Subquantile Minimization for Kernel Learning in the Huber ϵ-Contamination Model*

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Abstract

In this paper we propose Subquantile Minimization for learning with adversarial corruption in the training set. Superquantile objectives have been formed in the past in the context of fairness where one wants to learn an underrepresented distribution equally [LPMH21, RRM14]. Our intuition is to learn a more favorable representation of the *majority* class, thus we propose to optimize over the p-subquantile of the loss in the dataset. In particular, we study the Huber Contamination Problem for Kernel Learning where the distribution is formed as, $\hat{\mathbb{P}} = (1-\varepsilon)\mathbb{P} + \varepsilon\mathbb{Q}$, and we want to find the function $\inf_f \mathbb{E}_{\mathbf{x}\in\mathbb{P}}[\ell_f(\mathbf{x})]$, from the noisy distribution, $\hat{\mathbb{P}}$. We assume the adversary has knowledge of the true distribution of $\mathbb{P}$, and is able to corrupt the covariates and the labels of ε samples. To our knowledge, we are the first to study the problem of general kernel learning in the Huber Contamination Model. In our theoretical analysis, we analyze our non-convex concave objective function with the Moreau Envelope. We show (i) a stationary point with respect to the Moreau Envelope is a good point and (ii) we can reach a stationary point with gradient descent methods. Further, we analyze accelerated gradient methods for the non-convex concave minimax optimization problem. We empirically test Kernel Ridge Regression and Kernel Classification on various state of the art datasets and show Subquantile Minimization gives strong results. Furthermore, we run experiments on various datasets and compare with the state-of-the-art algorithms to show the superior performance of Subquantile Minimization.

*Preliminary Work

1 Introduction

There has been extensive study of algorithms to learn the target distribution from a Huber ε-Contaminated Model for a Generalized Linear Model (GLM), [DKK+19, ADKS22, LBSS21, OZS20, FB81] as well as for linear regression [BJKK17, MGJK19]. Robust Statistics has been studied extensively [DK23] for problems such as high-dimensional mean estimation [PBR19, CDGS20] and Robust Covariance Estimation [CDGW19, FWZ18]. Recently, there has been an interest in solving robust machine learning problems by gradient descent [PSBR18, DKK+19]. Subquantile minimization aims to address the shortcomings of standard ERM in applications of noisy/corrupted data [KLA18, JZL+18]. In many real-world applications, the covariates have a non-linear dependence on labels [AMMIL12, Section 3.4]. In which case it is suitable to transform the covariates to a different space utilizing kernels [HSS08]. Therefore, in this paper we consider the problem of Robust Learning for Kernel Learning.

Definition 1. (Huber ε-Contamination Model [HR09]). Given a corruption parameter $0 < \epsilon < 0.5$, a data matrix, $\mathbf{X}$ and labels $\mathbf{y}$. An adversary is allowed to inspect all samples and modify $n\varepsilon$ samples arbitrarily. The algorithm is then given the ϵ-corrupted data matrix $\mathbf{X}$ and $\mathbf{y}$ as training data.

Current approaches for robust learning across various machine learning tasks often use gradient descent over a robust objective, [LBSS21]. These robust objectives tend to not be convex and therefore do not have a strong analysis on the error bounds for general classes of models.

We similarly propose a robust objective which has a nonconvex-concave objective. This objective has also been proposed recently in [HYwL20] where there has been an analysis in the Binary Classification Task. We show Subquantile Minimization reduces to the same objective in [HYwL20]. We use theory from the weakly-convex concave optimization literature for our error bounds. We are able to levarage this theory by analyzing the asymptotic distribution of a softplus approximation of the Subquantile objective.

The study of Kernel Learning in the Gaussian Design is quite popular, [CLKZ21, Dic16]. In [CLKZ21], the feature space, $\phi(\mathbf{x}_i) \sim \mathcal{N}(0, \mathbf{\Sigma})$ where $\mathbf{\Sigma}$ is a diagonal matrix of dimension p, where p can be infinite. In this work, we adopt a similar framework, and with the power of Mercer's Theorem [Mer09], we are able to say $\mathrm{Tr}(\mathbf{\Sigma}) < \infty$. We use this fact extensively in our infinite-dimensional concentration inequalities.

Theorem 2. *(Informal). Let the dataset be given as $\{(\mathbf{x}_i, y_i)\}_{i=1}^n$ such that the labels and features of εn samples are arbitrarily corrupted by an adversary. Assume Subquantile Minimization returns $f_{\widehat{\mathbf{w}}}$ for $n \geq \frac{(1-2\varepsilon)(C_k\|\mathbf{\Sigma}\|_{\mathrm{op}}+\beta)}{(1-c_1)\lambda_{\min}(\mathbf{\Sigma})} + \sqrt{\beta}$ for a constant $c_1 \in (0,1)$ such that for*
Kernelized Regression:

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}} \|f_{\widehat{\mathbf{w}}} - f^*_{\mathbf{w}}\|_{\mathcal{H}} \leq O\left(\frac{\gamma\sigma}{\sqrt{\lambda_{\min}(\mathbf{\Sigma})}}\right) \tag{1}$$

where $\epsilon \to 0$ as number of gradient descenter iterations goes to ∞ and $\Sigma = \mathbb{E}[\phi(\mathbf{x}) \otimes \phi(\mathbf{x})]$.
Kernel Binary Classification:

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}} \|f_{\widehat{\mathbf{w}}} - f^*_{\mathbf{w}}\|_{\mathcal{H}} \leq O\left(\frac{\sqrt{\mathrm{Tr}(\mathbf{\Sigma})} + \sqrt{Q_k}}{\sqrt{n(1-2\varepsilon)}\lambda_{\min}(\mathbf{\Sigma})}\right) \tag{2}$$

Assume $n \geq (\frac{\mathrm{Tr}(\mathbf{\Sigma})}{\lambda_{\min}(\mathbf{\Sigma})(1-c)})^2$ for a constant $0 < c < 1$.
Kernel Multi-Class Classification:

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}} \left\|f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}\right\| \leq O\left(\frac{\sqrt{n(1-\varepsilon)\,\mathrm{Tr}(\mathbf{\Sigma})} + \sqrt{n\varepsilon Q_k}}{n(1-2\varepsilon)\lambda_{\min}(\mathbf{\Sigma})}\right) \tag{3}$$

1.1 Related Work

The idea of iterative thresholding algorithms for robust learning tasks dates back to 1806 by Legendre [Leg06]. From the popularity of Machine Learning, numerous algorithms have been developed in this idealogy. Therefore, we will dedicate this section to reviewing such works and to make clear our contributions to the iterative thresholding literature.

Robust Regression via Hard Thresholding [BJK15]. Bhatia et al. consider robust linear regression by considering an active set S, which contains the points with the lowest error. This set is updated each iteration in conjunction with either a full solve (Torrent-FC) or a gradient iteration (Torrent-GD). Torrent-GD is an unconstrained variant of our algorithm. The main limitation of this work is that only the case of label corruption is considered. We pick up the result of Theorem 9 and Theorem 11 in [BJK15] (up to constants) for linear regression with and without feature corruption, which is one of our key contributions.

Learning with bad training data via iterative trimmed loss minimization [SS19a]. This work considers optimizing over the bottom-k errors by choosing the αn points with smallest error and then updating the model from these αn. This general model is the same as ours. Theoretically, this work considers only general linear models. Experimentally, this work considers more general machine learning models such as GANS.

