1. $\int x \tan^{-1} x \, dx$

$$= (\tan^{-1} x)\,\frac{x^2}{2} - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2}\, dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{x^2+1-1}{x^2+1}\, dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int 1 - \frac{1}{x^2+1}\, dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}[x - \tan^{-1} x] + C.$$

2. $\int x \ln(1+x)\, dx$

$$= \ln(x+1) \cdot \frac{x^2}{2} - \int \frac{1}{x+1} \cdot \frac{x^2}{2}\, dx$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int \frac{x^2}{x+1}\, dx$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int \frac{x^2-1+1}{x+1}\, dx$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int \frac{x^2-1}{x+1} + \frac{1}{x+1}\, dx$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\left[\int\left((x-1) + \frac{1}{x+1}\right) dx\right]$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\left[\frac{x^2}{2} - x + \ln|x+1|\right] + C$$

3. $\int (p^3 + 6p)\sin p\, dp$

$$= -(p^3+6p)\cos p + \int (3p^2+6)\cos p\, dp$$

$$= -(p^3+6p)\cos p + (3p^2+6)\sin p - \int 6p \sin p\, dp$$

$$= -(p^3+6p)\cos p + (3p^2+6)\sin p$$

$$\left[-p\cos p + \int \cos p\, dp\right]$$

$$= -(p^3+6p)\cos p + (3p^2+6)\sin p + 6p\cos p$$

$$-6\sin p + C$$

$$= 3p^2 \sin p - (p^3+6p)\cos p + 6p\cos p + C$$

$$= 3p^2 \sin p - p^3 \cos p + C$$

$$= (3\sin p - p\cos p)\, p^2 + C.$$

4. $\int \sqrt[3]{\tan x}\, dx$

$$I = \int (\tan x)^{1/3}\, dx$$

$$\tan x = t^3 \Rightarrow \sec^2 x\, dx = 3t^2\, dt$$

$$= \int \frac{3t^3\, dt}{1+t^6}$$

$$t^2 = y \Rightarrow 2t\, dt = dy$$

$$= \frac{3}{2}\int \frac{y}{1+y^3}\, dy$$

$$I_1 = \int \frac{y+1-1}{y^3+1}\, dy$$

$$= \int \frac{dy}{y^2-y+1} - \int \frac{dy}{y^3+1}$$

$$= \int \frac{dy}{\left(y-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \int \frac{y^2-(y^2-1)}{y^3+1}\, dy$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2y-1}{\sqrt{3}}\right) - \frac{1}{3}\ln(y^3+1)$$

$$+ \int \frac{y-1}{y^2-y+1}.$$

$$\int \frac{y-1}{y^2-y+1}\,dy = \frac{1}{2}\int \frac{(2y-1)-1}{y^2-y+1}$$

$$= \frac{1}{2}\ln(y^2-y+1) - \frac{1}{2}\int \frac{dy}{y^2-y+1}$$

$$\Rightarrow I_1 = \frac{1}{\sqrt{3}}\tan^{-1}\frac{2y-1}{\sqrt{3}} - \frac{1}{3}\ln(y^3+1)$$

$$+\frac{1}{2}\ln(y^2-y+1)+C.$$

5. $\int \frac{1}{(e^x-1)^2}\,dx$

6. $\int \frac{\tan^{-1}x}{x^4}\,dx.$

$$I = \int \frac{\tan^{-1}}{x^4}dx = \int \tan^{-1}x.\frac{1}{x^4}dx$$

$$= (\tan^{-1}x)\left(-\frac{1}{3x^2}\right) - \int \frac{1}{1+x^2}\cdot\frac{1}{(-3x^3)}dx$$

$$= -\frac{\tan^{-1}x}{3x^3} + \frac{1}{3}\int \frac{dx}{x^3(1+x^2)},$$

$1+x^2=t$

$2x\,dx=dt$

$$= -\frac{\tan^{-1}x}{3x^3} + \frac{1}{6}\int \frac{dt}{(t-1)^2.t}$$

$$I = -\frac{\tan^{-1}x}{3x^3} + \frac{1}{6}I_1 \qquad ...(1)$$

$$I_1=\int\frac{1}{(1-t)^2.t}dt=\int\left\{\frac{A}{t-1}+\frac{B}{(t-1)^2}+\frac{C}{t}\right\}dt$$

$A=-1, B=1, C=1$

$$I_1=\int\left\{-\frac{1}{(t-1)}+\frac{1}{(t-1)^2}+\frac{1}{t}\right\}dt$$

$$=-\ln|t-1|-\frac{1}{(t-1)}+\ln|t| \qquad ...(2)$$

(1) ve (2) den

$$I=\frac{\tan^{-1}x}{3x^3}+\frac{1}{6}\left\{-\ln x^2-\frac{1}{x^2}+\ln(1+x^2)\right\}+C$$

$$I=-\frac{\tan^{-1}x}{3x^3}+\frac{1}{6}\ln\left|\frac{x^2+1}{x^2}\right|-\frac{1}{6x^2}+C.$$

