\text{BH}}$. Using the foregoing relations, the time-dependent scalar virial theorem and the scalar virial theorem can be rewritten as

$$\frac{1}{2}\ddot{I} + \frac{1}{2}\xi\dot{I} + \omega_0^2 I = 2(\Theta_c + \Theta_Q) + d \int P d\mathbf{r} + W_{ii} + sW_A - BM, \quad (\text{B24})$$

$$2\Theta_Q + d \int P d\mathbf{r} + W_{ii} - \omega_0^2 I + sW_A - BM = 0. \quad (\text{B25})$$

In the case of a polytropic equation of state, the integral $\int P d\mathbf{r}$ appearing in the virial theorem can be related to the internal energy U by using Eq. (B14).

Remark: If we consider the nongravitational limit ($G = 0$) and the dissipationless case ($\xi = 0$) where the free energy F is conserved, we can combine Eqs. (B12) and (B24) to obtain

$$\frac{1}{2}\ddot{I} + 2\omega_0^2 I = 2F - 2U + d \int P d\mathbf{r} + (s - 2)W_A - 2W_L - BM. \quad (\text{B26})$$

For a polytropic equation of state $P = K\rho^\gamma$, using Eq. (B14), we get

$$\frac{1}{2}\ddot{I} + 2\omega_0^2 I = 2F + [d(\gamma - 1) - 2]U + (s - 2)W_A - 2W_L - BM. \quad (\text{B27})$$

For the critical polytropic index $\gamma_c = 1 + 2/d$ (i.e. $n_c = d/2$) [103], for an algebraic potential of index $s = 2$ (this potential turns out to have special properties), and in the absence of logarithmic potential ($B = 0$), we get the closed equation

$$\frac{1}{2}\ddot{I} + 2\omega_0^2 I = 2F. \quad (\text{B28})$$

It has the solution $I(t) = A \cos(2\omega_0 t + \phi) + F/\omega_0^2$. This result is valid for a repulsive ($K > 0$) or an attractive ($K < 0$) self-interaction. In $d = 2$ dimensions, the critical index is $\gamma_c = 2$, corresponding to the standard BEC.

Remark: If we consider the strong friction limit ($\xi \rightarrow +\infty$), the TF approximation ($\hbar = 0$), a two-dimensional system ($d = 2$) and an isothermal equation of state $P = \rho k_B T/m$, the scalar virial theorem (B24) reduces, in the absence of algebraic potential ($A = 0$), to

$$\frac{1}{2}\xi \dot{I} + \omega_0^2 I = 2Nk_B T - \frac{GM^2}{2} - BM. \quad (\text{B29})$$

This is a closed equation. At equilibrium, we get the identity

$$\omega_0^2 I = 2Nk_B T - \frac{GM^2}{2} - BM. \quad (\text{B30})$$

With respect to the study performed in Sec. 5.1.5 of Ref. [82], the external logarithmic potential simply shifts the critical temperature to the value

$$k_B T_c = \frac{GMm}{4} + \frac{Bm}{2}. \quad (\text{B31})$$

5. Eigenenergy

If we consider a wave function of the form

$$\psi(\mathbf{r}, t) = \phi(\mathbf{r})e^{-iEt/\hbar}, \quad (\text{B32})$$

where $\phi(\mathbf{r}) = \sqrt{\rho(\mathbf{r})}$ is real, and substitute Eq. (B32) into Eqs. (B1) and (B2), we obtain the time-independent generalized GPP equations

$$-\frac{\hbar^2}{2m}\Delta\phi + m(\Phi + V'(\rho) + \Phi_{\text{ext}})\phi = E\phi, \quad (\text{B33})$$

$$\Delta\Phi = S_d G \phi^2. \quad (\text{B34})$$

Equations (B33) and (B34) define a nonlinear eigenvalue problem for the wave function $\phi(\mathbf{r})$ where the eigenvalue E is the energy (eigenenergy). Dividing Eq. (B33) by $\phi(\mathbf{r})$ and using $\rho = \phi^2$, we get

$$m\Phi + mV'(\rho) + m\Phi_{\text{ext}} + Q = E. \quad (\text{B35})$$

This equation coincides with Eq. (B15) provided that we make the identification $E = \mu$. Taking the gradient of this relation, one recovers the condition of quantum hydrostatic equilibrium from Eq. (28). Finally, multiplying Eq. (B35) by ρ and integrating over the whole configuration, we obtain the identity

$$NE = 2W + \int \rho V'(\rho) d\mathbf{r} + W_{\text{ext}} + \Theta_Q. \quad (\text{B36})$$

Remark: In the case of a power-law potential $V(\rho) = K\rho^\gamma/(\gamma - 1)$, leading to a polytropic equation of state $P = K\rho^\gamma$, Eqs. (B35) and (B36) take the form

$$\Phi + \frac{K\gamma}{\gamma - 1}\rho^{\gamma-1} + \Phi_{\text{ext}} + \frac{Q}{m} = \frac{E}{m}, \quad (\text{B37})$$

$$NE = 2W + \gamma U + W_{\text{ext}} + \Theta_Q, \quad (\text{B38})$$

where U is given by Eq. (B14).

6. The Gaussian ansatz

Making a Gaussian ansatz for the wave function, we find that the potential energy associated with the algebraic potential (B3) is given by

$$W_A = -\lambda \frac{MA}{R^s} \quad (\text{B39})$$

with

$$\lambda = \frac{1}{\Gamma(d/2)} \int_0^{+\infty} e^{-t} t^{(d-2-s)/2} dt = \frac{\Gamma[(d-s)/2]}{\Gamma(d/2)}. \quad (\text{B40})$$

For the logarithmic potential (B4), we find that

$$W_L = MB \ln R + D \quad (\text{B41})$$

with

$$D = \frac{MB}{2\Gamma(d/2)} \int_0^{+\infty} e^{-t} \ln(t) t^{(d-2)/2} dt = \frac{MB}{2} \psi\left(\frac{d}{2}\right), \quad (\text{B42})$$

where $\psi(z)$ is the digamma function. We recall that $\psi(1) = -\gamma_E = -0.577216\dots$ where γ_E is the Euler constant. For the harmonic potential (B5), denoting λ_H by α in order to be consistent with the notations from Ref. [82], we get

$$W_H = \frac{1}{2} \alpha \omega_0^2 M R^2 \quad \text{with} \quad \alpha = \frac{d}{2}. \quad (\text{B43})$$

For the BH potential (B6) and (B7), we get

$$W_{\text{BH}} = -\frac{\lambda_{\text{BH}}}{d-2} \frac{GM_{\text{BH}}M}{R^{d-2}} \quad \text{with} \quad \lambda_{\text{BH}} = \frac{1}{\Gamma(d/2)} \quad (d \neq 2), \quad (\text{B44})$$

$$W_{\text{BH}} = GM_{\text{BH}}M \ln R + D \quad \text{with} \quad D = \frac{GM_{\text{BH}}M}{2} \psi(1) \quad (d = 2). \quad (\text{B45})$$

In particular, in $d = 3$, we get $W_{\text{BH}} = -\lambda_{\text{BH}} GM_{\text{BH}}M/R$ with $\lambda_{\text{BH}} = 2/\pi^{1/2}$.

