SI140A Probability and Statistics Final Project: Performance Evaluation of Bandit Learning Algorithms

Jingran Fan Anrui Wang Zhao Lu

Due date: 2025/1/12 10:59pm

Abstract

Multi-armed bandit problems are fundamental models in sequential decision-making under uncertainty, wherein an agent must choose from several options (arms) over repeated trials to maximize cumulative rewards. These problems capture the delicate balance between exploration—seeking information about less-known options—and exploitation—leveraging current knowledge to select the best option. Classical bandit learning algorithms, such as ϵ-greedy, Upper Confidence Bound (UCB), and Thompson Sampling (TS), have been extensively studied and form the cornerstone of modern reinforcement learning techniques. More recently, Bayesian approaches that incorporate prior beliefs and update them with observed data have gained traction, offering theoretical elegance and robust performance in a variety of settings.

This project focuses on evaluating the performance of well-known bandit algorithms through numerical experiments. In Part I, we consider classical bandit algorithms operating on Bernoulli arms with parameters provided by an oracle. Although the oracle's parameters and optimal attainable reward (the "oracle value") are unknown to the algorithms, they provide a ground truth reference for performance comparison. We implement and benchmark ϵ-greedy (with various ϵ-values), UCB (with different confidence scales), and TS (with varying Beta priors) under identical experimental conditions. By analyzing their regret—defined as the gap between the algorithm's cumulative reward and the oracle value—we investigate how tuning parameters and prior knowledge impacts the exploration-exploitation trade-off.

In the optional Part II, we extend our analysis to a Bayesian bandit setting with discounted rewards, where prior distributions on arm parameters are continuously updated as more data is observed. We examine intuitive heuristics and discuss why these heuristics may fail to achieve optimality. Furthermore, we explore the derivation of optimal policies through recursive equations and investigate practical methods for exact and approximate solutions.

Overall, this project aims to provide a rigorous empirical evaluation of both classical and Bayesian bandit algorithms. By comparing their performance and understanding their underlying trade-offs, we gain deeper insights into bandit learning theory and develop intuition for selecting and designing effective strategies in diverse applications.

Contents

Introduction

In many real-world scenarios—from online advertising to medical trials—decision-makers must choose actions to maximize cumulative rewards. However, these decisions also provide valuable information that can guide future actions. This creates a fundamental tension between exploiting what we already know to gain immediate benefits and exploring new options to potentially improve future outcomes. This challenge is central to the field of reinforcement learning and is known as the exploration-exploitation trade-off.

A classic illustration is the multi-armed bandit problem. Imagine walking into a casino and facing a slot machine with multiple arms, each offering a different, unknown payoff distribution. Your goal is to pull the arms over a sequence of trials to earn as many rewards as possible. Since you do not know which arm is best, you must try them out (exploration) while continuing to play the arm that seems most promising (exploitation). Crucially, the reward probabilities remain fixed but hidden, and the only way to learn them is by experimenting.

This report examines three classical strategies for solving the multi-armed bandit problem—ϵ-greedy, Upper Confidence Bound (UCB), and Thompson Sampling (TS). By comparing their performance, we gain insight into how they balance exploration and exploitation and how well they adapt to uncertain, reward-driven environments.

Part I: Classical Bandit Algorithms

Problem 1

Choose $N = 5000$ and compute the theoretically maximized expectation of aggregate rewards over N time slots. Suppose we have an oracle that provides the parameters of the Bernoulli distributions for three arms as follows:

$$\theta_1 = 0.7, \quad \theta_2 = 0.5, \quad \theta_3 = 0.4.$$

We choose the time horizon as $N = 5000$ time steps. If we know these parameters beforehand (as the oracle does), the strategy to maximize the expected total reward is to always pull the arm with the highest success probability, which in this case is arm 1 (with $\theta_1 = 0.7$).

