SI140A Probability and Statistics Final Project: Performance Evaluation of Bandit Learning Algorithms

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Abstract

Multi-armed bandit problems are fundamental models in sequential decision-making under uncertainty, wherein an agent must choose from several options (arms) over repeated trials to maximize cumulative rewards. These problems capture the delicate balance between exploration—seeking information about less-known options—and exploitation—leveraging current knowledge to select the best option. Classical bandit learning algorithms, such as ϵ-greedy, Upper Confidence Bound (UCB), and Thompson Sampling (TS), have been extensively studied and form the cornerstone of modern reinforcement learning techniques. More recently, Bayesian approaches that incorporate prior beliefs and update them with observed data have gained traction, offering theoretical elegance and robust performance in a variety of settings.

This project focuses on evaluating the performance of well-known bandit algorithms through numerical experiments. In Part I, we consider classical bandit algorithms operating on Bernoulli arms with parameters provided by an oracle. Although the oracle's parameters and optimal attainable reward (the "oracle value") are unknown to the algorithms, they provide a ground truth reference for performance comparison. We implement and benchmark ϵ-greedy (with various ϵ-values), UCB (with different confidence scales), and TS (with varying Beta priors) under identical experimental conditions. By analyzing their regret—defined as the gap between the algorithm's cumulative reward and the oracle value—we investigate how tuning parameters and prior knowledge impacts the exploration-exploitation trade-off.

In the optional Part II, we extend our analysis to a Bayesian bandit setting with discounted rewards, where prior distributions on arm parameters are continuously updated as more data is observed. We examine intuitive heuristics and discuss why these heuristics may fail to achieve optimality. Furthermore, we explore the derivation of optimal policies through recursive equations and investigate practical methods for exact and approximate solutions.

Overall, this project aims to provide a rigorous empirical evaluation of both classical and Bayesian bandit algorithms. By comparing their performance and understanding their underlying trade-offs, we gain deeper insights into bandit learning theory and develop intuition for selecting and designing effective strategies in diverse applications.

Contents

Introduction

In many real-world scenarios—from online advertising to medical trials—decision-makers must choose actions to maximize cumulative rewards. However, these decisions also provide valuable information that can guide future actions. This creates a fundamental tension between exploiting what we already know to gain immediate benefits and exploring new options to potentially improve future outcomes. This challenge is central to the field of reinforcement learning and is known as the exploration-exploitation trade-off.

A classic illustration is the multi-armed bandit problem. Imagine walking into a casino and facing a slot machine with multiple arms, each offering a different, unknown payoff distribution. Your goal is to pull the arms over a sequence of trials to earn as many rewards as possible. Since you do not know which arm is best, you must try them out (exploration) while continuing to play the arm that seems most promising (exploitation). Crucially, the reward probabilities remain fixed but hidden, and the only way to learn them is by experimenting.

This report examines three classical strategies for solving the multi-armed bandit problem—ϵ-greedy, Upper Confidence Bound (UCB), and Thompson Sampling (TS). By comparing their performance, we gain insight into how they balance exploration and exploitation and how well they adapt to uncertain, reward-driven environments.

Part I: Classical Bandit Algorithms

Problem 1

Choose $N = 5000$ and compute the theoretically maximized expectation of aggregate rewards over N time slots. Suppose we have an oracle that provides the parameters of the Bernoulli distributions for three arms as follows:

$$\theta_1 = 0.7, \quad \theta_2 = 0.5, \quad \theta_3 = 0.4.$$

We choose the time horizon as $N = 5000$ time steps. If we know these parameters beforehand (as the oracle does), the strategy to maximize the expected total reward is to always pull the arm with the highest success probability, which in this case is arm 1 (with $\theta_1 = 0.7$).

The expected reward is:

$$\max_{I(t),t=1,\ldots,N} \mathbb{E}\left[\sum_{t=1}^{N} r_{I(t)}\right] = N \times \theta_1 = 5000 \times 0.7 = 3500.$$

Thus, the theoretically maximized expectation is: $\boxed{3500.}$

Problem 2

Imports and parameters:

```python
import matplotlib.pyplot as plt
import numpy as np
import random, math, copy

np.random.seed(42)
num_arms = 3
theta = np.array([0.7, 0.5, 0.4])
```

Implementation of ϵ-greedy algorithm:

```python
def epsilon_greedy(epsilon, N, theta):
    Q = np.zeros(num_arms)  # Estimated values for each arm
    counts = np.zeros(num_arms)  # Count of how many times each arm is pulled
    total_reward = 0  # Total reward tracker

    # Initialization: Pull each arm once
    for arm in range(num_arms):
        reward = 1 if np.random.rand() < theta[arm] else 0
        counts[arm] = 1
        Q[arm] = reward
        total_reward += reward

    # Main loop: Epsilon-greedy exploration and exploitation
    for t in range(num_arms, N):
        if np.random.rand() < epsilon:
            # Exploration: choose a random arm
            arm = np.random.randint(num_arms)
        else:
            # Exploitation: choose the arm with the highest estimated value
            arm = np.argmax(Q)

        # Simulate pulling the chosen arm
        reward = 1 if np.random.rand() < theta[arm] else 0

        counts[arm] += 1
        Q[arm] += (1 / counts[arm]) * (reward - Q[arm])

        total_reward += reward
```

```python
    return total_reward
```

Implementation of UCB algorithm:

```python
def ucb(c, N, theta):
    Q = np.zeros(num_arms)
    counts = np.zeros(num_arms)
    total_reward = 0

    for arm in range(num_arms):
        reward = 1 if np.random.rand() < theta[arm] else 0
        Q[arm] = reward
        counts[arm] = 1
        total_reward += reward

    for t in range(num_arms+1, N+1):
        ucb_values = Q + c * np.sqrt((2*np.log(t))/counts)
        arm = np.argmax(ucb_values)
        reward = 1 if np.random.rand() < theta[arm] else 0
        counts[arm] += 1
        Q[arm] += (1/counts[arm])*(reward - Q[arm])
        total_reward += reward
    return total_reward
```

Implementation of Thompson Sampling algorithm:

```python
from scipy.stats import beta

def thompson_sampling(N, theta, alpha_init, beta_init):
    alpha = alpha_init.copy()
    beta_ = beta_init.copy()
    total_reward = 0
    for t in range(N):
        sampled_thetas = [np.random.beta(alpha[j], beta_[j])
                          for j in range(num_arms)]
        arm = np.argmax(sampled_thetas)
        reward = 1 if np.random.rand() < theta[arm] else 0
        total_reward += reward
        alpha[arm] += reward
        beta_[arm] += 1 - reward
    return total_reward
```

Problem 3

The results for the three algorithms with various parameters, averaged over 200 trials and 5000 time slots, are presented below:

ε-Greedy

- $\varepsilon = 0.1$: **3408.44**
- $\varepsilon = 0.5$: **3085.66**
- $\varepsilon = 0.9$: **2748.22**

Upper Confidence Bound (UCB)

- $c = 1$: **3408.32**
- $c = 5$: **2979.74**
- $c = 10$: **2829.24**

Thompson Sampling (TS)

- $(\alpha_1, \beta_1) = (1, 1), (\alpha_2, \beta_2) = (1, 1), (\alpha_3, \beta_3) = (1, 1)$: **3480.75**
- $(\alpha_1, \beta_1) = (601, 401), (\alpha_2, \beta_2) = (401, 601), (\alpha_3, \beta_3) = (2, 3)$: **3492.41**

Problem 4

In this analysis, we compare the performance of three popular multi-armed bandit algorithms: ε-Greedy, Upper Confidence Bound (UCB), and Thompson Sampling (TS). The goal is to compute the gaps between the algorithm outputs (aggregated rewards over $N = 5000$ time slots) and the oracle value, and determine which algorithm performs best.

The theoretical best reward is calculated under the assumption that we know the parameters of the Bernoulli distributions for three arms as follows:

$$\theta_1 = 0.7, \quad \theta_2 = 0.5, \quad \theta_3 = 0.4.$$

Thus, the theoretically maximized expectation is:

$$\max_{I(t),t=1,\ldots,N} \mathbb{E}\left[\sum_{t=1}^{N} r_{I(t)}\right] = N \times \theta_1 = 5000 \times 0.7 = 3500.$$

Gap Calculation

The gap between the algorithm reward and the oracle reward of 3500 is calculated as:

$$\text{Gap} = \text{Oracle Value} - \text{Algorithm Reward}$$

1. ε-Greedy:

ε	Algorithm Reward	Gap
0.1	3408.44	91.56
0.5	3085.66	414.34
0.9	2748.22	751.78

2. Upper Confidence Bound (UCB):

c	Algorithm Reward	Gap
1	3408.32	91.68
5	2979.74	520.26
10	2829.24	670.76

3. Thompson Sampling (TS):

$(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)$	Algorithm Reward	Gap
$(1, 1), (1, 1), (1, 1)$	3480.75	19.25
$(601, 401), (401, 601), (2, 3)$	3492.41	7.59

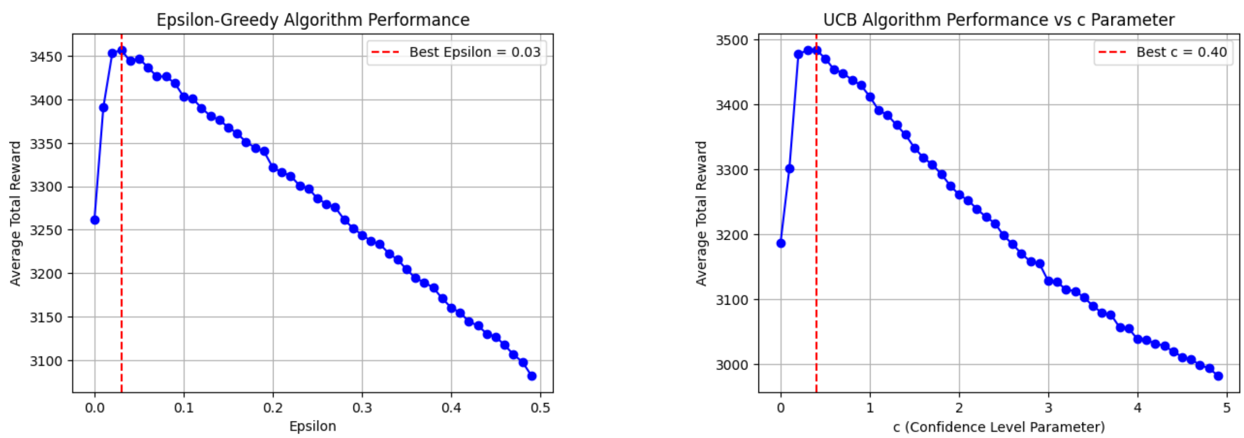
By optimizing the parameters of each algorithm, we can achieve better performance.

After performing the following process for ε-Greedy:

1. Sweep ε from 0 to 0.5 in increments of 0.01.
2. Runs 100 trials for each epsilon value.
3. Averages the total rewards over these 100 trials.
4. Identifies the ε that yields the highest average reward.

we find that the the best ε to maximize the total reward is 0.03, which yields the maximized rewards of 3457.02.

Similarly, for UCB, we sweep the confidence scale c from 0 to 5 in increments of 0.1, and find that the best c is 0.40, which yields the maximized rewards of 3483.90.



(a) ε-Greedy Results

(b) UCB Results

Figure 1: Algorithm Performance Analysis

But even with the optimized parameters, the gap of the first two algorithms (42.98 and 16.10 respectively) is still larger than **Thompson Sampling**, which has the smallest gap of 7.59. Therefore, the **Thompson Sampling** algorithm is the best-performing algorithm in this scenario.

Parameter Impact Analysis

1. ε-Greedy Algorithm

   The ε-Greedy algorithm explores with probability ε and exploits with probability $1 - \varepsilon$. The parameter ε controls the level of exploration versus exploitation.

   - **Low** ε **:**
     - Pros: Prefer exploitation to exploration, leading to higher rewards when the current best action is optimal.
     - Cons: Limited exploration may prevent discovery of better actions, especially in complex environments.

- **High ε (e.g., $\varepsilon = 0.5$):**
  - Pros: Increased exploration, potentially leading to the discovery of optimal actions.
  - Cons: Excessive exploration dilutes the focus on known good actions, potentially lowering overall rewards.
- **Optimal ε:** Based on experiments, we find that $\varepsilon = 0.03$ yields the highest average reward of 3457.02, indicating a good balance between exploration and exploitation. And when $\varepsilon > 0.03$, the average reward decreases as the ε (exploration) increases.

2. Upper Confidence Bound (UCB)

   The UCB formula consists of two terms:

   - **Exploitation ($\hat{\theta}(j)$):** This term represents the current mean reward of action j, which is used to exploit the known best action. The algorithm favors actions with a higher expected reward, leading to exploitation.
   - **Exploration ($c \cdot \sqrt{\frac{2\ln(t)}{\mathbf{count}(j)}}$):** This term encourages exploration by adding a bonus to actions that have been chosen fewer times. The bonus is larger for actions that are less tested, which helps to balance exploration with the exploitation of known rewards. The parameter c controls the size of the exploration term. A higher c increases exploration, and a smaller c favors exploitation.

   So the parameter c controls the trade-off between exploration and exploitation in the UCB algorithm, which is discussed as follows:

   - **Low c:**
     - Pros: Promotes exploitation as the confidence bounds become tighter.
     - Cons: Reduced exploration can prevent the algorithm from discovering better actions.
   - **High c:**
     - Pros: Increases exploration by enlarging the confidence bounds, particularly for arms that have been pulled fewer times.
     - Cons: Excessive exploration can reduce the focus on exploitation, potentially leading to suboptimal rewards.
   - **Optimal c:** In our experiments, $c = 0.40$ provides the best performance, yielding a maximum reward of 3483.90. And when $c > 0.40$, the average reward decreases as the c (exploration) increases.

3. Thompson Sampling

   The Thompson Sampling algorithm is a Bayesian method that uses prior and posterior beliefs to update the probability distribution for each action's reward. It assumes a Beta distribution as the prior for the reward probability of each arm, and updates the parameters α_j and β_j based on the observed outcomes.

   Prior Belief:

   - α_j and β_j represent our prior belief about the reward distribution for action j.

- For example, $\alpha_j = 1$ and $\beta_j = 1$ implies that we expect the reward probability for action j to be 50%, but we have low confidence in this belief (uniform distribution).
- A higher value of α_j relative to β_j means that we believe the reward probability for action j is higher, whereas a lower value of α_j relative to β_j suggests that we believe the reward probability is lower.

Effect of α_j and β_j:

- α_j and β_j are updated after each trial, based on whether the reward for action j was successful or not.
- When α_j is large and β_j is small (e.g., $\alpha_j = 2000$, $\beta_j = 8000$), we have a strong belief that the probability of a reward is low (20%), and the algorithm exploits this information.
- When both α_j and β_j are small (e.g., $\alpha_j = 1$, $\beta_j = 1$), there is high uncertainty about the reward distribution, encouraging more exploration of different actions.

Problem 5

The exploration-exploitation trade-off is a central challenge in decision-making processes, particularly in bandit algorithms. It arises when an agent, tasked with maximizing some reward, must decide how much to explore new options (i.e., gather more data) versus exploiting known, high-reward options based on the information it already has.

Exploration involves trying out different actions to gather more information about their outcomes, even if these actions might not seem optimal in the short term. This is essential in the early stages when little is known about the environment.

Exploitation involves choosing the option that has historically provided the highest reward, based on the information the agent has collected. The idea is to maximize immediate rewards using known information.

The trade-off occurs because if the agent always explores, it might fail to capitalize on known high-reward actions. On the other hand, if it always exploits, it risks missing potentially better actions that could be discovered through exploration. Thus, the challenge is in balancing exploration and exploitation over time to optimize overall rewards.

Algorithms for Addressing the Exploration-Exploitation Trade-off

1. Epsilon-Greedy Algorithm: The epsilon-greedy algorithm provides a simple way to balance exploration and exploitation:

   - With probability ε, the agent explores (chooses a random action).
   - With probability $1-\varepsilon$, the agent exploits (chooses the action that has provided the highest reward so far).

   The parameter ε controls the balance between exploration and exploitation. If ε is high, the agent explores more, while if it is low, it exploits more.

   Challenge: The main limitation of the epsilon-greedy approach is that it uses a fixed ε throughout the process. Over time, an agent might need to explore less and exploit more, but a fixed ε might not reflect this need. Adjusting ε over time (e.g., decreasing ε as the agent learns more) can improve performance, where more exploration is done at the beginning and exploitation increases as certainty builds.

2. Upper Confidence Bound (UCB) Algorithm: The UCB algorithm is based on the idea of balancing exploration and exploitation by considering both the average reward of each action and the uncertainty in the estimate of that reward:

   - For each arm (action), UCB calculates an upper bound on the potential reward based on how many times the arm has been selected and the variance in its reward.
   - The agent then selects the arm with the highest upper bound, balancing the need to exploit the best-known action and explore those with high uncertainty.

   The UCB algorithm relies on Hoeffding's inequality to estimate the confidence intervals for each arm's expected reward. The algorithm rewards actions with high uncertainty to ensure that they are explored adequately while still exploiting actions with the highest observed reward.

Advantage: As time progresses, the UCB algorithm places more weight on exploitation as uncertainty decreases, gradually refining the agent's knowledge. It inherently balances exploration and exploitation without requiring manual adjustment of the exploration rate.

3. Thompson Sampling Algorithm Thompson Sampling takes a probabilistic approach to the exploration-exploitation trade-off, using Bayesian inference:

    - Each arm is modeled by a Beta distribution (since Beta is the conjugate prior for Bernoulli/binomial likelihood), and the parameters of this distribution represent the belief about the arm's reward.
    - The agent samples from the Beta distributions and selects the arm with the highest sampled reward.
    - After each trial, the agent updates the Beta distribution based on the observed reward, refining its belief about the arm's expected reward.

    The agent uses prior knowledge (if available) and updates its beliefs about the rewards using the observed data. In this way, the exploration-exploitation trade-off is handled naturally by the sampling process: arms with higher uncertainty (higher variance in their Beta distribution) are explored more, while arms with lower uncertainty are exploited.

    Advantage: Thompson Sampling is highly effective and tends to outperform epsilon-greedy and UCB in many scenarios, particularly when the true reward distributions are well-modeled by Beta distributions. The algorithm's probabilistic nature makes it flexible and robust across different situations.

Ultimately, the exploration-exploitation trade-off is about finding a strategy that maximizes cumulative rewards over time. While epsilon-greedy is easy to implement and useful for simpler environments, UCB and Thompson Sampling are more sophisticated and provide better performance in many complex scenarios, especially when the agent's knowledge about the environment is continuously evolving.

Problem 6

Problem Settings (Dependent Case)

We examine a multi-armed bandit problem featuring three interdependent arms. This scenario introduces a dependency between the arms, which is set as follows:

After each arm pull, the probabilities are adjusted based on the outcome:

- **If a reward is obtained** from pulling arm j (reward = 1):

$$\theta_j \leftarrow \max(\theta_j - p, 0)$$
$$\theta_k \leftarrow \min(\theta_k + \frac{p}{2}, 1) \quad \forall k \neq j$$

- **If no reward is obtained** from pulling arm j (reward = 0):

$$\theta_j \leftarrow \min(\theta_j + p, 1)$$
$$\theta_k \leftarrow \max(\theta_k - \frac{p}{2}, 0) \quad \forall k \neq j$$

These adjustments ensure that the reward probabilities remain within the valid range $[0, 1]$.

Experimental Setup To evaluate the performance of the algorithms under the independent settings, the following parameters are used:

- **Number of Time Steps** (N): 5000.
- **Number of Trials** (repeat_time): 100.
- **Adjustment Parameter** (p): 0.005.

Objective The primary goal is to determine the optimal algorithmic parameters that maximize the average total reward over N time steps across multiple trials.

