**CC9 – Discrete Structures**
**Mathematical Induction**

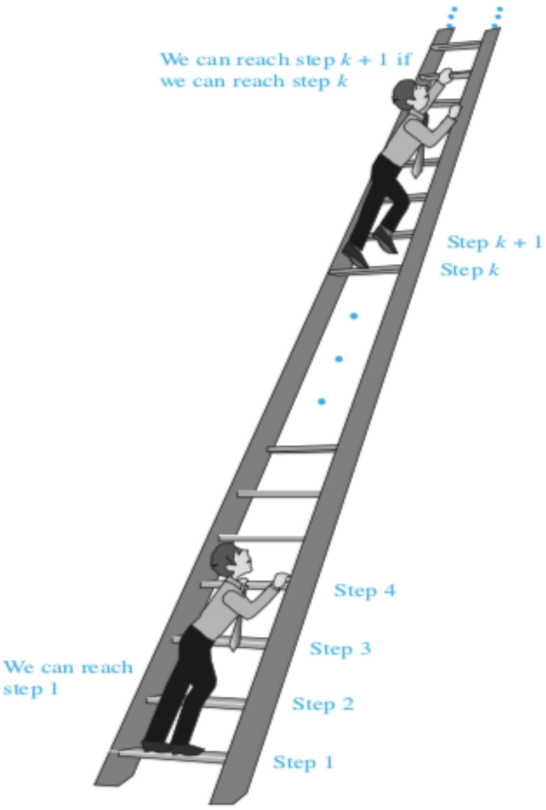
# Lesson

## INTRODUCTION

Suppose that we have an infinite ladder, as shown in the given figure, and we want to know whether we can reach every step on this ladder. We know two things;

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

**Figure: Climbing the infinite ladder**

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Can we conclude that we can reach every rung? By (1), we know that we can reach the first rung of the ladder. Moreover, because we can reach the first rung, by (2), we can also reach the second rung; it is the next rung after the first rung. Applying (2) again, because we can reach the second rung, we can also reach the third rung. Continuing in this way, we can show that we can reach the fourth rung, the fifth rung, and so on. If after 100 uses of (2) , that we can reach the $101^{st}$ rung, then we can conclude that we are able to reach every rung in the infinite ladder. We can verify this using an important proof technique called **mathematical induction**. That is, we can show that *P(n)* is true for every positive integer *n*, where *P(n)* is the statement that can reach the *nth* rung of the ladder.

Mathematical induction is an extremely important proof technique that can be used to prove assertions. It is extensively used to prove results about a large variety of discrete objects. For example, it used to prove results about the complexity of algorithms, the correctness of certain types of computer programs, theorems about graphs and trees, as well as a wide range of identities and inequalities.

## THE SUMMATION NOTATION

$$\sum_{i=m}^{n} \rightarrow sigma$$

summation notation is used to express the sum of terms.

i – index of summation

m – lower limit

n – upper limit

Examples/Exercises: Evaluate the ff. expressions.

1. $\sum_{i=0}^{5} i = 0 + 1 + 2 + 3 + 4 + 5$

   $\boxed{= \mathbf{15}}$

2. $\sum_{i=2}^{4} (i^2 + 2)$

   $= (2^2 + 2) + (3^2 + 2) + (4^2 + 2)$

   $= (4 + 2) + (9 + 2) + (16 + 2)$

   $= 6 + 11 + 18$

   $\boxed{= \mathbf{35}}$

3. $\sum_{i=1}^{5} 3i^2$ $= 3(1)^2 + 3(2)^2 + 3(3)^2 + 3(4)^2 + 3(5)^2$

$= 3 + 12 + 27 + 48 + 75$

$\boxed{= \mathbf{165}}$

4. $\sum_{i=1}^{n} 3i$ $= 3(1) + 3(2) + 3(3) + 3(4) + \ldots + 3(n)$

$\boxed{\mathbf{= 3 + 6 + 9 + 12 + \ldots + 3n}}$

## MATHEMATICAL INDUCTION

Mathematical induction can be used to prove statements that assert that p(n) is true for all positive integers n, where P(n) is a propositional function. A proof by mathematical induction has two parts, a **basis step**, where we show that *P(1)* is true, and an **inductive step**, where we show that for all positive integers *k*, if *P(k)* is true, then *P(k+1)* is true.

### PRINCIPLE OF MATHEMATICAL INDUCTION

To prove that *P(n)* is true for all positive integers *n*, where *P(n)* is a propositional function, we complete two steps:

***BASIS STEP:*** We verify that *P(1)* is true.

***INDUCTIVE STEP:*** We show that the conditional statement *P(k)* $\rightarrow$ *P(k+1)* is true for all positive integers *k*.

To complete the inductive step of a proof using the principle of mathematical induction, we assume that *P(k)* is true for an arbitrary positive integer *k* and show that under this assumption, *P(k+1)* must also be true. The assumption that *P(k)* is true is called the **inductive hypothesis**.

Once we complete both steps in a proof by mathematical induction, we have shown that P(n) is true for all positive integers, that is, we have shown that $\forall n P(n)$ is true wher the quantification is over the set of positive integers.

In the inductive step, we show that $\forall k(P(K) \rightarrow P(k+1))$ is true, where again, the domain is the set of positive integers.

Expressed as a rule of inference, this proof technique can be stated as

$$(P(1) \wedge \forall k(P(K) \rightarrow P(k+1))) \rightarrow \forall nP(n),$$

when the domain is the set of positive integers.

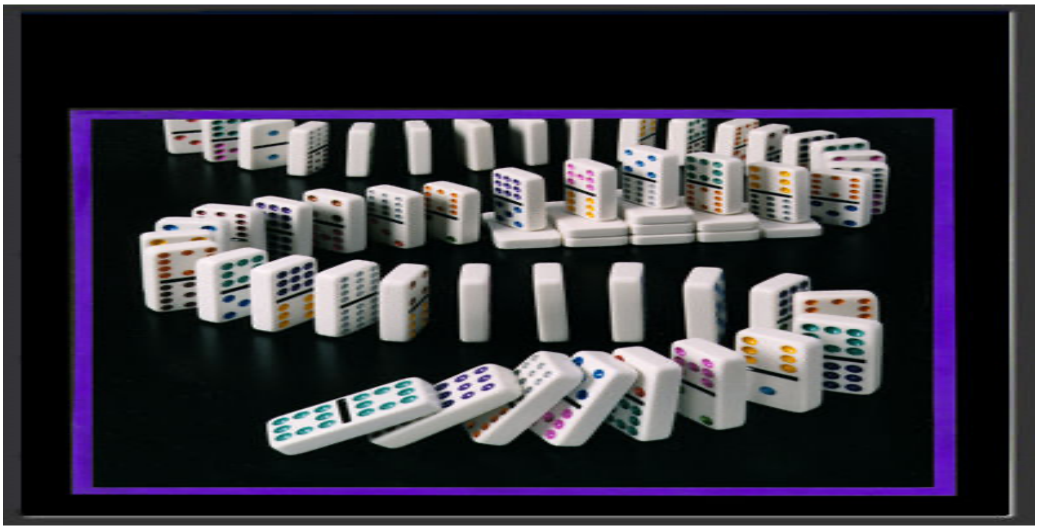
**ILLUSTRATION**

Consider the the following propostion over the positive integers, which we will label P(n).

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

For example, P(5): the sum of the positive integers from 1 to 5 is $\frac{5(5+1)}{2}$. Indeed, 1 + 2 + 3 + 4 + 5 = $\frac{5(5+1)}{2}$, which is equal to 15. Unfortunately, this doesn't serve as a proof that P(n) is a tautology. All that we've established is that 5 is in the set of P. Since the postive integers are infinite, we certainly can't use this approach to prove the formula.

**An Analogy:** Mathematical induction is often useful in overcoming a problem such as this. A proof by mathematical induction is similar to the knocking over a row of closely spaced dominions that are standing on an end. To knock over the five dominoes, all you need to do is push Domino 1 to the right. To be assured that they all will be knocked over, some work must be done ahead of time. The dominoes must be positioned so that if any domino is pushed to the right, it will push the next domino in the line.



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Now imagine the propositions *P(1), P(2), P(3)*, … to be an infinte line of dominoes. Let's see if the propositions are in the same formation as the dominoes were. First, we will focus on one specific point on the line: P(99) and P(100). We are not going to prove that either of these propositions is true, just that the truth of P(99) implies the truth of P(100). In terms of our analogy, if P(99) is knocked over, it will knock over P(100).