We now consider the generalized model of BECDM halos of Ref. [82] corresponding to an equation of state which is the sum of an isothermal equation of state and a polytropic equation of state: $P = \rho k_B T/m + K\rho^\gamma$. The free energy functional (B12) can be written as a function of R and \dot{R} (for a fixed mass M) as

$$F = \frac{1}{2} \alpha M \left(\frac{dR}{dt} \right)^2 + V(R), \quad (\text{B46})$$

with

$$V(R) = \sigma \frac{\hbar^2 M}{m^2 R^2} - \frac{\nu}{d-2} \frac{GM^2}{R^{d-2}} + \frac{1}{2} \omega_0^2 \alpha M R^2 + \frac{\zeta}{\gamma-1} \frac{KM^\gamma}{R^{d(\gamma-1)}} - d \frac{M k_B T}{m} \ln R + C - \lambda \frac{MA}{R^s} + MB \ln R + D \quad (d \neq 2), \quad (\text{B47})$$

$$V(R) = \sigma \frac{\hbar^2 M}{m^2 R^2} + \frac{1}{2} G M^2 \ln R + \frac{1}{2} \omega_0^2 \alpha M R^2 + \frac{\zeta}{\gamma - 1} \frac{K M^\gamma}{R^{2(\gamma-1)}} - 2 \frac{M k_B T}{m} \ln R + W_0 - \lambda \frac{M A}{R^s} + M B \ln R + D \quad (d = 2). \quad (\text{B48})$$

Equation (B46) can be interpreted as the total energy of a fictive particle with effective mass αM and position R moving in a potential $V(R)$. The first term is the classical kinetic energy Θ_c and the second term is the potential energy V including the quantum kinetic energy Θ_Q , the gravitational potential energy W , the potential energy W_H associated with the harmonic external potential, the internal energy U associated with the polytropic equation of state, the internal energy U_B associated with the isothermal equation of state, the potential energy W_A associated with the algebraic external potential, and the potential energy W_L associated with the logarithmic external potential. An equilibrium state is an extremum of $V(R)$. This leads to the general mass-radius relation

$$-2\sigma \frac{\hbar^2 M}{m^2 R^3} + \nu \frac{G M^2}{R^{d-1}} + \omega_0^2 \alpha M R - d \zeta \frac{K M^\gamma}{R^{d(\gamma-1)+1}} - d \frac{M k_B T}{m R} + \lambda s \frac{M A}{R^{s+1}} + \frac{M B}{R} = 0. \quad (\text{B49})$$

This relation can also be obtained from the virial theorem [82]. The complex pulsation $\omega^2 = (1/\alpha M) V''(R)$ describing the evolution of a small perturbation about equilibrium is given by

$$\omega^2 = \omega_0^2 + \frac{6\sigma}{\alpha} \frac{\hbar^2}{m^2 R^4} + [d(\gamma - 1) + 1] \frac{d\zeta}{\alpha} \frac{K M^{\gamma-1}}{R^{d(\gamma-1)+2}} - \frac{(d-1)\nu}{\alpha} \frac{G M}{R^d} + \frac{d}{\alpha} \frac{k_B T}{m R^2} - \frac{\lambda}{\alpha} s(s+1) \frac{A}{R^{s+2}} - \frac{B}{\alpha R^2}. \quad (\text{B50})$$

It can be expressed under the form

$$\omega^2 = \frac{6\Theta_Q + [d(\gamma - 1) + 1]d(\gamma - 1)U + (d-1)W_{ii} + \omega_0^2 I + dNk_B T + s(s+1)W_A - MB}{I}. \quad (\text{B51})$$

Alternative expressions of the pulsation can be obtained by combining Eq. (B51) with the equilibrium free energy (B12) in the case where the free energy is conserved ($\xi = 0$), or with the equilibrium virial theorem (B25). We also note the identity

$$\omega^2(R) = -\frac{1}{\alpha M} \left(\frac{2\sigma \hbar^2}{m^2 R^3} - \omega_0^2 \alpha R + \frac{dK\zeta(2-\gamma)M^{\gamma-1}}{R^{d(\gamma-1)+1}} + \frac{dk_B T}{m R} - \lambda s \frac{A}{R^{s+1}} - \frac{B}{R} \right) \frac{dM}{dR}, \quad (\text{B52})$$

which is related to the Poincaré turning point criterion. Let us consider particular cases of Eq. (B51).

For classical polytropes ($\Theta_Q = T = 0$), the virial theorem reduces to $d(\gamma - 1)U + W_{ii} - \omega_0^2 I + sW_A - BM = 0$ and the complex pulsation can be written as

$$\omega^2 = \frac{(2d - 2 - d\gamma)W_{ii} + (d\gamma - d + 2)\omega_0^2 I + s(s + d - d\gamma)W_A + d(\gamma - 1)MB}{I}. \quad (\text{B53})$$

For $d = 3$ and $\omega_0 = A = B = 0$, using $W_{ii} = W$, we recover the Ledoux formula $\omega^2 = (4 - 3\gamma)W/I$ [104] (see Refs. [105, 106] for generalizations).

For classical isothermal spheres ($\Theta_Q = U = 0$), the virial theorem reduces to $W_{ii} - \omega_0^2 I + dNk_B T + sW_A - BM = 0$ and the complex pulsation can be written as

$$\omega^2 = \frac{(d-2)W_{ii} + 2\omega_0^2 I + s^2 W_A}{I} \quad \text{or} \quad \omega^2 = \frac{(2-d)dNk_B T + d\omega_0^2 I + (s+2-d)sW_A + (d-2)MB}{I}. \quad (\text{B54})$$

For $d = 2$ and $A = 0$, we obtain $\omega^2 = 2\omega_0^2$ and the virial theorem leads to identity (B30).

In the noninteracting case ($U = 0$), the virial theorem reduces to $2\Theta_Q + W_{ii} - \omega_0^2 I + dNk_B T + sW_A - BM = 0$ and the complex pulsation can be written as

$$\omega^2 = \frac{(d-4)W_{ii} + 4\omega_0^2 I - 2dNk_B T + s(s-2)W_A + 2BM}{I}. \quad (\text{B55})$$

For nongravitational ($G = 0$) polytropes ($T = 0$), the virial theorem reduces to $2\Theta_Q + d(\gamma - 1)U - \omega_0^2 I + sW_A - BM = 0$ and the complex pulsation can be written as

$$\omega^2 = \frac{d(\gamma - 1)[d(\gamma - 1) - 2]U + 4\omega_0^2 I + s(s-2)W_A + 2BM}{I}. \quad (\text{B56})$$

For the critical index $\gamma_c = 1 + 2/d$ [103], for $s = 2$ and for $B = 0$, we obtain $\omega^2 = 4\omega_0^2$ in agreement with Eq. (B28). In the TF approximation ($\Theta_Q = 0$), the virial theorem reduces to $d(\gamma - 1)U - \omega_0^2 I + sW_A - BM = 0$ and the complex pulsation becomes

$$\omega^2 = \frac{[d(\gamma - 1) + 2]\omega_0^2 I + s[s - d(\gamma - 1)]W_A + d(\gamma - 1)BM}{I}. \quad (\text{B57})$$

Appendix C: Gravitational energy of a polytropic sphere with an external potential

A simple analytical formula due to Betti and Ritter [102] can be obtained for the gravitational energy of a polytropic sphere:

$$W = -\frac{3}{5-n} \frac{GM^2}{R} \quad (d=3). \quad (\text{C1})$$

In this Appendix, we determine the proper generalization of this formula in the case where the polytrope is submitted to an arbitrary external potential. Simplifications are given for the algebraic potential (including the harmonic potential and the BH potential) and for the logarithmic potential.