Algorithm Design

For the dependent case, we test the three original algorithms on dependent arms. Then, we implement a new algorithm, `Dependency-Aware Thompson Sampling` (DATS), which adapts the Thompson Sampling algorithm to account for the interdependence between arms to obtain a better result.

Algorithm Description: The Dependency-Aware Thompson Sampling with Dynamic Environment Updates algorithm is designed to handle non-stationary environments where arm probabilities change over time and there are potential dependencies between arms. The algorithm operates as follows:

1. **Initialization:**
    - Initialize probabilities (θ) for each arm.

- Set initial Beta distribution parameters (α and β) for each arm.
- Define parameters: N (number of iterations), p (probability update rate), ϵ (exploration rate), γ (dependency factor).

2. **Arm Selection:** For each iteration t from 1 to N:
    - With probability ϵ, explore by choosing a random arm.
    - Otherwise (probability $1 - \epsilon$), exploit using Thompson Sampling:
        - For each arm i, sample a value from Beta(α_i, β_i).
        - Choose the arm with the highest sampled value.
3. **Reward Observation:**
    - Observe a reward (0 or 1) based on the current probability of the chosen arm.
    - Add the reward to the total reward.
4. **Beta Distribution Update:**
    - If reward = 1:
        - Increment α of the chosen arm by 1.
        - If $\gamma > 0$, increment α of all other arms by γ.
    - If reward = 0:
        - Increment β of the chosen arm by 1.
        - If $\gamma > 0$, increment β of all other arms by γ.
5. **Return:** Total accumulated reward over all iterations.

Key Features: This approach combines several strategies to handle the challenges of a non-stationary, dependent arm environment:

- The ϵ-greedy method ensures continued exploration, which is crucial for detecting changes in the environment.
- Thompson Sampling provides a balance between exploration and exploitation based on the current beliefs about arm probabilities.

Algorithm 1 Dependency-Aware Thompson Sampling

Ensure: Total cumulative reward total_reward

$\alpha \leftarrow \alpha_{\text{init}}$
$\beta \leftarrow \beta_{\text{init}}$
$K \leftarrow \text{length}(\alpha)$
$\theta_{\text{current}} \leftarrow \theta$
total_reward $\leftarrow 0$
for $t \leftarrow 1$ to N **do**
    if $\text{Uniform}(0, 1) < \epsilon$ **then**
        chosen_arm $\leftarrow \text{Random}(\{1, \ldots, K\})$
    else
        for $i \leftarrow 1$ to K **do**
            samples$[i] \leftarrow \text{Beta}(\alpha[i], \beta[i])$
        end for
        chosen_arm $\leftarrow \arg\max(\text{samples})$
    end if
    reward $\leftarrow \nVdash[\text{Uniform}(0, 1) < \theta_{\text{current}}[\text{chosen\_arm}]]$
    total_reward $\leftarrow$ total_reward $+$ reward
    if reward $= 1$ **then**
        $\alpha[\text{chosen\_arm}] \leftarrow \alpha[\text{chosen\_arm}] + 1$
        for other_arm $\in \{1, \ldots, K\} \setminus \{\text{chosen\_arm}\}$ **do**
            $\alpha[\text{other\_arm}] \leftarrow \alpha[\text{other\_arm}] + \gamma$
        end for
    else
        $\beta[\text{chosen\_arm}] \leftarrow \beta[\text{chosen\_arm}] + 1$
        for other_arm $\in \{1, \ldots, K\} \setminus \{\text{chosen\_arm}\}$ **do**
            $\beta[\text{other\_arm}] \leftarrow \beta[\text{other\_arm}] + \gamma$
        end for
    end if
end for
return total_reward

Results and Analysis

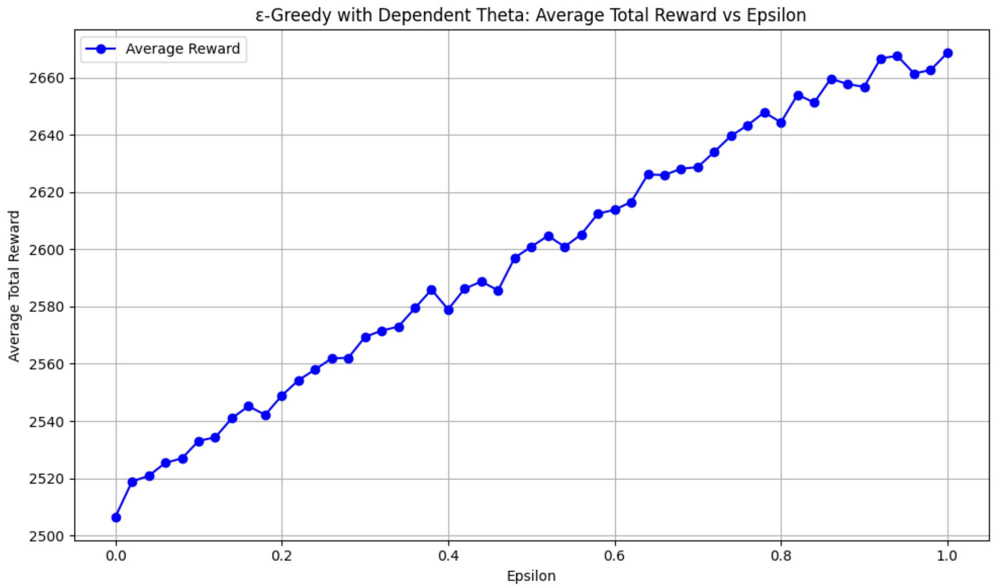


Figure 2: Dependent ε-Greedy Performance

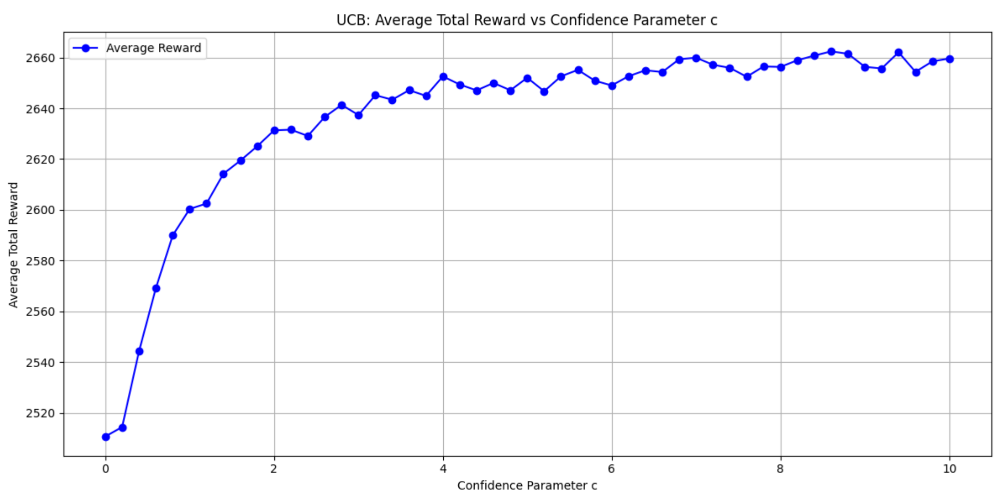


Figure 3: Dependent UCB Performance

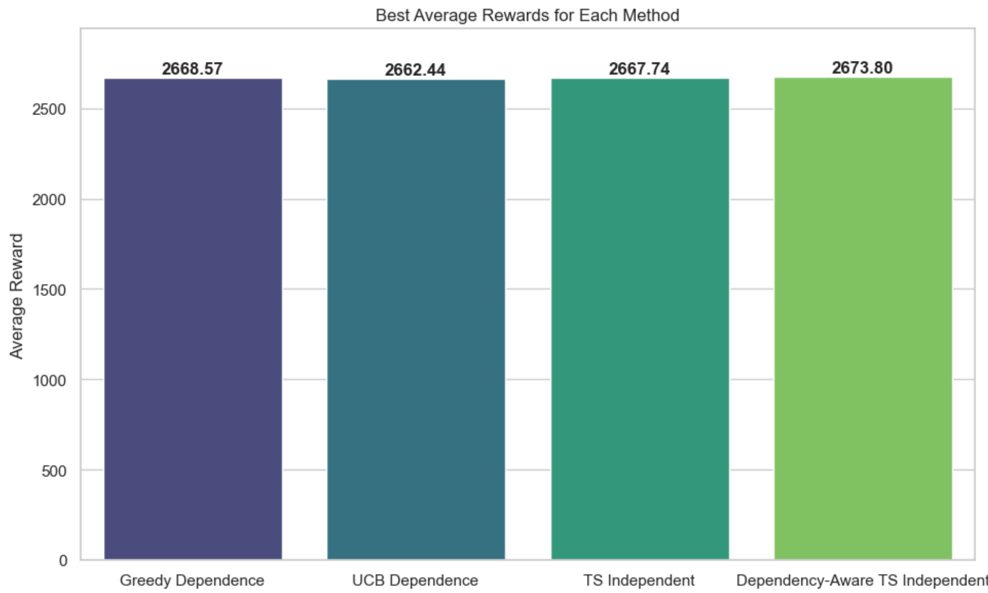


Figure 4: Comparison of TS and DATS with Different Alpha1

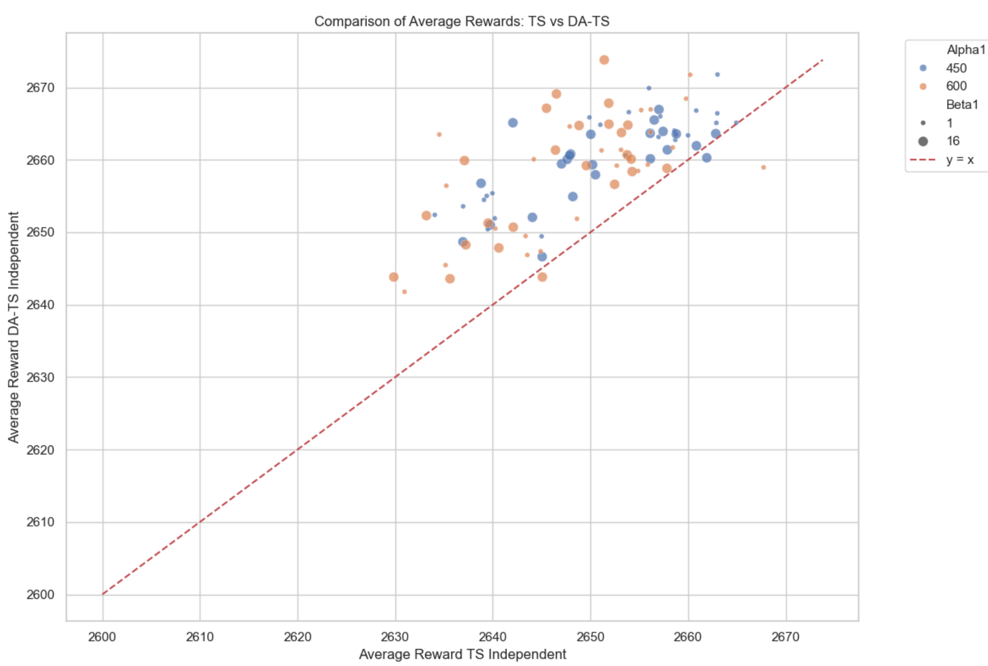


Figure 5: Comparison of TS and DATS

By experimenting, we receive the following results:

Algorithm	**Best Parameters**	**Maximum Reward**
ε-Greedy	$\varepsilon = 1.00$	2668.57
UCB	$c = 8.60$	2662.44
Thompson Sampling	$\alpha = [600, 300, 450]$ $\beta = [1, 1, 31]$	2667.74
Dependency-Aware TS	$\alpha = [600, 450, 450]$ $\beta = [16, 16, 31]$ $\epsilon = 0.001$ $\gamma = 10^{-6}$	2673.80

Table 1: Performance Comparison of Bandit Algorithms

The experimental results presented in Figures 2, 3, 4, 5, and Table 1 provide compelling evidence for the superiority of our proposed Dependency-Aware Thompson Sampling (DATS) algorithm over the other four methods examined: Greedy Dependence, UCB Dependence, and TS Independent.

1. **Superior Performance:** As shown in Figure 4 and Table 1, DATS achieves the highest average reward (2673.80) among all methods tested. This outperforms Greedy Dependence (2668.57), UCB Dependence (2662.44), and TS Independent (2667.74), demonstrating DATS's ability to make more informed decisions in multi-armed bandit problems with dependent arms.

2. **Consistent Outperformance:** The scatter plot in Figure 5 illustrates that DATS consistently outperforms the TS Independent method across various parameter settings. The majority of points lie above the $y = x$ line, indicating that DATS yields higher average rewards in most scenarios.

3. **Robustness to Hyperparameters:** Unlike the Greedy Dependence (Figure 2) and UCB Dependence (Figure 3) methods, which show high sensitivity to their respective hyperparameters (ϵ and confidence parameter c), DATS demonstrates more stable performance across different settings. This robustness is a crucial advantage in real-world applications where optimal hyperparameter tuning may be challenging.

Part II: Bayesian Bandit Algorithms

Problem 1

Experiment Settings

The simulation was conducted with the following parameters:

- **True Success Probabilities:** The true success probabilities for the two arms were set to: $\theta_{\text{true}} = [0.7, 0.5]$
- **Beta Distribution Priors:** The prior parameters for the Beta distributions were initialized as: $\alpha_{\text{prior}} = [1, 1], \quad \beta_{\text{prior}} = [1, 1]$
- **Gamma Values:** The discount factor γ was varied linearly from 0.95 to 1.0 in increments of 0.01, resulting in 50 values: $\gamma \in \text{linspace}(0.95, 1.0, 50)$
- **Number of Time Steps:** Each trial consisted of 5000 time steps: $T = 5000$
- **Number of Trials per Gamma:** The experiment was repeated 50 times for each γ value: $\text{Repeats} = 50$

Results

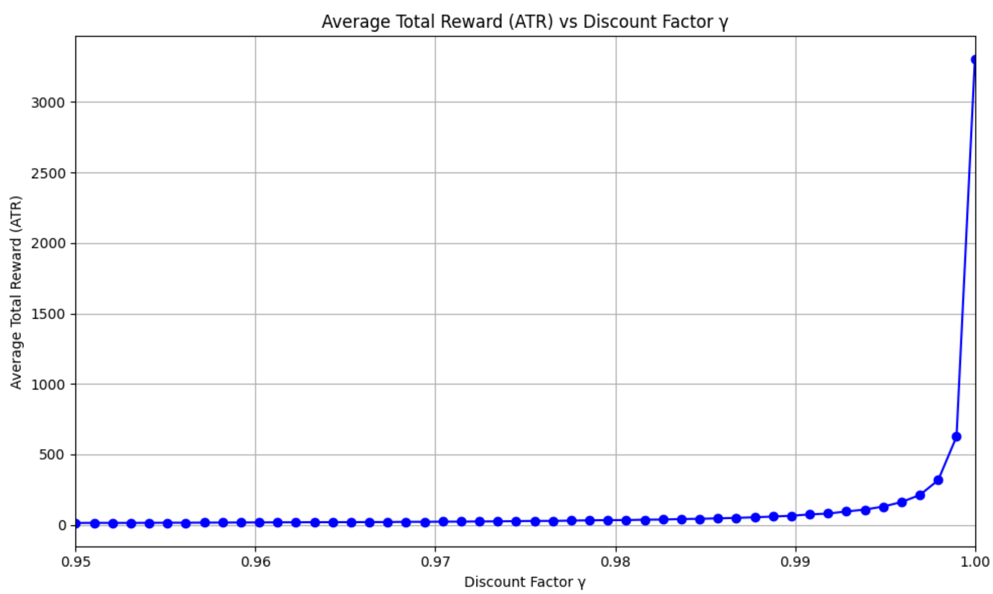


Figure 6: Intuitive Outcomes with Different γ Values

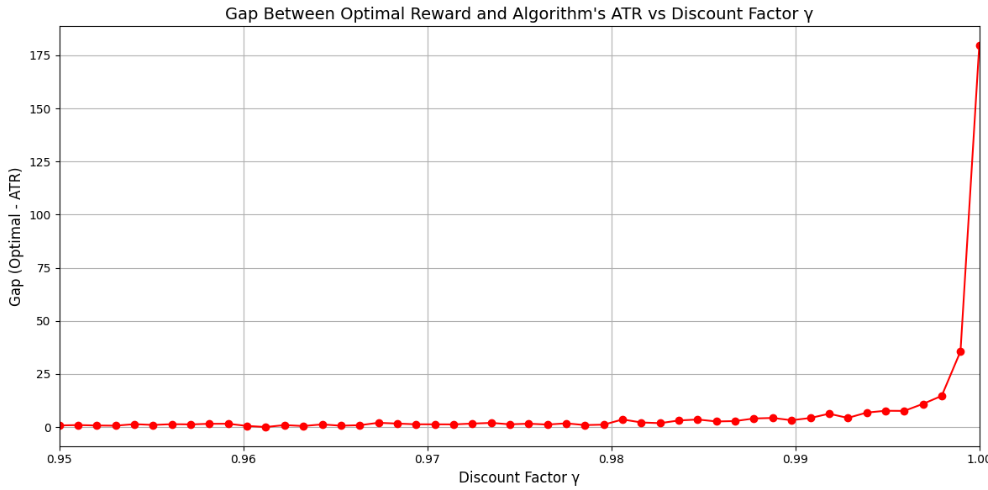


Figure 7: Gaps Between Optimal and Intuitive Algorithm Outcomes with Different γ Values

In this part, we implemented an intuitive algorithm and calculated the theoretical optimal rewards.

The theoretical optimal reward is calculated as:

$$\text{Optimal Reward} = \theta_{\text{best}} \times \frac{1 - \gamma^{\text{time\_steps}}}{1 - \gamma}$$

where θ_{best} is the highest success probability among the arms, γ is the discount factor, and *time_steps* is the total number of pulls in a trial.

The results of our experiments are visualized in the two figures:

From Figure 7, we observe that the gap between the optimal reward and the Average Total Reward (ATR) produced by the intuitive algorithm remains small for most values of the discount factor γ. This demonstrates that the intuitive algorithm behaves very well in most cases.

Additionally, as shown in Figure 6, the ATR increases as γ approaches 1.0, which aligns with our expectations since the algorithm becomes more conservative, favoring long-term rewards.

Problem 2

Situation when the intuitive algorithm fails to perform optimally

We set the theta values for the two arms as $\theta_{\text{true}} = [0.3, 0.6]$. By running the experiment with the same other settings as in Problem 1, we get the following results:

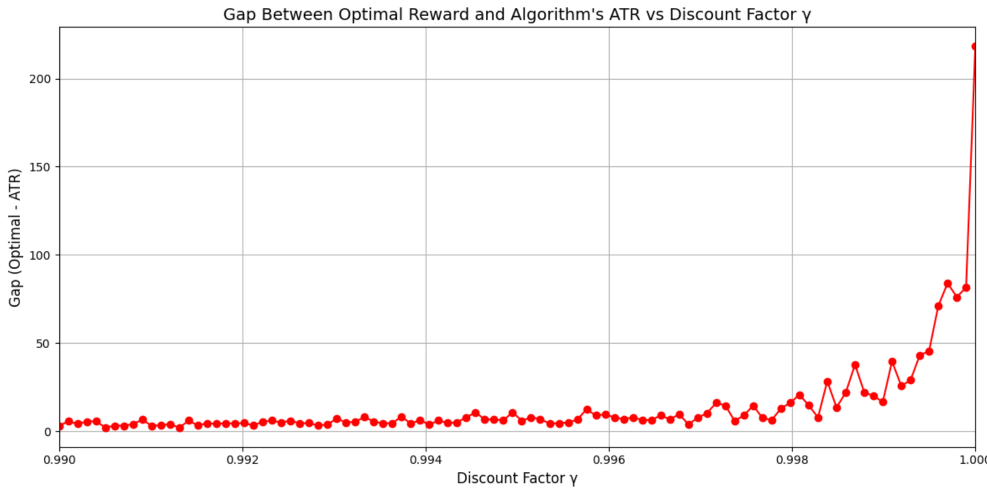


Figure 8: Gaps Between Optimal and Intuitive Algorithm Outcomes with Different γ Values

Since the $\gamma \in [0, 0.99]$ shows no significant difference in the gaps between the optimal and intuitive algorithm outcomes, we simply omit testing these values.

