

# Subquantile Minimization for Kernel Learning in the Huber $\epsilon$ -Contamination Model

author names withheld

**Editor:** Under Review for COLT 2024

## Abstract

In this paper we study Subquantile Minimization for learning the Huber- $\epsilon$  Contamination Problem for Kernel Learning. We assume the adversary has knowledge of the true distribution of  $\mathcal{P}$ , and is able to corrupt the covariates and the labels of  $\epsilon n$  samples for  $\epsilon \in [0, 0.5)$ . The distribution is formed as  $\hat{\mathcal{P}} = (1 - \epsilon)\mathcal{P} + \epsilon\mathcal{Q}$ , and we want to find the function  $f^* = \mathbb{E}_{\mathcal{D} \sim \mathcal{P}} [\ell(f; \mathcal{D})]$ , from the noisy distribution,  $\hat{\mathcal{P}}$ . Superquantile objectives have been studied extensively to reduce the risk of the tail [Laguel et al. \(2021\)](#); [Rockafellar et al. \(2014\)](#). We consider the contrasting case where we want to minimize the body of the risk. To our knowledge, we are the first to study the problem of general kernel learning in the Huber Contamination Model. We study a gradient-descent approach to solve a variational representation of the Subquantile Objective.

## 1. Introduction

There has been extensive study of algorithms to learn the target distribution from a Huber  $\epsilon$ -Contaminated Model for a Generalized Linear Model (GLM), ([Diakonikolas et al., 2019](#); [Awasthi et al., 2022](#); [Li et al., 2021](#); [Osama et al., 2020](#); [Fischler and Bolles, 1981](#)) as well as for linear regression [Bhatia et al. \(2017\)](#); [Mukhoty et al. \(2019\)](#). Robust Statistics has been studied extensively [Diakonikolas and Kane \(2023\)](#) for problems such as high-dimensional mean estimation [Prasad et al. \(2019\)](#); [Cheng et al. \(2020\)](#) and Robust Covariance Estimation [Cheng et al. \(2019\)](#); [Fan et al. \(2018\)](#). Recently, there has been an interest in solving robust machine learning problems by gradient descent [Prasad et al. \(2018\)](#); [Diakonikolas et al. \(2019\)](#). Subquantile minimization aims to address the shortcomings of standard ERM in applications of noisy/corrupted data ([Khetan et al., 2018](#); [Jiang et al., 2018](#)). In many real-world applications, the covariates have a non-linear dependence on labels ([Abu-Mostafa et al., 2012](#), Section 3.4). In which case it is suitable to transform the covariates to a different space utilizing kernels ([Hofmann et al., 2008](#)). Therefore, in this paper we consider the problem of Robust Learning for Kernel Learning.

**Definition 1 (Huber  $\epsilon$ -Contamination Model [Huber and Ronchetti \(2009\)](#))** *Given a corruption parameter  $0 < \epsilon < 0.5$ , a data matrix,  $\mathbf{X}$  and labels  $\mathbf{y}$ . An adversary is allowed to inspect all samples and modify  $\epsilon n$  samples arbitrarily. The algorithm is then given the  $\epsilon$ -corrupted data matrix  $\mathbf{X}$  and  $\mathbf{y}$  as training data.*

Current approaches for robust learning across various machine learning tasks often use gradient descent over a robust objective, ([Li et al., 2021](#)). These robust objectives tend to not be convex and therefore do not have a strong analysis on the error bounds for general classes of models.

We similarly propose a robust objective which has a nonconvex-concave objective. This objective has also been proposed recently in [Hu et al. \(2020\)](#) where there has been an analysis in the Binary Classification Task. We show Subquantile Minimization reduces to the same objective in [Hu](#)

et al. (2020). We use theory from the weakly-convex concave optimization literature for our error bounds. We are able to leverage this theory by analyzing the asymptotic distribution of a softplus approximation of the Subquantile objective.

The study of Kernel Learning in the Gaussian Design is quite popular, (Cui et al., 2021; Dicker, 2016). In (Cui et al., 2021), the feature space,  $\phi(\mathbf{x}_i) \sim \mathcal{N}(0, \Sigma)$  where  $\Sigma$  is a diagonal matrix of dimension  $p$ , where  $p$  can be infinite. In this work, we adopt a similar framework, and with the power of Mercer’s Theorem (Mercer, 1909), we are able to say  $\text{Tr}(\Sigma) < \infty$ . We use this fact extensively in our infinite-dimensional concentration inequalities.

**Theorem 2 (Informal).** *Let the dataset be given as  $\{(\mathbf{x}_i, y_i)\}_{i=1}^n$  such that the labels and covariates of  $\epsilon n$  samples are arbitrarily corrupted by an adversary.*

*Kernelized Regression:*

$$\|\hat{f} - f^*\|_{\mathcal{H}} \leq \varepsilon + O(\sigma)$$

*Kernel Binary Classification:*

$$\|\hat{f} - f^*\|_{\mathcal{H}} \leq \varepsilon + \tilde{O}\left(\frac{\mathcal{E}_{\text{OPT}}}{n(1-\epsilon)}\right) + \tilde{O}\left(\frac{1}{n^\beta(1-\epsilon)^\beta}\right)$$

## 1.1. Contributions

We will now state our main contributions clearly.

1. We provide rigorous error bounds for subquantile minimization in the kernel regression, kernel binary classification, and kernel multi-class classification. Furthermore, we provide our bounds for both label and feature corruption with a general Gaussian Design.

## 2. Preliminaries

**Notation.** We denote  $[T]$  as the set  $\{1, 2, \dots, T\}$ . We define  $(x)^+ \triangleq \max(0, x)$  as the Rectified Linear Unit (ReLU) function. We say  $y = O(x)$  if there exists  $x_0$  s.t. for all  $x \geq x_0$  there exists  $C$  s.t.  $y \leq Cx$ . We denote  $\tilde{O}$  to ignore log factors. We say  $y = \Omega(x)$  if there exists  $x_0$  s.t. for all  $x \geq x_0$  there exists  $C$  s.t.  $y \geq Cx$ .

### 2.1. Reproducing Kernel Hilbert Spaces

Let the function  $\phi : \mathbb{R}^d \rightarrow \mathcal{H}$  represent the Hilbert Space Representation or ‘feature transform’ from a vector in the original covariate space to the RKHS. We define  $k : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  as  $k(\mathbf{x}, \mathbf{x}) \triangleq \langle \phi(\mathbf{x}), \phi(\mathbf{x}) \rangle_{\mathcal{H}}$ . For a function in a RKHS,  $f \in \mathcal{H}$ , it follows for a function  $f$  parameterized by weights  $\mathbf{w} \in \mathbb{R}^n$ , that the point evaluation function is given as  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  and defined  $f(\cdot) \triangleq \sum_{i \in [n]} w_i k(\mathbf{x}_i, \cdot)$ .

### 2.2. Tensor Products

Let  $\mathcal{H}, \mathcal{K}$  be Hilbert Spaces, then  $\mathcal{H} \otimes \mathcal{K}$  is the tensor product space and is also a Hilbert Space (Ryan and a Ryan, 2002). For  $\phi_1, \psi_1 \in \mathcal{H}$  and  $\phi_2, \psi_2 \in \mathcal{K}$ , the inner product is defined as  $\langle \phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2 \rangle_{\mathcal{H} \otimes \mathcal{K}} = \langle \phi_1, \psi_1 \rangle_{\mathcal{H}} \langle \phi_2, \psi_2 \rangle_{\mathcal{K}}$ . We will utilize tensor products when we discuss infinite dimensional covariance estimation.

### 2.3. Distribution

In this paper we sample  $\mathbf{x} \sim \mathcal{X}$  such that  $\phi(\mathbf{x}) \sim \mathcal{P}$  is sub-Gaussian in the Hilbert Space where  $\mathbf{E}[\phi(\mathbf{x}_i)] = \mathbf{0}$  and  $\mathbf{E}[\phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i)] = \mathbf{\Gamma}$  where  $\text{Tr}(\mathbf{\Gamma}) < \infty$ . We have  $X$  is a centered Hilbert Space sub-Gaussian random function if for all  $\theta > 0$ ,

$$\mathbf{E}_{X \sim \mathcal{P}} [\exp(\theta \langle X, v \rangle_{\mathcal{H}})] \leq \exp\left(\frac{\theta^2 \langle v, \mathbf{\Gamma} v \rangle_{\mathcal{H}}}{2}\right) \quad (1)$$

The Gaussian Design for the Feature Space has gained popularity in the study of kernel learning (Cui et al., 2021).

### 2.4. Related Work

The idea of iterative thresholding algorithms for robust learning tasks dates back to 1806 by Legendre (Legendre, 1806). Iterative thresholding have been studied theoretically and tested empirically in various machine learning domains (Hu et al., 2023; Mukhoty et al., 2019). Therefore, we will dedicate this subsection to reviewing such works and to make clear our contributions to the iterative thresholding literature.

Bhatia et al. (2015) study iterative thresholding for least squares regression / sparse recovery. In particular, one part of their study is of a gradient descent algorithm when the data  $\mathcal{P} = \mathcal{Q} = \mathcal{N}(\mathbf{0}, \mathbf{I})$  or multivariate sub-Gaussian with proxy  $\mathbf{I}$ . Their proof of optimality relies on the fact that  $\lambda_{\min}(\mathbf{\Sigma}) = \lambda_{\max}(\mathbf{\Sigma})$  and with sufficient data,  $\lambda_{\max}(\mathbf{X})/\lambda_{\min}(\mathbf{X}) \searrow 1$ . This is similar to the study by Awasthi et al. (2022), where the iterative trimmed maximum likelihood estimator is studied for General Linear Models. The algorithm studied by Awasthi et al. (2022) utilizes a filtering algorithm with the sketching matrix  $\mathbf{\Sigma}^{-1/2}$  so the columns of  $\mathbf{X}$  are sampled from a multivariate sub-Gaussian Distribution with proxy  $\mathbf{I}$  before running the iterative thresholding procedure.

This does not generalize to kernel learning where we are given a matrix  $\mathbf{K}$  which is equivalent to inner product of the quasimatrix<sup>1</sup>,  $\Phi$ , with itself. In this case is not possible to sketch (Woodruff et al., 2014) the input matrix to have well-conditioned covariates. Thus, we are left with  $\Phi$  where the columns are sampled from a sub-Gaussian Distribution with proxy  $\mathbf{\Gamma}$  is a trace-class operator, which implies the eigenvalues tend to zero, i.e.  $\lambda_{\inf}(\mathbf{\Gamma}) = 0$ , and there is no longer a notion of  $\lambda_{\min}(\mathbf{\Gamma})$ .

### 2.5. Our Technique

We first reduce minimizing over the subquantile to an iterative thresholding algorithm by utilizing previous results in the non-convex-concave optimization literature. Suppose our algorithm returns a function  $\hat{f}$ . We analyze  $\|\hat{f} - f^*\|_{\mathcal{H}}$  through  $\|\hat{f} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}$ . We consider minimizing the term  $\Lambda^{(t)} \triangleq \|f - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 + \sum_{i \in S^{(t)}} (f(\mathbf{x}_i) - y_i)^2 + C \|f\|_{\mathcal{H}}^2$ . We utilize the strong convexity of kernel ridge regression to show we can decrease the second term, and use standard techniques in gradient descent to show decrease in the first term. We prove  $\Lambda^{(t)} \leq (1 - \Omega(L^{-1})) \cdot \Lambda^{(t)} + E$  where  $E \rightarrow 0$  as  $n \rightarrow \infty$ .

---

1. A quasimatrix is an infinite-dimensional analogue of a tall-skinny matrix that represents an ordered set of functions in  $\ell_2$  (see e.g. Townsend and Trefethen (2015)).

### 3. Subquantile Minimization

We propose to optimize over the subquantile of the risk. The  $p$ -quantile of a random variable,  $U$ , is given as  $\mathcal{Q}_p(U)$ , this is the largest number,  $t$ , such that the probability of  $U \leq t$  is at least  $p$ .

$$\mathcal{Q}_p(U) \leq t \iff \mathbf{Pr}\{U \leq t\} \geq p$$

The  $p$ -subquantile of the risk is then given by

$$\mathbb{L}_p(U) = \frac{1}{p} \int_0^p \mathcal{Q}_p(U) dq = \mathbf{E}[U|U \leq \mathcal{Q}_p(U)] = \max_{t \in \mathbb{R}} \left\{ t - \frac{1}{p} \mathbf{E}(t - U)^+ \right\}$$

Given an objective function,  $\ell$ , the kernelized learning problem becomes:

$$\min_{f \in \mathcal{K}} \max_{t \in \mathbb{R}} \left\{ g(t, f) \triangleq t - \sum_{i=1}^n (t - \ell(f; \mathbf{x}_i, y_i))^+ \right\}$$

where  $t$  is the  $p$ -quantile of the empirical risk. Note that for a fixed  $t$  therefore the objective is not concave with respect to  $\mathbf{w}$ . Thus, to solve this problem we use the iterations from Equation 11 in (Razaviyayn et al., 2020). Let  $\text{Proj}_{\mathcal{K}}$  be the projection of a function on to the convex set  $\mathcal{K} \triangleq \{f \in \mathcal{H} : \|f\|_{\mathcal{H}} \leq R\}$ , then our update steps are

$$t^{(k+1)} = \arg \max_{t \in \mathbb{R}} g(f^{(k)}, t)$$

$$f^{(k+1)} = \text{Proj}_{\mathcal{K}} \left[ f^{(k)} - \eta \nabla_f g(f^{(k)}, t^{(k+1)}) \right]$$

The proof of convergence for the above algorithm was given in Jin et al. (2020)[Theorem 35]. The sufficient condition for convergence is  $g(f, t)$  is concave with respect to  $t$ , which for the subquantile objective is simple to show.

#### 3.1. Reduction to Iterative Thresholding

To consider theoretical guarantees of Subquantile Minimization, we first analyze the inner and outer optimization problems. We first analyze kernel learning in the presence of corrupted data. Next, we provide error bounds for the two most important kernel learning problems, kernel ridge regression, and kernel classification. Now we will give our first result regarding kernel learning in the Huber  $\epsilon$ -contamination model. Now we will analyze the two-step minimax optimization steps described in Section 3.