1. General expression

For classical self-gravitating systems, or for self-gravitating BECs in the TF approximation, the condition of hydrostatic equilibrium can be written as

$$\nabla P + \rho \nabla \Phi + \rho \nabla \Phi_{\text{ext}} = \mathbf{0}. \quad (\text{C2})$$

For a polytropic equation of state of the form

$$P = K\rho^\gamma \quad \text{with} \quad \gamma = 1 + \frac{1}{n}, \quad (\text{C3})$$

we have

$$\frac{\nabla P}{\rho} = (n+1) \nabla \left(\frac{P}{\rho} \right). \quad (\text{C4})$$

As a result, the condition of hydrostatic equilibrium (C2) can be integrated into

$$(n+1) \frac{P}{\rho} + \Phi + \Phi_{\text{ext}} = \frac{E}{m}, \quad (\text{C5})$$

where E is a constant of integration representing the eigenenergy of the BEC (see Sec. B 5). Multiplying Eq. (C5) by ρ and integrating over the whole configuration, we obtain the identity

$$NE = (n+1) \int P d\mathbf{r} + 2W + W_{\text{ext}}. \quad (\text{C6})$$

Assuming $\gamma > 1$ (i.e. $0 \leq n < +\infty$) so that $P/\rho = 0$ on the boundary of the system $r = R$ where the density vanishes, we find from Eq. (C5) that

$$\frac{E}{m} = \Phi(R) + \Phi_{\text{ext}}(R). \quad (\text{C7})$$

This equation determines the eigenenergy E if we recall that [82]

$$\Phi(R) = -\frac{1}{d-2} \frac{GM}{R^{d-2}} \quad (d \neq 2), \quad (\text{C8})$$

$$\Phi(R) = GM \ln R \quad (d = 2). \quad (\text{C9})$$

As a result, Eq. (C6) can be rewritten as

$$(n+1) \int P d\mathbf{r} = M\Phi(R) + M\Phi_{\text{ext}}(R) - 2W - W_{\text{ext}}. \quad (\text{C10})$$

Combining this relation with the equilibrium scalar virial theorem (see Sec. B 4)

$$d \int P d\mathbf{r} + W_{ii} + W_{ii}^{\text{ext}} = 0, \quad (\text{C11})$$

we obtain the general identity

$$(n+1)W_{ii} - 2dW = -dM\Phi(R) - dM\Phi_{\text{ext}}(R) + dW_{\text{ext}} - (n+1)W_{ii}^{\text{ext}} \quad (\text{C12})$$

determining the gravitational energy W of a classical polytropic sphere submitted to an external potential. More explicit expressions are given below.

2. $d \neq 2$

When $d \neq 2$, using $W_{ii} = (d-2)W$ [82] and Eq. (C8), we find from Eq. (C12) that the gravitational energy is given by

$$W = \frac{1}{(d-2)n - (d+2)} \left[\frac{d}{d-2} \frac{GM^2}{R^{d-2}} - dM\Phi_{\text{ext}}(R) + dW_{\text{ext}} - (n+1)W_{ii}^{\text{ext}} \right]. \quad (\text{C13})$$

In the absence of external potential, we recover the Betti-Ritter formula in d dimensions [107]:

$$W = \frac{d}{(d-2)n - (d+2)} \frac{GM^2}{(d-2)R^{d-2}}. \quad (\text{C14})$$

For the algebraic potential (B3), using Eq. (B20), Eq. (C13) takes the form

$$W = \frac{1}{(d-2)n - (d+2)} \left[\frac{d}{d-2} \frac{GM^2}{R^{d-2}} + \frac{dMA}{R^s} + [d - s(n+1)] W_A \right]. \quad (\text{C15})$$

In particular, for the harmonic potential (B5), using Eq. (B10), we get

$$W = \frac{1}{(d-2)n - (d+2)} \left[\frac{d}{d-2} \frac{GM^2}{R^{d-2}} - \frac{dM\omega_0^2}{2} R^2 + \frac{1}{2}(d+2+2n)\omega_0^2 I \right]. \quad (\text{C16})$$

For the BH potential (B6) we get

$$W = \frac{1}{(d-2)n - (d+2)} \left[\frac{d}{d-2} \frac{GM^2}{R^{d-2}} + \frac{d}{d-2} \frac{GM_{\text{BH}}M}{R^{d-2}} + [d - (d-2)(n+1)] W_{\text{BH}} \right]. \quad (\text{C17})$$

In particular, in $d = 3$, we obtain

$$W = \frac{1}{n-5} \left[\frac{3GM^2}{R} + \frac{3GM_{\text{BH}}M}{R} + (2-n)W_{\text{BH}} \right]. \quad (\text{C18})$$

A closed expression is obtained for $n = 2$. Finally, for the logarithmic potential (B4), using Eq. (B20), Eq. (C13) takes the form

$$W = \frac{1}{(d-2)n - (d+2)} \left[\frac{d}{d-2} \frac{GM^2}{R^{d-2}} - dMB \ln R + dW_L + (n+1)BM \right]. \quad (\text{C19})$$

3. $d = 2$

When $d = 2$, using $W_{ii} = -GM^2/2$ [82] and Eq. (C9), we find from Eq. (C12) that the gravitational energy is given by

$$W = -\frac{n+1}{8}GM^2 + \frac{1}{2}GM^2 \ln R + \frac{1}{2}M\Phi_{\text{ext}}(R) - \frac{1}{2}W_{\text{ext}} + \frac{1}{4}(n+1)W_{ii}^{\text{ext}}. \quad (\text{C20})$$

For the algebraic potential (B3), using Eq. (B20), it takes the form

$$W = -\frac{n+1}{8}GM^2 + \frac{1}{2}GM^2 \ln R - \frac{MA}{2R^s} - \frac{1}{4}[2 - s(n+1)] W_A. \quad (\text{C21})$$

In particular, for the harmonic potential (B5), using Eq. (B10), we get

$$W = -\frac{n+1}{8}GM^2 + \frac{1}{2}GM^2 \ln R + \frac{M\omega_0^2}{4}R^2 - \frac{1}{4}(2+n)\omega_0^2 I. \quad (\text{C22})$$

On the other hand, for the logarithmic potential (B4), using Eq. (B20), we obtain

$$W = -\frac{n+1}{8}GM^2 + \frac{1}{2}GM^2 \ln R + \frac{1}{2}MB \ln R - \frac{1}{2}W_L - \frac{1}{4}(n+1)BM. \quad (\text{C23})$$

In particular, for the BH potential (B7), we get

$$W = -\frac{n+1}{8}GM^2 + \frac{1}{2}GM^2 \ln R + \frac{1}{2}GM_{\text{BH}}M \ln R - \frac{1}{2}W_{\text{BH}} - \frac{1}{4}(n+1)GM_{\text{BH}}M. \quad (\text{C24})$$

Appendix D: Exact nongravitational + noninteracting case with the BH potential

In the nongravitational + noninteracting case ($G = a_s = 0$), the wave function of the BEC is determined by the Schrödinger equation with the BH potential:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi - \frac{GM_{\text{BH}}m}{r} \psi. \quad (\text{D1})$$

This equation is similar to the Schrödinger equation with a Coulombian potential describing the Bohr atom. Therefore, a BEC in the nongravitational + noninteracting case is equivalent to a gravitational Bohr atom. Considering a stationary solution of the form

$$\psi(\mathbf{r}, t) = \phi(\mathbf{r}) e^{-iEt/\hbar}, \quad (\text{D2})$$

where E is real we obtain the eigenvalue equation

$$-\frac{\hbar^2}{2m} \Delta \phi - \frac{GM_{\text{BH}}m}{r} \phi = E\phi \quad (\text{D3})$$

or, equivalently,

$$\Delta \phi + \frac{2m}{\hbar^2} \left(E + \frac{GM_{\text{BH}}m}{r} \right) \phi = 0. \quad (\text{D4})$$