As we can see in the Figure 8, the gap between the optimal reward and the Average Total Reward (ATR) produced by the intuitive algorithm is significantly larger when $\gamma \in [0.998, 1.000]$. This indicates that the intuitive algorithm fails to perform optimally in this scenario.

Comparison between the intuitive algorithm and TS

We compare the performance of the intuitive algorithm with that of Thompson Sampling (TS) by plotting the Average Total Reward (ATR) for both algorithms across different values of the discount factor γ. The settings are the same as above. We get the following results:

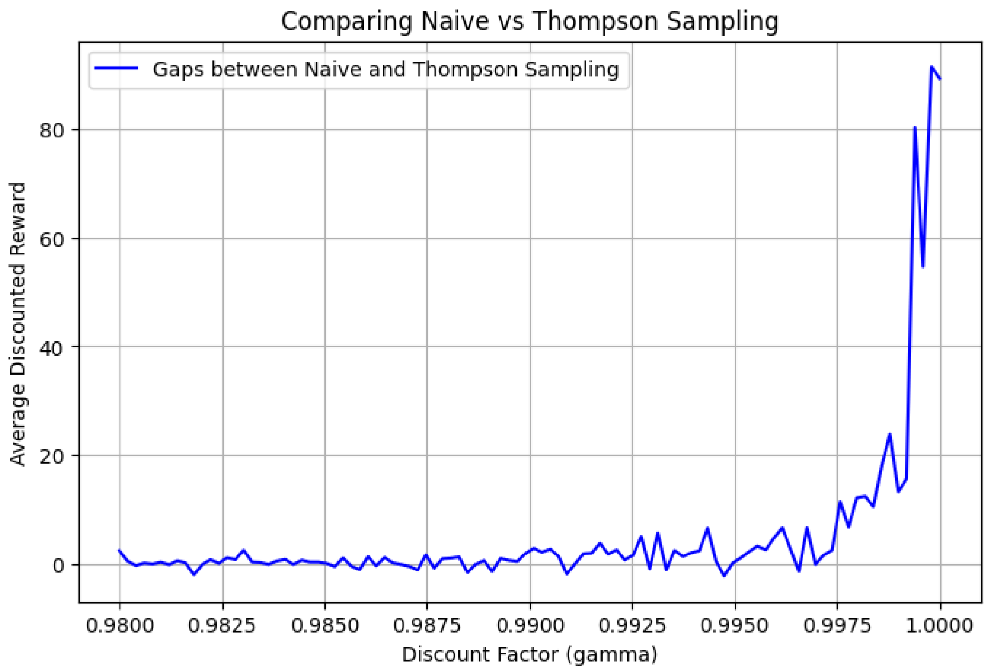


Figure 9: Comparison of Intuitive Algorithm and Thompson Sampling

As shown in Figure 9, the Average Total Reward (ATR) produced by Thompson Sampling (TS) is significantly higher than that of the intuitive algorithm across $\gamma \in [0.9875, 1.0000]$. This indicates that Thompson Sampling outperforms the intuitive algorithm in this scenario.

Therefore, by these two experiments, we can conclude that the intuitive algorithm may fail to perform optimally in certain situations, and Thompson Sampling (TS) can provide better results in such cases.

Problem 3

Problem Statement

For the expected total reward under an optimal policy, show that the following recurrence equation holds:

$$
\begin{aligned}
R_1(\alpha_1,\beta_1) &= \frac{\alpha_1}{\alpha_1+\beta_1}[1+\gamma R(\alpha_1+1,\beta_1,\alpha_2,\beta_2)] + \frac{\beta_1}{\alpha_1+\beta_1}[\gamma R(\alpha_1,\beta_1+1,\alpha_2,\beta_2)];\\
R_2(\alpha_2,\beta_2) &= \frac{\alpha_2}{\alpha_2+\beta_2}[1+\gamma R(\alpha_1,\beta_1,\alpha_2+1,\beta_2)] + \frac{\beta_2}{\alpha_2+\beta_2}[\gamma R(\alpha_1,\beta_1,\alpha_2,\beta_2+1)];\\
R(\alpha_1,\beta_1,\alpha_2,\beta_2) &= \max\{R_1(\alpha_1,\beta_1), R_2(\alpha_2,\beta_2)\}.
\end{aligned}
$$

Proof

At time $t=0$, the parameters θ_1 and θ_2 are assumed to follow independent Beta distributions with parameters (α_1,β_1) and (α_2,β_2), respectively.

Pull the First Arm

When arm 1 is pulled at time t, the reward is determined by the Bernoulli distribution $\text{Bern}(\theta_1)$:

- With probability θ_1, a success occurs, yielding an immediate reward of 1 and resulting in a posterior distribution $\text{Beta}(\alpha_1+1,\beta_1)$. The future reward is discounted by γ, leading to a total future reward of $\gamma R(\alpha_1+1,\beta_1,\alpha_2,\beta_2)$.
- With probability $1-\theta_1$, a failure occurs, yielding an immediate reward of 0 and resulting in a posterior distribution $\text{Beta}(\alpha_1,\beta_1+1)$. The total future reward in this case is $\gamma R(\alpha_1,\beta_1+1,\alpha_2,\beta_2)$.

Combining these outcomes, the expected reward from pulling arm 1 is:

$$R_1(\alpha_1,\beta_1) = \theta_1[1+\gamma R(\alpha_1+1,\beta_1,\alpha_2,\beta_2)] + (1-\theta_1)[\gamma R(\alpha_1,\beta_1+1,\alpha_2,\beta_2)].$$

Using the expectation of θ_1 under its Beta distribution, where:

$$E[\theta_1] = \frac{\alpha_1}{\alpha_1+\beta_1}, \quad E[1-\theta_1] = \frac{\beta_1}{\alpha_1+\beta_1},$$

we can rewrite $R_1(\alpha_1,\beta_1)$ as:

$$R_1(\alpha_1,\beta_1) = \frac{\alpha_1}{\alpha_1+\beta_1}[1+\gamma R(\alpha_1+1,\beta_1,\alpha_2,\beta_2)] + \frac{\beta_1}{\alpha_1+\beta_1}[\gamma R(\alpha_1,\beta_1+1,\alpha_2,\beta_2)].$$

Pull the Second Arm

Similarly, when arm 2 is pulled, the reward is determined by the Bernoulli distribution $\text{Bern}(\theta_2)$. The outcomes are:

- With probability θ_2, a success occurs, yielding an immediate reward of 1 and resulting in a posterior distribution $\text{Beta}(\alpha_2 + 1, \beta_2)$. The total future reward is $\gamma R(\alpha_1, \beta_1, \alpha_2 + 1, \beta_2)$.

- With probability $1 - \theta_2$, a failure occurs, yielding an immediate reward of 0 and resulting in a posterior distribution $\text{Beta}(\alpha_2, \beta_2 + 1)$. The total future reward in this case is $\gamma R(\alpha_1, \beta_1, \alpha_2, \beta_2 + 1)$.

Combining these outcomes, the expected reward from pulling arm 2 is:

$$R_2(\alpha_2, \beta_2) = \frac{\alpha_2}{\alpha_2 + \beta_2}[1 + \gamma R(\alpha_1, \beta_1, \alpha_2 + 1, \beta_2)] + \frac{\beta_2}{\alpha_2 + \beta_2}[\gamma R(\alpha_1, \beta_1, \alpha_2, \beta_2 + 1)].$$

The expected total reward under the optimal policy is the maximum of the rewards from pulling either arm 1 or arm 2:

$$R(\alpha_1, \beta_1, \alpha_2, \beta_2) = \max\{R_1(\alpha_1, \beta_1), R_2(\alpha_2, \beta_2)\}.$$

Combining all results, we have proved that the recurrence equations are:

$$\begin{aligned}
R_1(\alpha_1, \beta_1) &= \frac{\alpha_1}{\alpha_1 + \beta_1}[1 + \gamma R(\alpha_1 + 1, \beta_1, \alpha_2, \beta_2)] + \frac{\beta_1}{\alpha_1 + \beta_1}[\gamma R(\alpha_1, \beta_1 + 1, \alpha_2, \beta_2)], \\
R_2(\alpha_2, \beta_2) &= \frac{\alpha_2}{\alpha_2 + \beta_2}[1 + \gamma R(\alpha_1, \beta_1, \alpha_2 + 1, \beta_2)] + \frac{\beta_2}{\alpha_2 + \beta_2}[\gamma R(\alpha_1, \beta_1, \alpha_2, \beta_2 + 1)], \\
R(\alpha_1, \beta_1, \alpha_2, \beta_2) &= \max\{R_1(\alpha_1, \beta_1), R_2(\alpha_2, \beta_2)\}.
\end{aligned}$$

Problem 4

Problem Statement

How to solve the recurrence equations exactly or approximately?

Solution Approach

Dynamic Programming (DP) is a powerful method that, in theory, can provide exact solutions to the optimal policy. The principle of optimality, which underlies DP, ensures that if we can evaluate all possible states, we will find the globally optimal solution. However, the transition from theory to practice introduces several challenges that necessitate an approximate solution:

Infinite State Space The Beta-Bernoulli bandit has an infinite state space. Each arm's Beta distribution parameters (α and β) can grow indefinitely as we observe more outcomes. In theory, DP would require us to compute and store values for every possible combination of $(\alpha_1, \beta_1, \alpha_2, \beta_2)$, which is infeasible.

Computational Constraints Even if we could store an infinite number of states, computing the optimal value for each state would require an infinite number of operations. Real-world computers have finite processing capabilities, making exact computation impossible.

Memory Limitations Storing values for an infinite number of states would require infinite memory, which is not available in practice.

To address these challenges, we employ an approximation strategy:

1. **State Space Truncation:** We introduce a maximum value N_{max} (denoted as M in our algorithm) for each parameter. This effectively "truncates" our infinite state space to a finite one.

2. **Boundary Conditions:** We define boundary conditions for our truncated space (e.g., setting the value to 0 when $\alpha_i + \beta_i = N_{max}$ for either arm).

3. **Value Iteration:** We use iterative updates to approximate the value function, stopping when changes become smaller than a predefined tolerance or after a maximum number of iterations.

This approach allows us to find an approximate solution that is computationally feasible. The quality of this approximation depends on several factors:

- The choice of N_{max}: Larger values allow for a more accurate approximation but increase computational cost.

- The convergence tolerance: Smaller tolerances can provide more accurate results but may require more iterations.
- The discount factor γ: Values closer to 1 consider long-term rewards more heavily but may slow convergence.

In practice, these parameters are often tuned to balance solution quality with computational efficiency. While we sacrifice theoretical exactness, this approach often provides solutions that are "good enough" for practical applications, capturing the essential behavior of the optimal policy within a tractable computation framework.

Pseudocode

The following algorithm implements our solution approach:

Algorithm 2 Solve 2-Armed Beta-Bernoulli Bandit

```
procedure SOLVE_2ARMED_BANDIT_DP($M, \gamma, tol, max\_iter$)
    Input:
        $M$: Truncation level for $\alpha_i, \beta_i$
        $\gamma$: Discount factor ($0 < \gamma < 1$)
        $tol$: Convergence tolerance
        $max\_iter$: Maximum number of iterations
    Initialize $R$ and $policy$ as 4D arrays of size $(M+1) \times (M+1) \times (M+1) \times (M+1)$
    for $it = 1$ to $max\_iter$ do
        $delta \leftarrow 0$                                ▷ Track maximum change in this iteration
        for $\alpha_1 = 1$ to $M$ do
            for $\beta_1 = 1$ to $M$ do
                for $\alpha_2 = 1$ to $M$ do
                    for $\beta_2 = 1$ to $M$ do
                        Compute $R_1(\alpha_1, \beta_1)$ and $R_2(\alpha_2, \beta_2)$ using recurrence relations
                        $new\_val \leftarrow \max(R_1, R_2)$
                        $old\_val \leftarrow R[\alpha_1, \beta_1, \alpha_2, \beta_2]$
                        $R[\alpha_1, \beta_1, \alpha_2, \beta_2] \leftarrow new\_val$
                        $policy[\alpha_1, \beta_1, \alpha_2, \beta_2] \leftarrow \arg\max(R_1, R_2)$
                        $delta \leftarrow \max(delta, |new\_val - old\_val|)$
                    end for
                end for
            end for
        end for
        if $delta < tol$ then
            break                                           ▷ Convergence achieved
        end if
    end for
    return $R, policy$
end procedure
```

This algorithm iteratively computes the value function R and the optimal policy. The policy array stores the optimal action (0 for arm 1, 1 for arm 2) for each state. The algorithm terminates when either the solution converges (change in values less than tol) or the maximum number of iterations is reached.

Problem 5

In this problem, we implement the dynamic programming algorithm to solve the 2-armed Beta-Bernoulli bandit problem with the given recurrence relations. We set the parameters as follows:

1. Discount factors (γ): 100 evenly spaced values from 0.9 to 1
   The experiment tests a range of discount factors to explore their impact on performance.

2. Truncation level for Dynamic Programming (M): 17
   This parameter limits the state space for the DP algorithm, balancing computational feasibility with solution accuracy.

3. Tolerance (tol): 10^{-8}
   The algorithm is considered converged when the maximum change in the value function between iterations falls below this threshold.

4. Maximum iterations (max_iter): 50
   This sets an upper limit on the number of iterations for the DP algorithm, ensuring termination even if the tolerance-based convergence is not reached.

The experiment compares two algorithms:

Algorithm	**Description**
4D Dynamic Programming	Solves the problem exactly (up to the truncation level)
Thompson Sampling	Approximate method using Beta distributions

Table 2: Comparison of algorithms used in the experiment

For each γ value, both algorithms are evaluated over multiple trials. The performance metric is the discounted cumulative reward, defined as:

$$R_{\text{total}} = \sum_{t=1}^{T} \gamma^{t-1} r_t$$

where r_t is the reward at time step t, and T is the total number of time steps (5000 in this experiment).

The results are visualized to compare the algorithms' performance across different discount factors and to analyze the performance gap between them.

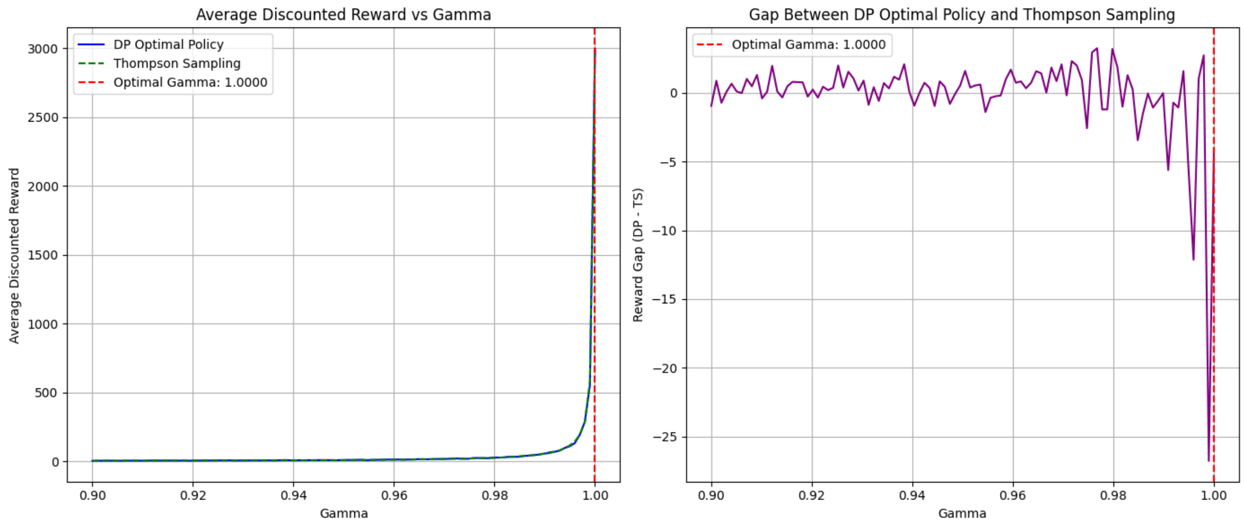


Figure 10: Dynamic Programming vs. Thompson Sampling: Optimal Performance

From Figure 10, we observe that the Dynamic Programming (DP) algorithm outperforms Thompson Sampling across various discount factors (γ), despite that the gap shows a sharp decrease as γ approaches 1. Interestingly, the optimal γ appears to be exactly 1.0000, suggesting that in this scenario, fully prioritizing long-term rewards yields the best performance. The fluctuations in the performance gap for high γ values (0.98-1.00) reveal complex dynamics in the relative efficacy of these algorithms as the planning horizon extends, warranting further investigation into the underlying mechanisms driving these differences.

Conclusion

This study provides a comprehensive evaluation of various bandit learning algorithms, offering valuable insights into their performance characteristics and the nuances of the exploration-exploitation trade-off. Our investigation encompassed both classical and Bayesian approaches, revealing important findings that contribute to the broader understanding of sequential decision-making under uncertainty.

In Part I, we examined the performance of ε-greedy, Upper Confidence Bound (UCB), and Thompson Sampling (TS) algorithms in a classical multi-armed bandit setting. Our results demonstrated that Thompson Sampling consistently outperformed the other algorithms across various parameter settings, achieving the smallest gap from the oracle value. This superior performance underscores the effectiveness of probabilistic methods in balancing exploration and exploitation.

The introduction of arm dependencies in our experiments highlighted the adaptability of these algorithms to more complex environments. Notably, our proposed Dependency-Aware Thompson Sampling (DATS) algorithm showed improved performance in this setting, illustrating the potential for tailored approaches in specific problem domains.

Part II of our study delved into Bayesian bandit algorithms, focusing on discounted reward scenarios. We derived and implemented a dynamic programming solution for the optimal policy, providing a benchmark for comparison with more computationally efficient heuristics. The results revealed that while intuitive algorithms perform well in many cases, they can fail to achieve optimality under certain conditions, particularly as the discount factor approaches 1.

Our analysis of the recurrence equations for the expected total reward under an optimal policy offers theoretical insights into the structure of the problem. The approximate solution method we developed, using state space truncation and value iteration, provides a practical approach to solving these complex problems within computational constraints.

The comparative analysis between the dynamic programming solution and Thompson Sampling across different discount factors yielded intriguing results. The DP approach consistently outperformed TS, the performance gap narrowed significantly (though TS outperforms DP significantly as γ is very close to 1), suggesting that simpler heuristics may be nearly optimal.

These findings have important implications for real-world applications of bandit algorithms. The superior performance of Thompson Sampling in various settings suggests its potential as a robust default choice for many problems. However, the success of the DATS algorithm in dependent arm scenarios highlights the value of domain-specific adaptations. Furthermore, the near-optimality of heuristic methods indicates that computationally intensive exact solutions may not always be necessary in practice.

Future research could explore several promising directions:

- Developing more sophisticated dependency models and corresponding algorithms to handle complex real-world scenarios.

- Investigating the theoretical properties of the Dependency-Aware Thompson Sampling algorithm and deriving bounds on its regret.
- Improve the accuracy and efficiency of the DP algorithm under the circumstances where the discount factor approaches 1.
- Exploring the application of these algorithms to specific domains such as online advertising, clinical trials, or recommendation systems.