The expected reward is:

$$\max_{I(t),t=1,\ldots,N} \mathbb{E}\left[\sum_{t=1}^{N} r_{I(t)}\right] = N \times \theta_1 = 5000 \times 0.7 = 3500.$$

Thus, the theoretically maximized expectation is: $\boxed{3500.}$

Problem 2

Imports and parameters:

```python
import matplotlib.pyplot as plt
import numpy as np
import random, math, copy

np.random.seed(42)
num_arms = 3
theta = np.array([0.7, 0.5, 0.4])
```

Implementation of ϵ-greedy algorithm:

```python
def epsilon_greedy(epsilon, N, theta):
    Q = np.zeros(num_arms)  # Estimated values for each arm
    counts = np.zeros(num_arms)  # Count of how many times each arm is pulled
    total_reward = 0  # Total reward tracker

    # Initialization: Pull each arm once
    for arm in range(num_arms):
        reward = 1 if np.random.rand() < theta[arm] else 0
        counts[arm] = 1
        Q[arm] = reward
        total_reward += reward

    # Main loop: Epsilon-greedy exploration and exploitation
    for t in range(num_arms, N):
        if np.random.rand() < epsilon:
            # Exploration: choose a random arm
            arm = np.random.randint(num_arms)
        else:
            # Exploitation: choose the arm with the highest estimated value
            arm = np.argmax(Q)

        # Simulate pulling the chosen arm
        reward = 1 if np.random.rand() < theta[arm] else 0

        counts[arm] += 1
        Q[arm] += (1 / counts[arm]) * (reward - Q[arm])

        total_reward += reward
```

```
    return total_reward
```

Implementation of UCB algorithm:

```
def ucb(c, N, theta):
    Q = np.zeros(num_arms)
    counts = np.zeros(num_arms)
    total_reward = 0

    for arm in range(num_arms):
        reward = 1 if np.random.rand() < theta[arm] else 0
        Q[arm] = reward
        counts[arm] = 1
        total_reward += reward

    for t in range(num_arms+1, N+1):
        ucb_values = Q + c * np.sqrt((2*np.log(t))/counts)
        arm = np.argmax(ucb_values)
        reward = 1 if np.random.rand() < theta[arm] else 0
        counts[arm] += 1
        Q[arm] += (1/counts[arm])*(reward - Q[arm])
        total_reward += reward
    return total_reward
```

Implementation of Thompson Sampling algorithm:

```
from scipy.stats import beta

def thompson_sampling(N, theta, alpha_init, beta_init):
    alpha = alpha_init.copy()
    beta_ = beta_init.copy()
    total_reward = 0
    for t in range(N):
        sampled_thetas = [np.random.beta(alpha[j], beta_[j])
                          for j in range(num_arms)]
        arm = np.argmax(sampled_thetas)
        reward = 1 if np.random.rand() < theta[arm] else 0
        total_reward += reward
        alpha[arm] += reward
        beta_[arm] += 1 - reward
    return total_reward
```

Problem 3

The results for the three algorithms with various parameters, averaged over 200 trials and 5000 time slots, are presented below:

ε-Greedy

- $\varepsilon = 0.1$: **3408.44**
- $\varepsilon = 0.5$: **3085.66**
- $\varepsilon = 0.9$: **2748.22**

Upper Confidence Bound (UCB)

- $c = 1$: **3408.32**
- $c = 5$: **2979.74**
- $c = 10$: **2829.24**

Thompson Sampling (TS)

- $(\alpha_1, \beta_1) = (1, 1), (\alpha_2, \beta_2) = (1, 1), (\alpha_3, \beta_3) = (1, 1)$: **3480.75**
- $(\alpha_1, \beta_1) = (601, 401), (\alpha_2, \beta_2) = (401, 601), (\alpha_3, \beta_3) = (2, 3)$: **3492.41**

Problem 4

In this analysis, we compare the performance of three popular multi-armed bandit algorithms: ε-Greedy, Upper Confidence Bound (UCB), and Thompson Sampling (TS). The goal is to compute the gaps between the algorithm outputs (aggregated rewards over $N = 5000$ time slots) and the oracle value, and determine which algorithm performs best.