## EXAMPLES OF PROOFS BY MATHEMATICAL INDUCTION

1. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

**Solution:**

**Basis Step:** P(1)

$$\sum_{n=1}^{1} i = \frac{n(n+1)}{2}$$

$$1 = \frac{1(1+1)}{2}$$

$$1 = \frac{1(2)}{2}$$

$$1 = 1$$

$\therefore$ True

**Inductive Hypothesis**: P(k)

$$\sum_{n=1}^{k} i = \frac{n(n+1)}{2}$$

$$1 + 2 + 3 + 4 + \ldots + k = \frac{k(k+1)}{2}$$

**Inductive Step**: P(k + 1)

$$\sum_{n=1}^{k+1} i = \frac{n(n+1)}{2}$$

$$1 + 2 + 3 + 4 + \ldots + k + (k + 1) = \frac{(k+1)(k+1+1)}{2}$$

$$\frac{k(k+1)}{2} + (k + 1) = \frac{(k+1)(k+2)}{2}$$

$\frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$ *(simplify the left side by adding the fractions)*

$$\frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)}{2}$$ *(factor the left side)*

$\therefore$ Valid

2. $\sum_{i=1}^{n}(2i-1) = n^2$

**Solution:**

**Basis Step:** P(1)

$$\sum_{i=1}^{1} 2i - 1 = n^2$$

[2(1) – 1] = $(1)^2$

2 – 1 = 1

1 = 1

$\therefore$ True

**Inductive Hypothesis**: P(k)

$$\sum_{n=1}^{k} 2i - 1 = n^2$$

[2(1) – 1] + [2(2) – 1] + [2(3) – 1] + [2(4) – 1] +…+ [2(k) – 1] = $k^2$

1 + 3 + 5 + 7 +…+ (2k – 1) = $k^2$

**Inductive Step**: P(k + 1)

$$\sum_{n=1}^{k+1} 2i - 1 = n^2$$

1 + 3 + 5 + 7 +…+ (2k – 1) + [2(k + 1) – 1] = $(k + 1)^2$

$k^2$ + (2k + 1) = $(k + 1)^2$

$k^2$ + 2k + 1 = $(k + 1)^2$

$(k + 1)^2 = (k + 1)^2$

$\therefore$ Valid

3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

**Solution:**

**Basis Step:** P(1)

$$\sum_{i=1}^{1} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$(1)^2 = \frac{1(1+1)[2(1)+1]}{6}$

$(1)^2 = \frac{1(2)(3)}{6}$

1 = 1

∴ True

**Inductive Hypothesis**: P(k)

$$\sum_{i=1}^{k} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$(1)^2 + (2)^2 + (3)^2 + (4)^2 + \ldots + (k)^2 = \frac{k(k+1)(2k+1)}{6}$

$1 + 4 + 9 + 16 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$

**Inductive Step:** P(k + 1)

$$\sum_{i=1}^{k+1} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$1 + 4 + 9 + 16 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)(k+1+1)[2(k+1)+1]}{6}$

$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

$\frac{k(k+1)(2k+1)+ 6(k+1)^2}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$

$\frac{(k+1)[k\,(2k+1)+6(k+1)]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$

$\frac{(k+1)[2k^2 +k+6k+6]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$

$\frac{(k+1)(\,2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$

$$\frac{(k+1)(2k+3)(k+2)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$\therefore$ Valid

Video Lessons:

1. Proof by Induction (9 minutes)
2. Mathematical Induction Practice Problems (18 minutes)
3. Mathematical Induction (14 minutes)

Supplementary Readings:

Principle of Mathematical Induction

Mathematical Induction

Mathematical Induction

## References


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3. Mathematical Induction. https://www.tutorialspoint.com/discrete_mathematics/discrete_mathematical_induction.htm
4. Principle of Mathematical Induction. http://ncert.nic.in/ncerts/l/keep204.pdf
5. Mathematical Induction. https://web.stanford.edu/class/archive/cs/cs103/cs103.1126/lectures/03/Slides03.pdf
6. Mathematical Induction Practice Problems. https://www.youtube.com/watch?v=tHNVX3e9zd0
7. Mathematical Induction. https://www.youtube.com/watch?v=Tm2PJPvAULs
8. Proof by Induction. https://www.youtube.com/watch?v=wblW_M_HVQ8



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**CC9 – Discrete Structures**
**Introduction to Set Theory**

# Lesson

## INTRODUCTION

The term set is intuitively understood by most people to mean a collection of objects that are called elements (of the set). Sets are used to group objects together. Often, but not always, the objects in a set have similar properties. For instance, all students who are currently enrolled in UC n] make up a set. Likewise, all students taking a course in discrete mathematics at any school make up a set. In addition, those students enrolled in your school who are taking a course in discrete mathematics form a set that can be be obtained by taking the elements common to the first two collections. This concept of sets is the starting point on which we will build more complex ideas, much as in-geometry where the concepts of a point and line are left undefined.

Because a set is such a simple notion, we maybe surprised to learn that it is one of the most difficult concepts for mathematicians to define. For example, the description above is not a proper defintion because it requires the definition of collection. (How would you define "collection"?). Even deeper problems arise when you consider the possibility that a set could contain itself. Although these problems are real concerns to some mathematicians, they will not be any concern to us.

## LESSON PROPER

**Definition:** A ***set*** is an unordered collection of well-defined ojects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A. (Order of elements does not matter and repetitions is ignored).

It is common for sets to be denoted using **uppercase letters**. **Lowercase letters** are usually used to denote elements of sets. The elements of the set are enclosed within barces **{ }** (curly brackets).

*Examples of a set:* A = {1, 2, 3, 4, 5},
B = {a, b, c, d, e, f, …, z}

More examples: V = set of vowels of the English alphabet
O = set of odd numbers less than 10
E = set of even numbers less than 100
N = set of natural numbers
X = set of provinces of Region 1

There are several ways to describe a set. One way is to list all the members of a set, when this is possible. This way of describing a set is known as the **roster method.**

*Example 1*. V = set of vowels of the English alphabet

**V = {a, e, i, o, u}**

*Example 2*. O = set of odd numbers less than 10

**O = {1, 3, 5, 7, 9}**

*Example 3*. N = set of natural numbers

**N = {1, 2, 3, 4, 5, …}**

*Example 4*. X = set of provinces of Region 1

**X = {Ilocos Sur, Ilocos Norte, La Union, Pangasinan}**

*Example 5*. E = set of even numbers less than 100

**E = {2, 4, 6, 8, 10, … 98}**

Another way to describe a set is to use **set builder** notation. We characterize all those elements in the set by stating the property or properties they must have to be members.

*Example 1*. V = set of vowels of the English alphabet

**V = {x|x is a vowel of the English alphabet}**

*Example 2.* O = set of odd numbers less than 10

**O = {x|x is an odd number less than 10}**

*Example 3.* N = set of natural numbers

**N = {x|x is a natural number}**

*Example 4.* X = set of provinces of Region 1

**X = {x|x is a province of Region 1}**

*Example 5.* E = set of even numbers less than 100

**E = {x|x is an even number less than 100}**

### Important sets in mathematics

**N** = {1, 2, 3, 4, 5, …}, the set of **natural numbers**
**Z** = {…, –3, –2, –1, 0, 1, 2, 3,…}, the set of **integers**
**Z⁺** = {1, 2, 3, 4, …}, the set of **positive integers**
**Q** = {a/b | a ∈ Z, q ∈, and b ≠ 0 }, the set of **rational numbers**
**R** , the set of **real numbers**
**R⁺**, the set of **positive real numbers**
**C**, the set of **complex numbers**

### Notation for intervals of real numbers

When *a* and *b* are real numbers with *a* < *b*, we write

$[a, b] = \{x \mid a \leq x \leq b\}$
$[a, b) = \{x \mid a \leq x < b\}$
$(a, b] = \{x \mid a < x \leq b\}$
$(a, b) = \{x \mid a < x < b\}$

** Note that [a, b] is called the **closed interval** from a to b and (a, b) is called the **open interval** from a to b.*

**Definition**: **Finite Set**. A set is finite set if it has a finite number of elements. Any set that is not finite is an **infinite set**.

**Definition**: **Empty Set**, or **Null Set**. It is a set that has no elements. It is denoted by ∅. The empty set can also be denoted by { } (that is, we represent the empty set with a pair of braces that encloses all the elements in this set). Example is the set of natural numbers between 1 and 2. A set whose element is the null set {∅} itself is called a **singleton set**.

## SET RELATIONS

**Definition:** Two sets are **equal** if and only if they have the same elements. We write A = B if each element of A is also an element of B.

*Example*: A = {a, b, c, d}
B = {b, d, a, c}
∴ A = B

**Definition**: Two sets are **equivalent** if their elements can be placed in a one-to-one correspondence. We write $A \leftrightarrow B$.

*Example*: A = {a, b, c, d, e, f}
B = {2, 3, 4, 5, 6, 7}
∴ $A \leftrightarrow B$

**Definition**: Let a be a finite set. The number of different elements in A is called its **cardinality** and is denoted by $|A|$.

*Example*: Let S be the set of letters of the English alphabet. Then $|S| = 26$

*Example*: Let A be the set of even numbers less than 10. Then $|A| = 4$

### SUBSETS

**Definition**: The set A is a **subset** of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

Showing that A is a subset of B  To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B.

Showing that A is  Not a subset of B  To that $A \not\subseteq B$, find a sinle $x \notin A$ such that $x \notin B$.

*Example*: The set of all odd positive integers less than 10 is a subset of the set of all positive integers less than 10.

