Quantum Amplitude Estimation – Applications to Derivative Pricing

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Moreover, by the central limit theorem,

$$\|\tilde{\mu}_N - \mu\|_2 = \sqrt{\mathbb{V}(\tilde{\mu}_N)} = \frac{\mathbb{V}(f(S_T))}{\sqrt{N}} \sim \mathcal{O}\left(\frac{1}{\sqrt{N}}\right).$$

Can a quantum computer do better?

Encoding the problem

Björn showed us how to construct a unitary operator $\mathcal{U}$ encoding the desired quantity μ as

$$\mathcal{U}\left|0\right\rangle_{n+1} = \sqrt{1-a}\left|\psi_0\right\rangle_n\left|0\right\rangle + \sqrt{a}\left|\psi_1\right\rangle_n\left|1\right\rangle .$$

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Can we estimate a?

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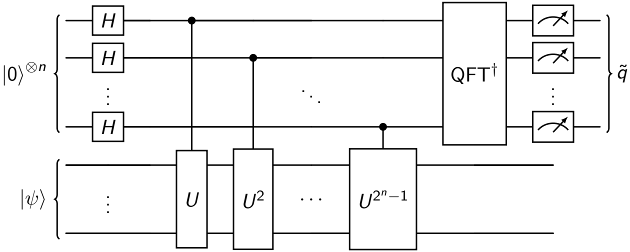

Figure: Phase estimation circuit.

Phase estimation

Theorem (Quantum phase estimation)

With probability $1-\delta$, the estimate $\tilde{q}$ provided by QPE satisfies

$$|\tilde{q} - q| \leq \frac{2}{N}\left(2 + \frac{1}{2\delta}\right) \sim \mathcal{O}\left(\frac{1}{N}\right),$$

where $N = 2^n$.

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Note'. For the general case $q \in [0, 2^n)$, see [Kit95], [MW23].

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Given: unitary operator $\mathcal{U}$ acting on $n+1$ qubits as

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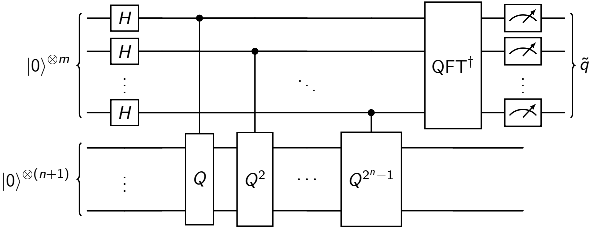

Figure: Amplitude estimation circuit.

Where $\mathcal{Q}$ is the Grover operator

$$\mathcal{Q} = \mathcal{U}\mathcal{S}_0\mathcal{U}^\dagger\mathcal{S}_{\psi_0},$$

and

$$\mathcal{S}_{\psi_0} = \mathbb{I} - 2\left|\psi_0\right\rangle_n \left\langle\psi_0\right|_n \otimes \left|0\right\rangle\left\langle 0\right|, \quad \mathcal{S}_0 = \mathbb{I} - 2|0\rangle_{n+1}\langle 0|_{n+1}.$$

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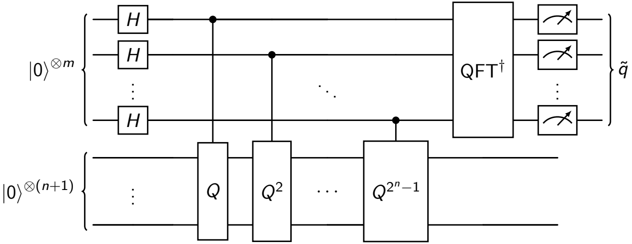

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The measured number $\tilde{q} \in \{0, 1, ..., 2^m - 1\}$ is then mapped to the estimate

$$\tilde{a} = \sin^2\left(\tilde{\theta}_a\right),$$

where

$$\tilde{\theta}_a = \frac{\pi\tilde{q}}{M}, \quad M = 2^m.$$

Amplitude estimation

Theorem (Quantum amplitude estimation)

With probability $8/\pi^2 \approx 0.81$*, the estimate* $\tilde{a}$ *provided by QAE satisfies*

$$|\tilde{a} - a| \leq \frac{2\pi\sqrt{a(1-a)}}{M} + \frac{\pi^2}{M^2} \sim \mathcal{O}\left(\frac{1}{M}\right).$$

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- Monte Carlo: $\mathcal{O}(1/\sqrt{N})$,
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In practice: run QAE k times and take the median value. This boost success probability to $1 - (1 - 8/\pi^2)^k$

Amplitude estimation

“Proof idea”

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"Proof idea"
Use the QPE theorem and the fact that the Grover operator acts as

$$\mathcal{Q}^k\mathcal{U}\left|0\right\rangle^{\otimes(n+1)} = \cos((2k+1)\theta_a)|\psi_0\rangle_n\left|0\right\rangle + \sin((2k+1)\theta_a)|\psi_1\rangle_n\left|1\right\rangle,$$

which implies

$$\mathbb{P}\left(|1\rangle\right) = \sin^2((2k+1)\theta_a),$$

where $\theta_a = \arcsin\left(\sqrt{a}\right)$.