**Lemma 3** *Let  $\ell : \mathcal{H} \times \mathcal{D} \rightarrow \mathbb{R}$  be a loss function (not necessarily convex). Let  $\mathbf{x}_1, \dots, \mathbf{x}_n$  denote the  $n$  data points ordered w.l.o.g such that  $\ell(f; \mathbf{x}_1, y_1) \leq \dots \leq \ell(f; \mathbf{x}_n, y_n)$ . If we denote  $\hat{v}_i \triangleq \ell(f; \mathbf{x}_i, y_i)$ , it then follows  $\hat{v}_{n(1-\epsilon)} \in \arg \max_{t \in \mathbb{R}} g(t, f)$ .*

**Proof.** First we can note, the max value of  $t$  for  $g$  is equivalent to the min value of  $t$  for the convex w.r.t  $t$  function  $-g$ . We can now find the Fermat Optimality Conditions for  $g$ .

$$\partial(-g(t, f)) = \partial \left( -t + \frac{1}{n(1-\epsilon)} \sum_{i=1}^n (t - \hat{v}_i)^+ \right) = -1 + \frac{1}{n(1-\epsilon)} \sum_{i=1}^{n(1-\epsilon)} \begin{cases} 1 & \text{if } t > \hat{v}_i \\ 0 & \text{if } t < \hat{v}_i \\ [0, 1] & \text{if } t = \hat{v}_i \end{cases}$$

We observe when setting  $t = \hat{\nu}_{n(1-\epsilon)}$ , it follows that  $0 \in \partial(-g(t, f))$ . This is equivalent to the  $(1 - \epsilon)$ -quantile of the Empirical Risk.  $\blacksquare$

From Lemma 3, we see that  $t$  will be greater than or equal to the errors of exactly  $n(1 - \epsilon)$  points. Thus, we are continuously updating over the  $n(1 - \epsilon)$  minimum errors.

**Lemma 4** *Let  $\hat{\nu}_i \triangleq \ell(f; \mathbf{x}_i, y_i)$  s.t.  $\hat{\nu}_{i-1} \leq \hat{\nu}_i \leq \hat{\nu}_{i+1}$ , if we choose  $t^{(k+1)} = \hat{\nu}_{n(1-\epsilon)}$  as by Lemma 3, it then follows  $\nabla_{\mathbf{w}} g(t^{(k)}, f^{(k)}) = \frac{1}{n(1-\epsilon)} \sum_{i=1}^{n(1-\epsilon)} \nabla f(\mathbf{x}_i; f^{(k)}, y_i)$*

**Proof.** By our choice of  $t^{(k+1)}$ , it follows:

$$\begin{aligned} \nabla_f g(t^{(k+1)}, f^{(k)}) &= \nabla_f \left( t^{(k+1)} - \frac{1}{n(1-\epsilon)} \sum_{i=1}^n \left( t^{(k+1)} - \ell(f^{(k)}; \mathbf{x}_i, y_i) \right)^+ \right) \\ &= -\frac{1}{n(1-\epsilon)} \sum_{i=1}^{n(1-\epsilon)} \nabla_f \left( t^{(k+1)} - \ell(f^{(k)}; \mathbf{x}_i, y_i) \right)^+ = \frac{1}{n(1-\epsilon)} \sum_{i=1}^n \nabla_f \ell(f^{(k)}; \mathbf{x}_i, y_i) \begin{cases} 1 & \text{if } t > \hat{\nu}_i \\ 0 & \text{if } t < \hat{\nu}_i \\ [0, 1] & \text{if } t = \hat{\nu}_i \end{cases} \end{aligned}$$

Now we note  $\hat{\nu}_{n(1-\epsilon)} \leq t^{(k+1)} \leq \hat{\nu}_{n(1-\epsilon)+1}$ . Then, we have

$$\nabla_f g(t^{(k+1)}, f^{(k)}) = \frac{1}{n(1-\epsilon)} \sum_{i=1}^{n(1-\epsilon)} \nabla_f \ell(f^{(k)}; \mathbf{x}_i, y_i)$$

This concludes the proof.  $\blacksquare$

We have therefore shown that the two-step optimization of Subquantile Minimization is equivalent to solving the iterative thresholding algorithm.

## 4. Convergence

In this section we give the algorithm for subquantile minimization for both kernelized ridge regression and kernelized binary classification. Then we give our convergence results.

### 4.1. Kernelized Regression

The loss for the Kernel Ridge Regression problem for a single training pair  $(\mathbf{x}_i, y_i) \in \mathcal{D}$  is given by the following equation

$$\ell(f; \mathbf{x}_i, y_i) = (f(\mathbf{x}_i) - y_i)^2$$

We will now give the algorithm. Our goals throughout the proofs will be to obtain approximation bounds for infinite-dimensional kernels. The key challenge is the obvious undetermined problem, i.e. considering an infinite eigenfunction basis, we require infinite samples to obtain an accurate approximation. Instead, we will calculate the approximation bounds for the rank- $m$  approximation of  $f^*$  and push  $m \rightarrow \infty$ .

**Theorem 5 (Subquantile Minimization for Kernelized Regression)** *Algorithm 2 run on a dataset  $\mathcal{D} \sim \hat{\mathcal{P}}$  and return  $\hat{f}$ . Then with probability exceeding  $1 - \delta$ ,*

$$\|\hat{f} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \leq \varepsilon + \tilde{O} \left( \frac{\|\text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}}{\lambda_{\max}(\Sigma) \sqrt{n}} + \sigma \sqrt{\frac{\text{Tr}(\Sigma) \log n}{n \lambda_{\max}^2(\Sigma)}} + \lambda_m(\Sigma) R^2 \right)$$

**Input:** Data Matrix:  $\mathbf{X} \in \mathbb{R}^{n \times d}$ ,  $n \gg d$ ; Labels:  $\mathbf{y} \in \mathbb{R}^n$ , Closed and Convex set  $\mathcal{K} \subset \mathcal{H}$   
**Output:** Function in  $\mathcal{H}$ :  $\hat{f}$

1. Calculate the Lipschitz Constant

$$L = \frac{2}{n(1-\epsilon)} \left( R \|\mathbf{K}\| + \sqrt{n} \|\mathbf{y}\|_{\infty} \sqrt{\text{Tr}(\mathbf{K})} \right)$$

2. Set the step-size  $\eta = 1/L$ .
3. Set the number of iterations

$$T = \tilde{O} \left( \log \left( \left( \frac{\lambda_{\max}(\mathbf{\Sigma}) \|f^*\|_{\mathcal{H}}}{\sqrt{n}} \right) \frac{1}{\epsilon} \right) \right)$$

4. **for**  $k = 1, 2, \dots, T$  **do**

3. Find the Subquantile denoted as  $S^{(k)}$  as the set of  $(1-\epsilon)n$  elements with the lowest error with respect to the loss function.
4. Calculate the gradient update.

$$\nabla_f g(t^{(k+1)}, f^{(k)}) \leftarrow \frac{2}{n(1-\epsilon)} \sum_{i \in S^{(k)}} (f^{(k)}(\mathbf{x}_i) - y_i) \cdot \phi(\mathbf{x}_i)$$

5. Perform Projected Gradient Descent Iteration with Lemma 23.

$$f^{(k+1)} \leftarrow \text{Proj}_{\mathcal{K}} \left[ f^{(k)} - \eta \nabla g(f^{(k)}, t^{(k+1)}) \right]$$

**Return:** Function in  $\mathcal{H}$ :  $f^{(T)}$

**Algorithm 1:** Subquantile Minimization for Kernelized Regression

Full proof with explicit constants is given in Appendix C.1. A direct application of Theorem 5 is that learning an infinite dimensional function  $f^*$  to within  $\epsilon$  error in the Hilbert Space Norm requires infinite data. Furthermore, we see that given covariate noise and label noise, our bound requires more iterations dependent on the magnitude of the corruption. Such a result is corroborated in Schmidt et al. (2018). For the linear and polynomial kernel, we then have  $\beta$  increases, therefore to obtain the same bound on  $\eta$  as with no feature noise, we simply need more data. The effect of ?? can be seen in the denominator of both terms. Instead of  $\lambda_{\min}(\mathbf{\Sigma})$  we have  $c_4 \lambda_m$  for a finite  $m$ . Let us note for the case where  $p$  is finite, i.e. the feature mapping is finite-dimensional, e.g. linear or polynomial kernel. Then we have that  $\text{Proj}_{\Psi_m^\perp}$  where  $m = p$  is equal to zero as  $\{\varphi_i\}_{i=1}^m$  spans the finite-dimensional space, in which we case we have the absolute constant given in ?? is equal to zero. It is important to note in all our bounds,  $\gamma \leq \sqrt{\frac{\epsilon}{1-2\epsilon}}$  is a theoretical worst case bound when

the Subquantile contains the minimum possible number of uncorrupted points. In other words, we have  $\gamma \triangleq \frac{|P \setminus S|}{|S \cap P|} \leq \frac{n\epsilon}{n(1-2\epsilon)} = \frac{\epsilon}{1-2\epsilon}$ . So, as  $|S \cap P|$  increases, we have a better error bound as  $|P \setminus S|$  decreases. As is typical in the robust statistics literature, we make no assumptions on the distribution of the corrupted data so we cannot say anything about  $|S \cap P|$ . We will have  $\gamma$  decreases if stationary points give high error for corrupt points as our optimization procedure moves toward a stationary point.

#### 4.2. Kernelized Binary Classification

The Negative Log Likelihood for the the Kernel Classification problem is given by the following equation for a single training pair  $(\mathbf{x}_i, y_i)$

$$\ell(\mathbf{x}_i, y_i; f) = -y_i \log(\sigma(f(\mathbf{x}_i))) - (1 - y_i) \log(1 - \sigma(f(\mathbf{x}_i)))$$

We will now give our algorithm.

**Input:** Data Matrix:  $\mathbf{X} \in \mathbb{R}^{n \times d}$ ,  $n \gg d$ ; Labels:  $\mathbf{y} \in \mathbb{R}^n$ , Closed and Convex set  $\mathcal{K} \subset \mathcal{H}$

**Output:** Function in  $\mathcal{H}$ :  $\hat{f}$

1. Set the step-size

$$\eta \leq O\left(\frac{\lambda_{\min}(\Sigma)}{\text{Tr}(\Sigma)}\right)$$

2. Set the number of iterations

$$T = O\left(\log\left(\left(\frac{\lambda_{\max}(\Sigma) \|f^*\|_{\mathcal{H}}}{\sqrt{n}}\right) \frac{1}{\epsilon}\right)\right)$$

3. **for**  $k = 1, 2, \dots, T$  **do**

3. Find the Subquantile denoted as  $S^{(k)}$  as the set of  $(1 - \epsilon)n$  elements with the lowest error with respect to the loss function.

4. Calculate the gradient update.

$$\nabla_f g(t^{(k+1)}, f^{(k)}) \leftarrow \frac{2}{n(1 - \epsilon)} \sum_{i \in S^{(k)}} (\sigma(f^{(k)}(\mathbf{x}_i)) - y_i) \cdot \phi(\mathbf{x}_i)$$

5. Perform Projected Gradient Descent Iteration with Lemma 23.

$$f^{(k+1)} \leftarrow \text{Proj}_{\mathcal{K}} \left[ f^{(k)} - \eta \nabla g(f^{(k)}, t^{(k+1)}) \right]$$

**Return:** Function in  $\mathcal{H}$ :  $f^{(T)}$

**Algorithm 2:** Subquantile Minimization for Binary Classification

**Theorem 6 (Subquantile Minimization for Binary Classification is Good with High Probability)**

Let Algorithm 2 be run on a dataset  $\mathcal{D} \sim \hat{\mathcal{P}}$  with learning rate  $\eta \triangleq \Omega(L^{-1})$ . Then after  $O\left(\log\left(\frac{\|f^*\|_{\mathcal{H}}}{\varepsilon}\right)\right)$  gradient descent iterations, with probability exceeding  $1 - \delta$ ,

$$\|f^{(T)} - f^*\|_{\mathcal{H}} \leq \varepsilon + \|\text{Proj}_{\Psi_m^\perp} f^*\|_{\mathcal{H}}$$

where  $C = \exp(-RP_k)$ .

**Proof.** The proof is deferred to Appendix D.1. ■

In Theorem 6, we introduce  $\mathcal{E}_{\text{OPT}}$ , which says we are only able to learn up to the intrinsic noise within the target function.

## 5. Discussion

The main contribution of this paper is the study of a nonconvex-concave formulation of Subquantile minimization for the robust learning problem for kernel ridge regression and kernel classification. We present an algorithm to solve the nonconvex-concave formulation and prove rigorous error bounds which show that the more good data that is given decreases the error bounds. We also present accelerated gradient methods for the two-step algorithm to solve the nonconvex-concave optimization problem and give novel theoretical bounds.

**Theory.** We develop strong theoretical bounds on the normed difference between the function returned by Subquantile Minimization and the optimal function for data in the target distribution,  $\mathcal{P}$ , in the sub-Gaussian Design. We are able to show if the number of inliers is sufficiently small, then the kernelized binary classification problem with binary cross-entropy loss is consistent.

**General Assumptions.** The general assumption is the majority of the data should inliers. This is not a very strong assumption, as by the definition of outlier it should be in the minority. Furthermore, we assume the feature maps have a Gaussian Design. Such a design in many prior works in kernel learning and we therefore find it suitable.

**Future Work.** The analysis of Subquantile Minimization can be extended to neural networks as kernel learning can be seen as a one-layer network. This generalization will be appear in subsequent work. Another interesting direction work in optimization is for accelerated methods for optimizing non-convex concave min-max problems with a maximization oracle. The current theory analyzes standard gradient descent for the minimization. Ideas such as Momentum and Nesterov Acceleration in conjunction with the maximum oracle are interesting and can be analyzed in future work.

## References

- Yaser S Abu-Mostafa, Malik Magdon-Ismail, and Hsuan-Tien Lin. *Learning from data*, volume 4. AMLBook New York, 2012.
- Pranjal Awasthi, Abhimanyu Das, Weihao Kong, and Rajat Sen. Trimmed maximum likelihood estimation for robust generalized linear model. In Alice H. Oh, Alekh Agarwal, Danielle Belgrave, and Kyunghyun Cho, editors, *Advances in Neural Information Processing Systems*, 2022.
- Kush Bhatia, Prateek Jain, and Purushottam Kar. Robust regression via hard thresholding. In C. Cortes, N. Lawrence, D. Lee, M. Sugiyama, and R. Garnett, editors, *Advances in Neural Information Processing Systems*, volume 28. Curran Associates, Inc., 2015.