## Proper Subset

**Definition**: If each element of A is an element of B, then A is a subset of B. If A is a subset of B, and if B has one or more elements not belonging to A, then A is a proper subset of B. We write $A \subset B$.
*Example*: The ff are some subsets of the set {a, b, c, d}

$\{a, b, c\} \subset \{a, b, c, d\}$
$\{a\} \subset \{a, b, c, d\}$
$\{b, c, d\} \subset \{a, b, c, d\}$
$\{a, c\} \subset \{a, b, c, d\}$
$\{a, b, c, d\} \subseteq \{a, b, c, d\}$

**Theorem**: For every set S (i) $\varnothing \subseteq S$ and (ii) $S \subseteq S$.

***Proof***: We will prove (i) and leave the proof of (ii) as an exercise.
Let S be a set. To show that $\varnothing \subseteq S$, we must show that $\forall x(x \in \varnothing \rightarrow x \in S)$ is true. Because the empty set contains no elements, it follows that $x \in \varnothing$ is always false. It follows that the conditional statement $x \in \varnothing \rightarrow x \in S$ is always true because its hypothesis is always false and a conditional statement with a false hypothesis is true. Therefore, $\forall x(x \in \varnothing \rightarrow x \in S)$ is true. Note that this is an example of a vacouos proof.

## POWER SETS

**Definition**: Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

*Example*: What is the power set of the set {0, 1, 2}?
$\mathcal{P}(\{0, 1, 2\}) = \{\varnothing, \{0\}, \{1\}, \{2\}, \{3\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$
Note: Empty set and the set itself are members of this set of subsets.

*Example*: What is the power set of the empty set?
$\mathcal{P}(\varnothing) = \{\varnothing\}$

Example: What is the power set of the set $\{\varnothing\}$?
$\mathcal{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$

**Note**: If a set has $n$ elements, then its power set has $2^n$ elements.

# SET OPERATIONS

## INTRODUCTION

Two or more sets can be combined inmany different ways. For instance, the set of students enrolled in Discrete Mathematics this trimester and the set of students enrolled in Systems Analysis and Design, we can form the set of students enrolled in Discrete Mathematics or Systems Analysis and Design, the set of students enrolled in Discrete Mathematics and Systems Analysis and Design, and so on.

**Definition**: Let A and B be sets. The ***union*** of the sets A and B denoted by $A \cup B$ is the set of all elements which belongs to A or B or both A and B.

$$A \cup B = \{x \mid x \in A \ \vee \ x \in B\}$$

*Example 1:* $A = \{a, b, c, d, e\}$

$B = \{a, e, i, o\}$

$\therefore A \cup B = \{a, b, c, d, e, i, o\}$

*Example 2:* $C = \{1, 2, 3, 4\}$

$D = \{w, x, y, z\}$

$\therefore C \cup D = \{1, 2, 3, 4, w, x, y, z\}$

**Definition:** Let A and B be sets. The ***intersection*** of the sets A and B, denoted by $A \cap B$ is the set of all elements that belongs to both A and to B.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

*Example 1*: $A = \{a, b, c, d, e\}$

$B = \{a, e, i, o\}$

$\therefore A \cap B = \{a, e\}$

*Example 2*: $C = \{1, 2, 3, 4\}$

$D = \{w, x, y, z\}$

$\therefore C \cap D = \{\ \}$ or $\varnothing$

**Definition**: Two sets are called ***disjoint*** if their intersection is the empty set.

**Definition:** Let A and B be sets. The **difference** of A and B denoted by A – B, is the set of all elements in A but not in B. The difference of A and B is the ***relative complement*** of B with respect to A. It is also called ***set difference***.

$A - B = \{x \mid x \in A \wedge x \notin B\}$

*Examples*: Given: Let: A = {1, 2, 3, 4}

B = {2, 4, 5, 6}

C = {3, 7, 9}

Find:

1. A – B = {1, 3}
2. B – A = {5, 6}
3. A – C = {1, 2, 4}
4. B – C = {2, 4, 5, 6}

*Note: Set difference is not commutative.*

**Definition**: Let A and B be sets. The ***symmetric difference*** of A and B written as A Δ B, is the set of all elements found in A or in B, but not in both A or B.

$A \Delta B = \{x \mid (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$

To find elements: $A \Delta B = \{A - B\} \cup \{B - A\}$ or,
$A \Delta B = \{A \cup B\} - \{A \cap B\}$

*Examples:(and exercises)*

Given; Let: A = {1, 3, 5, 7}

B = {1, 2, 4, 6, 8}

C = {1, 2, 3, 4, 5}

D = {2, 3, 5, 7}

E = {1, 2, 3, 4, 5, 6, 7, 8, 9}

Find the elements of the following sets:

1. $A \Delta B = \{A - B\} \cup \{B - A\}$
$= \{3, 5, 7\} \cup \{2, 4, 6, 8\}$
$= \{2, 3, 4, 5, 6, 7, 8\}$

2. $C \Delta D = \{C - D\} \cup \{D - C\}$
$= \{1, 4\} \cup \{7\}$
$= \{1, 4, 7\}$
3. $D \Delta A =$
4. $A \cap C =$
5. $B \cup C =$
6. $D - E =$
7. $(A - B) \cap (C - D) =$
8. $(A \Delta C) \cup (B \Delta E) =$
9. $(B \cap C) - (D \cap E) =$
10. $(A \cap B) \cap C =$

**Definition**: Let U be the universal set. The ***absolute complement*** of the set A, denoted by **A'**, is the complement of A with respect to U. Therefore, the complement of the set A is U – A.

$A' = \{x \mid x \in U \wedge x \notin A\}$

*Examples: (and exersices)*

Given: Let: $U = \{a, b, c, d, e, f, g, h, i, j\}$
$A = \{a, e, i\}$
$B = \{b, c, d, f, g, h, j\}$
$C = \{a, d, g, j\}$
$D = \{c, e, f, j\}$
$E = \{d, e, h, j\}$

Find the elements of the following set.

1. $A' = \{b, c, d, f, g, h, j\}$
2. $B' = \{a, e, i\}$
3. $D' - C' = \{a, b, d, g, h, i\} - \{b, c, e, f, h, i\}$
$= \{a, d, g\}$
4. $A' \Delta B' =$
5. $E' \cap C' =$
6. $(B')' =$
7. $A \cup C' =$
8. $E' - D' =$
9. $U' =$
10. $C' \cap D' =$

**Definition:** Let A and B be sets. The ***Cartesian product*** of A and B, denoted by A x B, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

*Examples:*

Given: Let: A = {1, 2, 3}
B = {4, 5}

Find the elements of the following sets.

1. A x B = {(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)}
2. B x A = {(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)}

**Definition:** The *Cartesian product* of the sets $A_1$, $A_2$ …$A_n$, denoted by $A_1$ x $A_2$ x …x $A_n$, is the set of ordered n-tuples ($a_1$, $a_2$, …$a_n$), where $a_i$ belongs to $A_i$ for i = 1, 2, …,n. In other words,

$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$.

*Example:* What is the Cartesian product A x B x C, where A = {0, 1}, B = {1, 2}, and C= {0, 1, 2}?

A x B x C = ((0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2))

Video Lessons:
1. Introduction to Sets (17 minutes)
2. Set Theory - Introduction (30 minutes)
3. Introduction to Set Theory (7 minutes)

Supplementary Readings:
Set Theory
Basic Set Theory
Introduction to Elementary Set Theory

## References


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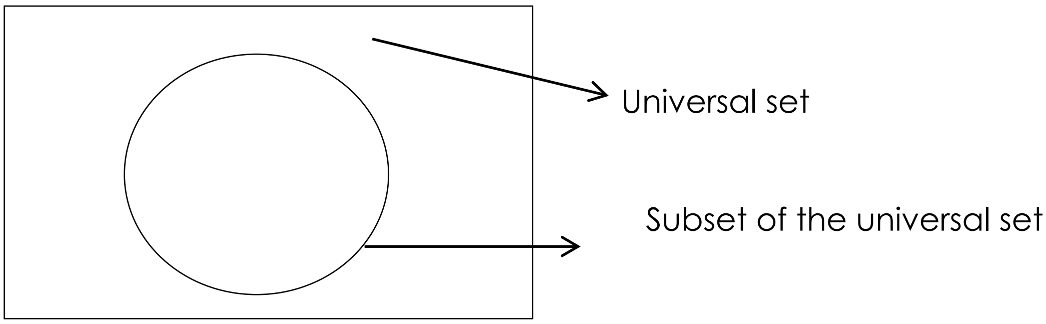
3. Set Theory. http://www1.spms.ntu.edu.sg/~frederique/dm4.pdf
4. Basic Set Theory. https://www.math.uh.edu/~dlabate/settheory_Ashlock.pdf
5. An introduction to elementary set theory. https://www.maa.org/sites/default/files/images/upload_library/46/Pengelley_projects/Project-5/set_theory_project.pdf
6. Introduction to sets. https://www.youtube.com/watch?v=tyDKR4FG3Yw
7. Set Theory – Introduction. https://www.youtube.com/watch?v=yCwnifwVjlg
8. Introduction to Set Theory. https://www.youtube.com/watch?v=vGelH3Jibt4
9. Russell's Paradox. https://www.youtube.com/watch?v=GpVRePLMLbU
10. Intuitionism in the Philosophy of Mathematics. https://plato.stanford.edu/entries/intuitionism/