7. $$\int\frac{x^3-1}{4x^3-x}\,dx$$

$$\int\frac{2x^2+41x-91}{(x-1)(x+3)(x-4)}dx$$

8. (i) $$\int\frac{x}{(x-1)(x^2+4)}dx$$

(ii) $$\int\frac{x^3dx}{x^4+3x^2+2}$$ (iii) $$\int\frac{x^3-1}{x^3+x}dx$$

(iv) $$\int\frac{x^4-2x^3+3x^2-x+3}{x^3-2x^2+3x}dx$$

9. $$\int\frac{dx}{x^2\sqrt{(x+1)}}.$$

$(x+1)=t^2$

$dx=2t\,dt$

$x=t^2-1$

$$\int\frac{dx}{x^2\sqrt{(x+1)}}=\int\frac{2t\,dt}{(t^2-1)^2.t}=2\int\frac{dt}{(t+1)^2(t-1)^2}$$

$$=\int\frac{1}{2}\left[\frac{1}{(t+1)^2}+\frac{1}{(t+1)}+\frac{1}{(t-1)^2}-\frac{1}{(t-1)}\right]dt,$$

$$=\frac{1}{2}\int\frac{dt}{(t+1)^2}+\frac{1}{2}\int\frac{dt}{(t+1)}+\frac{1}{2}\int\frac{dt}{(t-1)^2}-\frac{1}{2}\int\frac{dt}{(t-1)}$$

$$=-\frac{1}{2(t+1)}+\frac{1}{2}\ln|t+1|-\frac{1}{2(t-1)}-\frac{1}{2}\ln|t-1|+C$$

$$=-\frac{1}{2}\left[\frac{1}{(t+1)}+\frac{1}{(t-1)}\right]+\frac{1}{2}\ln\left|\frac{t+1}{t-1}\right|+C$$

$$=-\frac{\sqrt{(x+1)}}{x}+\frac{1}{2}\ln\left|\frac{\sqrt{(x+1)}+1}{\sqrt{(x+1)}-1}\right|+C.$$

10.

$$\int\sqrt{\left(\frac{x+1}{x+2}\right)}\frac{dx}{x+3}.$$

$$\frac{x+1}{x+2}=z^2 \quad\Rightarrow\quad x=\frac{2z^2-1}{1-z^2}=-2+\frac{1}{1-z^2}$$

$$x+3=\frac{2-z^2}{1-z^2}$$

$$dx=\frac{2zdz}{(1-z^2)^2}$$

$$I=\int z.\frac{2zdz}{(1-z^2)^2}\frac{(1-z^2)}{(2-z^2)}=\int\frac{2z^2dz}{(2-z^2)(1-z^2)}$$

$$=\int\left(\frac{2}{1-z^2}-\frac{4}{2-z^2}\right)dz$$

$$=\ln\left|\frac{1+z}{1-z}\right|-\sqrt{2}\ln\left|\frac{\sqrt{2}+z}{\sqrt{2}-z}\right|+C,$$

11.

$$\int\frac{x^4}{x^2+1}dx$$

$$\int\frac{x^4}{x^2+1}dx=\int\frac{x^4-1+1}{x^2+1}\,dx$$

$$=\int\frac{x^4-1}{x^2+1}+\frac{1}{x^2+1}\,dx$$

$$=\int(x^2-1)\,dx+\int\frac{1}{x^2+1}\,dx$$

$$=\frac{x^3}{3}-x+\tan^{-1}x+C$$

12. $\displaystyle\int \frac{x^2+3}{x^6(x^2+1)}\,dx$

$$\int \frac{(x^2+1)+2}{x^6(x^2+1)}\,dx = \int \frac{1}{x^6}\,dx + \int \frac{2}{x^6(x^2+1)}\,dx$$

$$= \int \frac{1}{x^6}\,dx + 2\int \frac{(x^6+1)-x^6}{x^6(x^2+1)}\,dx$$

$$= \int \frac{1}{x^6}\,dx + 2\int \frac{x^4-x^2+1}{x^6}\,dx - 2\int \frac{1}{x^2+1}\,dx$$

$$= \int \frac{1}{x^6}\,dx + 2\int \left(\frac{1}{x^2}-\frac{1}{x^4}+\frac{1}{x^6}\right)dx - 2\int \frac{1}{1+x^2}\,dx$$

$$= \frac{-1}{5x^5} + 2\left(-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}\right) - 2\tan^{-1}x + C.$$