- Kush Bhatia, Prateek Jain, Parameswaran Kamalaruban, and Purushottam Kar. Consistent robust regression. In I. Guyon, U. Von Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, editors, *Advances in Neural Information Processing Systems*, volume 30. Curran Associates, Inc., 2017.
- Xiaohui Chen and Yun Yang. Hanson–Wright inequality in Hilbert spaces with application to  $K$ -means clustering for non-Euclidean data. *Bernoulli*, 27(1):586 – 614, 2021. doi: 10.3150/20-BEJ1251.
- Yu Cheng, Ilias Diakonikolas, Rong Ge, and David P. Woodruff. Faster algorithms for high-dimensional robust covariance estimation. In Alina Beygelzimer and Daniel Hsu, editors, *Proceedings of the Thirty-Second Conference on Learning Theory*, volume 99 of *Proceedings of Machine Learning Research*, pages 727–757. PMLR, 25–28 Jun 2019.
- Yu Cheng, Ilias Diakonikolas, Rong Ge, and Mahdi Soltanolkotabi. High-dimensional robust mean estimation via gradient descent. In Hal Daumé III and Aarti Singh, editors, *Proceedings of the 37th International Conference on Machine Learning*, volume 119 of *Proceedings of Machine Learning Research*, pages 1768–1778. PMLR, 13–18 Jul 2020.
- Thomas H Cormen, Charles E Leiserson, Ronald L Rivest, and Clifford Stein. *Introduction to algorithms*. MIT press, 2022.
- Hugo Cui, Bruno Loureiro, Florent Krzakala, and Lenka Zdeborová. Generalization error rates in kernel regression: The crossover from the noiseless to noisy regime. *Advances in Neural Information Processing Systems*, 34:10131–10143, 2021.
- Ilias Diakonikolas and Daniel M Kane. *Algorithmic high-dimensional robust statistics*. Cambridge University Press, 2023.
- Ilias Diakonikolas, Gautam Kamath, Daniel M. Kane, Jerry Li, Jacob Steinhardt, and Alistair Stewart. Sever: A robust meta-algorithm for stochastic optimization. In *Proceedings of the 36th International Conference on Machine Learning, ICML ’19*, pages 1596–1606. JMLR, Inc., 2019.
- Lee H Dicker. Ridge regression and asymptotic minimax estimation over spheres of growing dimension. 2016.
- Jianqing Fan, Weichen Wang, and Yiqiao Zhong. An  $l_1$  eigenvector perturbation bound and its application to robust covariance estimation. *Journal of Machine Learning Research*, 18(207): 1–42, 2018.
- Martin A. Fischler and Robert C. Bolles. Random sample consensus: A paradigm for model fitting with applications to image analysis and automated cartography. *Commun. ACM*, 24(6):381–395, jun 1981. ISSN 0001-0782. doi: 10.1145/358669.358692.
- Arthur Gretton. Introduction to rkhs, and some simple kernel algorithms. *Adv. Top. Mach. Learn. Lecture Conducted from University College London*, 16(5-3):2, 2013.
- Arthur Gretton. Notes on mean embeddings and covariance operators, 2015.

- Thomas Hofmann, Bernhard Schölkopf, and Alexander J. Smola. Kernel methods in machine learning. *The Annals of Statistics*, 36(3):1171 – 1220, 2008. doi: 10.1214/0090536070000000677.
- O. Hölder. Ueber einen mittelwerthabsatz. *Nachrichten von der Königl. Gesellschaft der Wissenschaften und der Georg-Augusts-Universität zu Göttingen*, 1889:38–47, 1889.
- Shu Hu, Yiming Ying, xin wang, and Siwei Lyu. Learning by minimizing the sum of ranked range. In H. Larochelle, M. Ranzato, R. Hadsell, M.F. Balcan, and H. Lin, editors, *Advances in Neural Information Processing Systems*, volume 33, pages 21013–21023. Curran Associates, Inc., 2020.
- Shu Hu, Zhenhuan Yang, Xin Wang, Yiming Ying, and Siwei Lyu. Outlier robust adversarial training. *arXiv preprint arXiv:2309.05145*, 2023.
- Peter J. Huber and Elvezio Ronchetti. *Robust statistics*. Wiley series in probability and statistics. Wiley, Hoboken, N.J., 2nd ed. edition, 2009.
- Johan Ludwig William Valdemar Jensen. Sur les fonctions convexes et les inégalités entre les valeurs moyennes. *Acta mathematica*, 30(1):175–193, 1906.
- Lu Jiang, Zhengyuan Zhou, Thomas Leung, Li-Jia Li, and Li Fei-Fei. Mentornet: Learning data-driven curriculum for very deep neural networks on corrupted labels. In *ICML*, 2018.
- Chi Jin, Praneeth Netrapalli, and Michael Jordan. What is local optimality in nonconvex-nonconcave minimax optimization? In Hal Daumé III and Aarti Singh, editors, *Proceedings of the 37th International Conference on Machine Learning*, volume 119 of *Proceedings of Machine Learning Research*, pages 4880–4889. PMLR, 13–18 Jul 2020.
- Ashish Khetan, Zachary C. Lipton, and Anima Anandkumar. Learning from noisy singly-labeled data. In *International Conference on Learning Representations*, 2018.
- Yassine Laguel, Krishna Pillutla, Jérôme Malick, and Zaid Harchaoui. Superquantiles at work: Machine learning applications and efficient subgradient computation. *Set-Valued and Variational Analysis*, 29(4):967–996, Dec 2021. ISSN 1877-0541. doi: 10.1007/s11228-021-00609-w.
- Michel Ledoux and Michel Talagrand. *Probability in Banach Spaces: isoperimetry and processes*. Springer Science & Business Media, 2013.
- Adrien M Legendre. *Nouvelles methodes pour la determination des orbites des cometes: avec un supplement contenant divers perfectionnemens de ces methodes et leur application aux deux cometes de 1805*. Courcier, 1806.
- Tian Li, Ahmad Beirami, Maziar Sanjabi, and Virginia Smith. Tilted empirical risk minimization. In *International Conference on Learning Representations*, 2021.
- Colin McDiarmid et al. On the method of bounded differences. *Surveys in combinatorics*, 141(1): 148–188, 1989.
- James Mercer. Xvi. functions of positive and negative type, and their connection the theory of integral equations. *Philosophical transactions of the royal society of London. Series A, containing papers of a mathematical or physical character*, 209(441-458):415–446, 1909.

- Bhaskar Mukhoty, Govind Gopakumar, Prateek Jain, and Purushottam Kar. Globally-convergent iteratively reweighted least squares for robust regression problems. In Kamalika Chaudhuri and Masashi Sugiyama, editors, *Proceedings of the Twenty-Second International Conference on Artificial Intelligence and Statistics*, volume 89 of *Proceedings of Machine Learning Research*, pages 313–322. PMLR, 16–18 Apr 2019.
- Muhammad Osama, Dave Zachariah, and Petre Stoica. Robust risk minimization for statistical learning from corrupted data. *IEEE Open Journal of Signal Processing*, 1:287–294, 2020.
- Ioşif F Pinelis and Aleksandr Ivanovich Sakhanenko. Remarks on inequalities for large deviation probabilities. *Theory of Probability & Its Applications*, 30(1):143–148, 1986.
- Adarsh Prasad, Arun Sai Suggala, Sivaraman Balakrishnan, and Pradeep Ravikumar. Robust estimation via robust gradient estimation. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 82, 2018.
- Adarsh Prasad, Sivaraman Balakrishnan, and Pradeep Ravikumar. A unified approach to robust mean estimation. *arXiv preprint arXiv:1907.00927*, 2019.
- Meisam Razaviyayn, Tianjian Huang, Songtao Lu, Maher Nouiehed, Maziar Sanjabi, and Mingyi Hong. Nonconvex min-max optimization: Applications, challenges, and recent theoretical advances. *IEEE Signal Processing Magazine*, 37(5):55–66, 2020. doi: 10.1109/MSP.2020.3003851.
- R.T. Rockafellar, J.O. Royset, and S.I. Miranda. Superquantile regression with applications to buffered reliability, uncertainty quantification, and conditional value-at-risk. *European Journal of Operational Research*, 234(1):140–154, 2014. ISSN 0377-2217. doi: <https://doi.org/10.1016/j.ejor.2013.10.046>.
- Raymond A Ryan and R a Ryan. *Introduction to tensor products of Banach spaces*, volume 73. Springer, 2002.
- Ludwig Schmidt, Shibani Santurkar, Dimitris Tsipras, Kunal Talwar, and Aleksander Madry. Adversarially robust generalization requires more data. *Advances in neural information processing systems*, 31, 2018.
- Ilya Tolstikhin, Bharath K Sriperumbudur, Krikamol Mu, et al. Minimax estimation of kernel mean embeddings. *Journal of Machine Learning Research*, 18(86):1–47, 2017.
- Alex Townsend and Lloyd N Trefethen. Continuous analogues of matrix factorizations. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 471(2173):20140585, 2015.
- Hermann Weyl. Das asymptotische verteilungsgesetz der eigenwerte linearer partieller differentialgleichungen (mit einer anwendung auf die theorie der hohlraumstrahlung). *Mathematische Annalen*, 71(4):441–479, 1912.
- David P Woodruff et al. Sketching as a tool for numerical linear algebra. *Foundations and Trends® in Theoretical Computer Science*, 10(1–2):1–157, 2014.

Rahul Yedida, Snehanshu Saha, and Tejas Prashanth. Lipschitzlr: Using theoretically computed adaptive learning rates for fast convergence. *Applied Intelligence*, 51:1460–1478, 2021.

Nicholas Young. *An introduction to Hilbert space*. Cambridge university press, 1988.

William Henry Young. On classes of summable functions and their fourier series. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 87(594):225–229, 1912.

## Appendix A. Probability Theory

In this section we will give various concentration inequalities on the inlier data for functions in the Reproducing Kernel Hilbert Space. We will first give our assumptions for robust kernelized regression.

**Assumption 7 (Gaussian Design)** We assume for  $\mathbf{x}_i \sim \mathcal{P} \in \mathcal{X}$ , then it follows for the feature map,  $\phi(\cdot) : \mathcal{X} \rightarrow \mathcal{H}$ ,

$$\mathbf{x}_i \sim \mathcal{P}$$

where  $\Gamma$  is a possibly infinite dimensional covariance operator.

**Assumption 8 (Bounded Functions)** We assume for  $\mathbf{x}_i \sim \mathcal{P} \in \mathcal{X}$ , then it follows for the feature map,  $\phi(\cdot) : \mathcal{X} \rightarrow \mathcal{H}$ ,

$$\sup_{\mathbf{x} \in \mathcal{X}} \|\phi(\mathbf{x})\|_{\mathcal{H}}^2 \leq P_k < \infty$$

where  $\mathcal{H}$  is a Reproducing Kernel Hilbert Space.

**Assumption 9 (Normal Residuals)** The residual is defined as  $\mu_i \triangleq f^*(\mathbf{x}_i) - y_i$ . Then we assume for some  $\sigma > 0$ , it follows

$$\mu_i \sim \mathcal{N}(0, \sigma^2)$$

### A.1. Finite Dimensional Concentrations of Measure

**Proposition 10** Let  $\mu_1, \dots, \mu_n \sim \mathcal{N}(0, \sigma^2)$  for some  $\sigma > 0$ , then it follows for any  $s \geq 1$

$$\Pr \left\{ \max_{i \in [n]} |\mu_i| \geq \sigma \sqrt{2 \log n} \cdot s \right\} \leq \frac{\sqrt{2}}{\log n} e^{-s^2}$$

**Proof.** Let  $C$  be a positive constant to be determined.

$$\begin{aligned} \Pr_{\mu_i \sim \mathcal{N}(0, \sigma^2)} \left\{ \max_{i \in [n]} |\mu_i| \geq C \cdot s \right\} &\stackrel{(i)}{=} 2n \Pr_{\mu \sim \mathcal{N}(0, \sigma^2)} \{ \mu \geq C \cdot s \} = \frac{2n}{\sigma \sqrt{2\pi}} \int_{C \cdot s}^{\infty} e^{-\frac{1}{2} \left( \frac{x}{\sigma} \right)^2} dx \\ &\leq 2\sigma n \left( \frac{1}{C \cdot s} \right) e^{-\frac{1}{2} \left( \frac{C \cdot s}{\sigma} \right)^2} \leq \frac{\sqrt{2} n^{1-s^2}}{s \log n} \leq \frac{\sqrt{2}}{\log n} e^{-s^2} \end{aligned}$$

(i) follows from a union bound and noting for a i.i.d sequence of random variables  $\{X_i\}_{i \in [n]}$  and a constant  $C$ , it follows  $\Pr\{\max_{i \in [n]} X_i \geq C\} = n \Pr\{X \geq C\}$ . In the second to last inequality, we plug in  $C \triangleq \sigma \sqrt{2 \log n}$ . Our proof is now complete.  $\blacksquare$

**Proposition 11** Let  $\mu_1, \dots, \mu_n \sim \mathcal{N}(0, \sigma^2)$  for some  $\sigma > 0$ , then it follows for any  $s \geq 1$ ,

$$\Pr \left\{ \sum_{i=1}^n \mu_i^2 \geq 8n\sigma^2 \cdot s \right\} \leq 4e^{-s}$$

**Proof.** Concatenate all the samples  $\mu_i$  into a vector  $\boldsymbol{\mu} \in \mathbb{R}^n$ . Our proof generalizes for a  $\boldsymbol{\mu} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$  where  $\boldsymbol{\Sigma} \triangleq \mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^\top$  for a unitary  $\mathbf{U}$  and positive diagonal  $\boldsymbol{\Lambda}$ . Let  $C$  be a positive to be determined constant, we then have

$$\Pr_{\boldsymbol{\mu} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})} \left\{ \|\boldsymbol{\mu}\|^2 \geq C \cdot s \right\} = \Pr_{\boldsymbol{\mu} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})} \left\{ \|\boldsymbol{\mu}\| \geq \sqrt{C \cdot s} \right\} \leq 4 \exp \left( -\frac{C \cdot s}{8 \text{Tr}(\boldsymbol{\Sigma})} \right)$$

where the last inequality follows from Proposition 12. Now choosing  $C \triangleq 8 \text{Tr}(\boldsymbol{\Sigma})$  completes the proof.  $\blacksquare$

## A.2. Hilbert Space Concentrations of Measure

**Proposition 12 (Gaussian Concentration (Ledoux and Talagrand, 2013))** Suppose  $X$  is a Gaussian Variable in a Banach Space. Then,

$$\Pr \{ \|X\| > t \} \leq 4 \exp \left( -\frac{t^2}{8 \mathbf{E} \|X\|^2} \right)$$

**Proposition 13 (Gaussian Concentration from Mean (Pinelis and Sakhanenko, 1986))** Suppose  $X$  is a Gaussian Variable in a Banach Space. Then,

$$\Pr \{ \|X\| - \mathbf{E} \|X\| \geq t \} \leq \exp \left( -\frac{t^2}{2 \mathbf{E} \|X\|^2} \right)$$

Noting that all Hilbert Spaces are Banach Spaces (Young, 1988), we will use this proposition throughout the section.