## CC9 – Discrete Structures
## Venn Diagram and Set Identities (Laws of Operations on Sets)

# Lesson

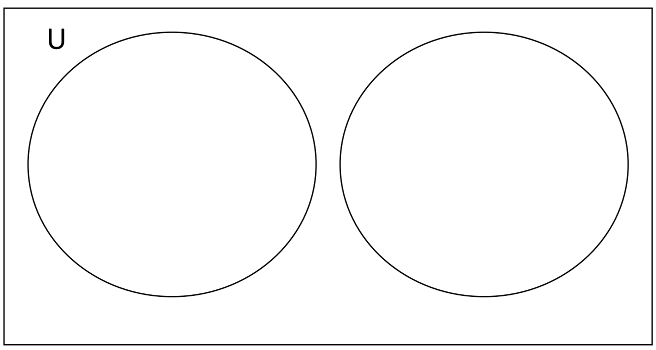
### VENN DIAGRAMS

When working with sets, as in other branches of mathematics, it is often quite useful to be able to draw a picture or diagrams of the situation under consideration. A diagram is called a Venn diagram. The universal set U is represented by the interior of a rectangle and the sets by circles inside the rectangle.

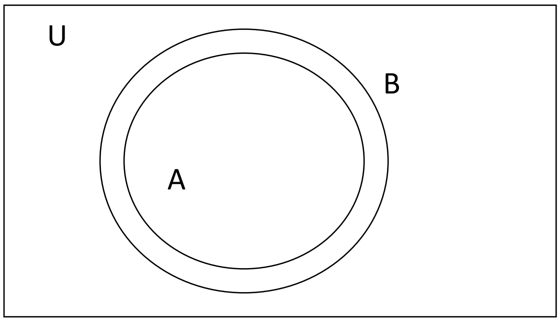



### ILLUSTRATIONS OF SET RELATIONS and OPERATIONS

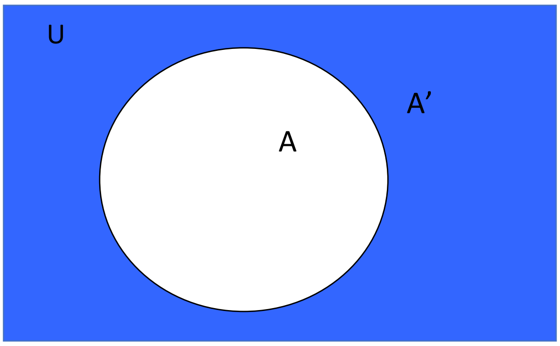
**Disjoint Set**



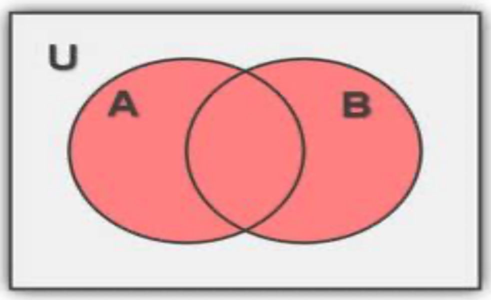
**Subsets (A ⊆ B)**



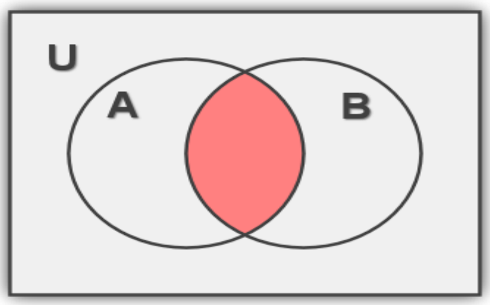

**Absolute Complement (A')**



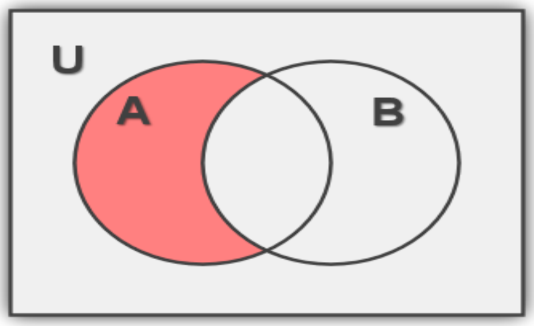

**Union of Sets (A ∪ B)**



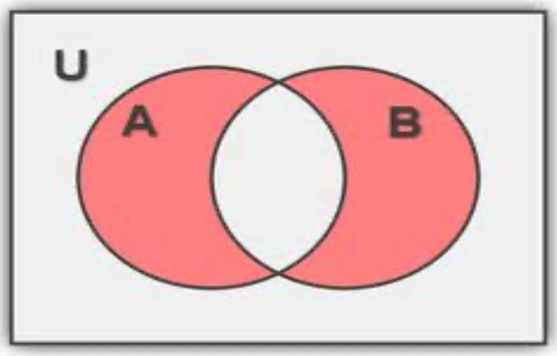

**Intersection of Sets (A ∩ B)**



**Relative Complement (A – B)**




**Symmetric Difference (A ∆ B)**

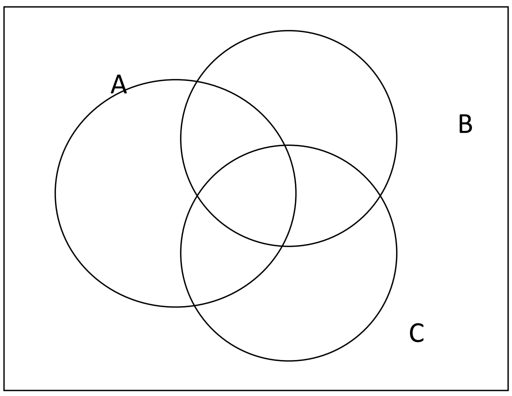



Examples and exercises:

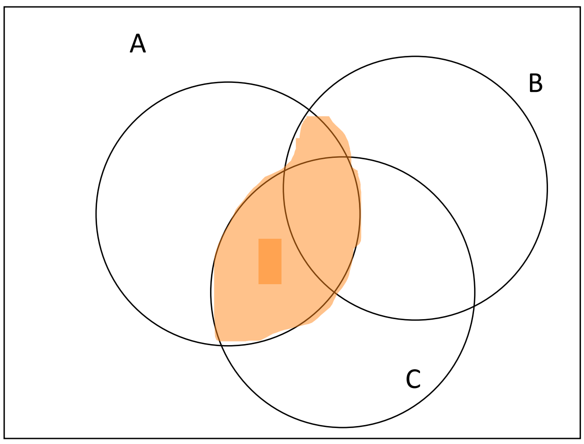
A. Using Venn diagram, shade the region that represents each of the ff:

1. **A ∩ (B ∪ C)**

Solution:




Answer:

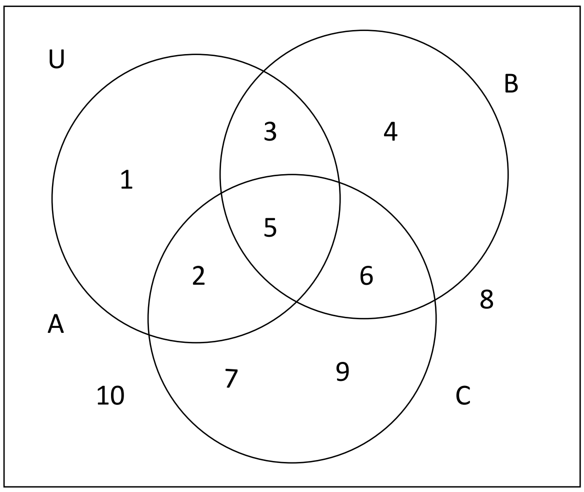


2. **(A ∩ B) ∩ C**
3. **(A ∪ B) ∩ C**
4. **(A ∩ C) ∪ (B ∩ C)**
5. **(A ∪ B) ∩ (B ∪ C)**

B. Using Venn diagram, place the ff. elements on their proper region.

Given: U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
A = {1, 2, 3, 5}
B = {3, 4, 5, 6}
C = {2, 5, 6, 7, 9}

Solution: Find;
A ∩ B = {3, 5}
A ∩ C = {2, 5}
B ∩ C = {5, 6}
A ∩ B ∩ C = {5}



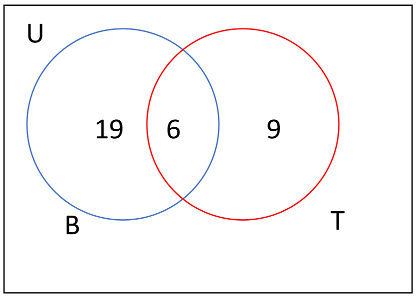

C. Application. Use Venn diagram to answer the ff. problem.

1. The number of boys on a certain baseball team is 25 and the track team is 15. Six boys are members of both teams. a) how many boys are members of at least one of these teams? b) how many are members of the track team only? c) how many are members of the baseball team only?