Trimmed Maximum Likelihood Estimation for Robust Generalized Linear Model [ADKS22]. This work studies a different class of generalized linear models. Interestingly, they show for Gaussian Regression the iterative trimmed maximum likelihood estimator is able to achieve near minimax optimal error. This work does not consider feature corruption and primarily focuses on the covariates sampled with Gaussian Design from Identity covariance.

Sum of Ranked Range Loss for Supervised Learning [HYwL20]. Hu et al. proposed learning over the bottom k losses, this is an alternative formulation of our algorithm. This is an extension of previous work studying the learning of the top k losses, [FLYH17]. They solve their optimization problem with difference of sums convex solvers. This work considers only the classification task and does not give rigorous error bounds. Subsequent work on analyzing the middle k losses is analyzed in [HYW+23].

The iterative trimmed loss framework with batch Stochastic Gradient Descent (SGD) is analyzed in [SS19b]. They experimentally test their design in deep learning applications such as image classification and Generative Adversarial Networks (GANs).

1.2 Contributions

We will now state our main contributions clearly.

1. We provide a novel theoretical framework using the Moreau Envelope for analyzing the iterative trimmed estimator for machine learning tasks.
2. We provide rigorous error bounds for subquantile minimization in the kernel regression, kernel binary classification, and kernel multi-class classification. Furthermore, we provide our bounds for both label and feature corruption with a general Gaussian Design.
3. We perform experiments on state-of-the-art matrices and show the effectiveness of our algorithm compared to other robust learning procedures. Furthermore, the experiments support the theory.

2 Subquantile Minimization

We propose to optimize over the subquantile of the risk. The p-quantile of a random variable, U, is given as $\mathcal{Q}_p(U)$, this is the largest number, t, such that the probability of $U \leq t$ is at least p.

$$\mathcal{Q}_p(U) \leq t \iff \mathbb{P}\left\{U \leq t\right\} \geq p \tag{4}$$

The p-subquantile of the risk is then given by

$$\mathbb{L}_p\left(U\right) = \frac{1}{p}\int_0^p \mathcal{Q}_p\left(U\right) dq = \mathbb{E}\left[U | U \leq \mathcal{Q}_p\left(U\right)\right] = \max_{t\in\mathbb{R}}\left\{t - \frac{1}{p}\mathbb{E}\left(t-U\right)^+\right\} \tag{5}$$

Given an objective function, ℓ, the kernelized learning poblem becomes:

$$f_{\widehat{\mathbf{w}}} = \arg\min_{f_\mathbf{w}\in\mathcal{K}}\max_{t\in\mathbb{R}}\left\{g(t, f_\mathbf{w}) \triangleq t - \sum_{i=1}^n \left(t - (f_\mathbf{w}(\mathbf{x}_i) - y_i)^2\right)^+\right\} \tag{6}$$

where t is the p-quantile of the empirical risk. Note that for a fixed t therefore the objective is not concave with respect to $\mathbf{w}$. Thus, to solve this problem we use the iterations from equation 11 in [RHL+20]. Let $\Pi_\mathcal{K}$ be the projection of a vector on to the convex set $\mathcal{K} \triangleq \{f \in \mathcal{H} : \|f\|_\mathcal{H} \leq R\}$, then our update steps are

$$t^{(k+1)} = \arg\max_{t\in\mathbb{R}} g(f_\mathbf{w}^{(k)}, t) \tag{7}$$

$$f_{\mathbf{w}}^{(k+1)} = \text{Proj}_{\mathcal{K}}\left(f_{\mathbf{w}}^{(k)} - \alpha \nabla_f g(f_{\mathbf{w}}^{(k)}, t^{(k+1)})\right) \tag{8}$$

We provide an algorithm for Subquantile Minimization of the ridge regression and classification kernel learning algorithm. **??** is applicable to both kernel ridge regression and kernel classication.

Input: Iterations: T, Quantile: p; Data Matrix: $\mathbf{X} \in \mathbb{R}^{n\times d}, n \gg d$; Labels: $\mathbf{y} \in \mathbb{R}^{n\times 1}$; Learning Rate schedule: $\alpha_1, \cdots, \alpha_T$
Output: Trained Parameters: $f_{\mathbf{w}}^{(t)}$

(1) Initialize the weights $\mathbf{w}_i^{(0)} \sim \text{Unif}\left[-\sqrt{\frac{6}{n}}, \sqrt{\frac{6}{n}}\right]$ for all $i \in [n]$.
(2) **for** $k = 1, 2, \ldots, T$ **do**
  (3) Find the Subquantile denoted as $S^{(k)}$ as the set of $(1-\varepsilon)n$ elements with the lowest error with respect to the loss function.
  (4) Update the gradient in accordance with the kernel learning problem.

$$\nabla_f g\left(t^{(k+1)}, f_{\mathbf{w}}^{(k)}\right) \leftarrow \sum_{i\in S^{(k)}} \left(f_{\mathbf{w}}^{(k)}(\mathbf{x}_i) - y_i\right) \cdot k(\mathbf{x}_i, \cdot) \tag{Regression}$$

$$\nabla_f g\left(t^{(k+1)}, f_{\mathbf{w}}^{(k)}\right) \leftarrow \sum_{i\in S^{(k)}} \left(\sigma\left(f_{\mathbf{w}}^{(k)}(\mathbf{x}_i)\right) - y_i\right) \cdot k(\mathbf{x}_i, \cdot) \tag{Binary Classification}$$

$$\nabla_f g\left(t^{(k+1)}, f_{\mathbf{W}}^{(k)}\right) \leftarrow \sum_{i\in S^{(k)}} \left(\text{softmax}\left(f_{\mathbf{W}}^{(k)}(\mathbf{x}_i)\right) - \mathbf{y}_i\right) \odot k(\mathbf{x}_i, \cdot) \tag{Multi-Class}$$

  (5) Perform Projected Standard Gradient Descent to find the next iterate

$$f_{\mathbf{w}}^{(k+1)} \leftarrow \text{Proj}_{\mathcal{K}}\left(f_{\mathbf{w}}^{(k)} - \alpha_{(k)} \nabla_f g\left(t^{(k+1)}, f_{\mathbf{w}}^{(k)}\right)\right)$$

(6) Pick t uniformly at random from $[T]$
(7) **Return:** $f_{\mathbf{w}}^{(t)}$

Algorithm 1: Subquantile Minimization for Kernel Learning

3 Theory

To consider theoretical guarantees of Subquantile Minimization, we first analyze the inner and outer optimization problems. We first analyze kernel learning in the presence of corrupted data. Next, we provide error bounds for the two most important kernel learning problems, kernel ridge regression, and kernel classification. Now we will give our first result regarding kernel learning in the Huber ϵ-contamination model. Now we will analyze the two-step minimax optimization steps described in Equations (7) and (8).

Lemma 3. *Let $f(\mathbf{x}; \mathbf{w})$ be a convex loss function. Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ denote the n data points ordered such that $f(\mathbf{x}_1; \mathbf{w}, y_1) \leq f(\mathbf{x}_2; \mathbf{w}, y_2) \leq \cdots \leq f(\mathbf{x}_n; \mathbf{w}, y_n)$. If we denote $\hat{\nu}_i \triangleq f(\mathbf{x}_i; \mathbf{w}, y_i)$, it then follows $\hat{\nu}_{np} \in \arg\max_{t\in\mathbb{R}} g(t, \mathbf{w})$.*

Proof is given in **??**.