**Lemma 14 (Sum of Sub-Gaussian Hilbert Space Functions)** Suppose  $X_1, \dots, X_{n(1-\epsilon)} \sim \mathcal{P}$  and  $X_{n(1-\epsilon)+1}, \dots, Z_n \sim \mathcal{Q}$  where  $\mathcal{P}$  and  $\mathcal{Q}$  are sub-Gaussian with proxy trace class operators,  $\mathbf{\Gamma}$  and  $\mathbf{\Lambda}$ , respectively. Let  $a_1, \dots, a_n$  be a fixed set of numbers in  $\mathbb{R}$ . Then  $\sum_{i=1}^n a_i X_i$  is sub-Gaussian with proxy  $(\sum_{i=1}^n a_i^2) (\mathbf{\Gamma} + \mathbf{\Lambda})$ .

**Proof.** Let  $v \in \mathcal{H}$  such that  $\|v\|_{\mathcal{H}} = 1$ . Then, we have for a  $\theta > 0$ ,

$$\begin{aligned} \mathbf{E} \left[ \exp \left( \theta \left\langle \sum_{i=1}^n a_i X_i, v \right\rangle_{\mathcal{H}} \right) \right] &= \mathbf{E} \left[ \prod_{i=1}^n \exp(\theta \langle a_i X_i, v \rangle_{\mathcal{H}}) \right] \stackrel{(i)}{\leq} \prod_{i=1}^n \mathbf{E} [\exp(n a_i \theta \langle X_i, v \rangle_{\mathcal{H}})]^{1/n} \\ &\leq \prod_{i=1}^{n(1-\epsilon)} \exp \left( \frac{\theta^2 a_i^2 \langle v, \mathbf{\Gamma} v \rangle_{\mathcal{H}}}{2} \right) \prod_{i=n(1-\epsilon)+1}^n \exp \left( \frac{\theta^2 a_i^2 \langle v, \mathbf{\Lambda} v \rangle_{\mathcal{H}}}{2} \right) \\ &\leq \exp \left( \frac{\theta^2 (\sum_{i=1}^n a_i^2) \langle v, (\mathbf{\Gamma} + \mathbf{\Lambda}) v \rangle_{\mathcal{H}}}{2} \right) \end{aligned}$$

where (i) follows from Hölder's Inequality for a product of functions (Hölder, 1889). We see the resultant variance proxy is  $n(1-\epsilon)\mathbf{\Gamma} + n\epsilon\mathbf{\Lambda}$  and the proof is complete.  $\blacksquare$

**Theorem 15 (Hilbert Space Hanson Wright (Chen and Yang, 2021))** Let  $X_i$  be a i.i.d sequence of sub-Gaussian random variables in  $\mathcal{H}$  such that  $\mathbf{E}[X_i] = 0$  and  $\mathbf{E}[X_i \otimes X_i] = \mathbf{\Gamma}$ . Then there exists a universal constant  $C > 0$  s.t. for any  $t > 0$ ,

$$\Pr \left\{ \sum_{i=1}^n \langle X_i, X_i \rangle_{\mathcal{H}} \geq n \text{Tr}(\mathbf{\Gamma}) + t \right\} \leq 2 \exp \left[ -C \min \left( \frac{t^2}{n \|\mathbf{\Gamma}\|_{\text{HS}}^2}, \frac{t}{\|\mathbf{\Gamma}\|_{\text{op}}} \right) \right]$$

From Theorem 15, it follows that the LHS is less than  $\delta \in (0, 1)$  when

$$t \geq \frac{1}{C} \|\mathbf{\Gamma}\|_{\text{op}} \log(2/\delta) \vee \sqrt{\frac{1}{C} n \|\mathbf{\Gamma}\|_{\text{HS}}^2 \log(2/\delta)}$$

Furthermore, we have when

$$\delta \leq 2 \exp \left[ -nC \left( \frac{\|\mathbf{\Gamma}\|_{\text{HS}}}{\|\mathbf{\Gamma}\|_{\text{op}}} \right)^2 \right]$$

it follows

$$t \geq \frac{1}{C} \|\mathbf{\Gamma}\|_{\text{op}} \log(2/\delta)$$

In other words, when the failure probability is sufficiently small we can use the above bound. We will reference this idea throughout this section.

**Proposition 16 (Probabilistic Maximum  $P_k$ )** *Let  $\mathbf{x}_i \sim \mathcal{P}$  such that  $\phi(\mathbf{x}_i) \sim \mathcal{N}(\mathbf{0}, \mathbf{\Gamma})$  (Assumption 7). Then it follows for any  $s \in \mathbb{R}_+$*

$$\Pr_{\mathbf{x}_i \sim \mathcal{P}} \left\{ \max_{i \in [n]} \|\phi(\mathbf{x}_i)\|_{\mathcal{H}} \geq \sqrt{2 \text{Tr}(\mathbf{\Gamma}) \log n} \cdot s \right\} \leq e^{-s^2}$$

**Proof.** Let  $C$  be a positive to be determined constant.

$$\begin{aligned} \Pr_{\mathbf{x}_i \sim \mathcal{P}} \left\{ \max_{i \in [n]} \|\phi(\mathbf{x}_i)\|_{\mathcal{H}} \geq C \cdot s \right\} &\stackrel{(i)}{\leq} n \Pr_{\mathbf{x} \sim \mathcal{P}} \{ \|\phi(\mathbf{x})\|_{\mathcal{H}} \geq C \cdot s \} \\ &\stackrel{(ii)}{\leq} n \inf_{\theta > 0} \mathbf{E} [\exp(\theta \|\phi(\mathbf{x})\|_{\mathcal{H}})] \exp(-\theta C \cdot s) \leq n \inf_{\theta \geq 0} \exp\left(\frac{\theta^2 \|\mathbf{\Gamma}\|_{\text{op}}}{2} - \theta C \cdot s\right) \\ &= n \exp\left(-\frac{C^2 \cdot s^2}{2 \|\mathbf{\Gamma}\|_{\text{op}}}\right) \end{aligned}$$

See (i) from the proof of Proposition 10. In (ii) we use the definition of a sub-Gaussian element of  $\mathcal{H}$  from  $\mathcal{P}$ . Setting  $C \triangleq \sqrt{2 \text{Tr}(\mathbf{\Gamma}) \log n}$  completes the proof.  $\blacksquare$

**Fact 17 (Sum of Binomial Coefficients (Cormen et al., 2022))** *Let  $k, n \in \mathbb{N}$  such that  $k \leq n$ , then*

$$\sum_{i=0}^k \binom{n}{i} \leq \left(\frac{en}{k}\right)^k$$

**Proposition 18 (Jensen's Inequality (Jensen, 1906))** *Suppose  $\varphi$  is a convex function, then for a random variable  $X$ , it holds*

$$\varphi(\mathbf{E}[X]) \leq \mathbf{E}[\varphi(X)]$$

*The inequality is reversed for  $\varphi$  concave.*

**Proposition 19 (RKHS Norm of Functions in the Reproducing Kernel Hilbert Space)** *Let  $\mathbf{x}_i \sim \mathcal{P}$  such that  $\mathbf{x}_i \sim \mathcal{P}$  (Assumption 7). Denote  $\mathcal{S}$  as all subsets of  $[n(1-\epsilon)]$  with size  $n(1-2\epsilon)$  for  $\epsilon < 0.5$  and  $P_B = \sum_{i=1}^n$ . Then it follows with probability exceeding  $1-\delta$ ,*

$$\max_{B \in \mathcal{S}} \left\| \int_{\mathcal{X}} r(X) dP_B(x) \right\|_{\mathcal{H}}^2 \leq n(1-2\epsilon) \left( \|\mathbf{\Gamma}\|_{\text{Tr}} + \frac{e}{C} \|\mathbf{\Gamma}\|_{\text{op}} \log \frac{1}{2\delta} \right)$$

**Proof.** We will use a standard symmetrization argument to obtain the expectation.

$$\mathbf{E}_{\mathbf{x}_i \sim \mathcal{P}} \left\| \sum_{i=1}^{n(1-2\epsilon)} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 = \mathbf{E}_{\mathbf{x}_i \sim \mathcal{P}} \mathbf{E}_{\xi_i \sim \mathcal{R}} \left\| \sum_{i=1}^{n(1-2\epsilon)} \xi_i \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2$$

$$= \mathbf{E}_{\mathbf{x}_i \sim \mathcal{P}} \mathbf{E}_{\xi_i \sim \mathcal{R}} \sum_{i=1}^{n(1-2\epsilon)} \sum_{j=1}^{n(1-2\epsilon)} \xi_i \xi_j k(\mathbf{x}_i, \mathbf{x}_j) \stackrel{(i)}{=} n(1-2\epsilon) \text{Tr}(\mathbf{\Gamma})$$

In (i) we note  $\mathbf{E} \|\Phi\|_{\text{HS}}^2 = \mathbf{E} \text{Tr}(\Phi \otimes \Phi) = n(1-2\epsilon) \text{Tr}(\mathbf{\Gamma})$ . From Proposition 13, we then obtain for sufficiently large  $\delta$ , it falls with probability at least  $1 - \delta$ ,

$$\left\| \sum_{i=1}^{n(1-2\epsilon)} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}} \geq n(1-2\epsilon) \text{Tr}(\mathbf{\Gamma}) + \frac{1}{C} \|\mathbf{\Gamma}\|_{\text{op}} \log(2/\delta)$$

We next apply a union bound over  $\mathcal{S}$  with Fact 17, we have

$$\begin{aligned} \max_{B \in \mathcal{S}} \left\| \sum_{i \in B} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 &\geq n(1-2\epsilon) \text{Tr}(\mathbf{\Gamma}) + \frac{1}{C} \|\mathbf{\Gamma}\|_{\text{op}} n(1-2\epsilon) \log \frac{e(1-2\epsilon)}{(1-\epsilon)} + \|\mathbf{\Gamma}\|_{\text{op}} \frac{1}{C} \log \frac{1}{2\delta} \\ &= n(1-2\epsilon) \left( \text{Tr}(\mathbf{\Gamma}) + \frac{e}{C} \|\mathbf{\Gamma}\|_{\text{op}} \right) + \frac{1}{C} \|\mathbf{\Gamma}\|_{\text{op}} \log \frac{1}{2\delta} \end{aligned}$$

This completes our proof.  $\blacksquare$

We will now study the covariance approximation problem. Our main probabilistic tool will be McDiarmid's Inequality.

**Proposition 20 (McDiarmid's Inequality (McDiarmid et al., 1989))** *Suppose  $f : \mathcal{X}_1 \times \dots \times \mathcal{X}_n \rightarrow \mathbb{R}$ . Consider i.i.d  $X_1, \dots, X_n$  where  $X_i \in \mathcal{X}_i$  for all  $i \in [n]$ . If there exists constants  $c_1, \dots, c_n$ , such that for all  $x_i \in \mathcal{X}_i$  for all  $i \in [n]$ , it holds*

$$\sup_{\tilde{X}_i \in \mathcal{X}_i} \left| f(X_1, \dots, X_{i-1}, X_i, X_{i+1}, \dots, X_n) - f(X_1, \dots, X_{i-1}, \tilde{X}_i, X_{i+1}, \dots, X_n) \right| \leq c_i$$

Then for any  $t > 0$ , it holds

$$\Pr \{ f(X_1, \dots, X_n) - \mathbf{E} [f(X_1, \dots, X_n)] \geq t \} \leq \exp \left( - \frac{2t^2}{\sum_{i=1}^n c_i^2} \right)$$

**Theorem 21 (Mean Estimation in the Hilbert Space (Tolstikhin et al., 2017))** *Define  $P_n \triangleq \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$  and  $P$  be the distribution of the covariates in  $\mathcal{X}$ . Suppose  $r : \mathcal{X} \rightarrow \mathcal{H}$  is a continuous function such that  $\sup_{X \in \mathcal{X}} \|r(X)\|_{\mathcal{H}}^2 \leq C_k < \infty$ . Then with probability at least  $1 - \delta$ ,*

$$\left\| \int_{\mathcal{X}} r(x) dP_n(x) - \int_{\mathcal{X}} r(x) dP(x) \right\| \leq \sqrt{\frac{C_k}{n}} + \sqrt{\frac{2C_k \log(1/\delta)}{n}}$$

We will strengthen upon the result by Tolstikhin et al. (2017) by using knowledge of the distribution to first derive the expectation.

