SABİT KATSAYILI LİNEER DİF. DENK. SİSTEMLERİNİN LAPLACE DÖNÜŞÜMLERİ İLE ÇÖZÜMÜ.

Laplace dönüşümü sabit katsayılı lineer dif. denklemlerin çözümünde de kullanılır. Burada bir cebirsel denklem sistemi karşılık gelir.

Soru Aşağıdaki diferensiyel denklem sistemlerinin çözümünü bulunuz.

1. $y'+z=x$, $y(0)=1$
   $z'+4y=0$, $z(0)=-1$

2. $z''+y'=\cos x$, $z(0)=-1$
   $y''-z=\sin x$, $z'(0)=-1$
   $y(0)=1$, $y'(0)=0$

3. $w'+y=\sin x$, $w(0)=0$
   $y'-z'=e^x$, $y(0)=1$
   $z'+w+y=1$, $z(0)=1$

4. $y''+z+y=0$, $y(0)=0$
   $z'+y'=0$, $y'(0)=0$
   $z(0)=1$

çözümler

1. $y'+z=x$
   $z'+4y=0$ $\quad y(0)=1,\ z(0)=-1$

$$\mathcal{L}\{y'+z\}=\mathcal{L}\{x\} \Rightarrow \mathcal{L}\{y'\}+\mathcal{L}\{z\}=\mathcal{L}\{x\}$$

$$\mathcal{L}\{z'+4y\}=\mathcal{L}\{0\} \Rightarrow \mathcal{L}\{z'\}+4\mathcal{L}\{y\}=\mathcal{L}\{0\}.$$

$$[sY(s)-y(0)]+Z(s)=\frac{1}{s^2} \Rightarrow sY(s)-1+Z(s)=\frac{1}{s^2}$$

$$[sZ(s)-z(0)]+4Y(s)=0 \Rightarrow sZ(s)+1+4Y(s)=0$$

$$\Rightarrow \left.\begin{aligned} sY(s)+Z(s)&=\frac{1+s^2}{s^2} \\ 4Y(s)+sZ(s)&=-1 \end{aligned}\right\} \Rightarrow \to \mathrm{II}$$

$$\mathrm{II} \to sY(s)+\left[-\frac{1}{s}\left(1+4Y(s)\right)\right]=\frac{1+s^2}{s^2}$$

$$sY(s)-\frac{4}{s}Y(s)=\frac{1+s^2}{s^2}+\frac{1}{s}$$

$$\Rightarrow \left(s-\frac{4}{s}\right)Y(s)=\frac{s+s^3+s^2}{s^3}$$

$$Y(s) = \frac{s^3+s^2+s}{s^3}\,\frac{s}{s^2-4} = \frac{s^4+s^3+s^2}{s^3(s^2-4)} = \frac{s^2+s+1}{s(s^2-4)}$$

$$\Rightarrow Y(s) = \frac{s^2+s+1}{s(s^2-4)}$$

$$Z(s) = -\frac{1}{s}\left[1+4Y(s)\right] = -\frac{1}{s} - \frac{4(s^2+s+1)}{s^2(s^2-4)}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s^2+s+1}{s(s^2-4)}\right\}.$$

$$\frac{s^2+s+1}{s(s^2-4)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+2} = \frac{A(s^2-4)+Bs(s+2)+Cs(s-2)}{s(s^2-4)}$$

$$\Rightarrow s^2+s+1 = (A+B+C)s^2 + (2B-2C)s - 4A$$

$$\left.\begin{aligned} A+B+C &= 1 \\ 2B-2C &= 1 \\ -4A &= 1 \end{aligned}\right\} \Rightarrow \quad A = -\frac{1}{4},\quad B = \frac{7}{8},\quad C = \frac{3}{8}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{7}{8}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{3}{8}\mathcal{L}^{-1}\left\{\frac{1}{s-(-2)}\right\}$$

$$y(x) = -\frac{1}{4} + \frac{7}{8}e^{2x} + \frac{3}{8}e^{-2x}$$

$$\mathcal{L}^{-1}\{Z(s)\} = \mathcal{L}^{-1}\left\{-\frac{1}{s} - \frac{4(s^2+s+1)}{s^2(s^2-4)}\right\}$$