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whose powers are very easy to calculate:

$$\mathcal{Q}^k = R_Y(2k\theta_p).$$

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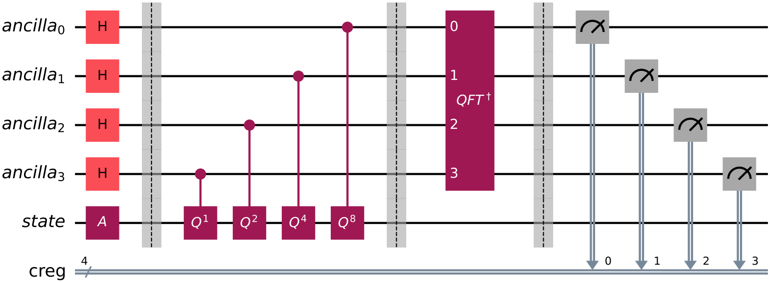


Figure: QAE circuit for binomial random variable, implemented in Qiskit.

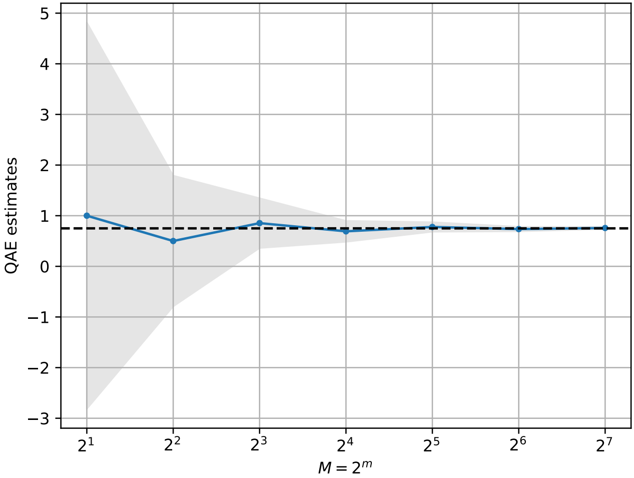


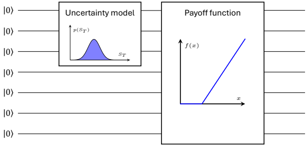
Figure: QAE estimates

QAE for derivative pricing

We saw how to encode the expected payoff μ of a european call option into an operator $\mathcal{U}$

$$\mathcal{U}\left|0\right\rangle_{n+1} = \sqrt{1-a}\left|\psi_0\right\rangle_n\left|0\right\rangle + \sqrt{a}\left|\psi_1\right\rangle_n\left|1\right\rangle,$$

after an appropriate re-scaling to the interval $[0, 1]$.



Amplitude estimation can be used to approximate a.

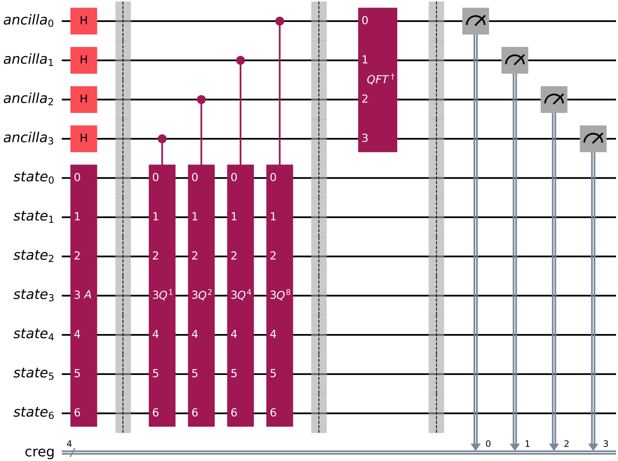


Figure: QAE circuit for option pricing. Implemented in Qiskit.

Notebook time!

Is QAE NISQ-ready?

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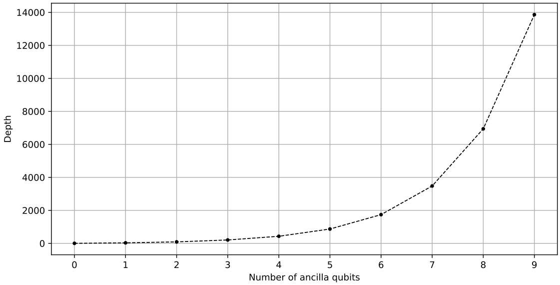

Figure: Depth vs ancillas.