Interpretation 4. From Lemma 3, we see the t will be greater than or equal to the errors of exactly np points. Thus, we are continuously updating over the np minimum errors.

Lemma 5. *Let $\hat{\nu}_i \triangleq f(\mathbf{x}_i; \mathbf{w}, y_i)$ s.t. $\hat{\nu}_{i-1} \leq \hat{\nu}_i \leq \hat{\nu}_{i+1}$, if we choose $t^{(k+1)} = \hat{\nu}_{np}$ as by Lemma 3, it then follows $\nabla_{\mathbf{w}} g(t^{(k)}, f_{\mathbf{w}}^{(k)}) = \frac{1}{np}\sum_{i=1}^{np} \nabla f(\mathbf{x}_i; f_{\mathbf{w}}^{(k)}, y_i)$*

Proof is given in Appendix B.2.

3.1 On the Softplus Approximation

It is clear our objective function is non-smooth. Thus we propose to use the Softplus approximation to smooth the function. The main ideas is to *first* approximate ReLU, consider the theory with respect to the approximation, and then take the limit as the approximation goes to the ReLU. The softplus approximation is given as follows,

$$\zeta_\lambda(x) = \frac{1}{\lambda}\log\left(1+e^{\lambda x}\right) \tag{9}$$

We then have the approximation of g as

$$\begin{aligned}\tilde{g}_\lambda(t, f_{\mathbf{w}}) &\triangleq t - \sum_{i=1}^{n} \zeta_\lambda\left(t - \ell\left(f_{\mathbf{w}}; \mathbf{x}_i, y_i\right)\right) \\ &= t - \frac{1}{np}\sum_{i=1}^{n}\frac{1}{\lambda}\log\left(1+\exp\left(\lambda\left(t-\ell\left(f_{\mathbf{w}}; \mathbf{x}_i, y_i\right)\right)\right)\right)\end{aligned} \tag{10}$$

More details on the Softplus Approximation such as exact computations can be found in Appendix B.3. We can then calculate the Lipschitz constant of the approximation function with respect to $f_{\mathbf{w}}$.

Lemma 6 (Lipschitz continuous gradient)**.** *Let $f_{\mathbf{w}}, f_{\tilde{\mathbf{w}}} \in \mathcal{K}$, then we have for any $\lambda > 0$,*

$$\left|\nabla_f \tilde{g}_\lambda(t, f_{\mathbf{w}}) - \nabla_f \tilde{g}_\lambda(t, f_{\tilde{\mathbf{w}}})\right| \leq \beta \left\|f_{\mathbf{w}} - f_{\tilde{\mathbf{w}}}\right\|_{\mathcal{H}} \tag{11}$$

where

$$\beta = \frac{1}{np}\sum_{i=1}^{n}\left\|\nabla_f^2 \ell\left(f_{\mathbf{w}}; \mathbf{x}_i, y_i\right)\right\|_{\mathrm{op}} \tag{12}$$

and β has no dependence on λ.

Proof is in Appendix B.4. This lemma is important as it states the β-smoothness constant is independent of the approximation term, λ. We will use this lemma in the next section by pushing $\lambda \to \infty$ and analyzing the resultant function.

3.2 Weakly Convex Concave Optimization Theory

With our smoothed function, we are now able to use the weakly-convex concave minimization literature to analyze g. The Moreau Envelope can be interpreted as an infimal convolution of the function f. When f is ρ-weakly convex, if $\lambda \leq \rho^{-1}$, then the Moreau Envelope is smooth.

Definition 7. (**Moreau Envelope on closed, convex set**, [Mor65]). Let f be proper lower semi-continuous convex function $\ell : \mathcal{K} \to \mathbb{R}$, where $\mathcal{K} \subset \mathcal{X}$ is a closed and convex set, then the Moreau Envelope is defined as:

$$\mathsf{M}_{\lambda\ell}(f_{\mathbf{w}}) \triangleq \inf_{f_{\hat{\mathbf{w}}}\in\mathcal{K}}\left\{\ell(f_{\hat{\mathbf{w}}}) + \frac{1}{2\rho}\left\|f_{\mathbf{w}} - f_{\hat{\mathbf{w}}}\right\|_{\mathcal{H}}^2\right\} \tag{13}$$

Definition 8. Define the function $\Phi(f_{\mathbf{w}}) \triangleq \max_{t\in\mathbb{R}} g\left(t, f_{\mathbf{w}}\right)$. This function is a L-weakly convex function in $\mathcal{K}$, i.e., $\Phi(f_{\mathbf{w}}) + \frac{L}{2}\left\|\mathbf{f}_{\mathbf{w}}\right\|_{\mathcal{H}}^2$ is a convex function over $\mathbf{w}$ in the convex and compact set $\mathcal{K}$.

Definition 9 (First Order Stationary Point)**.** Let $f_{\hat{\mathbf{w}}}$ be a first-order stationary point, then for any $f_{\mathbf{w}} \in \mathcal{K}$, it follows

$$\left\langle \nabla_f g\left(f_{\hat{\mathbf{w}}}\right), f_{\mathbf{w}} - f_{\hat{\mathbf{w}}}\right\rangle_{\mathcal{H}} \geq 0 \qquad \forall\, f_{\mathbf{w}} \in \mathcal{K} \tag{14}$$

Definition 10 (Stationary Point of Moreau Envelope)**.** A point $f_{\hat{\mathbf{w}}}$ is a stationary point of the Moreau Envelope defined in Definition 7 of Φ defined in Definition 8 if

$$f_{\hat{\mathbf{w}}} = \arg\inf_{f_{\mathbf{w}}\in\mathcal{K}}\left\{\Phi_\lambda\left(f_{\mathbf{w}}\right) + \frac{1}{2\rho}\left\|f_{\mathbf{w}} - f_{\hat{\mathbf{w}}}\right\|_{\mathcal{H}}^2\right\} \tag{15}$$

We will show that if a point $f_{\mathbf{w}}$ is a stationary point then this point is close to the optimal point for the uncorrupted distribution, i.e. $\left\|f_{\hat{\mathbf{w}}} - f_{\mathbf{w}}^*\right\|_{\mathcal{H}}$ is small.

Lemma 11 (Lower bound on distance from stationary point and optimal point)**.** *Let Φ_λ be defined as in Definition 8, then if $f_{\hat{\mathbf{w}}}$ is a stationary point as defined in Definition 10 and $g(t, f_{\mathbf{w}})$ has β-Lipschitz Gradient, then*

$$\lim_{\lambda\to\infty} \left(\Phi_\lambda\left(f_{\hat{\mathbf{w}}}\right) - \Phi_\lambda\left(f^*_{\mathbf{w}}\right)\right) \leq \beta \left\|f_{\hat{\mathbf{w}}} - f^*_{\mathbf{w}}\right\|^2_{\mathcal{H}} \tag{16}$$

We can now upper bound $\|f_{\hat{\mathbf{w}}} - f^*_{\mathbf{w}}\|_{\mathcal{H}}$. We proceed by contradiction, i.e. if a stationary point is sufficiently far from the optimal point, then this will break the stationary property proved in Lemma 11. This bound is different for each of the loss functions, so we must upper bound $\|f_{\hat{\mathbf{w}}} - f^*_{\mathbf{w}}\|_{\mathcal{H}}$ seperately for each loss function with the same high level overview.