*Given*: Let B = {x | x is a boy in the baseball team}
T = {x | x is a boy in the track team}
$B_{\text{(total members)}} = 25$
$T_{\text{(total members)}} = 15$
B ∩ T = 6

Solution: B(only) = 25 – 6
= 19
T(only) = 15 – 6
= 9

Venn Diagram:




Answers to the questions:

a. how many boys are members of at least one of these teams? **34**

b. b) how many are members of the track team only? **9**

c. how many are members of the baseball team only? **19**

2. In a certain high school, the total enrollment in Chemistry is 60, in Biology 75, and in Physics 70. Six students are enrolled in 3 courses. 20 students are enrolled in Chemistry and Biology, 38 students are enrolled in Chemistry and Physics, and 29 students are enrolled in Biology and Physics. Using Venn diagram, determine exactly how many students are enrolled in a) Physics only? b) Chemistry only c) Biology only? D) Chemistry and Biology but not Physics? e) Chemistry or Biology but not Physics?

Given: Let C = {x | x is a student enrolled in Chemistry}
B = {x | x is a student enrolled in Biology}
P = {x | x is a student enrolled in Physics}
$C_{(\text{total enrollees})} = 60$
$B_{(\text{total enrollees})} = 75$
$P_{(\text{total enrollees})} = 70$
$C \cap B \cap P = 6$
$C \cap B_{(\text{total})} = 20$
$C \cap P_{(\text{total})} = 38$
$B \cap P_{(\text{total})} = 29$

Solution: $C \cap B_{(\text{onlyl})} = 20 - 6$
$= 14$

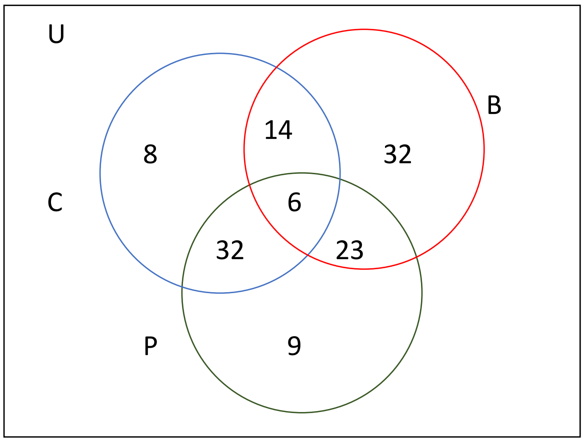
$C \cap P_{(\text{onlyl})} = 38 - 6$
$= 32$

$B \cap P_{(\text{total})} = 29 - 6$
$= 23$

$C_{(\text{only})} = 60 - (14 + 32 + 6)$
$= 8$

$B_{(only)}$ = 75 – (14 + 6 + 23)
= 32
$P_{(only)}$ = 70 – (32 + 6 + 23)
= 9

Venn Diagram:




Answers:

How many students are enrolled in

a) Physics only? **9**
b) Chemistry only? **8**
c) Bilology only? **32**
d) Chemistry and Biology but not Physics? **14**
e) Chemistry or Biology but not Physics? **54**

## SET IDENTIES (Laws of Operations on Set)

The table below lists the most important set identities. Students should note the similarity between these set identities and the logical equivalences. In fact, the set identities given can be proved from the corresponding logical equivalences.

One way to show that two sets are equal is to show that each is a subset of the other. To show that one set is a subset of a second, we can show that if an element belongs to the first set, then it must also belong to the second set. We generally use a direct proof to do this.

## SET IDENTITIES (LAWS OF OPERATTIONS ON SETS)

| ***Identity*** | ***Name*** |
|---|---|
| $A \cup \varnothing = A$<br>$A \cap U = A$ | Identity Laws |
| $A \cup U = U$<br>$A \cap \varnothing = \varnothing$<br>$\varnothing \cap A' = \varnothing$<br>$U \cup A' = U$ | Domination Laws |
| $A \cup A = A$<br>$A \cap A = A$<br>$U \cup U = U$<br>$U \cap U = U$ | Idempotent Laws |
| $(A')' = A$ | Involution Law |
| $A \cup B = B \cup A$<br>$A \cap B = B \cap A$ | Communtative Laws |
| $(A \cup B) \cup C = A \cup (B \cup C)$<br>$(A \cap B) \cap C = A \cap (B \cap C)$ | Associative Laws |
| $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$<br>$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ | Distributive Laws |
| $(A \cap B)' = A' \cup B'$<br>$(A \cup B)' = A' \cap B'$ | De Morgan's Laws |
| $A \cap (A \cup B) = A$<br>$A \cup (A \cap B) = A$ | Absorption Laws |
| $A \cup A' = U$<br>$A \cap A' = \varnothing$<br>$U' = \varnothing$ | Complement Laws |

***Examples***: Prove the ff. using the laws of operations on sets.

1. $(A' \cap B)' \cup A = A \cup B'$

simplify left side:

| **Steps** | **Reason/Justification** |
|---|---|
| $[(A')' \cup B' \cup A = A \cup B'$ | De Morgan's Law |
| $(A \cup B') \cup A = A \cup B'$ | Involution Law |
| $A \cup (A \cup B') = A \cup B'$ | Commutative Law |
| $(A \cup A) \cup B' = A \cup B'$ | Associative Law |
| $A \cup B' = A \cup B'$ | Idempotent law |

$\therefore$ Valid

2. **$(A' \cap B) \cup (A' \cap B') = A'$**

simplify left side:

| **Steps** | **Reason/Justification** |
|---|---|
| **$A' \cap (B \cup B') = A'$** | Distributive Law |
| **$A' \cap U = A'$** | Complement Law |
| **$A' = A'$** | Identity Law |

$\therefore$ Valid

3. **$(A \cup B)' \cap A = \varnothing$**

simplify left side:

| **Steps** | **Reason/Justification** |
|---|---|
| **$(A' \cap B') \cap A = \varnothing$** | De Morgan's Law |
| **$A \cap (A' \cap B') = \varnothing$** | Commutative Law |
| **$(A \cap A') \cap B' = \varnothing$** | Associative Law |
| **$\varnothing \cap B' = \varnothing$** | Complement Law |
| **$\varnothing = \varnothing$** | Domination Law |

$\therefore$ Valid

Video Lessons:
1. Venn Diagrams – An Introduction (6 minutes)
2. Shading Venn Diagrams Region (14 minutes)
3. Solving Problems with Venn Diagrams (6 minutes)

Supplementary Readings:
Venn Diagrams
Venn Diagram
Venn Diagrams and Set Operations

## References


1. Rosen, Kenneth D (2007). Discrete mathematics and its applications. 6th ed. Boston : McGraw Hill Higher Education.
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3. Set Theory. http://www1.spms.ntu.edu.sg/~frederique/dm4.pdf
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9. Shading Venn Diagram Regions. https://www.youtube.com/watch?v=XidkM5J3OQU
10. Solving Problems with Venn Diagrams. https://www.youtube.com/watch?v=MassxXy8iko

**CC9 – Discrete Structures**
**Relations and Functions**

# Lesson

## INTRODUCTION

The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. For this reason, sets of ordered pairs are called binary relations. We will introduce the basic terminology used to describe binary relations.

## RELATIONS

***Definition***: Let A and B be sets. A binary relation from A to B is a subset of A x B.

- A (binary) relation *R* from a set A to a set B, is any subset of the Cartesian product of A and B. If an ordered pair (x, y) $\in$ R, we write x *R* y and say that x is related to y. If A = B, then R is a binary relation on X.

The set x

{x $\in$ A | (x, y) $\in$ *R* for some y $\in$ B} is called the *domain* of *R*.

The set y

{y $\in$ B | (x, y) $\in$ *R* for some x $\in$ A} is called the *range* of *R*.

- In other words, a binary relation from A to B is a set *R* of ordered pair where the first element of each ordered pair comes from A and the second element comes from B.

*Example* 1. A marketing research firm classifies a person according to the ff. two criteria.

Gender: male (m), female (f)
Highest educational attainment: elementary school (e), high school (h), college (c), graduate school (g)

Find the relation defined on gender (G) and highest educational attainment (L) .

***Solution***: G x L = {(male, elementary), (male, high school), (male, graduate school), (female, elementary school), (female, high school), (female, graduate school)}

*Example* 2. Let X = {2, 3, 4} and Y = {3, 4, 5, 6, 7}. Find *R* from x to y, (x, y) ∈ *R*, such that x divides y. Find also the domain and the range.

**Solution:** *R* = {(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)}
*Domain*: {2. 3, 4}
*Range*: (3, 4, 6}

*Example* 3. Let *R* be the relation on X = {1, 2, 3, 4} defined by (x, y) ∈ *R* such that x ≤ y, (x, y) ∈ X.