Looking for a solution of the form

$$\phi = Ae^{-\gamma r}, \quad (\text{D5})$$

we find that

$$\gamma = \frac{GM_{\text{BH}}m^2}{\hbar^2} \quad \text{and} \quad E = -\frac{\hbar^2\gamma^2}{2m} = -\frac{G^2M_{\text{BH}}^2m^3}{2\hbar^2}. \quad (\text{D6})$$

Since the wave function (D5) has no node, it corresponds to the ground state of the gravitational Bohr atom. The total mass of the BEC is given by

$$M = \int_0^{+\infty} \phi^2 4\pi r^2 dr, \quad (\text{D7})$$

implying

$$A = \left(\frac{M\gamma^3}{\pi} \right)^{1/2} \quad \text{and} \quad \phi = \left(\frac{M}{\pi} \right)^{1/2} \gamma^{3/2} e^{-\gamma r}. \quad (\text{D8})$$

The density of the BEC is

$$\rho = \frac{M}{\pi} \gamma^3 e^{-2\gamma r}. \quad (\text{D9})$$

The central density is

$$\rho_0 = \frac{M}{\pi} \gamma^3 = \frac{G^3 M_{\text{BH}}^3 m^6 M}{\pi \hbar^6}. \quad (\text{D10})$$

Let R_{99} denote the radius containing 99% of the mass. It is given by $R_{99} = \xi_{99}/(2\gamma)$ where ξ_{99} is determined by the equation

$$\frac{\int_0^{\xi_{99}} e^{-\xi} \xi^2 d\xi}{\int_0^{+\infty} e^{-\xi} \xi^2 d\xi} = 0.99, \quad (\text{D11})$$

giving $\xi_{99} = 8.406\dots$. Therefore, the exact radius of the BEC is given by

$$R_{99} = 4.203 \frac{\hbar^2}{GM_{\text{BH}}m^2}. \quad (\text{D12})$$

This is the gravitational Bohr radius. We note that it is independent of the mass M of the BEC. It can be compared to the expression (49) obtained from the Gaussian ansatz. We note that the exact density profile ρ is exponential instead of being Gaussian.

In the nongravitational + noninteracting case, the exact equilibrium relations from Sec. IID reduce to

$$E_{\text{tot}} = \Theta_Q + W_{\text{BH}}, \quad NE = W_{\text{BH}} + \Theta_Q, \quad 2\Theta_Q + W_{\text{BH}} = 0. \quad (\text{D13})$$

We note that $E_{\text{tot}} = NE = -\Theta_Q = W_{\text{BH}}/2$. A direct calculation using Eq. (D9) gives

$$W_{\text{BH}} = -GM_{\text{BH}}M\gamma = -\frac{G^2M_{\text{BH}}^2Mm^2}{\hbar^2}, \quad (\text{D14})$$

$$\Theta_Q = \frac{\hbar^2M\gamma^2}{2m^2} = \frac{G^2M_{\text{BH}}^2Mm^2}{2\hbar^2}, \quad (\text{D15})$$

$$I = \frac{3M}{\gamma^2} = \frac{3M\hbar^4}{G^2M_{\text{BH}}^2m^4}. \quad (\text{D16})$$

We can check that the relations of Eq. (D13) are satisfied. On the other hand, Eq. (B51) reduces to

$$\omega^2 = \frac{6\Theta_Q + 2W_{\text{BH}}}{I}. \quad (\text{D17})$$

Using the virial theorem from Eq. (D13), we obtain

$$\omega^2 = -\frac{W_{\text{BH}}}{I} = \frac{2\Theta_Q}{I}. \quad (\text{D18})$$

Finally, using Eq. (D16), we get

$$\omega^2 = \frac{GM_{\text{BH}}\gamma^3}{3} = \frac{G^4M_{\text{BH}}^4m^6}{3\hbar^6}. \quad (\text{D19})$$

Appendix E: Exact nongravitational + TF case with the BH potential

In the nongravitational + TF case ($G = \hbar = 0$), the condition of hydrostatic equilibrium can be written as

$$\nabla P + \rho \nabla \Phi_{\text{ext}} = \mathbf{0}. \quad (\text{E1})$$

For a polytropic equation of state of the form

$$P = K\rho^\gamma \quad \text{with} \quad \gamma = 1 + \frac{1}{n} \quad (\text{E2})$$

we have

$$\frac{\nabla P}{\rho} = (n+1)\nabla\left(\frac{P}{\rho}\right). \quad (\text{E3})$$

As a result, the condition of hydrostatic equilibrium (E1) can be integrated into

$$(n+1)K\rho^{1/n} + \Phi_{\text{ext}} = \frac{E}{m}, \quad (\text{E4})$$

where E is a constant of integration representing the eigenenergy of the BEC (see Sec. B5). Assuming $\gamma > 1$ (i.e. $0 \leq n < +\infty$) so that $P/\rho = 0$ on the boundary of the system $r = R$ where the density vanishes, we find from Eq. (E4) that

$$\frac{E}{m} = \Phi_{\text{ext}}(R). \quad (\text{E5})$$

As a result, Eq. (E4) can be rewritten as

$$\rho(r) = \left[\frac{\Phi_{\text{ext}}(R) - \Phi_{\text{ext}}(r)}{(n+1)K} \right]^n. \quad (\text{E6})$$

This equation determines the density profile $\rho(r)$ of the BEC in the TF approximation for an arbitrary external potential. For the BH potential given by Eq. (12), assuming $K > 0$, we obtain

$$\rho(r) = \left[\frac{GM_{\text{BH}}}{K(n+1)} \right]^n \left(\frac{1}{r} - \frac{1}{R} \right)^n. \quad (\text{E7})$$

For $r \rightarrow 0$, the density behaves as

$$\rho(r) \sim \left[\frac{GM_{\text{BH}}}{K(n+1)} \right]^n \frac{1}{r^n}. \quad (\text{E8})$$

It is normalisable provided that $n < 3$. Multiplying Eq. (E7) by $4\pi r^2$ and integrating over the sphere of radius R we obtain the exact mass-radius relation

$$M = \frac{2}{3}(n-2)(n-1) \frac{n\pi^2}{\sin(n\pi)} \left[\frac{GM_{\text{BH}}}{K(n+1)} \right]^n R^{3-n}, \quad (\text{E9})$$

where we have used the identity

$$\int_0^1 \left(\frac{1}{x} - 1 \right)^n x^2 dx = \frac{1}{6}(n-2)(n-1) \frac{n\pi}{\sin(n\pi)} \quad (-1 < n < 3). \quad (\text{E10})$$

For the usual BEC corresponding to $n = 1$ and $K = 2\pi a_s \hbar^2 / m^3$, the density profile is

$$\rho(r) = \frac{GM_{\text{BH}}m^3}{4\pi a_s \hbar^2} \left(\frac{1}{r} - \frac{1}{R} \right). \quad (\text{E11})$$

On the other hand, the integral in Eq. (E10) is equal to $1/6$ so the exact mass-radius relation (E9) reduces to

$$M = \frac{GM_{\text{BH}}m^3}{6a_s \hbar^2} R^2. \quad (\text{E12})$$

This can be compared to the relation (51) obtained from the Gaussian ansatz. We note that the density profile $\rho(r)$ is very different from a Gaussian in the present case. In the nongravitational + TF case, the exact equilibrium relations from Sec. IID reduce to

$$E_{\text{tot}} = W_{\text{BH}} + U, \quad NE = 2U + W_{\text{BH}}, \quad 3U + W_{\text{BH}} = 0. \quad (\text{E13})$$