3.3 Kernelized Regression

The loss for the Kernel Ridge Regression problem for a single training pair $(\mathbf{x}_i, y_i) \in \mathcal{D}$ is given by the following equation

$$\ell\left(f_{\mathbf{w}}; \mathbf{x}_i, y_i,\right) = \left(f_{\mathbf{w}}(\mathbf{x}_i) - y_i\right)^2 \tag{17}$$

It is important to note that β is upper bounded as

$$\begin{aligned}\beta &= \frac{2}{np}\operatorname{Tr}(\mathbf{K}) \leq \frac{2}{np}(n(1-\varepsilon)\max_{i\in P} k(x_i, x_i) + n\varepsilon \max_{j\in Q} k(x_j, x_j) \\ &= 2p^{-1}((1-\varepsilon)P_k + \varepsilon Q_k)\end{aligned} \tag{18}$$

which is independent of n. For our bounds, to be useful, we require the *Strong Projection Property*.

Definition 12 (Strong Projection Property)**.** Let $f^*_{\mathbf{w}}$ be the optimal function for the uncorrupted dataset, $\mathbb{P}$. Then, we have for a finite m and an absolute constant $c > 0$,

$$\|\mathrm{Proj}_m f^*_{\mathbf{w}} - f^*_{\mathbf{w}}\|_{\mathcal{H}} = 0, \qquad \|\mathrm{Proj}_m f_{\mathbf{w}} - f^*_{\mathbf{w}}\| > c \tag{19}$$

The *Strong Projection Property* is important as $\lambda_{\min}(\mathbf{\Sigma})$ is not well defined for an infinite dimensional feature space, e.g. Gaussian Kernel. The implication of the *Strong Projection Property* is given in the following lemma.

Lemma 13 (Strong Projection Property Implication)**.** *Assume the Strong Projection Property (Definition 12) holds for $f^{(t)}_{\mathbf{w}}$ for all $t \in [T]$, where $f^{(t)}_{\mathbf{w}}$ are iterates from Algorithm 1. Then, it follows for a $m \in \mathbb{N}$ and a constant $C > 0$,*

$$\begin{aligned}&\left\langle \mathbf{\Sigma}, \left(f^{(t)}_{\mathbf{w}} - f^*_{\mathbf{w}}\right) \otimes \left(f^{(t)}_{\mathbf{w}} - f^*_{\mathbf{w}}\right)\right\rangle_{\mathrm{HS}} \geq C\lambda_m \left\|f^{(t)}_{\mathbf{w}} - f^*_{\mathbf{w}}\right\|^2_{\mathcal{H}}, \\ &C \triangleq \frac{\left\|f^{(t)}_{\mathbf{w}} - f^*_{\mathbf{w}}\right\|^2_{\mathcal{H}} - \left\|\mathrm{Proj}_{m,\perp} f_{\mathbf{w}}\right\|^2_{\mathcal{H}}}{\left\|f^{(t)}_{\mathbf{w}} - f^*_{\mathbf{w}}\right\|^2_{\mathcal{H}}}\end{aligned} \tag{20}$$

Proof is given in Appendix C.2. We will discuss the implication of Lemma 13 after stating the following theorem.

Theorem 14 (Stationary Point for Kernelized Regression is Good)**.** *Let $f_{\hat{\mathbf{w}}}$ be a stationary point defined in Definition 10 for the function Φ defined in Definition 8. Then for a constant $c_1 \in (0,1)$, if $n \geq \frac{8\operatorname{Tr}(\mathbf{\Sigma})^2}{\lambda_{\min}(\mathbf{\Sigma})(1-c_1)^2(1-2\varepsilon)} + \frac{8\beta}{(1-c_1)^2(1-2\varepsilon)}$,*

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}} \|f_{\hat{\mathbf{w}}} - f_{\mathbf{w}^*}\|_{\mathcal{H}} \leq \sqrt{\frac{\gamma\sigma}{c_1\lambda_{\min}\left(\mathbf{\Sigma}\right)}} + \frac{O\left(\sigma\sqrt{\gamma\log\left(n(1-2\varepsilon)\right)\operatorname{Tr}\left(\mathbf{\Sigma}\right)}\right)}{c_1\sqrt{n(1-2\varepsilon)}c\lambda_{\min}\left(\mathbf{\Sigma}\right)} \tag{21}$$

where β is the Lipschitz Gradient Constant given in Lemma 28.

In Theorem 14, we have an upper bound on the expected distance from a stationary point to the optimal point over the distance of the dataset. The numerator of the second term grows in $O(\sqrt{\log(n)})$ and the denominator grows in $O(\sqrt{n})$ as can be shown by choosing sufficiently large n. Asymptotically the second term will then go to 0. In the first term, we have both the numerator and denominator scale in $O(n)$. Furthermore, when we consider the case of feature noise, e.g. a large multiplicative term on the features, we simply require more data to obtain the same bounds. Such a result is corroborated in [SST+18]. For the linear and polynomial kernel, we then have β increases, therefore to obtain the same bound on η as with no feature noise, we simply need more data. The effect of Lemma 13 can be seen in the denominator of both terms. Instead of $\lambda_{\min}(\mathbf{\Sigma})$ we have $c_4\lambda_m$ for a finite m. This differene will be clear in the following corollary, where we utilize the theory develoepd for kernelized regression to imply a result for regularized linear regression.

Corollary 15 (Linear Regression Expected Erorr Bound)**.** *Consider Subquantile Minimization for Linear Regression on the data X with optimal parameters $\mathbf{w}^*$. Assume $\mathbf{x}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ for $i \in [n]$. Then after T iterations of Algorithm 1, we have the following error bounds for robust kernelized linear regression. Given sufficient data*

$$\mathbb{E}\left\|\mathbf{w}^{(T)} - \mathbf{w}^*\right\|_2 \leq O\left(\frac{\gamma\sigma}{\sqrt{\lambda_{\min}(\mathbf{\Sigma})}}\right) \tag{22}$$

Proof given in Appendix C.4. Let us note for the case where p is finite, i.e. the feature mapping is finite-dimensional, e.g. linear or polynomial kernel. Then we have that $\mathrm{Proj}_{m,\perp}$ where $m = p$ is equal to zero as $\{\varphi_i\}_{i=1}^m$ spans the finite-dimensional space, in which we case we have the absolute constant given in Definition 12 is equal to zero. It is important to note in all our bounds, $\gamma \leq \sqrt{\frac{\varepsilon}{1-2\varepsilon}}$ is a theoretical worst case bound when the Subquantile contains the minimum possible number of uncorrupted points. In other words, we have $\gamma \triangleq \frac{|P\setminus S|}{|S\cap P|} \leq \frac{n\varepsilon}{n(1-2\varepsilon)} = \frac{\varepsilon}{1-2\varepsilon}$. So, as $|S\cap P|$ increases, we have a better error bound as $|P\setminus S|$ decreases. As is typical in the robust statistics literature, we make no assumptions on the distribution of the corrupted data so we cannot say anything about $|S\cap P|$. We will have γ decreases if stationary points give high error for corrupt points as our optimization procedure moves toward a stationary point.