***Solution***: *R* = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}

*Example 4*. Let A = {1, 2, 3, 4, 5, 6}, find (x, y) ∈ *R* defined by
*R* = {(x, y) | x is a divisor of y}

***Solution***: *R* = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)}

## REPRESENTING RELATIONS

### Set Notation

- One way is to lists its ordered pairs, or use set builder notation.

*Example*: Let R be the relation on X = {1, 2, 3, 4} defined by (x, y) ∈ R such that x ≤ y, (x, y) ∈ X
*Solution*: *R* = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (4, 4)}

### Table

*Example*: Let R be the relation on X = {1, 2, 3, 4} defined by (x, y) ∈ R such that x ≤ y, (x, y) ∈ X

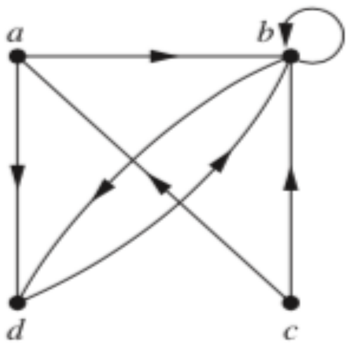
| *R* | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | x | x | x | x |
| 2 | | x | x | x |
| 3 | | | x | x |
| 4 | | | | x |

### Graphs

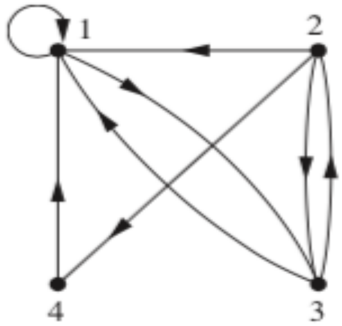
*Definition:* A directed graph, or digraph, consists of a set V of *vertices* ( or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex *a* is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.

Note: An edge of the form (a, a) is represented using an arc from the vertex a back to itself. Such an edge is called a loop.

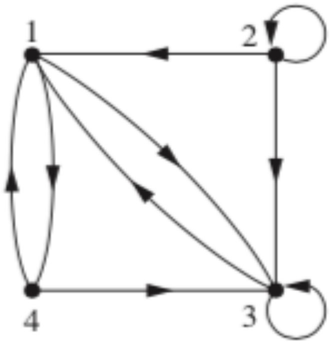
*Example* 1: The directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b).




*Example* 2. The directed graph of the relation
*R* = {(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)}




Example 3. What are the ordered pairs in the relation R represented by the directed graph below?




*R* = {(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)}

**Matrices**

The relation R can be represented by the matrix

$M_R = [m_{ij}]$

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R \\ 0 \text{ if } (a_i, b_j) \notin R \end{cases}$$

*Example* 1. $R$ = {(1, b), (1, d), (2, c), (3, c), (3, b), (4, a)} from X = {1, 2, 3, 4} to Y = {a, b, c, d}, what is the matrix representing $R$ if $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$ and $b_1 = b$, $b_2 = d$, $b_3 = c$, $b_4 = a$.

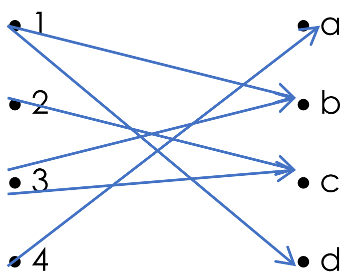
$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

*Example* 2: Find $R$ from A = {2, 3, 4} to B = {5, 6, 7, 8} relative to the orderings 2, 3, 4 and 5, 6, 7, 8 defined by (x, y) $\in$ R, if x divides y.

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Arrow Diagram**

Example: $R$ = {(1, b), (1, d), (2, c), (3, c), (3, b), (4, a)}




**Inverse of a Relation**

**Definition:** Let R be a relation from two non-empty sets X to Y. The Inverse of R denoted by $R^{-1}$ is the relation from Y to X defined by $R^{-1}$ = { (y, x ) | (x, y ) $\in$ $R$}

*Example* 1: Find $R^{-1}$

1. Let A = {2, 4, 6} and B = {5, 6, 7, 8, 9}. The relation defined on *R* is the set of ordered pairs

$R = \{(2, 6), (2, 7), (4, 6), (2, 5), (4, 9), (6, 9)\}$

The inverse of *R* is

$R^{-1} = \{(6, 2), (7, 2), (6, 4), (5, 2), (9, 4), (9, 6)\}$

*Example* 2: Find *R* from A = {2, 3, 4} to B = {5, 6, 7, 8} defined by $(x, y) \in R$, if x divides y.

$R = \{(2, 6), (2, 8), (3, 6), (4, 8)\}$

The inverse of *R* is

$R^{-1} = \{(6, 2), (8, 2), (6, 3), (8, 4)\}$

## PROPERTIES OF RELATIONS

There are several properties that are used to classify relations on a set. We will introduce the most important of these.

### Reflexive Property

*Definition*: A relation *R* on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

In other words a relation on A is reflexive if every element of A is related to itself.

*Example* 1. Let X = {1, 2, 3, 4}. The ff are some relations on X. Which of the relation is reflexive?

$R_1 = \{(1, 1), (1, 2), (2, 2), (3, 4), (4, 1), (4, 4)\}$

- not reflexive
- Not all elements of set X is related to itself

$R_2 = \{\underline{(1, 1)}, (1, 2), (1, 4), (2, 1), \underline{(2, 2)}, \underline{(3, 3)}, (4, 1), \underline{(4, 4)}\}$

- reflexive
- All elements of set X is related to itself.

*Example* 2. Let X = {1, 2, 3, 4}, defined by $(x, y) \in R$ if $x \leq y$, $(x, y) \in X$

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$

- reflexive
- All elements of set X is related to itself.

*Example* 3. Which of these relations are reflexive?

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
$R_2 = \{(1,1), (1,2), (2,1)\}$
$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$
$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$
$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$
$R_6 = \{(3,4)\}$

***Solution***: The relations $R_3$ and $R_5$ are reflexive since they both contain all pairs of the form (x, x), namely, (1, 1), (2, 2), (3, 3), and (4, 4).

**Symmetric Property**

*Definition:* A relation $R$ on a set A is called *symmetric* if (b, a) $\in R$ whenever (a, b) $\in R$, for all a, b $\in$ A.

In other words a relation $R$ on a set A is symmetric if for all a, b $\in$ A, if (a, b) $\in R$, then (b, a) $\in R$.

A relation is symmetric iff "a is related to b" implies that "b is related to a".

*Example* 1. Let A = {1, 2, 3, 4}, defined by (a, b) $\in$ R, if a divides b, (a, b) $\in$ A.

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

- It is not symmetric since $(1,2) \in R$ but $(2,1) \notin R$

*Example* 2. Which of these is symmetric?

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
$R_2 = \{(1,1), (\mathbf{1,2}), (\mathbf{2,1})\}$
$R_3 = \{(1,1), (\mathbf{1,2}), (\mathbf{1,4}), (\mathbf{2,1}), (2,2), (3,3), (\mathbf{4,1}), (4,4)\}$
$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$
$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$
$R_6 = \{(3,4)\}$

- The relations $R_2$ and $R_3$ are symmetric because in each case (y, x) belongs to the relation whenever (x, y) does.
- The other relations aren't symmetric.

**Antisymmetric Property**

*Definition:* A relation $R$ on a set A such that for all a, b $\in$ A, if (a, b) $\in$ R and (b, a) $\in R$, then a = b is called *antisymmetric*.

In other words a relation $R$ on a set A is antisymmetric if for all a, b $\in$ A, (a, b) $\in R$ and (b, a) $\notin R$ only if a = b.
A relation is antisymmetric iff there are no pairs of distinct elements with *a* related to *b* and *b* related to *a*. That is, the only way to have *a* related to *b* and *b* related to *a* is for *a* and *b* to be the same element.

- Symmetric and antisymmetric are NOT exactly opposites.

*Example* 1. Let A={1,2,3,4}, defined by (x, y) $\in$ R, such that x $\neq$ y, (x, y) $\in$ A.

R = {(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4), (4,1),(4,2),(4,3)}

- It is not antisymmetric since (1,2) ∈ R and (2,1) ∈ R but 1≠ 2.

*Example* 2. Which of these is antisymmetric?

$R_1$ = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
$R_2$ = {(1,1), (1,2), (2,1)}
$R_3$ = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)}
$R_4$ = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}
$R_5$ = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
$R_6$ = {(3,4)}

- The relations $R_4$, $R_5$ and $R_6$ are antisymmetric because there is no pair of elements x and *y* with x $\neq$ *y* such that both (*x,y*) and (*y,x*) belong to the relation.

**Transitive Property**
*Definition*: A relation $R$ on a set A is called *transitive* if whenever (a, b) $\in R$ and (b, c) $\in R$, then (a, c) $\in R$, for all a, b, c $\in$ A.

In other words a relation $R$ on a set A is transitive if for all a, b, c $\in$ A, if (a, b) and (b, c) $\in R$, then (a, c) $\in R$.