In conclusion, this study advances our understanding of bandit algorithms' behavior across various settings and parameter ranges. By rigorously comparing classical and Bayesian approaches, we have provided insights that can guide algorithm selection and design in practical applications. As sequential decision-making problems continue to grow in importance across numerous fields, the insights gained from this work contribute to the ongoing development of efficient and effective reinforcement learning strategies.

Contributions of Team Members

- **Anrui Wang:**
  - Part I
  - Part II Problems 4 & 5
  - LaTeX report writing
  - Code formatting
  - Roughly 50% workload
- **Zhao Lu:**
  - Part II brainstorming and coding
  - Roughly 35% workload
- **Jingran Fan:**
  - Part II problem 1 & 2 drafting, problem 5 brainstorming
  - Emotional Support
  - Roughly 15% workload

Appendix: PDF version of Jupyter Notebook

In [4]:
```
import matplotlib.pyplot as plt
import numpy as np
import random, math, copy
```

In [5]:
```
# Set random seed for reproducibility
np.random.seed(42)

# Parameters
num_arms = 3

# Oracle theta of each arm
theta = np.array([0.7, 0.5, 0.4])
```

Problem 2: Implement classical bandit algorithms

1. The epsilon-greedy Algorithm

In [80]:
```
def epsilon_greedy(epsilon, N, theta):
    """
    Implement the epsilon-greedy algorithm for a Bernoulli bandit problem.

    Parameters
    ----------
    epsilon : float
        The probability of exploration.
    N : int
        Number of time steps.
    theta : array-like
        True success probabilities of each arm.

    Returns
    -------
    total_reward : float
        Total reward accumulated over N time steps.
    """
    Q = np.zeros(num_arms)  # Estimated values for each arm
    counts = np.zeros(num_arms)  # Count of how many times each arm is pulled
    total_reward = 0  # Total reward tracker

    # Initialization: Pull each arm once
    for arm in range(num_arms):
        reward = 1 if np.random.rand() < theta[arm] else 0
        counts[arm] = 1
        Q[arm] = reward
        total_reward += reward

    # Main Loop: Epsilon-greedy exploration and exploitation
    for t in range(num_arms, N):
        if np.random.rand() < epsilon:
```

```
            # Exploration: choose a random arm
            arm = np.random.randint(num_arms)
        else:
            # Exploitation: choose the arm with the highest estimated value
            arm = np.argmax(Q)

        # Simulate pulling the chosen arm
        reward = 1 if np.random.rand() < theta[arm] else 0

        counts[arm] += 1
        Q[arm] += (1 / counts[arm]) * (reward - Q[arm])

        total_reward += reward

    return total_reward
```

2. The UCB (Upper Confidence Bound) Algorithm

In [81]:
```
def ucb(c, N, theta):
    """
    Implement the UCB (Upper Confidence Bound) algorithm for a Bernoulli bandit pro

    Parameters
    ----------
    c : float
        Confidence level parameter for the UCB algorithm.
    N : int
        Number of time steps.
    theta : array-like
        True success probabilities of each arm.

    Returns
    -------
    rewards_history : array
        The rewards obtained at each time step.
    """

    Q = np.zeros(num_arms)
    counts = np.zeros(num_arms)
    total_reward = 0

    # Initialize by pulling each arm once
    for arm in range(num_arms):
        reward = 1 if np.random.rand() < theta[arm] else 0
        Q[arm] = reward
        counts[arm] = 1
        total_reward += reward

    for t in range(num_arms+1, N+1):
        # Avoid division by zero because each arm was pulled once
        ucb_values = Q + c * np.sqrt((2*np.log(t))/counts)
        arm = np.argmax(ucb_values)
        reward = 1 if np.random.rand() < theta[arm] else 0
        counts[arm] += 1
```

```
        Q[arm] += (1/counts[arm])*(reward - Q[arm])
        total_reward += reward
    return total_reward
```

3. TS (Thompson Sampling) Algorithm

In [82]:
```
from scipy.stats import beta

def thompson_sampling(N, theta, alpha_init, beta_init):
    """
    Implement the Thompson Sampling (TS) algorithm for a Bernoulli bandit problem.

    Parameters
    ----------
    N : int
        Number of time steps.
    theta : array-like
        True success probabilities of each arm.
    alpha_init : array-like
        Initial alpha parameters for the Beta distributions of each arm.
    beta_init : array-like
        Initial beta parameters for the Beta distributions of each arm.

    Returns
    -------
    rewards_history : array
        The rewards obtained at each time step.
    """
    alpha = alpha_init.copy()
    beta_ = beta_init.copy()
    total_reward = 0
    for t in range(N):
        sampled_thetas = [np.random.beta(alpha[j], beta_[j]) for j in range(num_arm
        arm = np.argmax(sampled_thetas)
        reward = 1 if np.random.rand() < theta[arm] else 0
        total_reward += reward
        alpha[arm] += reward
        beta_[arm] += 1 - reward
    return total_reward
```

Problem 3: Each experiment lasts for $N = 5000$ time slots, and we run each experiment 200 trials. Results are averaged over these 200 independent trials.

In [83]:
```
# Parameters
N = 5000
num_trials = 200
epsilons = [0.1, 0.5, 0.9]
cs = [1, 5, 10]
```

```
# Two sets of prior parameters for TS
# Set 1: (1,1), (1,1), (1,1)
alpha_set_1 = np.array([1, 1, 1])
beta_set_1 = np.array([1, 1, 1])

# Set 2: (601,401), (401,601), (2,3)
alpha_set_2 = np.array([601, 401, 2])
beta_set_2 = np.array([401, 601, 3])

# True parameters of the arms (as per the oracle, but not known to the algorithm)
theta = np.array([0.7, 0.5, 0.4])
```

In [84]:

```
# Epsilon-greedy
print("Epsilon-greedy results:")
for eps in epsilons:
    rewards = []
    for _ in range(num_trials):
        rewards.append(epsilon_greedy(eps, N, theta))
    mean_reward = np.mean(rewards)
    print(f"Epsilon = {eps}, Average total reward over {num_trials} trials: {mean_r
```

```
Epsilon-greedy results:
Epsilon = 0.1, Average total reward over 200 trials: 3408.44
Epsilon = 0.5, Average total reward over 200 trials: 3085.66
Epsilon = 0.9, Average total reward over 200 trials: 2748.215
```

In [85]:

```
# UCB
print("\nUCB results:")
for c_val in cs:
    rewards = []
    for _ in range(num_trials):
        rewards.append(ucb(c_val, N, theta))
    mean_reward = np.mean(rewards)
    print(f"c = {c_val}, Average total reward over {num_trials} trials: {mean_rewar
```

```
UCB results:
c = 1, Average total reward over 200 trials: 3408.315
c = 5, Average total reward over 200 trials: 2979.74
c = 10, Average total reward over 200 trials: 2829.24
```

In [86]:

```
# Thompson Sampling
print("\nThompson Sampling results:")
rewards_set_1 = []
for _ in range(num_trials):
    rewards_set_1.append(thompson_sampling(N, theta, alpha_set_1, beta_set_1))
mean_set_1 = np.mean(rewards_set_1)
print(f"Set 1 Priors (1,1),(1,1),(1,1), Average total reward: {mean_set_1}")

rewards_set_2 = []
for _ in range(num_trials):
    rewards_set_2.append(thompson_sampling(N, theta, alpha_set_2, beta_set_2))
mean_set_2 = np.mean(rewards_set_2)
print(f"Set 2 Priors (601,401),(401,601),(2,3), Average total reward: {mean_set_2}"
```

```
Thompson Sampling results:
Set 1 Priors (1,1),(1,1),(1,1), Average total reward: 3480.75
Set 2 Priors (601,401),(401,601),(2,3), Average total reward: 3492.41
```

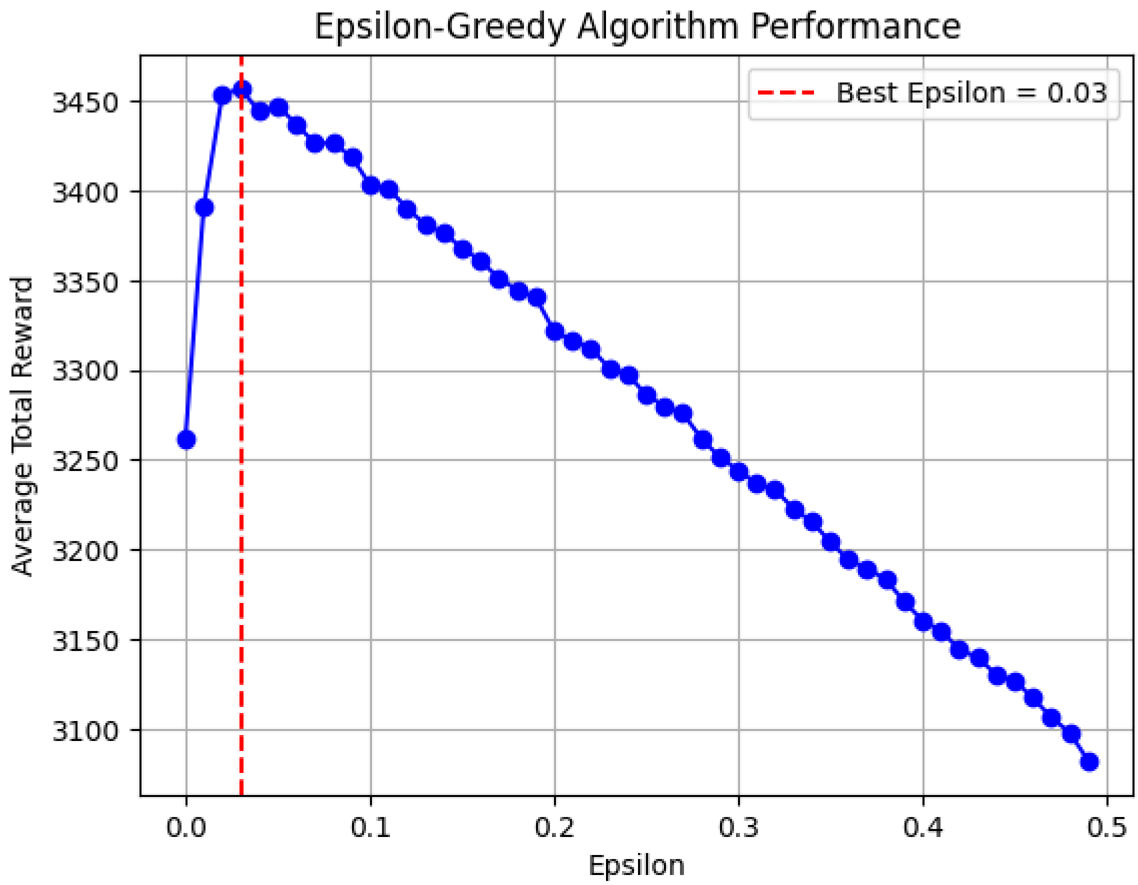
Problem 4

4.1 Find the optimal results for each algorithm

In [94]:

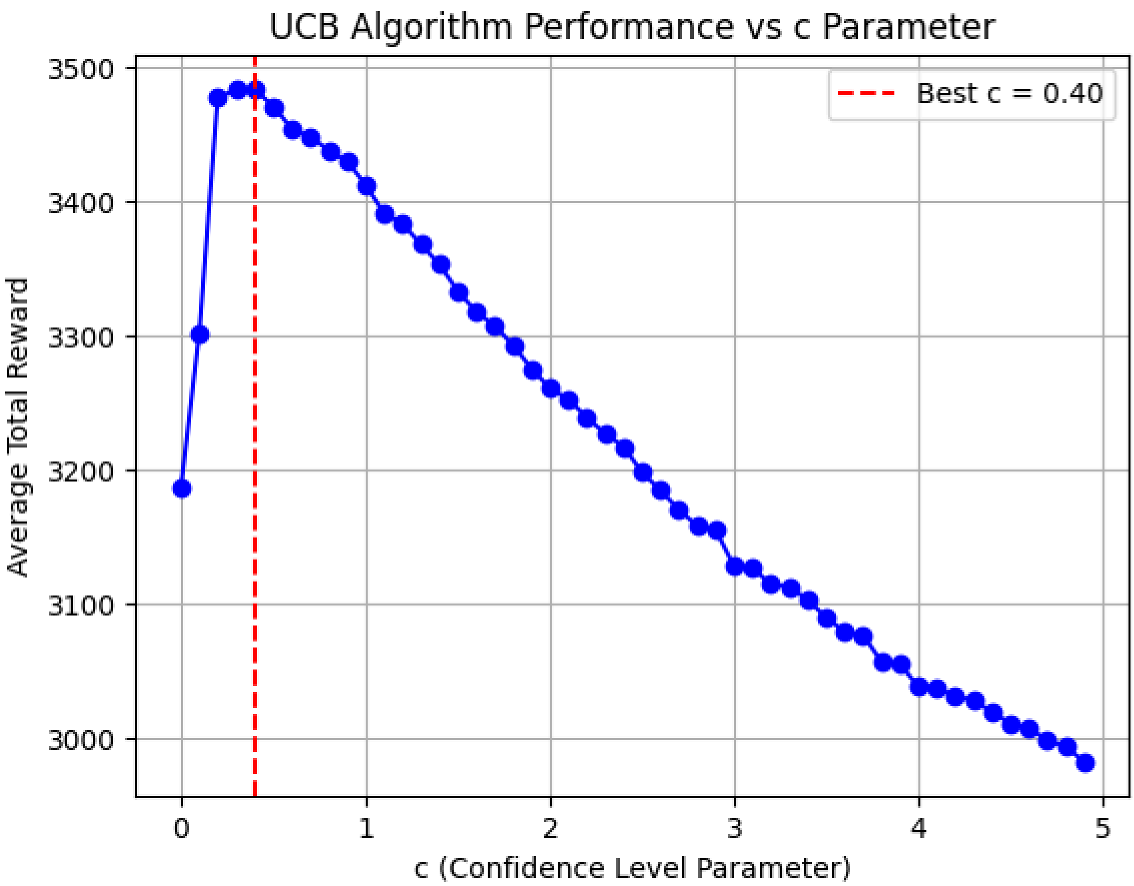
```
num_trials = 100

epsilon_values = np.arange(0, 0.5, 0.01)
average_rewards = []

for eps in epsilon_values:
    rewards = []
    for _ in range(num_trials):
        rewards.append(epsilon_greedy(eps, N, theta))
    average_rewards.append(np.mean(rewards))

# Find the best epsilon
best_epsilon = epsilon_values[np.argmax(average_rewards)]
print(f"Best epsilon: {best_epsilon:.2f}")
print(f"Maximum average total reward: {np.max(average_rewards):.2f}")

# Plot the results
import matplotlib.pyplot as plt

plt.plot(epsilon_values, average_rewards, marker='o', linestyle='-', color = 'b')
plt.axvline(x=best_epsilon, color='r', linestyle='--', label=f'Best Epsilon = {best
plt.xlabel('Epsilon')
plt.ylabel('Average Total Reward')
plt.title('Epsilon-Greedy Algorithm Performance')
plt.legend()
plt.grid(True)
plt.show()
```

```
Best epsilon: 0.03
Maximum average total reward: 3457.02
```



In [93]:
```python
c_values = np.arange(0, 5, 0.1)
average_rewards = []

# Run UCB for each value of c and compute the average reward over multiple trials
for c in c_values:
    rewards = []
    for _ in range(num_trials):
        total_reward = ucb(c, N, theta)
        rewards.append(total_reward)
    average_rewards.append(np.mean(rewards))

# Identify the best c
best_c_index = np.argmax(average_rewards)
best_c = c_values[best_c_index]
best_average_reward = average_rewards[best_c_index]

print(f"Best c value: {best_c:.2f}")
print(f"Maximum average total reward: {best_average_reward:.2f}")

# Plot the results
plt.plot(c_values, average_rewards, marker='o', linestyle='-', color='b')
plt.axvline(x=best_c, color='r', linestyle='--', label=f'Best c = {best_c:.2f}')
plt.xlabel('c (Confidence Level Parameter)')
plt.ylabel('Average Total Reward')
plt.title('UCB Algorithm Performance vs c Parameter')
plt.legend()
plt.grid(True)
plt.show()
```

```
Best c value: 0.40
Maximum average total reward: 3483.90
```



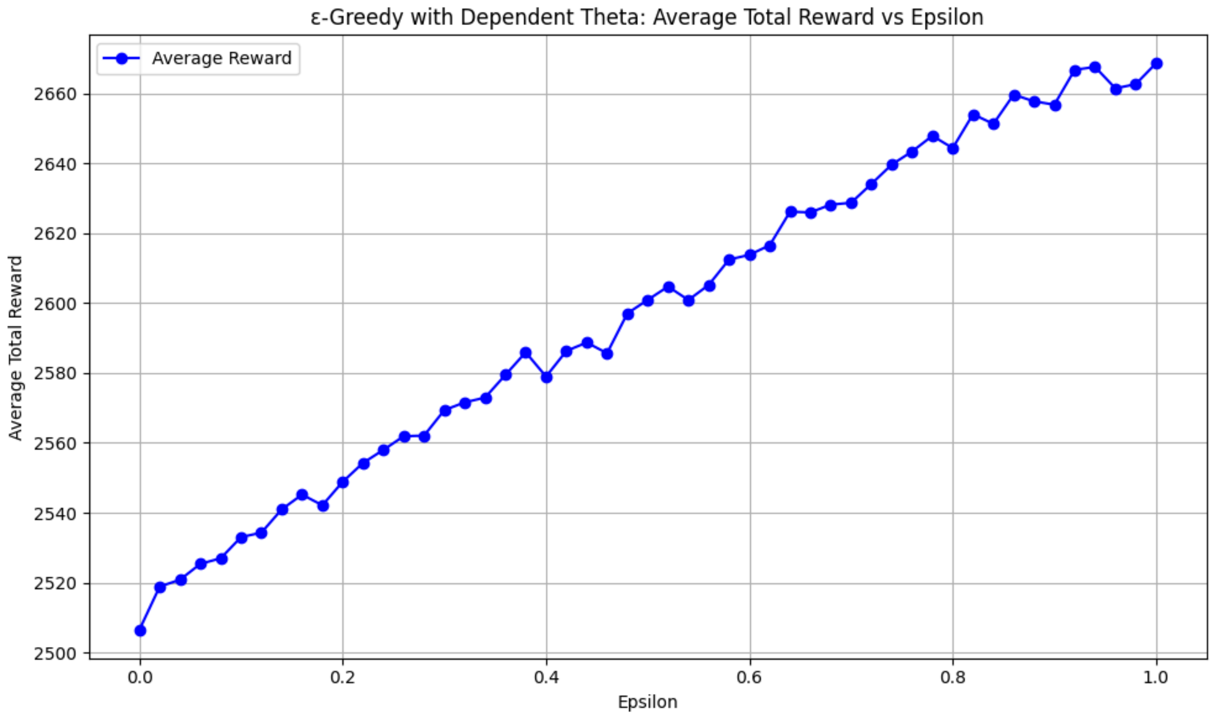
Problem 6

In [2]:
```python
import numpy as np
import random
import matplotlib.pyplot as plt
from itertools import product
import seaborn as sns
import pandas as pd

num_arms = 3
```