**Proposition 22 (Probabilistic Bound on Infinite Dimensional Covariance Estimation)** *Let  $\mathbf{x}_1, \dots, \mathbf{x}_n$  be i.i.d sampled from  $\mathcal{P}$  such that  $\phi(\mathbf{x}_i) \sim \mathcal{P}$  (Assumption 7). Denote  $\mathcal{S}$  as all subsets of  $[n]$  with size from  $n(1-2\epsilon)$  to  $n(1-\epsilon)$ . We then have simultaneously with probability exceeding  $1 - \delta$ ,*

$$\left\| \frac{1}{n} \sum_{i=1}^n \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) - \mathbf{\Gamma} \right\|_{\text{HS}} \leq \sqrt{\frac{8}{n}} \|\mathbf{\Gamma}\|_{\text{op}} + \sqrt{\frac{2 \log(2/\delta)}{n}} P_k$$



$$\max_{A \in \mathcal{S}} \left\| \frac{1}{n(1-\epsilon)} \sum_{i \in A} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) - \Gamma \right\|_{\text{HS}} \leq \sqrt{\frac{8}{n(1-\epsilon)}} \|\Gamma\|_{\text{op}} + \sqrt{\frac{2P_k^2 \log(2/\delta)}{n(1-\epsilon)}} + P_k \sqrt{2\epsilon \log \frac{e}{\epsilon}}$$

**Proof.** We will calculate the mean operator in the Hilbert Space  $\mathcal{H} \otimes \mathcal{H}$  and use the  $\sqrt{n}$ -consistency of estimating the mean-element in a Hilbert Space to obtain the probability bounds.

$$\begin{aligned} & \mathbf{E}_{\phi(\mathbf{x}_i) \sim \mathcal{P}} \left\| \frac{1}{n(1-\epsilon)} \sum_{i=1}^{n(1-\epsilon)} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) - \Gamma \right\|_{\text{HS}} \\ & \stackrel{(ii)}{\leq} \mathbf{E}_{\phi(\mathbf{x}_i) \sim \mathcal{P}} \mathbf{E}_{\phi(\tilde{\mathbf{x}}_i) \sim \mathcal{P}} \left\| \frac{1}{n(1-\epsilon)} \sum_{i=1}^{n(1-\epsilon)} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) - \phi(\tilde{\mathbf{x}}_i) \otimes \phi(\tilde{\mathbf{x}}_i) \right\|_{\text{HS}} \\ & \stackrel{(iii)}{=} \mathbf{E}_{\phi(\mathbf{x}_i) \sim \mathcal{P}} \mathbf{E}_{\phi(\tilde{\mathbf{x}}_i) \sim \mathcal{P}} \mathbf{E}_{\xi_i \sim \mathcal{R}} \left\| \frac{1}{n(1-\epsilon)} \sum_{i=1}^{n(1-\epsilon)} \xi_i (\phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) - \phi(\tilde{\mathbf{x}}_i) \otimes \phi(\tilde{\mathbf{x}}_i)) \right\|_{\text{HS}} \\ & \leq \mathbf{E}_{\phi(\mathbf{x}_i) \sim \mathcal{P}} \mathbf{E}_{\xi_i \sim \mathcal{R}} \left\| \frac{2}{n(1-\epsilon)} \sum_{i=1}^{n(1-\epsilon)} \xi_i \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) \right\|_{\text{HS}} \\ & \leq \frac{2}{n(1-\epsilon)} \mathbf{E}_{\phi(\mathbf{x}_i) \sim \mathcal{P}} \left( \mathbf{E}_{\xi_i \sim \mathcal{R}} \left\| \sum_{i=1}^{n(1-\epsilon)} \xi_i \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) \right\|_{\text{HS}}^2 \right)^{1/2} \end{aligned}$$

In (ii) we apply a union bound. In (ii) we note that  $\phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) - \Gamma$  is a mean  $\mathbf{0}$  operator in the tensor product space  $\mathcal{H} \otimes \mathcal{H}$ . Then for  $X, Y \in \mathcal{H} \otimes \mathcal{H}$  s.t.  $\mathbf{E}[Y] = \mathbf{0}$  it follows  $\|X\|_{\text{HS}} = \|X - \mathbf{E}[Y]\|_{\text{HS}} = \|\mathbf{E}[X - Y]\|_{\text{HS}}$  and finally we apply Jensen's Inequality. Let  $e_k$  for  $k \in [p]$  ( $p$  possibly infinite) represent a complete orthonormal basis for the image of  $\Gamma$ . By expanding out the Hilbert-Schmidt Norm, we then have

$$\begin{aligned} & \frac{2}{n(1-\epsilon)} \left( \mathbf{E}_{\phi(\mathbf{x}_i) \sim \mathcal{P}} \mathbf{E}_{\xi_i \sim \mathcal{R}} \left\| \sum_{i=1}^{n(1-\epsilon)} \xi_i \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) \right\|_{\text{HS}}^2 \right)^{1/2} \\ & = \frac{2}{n(1-\epsilon)} \left( \mathbf{E}_{\phi(\mathbf{x}_i) \sim \mathcal{P}} \mathbf{E}_{\xi_i \sim \mathcal{R}} \sum_{k=1}^p \left\langle \sum_{i=1}^{n(1-\epsilon)} \xi_i \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) e_k, \sum_{j=1}^{n(1-\epsilon)} \xi_j \phi(\mathbf{x}_j) \otimes \phi(\mathbf{x}_j) e_k \right\rangle_{\text{HS}} \right)^{1/2} \\ & = \frac{2}{n(1-\epsilon)} \left( \mathbf{E}_{\phi(\mathbf{x}_i) \sim \mathcal{P}} \mathbf{E}_{\xi_i \sim \mathcal{R}} \sum_{k=1}^p \sum_{i=1}^{n(1-\epsilon)} \sum_{j=1}^{n(1-\epsilon)} \xi_i \xi_j \langle \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) e_k, \phi(\mathbf{x}_j) \otimes \phi(\mathbf{x}_j) e_k \rangle_{\text{HS}} \right)^{1/2} \\ & \stackrel{(iv)}{\leq} \frac{2}{n(1-\epsilon)} \left( \mathbf{E}_{\phi(\mathbf{x}_i) \sim \mathcal{P}} \sum_{k=1}^p \sum_{i=1}^{n(1-\epsilon)} \langle \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) e_k, \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) e_k \rangle_{\text{HS}} \right)^{1/2} \\ & = \frac{2}{n(1-\epsilon)} \left( \sum_{i=1}^{n(1-\epsilon)} \mathbf{E}_{\phi(\mathbf{x}_i) \sim \mathcal{P}} \|\phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i)\|_{\text{HS}}^2 \right)^{1/2} \\ & \stackrel{(v)}{=} \frac{2}{\sqrt{n(1-\epsilon)}} \left( \mathbf{E}_{\phi(\mathbf{x}_i) \sim \mathcal{P}} \|\phi(\mathbf{x}_i)\|_{\mathcal{H}}^4 \right)^{1/2} \end{aligned}$$

(iv) follows from noticing  $\mathbf{E}_{\xi_i, \xi_j \sim \mathcal{R}} [\xi_i \xi_j] = \delta_{ij}$ . (v) follows from expanding the Hilbert-Schmidt Norm and applying Parseval's Identity. We have

$$\mathbf{E}_{\mathbf{x} \sim \mathcal{X}} [\|\phi(\mathbf{x})\|_{\mathcal{H}}^4] = \int_0^\infty \Pr \left\{ \|\phi(\mathbf{x})\|_{\mathcal{H}}^4 \geq t \right\} dt = \int_0^\infty \Pr \left\{ \|\phi(\mathbf{x})\|_{\mathcal{H}} \geq t^{1/4} \right\} dt$$

$$\begin{aligned}
&\stackrel{(vi)}{\leq} \int_0^\infty \inf_{\theta>0} \mathbf{E}_{\mathbf{x} \sim \mathcal{X}} [\exp(\theta \|\phi(\mathbf{x})\|_{\mathcal{H}})] \exp(-\theta t^{1/4}) dt \leq \int_0^\infty \inf_{\theta>0} \exp\left(\frac{\theta^2 \|\mathbf{\Gamma}\|_{\text{op}}}{2} - \theta t^{1/4}\right) dt \\
&= \int_0^\infty \exp\left(-\frac{\sqrt{t}}{\|\mathbf{\Gamma}\|_{\text{op}}}\right) dt = 2 \|\mathbf{\Gamma}\|_{\text{op}}^2
\end{aligned}$$

In (vi) we apply Markov's Inequality. From which we obtain,

$$\mathbf{E}_{\phi(\mathbf{x}_i) \sim \mathcal{P}} \left\| \frac{1}{n(1-\epsilon)} \sum_{i=1}^{n(1-\epsilon)} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) - \mathbf{\Gamma} \right\|_{\text{HS}} \leq \sqrt{\frac{8}{n(1-\epsilon)}} \|\mathbf{\Gamma}\|_{\text{op}}$$

Then, define the function  $r(\mathbf{x}) : \mathcal{X} \rightarrow \mathcal{H} \otimes \mathcal{H}$ ,  $\mathbf{x} \rightarrow \phi(\mathbf{x}) \otimes \phi(\mathbf{x})$ . From Assumption 8, we have  $r(\mathbf{x}) = \|\phi(\mathbf{x}) \otimes \phi(\mathbf{x})\|_{\text{HS}} \leq \|\phi(\mathbf{x})\|_{\mathcal{H}}^2 \leq P_k$ . We will use McDiamard's Inequality, consider  $\tilde{P} \triangleq \delta_{X_i}$  with one modified element. Then consider the equation  $f(x_1, \dots, x_n) : \mathcal{X} \times \dots \times \mathcal{X} \rightarrow \mathcal{H} \otimes \mathcal{H} \times \dots \times \mathcal{H} \otimes \mathcal{H}$ ,  $x_1, \dots, x_n \rightarrow \|\int_{\mathcal{X}} r(x) dP_B(x) - \int_{\mathcal{X}} r(x) dP(x)\|_{\text{HS}}$ .

$$\begin{aligned}
&\left\| \int_{\mathcal{X}} r(x) dP_B(x) dx - \int_{\mathcal{X}} r(x) dP(x) dx \right\|_{\text{HS}} - \left\| \int_{\mathcal{X}} r(x) dP_{\tilde{B}}(x) dx - \int_{\mathcal{X}} r(x) dP(x) dx \right\|_{\text{HS}} \\
&\leq \frac{1}{n(1-\epsilon)} (\|r(x_i)\|_{\text{HS}} + \|r(\tilde{x}_i)\|_{\text{HS}}) \leq \frac{2P_k}{n(1-\epsilon)}
\end{aligned}$$

Then, we have from McDiarmid's inequality (Proposition 20),

$$\mathbf{Pr} \left\{ \left\| \int_{\mathcal{X}} r(x) dP_B(x) - \int_{\mathcal{X}} r(x) dP(x) \right\|_{\text{HS}} - \sqrt{\frac{8}{n(1-\epsilon)}} \|\mathbf{\Gamma}\|_{\text{op}} \geq t \right\} \leq \exp \left( -\frac{2t^2 n(1-\epsilon)}{P_k^2} \right)$$

We then have our first claim with probability exceeding  $1 - \delta$ ,

$$\left\| \int_{\mathcal{X}} r(x) dP_B(x) - \int_{\mathcal{X}} r(x) dP(x) \right\|_{\text{HS}} \leq \sqrt{\frac{8}{n(1-\epsilon)}} \|\mathbf{\Gamma}\|_{\text{op}} + \sqrt{\frac{2P_k^2 \log(2/\delta)}{n(1-\epsilon)}}$$

Next, applying a union bound over  $\mathcal{S}$  with Fact 17, we have

$$\max_{B \in \mathcal{S}} \left\| \int_{\mathcal{X}} r(x) dP_B(x) - \int_{\mathcal{X}} r(x) dP(x) \right\|_{\text{HS}} \leq \sqrt{\frac{8}{n(1-\epsilon)}} \|\mathbf{\Gamma}\|_{\text{op}} + \sqrt{\frac{2P_k^2 \log(2/\delta)}{n(1-\epsilon)}} + 2P_k^2 \epsilon \log \frac{e}{\epsilon}$$

Simplifying the resultant bound completes the proof.  $\blacksquare$

## Appendix B. Proofs for Structural Results

In this section we give the deferred proofs of our main structural results of the subquantile objective function.

### B.1. Projection onto a Norm Ball

In this section we show normalizing on to a norm-ball in the RKHS can be implemented efficiently.

**Lemma 23** Let  $\mathcal{K} \triangleq \{f : \|f\|_{\mathcal{H}} \leq R\}$ . Then, for a  $\hat{f} \notin \mathcal{K}$ , it follows

$$\text{Proj}_{\mathcal{K}} \hat{f} = \left( \frac{R}{\|\hat{f}\|} \right) \hat{f}$$

**Proof.** We will formulate the dual problem and then find the corresponding  $f_{\mathbf{w}}$  that solves the dual.

$$\begin{aligned} \text{Proj}_{\mathcal{K}} \hat{f} &= \arg \min_{f \in \mathcal{K}} \|f - \hat{f}\|_{\mathcal{H}}^2 = \arg \min_{f \in \mathcal{K}} \|f\|_{\mathcal{H}}^2 + \|\hat{f}\|_{\mathcal{H}}^2 - 2\langle f, \hat{f} \rangle_{\mathcal{H}} \\ &= \arg \min_{f \in \mathcal{K}} \|f\|_{\mathcal{H}}^2 - 2\langle f, \hat{f} \rangle_{\mathcal{H}} \end{aligned}$$

From here we can solve the dual problem. The Lagrangian is given by,

$$\mathcal{L}(f, u) \triangleq \|f\|_{\mathcal{H}}^2 - 2\langle f, \hat{f} \rangle + u (\|f\|_{\mathcal{H}}^2 - R^2)$$

Then, we have dual problem as  $\theta(u) = \min_{f \in \mathcal{H}} \mathcal{L}(f, u)$ . Taking the derivative of the Lagrangian and setting it to zero, we obtain  $\arg \min_{f \in \mathcal{H}} \mathcal{L}(f, u) = (1 + u)^{-1} \hat{f}$ . With some more work, we obtain  $\arg \max_{u > 0} \theta(u) = R^{-1} \|\hat{f}\| - 1$ . We then have  $f$  at  $u^*$  as  $f = R \|\hat{f}\|_{\mathcal{H}}^{-1} \hat{f}$ . Since  $\|\hat{f}\| > R$  as  $\hat{f} \notin \mathcal{K}$  by assumption, our proof is complete.  $\blacksquare$

### Appendix C. Proofs for Kernelized Regression

We will first give a simple calculation of the  $\beta$ -smoothness parameter of the subquantile objective. We then will give proofs for our approximation error bounds.