*Example* 1. Let A = {1, 2, 3, 4}, defined by (x, y) $\in$ R, if x divides y, (x, y) $\in$ A.

$R$ = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
-transitive because (1,2), (2,4) and (1, 4) $\in$ R

Example 2. Which of these is transitive?

$R_1$ = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
$R_2$ = {(1,1), (1,2), (2,1)}
$R_3$ = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)}
$R_4$ = { ***(2,1)***, **(3,1)**, **<u>(3,2)</u>**, *(4,1)*, *<u>(4,2)</u>* , <u>(4,3)</u>}
$R_5$ = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}

$R_6 = \{(3,4)\}$

- The relations $R_4$, $R_5$ and $R_6$ are transitive because if $(x,y)$ and $(y,z)$ belong to the relation, then $(x,z)$ does also.
- The other relations aren't transitive.

## COMBINING RELATIONS

Since relations from A to B are subsets of A x B, two relations from A to B can be combined in any way two sets can be combined.

Relations are sets → combinations via set operations

- Set operations of: union, intersection, set difference and symmetric difference.

*Example*: Let A = {1,2,3} and B = {u,v} and

$R_1 = \{(1,u), (2,u), (2,v), (3,u)\}$

$R_2 = \{(1,v),(3,u),(3,v)\}$

What is:

$R_1 \cup R_2 = \{(1, u), (1, v), (2, u), (2, v), (3, u), (3, v)\}$

$R_1 \cap R_2 = \{(3, u)\}$

$R_1 - R_2 = \{(1, u), (2, u), (2, v)\}$

$R_2 - R_1 = \{(1, v), (3, v)\}$

$R_1 \Delta R_2 = \{(1, u), (1, v), (2, u), (2, v), 3, v)\}$

## COMPOSITION OF RELATIONS

*Definition*: Let $R$ be a relation from set $A$ to a set $B$ and $S$ a relation from set $B$ to a set $C$. The composite of $R$ and $S$ is the relation consisting of ordered pairs (a, c) where $a \in A$, $c \in C$, and for which there exist an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of $R$ and $S$ by $S \circ R$ .

- ✓ Computing the composite of two relations requires that we find elements that are the second element of ordered pairs in the first relation and the first element of ordered pairs in the second relation.

*Example:* 1. What is the composite of the relations $R$ and $S$, where R is the relation from {1, 2, 3} to {1, 2, 3, 4} with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3,4)\}$ and $S$ is the relation from {1, 2, 3, 4} to {0, 1, 2} with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4,1)\}$

Solution: $S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}$

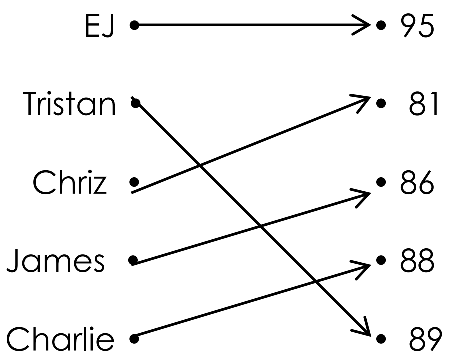
*Example* 2. Let $R = \{(1,1), (2, 1), (3,2), (4,3)\}$. Find $R \circ R$

Solution: $R \circ R = \{(1,1), (2,1), (3, 1), (4, 2)\}$

**FUNCTIONS**

The concept of function is extremely important in mathematics and computer science. Functions are used to represent how long it takes a computer to solve problems of a given size. Many computer programs and subroutines are designed to calculate values of functions. Recursive functions, which are functions defined in terms of themselves, are used throughout computer science. This section reviews the basic concepts involving functions needed in discrete mathematics.

*Example:* Grades of students in Discrete Mathematics




*Definition*: Let A and B be nonempty sets. A *function* $f$ from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is unique element of B assigned by the function $f$ to the element a of A. If $f$ is a function from A to B, we write $f: A \to B$.

- In other words, a function is a relation when each element of the domain is paired with exactly one element of the range.

- ❑ For every x there is exactly one y.
- ❑ The x-coordinate cannot repeat.
- ❑ The set of all values of x is called the domain, and the set of all values of y is called the range.

***Remark***: Functions are sometimes also called **mappings** or **transformation.**

*Example*: 1. Let $f$: **Z** → **Z** assign the square of an integer to this integer. Then, $f(x) = x^2$ , where the domain of $f$ is the set of all integers, and the range of $f$ is the set of all integers that are perfect squares, namely, {0, 1, 4, 9, 16, …}

*Example*: 2. Determine the domain and the range of the function:

$f(x) = \frac{2}{x-1}$

the function can be written as: $y = \frac{2}{x-1}$

***Solution***: In determining the domain and the range of the function, what is considered is the set of real numbers and that there are two things one has to remember:

a) any number divided ny zero is undefined.
b) The square root of a number less than zero is not a real number.

Note that at x = 1, the given function shall be $\frac{2}{0}$ which is udefined, hence the domain D is: D = {x | x ∈ *R*; x ≠ 1}

To determine the range, let us put x in terms of y by manipulating the function, which will result to: $x = \frac{2+y}{y}$

at y = 0, the equation is undefined since a number divided by zero does not exist. Hence the range R of the function is:

R = {y | y ∈ *R*, y ≠ 0}

*Example*: 3. Let $f(x) = \frac{x}{x+1}$ ; Find:

a. $f(2)$ b) $f(0)$ c) $f(x - 1)$ d) $f(x + h)$ e) $f(\frac{2}{3})$

**Solution**: a) $f(2)$

$$f(2) = \frac{2}{2+1} = \frac{2}{3}$$

c) $f(x-1) = \frac{x-1}{(x-1)+1}$

$$= \frac{x-1}{x}$$

The three other numbers is left for students for their practice exercises.

Video Lessons:
1. Relations (15 minutes)
2. Relations and Functions (7 minutes)
3. Relations (57 minutes)

Supplementary Readings:
Relations and Functions: Basics
Lecture Notes on Relations and Functions
Relations

## References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5. Lecture Notes on Relations and Functions. http://math.uga.edu/~pete/3200relationsfunctions.pdf
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11. Relations. https://www.youtube.com/watch?v=_BIKq9Xo_5A&list=PL0862D1A947252D20&index=14&t=0s

**CC9 – Discrete Structures**
**Matrices and Boolean Algebra**

# Lesson

## BASIC DEFINITIONS

## MATRIX

***Definition:*** A *matrix* is a rectangular array of numbers. A matrix with *m* rows and *n* columns is called an *m x n* matrix. The plural of matrix is matrices.

Example: 1. The matrix $\begin{bmatrix} 1 & -2 & 0 \\ 4 & 2 & 3 \end{bmatrix}$ is a 2 x 3 matrix.

Example: 2. The matix $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & -1 \\ 5 & -2 & 0 \end{bmatrix}$ is a 3 x 3 matrix.

***Definition:*** A matrix with the same number of rows as columns is called *square matrix.*

Example: 3. The matrix $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & -1 \\ 5 & -2 & 0 \end{bmatrix}$ is a square matrix.

***Definition:*** Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

***Definition:*** A *row matrix* has only one row.

Example: 4. The matrix $\begin{bmatrix} 1 & 2 & 5 & -9 \end{bmatrix}$ is a row matrix.

***Definition:*** A *column matrix* has only one column.

Example: 5. The matrix $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a column matrix.

***Definition:*** A *zero matrix* contains zeros.

Example: 6. The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a zero matrix

**Note**: Boldface uppercase letters will be used to represent matrices.

Defintion: Let *m* and *n* be positive integers and let

$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix}.$$

The *i*th row of **A** is the 1 x n matrix $\begin{bmatrix} a_{i1} & a_{i2} & . & . & a_{in} \end{bmatrix}$. The *j*th column of **A** is the m x 1 matrix $\begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ a_{mj} \end{bmatrix}$.

The (***i, j***)th element or entry of **A** is the element $a_{ij}$, that is, the number in the *i*th row and *j*th column of **A.** A convenient shorthand notation for expressing the matrix **A** is to write **A** = [a*ij*], which indicates that **A** is the matrix with its (*i, j*)th element equal to a*ij*.

## MATRIX ARITHMETIC

The basic operations of matrix arithmetic will now be discussed, beginning with a definition of matrix addition (subtraction).

***Definition***: Let **A** = [a*ij*] and **B** = [b*ij*] be *m* x *n* matrices. The sum of **A** and **B**, denoted by **A + B**, is the *m* x *n* matrix that has a*ij* + b*ij* as its (*i, j*)th element. In other words, **A + B** = [a*ij* + b*ij*].

The sum(difference) of two matrices of the same size is obtained by adding elements in the corresponding positions. Matrices of different sizes cannot be added (subtracted), because the sum(difference) of two matrices is defined only when both matrices have the same number of rows and the same number of columns.