3.4 Kernelized Binary Classification

The Negative Log Likelihood for the the Kernel Classification problem is given by the following equation for a single training pair $(\mathbf{x}_i, y_i)$

$$\ell(\mathbf{x}_i, y_i; f_{\mathbf{w}}) = -\left(y_i \log\left(\sigma\left(f_{\mathbf{w}}(\mathbf{x}_i)\right)\right) - (1-y_i)\log\left(1-\sigma\left(f_{\mathbf{w}}(\mathbf{x}_i)\right)\right)\right) \tag{23}$$

Theorem 16. *[A stationary point is good for kernel binary classification] Let $f_{\hat{\mathbf{w}}}$ be a statoinary point defined in Definition 9 for the function Φ defined in Definition 8. Then for a constant $c_4 \in (0,1)$, if $n \geq \frac{4\,\mathrm{Tr}(\mathbf{\Sigma})}{\lambda_{\min}(\mathbf{\Sigma})(1-2\varepsilon)(1-c_4)}$, then in expectation over the dataset distribution,*

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}}\|f_{\hat{\mathbf{w}}} - f^*_{\mathbf{w}}\|_{\mathcal{H}} \leq O\left(\frac{\sqrt{\mathrm{Tr}(\mathbf{\Sigma})} + \sqrt{Q_k}}{\sqrt{n(1-2\varepsilon)}\exp\left(-R\left(\mathrm{Tr}\left(\mathbf{\Sigma}\right) + \log n\right)\right)\lambda_{\min}(\mathbf{\Sigma})}\right) \tag{24}$$

Proof is given in Appendix D.2. This result although shows consistency, i.e. when $n \to \infty$, then we have in expectation $\|f_{\mathbf{w}} - f^*_{\mathbf{w}}\| \to 0$, however it does crucially rely on the fact that Q_k is bounded, and in general when n is not large, a large Q_k does affect the error bounds. To mitigate the effect of a large Q_k, a filtering algorithm can be used to remove points, $\mathbf{x}_i$, such that $k(\mathbf{x}_i, \mathbf{x}_i)$ is far from the mean.

3.5 Kernelized Multi-Class Classification

The Negative Log-Likelihood Loss for the the Kernel Multi-Class Classification problem is given by the following equation for a single training pair $(\mathbf{x}_i, y_i)$, note $\mathbf{W}$ is now a matrix

$$\ell(\mathbf{x}_i, y_i; f_{\mathbf{W}}) = -\sum_{j=1}^{|\mathcal{Y}|}\mathbb{I}\{j = y_i\}\log\left(\frac{\exp\left(f_{\mathbf{w}_j}(\mathbf{x}_i)\right)}{\sum_{k=1}^{|\mathcal{Y}|}\exp\left(f_{\mathbf{w}_k}(\mathbf{x}_i)\right)}\right) \tag{25}$$

Theorem 17 (Stationary Point for Kernelized Multi-Class Classification is Good)**.** *Let $f_{\hat{\mathbf{w}}}$ be a stationary point defined in Definition 10 for the function Φ defined in Definition 8. Then for a constant $c_1 \in (0,1)$, if $n \geq \frac{8\,\mathrm{Tr}(\boldsymbol{\Sigma})^2}{\lambda_{\min}(\boldsymbol{\Sigma})(1-c_1)^2(1-2\varepsilon)} + \frac{8\beta}{(1-c_1)^2(1-2\varepsilon)}$,*

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}} \left\| f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}} \right\|_{\mathcal{H}} \leq O\left(\frac{\sqrt{n(1-\varepsilon)\,\mathrm{Tr}(\boldsymbol{\Sigma})} + \sqrt{n\varepsilon Q_k}}{n(1-2\varepsilon)\lambda_{\min}(\boldsymbol{\Sigma}) - \sqrt{n(1-2\varepsilon)}\,\mathrm{Tr}(\boldsymbol{\Sigma})} \right) \tag{26}$$

where β is the Lipschitz Gradient Constant given in Lemma 32.

3.6 Optimization

In practice, however, it is important to note that solving for $\|\nabla\Phi_\lambda\|_{\mathcal{H}} = 0$ is NP-Hard. Thus, we will analyze the approximate stationary point.

Lemma 18 ([Roc70, DD19])**.** *Assume the function Φ is β-weakly convex. Let $\lambda < \frac{1}{\beta}$, and let $f_{\hat{\mathbf{w}}} = \arg\min_{f_{\mathbf{w}}\in\mathcal{K}} (\Phi(f_{\mathbf{w}}) + \frac{1}{2\lambda}\|f_{\mathbf{w}} - f_{\hat{\mathbf{w}}}\|^2_{\mathcal{H}})$, then $\|\nabla\Phi_\lambda(f_{\mathbf{w}})\|_{\mathcal{H}} \leq \epsilon$ implies:*

$$\|f_{\hat{\mathbf{w}}} - f_{\mathbf{w}}\|_{\mathcal{H}} = \lambda\epsilon \quad \textit{and} \quad \min_{\mathbf{g}\in\partial\Phi(f_{\hat{\mathbf{w}}})+\partial\mathcal{I}_{\mathcal{K}}(f_{\hat{\mathbf{w}}})} \|\mathbf{g}\|_{\mathcal{H}} \leq \epsilon \tag{27}$$

With Lemma 18 in hand, it suffices to show that $\|\nabla\Phi_\lambda(f_{\mathbf{w}})\|_{\mathcal{H}}$ is small, as it then follows that $f_{\mathbf{w}}$ is close to a stationary point of the Moreau Envelope. It has been shown in optimization theory that utilizing standard gradient descent, $\|\nabla\Phi_\lambda(f_{\mathbf{w}})\|_{\mathcal{H}}$ decreases at a rate of $O(T^{-1/2})$. The exact theorem and proof can be seen in [DD19] and a proof where the maximimum of the inner problem can be calculated to within $(1+\epsilon)$ optimality can be seen in [JNJ20] and [CDGS20].

4 Discussion

The main contribution of this paper is the study of a nonconvex-concave formulation of Subquantile minimization for the robust learning problem for kernel ridge regression and kernel classification. We present an algorithm to solve the nonconvex-concave formulation and prove rigorous error bounds which show that the more good data that is given decreases the error bounds. We also present accelerated gradient methods for the two-step algorithm to solve the nonconvex-concave optimization problem and give novel theoretical bounds.

Theory. We develop strong theoretical bounds on the normed difference between the function returned by Subquantile Minimization and the optimal function for data in the target distribution, $\mathbb{P}$, in the Gaussian Design. In expectation and with high probability, given sufficient data dependent on the kernel, we obtain a near minimax optimal error bound for a general positive definite continuous kernel. Our theoretical analysis is novel in that it utilizes the Moreau Envelope from a min-max formulation of the iterative thresholding algorithm.

Experiments. From our experiments, we see Subquantile Minimization is competitive with algorithms developed solely for robust linear regression as well as other meta-algorithms. Our theoretical analysis is through the lens of kernel-learning, but the generalization to linear regression from a non-kernel perspective can be done. In kernelized regression, we see SUBQUANTILE is the strongest of the meta-algorithms. Furthemore, in binary and multi-class classification, SUBQUANTILE is very strong. Thus, we can see empirically SUBQUANTILE is the strongest meta-algorithm across all kernelized regression and classification tasks and also the strongest algorithm in linear regression.

Interpretability. One of the strengths in Subquantile Optimization is the high interpretability. Once training is finished, we can see the $n(1-p)$ points with highest error to find the outliers and the features follow Gaussian Design. Furthermore, there is only hyperparameter p, which should be chosen to be approximately the percentage of inliers in the data and thus is not very difficult to tune for practical purposes. Our theory suggests for a problem where the amount of corruptions is unknown,

General Assumptions. The general assumption is the majority of the data should inliers. This is not a very strong assumption, as by the definition of outlier it should be in the minority. Furthermore, we assume the feature maps have a Gaussian Design. Such a design in many prior works in kernel learning and we therefore find it suitable.