We note that $E_{\text{tot}} = 2NE = -2U = (2/3)W_{\text{BH}}$. A direct calculation using Eq. (E11) gives

$$U = \frac{G^2 M_{\text{BH}}^2 m^3 R}{6a_s \hbar^2}, \quad W_{\text{BH}} = -\frac{G^2 M_{\text{BH}}^2 m^3 R}{2a_s \hbar^2}, \quad NE = -\frac{GM_{\text{BH}}M}{R}, \quad I = \frac{GM_{\text{BH}}m^3 R^4}{20a_s \hbar^2}. \quad (\text{E14})$$

We can check that the relations of Eq. (E13) are satisfied. On the other hand, Eq. (B51) reduces to

$$\omega^2 = \frac{12U + 2W_{\text{BH}}}{I}. \quad (\text{E15})$$

Using the virial theorem from Eq. (E13), we obtain

$$\omega^2 = -\frac{2W_{\text{BH}}}{I} = \frac{6U}{I}. \quad (\text{E16})$$

Finally, using Eq. (E14), we get

$$\omega^2 = \frac{20GM_{\text{BH}}}{R^3}. \quad (\text{E17})$$

Appendix F: Exact nongravitational case without BH

In the nongravitational case without BH, the exact equilibrium relations from Sec. IID reduce to

$$E_{\text{tot}} = \Theta_Q + U, \quad NE = 2U + \Theta_Q, \quad 2\Theta_Q + 3U = 0. \quad (\text{F1})$$

We note that $E_{\text{tot}} = -NE = -U/2 = \Theta_Q/3$. On the other hand, Eq. (B51) reduces to

$$\omega^2 = \frac{6\Theta_Q + 12U}{I}. \quad (\text{F2})$$

Using the virial theorem from Eq. (F1), we obtain

$$\omega^2 = -\frac{2\Theta_Q}{I} = \frac{3U}{I}. \quad (\text{F3})$$

We also recall the exact results [32]:

$$R_{99} = 3.64 \frac{|a_s|}{m} M, \quad NE = -0.435 \frac{\hbar^2}{M a_s^2}. \quad (\text{F4})$$

Appendix G: Exact noninteracting case without BH

In the noninteracting case without BH, the exact equilibrium relations from Sec. IID reduce to

$$E_{\text{tot}} = \Theta_Q + W, \quad NE = 2W + \Theta_Q, \quad 2\Theta_Q + W = 0. \quad (\text{G1})$$

We note that $E_{\text{tot}} = NE/3 = W/2 = -\Theta_Q$. On the other hand, Eq. (B51) reduces to

$$\omega^2 = \frac{6\Theta_Q + 2W}{I}. \quad (\text{G2})$$

Using the virial theorem from Eq. (G1), we obtain

$$\omega^2 = \frac{2\Theta_Q}{I} = -\frac{W}{I}. \quad (\text{G3})$$

We also recall the exact results [3, 31, 32]:

$$R_{99} = 9.946 \frac{\hbar^2}{GMm^2}, \quad NE = -0.1628 \frac{G^2 M^3 m^2}{\hbar^2}. \quad (\text{G4})$$

Appendix H: Exact TF limit without BH

In the TF approximation ($\hbar = 0$), the differential equation determining the density profile of the BEC without central BH is (see Sec. IIC)

$$\Delta\rho + \frac{Gm^3}{a_s \hbar^2} \rho = 0. \quad (\text{H1})$$

The solution of this equation is

$$\rho(r) = \rho_0 \frac{\sin(\pi r/R)}{\pi r/R}, \quad (\text{H2})$$

where

$$R = \pi \left(\frac{a_s \hbar^2}{Gm^3} \right)^{1/2} \quad (\text{H3})$$

is the radius at which the density vanishes and ρ_0 is the central density. It is determined by the mass according to the relation

$$\rho_0 = \frac{\pi M}{4R^3} = \frac{M}{4\pi^2} \left(\frac{Gm^3}{a_s \hbar^2} \right)^{3/2}. \quad (\text{H4})$$

We note that the radius has a constant value independent of the mass. In the TF limit without BH, the exact equilibrium relations from Sec. IID reduce to

$$E_{\text{tot}} = W + U, \quad NE = 2W + 2U, \quad 3U + W = 0. \quad (\text{H5})$$

We note that $E_{\text{tot}} = (1/2)NE = -2U = (2/3)W$. We can determine the eigenenergy E by applying the relation [see Eq. (B37) with $Q = \Phi_{\text{ext}} = 0$]

$$\Phi + \frac{4\pi a_s \hbar^2}{m^3} \rho = \frac{E}{m} \quad (\text{H6})$$

at $r = R$, giving

$$NE = -\frac{GM^2}{R}. \quad (\text{H7})$$

On the other hand, a direct calculation using Eq. (H2) gives

$$U = \frac{GM^2}{4R}, \quad I = \frac{(\pi^2 - 6)MR^2}{\pi^2}. \quad (\text{H8})$$

Finally, according to the results of Appendix C, we have

$$W = -\frac{3GM^2}{4R}. \quad (\text{H9})$$

We can check that the relations from Eq. (H5) are satisfied. On the other hand, Eq. (B51) reduces to

$$\omega^2 = \frac{12U + 2W}{I}. \quad (\text{H10})$$

Using the virial theorem from Eq. (H5) we obtain

$$\omega^2 = \frac{6U}{I} = -\frac{2W}{I}. \quad (\text{H11})$$

Finally, using Eq. (H8), we get

$$\omega^2 = \frac{3\pi^2 GM}{2(\pi^2 - 6)R^3}. \quad (\text{H12})$$

This returns the results from [31].

Appendix I: Exact TF limit with the BH potential

In the TF approximation ($\hbar = 0$), the differential equation determining the density profile of the BEC in the presence of a central BH is (see Sec. IIC)

$$-\frac{4\pi a_s \hbar^2}{m^3} \Delta \rho = 4\pi G \rho + 4\pi G M_{\text{BH}} \delta(\mathbf{r}). \quad (\text{I1})$$

For $r \neq 0$, it reduces to

$$\Delta \rho + \frac{Gm^3}{a_s \hbar^2} \rho = 0. \quad (\text{I2})$$

The general solution of this equation is

$$\rho = A \frac{\sin(kr)}{r} + B \frac{\cos(kr)}{r}, \quad (\text{I3})$$

where we have defined

$$k = \left(\frac{Gm^3}{a_s \hbar^2} \right)^{1/2}. \quad (\text{I4})$$

Integrating Eq. (I1) over a sphere of radius r , using the Gauss-Ostrogradsky theorem to convert a volume integral into a surface integral, and letting $r \rightarrow 0$, we get

$$-\frac{4\pi a_s \hbar^2}{m^3} \oint_{S_r} \nabla \rho \cdot d\mathbf{S} = 4\pi G M_{\text{BH}}, \quad (\text{I5})$$

implying

$$\frac{d\rho}{dr} \sim -\frac{Gm^3 M_{\text{BH}}}{4\pi a_s \hbar^2 r^2} \quad (r \rightarrow 0). \quad (\text{I6})$$

Therefore, when $r \rightarrow 0$, the density behaves as⁷

$$\rho \sim \frac{Gm^3 M_{\text{BH}}}{4\pi a_s \hbar^2 r}. \quad (\text{I7})$$

This diverging behavior determines the constant B in Eq. (I3). We get

$$B = \frac{Gm^3 M_{\text{BH}}}{4\pi a_s \hbar^2} = \frac{k^2 M_{\text{BH}}}{4\pi}. \quad (\text{I8})$$

On the other hand, if we call R the value of the radial distance at which the density vanishes, we find that the constant A in Eq. (I3) is given by

$$A = -\frac{B}{\tan(kR)}. \quad (\text{I9})$$

As a result, the density profile can be written as

$$\rho = \frac{k^2 M_{\text{BH}}}{4\pi r} \left[\cos(kr) - \frac{\sin(kr)}{\tan(kR)} \right]. \quad (\text{I10})$$

It is plotted in Fig. 15. The total mass is given by

$$\frac{M}{M_{\text{BH}}} = \int_0^{kR} \left[\cos(x) - \frac{\sin(x)}{\tan(kR)} \right] x dx. \quad (\text{I11})$$

Using the identities

$$\int_0^{kR} \sin(x) x dx = \sin(kR) - kR \cos(kR), \quad (\text{I12})$$

and

$$\int_0^{kR} \cos(x) x dx = \cos(kR) + kR \sin(kR) - 1, \quad (\text{I13})$$

⁷ We can also obtain this result by taking the limit $r \rightarrow 0$ in Eq. (I16), using $\Phi \rightarrow 0$ when $r \rightarrow 0$.