In [2]:
```python
# Initialize global variables for counts and estimated thetas
count = [0, 0, 0]  # Corresponds to Arm 1, Arm 2, Arm 3
theta = [0.0, 0.0, 0.0]  # Estimated thetas for Arm 1, Arm 2, Arm 3

def init_greedy():
    """
    Initializes the counts and estimated thetas for the greedy algorithm.
    """
    global count, theta
    count = [0, 0, 0]  # Reset counts for Arms 1, 2, 3
    theta = [0.0, 0.0, 0.0]  # Reset estimated thetas

def greedy_dependence(n, epsilon, initial_theta_oracled, p):
```

```
    """
    Greedy algorithm with dependency in theta_oracled.

    Parameters:
    - n: Number of time steps
    - epsilon: Exploration rate
    - initial_theta_oracled: Initial probabilities for each arm [θ1, θ2, θ3]
    - p: Probability adjustment parameter
    """
    global count, theta
    init_greedy()  # Initialize counts and estimates
    total_reward = 0  # Total actual rewards obtained

    # Deep copy to avoid modifying the original initial_theta_oracled
    current_theta = initial_theta_oracled.copy()

    for t in range(n):
        prob = random.random()  # Generate a random number in [0,1)

        if prob < epsilon:
            # Explore: choose a random arm from {0,1,2} corresponding to Arm 1, 2,
            arm = random.randint(0, 2)
        else:
            # Exploit: choose the arm with the highest estimated theta
            arm = np.argmax(theta)
            if theta[arm] == 0:
                # If all estimated thetas are 0, choose a random arm
                arm = random.randint(0, 2)

        # Simulate pulling the chosen arm: reward is 1 with probability current_the
        r_i = np.random.binomial(1, current_theta[arm])

        # Accumulate the actual reward
        total_reward += r_i

        # Update counts and estimated thetas using incremental averaging
        count[arm] += 1
        theta[arm] += (r_i - theta[arm]) / count[arm]

        # Update theta_oracled based on the outcome
        if r_i == 1:
            # If reward obtained, decrease theta of pulled arm and increase others
            current_theta[arm] = max(current_theta[arm] - p, 0.0)
            for other_arm in range(3):
                if other_arm != arm:
                    current_theta[other_arm] = min(current_theta[other_arm] + p / 2
        else:
            # If no reward, increase theta of pulled arm and decrease others
            current_theta[arm] = min(current_theta[arm] + p, 1.0)
            for other_arm in range(3):
                if other_arm != arm:
                    current_theta[other_arm] = max(current_theta[other_arm] - p / 2

    return total_reward

# Define the initial true reward probabilities (unknown to the algorithm)
```

```python
initial_theta_oracled = [0.7, 0.5, 0.4]  # ϑ1=0.7, ϑ2=0.5, ϑ3=0.4

# Experiment Parameters
epsilon_values = np.arange(0, 1.02, 0.02)  # Epsilon from 0 to 1 in steps of 0.02
repeat_time = 100  # Number of trials for each epsilon
N = 5000  # Number of time steps per trial
p = 0.005  # Probability adjustment parameter

rewards = np.zeros(len(epsilon_values))  # Average rewards for each epsilon

# Run experiments for each epsilon
for i, eps in enumerate(epsilon_values):
    for trial in range(repeat_time):
        # For each trial, reset the initial theta_oracled
        theta_oracled = initial_theta_oracled.copy()
        reward = greedy_dependence(N, eps, theta_oracled, p)
        rewards[i] += reward / repeat_time

# Plot the results
plt.figure(figsize=(10, 6))

# Plot Average Total Reward vs. Epsilon
plt.plot(epsilon_values, rewards, marker='o', linestyle='-', color='blue', label='A
plt.scatter(epsilon_values, rewards, color='red', s=10)
plt.xlabel('Epsilon')
plt.ylabel('Average Total Reward')
plt.title('ε-Greedy with Dependent Theta: Average Total Reward vs Epsilon')
plt.grid(True)
plt.legend()

plt.tight_layout()
plt.show()

# Identify and print the best epsilon based on rewards
best_index = np.argmax(rewards)
best_epsilon = epsilon_values[best_index]
best_reward = rewards[best_index]
print(f"Best epsilon: {best_epsilon:.2f}")
print(f"Maximum average total reward: {best_reward:.2f}")
```



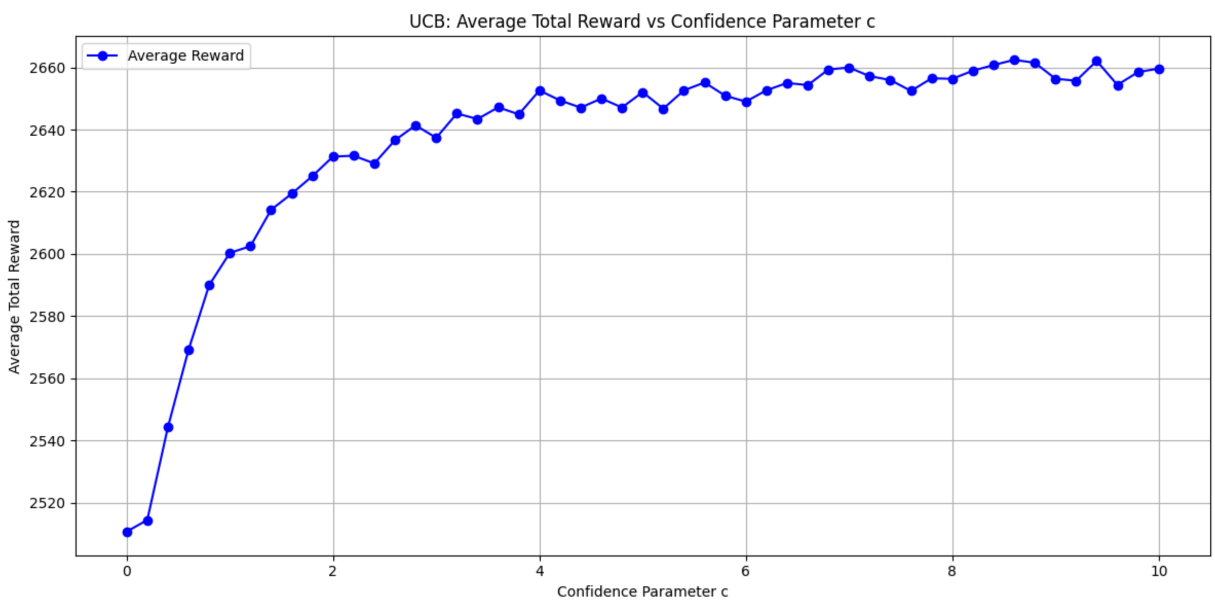

```
Best epsilon: 1.00
Maximum average total reward: 2668.57
```

In [3]:

```
def ucb_dependence(c, N, initial_theta_oracled, p=0.005):
    """
    UCB algorithm with independent arms.

    Parameters:
    - c: Confidence parameter for UCB
    - N: Number of time steps
    - initial_theta_oracled: List of initial true reward probabilities for each arm
    - p: Probability adjustment parameter
    """
    num_arms = 3
    Q = np.zeros(num_arms)          # Estimated rewards for each arm
    counts = np.zeros(num_arms)     # Number of times each arm has been pulled
    total_reward = 0                 # Total accumulated reward

    # Deep copy to avoid modifying the original initial_theta_oracled
    theta_oracled = initial_theta_oracled.copy()

    # Initialize by pulling each arm once
    for arm in range(num_arms):
        reward = 1 if random.random() < theta_oracled[arm] else 0
        Q[arm] = reward
        counts[arm] = 1
        total_reward += reward

        # Update theta_oracled based on the outcome
        if reward == 1:
            # If reward obtained, decrease the probability of the pulled arm and in
            theta_oracled[arm] = max(theta_oracled[arm] - p, 0.0)
            for other_arm in range(num_arms):
                if other_arm != arm:
```

```
                theta_oracled[other_arm] = min(theta_oracled[other_arm] + p / 2
        else:
            # If no reward, increase the probability of the pulled arm and decrease
            theta_oracled[arm] = min(theta_oracled[arm] + p, 1.0)
            for other_arm in range(num_arms):
                if other_arm != arm:
                    theta_oracled[other_arm] = max(theta_oracled[other_arm] - p / 2

    # Run UCB algorithm for remaining time steps
    for t in range(num_arms, N):
        # Compute UCB values for each arm
        ucb_values = Q + c * np.sqrt((2 * np.log(t + 1)) / counts)
        arm = np.argmax(ucb_values)

        # Pull the selected arm and observe the reward
        reward = 1 if random.random() < theta_oracled[arm] else 0
        total_reward += reward

        # Update counts and estimated rewards
        counts[arm] += 1
        Q[arm] += (reward - Q[arm]) / counts[arm]

        # Update theta_oracled based on the outcome
        if reward == 1:
            # If reward obtained, decrease the probability of the pulled arm and in
            theta_oracled[arm] = max(theta_oracled[arm] - p, 0.0)
            for other_arm in range(num_arms):
                if other_arm != arm:
                    theta_oracled[other_arm] = min(theta_oracled[other_arm] + p / 2
        else:
            # If no reward, increase the probability of the pulled arm and decrease
            theta_oracled[arm] = min(theta_oracled[arm] + p, 1.0)
            for other_arm in range(num_arms):
                if other_arm != arm:
                    theta_oracled[other_arm] = max(theta_oracled[other_arm] - p / 2

    return total_reward

# Define the initial true reward probabilities (unknown to the algorithm)
initial_theta_oracled = [0.7, 0.5, 0.4]  # [ϑ1, ϑ2, ϑ3]

# Experiment Parameters
c_values = np.arange(0.0, 10.2, 0.2)  # Confidence parameter c from 0 to 10 in step
repeat_time = 100  # Number of trials for each c
N = 5000  # Number of time steps per trial
p = 0.005  # Probability adjustment parameter

average_rewards = np.zeros(len(c_values))  # Average rewards for each c

# Run experiments for each c
for i, c in enumerate(c_values):
    for trial in range(repeat_time):
        # For each trial, reset the initial_theta_oracled
        theta_oracled = initial_theta_oracled.copy()
        reward = ucb_dependence(c, N, theta_oracled, p)
        average_rewards[i] += reward / repeat_time
```

```
# Plot the results
plt.figure(figsize=(12, 6))

# Plot Average Total Reward vs. Confidence Parameter c
plt.plot(c_values, average_rewards, marker='o', linestyle='-', color='blue', label=
plt.scatter(c_values, average_rewards, color='red', s=10)
plt.xlabel('Confidence Parameter c')
plt.ylabel('Average Total Reward')
plt.title('UCB: Average Total Reward vs Confidence Parameter c')
plt.grid(True)
plt.legend()

plt.tight_layout()
plt.show()

# Identify and print the best c based on rewards
best_index = np.argmax(average_rewards)
best_c = c_values[best_index]
best_reward = average_rewards[best_index]
print(f"Best c: {best_c:.2f}")
print(f"Maximum average total reward: {best_reward:.2f}")
```




```
Best c: 8.60
Maximum average total reward: 2662.44
```

In [3]:

```
def thompson_sampling_dependence(N, theta_oracled, alpha_init, beta_init, p=0.005):
    num_arms = 3
    alpha = alpha_init.copy()
    beta = beta_init.copy()
    total_reward = 0.0

    # Deep copy to avoid modifying the original theta_oracled
    theta_current = theta_oracled.copy()

    for t in range(N):
        # Sample theta from Beta distributions for each arm
        sampled_thetas = [np.random.beta(alpha[j], beta[j]) for j in range(num_arms
```

```
        # Select the arm with the highest sampled theta
        arm = np.argmax(sampled_thetas)

        # Simulate pulling the selected arm: reward is 1 with probability theta_cur
        reward = 1 if np.random.random() < theta_current[arm] else 0
        total_reward += reward

        # Update the Beta distribution parameters for the selected arm
        alpha[arm] += reward
        beta[arm] += (1 - reward)

        # Update theta_oracled based on the outcome
        if reward == 1:
            # If reward obtained, decrease theta of pulled arm and increase others
            theta_current[arm] = max(theta_current[arm] - p, 0.0)
            for other_arm in range(num_arms):
                if other_arm != arm:
                    theta_current[other_arm] = min(theta_current[other_arm] + p / 2
        else:
            # If no reward, increase theta of pulled arm and decrease others
            theta_current[arm] = min(theta_current[arm] + p, 1.0)
            for other_arm in range(num_arms):
                if other_arm != arm:
                    theta_current[other_arm] = max(theta_current[other_arm] - p / 2

    return total_reward
```

In [4]:
```
def dependency_aware_thompson_sampling(N, theta, alpha_init, beta_init, p=0.005, ep
    alpha = alpha_init.copy()
    beta  = beta_init.copy()
    K = len(alpha)
    theta_current = theta.copy()
    total_reward = 0

    for t in range(N):
        # Epsilon-greedy: with prob epsilon, pick a random arm
        if np.random.rand() < epsilon:
            chosen_arm = np.random.choice(K)
        else:
            # Otherwise, Thompson sample from each arm's Beta posterior
            samples = [np.random.beta(alpha[i], beta[i]) for i in range(K)]
            chosen_arm = np.argmax(samples)

        # Observe reward from environment
        reward = (np.random.rand() < theta_current[chosen_arm])
        total_reward += reward

        # --- Update Beta posterior for chosen arm and other arms ---
        if reward:
            # Chosen arm gets a standard Beta update
            alpha[chosen_arm] += 1

            # If gamma > 0, nudge alpha of the other arms
            for other_arm in range(K):
                if other_arm != chosen_arm:
```

```
                alpha[other_arm] += gamma
        else:
            # Chosen arm gets a standard Beta update
            beta[chosen_arm] += 1

            # If gamma > 0, nudge beta of the other arms
            for other_arm in range(K):
                if other_arm != chosen_arm:
                    beta[other_arm] += gamma

        # environment update
        if reward:
            theta_current[chosen_arm] = max(theta_current[chosen_arm] - p, 0.0)
            for other_arm in range(K):
                if other_arm != chosen_arm:
                    theta_current[other_arm] = min(theta_current[other_arm] + p/2,
        else:
            theta_current[chosen_arm] = min(theta_current[chosen_arm] + p, 1.0)
            for other_arm in range(K):
                if other_arm != chosen_arm:
                    theta_current[other_arm] = max(theta_current[other_arm] - p/2,

    return total_reward
```

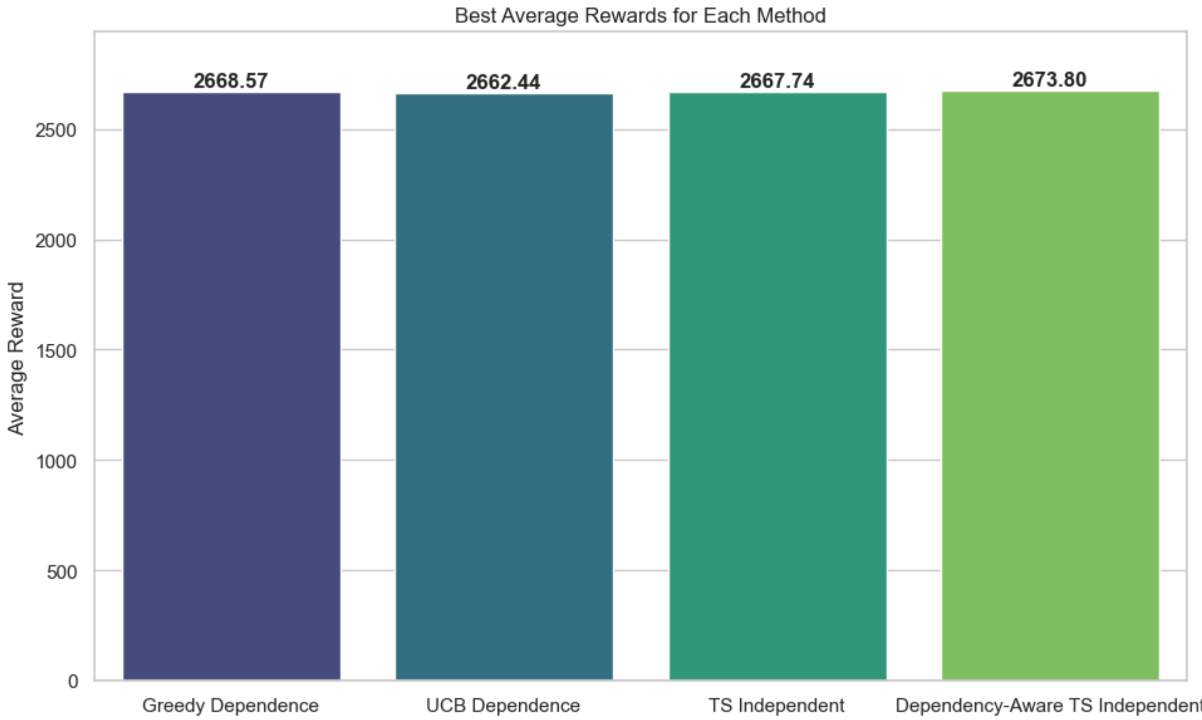
In [6]:
```
# Define the true reward probabilities (independent arms)
theta1_true = 0.5
theta2_true = 0.4
theta3_true = 0.7
theta = [theta3_true, theta1_true, theta2_true]  # [ϑ1, ϑ2, ϑ3] = [0.7, 0.5, 0.4]

# ---------- Parameter Ranges for Three Arms ----------
alpha1_values = [450, 600]
beta1_values  = [1, 16]
alpha2_values = [300, 450]
beta2_values  = [1, 16]
alpha3_values = [150, 300, 450]
beta3_values  = [16, 31]

N = 5000
repeat_time = 50

# ---------- Generate All Parameter Combinations for Three Arms ----------
parameter_combinations = list(product(
    alpha1_values, alpha2_values, alpha3_values,
    beta1_values, beta2_values, beta3_values
))

# ---------- Initialize Result Lists for Both Algorithms ----------
results_ts_independent = []  # For thompson_sampling_independent
results_da_ts_independent = []  # For dependency_aware_thompson_sampling (now indep

# ---------- Running Both Algorithms Across All Parameter Combinations ----------
for idx, (alpha1_val, alpha2_val, alpha3_val, beta1_val, beta2_val, beta3_val) in e
    alpha_init_ts = [alpha1_val, alpha2_val, alpha3_val]
    beta_init_ts = [beta1_val, beta2_val, beta3_val]
```