**Lemma 24 ( $L$ -Lipschitz of  $g(t, f)$  w.r.t  $f$ )** Let  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \sim \hat{\mathcal{P}}$ . It then follows:

$$|g(t, f) - g(t, \hat{f})| \leq L \|f - \hat{f}\|_{\mathcal{H}}$$

where

$$L = \frac{2}{n(1 - \epsilon)} \left( R \|\mathbf{K}\| + \sqrt{n} \|\mathbf{y}\|_{\infty} \sqrt{\text{Tr}(\mathbf{K})} \right)$$

Over the distribution of the target distribution, if

$$n \geq (1 - \epsilon)^{-1} \left( 450 + 100 \left( \frac{P_k}{\lambda_{\max}(\mathbf{\Gamma})} \right)^2 \log(2/\delta) \right)$$

we have with probability  $1 - \delta$ ,

$$R \|\mathbf{\Gamma}\|_{\text{op}} + \|\mathbf{y}\|_{\infty} \sqrt{\|\mathbf{\Gamma}\|_{\text{Tr}}} \leq L \leq R \left( \sqrt{2} \|\mathbf{\Gamma}\|_{\text{op}} + \epsilon(1 - \epsilon)^{-1} Q_k \right) + \|\mathbf{y}\|_{\infty} \sqrt{\text{Tr}(\mathbf{\Gamma}) + \epsilon Q_k}$$

**Proof.** For any  $f, \hat{f} \in \mathcal{K}$ , we will first show the gradient is bounded.

$$|g(t, f) - g(t, \hat{f})| = \left| \int_0^1 \nabla_f g(t, (1 - \lambda)f + \lambda\hat{f})(f - \hat{f}) d\lambda \right|$$

$$\leq \|f - \hat{f}\|_{\mathcal{H}} \left| \int_0^1 \nabla_f g(t, (1-\lambda)f + \lambda\hat{f}) d\lambda \right| \stackrel{(i)}{\leq} \|f - \hat{f}\|_{\mathcal{H}} \max_{\tilde{f} \in \mathcal{K}} \|\nabla_f g(t, \tilde{f})\|_{\mathcal{H}}$$

In (i), we note that since  $\mathcal{K}$  is convex, then by definition as  $f, \hat{f} \in \mathcal{K}$ , we have for any  $\lambda \in [0, 1]$ , the convex combination  $(1-\lambda)f + \lambda\hat{f} \in \mathcal{K}$ . We use the  $\mathcal{H}$  norm of the gradient to bound  $L$  from above for an element in the convex closed set  $\mathcal{K}$ .

$$\|\nabla g(t, f)\|_{\mathcal{H}} = \left\| \frac{2}{n(1-\epsilon)} \sum_{i=1}^n \mathbb{I}\{t \geq (f(\mathbf{x}_i) - y_i)^2\} (f(\mathbf{x}_i) - y_i) \cdot \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}$$

W.L.O.G, let  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$  where  $0 \leq m \leq n$ , represent the data vectors such that  $t \geq (f(\mathbf{x}_i) - y_i)^2$ .

$$\begin{aligned} &= \left\| \frac{2}{n(1-\epsilon)} \sum_{i=1}^m (f(\mathbf{x}_i) - y_i) \cdot k(\mathbf{x}_i, \cdot) \right\|_{\mathcal{H}} \leq \frac{2}{n(1-\epsilon)} \left( \left\| \sum_{i=1}^m f(\mathbf{x}_i) \cdot k(\mathbf{x}_i, \cdot) \right\|_{\mathcal{H}} + \left\| \sum_{i=1}^m y_i k(\mathbf{x}_i, \cdot) \right\|_{\mathcal{H}} \right) \\ &\stackrel{(i)}{\leq} \frac{2}{n(1-\epsilon)} \left( \left\| \sum_{i=1}^m \left\langle \sum_{j=1}^n w_j \phi(\mathbf{x}_j), \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \cdot \phi(\mathbf{x}_i) \right\|_{\mathcal{H}} + \|\mathbf{y}\|_{\infty} \sum_{i=1}^n \sqrt{k(\mathbf{x}_i, \mathbf{x}_i)} \right) \\ &\leq \frac{2}{n(1-\epsilon)} \left( \|f\|_{\mathcal{H}} \left\| \sum_{i=1}^n \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) \right\|_{\text{op}} + \|\mathbf{y}\|_{\infty} \left( \sum_{i=1}^n \sqrt{k(\mathbf{x}_i, \mathbf{x}_i)} \right) \right) \\ &\leq \frac{2}{n(1-\epsilon)} \left( R \|\mathbf{K}\| + \|\mathbf{y}\|_{\infty} \sqrt{n \|\mathbf{K}\|_{\text{Tr}}} \right) \end{aligned}$$

(i) follows from the reproducing property for RKHS (Gretton, 2013). Similar results for the Lipschitz Constant for non-kernelized learning algorithms can be seen in Yedida et al. (2021).

**Lower Bound.** We will first lower bound the trace

$$\begin{aligned} \|\mathbf{K}\| &= \|\Phi \otimes \Phi\|_{\text{op}} \geq \|\Phi_P \otimes \Phi_P\|_{\text{op}} = \|n(1-\epsilon)\mathbf{\Gamma} + \Phi_P \otimes \Phi_P - n(1-\epsilon)\mathbf{\Gamma}\|_{\text{op}} \\ &\geq n(1-\epsilon)\lambda_{\max}(\mathbf{\Gamma}) - \|\Phi_P \otimes \Phi_P - n(1-\epsilon)\mathbf{\Gamma}\|_{\text{op}} \\ &\geq n(1-\epsilon)\lambda_{\max}(\mathbf{\Gamma}) - \sqrt{n(1-\epsilon)} \left( \sqrt{8}\lambda_{\max}(\mathbf{\Gamma}) + \sqrt{2P_k^2 \log(2/\delta)} \right) \\ &\geq 0.5n(1-\epsilon)\lambda_{\max}(\mathbf{\Gamma}) \end{aligned}$$

when  $n \geq (1-\epsilon)^{-1} \left( 64 + 16 \left( \frac{P_k}{\lambda_{\max}(\mathbf{\Sigma})} \right)^2 \log(2/\delta) \right)$  with probability exceeding  $1 - \delta$ . We find a similar bound for  $\|\mathbf{K}\|_{\text{Tr}}$  by noting  $\|\mathbf{K}\|_{\text{Tr}} \geq \|\Phi_P \otimes \Phi_P\|_{\text{HS}}$ .

$$\sqrt{n \|\mathbf{K}\|_{\text{Tr}}} \geq 0.5n(1-\epsilon)\sqrt{\text{Tr}(\mathbf{\Gamma})}$$

when  $n \geq (1-\epsilon)^{-1} \left( 64 \left( \frac{\|\mathbf{\Gamma}\|_{\text{op}}}{\|\mathbf{\Gamma}\|_{\text{HS}}} \right)^2 + 16 \left( \frac{P_k}{\|\mathbf{\Gamma}\|_{\text{HS}}} \right)^2 \log(2/\delta) \right)$ . This completes our proof of the lower bound.

**Upper Bound.** We will upper bound the operator norm and extend the bound to the Trace.

$$\|\mathbf{K}\| = \|\Phi \otimes \Phi\|_{\text{op}} \leq n(1-\epsilon)\lambda_{\max}(\mathbf{\Gamma}) + \|\Phi_P \otimes \Phi_P - n(1-\epsilon)\mathbf{\Gamma}\|_{\text{HS}} + n\epsilon Q_k$$

$$\begin{aligned} &\leq n(1-\epsilon)\lambda_{\max}(\mathbf{\Gamma}) + \sqrt{n(1-\epsilon)} \left( \sqrt{8}\lambda_{\max}(\mathbf{\Gamma}) + \sqrt{2P_k^2 \log(2/\delta)} \right) + n\epsilon Q_k \\ &\leq 2n(1-\epsilon)\lambda_{\max}(\mathbf{\Gamma}) + n\epsilon Q_k \end{aligned}$$

when  $n \geq (1-\epsilon)^{-1} \left( 16 + 4 \left( \frac{P_k}{\|\mathbf{\Gamma}\|} \right)^2 \log(2/\delta) \right)$  with probability exceeding  $1 - \delta$ .

$$\begin{aligned} \text{Tr}(\mathbf{K}) &\leq \sum_{i \in [P]} k(\mathbf{x}_i, \mathbf{x}_i) + n\epsilon Q_k \leq n(1-\epsilon) \text{Tr}(\mathbf{\Gamma}) + C^{-1} \|\mathbf{\Gamma}\|_{\text{op}} \log(2/\delta) + n\epsilon Q_k \\ &\leq 2n(1-\epsilon) \text{Tr}(\mathbf{\Gamma}) + n\epsilon Q_k \end{aligned}$$

when  $n \geq (1-\epsilon)^{-1} C^{-1} \log(2/\delta) \left( \frac{\|\mathbf{\Gamma}\|_{\text{op}}}{\|\mathbf{\Gamma}\|_{\text{Tr}}} \right)$ . Rearranging completes the proof.  $\blacksquare$

We now give the necessary results to prove our main approximation bound.

**Proposition 25 (Young's Inequality (Young, 1912))** *For all  $a, b \in \mathbb{R}$ , it holds*

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

**Lemma 26 (Smooth Descent Lemma)** *Suppose  $\alpha \leq \|\nabla^2 f(x)\|_{\text{op}} \leq \beta$  for all  $x \in \mathcal{X}$ , then for a stepsize  $\eta \leq 1/L$ , it follows for all  $x \in \mathcal{X}$ ,*

$$f(x - \eta \nabla f(x)) \leq \left( 1 - \frac{\mu}{L} \right) (f(x) - f^*(x))$$

We are now ready to prove our main approximation bound.

### C.1. Proof of Theorem 5 (In Progress)

**Proof.** From Algorithm 1, we have for kernelized linear regression the following update,

$$f^{(t+1)} = \text{Proj}_{\mathcal{K}} \left[ f^{(t)} - \frac{2\eta}{n(1-\epsilon)} \sum_{i \in S^{(t)}} (f^{(t)}(\mathbf{x}_i) - y_i) \cdot \phi(\mathbf{x}_i) - C f^{(t)} \right] \quad (2)$$

Next, we note that we can partition  $S = (S \cap P) \cup (S \cap Q) \triangleq \text{TP} \cup \text{FP}$ . Then we have

$$\begin{aligned} \|f^{(t+1)} - f^*\|_{\mathcal{H}}^2 &= \|\text{Proj}_{\mathcal{K}} [f^{(t)} - \nabla_f g(f^{(t)}, t^*)] - f^*\|_{\mathcal{H}}^2 \\ &\stackrel{(i)}{\leq} \|f^{(t)} - \nabla_f g(f^{(t)}, t^*) - f^*\|_{\mathcal{H}}^2 \\ &\leq 2\|f^{(t)} - \nabla_f g(f^{(t)}, t^*) - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 + 2\|\text{Proj}_{\Psi_m^\perp} f^*\|^2 \\ &= 2\|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 - 4\eta \langle \nabla_f g(f^{(t)}, t^*), f^{(t)} - \text{Proj}_{\Psi_m} f^* \rangle_{\mathcal{H}} \\ &\quad + 2\eta^2 \|\nabla_f g(f^{(t)}, t^*)\|_{\mathcal{H}}^2 + 2\|\text{Proj}_{\Psi_m^\perp} f^*\|^2 \end{aligned} \quad (3)$$

where (i) follows from noting the projection is a contraction. We will dedicate the rest of the proof to upper bounding the first three terms in Equation (3). We will first bound the second term in Equation (3) by splitting it into terms using the following relation,

$$2\eta \langle \nabla_f g(f^{(t)}, t^*), f^{(t)} - \text{Proj}_{\Psi_m} f^* \rangle_{\mathcal{H}}$$

$$\begin{aligned}
&\stackrel{(2)}{=} 2\eta \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \frac{2}{n(1-\epsilon)} \sum_{i \in S^{(t)}} (f^{(t)}(\mathbf{x}_i) - y_i) \cdot \phi(\mathbf{x}_i) + C f^{(t)} \right\rangle_{\mathcal{H}} \\
&\stackrel{(i)}{=} \frac{4\eta}{n(1-\epsilon)} \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \sum_{i \in S^{(t)} \cap P} (f^{(t)}(\mathbf{x}_i) - f^*(\mathbf{x}_i) - \mu_i) \cdot \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \\
&\quad + \frac{4\eta}{n(1-\epsilon)} \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \sum_{i \in S^{(t)} \cap Q} (f^{(t)}(\mathbf{x}_i) - y_i) \cdot \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \\
&\quad + \frac{4\eta}{n(1-\epsilon)} \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, C f^{(t)} \right\rangle_{\mathcal{H}}
\end{aligned} \tag{4}$$

where (i) follows from Theorem 7. We will now lower bound the first term of Equation (4).

$$\begin{aligned}
&\frac{4\eta}{n(1-\epsilon)} \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \sum_{i \in S^{(t)} \cap P} (f^{(t)}(\mathbf{x}_i) - f^*(\mathbf{x}_i) - \mu_i) \cdot \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \\
&= \frac{4\eta}{n(1-\epsilon)} \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \left[ \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) \right] (f^{(t)} - \text{Proj}_{\Psi_m} f^*) \right\rangle_{\mathcal{H}} \\
&\quad + \frac{4\eta}{n(1-\epsilon)} \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \left[ \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) \right] (\text{Proj}_{\Psi_m^\perp} f^*) \right\rangle_{\mathcal{H}} \\
&\quad - \frac{4\eta}{n(1-\epsilon)} \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \sum_{i \in S^{(t)} \cap P} \mu_i \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \\
&\stackrel{(ii)}{\geq} 8\eta \frac{(1-2\epsilon)}{(1-\epsilon)} \|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \left( \lambda_m(\Gamma) - \left\| \frac{1}{n(1-\epsilon)} \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) - \Gamma \right\|_{\text{HS}} \right) \\
&\quad - \frac{4\eta^2}{n(1-\epsilon)} \|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \left\| \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) \right\|_{\text{op}} \left\| \text{Proj}_{\Psi_m^\perp} f^* \right\|_{\mathcal{H}} \\
&\quad - \frac{1}{n(1-\epsilon)} \left\| \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) \right\|_{\text{op}} \left\| \text{Proj}_{\Psi_m^\perp} f^* \right\|_{\mathcal{H}} - \frac{8\lambda_m(\Gamma)\eta(1-2\epsilon)}{(1-\epsilon)} \left\| \text{Proj}_{\Psi_m^\perp} f^{(t)} \right\|_{\mathcal{H}}^2 \\
&\quad - \frac{4\eta}{n(1-\epsilon)} \|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \left\| \sum_{i \in S^{(t)} \cap P} \mu_i \phi(\mathbf{x}_i) \right\|_{\mathcal{H}} - \frac{1}{n(1-\epsilon)} \left\| \sum_{i \in S^{(t)} \cap P} \mu_i \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}
\end{aligned} \tag{5}$$

where in (ii) we define  $\tilde{n} \triangleq |S^{(t)} \cap P|$  and use Cauchy-Schwarz on the last two terms, we then have the simple inequality  $\|\text{Proj}_{\Psi_m} [f^{(t)} - f^*]\|_{\mathcal{H}} = \|f^{(t)} - \text{Proj}_{\Psi_m} f^* - \text{Proj}_{\Psi_m^\perp} f^{(t)}\|_{\mathcal{H}}^2 \leq 2\|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 + 2\|\text{Proj}_{\Psi_m^\perp} f^{(t)}\|_{\mathcal{H}}^2$  and split the final two terms with Young's Inequality (Proposition 25). We will now upper bound the second and third terms of Equation (4) together.

$$\begin{aligned}
&\frac{4\eta}{n(1-\epsilon)} \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \sum_{i \in S^{(t)} \cap Q} (f^{(t)}(\mathbf{x}_i) - y_i) \cdot \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} + \frac{4\eta}{n(1-\epsilon)} \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, C f^{(t)} \right\rangle_{\mathcal{H}} \\
&\leq \frac{4\eta}{n(1-\epsilon)} \|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}} \left( \left\| \sum_{i \in S^{(t)} \cap Q} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}} \sqrt{\sum_{i \in S^{(t)} \cap Q} (f^{(t)}(\mathbf{x}_i) - y_i)^2} + C \|f^{(t)}\|_{\mathcal{H}} \right)
\end{aligned}$$

$$\stackrel{(iii)}{\leq} \frac{16C^2\eta}{[n(1-\epsilon)]^2} \|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \left\| \sum_{i \in S^{(t)} \cap Q} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 + \eta \left( \sum_{i \in S^{(t)}} (f^{(t)}(\mathbf{x}_i) - y_i)^2 + C \|f^{(t)}\|_{\mathcal{H}}^2 \right) \quad (6)$$

where (iii) follows from Young's Inequality (Proposition 25) for a  $\beta \in (0, 1)$ . We also assume  $C \geq 1$ . We will now upper bound the final term in Equation (3).