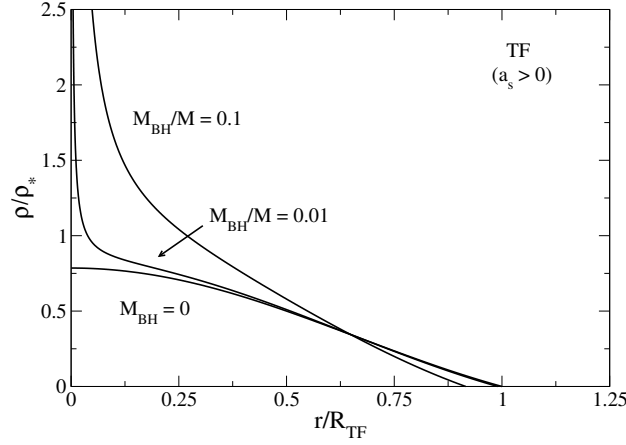


FIG. 15: Density profile of self-gravitating BECs with a repulsive self-interaction ($a_s > 0$) in the TF limit ($\hbar = 0$) in the presence of a central black hole. We have normalized the radius by R_{TF} given by Eq. (2) and the density by $\rho_* = M/R_{\text{TF}}^3$. We have plotted the profile without black hole [see Eq. (H2)] for comparison.

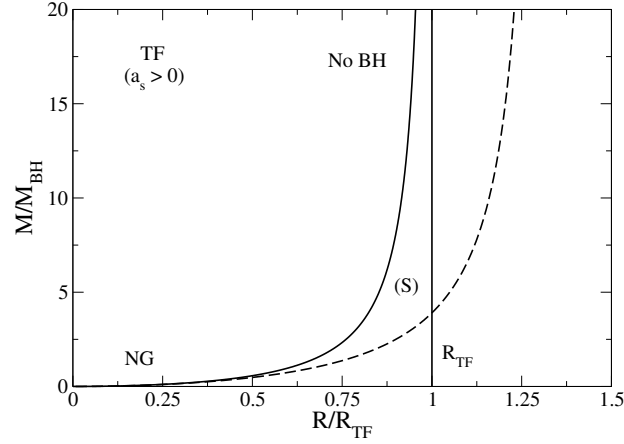


FIG. 16: Mass-radius relation of self-gravitating BECs with a repulsive self-interaction ($a_s > 0$) in the TF limit ($\hbar = 0$) in the presence of a central black hole. We have normalized the radius by R_{TF} given by Eq. (2) and the mass by M_{BH} . The solid line is the exact mass-radius relation from Eq. (I14) and the dashed line is the approximate mass-radius relation (83) obtained from the Gaussian ansatz (in that case, R on the figure represents the radius R_{99} containing 99% of the mass).

we obtain the exact mass-radius relation

$$\frac{M}{M_{\text{BH}}} = \frac{kR}{\sin(kR)} - 1. \quad (\text{I14})$$

It can be compared to the mass-radius relation (83) obtained from the Gaussian ansatz (see Fig. 16). We note that ρ is very different from a Gaussian in the present case.

In the TF limit, the exact equilibrium relations from Sec. IID reduce to

$$E_{\text{tot}} = W + W_{\text{BH}} + U, \quad NE = 2W + 2U + W_{\text{BH}}, \quad 3U + W + W_{\text{BH}} = 0. \quad (\text{I15})$$

We can determine the eigenenergy E by applying the relation [see Eq. (B37) with $Q = 0$]

$$\Phi + \frac{4\pi a_s \hbar^2}{m^3} \rho - \frac{GM_{\text{BH}}}{r} = \frac{E}{m} \quad (\text{I16})$$

at $r = R$, giving

$$NE = -\frac{GM(M_{\text{BH}} + M)}{R}. \quad (\text{I17})$$

On the other hand, a direct calculation using Eq. (I10) gives

$$W_{\text{BH}} = -GM_{\text{BH}}^2 k \frac{1 - \cos(kR)}{\sin(kR)}, \quad U = -\frac{1}{4} GkM_{\text{BH}}^2 \left[\frac{1}{\tan(kR)} - \frac{kR}{\sin^2(kR)} \right], \quad I = \frac{M_{\text{BH}}}{k^2} \left[6 + \frac{kR(k^2 R^2 - 6)}{\sin(kR)} \right]. \quad (\text{I18})$$

Finally, according to the results of Appendix C, we have

$$W = -\frac{1}{4} \left(\frac{3GM^2}{R} + \frac{3GM_{\text{BH}}M}{R} + W_{\text{BH}} \right). \quad (\text{I19})$$

We can check that the relations of Eq. (I15) are satisfied. On the other hand, Eq. (B51) reduces to

$$\omega^2 = \frac{12U + 2W + 2W_{\text{BH}}}{I}. \quad (\text{I20})$$

Using the virial theorem from Eq. (I15), we obtain

$$\omega^2 = \frac{6U}{I} = -\frac{2(W + W_{\text{BH}})}{I}. \quad (\text{I21})$$

Finally, using Eq. (I18), we get

$$\omega^2 = -\frac{3}{2} GM_{\text{BH}} k^3 \frac{\sin(kR) \cos(kR) - kR}{\sin(kR) [6 \sin(kR) + kR(k^2 R^2 - 6)]}. \quad (\text{I22})$$

The pulsation is plotted in Fig. 17 as a function of the BEC radius. It presents a minimum value ω_{min} .

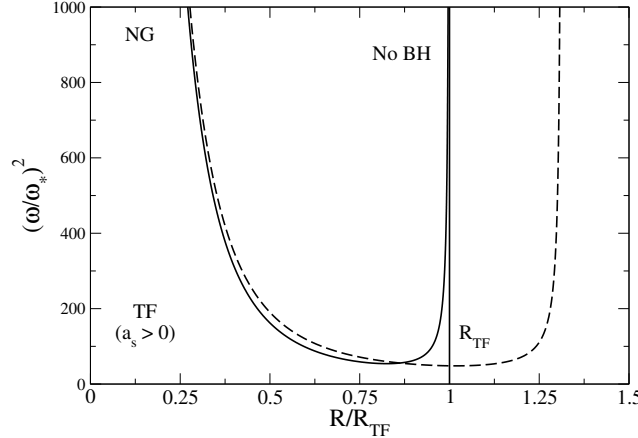


FIG. 17: Square pulsation of self-gravitating BECs with a repulsive self-interaction ($a_s > 0$) in the TF limit ($\hbar = 0$) in the presence of a central black hole as a function of the radius. We have normalized the radius by R_{TF} given by Eq. (2) and the pulsation by $\omega_* = (GM_{\text{BH}}/R_{\text{TF}}^3)^{1/2}$. The solid line is the exact relation from Eq. (I22) and the dashed line is the approximate relation (83) and (84) obtained from the Gaussian ansatz (in that case, R on the figure represents the radius R_{99} containing 99% of the mass). There is a minimum pulsation $\omega_{\text{min}} = 7.35 \omega_*$ (the Gaussian ansatz gives $\omega_{\text{min}} = 6.95 \omega_*$).