$$= -\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 4\mathcal{L}^{-1}\left\{\frac{s^2+s+1}{s^2(s^2-4)}\right\}.$$

$$\frac{s^2+s+1}{s^2(s^2-4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} + \frac{D}{s+2}.$$

$$s^2+s+1 = (A+C+D)s^3 + (B+2C-2D)s^2 - 4As - 4B$$

$$\left.\begin{aligned} A+D+C &= 0 \\ B+2C-2D &= 1 \\ -4A &= 1 \\ -4B &= 1 \end{aligned}\right\} \Rightarrow \quad A = -\frac{1}{4},\quad B = -\frac{1}{4},\quad C = \frac{7}{16},\quad D = -\frac{3}{16}$$

$$\mathcal{L}^{-1}\{Z(s)\} = -\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 4\left[-\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{7}{16}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{3}{16}\mathcal{L}^{-1}\left\{\frac{1}{s-(-2)}\right\}\right]$$

$$= -1 + 1 + x - \frac{7}{4}e^{2x} + \frac{3}{4}e^{-2x}$$

$$z(x) = x - \frac{7}{4}e^{2x} + \frac{3}{4}e^{-2x}$$

$$\Rightarrow \begin{cases} y(x) = -\frac{1}{4} + \frac{7}{8}e^{2x} + \frac{3}{8}e^{-2x} \\ z(x) = x - \frac{7}{4}e^{2x} + \frac{3}{4}e^{-2x} \end{cases}$$

Sistemin çözümüdür.

2. $z'' + y' = \cos x \qquad z(0) = -1, \quad z'(0) = -1$
$y'' - z = \sin x \qquad y(0) = 1, \quad y'(0) = 0$

$\mathcal{L}\{z'' + y'\} = \mathcal{L}\{\cos x\} \Rightarrow \mathcal{L}\{z''\} + \mathcal{L}\{y'\} = \mathcal{L}\{\cos x\}$

$\mathcal{L}\{y'' - z\} = \mathcal{L}\{\sin x\} \Rightarrow \mathcal{L}\{y''\} - \mathcal{L}\{z\} = \mathcal{L}\{\sin x\}$.

$$[s^2 Z(s) + s + 1] + sY(s) - 1 = \frac{s}{s^2+1}$$

$$s^2 Y(s) - s - Z(s) = \frac{1}{s^2+1}$$

$$s^2 Z(s) + sY(s) = \frac{s}{s^2+1} - s = \frac{s - s^3 - s}{s^2+1}$$

$$s^2 Y(s) - Z(s) = \frac{1}{s^2+1} + s = \frac{1 + s^3 + 1}{s^2+1}$$

$$s^2 Z(s) + sY(s) = -\frac{s^3}{s^2+1}$$

$$s^2 Y(s) - Z(s) = \frac{s^3 + s + 1}{s^2+1}$$

$$\Rightarrow Y(s) = \frac{s}{s^2+1}, \quad Z(s) = -\frac{s+1}{s^2+1}$$

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$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos x$$

$$\mathcal{L}^{-1}\{Z(s)\} = \mathcal{L}^{-1}\left\{-\frac{s+1}{s^2+1}\right\} = -\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}.$$

$$z(x) = -\cos x - \sin x$$

$$\Rightarrow \begin{cases} y(x) = \cos x \\ z(x) = -\cos x - \sin x \end{cases} \quad \text{dm}$$