$$\begin{aligned} \eta^2 \|\nabla f g(f^{(t)}, t^*)\|_{\mathcal{H}}^2 &= \frac{4\eta^2}{n^2(1-\epsilon)^2} \left\| \sum_{i \in S^{(t)}} (f^{(t)}(\mathbf{x}_i) - y_i) \cdot \phi(\mathbf{x}_i) + C f^{(t)} \right\|_{\mathcal{H}}^2 \\ &\leq \frac{8C\eta^2}{n^2(1-\epsilon)^2} \left\| \sum_{i \in S^{(t)}} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 \left( \sum_{i \in S^{(t)}} (f^{(t)}(\mathbf{x}_i) - y_i)^2 + C \|f^{(t)}\|_{\mathcal{H}}^2 \right) \end{aligned} \quad (7)$$

We can now complete the upper bound for  $\|f^{(t+1)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2$  combining (5)-(7).

$$\begin{aligned} \|f^{(t+1)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 &\leq \|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \\ &\cdot \left( 1 - \frac{8\eta(1-2\epsilon)}{(1-\epsilon)} \left( \lambda_m(\mathbf{\Gamma}) - \left\| \frac{1}{n(1-\epsilon)} \sum_{i \in S^{(t+1)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) - \mathbf{\Gamma} \right\|_{\text{HS}} \right) + \frac{16\eta}{[n(1-\epsilon)]^2} \left\| \sum_{i \in S^{(t)} \cap Q} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 \right. \\ &\quad \left. + \frac{4\eta^2}{n(1-\epsilon)} \left\| \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) \right\|_{\text{op}} \|\text{Proj}_{\Psi_m^\perp} f^*\|_{\mathcal{H}} + \frac{4\eta}{n(1-\epsilon)} \left\| \sum_{i \in S^{(t)} \cap P} \mu_i \phi(\mathbf{x}_i) \right\|_{\mathcal{H}} \right) \\ &\quad + \left( \eta + \frac{8\eta^2}{n^2(1-\epsilon)^2} \left\| \sum_{i \in S^{(t)}} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 \right) \left( \sum_{i \in S^{(t)}} (f^{(t)}(\mathbf{x}_i) - y_i)^2 + C \|f^{(t)}\|_{\mathcal{H}}^2 \right) + \lambda_m(\mathbf{\Gamma}) \|\text{Proj}_{\Psi_m^\perp} f^{(t)}\|_{\mathcal{H}}^2 \\ &\quad + \frac{4\eta}{n(1-\epsilon)} \left\| \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) \right\|_{\text{op}} \|\text{Proj}_{\Psi_m^\perp} f^*\|_{\mathcal{H}} + \frac{2\eta}{n(1-\epsilon)} \left\| \sum_{i \in S^{(t)} \cap P} \mu_i \phi(\mathbf{x}_i) \right\|_{\mathcal{H}} \\ &\triangleq \|f^{(t+1)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \leq \|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 (1 - I_1 + I_2 + I_3 + I_4) \\ &\quad (\eta + II_1) \left( \sum_{i \in S^{(t)}} (f^{(t)}(\mathbf{x}_i) - y_i)^2 + C \|f^{(t)}\|_{\mathcal{H}}^2 \right) + III_1 + III_2 + III_3 \end{aligned} \quad (8)$$

We will now look at the residual term. From noting that Subquantile Minimization is  $\beta$ -smooth and  $\alpha$  strongly convex for kernelized ridge regression (see § C), we have from the Descent Lemma (see Lemma 26).

$$\begin{aligned} \sum_{i \in S^{(t+1)}} (f^{(t+1)}(\mathbf{x}_i) - y_i)^2 + C \|f^{(t+1)}\|_{\mathcal{H}}^2 &\leq \sum_{i \in S^{(t)}} (f^{(t+1)}(\mathbf{x}_i) - y_i)^2 + C \|f^{(t+1)}\|_{\mathcal{H}}^2 \\ &\leq \left( 1 - \frac{C}{L} \right) \left( \sum_{i \in S^{(t)}} (f^{(t)}(\mathbf{x}_i) - y_i)^2 + C \|f^{(t)}\|_{\mathcal{H}}^2 \right) \end{aligned}$$

where the first inequality follows from the optimality of the subquantile set, and the second inequality follows from the Descent Lemma (Lemma 26). We denote the term parameterized by  $t \in [T]$ ,

$$\Lambda^{(t)} \triangleq \|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 + \sum_{i \in S^{(t)}} (f^{(t)}(\mathbf{x}_i) - y_i)^2 + C \|f^{(t)}\|_{\mathcal{H}}^2$$

We will show there exists  $n$  such that  $\Lambda^{(t+1)} \leq (1 - \Omega(L^{-1})) \cdot \Lambda^{(t)} + E$  where  $E$  is some small error such that  $E \searrow 0$  as  $n \rightarrow \infty$ . On a high-level, we have  $I_1 = O(\eta \lambda_m(\mathbf{\Gamma}))$ . We analyze  $II_1$  first. From the assumption that the corrupted covariates are centered, we have for a sufficiently small  $\delta$  that with probability at least  $1 - \delta$  from Proposition 19,

$$\begin{aligned} \left\| \sum_{i \in S^{(t)}} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 &\leq 2 \max_{\substack{\sigma \in \Pi \\ |\sigma| = n(1-\epsilon)}} \left\| \sum_{i=1}^{n(1-\epsilon)} \phi(\mathbf{x}_{\sigma(i)}) \right\|_{\mathcal{H}}^2 + 2 \left\| \sum_{i \in S^{(t)} \cap Q} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 \\ &\leq 2n(1-\epsilon) \left( \text{Tr}(\mathbf{\Gamma}) + \frac{e}{C} \|\mathbf{\Gamma}\|_{\text{op}} \log(2/\delta) + Q_k \right) \end{aligned}$$

where  $Q_k = \max_{i \in Q} k(\mathbf{x}_i, \mathbf{x}_i)$ . We then have with probability exceeding  $1 - \delta$ ,

$$II_1 \leq \frac{4\eta^2}{n(1-\epsilon)} \left( \text{Tr}(\mathbf{\Gamma}) + \frac{e}{C} \|\mathbf{\Gamma}\|_{\text{op}} \log(2/\delta) + Q_k \right) \leq \frac{1}{4L}$$

when  $n \geq 16L^{-1} (1-\epsilon)^{-1} \left( \text{Tr}(\mathbf{\Gamma}) + \frac{e}{C} \|\mathbf{\Gamma}\|_{\text{op}} \log(2/\delta) + Q_k \right)$ . This gives us  $II \leq 1 - \frac{C-2}{L}$ , then from our choice of  $C$ , we have  $II \leq 1 - L^{-1}$ . We will analyze the terms in  $I$  individually.

$$I_2 \triangleq \frac{4\eta}{[n(1-\epsilon)]^2} \left\| \sum_{i \in S^{(t)} \cap Q} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 \leq \frac{4Q_k \ln(1/\delta)}{Ln(1-\epsilon)}$$

We then see that  $I_2 \leq (1/4)L^{-1}$  when  $n \geq 16Q_k \ln(1/\delta)$ .

$$\begin{aligned} I_3 &\triangleq \frac{4\eta^2}{n(1-\epsilon)} \left\| \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) \right\|_{\text{op}} \left\| \text{Proj}_{\Psi_m^\perp} f^* \right\|_{\mathcal{H}} \\ &\leq 4\eta^2 \left\| \text{Proj}_{\Psi_m^\perp} f^* \right\|_{\mathcal{H}} \left( \lambda_{\max}(\mathbf{\Gamma}) + \left\| \frac{1}{n(1-2\epsilon)} \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) - \mathbf{\Gamma} \right\|_{\text{HS}} \right) \\ &\leq 4\eta^2 \left\| \text{Proj}_{\Psi_m^\perp} f^* \right\|_{\mathcal{H}} \left( \lambda_{\max}(\mathbf{\Gamma}) + \sqrt{\frac{8}{n(1-\epsilon)}} \lambda_{\max}(\mathbf{\Gamma}) + \sqrt{\frac{2P_k^2 \log(2/\delta)}{n(1-\epsilon)} + 2P_k^2 \epsilon \log \frac{e}{\epsilon}} \right) \end{aligned}$$

Next we analyze  $I_4$ , let  $\mathcal{S}$  be the set of all combinations of subsets of size  $[n(1-2\epsilon)]$  to  $[n(1-\epsilon)]$ . Then with probability exceeding  $1 - \delta$ , we have

$$I_4 \triangleq \left\| \sum_{i \in S^{(t)} \cap P} \mu_i \phi(\mathbf{x}_i) \right\|_{\mathcal{H}} \leq \sqrt{2\sigma^2 n(1-\epsilon) \log \frac{\sqrt{2}}{\delta \log n} \log^2 n(1-\epsilon) \left( \text{Tr}(\mathbf{\Gamma}) + \frac{e}{C} \|\mathbf{\Gamma}\|_{\text{op}} \log(1/2\delta) \right)}$$

## Appendix D. Proofs for Kernelized Binary Classification

In this section, we will prove error bounds for Subquantile Minimization in the Kernelized Binary Classification Problem.



### D.1. Proof of Theorem 6 (In Progress)

From Algorithm 2, we have for kernelized binary classification,

$$f^{(t+1)} = \text{Proj}_{\mathcal{K}} \left[ f^{(t)} - \frac{\eta}{n(1-\epsilon)} \sum_{i \in S^{(t)}} \left( \sigma(f^{(t)}(\mathbf{x}_i)) - y_i \right) \cdot \phi(\mathbf{x}_i) \right] \quad (9)$$

From which it follows,

$$\begin{aligned} \|f^{(t+1)} - f^*\|_{\mathcal{H}}^2 &= \left\| \text{Proj}_{\mathcal{K}} \left[ f^{(t)} - \frac{\eta}{n(1-\epsilon)} \nabla g(f^{(t)}, t^*) \right] - f^* \right\|_{\mathcal{H}}^2 \\ &\stackrel{(i)}{\leq} \left\| f^{(t)} - \frac{\eta}{n(1-\epsilon)} \nabla g(f^{(t)}, t^*) - f^* \right\|_{\mathcal{H}}^2 \\ &\leq 2 \|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 - \frac{4\eta}{n(1-\epsilon)} \left\langle \nabla f g(f^{(t)}, t^*), f^{(t)} - \text{Proj}_{\Psi_m} f^* \right\rangle_{\mathcal{H}} \\ &\quad + \frac{2\eta^2}{n^2(1-\epsilon)^2} \|\nabla f g(f^{(t)}, t^*)\|_{\mathcal{H}}^2 + 2 \|\text{Proj}_{\Psi_m^\perp} f^*\|_{\mathcal{H}}^2 \end{aligned} \quad (10)$$

where (i) follows from the contraction property of the projection operator onto norm ball  $\mathcal{K}$  and assuming  $f^* \in \mathcal{K}$ . We will expand the second term in Equation (10).

$$\begin{aligned} &\frac{2\eta}{n(1-\epsilon)} \left\langle \nabla f g(f^{(t)}, t^*), f^{(t)} - \text{Proj}_{\Psi_m} f^* \right\rangle_{\mathcal{H}} \\ &\stackrel{(9)}{=} \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \frac{2\eta}{n(1-\epsilon)} \sum_{i \in S^{(t)}} \left( \sigma(f^{(t)}(\mathbf{x}_i)) - y_i \right) \cdot \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \\ &= \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \frac{2\eta}{n(1-\epsilon)} \sum_{i \in S^{(t)}} \left( \sigma(f^{(t)}(\mathbf{x}_i)) - \sigma(f^*(\mathbf{x}_i)) \right) \cdot \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \\ &\quad + \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \frac{2\eta}{n(1-\epsilon)} \sum_{i \in S^{(t)}} \left( \sigma(f^*(\mathbf{x}_i)) - y_i \right) \cdot \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \end{aligned} \quad (11)$$

We first upper bound upper bound the second term in Equation (11). From the Cauchy-Schwarz Inequality and noting  $y_i \in \{0, 1\}$  and  $\text{range}(\sigma) \in (0, 1)$ , we have the following,

$$\begin{aligned} &\left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \frac{2\eta}{n(1-\epsilon)} \sum_{i \in S^{(t)}} \left( \sigma(f^*(\mathbf{x}_i)) - y_i \right) \cdot \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \\ &\leq \frac{2\eta}{n(1-\epsilon)} \|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}} \left\| \sum_{i \in S^{(t)}} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}} \max_{i \in S^{(t)}} |\sigma(f^*(\mathbf{x}_i)) - y_i| \\ &\stackrel{(ii)}{\leq} \frac{\eta^2}{n^{2-\beta}(1-\epsilon)^{2-\beta}} \|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \left\| \sum_{i \in S^{(t)}} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 + \frac{2}{n^\beta(1-\epsilon)^\beta} \end{aligned} \quad (12)$$

where (ii) follows from Young's Inequality (Proposition 25) and noting for a vector  $\mathbf{x} \in \mathbb{R}^d$  it holds  $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2$  and letting  $\beta \in [0, 1]$  be an undetermined constant. Let us now consider the function  $h : \mathcal{H} \rightarrow \mathbb{R}$  defined as  $h(f) \triangleq \sum_{i \in S \cap P} \log(1 + \exp(f(\mathbf{x}_i)))$ . We can then calculate the gradients by hand,  $\nabla h(f) = \sum_{i \in S \cap P} \sigma(f(\mathbf{x}_i)) \cdot \phi(\mathbf{x}_i)$  and  $\nabla^2 h(f) = \sum_{i \in S \cap P} \sigma(f(\mathbf{x}_i))(1 - \sigma(f(\mathbf{x}_i))) \cdot$