***Definition:*** *Scalar multiplication*. Let **A** be an *m x n* matrix and c a scalar. Then c**A** is the *m x n* matrix obtained by mutiplying c times each entry of **A**; that is (cA)*ij* = c*aij.*

Example: 1. Given: $A=\begin{bmatrix}1 & -8 & 3\\ -4 & 2 & 0\end{bmatrix}$; $B=\begin{bmatrix}2 & -3 & -5\\ 0 & 4 & 1\end{bmatrix}$; $C=\begin{bmatrix}-5 & 0 & 2\\ 3 & 7 & 1\end{bmatrix}$;

$D=\begin{bmatrix}7 & -9 & 3\\ -6 & 5 & -3\end{bmatrix}$

Find: 1. **A + B**
2. **B – C**
3. **A – C + B**
4. 2**A** – 3**B**
5. 4**A** + 2**B** – 5**C**
6. **– A** + 5**C** + 3**D**

*Solution:* 1. **A + B**

$$\begin{bmatrix}1 & -8 & 3\\ -4 & 2 & 0\end{bmatrix}+\begin{bmatrix}2 & -3 & -5\\ 0 & 4 & 1\end{bmatrix}=\begin{bmatrix}1+2 & -8+-3 & 3+-5\\ -4+0 & 2+4 & 0+1\end{bmatrix}$$

$$=\begin{bmatrix}3 & -11 & -2\\ -4 & 6 & 1\end{bmatrix}$$

4. 2**A** – 3**B**

$$2\begin{bmatrix}1 & -8 & 3\\ -4 & 2 & 0\end{bmatrix}-3\begin{bmatrix}2 & -3 & -5\\ 0 & 4 & 1\end{bmatrix}=\begin{bmatrix}2 & -16 & 6\\ -8 & 4 & 0\end{bmatrix}-\begin{bmatrix}6 & -9 & -15\\ 0 & 12 & 3\end{bmatrix}$$

$$=\begin{bmatrix}-4 & -7 & 21\\ -8 & -8 & -3\end{bmatrix}$$

The remaining numbers are left for the students as practice exercises.

## MULTIPLICATION OF MATRICES

***Definition***: Let **A** be an *m x n* matrix and **B** be a *n x k* matrix. The product of **A** and **B** denoted by **AB**, is the *m x k* matrix with its (*i, j*)th entry equal to the sum of the products of the corresponding elements from the *i*th row of **A** and the *j*th column of **B**. In other words, if **AB** = [cij], then

$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}b_{kj}$.

In other words, multiply each entry of row *i* of **A** by the corresponding entry of column *j* of **B** and add the results.

- This is called the dot product of row *i* of **A** and column *j* of **B**.

Multiplication is defined when the number of *columns* in **A** is the same as the number of *rows* in **B**.

Example: 1. Given: $A = \begin{bmatrix} 1 & 3 \\ 5 & -4 \end{bmatrix}; B = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}$

Find: 1. **AB**
2. **BA**

*Solution:* 1. $\mathbf{A}_{2\times 2}\,\mathbf{B}_{2\times 3}$ (multiplication is possible since the number of columns of **A** is equal to the number of rows in **B**. The product **AB** is 2 x 3 matrix.)

$$AB = \begin{bmatrix} 1(-1)+3(2) & 1(3)+3(1) & 1(2)+3(3) \\ 5(-1)+-4(2) & 5(3)+-4(1) & 5(2)+-4(3) \end{bmatrix}$$

$$= \begin{bmatrix} -1+6 & 3+3 & 2+9 \\ -5+-8 & 15+-4 & 10+-12 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6 & 11 \\ -13 & 11 & -2 \end{bmatrix}$$

2. $\mathbf{B}_{2\times 3}\,\mathbf{A}_{2\times 2}$ (multiplication is not possible)

Example: 2. Given: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$

Find: 1. **AB**
2. **BA**

*Solution*: 1. $\mathbf{A}_{1\times 4}\,\mathbf{B}_{4\times 1} = \mathbf{AB}_{1\times 1}$

$$AB = \begin{bmatrix} 1(5)+2(6)+3(7)+4(8) \end{bmatrix}$$

$$AB = \begin{bmatrix} 70 \end{bmatrix}$$

2. $\mathbf{B}_{4\times 1}\mathbf{A}_{1\times 4} = \mathbf{BA}_{4\times 4}$

$$BA = \begin{bmatrix} 5(1) & 5(2) & 5(3) & 5(4) \\ 6(1) & 6(2) & 6(3) & 6(4) \\ 7(1) & 7(2) & 7(3) & 7(4) \\ 8(1) & 8(2) & 8(3) & 8(4) \end{bmatrix}$$

$$BA = \begin{bmatrix} 5 & 10 & 15 & 20 \\ 6 & 12 & 18 & 24 \\ 7 & 14 & 21 & 28 \\ 8 & 16 & 24 & 32 \end{bmatrix}$$

Note: **AB ≠ BA**

Exercises: Given: $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 5 \\ 3 & 2 & 1 \end{bmatrix}; B = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 2 & 2 \\ -1 & 3 & -2 \end{bmatrix}; C = \begin{bmatrix} 2 & 1 \\ 3 & -4 \\ 0 & 2 \end{bmatrix}$

Compute if possible;

1. **AB**
2. **BC**
3. **AA**
4. **ABC**
5. $\mathbf{B^TA}$

## TRANSPOSE OF A MATRIX

*Definition*: Let **A** = [aij] be an *m* x *n* matrix. The transpose of **A**, denoted by $\mathbf{A^T}$ , is the n x m matrix obtained by interchanging the rows and columns of **A**. In other words, if $\mathbf{A^T}$ = [b*ij*], then b*ij* = a*ij* for *i* = 1, 2, …,*n* and *j* = 1, 2, …,*m*.

Example: 1. Find the tranpose of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

2. Find the transpose of the matrix $A=\begin{bmatrix} 1 & 0 & 4 \\ 3 & 6 & -5 \\ 4 & -9 & 8 \end{bmatrix}$

The second number is left for the students as practice exercise.

## ZERO-ONE MATRICES

A matrix all of whose entries are either 0 or 1 is called a **zero-one matrix**. Zero-one matrices are often used to represent discrete structures. Algorithms using these structures are based on Boolean arithmetic with zero-one matrices. This arithmetic is based on the Boolean operations $\wedge$ and $\vee$, which operate on pairs of bits, defined by

$$b_1 \wedge b_2 = \begin{cases} \mathbf{1 \text{ if } b_1 = b_2 = 1} \\ \mathbf{0 \text{ if otherwise}} \end{cases}$$

$$b_1 \vee b_2 = \begin{cases} \mathbf{1 \text{ if } b_1 = 1 \text{ or } b_2 = 1} \\ \mathbf{0 \text{ if otherwise}} \end{cases}$$

***Definition***: Let **A** = [a*ij*] and **B** = [b*ij*] be m x n zero-one matrices. Then the ***join*** of **A** and **B** is the zero-one matrix with (*i, j*)th entry a*ij* $\vee$ b*ij*. The join of **A** and **B** is denoted by **A** $\vee$ **B.** The ***meet*** of **A** and **B** is the zero-one matrix with (*i, j*)th a*ij* $\wedge$ b*ij*. The meet of **A** and **B** is denoted by **A** $\wedge$ **B**.

Example: 1. Find the join and meet of the zero-one matrices

$$A=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}; B=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

***Solution:*** We find the join of **A** and **B**;

$$A \vee B=\begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix}$$

$$A \vee B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The meet of **A** and **B** is:

$$AB = \begin{bmatrix} 1\wedge 0 & 0\wedge 1 & 1\wedge 0 \\ 0\wedge 1 & 1\wedge 1 & 0\wedge 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Example: 2. Find the join and the meet of the zero-one matrices;

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}; B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The second number is left for the students as practice exercise.

## BOOLEAN PRODUCT

***Definition***: Let **A** = [a*ij*] be an **m x k** zero-one matrix and **B** = [b*ij*] be a **k x n** zero-one matrix. The Boolean product of A and B, denoted by A ⊗ B is the m x n matrix with (i, j) the entry [cij] where

$$[c_{ij}] = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{j2}) \vee \ldots \vee (a_{ik} \wedge b_{kj})$$

Note that the Boolean product of A and B is obtained in an analogous way to the ordinary product of these matrices, but with addition replaced with the operation $\vee$ and with multiplication replaced with the operation $\wedge$.

Example: 1. Find the Boolean product of **A** and **B**, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}; B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

***Solution***:

$$A \otimes B = \begin{bmatrix} (1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 1) & (1\wedge 0)\vee(0\wedge 1) \\ (0\wedge 1)\vee(1\wedge 0) & (0\wedge 1)\vee(1\wedge 1) & (0\wedge 0)\vee(1\wedge 1) \\ (1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 1) & (1\wedge 0)\vee(0\wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Example: 2. Find the Boolean product of **A** and **B**;

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

The second number is left for the students as practice exercise.

Video Lessons:

1. Introuction to Matrices and Operations (20 minutes)
2. The Applications of Matrices (30 minutes)
3. Defined Matrix Operations (5 minutes)

Supplementary Readings:

Matrices

Matrix Algebra for Engineers

Matrices

## References


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