3 =
$$\begin{aligned} w' + y &= \sin x \\ y' - z &= e^x \\ z' + w + y &= 1 \end{aligned} \qquad \begin{aligned} w(0) &= 0 \\ y(0) &= 1 \\ z(0) &= 1 \end{aligned}$$

$$\mathcal{L}\{w' + y\} = \mathcal{L}\{\sin x\} \Rightarrow \mathcal{L}\{w'\} + \mathcal{L}\{y\} = \mathcal{L}\{\sin x\}$$
$$\mathcal{L}\{y' - z\} = \mathcal{L}\{e^x\} \Rightarrow \mathcal{L}\{y'\} - \mathcal{L}\{z\} = \mathcal{L}\{e^x\}$$
$$\mathcal{L}\{z' + w + y\} = \mathcal{L}\{1\} \Rightarrow \mathcal{L}\{z'\} + \mathcal{L}\{w\} + \mathcal{L}\{y\} = \mathcal{L}\{1\}.$$

$$\begin{aligned} sW(s) + Y(s) &= \frac{1}{s^2+1} \\ sY(s) - \cancel{1} - Z(s) &= \frac{1}{s-1} \\ sZ(s) - 1 + W(s) + Y(s) &= \frac{1}{s} \end{aligned} \qquad \begin{aligned} sW(s) + Y(s) &= \frac{1}{s^2+1} \\ -Z(s) + sY(s) &= \frac{s}{s-1} \\ W(s) + sZ(s) + Y(s) &= \frac{s+1}{s} \end{aligned}$$

den.

$$W(s) = \frac{\begin{vmatrix} 1/s^2+1 & 1 & 0 \\ s/s-1 & s & -1 \\ 1+s/s & 1 & s \end{vmatrix}}{\begin{vmatrix} s & 1 & 0 \\ 0 & s & -1 \\ 1 & 1 & s \end{vmatrix}} = \frac{\frac{1}{s^2+1}\left[s^2+1\right] - \left[\frac{s^2}{s-1} + \frac{1+s}{s}\right]}{s^3+s-1}$$

$$= \frac{1 - \left[\frac{s^3 + (s^2-1)}{s(s-1)}\right]}{s^3+s-1} = \frac{-(s^3+s-1)}{s(s-1)(s^3+s-1)}$$

$$W(s) = -\frac{1}{s(s-1)}$$

$$\Rightarrow s\left(-\frac{1}{s(s-1)}\right)+Y(s)=\frac{1}{s^2+1}$$

$$\Rightarrow Y(s)=\frac{1}{s^2+1}+\frac{1}{s-1}=\frac{s-1+s^2+1}{(s^2+1)(s-1)}=\frac{s^2+s}{(s^2+1)(s-1)}$$

$$Y(s)=\frac{s^2+s}{(s^2+1)(s-1)}$$

$$\Rightarrow -s\,Z(s)+\frac{s(s^2+s)}{(s^2+1)(s-1)}=\frac{s}{s-1}$$

$$\Rightarrow Z(s)=\frac{s(s^2+s)}{s(s^2+1)(s-1)}-\frac{s}{s(s-1)}=\frac{s^2+s-(s^2+1)}{(s^2+1)(s-1)}$$

$$=\frac{s-1}{(s^2+1)(s-1}=\frac{1}{s^2+1}$$

$$Z(s)=\frac{\begin{vmatrix} s & 1 & 1/s^2+1 \\ 0 & s & s/s-1 \\ 1 & 1 & s+1/s \end{vmatrix}}{s^3+s-1}=\frac{s(s^2-1)(s^2+1)+(s-s^2)(s^2+1)-s(s-1)}{(s-1)(s^2+1)(s^3+s-1}$$

$$=\frac{s}{s^2+1}$$

$$\Rightarrow \mathcal{L}^{-1}\{W(s)\}=\mathcal{L}^{-1}\left\{-\frac{1}{s(s-1)}\right\}=$$

$$\frac{1}{s(s-1)}=\frac{A}{s}+\frac{B}{s-1}=\frac{A(s-1)+Bs}{s(s-1)}=\frac{(A+B)s-A}{s(s-1)}$$

$$-A=1 \Rightarrow A=1 \quad , \quad A+B=0 \Rightarrow B=-1$$

$$\Rightarrow \mathcal{L}^{-1}\{W(s)\}=-\left[\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}\right]$$

$$W(x)=-\left[1-e^x\right]$$

$$W(x)=e^x-1$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{s^2+s}{(s^2+1)(s-1)}\right\}.$$

$$\frac{s^2+s}{(s^2+1)(s-1)}=\frac{\cancel{s^2+s}}{(}=\frac{As+B}{s^2+1}+\frac{C}{s-1}$$