The theoretical best reward is calculated under the assumption that we know the parameters of the Bernoulli distributions for three arms as follows:

$$\theta_1 = 0.7, \quad \theta_2 = 0.5, \quad \theta_3 = 0.4.$$

Thus, the theoretically maximized expectation is:

$$\max_{I(t),t=1,\ldots,N} \mathbb{E}\left[\sum_{t=1}^{N} r_{I(t)}\right] = N \times \theta_1 = 5000 \times 0.7 = 3500.$$

Gap Calculation

The gap between the algorithm reward and the oracle reward of 3500 is calculated as:

$$\text{Gap} = \text{Oracle Value} - \text{Algorithm Reward}$$

1. ε-Greedy:

ε	Algorithm Reward	Gap
0.1	3408.44	91.56
0.5	3085.66	414.34
0.9	2748.22	751.78

2. Upper Confidence Bound (UCB):

c	Algorithm Reward	Gap
1	3408.32	91.68
5	2979.74	520.26
10	2829.24	670.76

3. Thompson Sampling (TS):

$(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)$	Algorithm Reward	Gap
$(1,1),(1,1),(1,1)$	3480.75	19.25
$(601,401),(401,601),(2,3)$	3492.41	7.59

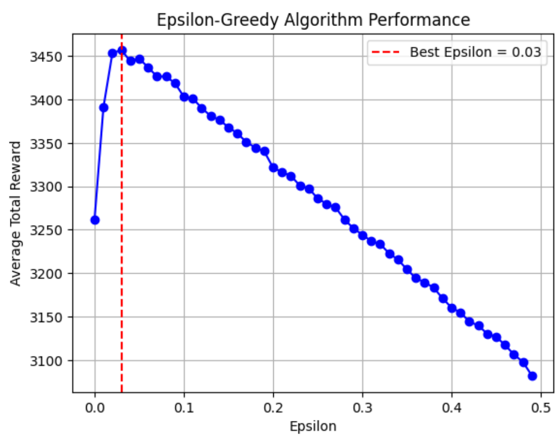
By optimizing the parameters of each algorithm, we can achieve better performance.

After performing the following process for ε-Greedy:

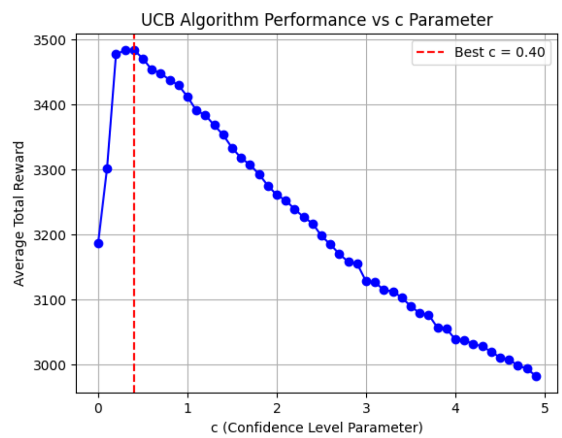
1. Sweep ε from 0 to 0.5 in increments of 0.01.
2. Runs 100 trials for each epsilon value.
3. Averages the total rewards over these 100 trials.
4. Identifies the ε that yields the highest average reward.

we find that the the best ε to maximize the total reward is 0.03, which yields the maximized rewards of 3457.02.

Similarly, for UCB, we sweep the confidence scale c from 0 to 5 in increments of 0.1, and find that the best c is 0.40, which yields the maximized rewards of 3483.90.



(a) ε-Greedy Results



(b) UCB Results

Figure 1: Algorithm Performance Analysis

But even with the optimized parameters, the gap of the first two algorithms (42.98 and 16.10 respectively) is still larger than **Thompson Sampling**, which has the smallest gap of 7.59. Therefore, the **Thompson Sampling** algorithm is the best-performing algorithm in this scenario.

Parameter Impact Analysis

1. ε-Greedy Algorithm

   The ε-Greedy algorithm explores with probability ε and exploits with probability $1-\varepsilon$. The parameter ε controls the level of exploration versus exploitation.