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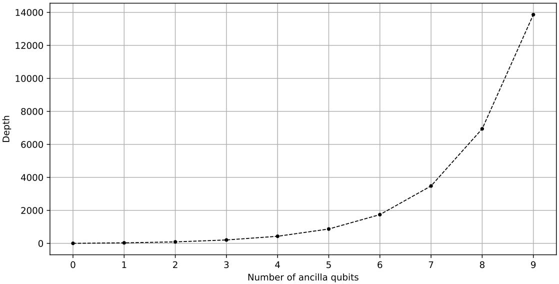

Figure: Depth vs ancillas.

IBM Heron can run ≈ 1800 gates within the coherence time of its qubits [IBM].

QAE alternatives

- Iterative quantum amplitude estimation [Gri+21].
- Faster amplitude estimation [Nak20].
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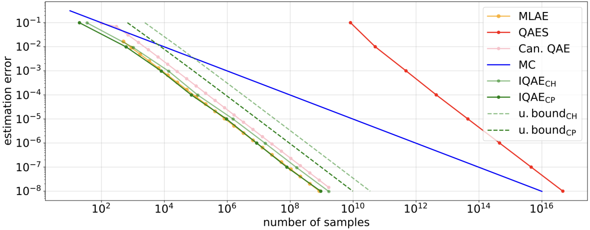

Figure: Convergence rates of different QAE algorithms (Figure 3 in [Gri+21]).

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- Not NISQ-ready, but active area of research.

More quantum finance

- [Sta+20]: Portfolios of options, basket options, path-dependent options.
- [WK24]: Asian and barrier options under Heston model.
- [ZLW19]: Option Pricing with qGANs.
- [DL21]: Quantum Support Vector Regression for Disability Insurance.

References I

[Bra+02] Gilles Brassard et al. *Quantum amplitude amplification and estimation*. 2002. DOI: 10.1090/conm/305/05215 (cit. on pp. 38, 39).

[DL21] Boualem Djehiche and Björn Löfdahl. "Quantum Support Vector Regression for Disability Insurance". In: *Risks* 9.12 (2021). ISSN: 2227-9091. DOI: 10.3390/risks9120216 (cit. on p. 64).

[Gri+21] Dmitry Grinko et al. "Iterative quantum amplitude estimation". In: *npj Quantum Information* 7.1 (Mar. 2021). ISSN: 2056-6387. DOI: 10.1038/s41534-021-00379-1 (cit. on pp. 59, 60).

[IBM] IBM. *IBM Quantum Summit 2023*. URL: https://www.ibm.com/quantum/summit-2023 (cit. on pp. 56, 57, 58).

References II

[Kit95] Alexei Y. Kitaev. "Quantum measurements and the Abelian Stabilizer Problem". In: *Electron. Colloquium Comput. Complex.* TR96 (1995) (cit. on pp. 29, 30, 31, 32, 33).

[MW23] Nikhil S. Mande and Ronald de Wolf. "Tight Bounds for Quantum Phase Estimation and Related Problems". In: (2023). arXiv: `2305.04908 [quant-ph]` (cit. on pp. 29, 30, 31, 32, 33).

[Nak20] Kouhei Nakaji. "Faster amplitude estimation". In: *Quantum Information and Computation* 20.13,14 (Nov. 2020), pp. 1109–1123. ISSN: 1533-7146. DOI: `10.26421/qic20.13-14-2` (cit. on pp. 59, 60).

References III

[Sta+20] Nikitas Stamatopoulos et al. "Option Pricing using Quantum Computers". In: *Quantum* 4 (July 2020), p. 291. ISSN: 2521-327X. DOI: 10.22331/q-2020-07-06-291 (cit. on p. 64).

[Suz+20] Yohichi Suzuki et al. "Amplitude estimation without phase estimation". In: *Quantum Information Processing* 19.2 (Jan. 2020). ISSN: 1573-1332. DOI: 10.1007/s11128-019-2565-2 (cit. on pp. 59, 60).

[WK24] Guoming Wang and Angus Kan. "Option pricing under stochastic volatility on a quantum computer". In: *Quantum* 8 (Oct. 2024), p. 1504. ISSN: 2521-327X. DOI: 10.22331/q-2024-10-23-1504 (cit. on p. 64).

References IV

[ZLW19] Christa Zoufal, Aurélien Lucchi, and Stefan Woerner. "Quantum Generative Adversarial Networks for learning and loading random distributions". en. In: *Npj Quantum Inf.* 5.1 (Nov. 2019) (cit. on p. 64).