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$$s^2+s = As(s-1)+B(s-1)+C(s^2+1)$$
$$\Rightarrow \quad = (A+C)s^2 + (-A+B)s - B + C$$

$$A+C=1 \qquad B=C \qquad \Rightarrow -A+C=1 \quad \Rightarrow A=0$$
$$-A+B=1 \qquad\qquad A+C=1 \qquad C=1$$
$$-B+C=0 \qquad\qquad\qquad B=1$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s^2+s}{(s^2+1)(s-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$\Rightarrow y(x) = \sin x + e^x .$$

$$Y(s) = \frac{\begin{vmatrix} s & 1/s^2+1 & 0 \\ 0 & +s/s-1 & -1 \\ 1 & 1+s/s & s \end{vmatrix}}{s^3+s-1} = \frac{s\left[\frac{s^2}{s-1} + \frac{1+s}{s}\right] - \frac{1}{s^2+1}}{s^3+s-1}$$

$$Y(s) = \frac{s^5+s^4+s^3-s}{(s^2+1)(s-1)(s^3+s-1)} = \frac{s^2+s}{(s^2+1)(s-1)}$$

$$s^5+s^4+s^3-s = s(s^4-1)+s^3(s+1) = s(s^2-1)(s^2+1)+s^3(s+1)$$
$$= s(s-1)(s+1)(s^2+1)+s^3(s+1)$$
$$= s(s+1)\left[(s^2+1)(s-1)+s^2\right]$$
$$= (s^2+s)\left[s^3-s^2+s-1+s^2\right]$$
$$= (s^2+s)\left[s^3+s-1\right].$$

$$\mathcal{L}^{-1}\{Z(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos x$$

$$\Rightarrow \begin{cases} w(x) = e^x - 1 \\ y(x) = \sin x + e^x \\ z(x) = \cos x \end{cases}$$

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4. $y'' + z + y = 0$
$z' + y' = 0$ $\quad y(0)=0 \quad y'(0)=0 \quad z(0)=1$ (49)

$\mathcal{L}\{y''+z+y\} = \mathcal{L}\{0\} \Rightarrow \mathcal{L}\{y''\} + \mathcal{L}\{z\} + \mathcal{L}\{y\} = \mathcal{L}\{0\}$

$\mathcal{L}\{z'+y'\} = \mathcal{L}\{0\} \Rightarrow \mathcal{L}\{z'\} + \mathcal{L}\{y'\} = \mathcal{L}\{0\}$

$\Rightarrow s^2Y(s) + Z(s) + Y(s) = 0 \Rightarrow s^2Y(s) + Z(s) + Y(s) = 0$

$sZ(s) - 1 + sY(s) = 0 \Rightarrow sY(s) + sZ(s) = 1$

$\Rightarrow (s^2+1)Y(s) = -Z(s) \Rightarrow Y(s) = -\dfrac{Z(s)}{s^2+1}$

$-\dfrac{s\,Z(s)}{s^2+1} + s\,Z(s) = 1 \Rightarrow \left[\dfrac{s(s^2+1)-s}{s^2+1}\right]Z(s) = 1$

$\Rightarrow Z(s) = \dfrac{s^2+1}{s(s^2+1)-s} = \dfrac{s^2+1}{s^3+s-s} = \dfrac{s^2+1}{s^3}$

$\Rightarrow Z(s) = \dfrac{1}{s} + \dfrac{1}{s^3} = \dfrac{s^2+1}{s^3}$

$\Rightarrow Y(s) = -\dfrac{s^2+1}{s^3(s^2+1)} = -\dfrac{1}{s^3}$

$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = -\dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{2}{s^3}\right\} = -\dfrac{1}{2}x^2$

$\mathcal{L}^{-1}\{Z(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s} + \dfrac{1}{s^3}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} + \dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{2}{s^3}\right\}$

$= 1 + \dfrac{1}{2}x^2$

$\Rightarrow \begin{cases} y(x) = -\dfrac{1}{2}x^2 \\ z(x) = 1 + \dfrac{x^2}{2} \end{cases}$

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