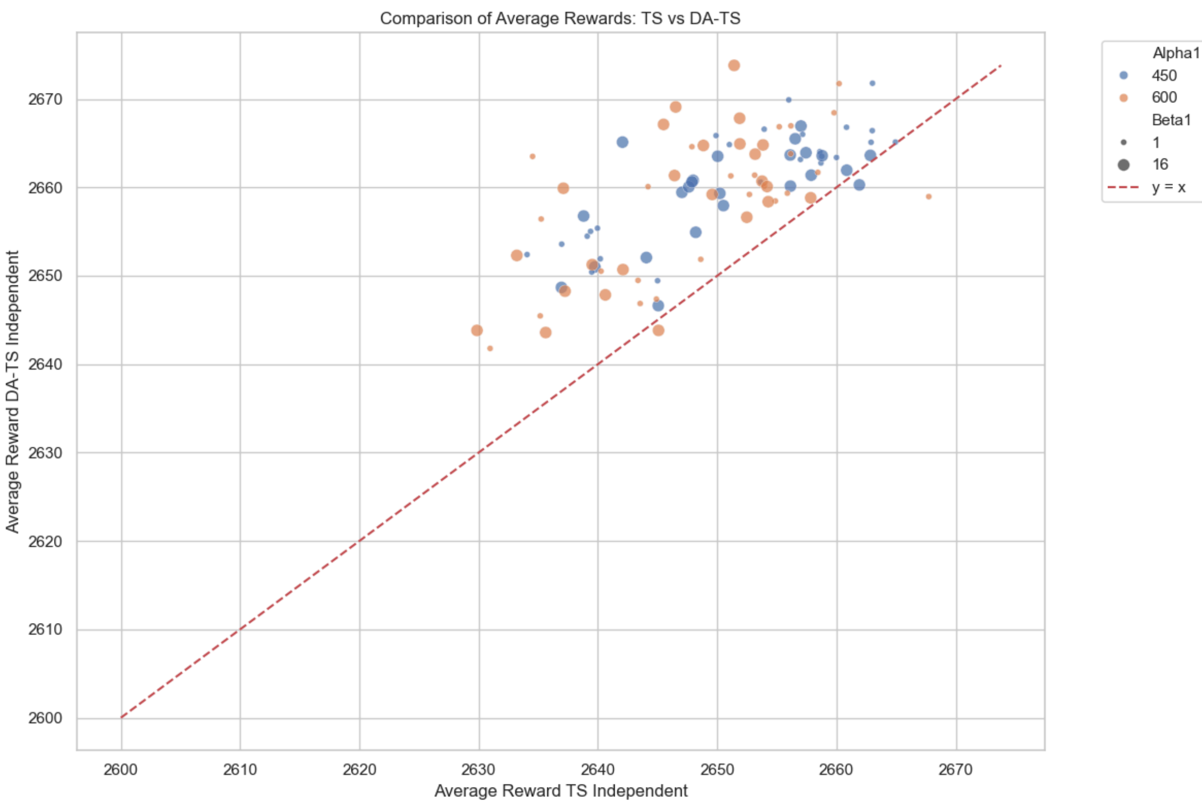
```
    alpha_init_da = [alpha1_val, alpha2_val, alpha3_val]
    beta_init_da = [beta1_val, beta2_val, beta3_val]

    # ---------- Run Trials for Thompson Sampling Independent ----------
    total_reward_ts = 0.0
    for _ in range(repeat_time):
        theta_oracled = [0.7, 0.5, 0.4]  # [ϑ1, ϑ2, ϑ3]
        reward = thompson_sampling_dependence(N, theta_oracled, alpha_init_ts, beta
        total_reward_ts += reward / repeat_time

    # ---------- Run Trials for Dependency-Aware Thompson Sampling ----------
    epsilon_values = [1e-2,7e-3, 5e-3, 3e-3, 1e-3]
    gamma_values = [1e-6, 1e-5, 1e-4, 1e-3, 1e-2]

    best_reward_da = -np.inf
    best_epsilon = None
    best_gamma = None

    for epsilon in epsilon_values:
        for gamma in gamma_values:
            total_reward_da = 0.0
            for _ in range(repeat_time):
                # Reset theta_oracled for each trial
                theta_oracled = [theta3_true, theta1_true, theta2_true]  # [ϑ1, ϑ2,
                # Run DA-TS with current epsilon and gamma
                reward = dependency_aware_thompson_sampling(
                    N, theta_oracled, alpha_init_da, beta_init_da,
                    p=0.005, epsilon=epsilon, gamma=gamma
                )
                total_reward_da += reward / repeat_time

            # Check if this (epsilon, gamma) pair yields a better reward
            if total_reward_da > best_reward_da:
                best_reward_da = total_reward_da
                best_epsilon = epsilon
                best_gamma = gamma

    # ---------- Store Results ----------
    results_ts_independent.append({
        'Alpha1': alpha1_val,
        'Alpha2': alpha2_val,
        'Alpha3': alpha3_val,
        'Beta1': beta1_val,
        'Beta2': beta2_val,
        'Beta3': beta3_val,
        'Avg Reward TS': total_reward_ts
    })
    results_da_ts_independent.append({
        'Alpha1': alpha1_val,
        'Alpha2': alpha2_val,
        'Alpha3': alpha3_val,
        'Beta1': beta1_val,
        'Beta2': beta2_val,
        'Beta3': beta3_val,
        'Best Epsilon DA-TS': best_epsilon,
        'Best Gamma DA-TS': best_gamma,
```

```
        'Avg Reward DA-TS': best_reward_da
    })

# ---------- Convert Results to DataFrames ----------
df_ts = pd.DataFrame(results_ts_independent)
df_da = pd.DataFrame(results_da_ts_independent)

# ---------- Merge DataFrames for Easier Comparison ----------
df_combined = pd.merge(df_ts, df_da, on=['Alpha1', 'Alpha2', 'Alpha3', 'Beta1', 'Be

# ---------- Find Best Outcomes for Each Method ----------

best_avg_ts = df_ts['Avg Reward TS'].max()
best_avg_da = df_da['Avg Reward DA-TS'].max()

# ---------- Enhanced Printing ----------

print("===== Best Average Rewards =====")
print(f"Thompson Sampling Independent: {best_avg_ts:.2f}")
print(f"Dependency-Aware Thompson Sampling Independent: {best_avg_da:.2f}")

# ---------- List Top 5 Parameter Combinations for Each Method ----------

print("\n===== Top 5 Parameter Combinations for Thompson Sampling Independent =====
top5_ts = df_ts.sort_values(by='Avg Reward TS', ascending=False).head(5)
print(top5_ts.to_string(index=False))

print("\n===== Top 5 Parameter Combinations for Dependency-Aware Thompson Sampling 
top5_da = df_da.sort_values(by='Avg Reward DA-TS', ascending=False).head(5)
print(top5_da.to_string(index=False))

# ---------- Plotting ----------

# Set the style for seaborn
sns.set(style="whitegrid")

# 1. Bar Plot of Best Average Rewards
greedy_dependence_avg = 2668.57
ucb_dependence_avg = 2662.44
plt.figure(figsize=(10, 6))
methods = ['Greedy Dependence', 'UCB Dependence', 'TS Independent', 'Dependency-Awa
avg_rewards = [greedy_dependence_avg, ucb_dependence_avg, best_avg_ts, best_avg_da]
# Fix deprecated palette usage in barplot
sns.barplot(x=methods,
            y=avg_rewards,
            hue=methods,  # Assign x to hue
            legend=False, # Hide redundant legend
            palette="viridis")
plt.ylabel('Average Reward')
plt.title('Best Average Rewards for Each Method')
plt.ylim(0, max(avg_rewards)*1.1)
for i, v in enumerate(avg_rewards):
    plt.text(i, v + max(avg_rewards)*0.01, f"{v:.2f}", ha='center', fontweight='bol
plt.tight_layout()
plt.show()
```

```python
# 2. Scatter Plot Comparing Both Methods with Reference Line
plt.figure(figsize=(12, 8))
scatter = sns.scatterplot(
    data=df_combined,
    x='Avg Reward TS',
    y='Avg Reward DA-TS',
    hue='Alpha1',
    size='Beta1',
    palette='deep',
    alpha=0.7
)
# Add reference line y = x
max_val = max(df_combined['Avg Reward TS'].max(), df_combined['Avg Reward DA-TS'].m
plt.plot([2600, max_val], [2600, max_val], 'r--', label='y = x')
plt.xlabel('Average Reward TS Independent')
plt.ylabel('Average Reward DA-TS Independent')
plt.title('Comparison of Average Rewards: TS vs DA-TS')
plt.legend(bbox_to_anchor=(1.05, 1), loc='upper left')
plt.tight_layout()
plt.show()

# 3. Optional: Save Plots
plt.savefig('pics/best_average_rewards.png')
plt.savefig('pics/comparison_scatter.png')
```

```
===== Best Average Rewards =====
Thompson Sampling Independent: 2667.74
Dependency-Aware Thompson Sampling Independent: 2673.80

===== Top 5 Parameter Combinations for Thompson Sampling Independent =====
 Alpha1  Alpha2  Alpha3  Beta1  Beta2  Beta3  Avg Reward TS
    600     300     450      1      1     31        2667.74
    450     450     300      1     16     16        2664.94
    450     450     450      1      1     31        2663.02
    450     450     450      1     16     31        2663.00
    450     300     450      1      1     31        2662.90

===== Top 5 Parameter Combinations for Dependency-Aware Thompson Sampling Independen
t =====
 Alpha1  Alpha2  Alpha3  Beta1  Beta2  Beta3  Best Epsilon DA-TS  Best Gamma DA-TS
Avg Reward DA-TS
    600     450     450     16     16     31               0.001          0.000001
2673.80
    450     450     450      1      1     31               0.007          0.000001
2671.78
    600     450     450      1      1     31               0.007          0.000001
2671.74
    450     300     300      1      1     31               0.010          0.010000
2669.90
    600     450     300     16      1     16               0.007          0.000100
2669.10
```






```
<Figure size 640x480 with 0 Axes>
```

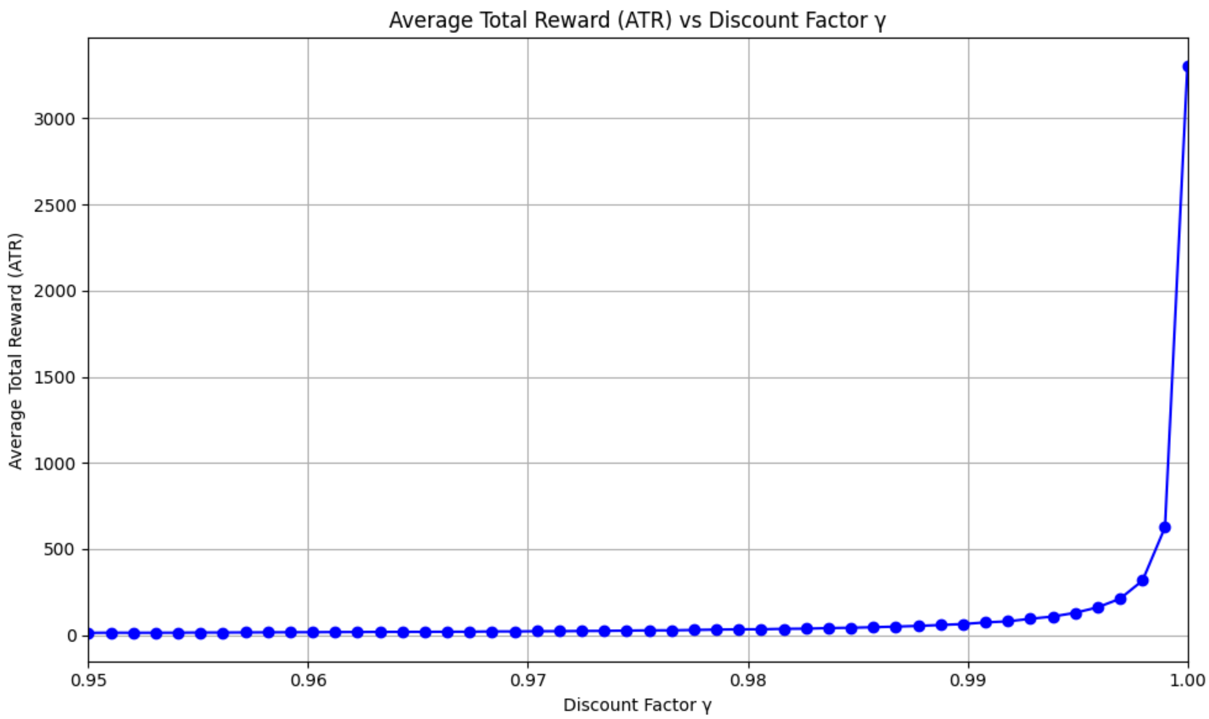
Part II

Problem 1: One intuitive policy suggests that in each time slot we should pull the arm for which the current expected value of θ_i is the largest. This policy behaves very good in most cases. Please design simulations to check the behavior of this policy

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import beta

np.random.seed(42)
```

In [9]:
```
# Simulation parameters
true_theta = [0.7, 0.5]        # True success probabilities for arm 1 and arm 2
alpha_prior = [1, 1]           # Prior alpha parameters for Beta distributions
beta_prior = [1, 1]            # Prior beta parameters for Beta distributions
gamma_values = np.linspace(0.95, 1.0, 50)  # Gamma values from 0.5 to 1.0 in increm
time_steps = 5000              # Number of pulls per trial
repeat_time = 50               # Number of trials per gamma

# Initialize array to store average total rewards for each gamma
average_total_rewards = []

# Iterate over each gamma value
for gamma in gamma_values:
    total_rewards = []  # To store total rewards for each trial

    # Repeat the trial 'repeat_time' times for averaging
    for trial in range(repeat_time):
        # Initialize Beta parameters for each arm
        alpha = alpha_prior.copy()
        beta_params = beta_prior.copy()

        cumulative_reward = 0  # Total reward for this trial

        for t in range(1, time_steps + 1):
            # Calculate expected theta for each arm
            expected_theta = [alpha[i] / (alpha[i] + beta_params[i]) for i in range

            # Select the arm with the highest expected theta
            chosen_arm = np.argmax(expected_theta)

            # Simulate a pull: success with probability true_theta[chosen_arm]
            success = np.random.rand() < true_theta[chosen_arm]

            # Update Beta posterior
            if success:
                alpha[chosen_arm] += 1
                reward = gamma**(t-1)
            else:
                beta_params[chosen_arm] += 1
                reward = 0

            # Update cumulative reward
            cumulative_reward += reward

        total_rewards.append(cumulative_reward)

    # Calculate average total reward for this gamma
    avg_reward = np.mean(total_rewards)
    average_total_rewards.append(avg_reward)
```

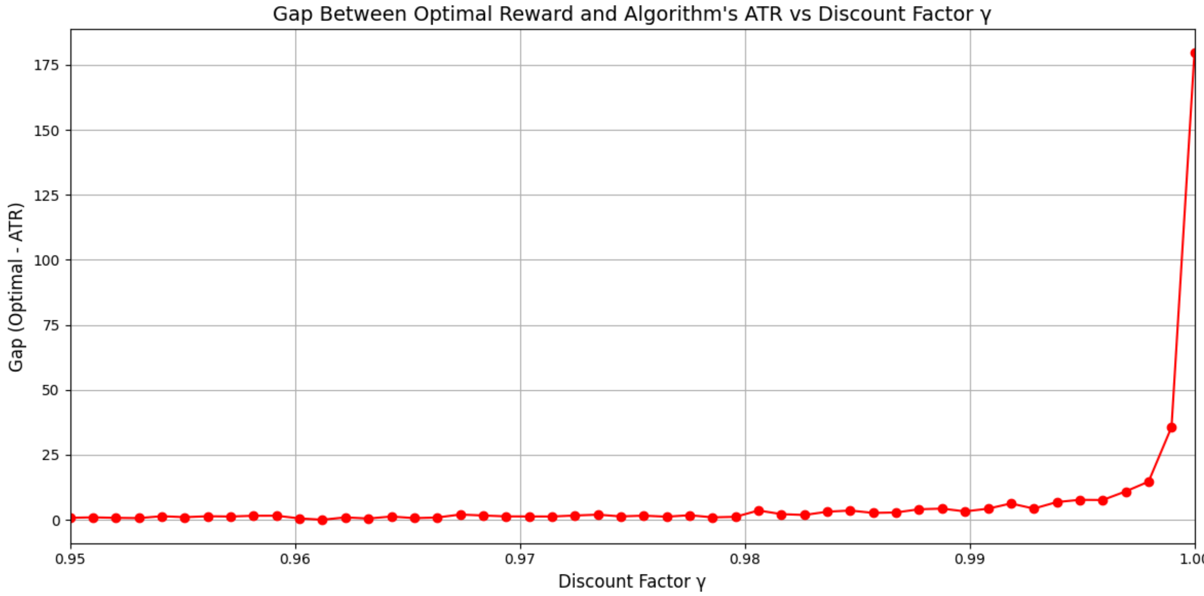
```
# Visualization: ATR vs Gamma
plt.figure(figsize=(10, 6))
plt.plot(gamma_values, average_total_rewards, marker='o', linestyle='-', color='blu
plt.title('Average Total Reward (ATR) vs Discount Factor γ')
plt.xlabel('Discount Factor γ')
plt.ylabel('Average Total Reward (ATR)')
plt.grid(True)
plt.xlim(0.95, 1.0)
plt.tight_layout()
plt.show()
```



In [10]:

```
# Identify the best arm (with the highest true_theta)
best_arm = np.argmax(true_theta)
theta_best = true_theta[best_arm]

# Initialize lists to store results
gamma_list = []
gap_list = []

# Iterate over each gamma value
for gamma in gamma_values:
    total_rewards = []  # To store total rewards for each trial

    # Repeat the trial 'repeat_time' times for averaging
    for trial in range(repeat_time):
        # Initialize Beta parameters for each arm
        alpha = alpha_prior.copy()
        beta_params = beta_prior.copy()

        cumulative_reward = 0.0  # Total reward for this trial

        for t in range(1, time_steps + 1):
            # Calculate expected theta for each arm using current Beta posterior
```

```
            expected_theta = [alpha[i] / (alpha[i] + beta_params[i]) for i in range

            # Select the arm with the highest expected theta
            chosen_arm = np.argmax(expected_theta)

            # Simulate a pull: success with probability true_theta[chosen_arm]
            success = np.random.rand() < true_theta[chosen_arm]

            # Update Beta posterior based on the outcome
            if success:
                alpha[chosen_arm] += 1
                reward = gamma**(t-1)
            else:
                beta_params[chosen_arm] += 1
                reward = 0.0

            # Accumulate the reward
            cumulative_reward += reward

        total_rewards.append(cumulative_reward)

    # Calculate average total reward for this gamma
    avg_reward = np.mean(total_rewards)

    # Compute theoretical optimal reward
    if gamma < 1.0:
        # Geometric series sum: theta_best * (1 - gamma^time_steps) / (1 - gamma)
        optimal_reward = theta_best * (1 - gamma**time_steps) / (1 - gamma)
    else:
        # Handle the case when gamma = 1.0
        optimal_reward = theta_best * time_steps

    # Compute the gap between optimal reward and algorithm's average reward
    gap = optimal_reward - avg_reward

    # Store the results
    gamma_list.append(gamma)
    gap_list.append(gap)

# Convert lists to numpy arrays for easier handling
gamma_array = np.array(gamma_list)
gap_array = np.array(gap_list)

# Visualization: Gap vs Gamma
plt.figure(figsize=(12, 6))
plt.plot(gamma_array, gap_array, marker='o', linestyle='-', color='red')
plt.title('Gap Between Optimal Reward and Algorithm\'s ATR vs Discount Factor γ', f
plt.xlabel('Discount Factor γ', fontsize=12)
plt.ylabel('Gap (Optimal - ATR)', fontsize=12)
plt.grid(True)
plt.xlim(0.95, 1.0)
plt.tight_layout()
plt.show()
```

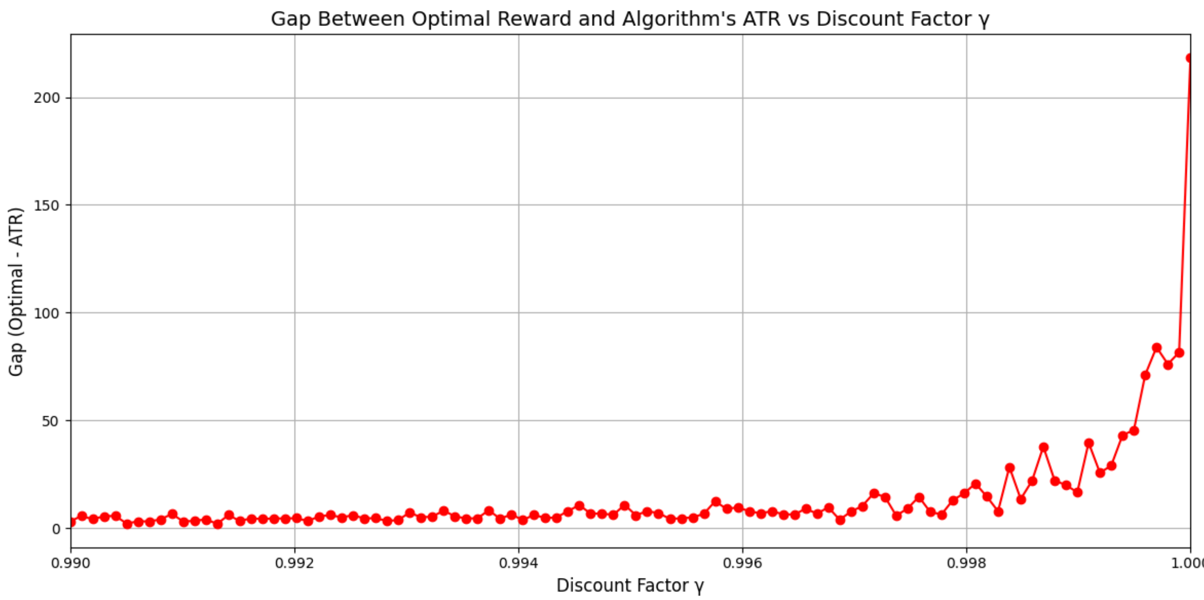


Problem 2

However, such intuitive policy is unfortunately not optimal. Please provide an example to show why such policy is not optimal.

In [2]:
```
# Simulation parameters
true_theta = [0.3, 0.6]        # True success probabilities for arm 1 and arm 2
alpha_prior = [1, 1]           # Prior alpha parameters for Beta distributions
beta_prior = [1, 1]            # Prior beta parameters for Beta distributions
gamma_values = np.linspace(0.99, 1.00, 100)  # Gamma values
time_steps = 5000              # Number of pulls per trial
repeat_time = 50               # Number of trials per gamma