For $M \rightarrow 0$ and $R \rightarrow 0$, we recover the nongravitational limit of Appendix E. For $M \rightarrow +\infty$ and $R \rightarrow R_{\text{TF}}$ we recover the no BH limit of Appendix H with the additional relations

$$\frac{M}{M_{\text{BH}}} \sim \frac{1}{1 - R/R_{\text{TF}}}, \quad \left(\frac{\omega}{\omega_*} \right)^2 \sim \frac{3\pi^2}{2(\pi^2 - 6)} \frac{1}{1 - R/R_{\text{TF}}}. \quad (\text{I23})$$

When $M_{\text{BH}} = 0$, we recover the results of Appendix H.

Appendix J: Gravitational TF model with a central BH

In this Appendix, we generalize the results of Appendix I and consider a self-gravitating polytropic sphere of index $\gamma > 1$ surrounding a central BH. As in Appendix I, we make the TF approximation which amounts to neglecting the quantum potential.

1. Exact results

For a polytropic equation of state $P = K\rho^\gamma$ with $\gamma = 1 + 1/n$, Eq. (31) takes the form

$$-K(n+1)\Delta\rho^{1/n} + \frac{\hbar^2}{2m^2}\Delta\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = 4\pi G\rho + 4\pi GM_{\text{BH}}\delta(\mathbf{r}). \quad (\text{J1})$$

In the TF approximation ($\hbar = 0$), it reduces to

$$-K(n+1)\Delta\rho^{1/n} = 4\pi G\rho + 4\pi GM_{\text{BH}}\delta(\mathbf{r}). \quad (\text{J2})$$

For $r \neq 0$, we get

$$-K(n+1)\Delta\rho^{1/n} = 4\pi G\rho. \quad (\text{J3})$$

As in the usual theory of self-gravitating polytropic spheres [102], we introduce the variables (ξ, θ) from the relations

$$\rho = \rho_0\theta^n \quad \text{and} \quad r = \left[\frac{K(n+1)\rho_0^{1/n-1}}{4\pi G} \right]^{1/2} \xi \equiv r_0\xi, \quad (\text{J4})$$

but we stress that ρ_0 is *not* the central density (which is infinite in the presence of a central point source). With these variables, Eq. (J3) reduces to the Lane-Emden equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n. \quad (\text{J5})$$

Integrating Eq. (J2) over a sphere of radius r , using the Gauss-Ostrogradsky theorem to convert a volume integral into a surface integral, and letting $r \rightarrow 0$, we get

$$-K(n+1) \oint_{S_r} \nabla(\rho^{1/n}) \cdot d\mathbf{S} = 4\pi GM_{\text{BH}}, \quad (\text{J6})$$

implying

$$\frac{d\rho^{1/n}}{dr} \sim -\frac{GM_{\text{BH}}}{K(n+1)r^2}. \quad (\text{J7})$$

Therefore, when $r \rightarrow 0$, the density behaves as⁸

$$\rho^{1/n} \sim \frac{GM_{\text{BH}}}{K(n+1)r}. \quad (\text{J9})$$

It is convenient to introduce the variable

$$u = \theta\xi. \quad (\text{J10})$$

In that case, the Lane-Emden equation (J5) is transformed into

$$\frac{d^2u}{d\xi^2} = -\frac{u^n}{\xi^{n-1}}. \quad (\text{J11})$$

⁸ We can also obtain this result from Eq. (B37) which, in the TF approximation, reduces to

$$(n+1)K\rho^{1/n} + \Phi - \frac{GM_{\text{BH}}}{r} = \frac{E}{m}. \quad (\text{J8})$$

Taking the limit $r \rightarrow 0$, and using $\Phi \rightarrow 0$, we recover Eq. (J9).

On the other hand, the asymptotic behavior of the density close to the origin [see Eq. (J9)] leads to the boundary condition

$$u(0) = \frac{GM_{\text{BH}}}{K(n+1)\rho_0^{1/n}r_0} = \frac{M_{\text{BH}}}{4\pi\rho_0r_0^3}, \quad (\text{J12})$$

where we have introduced the radius r_0 defined by Eq. (J4). We now choose the reference density ρ_0 such that

$$u(0) = 1. \quad (\text{J13})$$

This implies

$$\frac{M_{\text{BH}}}{4\pi\rho_0r_0^3} = 1, \quad (\text{J14})$$

leading to

$$r_0 = \left[\frac{K(n+1)}{4\pi G} \right]^{n/(3-n)} \left(\frac{M_{\text{BH}}}{4\pi} \right)^{(1-n)/(3-n)} \quad (\text{J15})$$

and

$$\rho_0 = \left[\frac{4\pi G}{K(n+1)} \right]^{3n/(3-n)} \left(\frac{M_{\text{BH}}}{4\pi} \right)^{2n/(3-n)}. \quad (\text{J16})$$

Using the Lane-Emden equation (J5) and the variable u defined by Eq. (J10), we find that the total mass of the configuration is given by⁹

$$M = \int_0^R \rho 4\pi r^2 dr = -4\pi\rho_0r_0^3 \left[\xi^2 \frac{d\theta}{d\xi} \right]_0^{\xi_1} = -4\pi\rho_0r_0^3 [\xi u' - u]_0^{\xi_1} = -4\pi\rho_0r_0^3 (\xi_1 u'_1 + 1), \quad (\text{J17})$$

where ξ_1 is the normalized distance at which the density vanishes ($\theta = u = 0$) and $u'_1 = u'(\xi_1)$. Using (J14), we obtain

$$\frac{M}{M_{\text{BH}}} = -\xi_1 u'_1 - 1. \quad (\text{J18})$$

On the other hand, the (physical) radius of the configuration is given by

$$\frac{R}{r_0} = \xi_1. \quad (\text{J19})$$

The previous equations allow us to obtain the density profile and the mass-radius relation for various polytropic index n . To that purpose, one has to solve the differential equation (J11) with the boundary condition $u(0) = 1$ and $u'(0) = a$ (for a given value of a) up to the normalized distance ξ_1 at which u vanishes. The density profile is then determined by Eqs. (J4) and (J10) while the mass and the radius of the configuration are given by Eqs. (J18) and (J19). By varying a we can obtain the complete mass-radius relation. In general, the differential equation (J11) must be solved numerically except for the particular index $n = 1$ (see Appendix I). Application of these results for different values of n will be given in a forthcoming paper [108]. In the following section, we present approximate analytical results obtained from the Gaussian ansatz.

Remark: In the electrostatic case, the previous equations with $n = 3/2$ correspond to the TF theory of atoms in which a central charge $+Q$ is surrounded by a cloud of opposite charges $-Ne$ in Coulombian interaction. In this analogy, the central charge is the equivalent of the BH and the charged cloud is the equivalent of the gravitational halo. The crucial difference¹⁰ is that the charges $-e$ are mutually repulsive (the atom being stabilized by the attraction of the central charge $+Q$) while the gravitational particles are mutually attractive (the BH having the tendency to reinforce their attraction and possibly destabilize the system). This analogy will be further developed in a forthcoming paper [108].

⁹ The integral converges for $r \rightarrow 0$ provided that $n < 3$. We will make this assumption in the following.

¹⁰ Another important difference is that, in the case of atoms, the number of charges N is small making the mean field approximation in general inaccurate. By contrast, for astrophysical systems where $N \gg 1$, the mean field approximation is excellent.