Future Work. The analysis of Subquantile Minimization can be extended to neural networks as kernel learning can be seen as a one-layer network. This generalization will be appear in subsequent work. Another interesting direction work in optimization is for accelerated methods for optimizing non-convex concave min-max problems with a maximization oracle. The current theory analyzes standard gradient descent for the minimization. Ideas such as Momentum and Nesterov Acceleration in conjunction with the maximum oracle are interesting and can be analyzed in future work.

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A Concentration Inequalities

In this section we will give various concentration inequalities on the inlier data for functions in the Reproducing Kernel Hilbert Space. We will first give our assumptions for robust kernelized regression.

Assumption 19 (Gaussian Design)**.** We assume for $\mathbf{x}_i \sim \mathbb{P} \in \mathcal{X}$, then it follows for the feature map, $\phi(\cdot) : \mathcal{X} \to \mathcal{H}$,

$$\phi(\mathbf{x}_i) \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}) \tag{28}$$

where $\mathbf{\Sigma}$ is a possibly infinite dimensional covariance operator.

Assumption 20 (Normal Residuals)**.** The residual is defined as $\mu_i \triangleq f^*_{\mathbf{w}}(\mathbf{x}_i) - y_i$. Then we assume for some $\sigma > 0$, it follows

$$\mu_i \sim \mathcal{N}(0, \sigma^2) \tag{29}$$

Lemma 21 (Maximum of Gaussians)**.** *Let $\mu_1, \ldots, \mu_n \sim \mathcal{N}(0, \sigma^2)$ for some $\sigma > 0$. Then it follows*

$$\mathop{\mathbb{E}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \max_{i \in [n]} |\mu_i| \leq O\left(\sigma\sqrt{\log n}\right) \tag{30}$$

Proof. We will integrate over the CDF to make our claim.

$$\mathop{\mathbb{E}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \max_{i \in [n]} |\mu_i| = \int_0^\infty \mathop{\mathbb{P}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \left\{\max_{i \in [n]} |\mu_i| > t\right\} dt \overset{(i)}{\leq} c_1 + n \int_{c_1}^\infty \mathop{\mathbb{P}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \left\{|\mu_i| \geq t\right\} dt \tag{31}$$

$$\overset{(ii)}{=} c_1 + 2n \int_{c_1}^\infty \mathop{\mathbb{P}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \left\{\mu_i \geq t\right\} dt = c_1 + 2n \int_{c_1}^\infty \int_t^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx dt \tag{32}$$

$$\leq c_1 + \frac{n}{\sigma}\sqrt{\frac{2}{\pi}} \int_{c_1}^\infty \int_t^\infty \left(\frac{x}{t}\right) e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx dt = c_1 + n\sigma\sqrt{\frac{2}{\pi}} \int_{c_1}^\infty \frac{e^{-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2}}{t} dt \tag{33}$$

$$\leq c_1 + n\sigma\sqrt{\frac{2}{\pi}} \int_{c_1}^\infty \left(\frac{t}{c_1}\right) e^{-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2} dt = c_1 + n\sigma^3\sqrt{\frac{2}{\pi}} \frac{e^{-\frac{1}{2}\left(\frac{c_1}{\sigma}\right)^2}}{c_1} \tag{34}$$

(i) follows from a union bound and noting for a i.i.d sequence of random variables $\{X_i\}_{i \in [n]}$ and a constant C, it follows $\mathbb{P}\{\max_{i \in [n]} X_i \geq C\} = n\mathbb{P}\{X \geq C\}$ where X is sampled from the same distribution as each X_i. (ii) follows from the symmetricity of the Gaussian distribution about zero. From here, we choose $c_1 \triangleq \sigma\sqrt{2\log n}$. Then we have,

$$\mathop{\mathbb{E}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \max_{i \in [n]} |\mu_i| \leq \sigma\sqrt{2\log n} + \frac{\sigma^2}{\sqrt{\pi \log n}} \tag{35}$$

This completes the proof. ∎

Lemma 22 (Maximum of Squared Gaussians)**.** *Let $\mu_1, \ldots, \mu_n \sim \mathcal{N}(0, \sigma^2)$ for $\sigma > 0$, $n > 1$. Then it follows*

$$\mathop{\mathbb{E}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \max_{i \in [n]} \mu_i^2 \leq O\left(\sigma^2 \log(n) + \frac{\sigma^3}{\log(n)}\right) \tag{36}$$

Proof. Our proof follows similarly to the proof for Lemma 21.

$$\mathop{\mathbb{E}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \max_{i \in \mathbb{N}} \mu_i^2 = \int_0^\infty \mathop{\mathbb{P}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \left\{\max_{i \in [n]} \mu_i^2 \geq t\right\} dt \leq c_2 + n \int_{c_2}^\infty \mathop{\mathbb{P}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \left\{|\mu_i| \geq \sqrt{t}\right\} dt \tag{37}$$

$$= c_2 + 2n \int_{c_2}^\infty \mathop{\mathbb{P}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \left\{\mu_i \geq \sqrt{t}\right\} dt = c_2 + 2n \int_{c_2}^\infty \int_{\sqrt{t}}^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx dt \tag{38}$$

$$= c_2 + n\sigma\sqrt{\frac{2}{\pi}} \int_{c_2}^\infty \frac{e^{-\frac{1}{2}\left(\frac{t}{\sigma^2}\right)}}{\sqrt{t}} dt \overset{(i)}{\leq} c_2 + n\sigma\sqrt{\frac{2}{\pi}} \int_{c_2}^\infty \left(\frac{t}{c_2}\right) e^{-\frac{1}{2}\left(\frac{t}{\sigma^2}\right)} dt \tag{39}$$

$$\leq c_2 + \left(\sqrt{\frac{2}{\pi}}\right) \frac{n\sigma\left(4\sigma^4 + 2c_2\sigma^2\right) e^{-\frac{c_2}{2\sigma^2}}}{c_2} \tag{40}$$

(i) holds for $c_2 > 1$. Then, setting $c_2 \triangleq 2\sigma^2 \log(n)$, we have

$$\mathop{\mathbb{E}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \max_{i \in [n]} \mu_i^2 \leq 2\sigma^2 \log(n) + \left(2\sigma^3 \sqrt{\frac{2}{\pi}}\right) \left(1 + \frac{1}{\log(n)}\right) \tag{41}$$

This completes the proof. ■

Lemma 23 ([1]Expected Maximum P_k). *Let $\mathbf{x}_i \sim \mathbb{P}$ such that $\phi(\mathbf{x}_i) \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ from Assumption 19. Then it follows*

$$\mathop{\mathbb{E}}_{\phi(\mathbf{x}_i) \sim \mathcal{N}(\mathbf{0},\mathbf{\Sigma})} \left[\max_{i \in [n]} k(\mathbf{x}_i, \mathbf{x}_i)\right] \leq O\left(\mathrm{Tr}\left(\mathbf{\Sigma}\right) + \log n\right) \tag{42}$$

Proof. We once integrate over the CDF to make our claim.