# Identify the best arm (with the highest true_theta)
best_arm = np.argmax(true_theta)
theta_best = true_theta[best_arm]

# Initialize lists to store results
gamma_list = []
gap_list = []

# Iterate over each gamma value
for gamma in gamma_values:
    total_rewards = []  # To store total rewards for each trial

    # Repeat the trial 'repeat_time' times for averaging
    for trial in range(repeat_time):
        # Initialize Beta parameters for each arm
        alpha = alpha_prior.copy()
        beta_params = beta_prior.copy()

        cumulative_reward = 0.0  # Total reward for this trial

        for t in range(1, time_steps + 1):
            # Calculate expected theta for each arm using current Beta posterior
            expected_theta = [alpha[i] / (alpha[i] + beta_params[i]) for i in range
```

```
            # Select the arm with the highest expected theta
            chosen_arm = np.argmax(expected_theta)

            # Simulate a pull: success with probability true_theta[chosen_arm]
            success = np.random.rand() < true_theta[chosen_arm]

            # Update Beta posterior based on the outcome
            if success:
                alpha[chosen_arm] += 1
                reward = gamma**(t-1)
            else:
                beta_params[chosen_arm] += 1
                reward = 0.0

            # Accumulate the reward
            cumulative_reward += reward

        total_rewards.append(cumulative_reward)

    # Calculate average total reward for this gamma
    avg_reward = np.mean(total_rewards)

    # Compute theoretical optimal reward
    if gamma < 1.0:
        # Geometric series sum: theta_best * (1 - gamma^time_steps) / (1 - gamma)
        optimal_reward = theta_best * (1 - gamma**time_steps) / (1 - gamma)
    else:
        # Handle the case when gamma = 1.0
        optimal_reward = theta_best * time_steps

    # Compute the gap between optimal reward and algorithm's average reward
    gap = optimal_reward - avg_reward

    # Store the results
    gamma_list.append(gamma)
    gap_list.append(gap)

# Convert lists to numpy arrays for easier handling
gamma_array = np.array(gamma_list)
gap_array = np.array(gap_list)

# Visualization: Gap vs Gamma
plt.figure(figsize=(12, 6))
plt.plot(gamma_array, gap_array, marker='o', linestyle='-', color='red')
plt.title('Gap Between Optimal Reward and Algorithm\'s ATR vs Discount Factor γ', f
plt.xlabel('Discount Factor γ', fontsize=12)
plt.ylabel('Gap (Optimal - ATR)', fontsize=12)
plt.grid(True)
plt.xlim(0.99, 1.0)
plt.tight_layout()
plt.show()
```



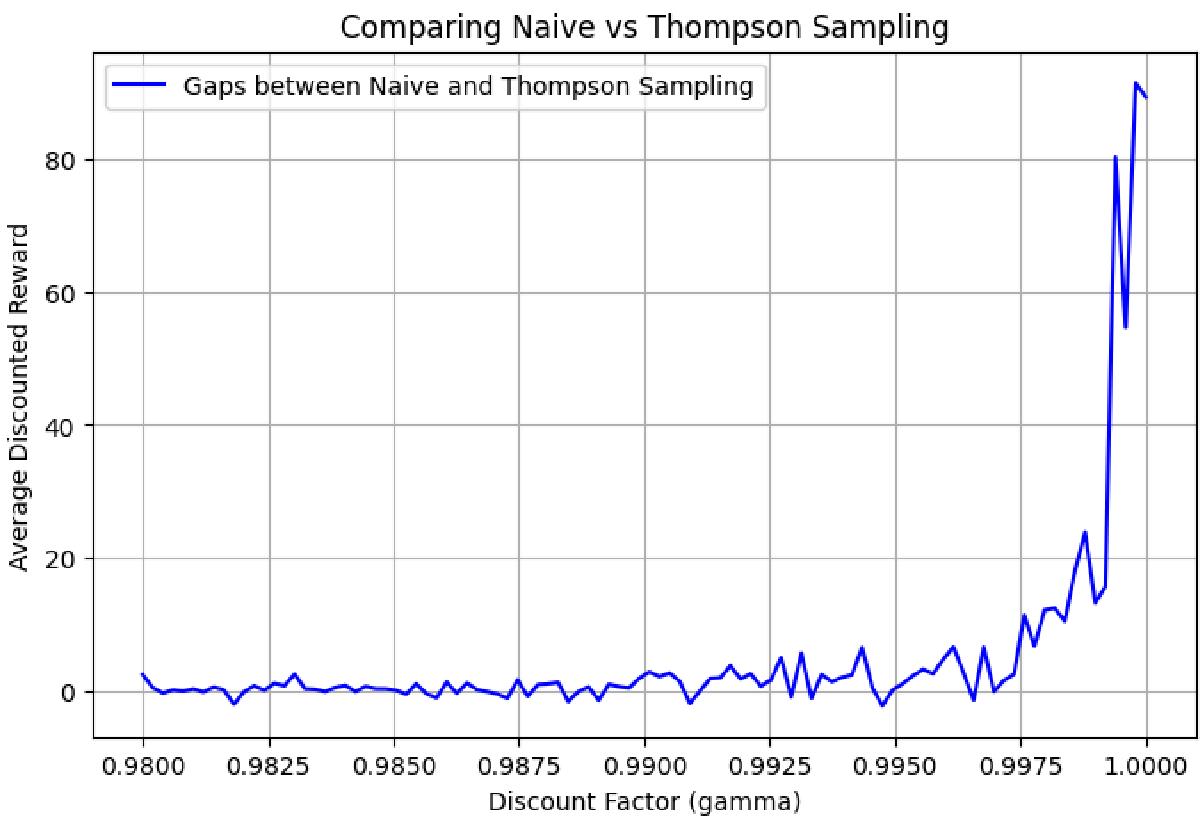
Compared with TS

In [6]:

```
# Simulation parameters
true_theta = [0.3, 0.6]         # True success probabilities for arm 1 and arm 2
alpha_prior = [1, 1]            # Prior alpha parameters for Beta distributions
beta_prior = [1, 1]             # Prior beta parameters for Beta distributions
gamma_values = np.linspace(0.98, 1.00, 100)  # Gamma values
time_steps = 5000               # Number of pulls per trial
repeat_time = 50                # Number of trials per gamma

# -------------------------------------------------------------
# Helper function: sample Bernoulli reward from an arm
# -------------------------------------------------------------
def draw_reward(arm_idx):
    """Simulate pulling arm_idx and return reward (1 or 0)."""
    return 1 if (np.random.rand() < true_theta[arm_idx]) else 0
# -------------------------------------------------------------
# Naive Strategy: Always pick arm with highest posterior mean
# -------------------------------------------------------------
def run_naive_strategy(gamma, alpha0, beta0):
    """
    Runs the naive strategy for 'time_steps' pulls with discount factor gamma.
    alpha0, beta0 are the prior parameters for each arm (list of length 2).
    Returns the total discounted reward.
    """
    # Initialize alpha, beta
    alpha = np.array(alpha0, dtype=float)
    beta  = np.array(beta0,  dtype=float)

    total_discounted_reward = 0.0
    discount_power = 0      # exponent for gamma^(t-1)

    for t in range(1, time_steps + 1):
        # Posterior means for each arm
        posterior_means = alpha / (alpha + beta)
```

```
        # Choose the arm with the highest posterior mean
        chosen_arm = np.argmax(posterior_means)

        # Draw a Bernoulli reward
        reward = draw_reward(chosen_arm)

        # Update posterior
        alpha[chosen_arm] += reward
        beta[chosen_arm]  += (1 - reward)

        # Accumulate discounted reward
        total_discounted_reward += (gamma ** discount_power) * reward
        discount_power += 1

    return total_discounted_reward
# --------------------------------------------------------------
# Thompson Sampling Strategy
# --------------------------------------------------------------
def thompson_sampling(gamma, alpha0, beta0):
    """
    Runs Thompson Sampling for 'time_steps' pulls with discount factor gamma.
    alpha0, beta0 are the prior parameters for each arm.
    Returns the total discounted reward.
    """
    alpha = np.array(alpha0, dtype=float)
    beta  = np.array(beta0,  dtype=float)

    total_discounted_reward = 0.0
    discount_power = 0

    for t in range(1, time_steps + 1):
        # Sample theta-hat from current posterior for each arm
        sampled_thetas = np.random.beta(alpha, beta)

        # Choose the arm that maximizes the sampled theta
        chosen_arm = np.argmax(sampled_thetas)

        # Draw reward
        reward = draw_reward(chosen_arm)

        # Update posterior
        alpha[chosen_arm] += reward
        beta[chosen_arm]  += (1 - reward)

        # Accumulate discounted reward
        total_discounted_reward += (gamma ** discount_power) * reward
        discount_power += 1

    return total_discounted_reward

# --------------------------------------------------------------
# Main Experiment Loop
# --------------------------------------------------------------
gap_means = []

for gamma in gamma_values:
```

```
    gap_results = []

    for _ in range(repeat_time):
        # Run Naive
        naive_reward = run_naive_strategy(
            gamma,
            alpha_prior,
            beta_prior
        )


        # Run Thompson Sampling
        ts_reward = thompson_sampling(
            gamma,
            alpha_prior,
            beta_prior
        )
        gap_results.append(ts_reward - naive_reward)

    gap_means.append(np.mean(gap_results))

# ---------------------------------------------------------------
# Plotting Results
# ---------------------------------------------------------------
plt.figure(figsize=(8, 5))
plt.plot(gamma_values, gap_means, 'b-', label='Gaps between Naive and Thompson Samp
plt.xlabel('Discount Factor (gamma)')
plt.ylabel('Average Discounted Reward')
plt.title('Comparing Naive vs Thompson Sampling')
plt.legend()
plt.grid(True)
plt.show()
```



Problem 5

Find the optimal policy (approximately).

In [2]:
```python
import numpy as np
import matplotlib.pyplot as plt

# -------------------------------------------------------------
# Dynamic Programming Function
# -------------------------------------------------------------
def solve_2armed_bandit_dp(M=10, gamma=0.95, tol=1e-8, max_iter=50):
    """
    Solve the 2-armed Beta-Bernoulli bandit using 4D dynamic programming.

    Arguments:
    ----------
    M : int
        Truncation level for alpha_i, beta_i.
    gamma : float
        Discount factor in (0,1).
    tol : float
        Convergence tolerance for the value iteration.
    max_iter : int
        Maximum number of iterations to run.

    Returns:
    --------
    R : 4D numpy array, shape (M+1, M+1, M+1, M+1)
```

```
        The approximate value function.
    policy : 4D numpy array of 0 or 1
        Optimal action: 0 for arm1, 1 for arm2.
    """
    # Initialize value function and policy arrays
    R = np.zeros((M+1, M+1, M+1, M+1), dtype=np.float64)
    policy = np.zeros((M+1, M+1, M+1, M+1), dtype=int)

    def clamp(x):
        return min(x, M)

    for it in range(max_iter):
        delta = 0.0  # Maximum change in this iteration

        # Iterate over all possible states
        for alpha1 in range(1, M+1):
            for beta1 in range(1, M+1):
                for alpha2 in range(1, M+1):
                    for beta2 in range(1, M+1):
                        # Compute expected reward for choosing arm 1
                        p1 = alpha1 / (alpha1 + beta1)
                        R_success_1 = R[clamp(alpha1 + 1), beta1, alpha2, beta2]
                        R_fail_1 = R[alpha1, clamp(beta1 + 1), alpha2, beta2]
                        R1 = p1 * (1.0 + gamma * R_success_1) + (1.0 - p1) * (gamma

                        # Compute expected reward for choosing arm 2
                        p2 = alpha2 / (alpha2 + beta2)
                        R_success_2 = R[alpha1, beta1, clamp(alpha2 + 1), beta2]
                        R_fail_2 = R[alpha1, beta1, alpha2, clamp(beta2 + 1)]
                        R2 = p2 * (1.0 + gamma * R_success_2) + (1.0 - p2) * (gamma

                        # Choose the action with the higher expected reward
                        new_val = max(R1, R2)

                        # Update the value function
                        old_val = R[alpha1, beta1, alpha2, beta2]
                        diff = abs(new_val - old_val)
                        if diff > delta:
                            delta = diff
                        R[alpha1, beta1, alpha2, beta2] = new_val

                        # Update the policy
                        if R1 > R2:
                            policy[alpha1, beta1, alpha2, beta2] = 0  # Choose arm1
                        else:
                            policy[alpha1, beta1, alpha2, beta2] = 1  # Choose arm2

        if delta < tol:
            break

    return R, policy

# -------------------------------------------------------------
# Simulation Parameters
# -------------------------------------------------------------
true_theta = [0.3, 0.6]        # True success probabilities for arm 1 and arm 2
```

```
alpha_prior = [1, 1]           # Prior alpha parameters for Beta distributions
beta_prior = [1, 1]            # Prior beta parameters for Beta distributions
gamma_values = np.linspace(0.9, 1, 100)  # Gamma values
time_steps = 5000              # Number of pulls per trial
repeat_time = 10               # Number of trials per gamma
M = 17                         # Truncation level for DP

# -----------------------------------------------------------------
# Simulation Function for DP-Based Policy
# -----------------------------------------------------------------
def simulate_trial(policy, M, true_theta, alpha_prior, beta_prior, gamma, time_step
    """
    Simulate a single trial of the bandit problem using the provided policy.

    Arguments:
    ----------
    policy : 4D numpy array
        Optimal policy derived from DP.
    M : int
        Truncation level.
    true_theta : list of float
        True success probabilities for each arm.
    alpha_prior : list of int
        Prior alpha parameters for Beta distributions.
    beta_prior : list of int
        Prior beta parameters for Beta distributions.
    gamma : float
        Discount factor.
    time_steps : int
        Number of pulls in the trial.

    Returns:
    --------
    total_reward : float
        Total discounted reward accumulated in the trial.
    """
    # Initialize Beta parameters
    alpha = [alpha_prior[0], alpha_prior[1]]
    beta = [beta_prior[0], beta_prior[1]]
    total_reward = 0.0
    current_gamma_power = 1.0  # gamma^{t-1}, starts at t=1

    for t in range(1, time_steps + 1):
        # Current state with truncation
        a1 = min(alpha[0], M)
        b1 = min(beta[0], M)
        a2 = min(alpha[1], M)
        b2 = min(beta[1], M)

        # Determine action from policy
        action = policy[a1, b1, a2, b2]

        # Pull the selected arm
        arm = action  # 0 or 1
        success = np.random.rand() < true_theta[arm]
        if success:
```

```
            total_reward += current_gamma_power  # Reward is gamma^{t-1}
            alpha[arm] += 1
        else:
            beta[arm] += 1

        # Update the discount factor for the next time step
        current_gamma_power *= gamma

    return total_reward

# -------------------------------------------------------------
# Thompson Sampling Simulation Function
# -------------------------------------------------------------
def thompson_sampling_simulation(true_theta, alpha_prior, beta_prior, gamma, time_s
    """
    Simulate a single trial of the bandit problem using Thompson Sampling.

    Arguments:
    ----------
    true_theta : list of float
        True success probabilities for each arm.
    alpha_prior : list of int
        Prior alpha parameters for Beta distributions.
    beta_prior : list of int
        Prior beta parameters for Beta distributions.
    gamma : float
        Discount factor.
    time_steps : int
        Number of pulls in the trial.

    Returns:
    --------
    total_reward : float
        Total discounted reward accumulated in the trial.
    """
    alpha = np.array(alpha_prior, dtype=float)
    beta = np.array(beta_prior, dtype=float)

    total_discounted_reward = 0.0
    discount_power = 0

    for t in range(1, time_steps + 1):
        # Sample theta-hat from current posterior for each arm
        sampled_thetas = np.random.beta(alpha, beta)

        # Choose the arm that maximizes the sampled theta
        chosen_arm = np.argmax(sampled_thetas)

        # Draw reward
        reward = 1 if np.random.rand() < true_theta[chosen_arm] else 0

        # Update posterior
        alpha[chosen_arm] += reward
        beta[chosen_arm] += (1 - reward)

        # Accumulate discounted reward
```

```
        total_discounted_reward += (gamma ** discount_power) * reward
        discount_power += 1

    return total_discounted_reward

# ------------------------------------------------------------
# Gamma Evaluation Function
# ------------------------------------------------------------
def evaluate_gamma(gamma):
    """
    Evaluate a single gamma value by solving DP and running simulations for both DP

    Arguments:
    ----------
    gamma : float
        Discount factor.