2. Gaussian ansatz

Using a Gaussian ansatz, the mass-radius relation corresponding to Eq. (J1) is [see Eq. (B49)]:

$$-2\sigma \frac{\hbar^2 M}{m^2 R^3} + \nu \frac{GM^2}{R^2} - 3\zeta \frac{KM^\gamma}{R^{3\gamma-2}} + \lambda \frac{GM_{\text{BH}}M}{R^2} = 0. \quad (\text{J20})$$

In the TF approximation, it reduces to

$$R^{3\gamma-4} = \frac{3\zeta KM^{\gamma-1}}{\nu GM + \lambda GM_{\text{BH}}}. \quad (\text{J21})$$

As an illustration, let us apply these results to a system of nonrelativistic self-gravitating fermions at $T = 0$ surrounding a central object (black hole). This could represent a model of fermionic dark matter halos.

We first consider nonrelativistic fermions at $T = 0$ that are described by an equation of state of the form [102]:

$$P = \frac{1}{20} \left(\frac{3}{\pi} \right)^{2/3} \frac{\hbar^2}{m^{8/3}} \rho^{5/3}. \quad (\text{J22})$$

This corresponds to a polytrope of index $\gamma = 5/3$ (i.e. $n = 3/2$). In that case, the mass-radius relation (J21) becomes

$$R = \frac{3\zeta}{20} \left(\frac{3}{\pi} \right)^{2/3} \frac{\hbar^2}{Gm^{8/3}} \frac{M^{2/3}}{\nu M + \lambda M_{\text{BH}}}. \quad (\text{J23})$$

When $M_{\text{BH}} = 0$, we recover the standard mass-radius relation of nonrelativistic fermion stars (within the Gaussian ansatz approximation):

$$R = \frac{3\zeta}{20\nu} \left(\frac{3}{\pi} \right)^{2/3} \frac{\hbar^2}{Gm^{8/3} M^{1/3}}. \quad (\text{J24})$$

The prefactor is 0.0539 (the exact prefactor is 0.114 [102] and we recall that $R_{99} = 2.38167R$ for the Gaussian profile). This relation is monotonic, the radius increasing as the mass decreases. When $M_{\text{BH}} \neq 0$, we find the existence of a maximum radius

$$R_{\text{max}} = \frac{\zeta}{20\lambda^{1/3}} \left(\frac{6}{\nu\pi} \right)^{2/3} \frac{\hbar^2}{Gm^{8/3} M_{\text{BH}}^{1/3}} \quad (\text{J25})$$

corresponding to a mass

$$M_* = \frac{2\lambda}{\nu} M_{\text{BH}}. \quad (\text{J26})$$

The prefactors are 0.0202 and 5.66 respectively. The mass-radius relation (J23) can be rewritten as

$$\frac{R}{R_{\text{max}}} = \frac{3 \left(\frac{M}{M_*} \right)^{2/3}}{2 \left(\frac{M}{M_*} \right) + 1}. \quad (\text{J27})$$

It is plotted in Fig. 18.

Let us now consider ultra-relativistic fermions at $T = 0$ that are described by an equation of state of the form [102]:

$$P = \frac{1}{8} \left(\frac{3}{\pi} \right)^{1/3} \frac{\hbar c}{m^{4/3}} \rho^{4/3}. \quad (\text{J28})$$

This corresponds to a polytrope of index $\gamma = 4/3$ (i.e. $n = 3$). In that case, the mass-radius relation (J21) becomes

$$\nu GM_e + \lambda GM_{\text{BH}} = \frac{3\zeta}{8} \left(\frac{3}{\pi} \right)^{1/3} \frac{\hbar c}{m^{4/3}} M_e^{1/3} \quad (\text{J29})$$

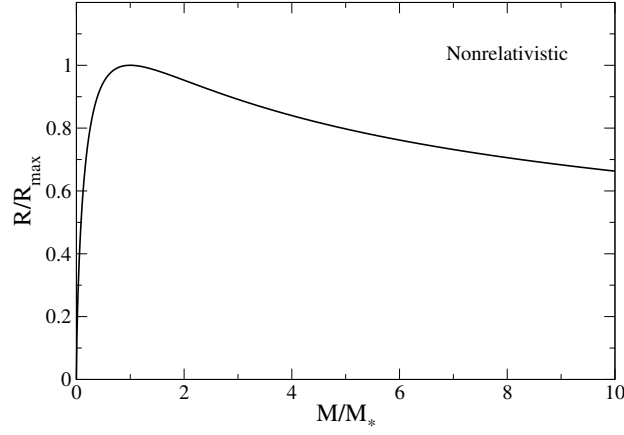


FIG. 18: Mass-radius relation (within the Gaussian ansatz) of nonrelativistic self-gravitating fermions at $T = 0$ surrounding a central black hole.

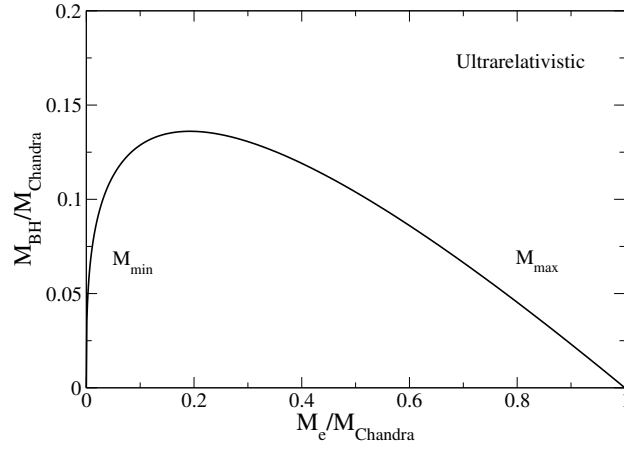


FIG. 19: Maximum and minimum masses (obtained in the ultrarelativistic limit within the Gaussian ansatz) of self-gravitating fermions at $T = 0$ surrounding a central black hole.

The mass is independent of the radius. This equation actually determines a maximum mass and a minimum mass as a function of the black hole mass. When $M_{\text{BH}} = 0$, we recover the Chandrasekhar maximum mass (within the Gaussian ansatz approximation):

$$M_{\text{Chandra}} = \left(\frac{3\zeta}{8\nu}\right)^{3/2} \left(\frac{3}{\pi}\right)^{1/2} \left(\frac{hc}{G}\right)^{3/2} \frac{1}{m^2}. \quad (\text{J30})$$

The prefactor is 0.197 which coincides (to the order of accuracy that we consider) with the exact prefactor [102]. We can then rewrite Eq. (J29) as

$$\frac{M_{\text{BH}}}{M_{\text{Chandra}}} = \frac{\nu}{\lambda} \left[\left(\frac{M_e}{M_{\text{Chandra}}} \right)^{1/3} - \frac{M_e}{M_{\text{Chandra}}} \right]. \quad (\text{J31})$$

The relation between the extremal masses M_e and the BH mass M_{BH} is plotted in Fig. 19. The maximum mass and the minimum mass become equal when

$$\frac{M_{\text{BH}}}{M_{\text{Chandra}}} = \frac{\nu}{\lambda} \frac{2}{3\sqrt{3}} = 0.136. \quad (\text{J32})$$

It that case, they have the value

$$\frac{M_e^*}{M_{\text{Chandra}}} = \frac{1}{3\sqrt{3}} = 0.192. \quad (\text{J33})$$

There is no possible equilibrium, for any mass M , when $M_{\text{BH}} > (2\nu/3\sqrt{3}\lambda)M_{\text{Chandra}} = 0.136 M_{\text{Chandra}}$. A more complete discussion of these results will be given elsewhere [108].

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