$$\begin{aligned}
\mathop{\mathbb{E}}_{\phi(\mathbf{x}_i) \sim \mathcal{N}(\mathbf{0},\mathbf{\Sigma})} \left[\max_{i \in [n]} k(\mathbf{x}_i, \mathbf{x}_i)\right] &= c_2 + \int_{c_2}^{\infty} \mathop{\mathbb{P}}_{\phi(\mathbf{x}_i) \sim \mathcal{N}(\mathbf{0},\mathbf{\Sigma})} \left\{\max_{i \in [n]} k(\mathbf{x}_i, \mathbf{x}_i) \geq t\right\} dt &(43)\\
&\overset{(i)}{\leq} c_2 + n \int_{c_2}^{\infty} \mathop{\mathbb{P}}_{\phi(\mathbf{x}) \sim \mathcal{N}(\mathbf{0},\mathbf{\Sigma})} \left\{k(\mathbf{x}, \mathbf{x}) \geq t\right\} dt &(44)\\
&\overset{(ii)}{\leq} c_2 + n \int_{c_2}^{\infty} \inf_{\theta > 0} e^{-t\theta} \mathbb{E}\left[e^{\theta k(\mathbf{x},\mathbf{x})}\right] dt &(45)\\
&= c_2 + n \int_{c_2}^{\infty} \inf_{\theta > 0} e^{-t\theta} \mathbb{E}\left[e^{\theta \sum_{i \in [d]} x_i^2}\right] dt &(46)\\
&= c_2 + n \int_{c_2}^{\infty} \inf_{0 < \theta < 1/2} (1 - 2\theta)^{-d/2} \exp\left[\frac{\theta\, \mathrm{Tr}\left(\mathbf{\Sigma}\right)}{1 - 2\theta} - t\theta\right] dt &(47)\\
&\leq c_2 + n \inf_{0 < \theta < 1/2} \int_{c_2}^{\infty} (1 - 2\theta)^{-d/2} \exp\left[\frac{\theta\, \mathrm{Tr}\left(\mathbf{\Sigma}\right)}{1 - 2\theta} - t\theta\right] dt &(48)\\
&\overset{(iii)}{\leq} c_2 + n \int_{c_2}^{\infty} \left(1 - 2^{1-p}\right)^{-d/2} \exp\left[\left(\frac{2^{-p}}{1 - 2^{1-p}}\right) \mathrm{Tr}\left(\mathbf{\Sigma}\right) - 2^{-p}\theta\right] dt &(49)\\
&= c_2 + n \left(1 - 2^{-p}\right)^{-d/2} \left(\frac{\exp\left[\left(\frac{2^{p}}{1 - 2^{1-p}}\right) \mathrm{Tr}\left(\mathbf{\Sigma}\right)\right]}{2^{-p} \exp\left[2^{-p} c_2\right]}\right) &(50)\\
&\overset{(iv)}{=} \left(1 - 2^{1-p}\right)^{-1} \mathrm{Tr}\left(\mathbf{\Sigma}\right) + 2^p \log n + \left(1 - 2^{1-p}\right)^{-d/2} &(51)
\end{aligned}$$

See (i) from the proof of Lemma 21. (ii) follows from a Chernoff bound [Che52]. (iii) follows from setting $\theta \triangleq 2^{-p}$ for $p \in \mathbb{R}_{++}$ such that $p > 1$ and $p < \infty$. (iv) follows from setting $c_2 \triangleq (1 - 2^{1-p})^{-1} \mathrm{Tr}(\mathbf{\Sigma}) + 2^p \log n + (1 - 2^{1-p})^{-d/2}$. Further optimization can be done over p dependent on $\mathrm{Tr}(\mathbf{\Sigma})$ and $\log(n)$. ■

Lemma 24 (Norm of Functions with Gaussian Design in the Reproducing Kernel Hilbert Space)**.** *Let $\mathbf{x}_i \sim \mathbb{P}$ such that $\phi(\mathbf{x}_i) \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ from Assumption 19 and Assumption 20. Then, it follows*

$$\mathop{\mathbb{E}}_{\phi(\mathbf{x}_i) \sim \mathcal{N}(\mathbf{0},\mathbf{\Sigma})} \mathop{\mathbb{E}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \left\|\sum_{i=1}^{n} \mu_i \phi(\mathbf{x}_i)\right\|_{\mathcal{H}} \leq O\left(\sigma \sqrt{n \log n\, \mathrm{Tr}\left(\mathbf{\Sigma}\right)}\right) \tag{52}$$

Proof. Our proof follows standard ideas from High-Dimensional Probability. Let ξ_i for $i \in [n]$ denote i.i.d Rademacher variables such that for $\xi_i \sim \mathcal{R}$, it follows $\mathbb{P}\{\xi_i = 1\} = \mathbb{P}\{\xi_i = -1\} = \frac{1}{2}$. We then have,

$$\begin{aligned}
\mathop{\mathbb{E}}_{\phi(\mathbf{x}_i) \sim \mathcal{N}(\mathbf{0},\mathbf{\Sigma})} \mathop{\mathbb{E}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \left\|\sum_{i=1}^{n} \mu_i \phi(\mathbf{x}_i)\right\|_{\mathcal{H}} &\leq \mathop{\mathbb{E}}_{\phi(\mathbf{x}_i) \sim \mathcal{N}(\mathbf{0},\mathbf{\Sigma})} \mathop{\mathbb{E}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \max_{i \in [n]} |\mu_i| \left\|\sum_{i=1}^{n} \phi(\mathbf{x}_i)\right\|_{\mathcal{H}} &(53)\\
&\overset{\text{lem. } 21}{\leq} O\left(\sigma \sqrt{\log n}\right) \mathop{\mathbb{E}}_{\phi(\mathbf{x}_i) \sim \mathcal{N}(\mathbf{0},\mathbf{\Sigma})} \mathop{\mathbb{E}}_{\xi_i \sim \mathcal{R}} \left\|\sum_{i=1}^{n} \xi_i \phi(\mathbf{x}_i)\right\|_{\mathcal{H}} &(54)
\end{aligned}$$

[1]In Progress

$$\overset{(i)}{\leq} O\left(\sigma\sqrt{\log n}\right)\left(\underset{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Sigma})}{\mathbb{E}}\underset{\xi_i\sim\mathcal{R}}{\mathbb{E}}\left\|\sum_{i=1}^n \xi_i\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}^2\right)^{1/2} \tag{55}$$

$$= O\left(\sigma\sqrt{\log n}\right)\left(\underset{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Sigma})}{\mathbb{E}}\underset{\xi_i\sim\mathcal{R}}{\mathbb{E}}\left\langle\sum_{i=1}^n \xi_i\phi(\mathbf{x}_i),\sum_{j=1}^n \xi_j\phi(\mathbf{x}_j)\right\rangle_{\mathcal{H}}\right)^{1/2} \tag{56}$$

$$\overset{(ii)}{=} O\left(\sigma\sqrt{\log n}\right)\left(\underset{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Sigma})}{\mathbb{E}}\underset{\xi_i\sim\mathcal{R}}{\mathbb{E}}\sum_{i=1}^n\sum_{j=1}^n \xi_i\xi_j k(\mathbf{x}_i,\mathbf{x}_j)\right)^{1/2} \tag{57}$$

$$\overset{(iii)}{=} O\left(\sigma\sqrt{\log n}\right)\left(\underset{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Sigma})}{\mathbb{E}}\sum_{i=1}^n k(x_i,x_i)\right)^{1/2} \tag{58}$$

$$= O\left(\sigma\sqrt{n\log n\,\mathrm{Tr}\left(\mathbf{\Sigma}\right)}\right) \tag{59}$$