    Returns:
    --------
    gamma : float
        The gamma value evaluated.
    average_reward_dp : float
        Average total discounted reward over all DP trials.
    average_reward_ts : float
        Average total discounted reward over all TS trials.
    """
    print(f"Evaluating gamma = {gamma:.4f}")

    # Solve DP to get the policy
    R, policy = solve_2armed_bandit_dp(M=M, gamma=gamma)

    # Initialize total rewards for all trials
    total_rewards_dp = np.zeros(repeat_time, dtype=np.float64)
    total_rewards_ts = np.zeros(repeat_time, dtype=np.float64)

    # Simulate all trials for DP-based policy
    for trial in range(1, repeat_time + 1):
        reward = simulate_trial(
            policy, M, true_theta, alpha_prior, beta_prior, gamma, time_steps
        )
        total_rewards_dp[trial - 1] = reward

    # Simulate all trials for Thompson Sampling policy
    for trial in range(1, repeat_time + 1):
        reward_ts = thompson_sampling_simulation(
            true_theta, alpha_prior, beta_prior, gamma, time_steps
        )
        total_rewards_ts[trial - 1] = reward_ts

    # Calculate average rewards
    average_reward_dp = np.mean(total_rewards_dp)
    average_reward_ts = np.mean(total_rewards_ts)

    print(f"Gamma={gamma:.4f}: DP Avg Reward={average_reward_dp:.2f}, TS Avg Reward

    return gamma, average_reward_dp, average_reward_ts
```

```
# ------------------------------------------------------------
# Main Evaluation Loop
# ------------------------------------------------------------
# Initialize lists to store results
results_dp = []
results_ts = []

# Total number of gamma values
total_gammas = len(gamma_values)

# Iterate over gamma_values and collect results
for idx, gamma in enumerate(gamma_values, 1):
    print(f"\nProcessing gamma {idx}/{total_gammas}: gamma = {gamma:.4f}")
    gamma_result = evaluate_gamma(gamma)
    _, avg_dp, avg_ts = gamma_result
    results_dp.append(avg_dp)
    results_ts.append(avg_ts)

print("\nAll gamma values have been evaluated.\n")

# Convert results to numpy arrays for easier processing
gamma_evaluated = np.array(gamma_values)
avg_rewards_dp = np.array(results_dp)
avg_rewards_ts = np.array(results_ts)

# Compute the gap between DP and TS
gap = avg_rewards_dp - avg_rewards_ts

# ------------------------------------------------------------
# Find the gamma with the highest average reward for DP
# ------------------------------------------------------------
optimal_index = np.argmax(avg_rewards_dp)
optimal_gamma = gamma_evaluated[optimal_index]
optimal_reward_dp = avg_rewards_dp[optimal_index]
optimal_reward_ts = avg_rewards_ts[optimal_index]

print(f"Optimal gamma for DP: {optimal_gamma:.4f}")
print(f"DP Reward at Optimal Gamma: {optimal_reward_dp:.2f}")
print(f"TS Reward at Optimal Gamma: {optimal_reward_ts:.2f}")

# ------------------------------------------------------------
# Plot the Results
# ------------------------------------------------------------
plt.figure(figsize=(14, 6))

# Plot Average Rewards for DP and TS
plt.subplot(1, 2, 1)
plt.plot(gamma_evaluated, avg_rewards_dp, linestyle='-', color='blue', label='DP Op
plt.plot(gamma_evaluated, avg_rewards_ts, linestyle='--', color='green', label='Tho
plt.xlabel('Gamma')
plt.ylabel('Average Discounted Reward')
plt.title('Average Discounted Reward vs Gamma')
plt.axvline(optimal_gamma, color='red', linestyle='--', label=f'Optimal Gamma: {opt
plt.legend()
plt.grid(True)
```

```
# Plot the Gap between DP and TS
plt.subplot(1, 2, 2)
plt.plot(gamma_evaluated, gap, linestyle='-', color='purple')
plt.xlabel('Gamma')
plt.ylabel('Reward Gap (DP - TS)')
plt.title('Gap Between DP Optimal Policy and Thompson Sampling')
plt.axvline(optimal_gamma, color='red', linestyle='--', label=f'Optimal Gamma: {opt
plt.legend()
plt.grid(True)

plt.tight_layout()
plt.show()
```

```
Processing gamma 1/100: gamma = 0.9000
Evaluating gamma = 0.9000
Gamma=0.9000: DP Avg Reward=4.23, TS Avg Reward=5.18

Processing gamma 2/100: gamma = 0.9010
Evaluating gamma = 0.9010
Gamma=0.9010: DP Avg Reward=5.87, TS Avg Reward=4.99

Processing gamma 3/100: gamma = 0.9020
Evaluating gamma = 0.9020
Gamma=0.9020: DP Avg Reward=5.06, TS Avg Reward=5.79

Processing gamma 4/100: gamma = 0.9030
Evaluating gamma = 0.9030
Gamma=0.9030: DP Avg Reward=6.04, TS Avg Reward=5.97

Processing gamma 5/100: gamma = 0.9040
Evaluating gamma = 0.9040
Gamma=0.9040: DP Avg Reward=5.80, TS Avg Reward=5.14

Processing gamma 6/100: gamma = 0.9051
Evaluating gamma = 0.9051
Gamma=0.9051: DP Avg Reward=5.22, TS Avg Reward=5.12

Processing gamma 7/100: gamma = 0.9061
Evaluating gamma = 0.9061
Gamma=0.9061: DP Avg Reward=5.56, TS Avg Reward=5.59

Processing gamma 8/100: gamma = 0.9071
Evaluating gamma = 0.9071
Gamma=0.9071: DP Avg Reward=6.20, TS Avg Reward=5.20

Processing gamma 9/100: gamma = 0.9081
Evaluating gamma = 0.9081
Gamma=0.9081: DP Avg Reward=6.30, TS Avg Reward=5.83

Processing gamma 10/100: gamma = 0.9091
Evaluating gamma = 0.9091
Gamma=0.9091: DP Avg Reward=6.32, TS Avg Reward=5.02

Processing gamma 11/100: gamma = 0.9101
Evaluating gamma = 0.9101
Gamma=0.9101: DP Avg Reward=5.29, TS Avg Reward=5.70

Processing gamma 12/100: gamma = 0.9111
Evaluating gamma = 0.9111
Gamma=0.9111: DP Avg Reward=6.39, TS Avg Reward=6.32

Processing gamma 13/100: gamma = 0.9121
Evaluating gamma = 0.9121
Gamma=0.9121: DP Avg Reward=7.09, TS Avg Reward=5.13

Processing gamma 14/100: gamma = 0.9131
Evaluating gamma = 0.9131
Gamma=0.9131: DP Avg Reward=6.68, TS Avg Reward=6.59
```

```
Processing gamma 15/100: gamma = 0.9141
Evaluating gamma = 0.9141
Gamma=0.9141: DP Avg Reward=6.42, TS Avg Reward=6.76

Processing gamma 16/100: gamma = 0.9152
Evaluating gamma = 0.9152
Gamma=0.9152: DP Avg Reward=6.82, TS Avg Reward=6.35

Processing gamma 17/100: gamma = 0.9162
Evaluating gamma = 0.9162
Gamma=0.9162: DP Avg Reward=6.66, TS Avg Reward=5.86

Processing gamma 18/100: gamma = 0.9172
Evaluating gamma = 0.9172
Gamma=0.9172: DP Avg Reward=6.40, TS Avg Reward=5.62

Processing gamma 19/100: gamma = 0.9182
Evaluating gamma = 0.9182
Gamma=0.9182: DP Avg Reward=7.05, TS Avg Reward=6.29

Processing gamma 20/100: gamma = 0.9192
Evaluating gamma = 0.9192
Gamma=0.9192: DP Avg Reward=5.76, TS Avg Reward=6.04

Processing gamma 21/100: gamma = 0.9202
Evaluating gamma = 0.9202
Gamma=0.9202: DP Avg Reward=6.67, TS Avg Reward=6.43

Processing gamma 22/100: gamma = 0.9212
Evaluating gamma = 0.9212
Gamma=0.9212: DP Avg Reward=6.55, TS Avg Reward=6.90

Processing gamma 23/100: gamma = 0.9222
Evaluating gamma = 0.9222
Gamma=0.9222: DP Avg Reward=7.24, TS Avg Reward=6.80

Processing gamma 24/100: gamma = 0.9232
Evaluating gamma = 0.9232
Gamma=0.9232: DP Avg Reward=7.34, TS Avg Reward=7.15

Processing gamma 25/100: gamma = 0.9242
Evaluating gamma = 0.9242
Gamma=0.9242: DP Avg Reward=7.07, TS Avg Reward=6.72

Processing gamma 26/100: gamma = 0.9253
Evaluating gamma = 0.9253
Gamma=0.9253: DP Avg Reward=8.15, TS Avg Reward=6.17

Processing gamma 27/100: gamma = 0.9263
Evaluating gamma = 0.9263
Gamma=0.9263: DP Avg Reward=8.09, TS Avg Reward=7.71

Processing gamma 28/100: gamma = 0.9273
Evaluating gamma = 0.9273
Gamma=0.9273: DP Avg Reward=8.47, TS Avg Reward=6.94
```

```
Processing gamma 29/100: gamma = 0.9283
Evaluating gamma = 0.9283
Gamma=0.9283: DP Avg Reward=7.42, TS Avg Reward=6.37

Processing gamma 30/100: gamma = 0.9293
Evaluating gamma = 0.9293
Gamma=0.9293: DP Avg Reward=7.28, TS Avg Reward=7.12

Processing gamma 31/100: gamma = 0.9303
Evaluating gamma = 0.9303
Gamma=0.9303: DP Avg Reward=7.80, TS Avg Reward=6.92

Processing gamma 32/100: gamma = 0.9313
Evaluating gamma = 0.9313
Gamma=0.9313: DP Avg Reward=7.46, TS Avg Reward=8.34

Processing gamma 33/100: gamma = 0.9323
Evaluating gamma = 0.9323
Gamma=0.9323: DP Avg Reward=8.42, TS Avg Reward=8.02

Processing gamma 34/100: gamma = 0.9333
Evaluating gamma = 0.9333
Gamma=0.9333: DP Avg Reward=7.52, TS Avg Reward=8.12

Processing gamma 35/100: gamma = 0.9343
Evaluating gamma = 0.9343
Gamma=0.9343: DP Avg Reward=8.76, TS Avg Reward=8.06

Processing gamma 36/100: gamma = 0.9354
Evaluating gamma = 0.9354
Gamma=0.9354: DP Avg Reward=8.03, TS Avg Reward=7.71

Processing gamma 37/100: gamma = 0.9364
Evaluating gamma = 0.9364
Gamma=0.9364: DP Avg Reward=8.45, TS Avg Reward=7.29

Processing gamma 38/100: gamma = 0.9374
Evaluating gamma = 0.9374
Gamma=0.9374: DP Avg Reward=9.14, TS Avg Reward=8.19

Processing gamma 39/100: gamma = 0.9384
Evaluating gamma = 0.9384
Gamma=0.9384: DP Avg Reward=10.54, TS Avg Reward=8.47

Processing gamma 40/100: gamma = 0.9394
Evaluating gamma = 0.9394
Gamma=0.9394: DP Avg Reward=8.79, TS Avg Reward=8.70

Processing gamma 41/100: gamma = 0.9404
Evaluating gamma = 0.9404
Gamma=0.9404: DP Avg Reward=8.27, TS Avg Reward=9.21

Processing gamma 42/100: gamma = 0.9414
Evaluating gamma = 0.9414
Gamma=0.9414: DP Avg Reward=9.20, TS Avg Reward=9.22
```

```
Processing gamma 43/100: gamma = 0.9424
Evaluating gamma = 0.9424
Gamma=0.9424: DP Avg Reward=9.35, TS Avg Reward=8.63

Processing gamma 44/100: gamma = 0.9434
Evaluating gamma = 0.9434
Gamma=0.9434: DP Avg Reward=9.59, TS Avg Reward=9.23

Processing gamma 45/100: gamma = 0.9444
Evaluating gamma = 0.9444
Gamma=0.9444: DP Avg Reward=9.20, TS Avg Reward=10.17

Processing gamma 46/100: gamma = 0.9455
Evaluating gamma = 0.9455
Gamma=0.9455: DP Avg Reward=10.27, TS Avg Reward=9.44

Processing gamma 47/100: gamma = 0.9465
Evaluating gamma = 0.9465
Gamma=0.9465: DP Avg Reward=9.82, TS Avg Reward=9.38

Processing gamma 48/100: gamma = 0.9475
Evaluating gamma = 0.9475
Gamma=0.9475: DP Avg Reward=9.56, TS Avg Reward=10.37

Processing gamma 49/100: gamma = 0.9485
Evaluating gamma = 0.9485
Gamma=0.9485: DP Avg Reward=9.20, TS Avg Reward=9.30

Processing gamma 50/100: gamma = 0.9495
Evaluating gamma = 0.9495
Gamma=0.9495: DP Avg Reward=10.23, TS Avg Reward=9.72

Processing gamma 51/100: gamma = 0.9505
Evaluating gamma = 0.9505
Gamma=0.9505: DP Avg Reward=11.19, TS Avg Reward=9.61

Processing gamma 52/100: gamma = 0.9515
Evaluating gamma = 0.9515
Gamma=0.9515: DP Avg Reward=10.85, TS Avg Reward=10.47

Processing gamma 53/100: gamma = 0.9525
Evaluating gamma = 0.9525
Gamma=0.9525: DP Avg Reward=11.53, TS Avg Reward=11.00

Processing gamma 54/100: gamma = 0.9535
Evaluating gamma = 0.9535
Gamma=0.9535: DP Avg Reward=12.21, TS Avg Reward=11.62

Processing gamma 55/100: gamma = 0.9545
Evaluating gamma = 0.9545
Gamma=0.9545: DP Avg Reward=9.53, TS Avg Reward=10.93

Processing gamma 56/100: gamma = 0.9556
Evaluating gamma = 0.9556
Gamma=0.9556: DP Avg Reward=11.30, TS Avg Reward=11.66
```

```
Processing gamma 57/100: gamma = 0.9566
Evaluating gamma = 0.9566
Gamma=0.9566: DP Avg Reward=12.15, TS Avg Reward=12.40

Processing gamma 58/100: gamma = 0.9576
Evaluating gamma = 0.9576
Gamma=0.9576: DP Avg Reward=12.47, TS Avg Reward=12.68

Processing gamma 59/100: gamma = 0.9586
Evaluating gamma = 0.9586
Gamma=0.9586: DP Avg Reward=13.73, TS Avg Reward=12.77

Processing gamma 60/100: gamma = 0.9596
Evaluating gamma = 0.9596
Gamma=0.9596: DP Avg Reward=14.52, TS Avg Reward=12.84

Processing gamma 61/100: gamma = 0.9606
Evaluating gamma = 0.9606
Gamma=0.9606: DP Avg Reward=14.34, TS Avg Reward=13.61

Processing gamma 62/100: gamma = 0.9616
Evaluating gamma = 0.9616
Gamma=0.9616: DP Avg Reward=14.11, TS Avg Reward=13.29

Processing gamma 63/100: gamma = 0.9626
Evaluating gamma = 0.9626
Gamma=0.9626: DP Avg Reward=14.08, TS Avg Reward=13.75

Processing gamma 64/100: gamma = 0.9636
Evaluating gamma = 0.9636
Gamma=0.9636: DP Avg Reward=15.20, TS Avg Reward=14.47

Processing gamma 65/100: gamma = 0.9646
Evaluating gamma = 0.9646
Gamma=0.9646: DP Avg Reward=16.59, TS Avg Reward=15.02

Processing gamma 66/100: gamma = 0.9657
Evaluating gamma = 0.9657
Gamma=0.9657: DP Avg Reward=17.22, TS Avg Reward=15.81

Processing gamma 67/100: gamma = 0.9667
Evaluating gamma = 0.9667
Gamma=0.9667: DP Avg Reward=15.93, TS Avg Reward=15.92

Processing gamma 68/100: gamma = 0.9677
Evaluating gamma = 0.9677
Gamma=0.9677: DP Avg Reward=18.08, TS Avg Reward=16.25

Processing gamma 69/100: gamma = 0.9687
Evaluating gamma = 0.9687
Gamma=0.9687: DP Avg Reward=18.17, TS Avg Reward=17.32

Processing gamma 70/100: gamma = 0.9697
Evaluating gamma = 0.9697
Gamma=0.9697: DP Avg Reward=18.94, TS Avg Reward=16.87
```

```
Processing gamma 71/100: gamma = 0.9707
Evaluating gamma = 0.9707
Gamma=0.9707: DP Avg Reward=18.94, TS Avg Reward=19.12

Processing gamma 72/100: gamma = 0.9717
Evaluating gamma = 0.9717
Gamma=0.9717: DP Avg Reward=21.93, TS Avg Reward=19.63

Processing gamma 73/100: gamma = 0.9727
Evaluating gamma = 0.9727
Gamma=0.9727: DP Avg Reward=21.67, TS Avg Reward=19.69

Processing gamma 74/100: gamma = 0.9737
Evaluating gamma = 0.9737
Gamma=0.9737: DP Avg Reward=20.85, TS Avg Reward=19.93

Processing gamma 75/100: gamma = 0.9747
Evaluating gamma = 0.9747
Gamma=0.9747: DP Avg Reward=19.46, TS Avg Reward=22.04

Processing gamma 76/100: gamma = 0.9758
Evaluating gamma = 0.9758
Gamma=0.9758: DP Avg Reward=25.31, TS Avg Reward=22.38

Processing gamma 77/100: gamma = 0.9768
Evaluating gamma = 0.9768
Gamma=0.9768: DP Avg Reward=25.60, TS Avg Reward=22.36

Processing gamma 78/100: gamma = 0.9778
Evaluating gamma = 0.9778
Gamma=0.9778: DP Avg Reward=24.24, TS Avg Reward=25.46

Processing gamma 79/100: gamma = 0.9788
Evaluating gamma = 0.9788
Gamma=0.9788: DP Avg Reward=24.13, TS Avg Reward=25.34

Processing gamma 80/100: gamma = 0.9798
Evaluating gamma = 0.9798
Gamma=0.9798: DP Avg Reward=28.80, TS Avg Reward=25.61

Processing gamma 81/100: gamma = 0.9808
Evaluating gamma = 0.9808
Gamma=0.9808: DP Avg Reward=28.95, TS Avg Reward=27.10

Processing gamma 82/100: gamma = 0.9818
Evaluating gamma = 0.9818
Gamma=0.9818: DP Avg Reward=29.40, TS Avg Reward=30.41

Processing gamma 83/100: gamma = 0.9828
Evaluating gamma = 0.9828
Gamma=0.9828: DP Avg Reward=34.18, TS Avg Reward=32.91

Processing gamma 84/100: gamma = 0.9838
Evaluating gamma = 0.9838
Gamma=0.9838: DP Avg Reward=34.37, TS Avg Reward=34.12
```

```
Processing gamma 85/100: gamma = 0.9848
Evaluating gamma = 0.9848
Gamma=0.9848: DP Avg Reward=34.12, TS Avg Reward=37.57

Processing gamma 86/100: gamma = 0.9859
Evaluating gamma = 0.9859
Gamma=0.9859: DP Avg Reward=39.85, TS Avg Reward=41.39

Processing gamma 87/100: gamma = 0.9869
Evaluating gamma = 0.9869
Gamma=0.9869: DP Avg Reward=43.05, TS Avg Reward=43.11

Processing gamma 88/100: gamma = 0.9879
Evaluating gamma = 0.9879
Gamma=0.9879: DP Avg Reward=46.48, TS Avg Reward=47.55

Processing gamma 89/100: gamma = 0.9889
Evaluating gamma = 0.9889
Gamma=0.9889: DP Avg Reward=50.42, TS Avg Reward=51.01

Processing gamma 90/100: gamma = 0.9899
Evaluating gamma = 0.9899
Gamma=0.9899: DP Avg Reward=57.54, TS Avg Reward=57.58

Processing gamma 91/100: gamma = 0.9909
Evaluating gamma = 0.9909
Gamma=0.9909: DP Avg Reward=61.79, TS Avg Reward=67.40

Processing gamma 92/100: gamma = 0.9919
Evaluating gamma = 0.9919
Gamma=0.9919: DP Avg Reward=70.39, TS Avg Reward=71.11

Processing gamma 93/100: gamma = 0.9929
Evaluating gamma = 0.9929
Gamma=0.9929: DP Avg Reward=78.76, TS Avg Reward=79.83

Processing gamma 94/100: gamma = 0.9939
Evaluating gamma = 0.9939
Gamma=0.9939: DP Avg Reward=97.94, TS Avg Reward=96.37

Processing gamma 95/100: gamma = 0.9949
Evaluating gamma = 0.9949
Gamma=0.9949: DP Avg Reward=110.69, TS Avg Reward=116.20

Processing gamma 96/100: gamma = 0.9960
Evaluating gamma = 0.9960
Gamma=0.9960: DP Avg Reward=133.00, TS Avg Reward=145.14

Processing gamma 97/100: gamma = 0.9970
Evaluating gamma = 0.9970
Gamma=0.9970: DP Avg Reward=194.02, TS Avg Reward=193.00

Processing gamma 98/100: gamma = 0.9980
Evaluating gamma = 0.9980
Gamma=0.9980: DP Avg Reward=289.31, TS Avg Reward=286.59
```

```
Processing gamma 99/100: gamma = 0.9990
Evaluating gamma = 0.9990
Gamma=0.9990: DP Avg Reward=555.59, TS Avg Reward=582.36

Processing gamma 100/100: gamma = 1.0000
Evaluating gamma = 1.0000
Gamma=1.0000: DP Avg Reward=2996.70, TS Avg Reward=3000.90

All gamma values have been evaluated.

Optimal gamma for DP: 1.0000
DP Reward at Optimal Gamma: 2996.70
TS Reward at Optimal Gamma: 3000.90
```