$\phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i)$ . From the properties of strong convexity, we have for any  $f, \hat{f} \in \mathcal{H}$ , there exists  $\tilde{f} \in \mathcal{H}$  such that,

$$\begin{aligned} \left\langle f - \text{Proj}_{\Psi_m} \hat{f}, \nabla h(f) - \nabla h(\hat{f}) \right\rangle_{\mathcal{H}} &= \left\langle f - \text{Proj}_{\Psi_m} \hat{f}, \nabla^2 h(\tilde{f})(f - \hat{f}) \right\rangle_{\mathcal{H}} \\ &\stackrel{(iii)}{=} \left\langle \nabla^2 h(\tilde{f}), (f - \hat{f}) \otimes (f - \hat{f}) \right\rangle_{\text{HS}} + \left\langle \nabla^2 h(\tilde{f}), (\text{Proj}_{\Psi_m^\perp} f^*) \otimes (f - \hat{f}) \right\rangle_{\mathcal{H}} \end{aligned} \quad (13)$$

where the equality in (iii) is given in (Gretton, 2015, Section 3.2). Then, from the strong convexity of  $h$ , there exists a constant  $C$  such that the following inequality holds,

$$\begin{aligned} &\left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \frac{2\eta}{n(1-\epsilon)} \sum_{i \in S^{(t)} \cap P} (\sigma(f^{(t)}(\mathbf{x}_i)) - \sigma(f^*(\mathbf{x}_i))) \cdot \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \\ &\stackrel{(13)}{\gtrsim} \frac{2\eta}{n(1-\epsilon)} \left\langle \sum_{i \in S^{(t)}} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i), \text{Proj}_{\Psi_m} [f^{(t)} - f^*] \otimes \text{Proj}_{\Psi_m} [f^{(t)} - f^*] \right\rangle_{\text{HS}} \\ &\quad - \frac{2\eta}{n(1-\epsilon)} \left\| \text{Proj}_{\Psi_m^\perp} f^* \right\|_{\mathcal{H}} \left\| \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}} \\ &\stackrel{(iv)}{\gtrsim} 4\eta \frac{(1-2\epsilon)}{(1-\epsilon)} \left( \lambda_m(\mathbf{\Gamma}) - \left\| \frac{1}{n(1-2\epsilon)} \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) - \mathbf{\Gamma} \right\|_{\text{HS}} \right) \|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \\ &\quad - \frac{2\eta}{n(1-\epsilon)} \left\| \text{Proj}_{\Psi_m^\perp} f^* \right\|_{\mathcal{H}} \left\| \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}} - 4\eta \gamma \lambda_m(\mathbf{\Gamma}) \left\| \text{Proj}_{\Psi_m^\perp} f^{(t)} \right\|_{\mathcal{H}}^2 \end{aligned} \quad (14)$$

where (iv) follows from Weyl's inequality (Weyl, 1912) and noting that  $|S^{(t)} \cap P| \geq n(1-2\epsilon)$ . We now briefly analyze the constant introduced in Equation (14).

$$C \triangleq \inf_{\mathbf{x} \in P} \sigma(f(\mathbf{x}_i))(1 - \sigma(f(\mathbf{x}_i))) \geq (1/2) \exp \left( -\max_{\mathbf{x} \in P} f(\mathbf{x}) \right) \geq (1/2) \exp \left( -R \max_{\mathbf{x} \in P} \|\phi(\mathbf{x})\|_{\mathcal{H}} \right) \quad (15)$$

The final inequality follows from (Gretton, 2013, Theorem 17). Then, from the bijectivity of the exponential function, we can invoke Proposition 16, and with probability exceeding  $1 - \delta$ , we have

$$C \geq (1/2) \exp \left( -R \sqrt{8 \text{Tr}(\mathbf{\Gamma}) \log n \log \frac{4e}{\delta}} \right)$$

. Next, let us briefly analyze  $\left\| \text{Proj}_{\Psi_m^\perp} f^{(t)} \right\|_{\mathcal{H}}^2$ .

$$\begin{aligned} \left\| \text{Proj}_{\Psi_m^\perp} f^{(t+1)} \right\|_{\mathcal{H}}^2 &\leq 2 \left\| \text{Proj}_{\Psi_m^\perp} f^{(t)} \right\|_{\mathcal{H}}^2 + 2 \left\| \frac{\eta}{n(1-\epsilon)} \sum_{i \in S^{(t)}} (\sigma(f(\mathbf{x}_i)) - y_i)^2 \cdot \text{Proj}_{\Psi_m^\perp} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 \\ &\leq 2 \left\| \text{Proj}_{\Psi_m^\perp} f^{(t)} \right\|_{\mathcal{H}}^2 + 4 \left\| \frac{\eta}{n(1-\epsilon)} \sum_{i \in Q} \text{Proj}_{\Psi_m^\perp} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 + 4 \left\| \frac{\eta}{n(1-\epsilon)} \sum_{i \in S^{(t)} \cap P} \text{Proj}_{\Psi_m^\perp} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 \\ &\stackrel{(v)}{\leq} 2 \left\| \text{Proj}_{\Psi_m^\perp} f^{(t)} \right\|_{\mathcal{H}}^2 + \frac{4\eta^2 (\text{Tr}(\mathbf{\Gamma}_{22}) (1 + \frac{\epsilon}{C} \|\mathbf{\Gamma}\|_{\text{op}} \log(2/\delta)) + Q_k)}{n(1-\epsilon)} \end{aligned} \quad (16)$$

when in (v) we partition  $\mathbf{\Gamma}$  into

$$\mathbf{\Gamma} = \begin{matrix} m & \infty \\ \infty & \end{matrix} \begin{bmatrix} \mathbf{\Gamma}_{11} & \mathbf{\Gamma}_{12} \\ \mathbf{\Gamma}_{12} & \mathbf{\Gamma}_{22} \end{bmatrix}$$

From the recursion in Equation (16) and noting  $\|\text{Proj}_{\Psi_m^\perp} f^{(t)}\|_{\mathcal{H}} = 0$ , we have

$$\|\text{Proj}_{\Psi_m^\perp} f^{(t)}\|_{\mathcal{H}}^2 \leq \frac{2^{t+2}\eta^2 \left( \|\mathbf{\Gamma}_{22}\|_{\text{Tr}} \left( 1 + \frac{e}{C} \|\mathbf{\Gamma}_{22}\|_{\text{op}} \log(2/\delta) \right) + Q_k \right)}{n(1-\epsilon)}$$

We will now bound the third term in Equation (10).

$$\begin{aligned} \|\nabla_f g(f^{(t)}, t^*)\|_{\mathcal{H}}^2 &= \left\| \frac{\eta}{n(1-\epsilon)} \sum_{i \in S^{(t)}} (\sigma(f^{(t)}(\mathbf{x}_i)) - y_i) \cdot \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 \\ &\leq \frac{\eta^2}{n^2(1-\epsilon)^2} \max_{i \in S^{(t)}} |\sigma(f^{(t)}(\mathbf{x}_i)) - y_i|^2 \cdot \left\| \sum_{i \in S^{(t)}} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 \stackrel{(v)}{\leq} \frac{\eta^2}{n^2(1-\epsilon)^2} \left\| \sum_{i \in S^{(t)}} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 \end{aligned} \quad (17)$$

where (v) follows from noting for any  $\mathbf{x} \in \mathbb{R}^d$  it holds  $\|\mathbf{x}\|_{\infty} \leq \|\mathbf{x}\|_2$ . Furthermore, we note that if  $-\log(\sigma(f(\mathbf{x}))) \leq -\log(\sigma(f(\hat{\mathbf{x}})))$ , then  $\sigma(f(\mathbf{x})) \geq \sigma(f(\hat{\mathbf{x}}))$ . Now, combining (10)-(17), we obtain

$$\begin{aligned} \|f^{(t+1)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 &\leq \|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \\ &\cdot \left( 1 - \frac{2C\eta(1-2\epsilon)}{(1-\epsilon)} \left( \lambda_m(\mathbf{\Gamma}) - \left\| \frac{1}{n(1-2\epsilon)} \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) - \mathbf{\Gamma} \right\|_{\text{HS}} \right) + \frac{\eta^2}{[n(1-\epsilon)]^{3/2}} \left\| \sum_{i \in S^{(t)}} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 \right) \\ &+ \frac{\eta^2}{n^2(1-\epsilon)^2} \left\| \sum_{i \in S^{(t)}} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 + \frac{2}{\sqrt{n(1-\epsilon)}} + \frac{2\eta}{n(1-\epsilon)} \|\text{Proj}_{\Psi_m^\perp} f^*\|_{\mathcal{H}}^2 \left\| \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}} \\ &+ 4\eta\gamma C \lambda_m(\mathbf{\Gamma}) \left\| \text{Proj}_{\Psi_m^\perp} f^{(t)} \right\|_{\mathcal{H}}^2 \\ &\triangleq \|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 (1 - IV_1 + IV_2) + V_1 + V_2 + V_3 + V_4 \end{aligned}$$

Denote  $\mathcal{S}$  as the set of combinations of  $[n(1-2\epsilon)]$  to  $[n(1-\epsilon)]$  probability at least  $1-\delta$ , we have

$$\max_{\substack{\sigma \in \Pi \\ |\sigma|=n(1-\epsilon)}} \left\| \sum_{i=1}^{n(1-\epsilon)} \phi(\mathbf{x}_{\sigma(i)}) \right\|_{\mathcal{H}} \leq \sqrt{n(1-\epsilon) \left( \text{Tr}(\mathbf{\Gamma}) + \frac{e}{C} \|\mathbf{\Gamma}\|_{\text{op}} \log(2/\delta) \right)}$$

Then, from the assumption that the corrupted covariates are centered, we have for a sufficiently small  $\delta$  that with probability at least  $1-\delta$  from Proposition 19,

$$\begin{aligned} \left\| \sum_{i \in S^{(t)}} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 &\leq 2 \max_{\substack{\sigma \in \Pi \\ |\sigma|=n(1-\epsilon)}} \left\| \sum_{i=1}^{n(1-\epsilon)} \phi(\mathbf{x}_{\sigma(i)}) \right\|_{\mathcal{H}}^2 + 2 \left\| \sum_{i \in S^{(t)} \cap Q} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 \\ &\leq 2n(1-\epsilon) \left( \text{Tr}(\mathbf{\Gamma}) + \frac{e}{C} \|\mathbf{\Gamma}\|_{\text{op}} \log(2/\delta) + Q_k \right) \end{aligned}$$

where  $Q_k = \max_{i \in Q} k(\mathbf{x}_i, \mathbf{x}_i)$ . Next, to obtain

$$\lambda_m(\mathbf{\Gamma}) - \left\| \frac{1}{n(1-2\epsilon)} \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) - \mathbf{\Gamma} \right\|_{\text{HS}} \geq 0.9\lambda_m(\mathbf{\Gamma})$$

with probability greater than  $1 - \delta$ . We utilize Proposition 22 **update to new bound with dependence on  $\epsilon$**  and require

$$n = \frac{1600 \|\mathbf{\Gamma}\|_{\text{op}}^2 (1-2\epsilon)^{-1}}{\lambda_m^2(\mathbf{\Gamma})} + 400P_k^2(1-2\epsilon)^{-1} \log(2/\delta) \epsilon \log \frac{e}{\epsilon}$$

Assume the above data requirement is true. Define  $\gamma \triangleq \frac{1-\epsilon}{1-2\epsilon}$ . To solve the quadratic equation  $III \leq 99/100$ , we also require

$$n = C_n \left[ \frac{\left( \text{Tr}(\mathbf{\Gamma}) + \frac{\epsilon}{C} \|\mathbf{\Gamma}\|_{\text{op}} \log(2/\delta) + Q_k \right)^2}{3.24^2(1-\epsilon)\lambda_m^4(\mathbf{\Gamma})C^4\gamma^4} \right]$$

where  $C_n \geq 16/10000$  and is dependent on the rate of decrease. From where  $III \leq 99/100$  when

$$\frac{1.8 - \sqrt{3.24 - 0.04C_n^{-1/2}}}{3.24C\lambda_m(\mathbf{\Gamma})\gamma} \leq \eta \leq \frac{1.8 + \sqrt{3.24 - 0.04C_n^{-1/2}}}{3.24C\lambda_m(\mathbf{\Gamma})\gamma}$$

Thus, we have  $III \leq 0.99$  with probability exceeding  $1 - \delta$ . We will now show all the terms in  $IV$  are small. With probability at least  $1 - \delta$ ,

$$\begin{aligned} \frac{\eta^2}{[n(1-\epsilon)]^2} \left\| \sum_{i \in S^{(t)}} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 &\leq \frac{2\eta^2 \left( \text{Tr}(\mathbf{\Gamma}) + \frac{\epsilon}{C} \|\mathbf{\Gamma}\|_{\text{op}} \log(2/\delta) + Q_k \right)}{n(1-\epsilon)} \\ &\leq \frac{2 \cdot 3.24^2 \lambda_m^2(\mathbf{\Gamma}) C^2 \gamma^2}{C_n \left( \text{Tr}(\mathbf{\Gamma}) + \frac{\epsilon}{C} \|\mathbf{\Gamma}\|_{\text{op}} \log(2/\delta) + Q_k \right)} = O \left( \frac{\lambda_m^2(\mathbf{\Gamma})}{C_n \text{Tr}(\mathbf{\Gamma})} \right) \quad (18) \end{aligned}$$

Using the above bound, we obtain

$$\begin{aligned} 4\eta\gamma C\lambda_m(\mathbf{\Gamma}) \left\| \text{Proj}_{\Psi_m^\perp} f^{(t)} \right\|_{\mathcal{H}}^2 &\leq \frac{3.24^2 \cdot 2^{t+5} \lambda_m^2(\mathbf{\Gamma}) C^2 \gamma^2 \left( \|\mathbf{\Gamma}_{22}\|_{\text{Tr}} + \frac{\epsilon}{C} \|\mathbf{\Gamma}_{22}\|_{\text{op}} \log(2/\delta) + Q_m \right)}{C_n \left( \text{Tr}(\mathbf{\Gamma}) + \frac{\epsilon}{C} \|\mathbf{\Gamma}\|_{\text{op}} \log(2/\delta) + Q_k \right)} \\ &= O \left( \frac{2^{t+5} \lambda_m^2(\mathbf{\Gamma}) (\|\mathbf{\Gamma}_{22}\|_{\text{Tr}} + Q_m)}{C_n (\text{Tr}(\mathbf{\Gamma}) + Q_k)} \right) \quad (19) \end{aligned}$$

In Equations 18 and 19, we see that  $C_n$  is in the denominator and goes to  $\infty$  as  $n \nearrow \infty$ . Thus we see  $IV \searrow 0$  as  $n \nearrow \infty$ . We then have

$$\|f^{(T)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}} \leq 0.99^T \|f^{(0)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}} + \sum_{k=0}^T 0.99^k (IV) \leq \frac{\epsilon}{2} + 100(IV)$$

after  $\log \left( \frac{2\|f^*\|_{\mathcal{H}}}{\epsilon} \right)$  iterations, our proof is complete ■