   - **Low** ε **:**
     - Pros: Prefer exploitation to exploration, leading to higher rewards when the current best action is optimal.
     - Cons: Limited exploration may prevent discovery of better actions, especially in complex environments.

- **High ε (e.g., $\varepsilon = 0.5$):**
  - Pros: Increased exploration, potentially leading to the discovery of optimal actions.
  - Cons: Excessive exploration dilutes the focus on known good actions, potentially lowering overall rewards.
- **Optimal ε:** Based on experiments, we find that $\varepsilon = 0.03$ yields the highest average reward of 3457.02, indicating a good balance between exploration and exploitation. And when $\varepsilon > 0.03$, the average reward decreases as the ε (exploration) increases.

2. Upper Confidence Bound (UCB)

   The UCB formula consists of two terms:

   - **Exploitation ($\hat{\theta}(j)$):** This term represents the current mean reward of action j, which is used to exploit the known best action. The algorithm favors actions with a higher expected reward, leading to exploitation.
   - **Exploration ($c \cdot \sqrt{\frac{2\ln(t)}{\mathbf{count}(j)}}$):** This term encourages exploration by adding a bonus to actions that have been chosen fewer times. The bonus is larger for actions that are less tested, which helps to balance exploration with the exploitation of known rewards. The parameter c controls the size of the exploration term. A higher c increases exploration, and a smaller c favors exploitation.

   So the parameter c controls the trade-off between exploration and exploitation in the UCB algorithm, which is discussed as follows:

   - **Low c:**
     - Pros: Promotes exploitation as the confidence bounds become tighter.
     - Cons: Reduced exploration can prevent the algorithm from discovering better actions.
   - **High c:**
     - Pros: Increases exploration by enlarging the confidence bounds, particularly for arms that have been pulled fewer times.
     - Cons: Excessive exploration can reduce the focus on exploitation, potentially leading to suboptimal rewards.
   - **Optimal c:** In our experiments, $c = 0.40$ provides the best performance, yielding a maximum reward of 3483.90. And when $c > 0.40$, the average reward decreases as the c (exploration) increases.

3. Thompson Sampling

   The Thompson Sampling algorithm is a Bayesian method that uses prior and posterior beliefs to update the probability distribution for each action's reward. It assumes a Beta distribution as the prior for the reward probability of each arm, and updates the parameters α_j and β_j based on the observed outcomes.

   Prior Belief:

   - α_j and β_j represent our prior belief about the reward distribution for action j.

- For example, $\alpha_j = 1$ and $\beta_j = 1$ implies that we expect the reward probability for action j to be 50
- A higher value of α_j relative to β_j means that we believe the reward probability for action j is higher, whereas a lower value of α_j relative to β_j suggests that we believe the reward probability is lower.

Effect of α_j and β_j:

- α_j and β_j are updated after each trial, based on whether the reward for action j was successful or not.
- When α_j is large and β_j is small (e.g., $\alpha_j = 2000$, $\beta_j = 8000$), we have a strong belief that the probability of a reward is low (20%), and the algorithm exploits this information.
- When both α_j and β_j are small (e.g., $\alpha_j = 1$, $\beta_j = 1$), there is high uncertainty about the reward distribution, encouraging more exploration of different actions.

Balancing Exploration and Exploitation:

- Thompson Sampling uses the Beta distribution to balance exploration and exploitation. Actions with higher posterior probability are more likely to be chosen, but there is still a chance of selecting suboptimal actions (exploration).
- The larger the parameters α_j and β_j, the more confident we are in the reward probability, and the less frequently our belief is updated. This reduces exploration and makes the algorithm focus more on exploitation.

Effect of Prior Distribution:

- If we set the prior α_j and β_j values to be small, the agent explores more, as the uncertainty is high. This might lead to more exploration but fewer immediate rewards.
- If we set the prior to values close to the true expected reward (the oracle value), the algorithm will converge quickly to the best action, leading to better performance.

Part II: Bayesian Bandit Algorithms