(i) follows from Jensen's Inequality. (ii) follows from the definition of the kernel [Gre13a]. (iii) holds as we have $\mathbb{E}[\xi_i\xi_j] = \delta_{i,j}$, where δ is the Kronecker Delta function. ∎

Lemma 25 (Infinite Dimensional Covariance Estimation in the Hilbert-Schmidt Norm)**.** *Let* $\mathbf{\Sigma} \triangleq \mathbb{E}_{\phi(\mathbf{x}_i)\sim\mathbb{P}}[\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)]$*. Then let* $\mathbf{x}_1,\ldots,\mathbf{x}_n$ *be i.i.d sampled from* $\mathbb{P}$ *such that* $\phi(\mathbf{x}_i)\sim\mathcal{N}(0,\mathbf{\Sigma})$ *from Assumption 19, we then have*

$$\underset{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Sigma})}{\mathbb{E}}\left\|\frac{1}{n}\sum_{i=1}^n\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)-\mathbf{\Sigma}\right\|_{\mathrm{HS}} \leq O\left(n^{-1/2}\,\mathrm{Tr}\left(\mathbf{\Sigma}\right)\right) \tag{60}$$

Proof. Our proof follows standard ideas from High-Dimensional Probability. Let ξ_i for $i\in[n]$ denote i.i.d Rademacher variables such that for $\xi_i\sim\mathcal{R}$, it follows $\mathbb{P}\{\xi_i=1\}=\mathbb{P}\{\xi_i=-1\}=\frac{1}{2}$. We then have,

$$\underset{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Sigma})}{\mathbb{E}}\left\|\frac{1}{n}\sum_{i=1}^n\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)-\mathbf{\Sigma}\right\|_{\mathrm{HS}}$$

$$\overset{(i)}{\leq}\underset{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Sigma})}{\mathbb{E}}\underset{\tilde{\phi}(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Sigma})}{\mathbb{E}}\left\|\frac{1}{n}\sum_{i=1}^n\left(\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)-\tilde{\phi}(\mathbf{x}_i)\otimes\tilde{\phi}(\mathbf{x}_i)\right)\right\|_{\mathrm{HS}} \tag{61}$$

$$=\underset{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Sigma})}{\mathbb{E}}\underset{\tilde{\phi}(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Sigma})}{\mathbb{E}}\underset{\xi_i\sim\mathcal{R}}{\mathbb{E}}\left\|\frac{1}{n}\sum_{i=1}^n\xi_i\left(\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)-\tilde{\phi}(\mathbf{x}_i)\otimes\tilde{\phi}(\mathbf{x}_i)\right)\right\|_{\mathrm{HS}} \tag{62}$$

$$\overset{(ii)}{\leq}\frac{2}{n}\underset{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Sigma})}{\mathbb{E}}\underset{\xi_i\sim\mathcal{R}}{\mathbb{E}}\left\|\sum_{i=1}^n\xi_i\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)\right\|_{\mathrm{HS}} \tag{63}$$

$$\overset{(iii)}{\leq}\frac{2}{n}\left(\underset{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Sigma})}{\mathbb{E}}\underset{\xi_i\sim\mathcal{R}}{\mathbb{E}}\left\|\sum_{i=1}^n\xi_i\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)\right\|_{\mathrm{HS}}^2\right)^{1/2} \tag{64}$$

(i) follows from noticing $\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)-\mathbf{\Sigma}$ is a mean $\mathbf{0}$ operator in $\mathcal{H}\otimes\mathcal{H}$, then for $X,Y\in\mathcal{H}\otimes\mathcal{H}$ s.t. $\mathbb{E}[Y]=\mathbf{0}$ it follows $\|X\|_{\mathrm{HS}}=\|X-\mathbb{E}[Y]\|_{\mathrm{HS}}=\|\mathbb{E}_Y[X-Y]\|_{\mathrm{HS}}$ and finally applying Jensen's Inequality. (ii) follows from the triangle inequality. (iii) follows from Jensen's Inequality. Let e_k for $k\in[p]$ represent an orthonormal basis for the Hilbert Space $\mathcal{H}$. By expanding out the Hilbert-Schmidt Norm, we then have

$$\frac{2}{n}\left(\underset{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Sigma})}{\mathbb{E}}\underset{\xi_i\sim\mathcal{R}}{\mathbb{E}}\left\|\sum_{i=1}^n\xi_i\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)\right\|_{\mathrm{HS}}^2\right)^{1/2}$$

$$=\frac{2}{n}\left(\underset{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Sigma})}{\mathbb{E}}\underset{\xi_i\sim\mathcal{R}}{\mathbb{E}}\sum_{k=1}^p\left\langle\sum_{i=1}^n\xi_i\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)e_k,\sum_{j=1}^n\xi_j\phi(\mathbf{x}_j)\otimes\phi(\mathbf{x}_j)e_k\right\rangle\right)^{1/2} \tag{65}$$

$$= \frac{2}{n}\left(\mathop{\mathbb{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\boldsymbol{\Sigma})}\mathop{\mathbb{E}}_{\xi_i\sim\mathcal{R}}\sum_{k=1}^{p}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\xi_j\left\langle\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)e_k,\phi(\mathbf{x}_j)\otimes\phi(\mathbf{x}_j)e_k\right\rangle\right)^{1/2} \tag{66}$$

$$\overset{(iv)}{\leq} \frac{2}{n}\left(\mathop{\mathbb{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\boldsymbol{\Sigma})}\sum_{k=1}^{p}\sum_{i=1}^{n}\left\langle\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)e_k,\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)e_k\right\rangle\right)^{1/2} \tag{67}$$

$$= \frac{2}{n}\left(\sum_{i=1}^{n}\mathop{\mathbb{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\boldsymbol{\Sigma})}\|\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)\|_{\mathrm{HS}}^2\right)^{1/2} \overset{(v)}{=} \frac{2}{n}\left(\sum_{i=1}^{n}\mathop{\mathbb{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\boldsymbol{\Sigma})}\|\phi(\mathbf{x}_i)\|_{\mathcal{H}}^4\right)^{1/2} \tag{68}$$

$$= \frac{2}{n}\left(\sum_{i=1}^{n}\mathop{\mathbb{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\boldsymbol{\Sigma})}\left[k^2(x_i,x_i)\right]\right)^{1/2} = \frac{2}{\sqrt{n}}\left(2\,\mathrm{Tr}\left(\boldsymbol{\Sigma}^2\right)+\mathrm{Tr}\left(\boldsymbol{\Sigma}\right)^2\right)^{1/2} \leq 2\sqrt{3}n^{-1/2}\,\mathrm{Tr}\left(\boldsymbol{\Sigma}\right) \tag{69}$$

(iv) follows from noticing $\mathbb{E}_{\xi_i,\xi_j\sim\mathcal{R}}[\xi_i\xi_j] = \delta_{i,j}$. (v) follows from expanding the Hilbert-Schmidt Norm and applying Parseval's Identity. We note $\mathrm{Tr}(\boldsymbol{\Sigma}) < \infty$ and therefore even though the covariance operator is infinite-dimensional we are able to get a finite bound on the covariance approximation. This completes the proof. ■

Lemma 26 (Finite Dimensional Covariate Estimation in the Spectral Norm)**.** *Let* $\mathbf{x}_1,\ldots,\mathbf{x}_n \sim \mathcal{N}(\mathbf{0},\boldsymbol{\Sigma})$*. It then follows,*

$$\mathop{\mathbb{E}}_{\mathbf{x}_i\sim\mathcal{N}(\mathbf{0},\boldsymbol{\Sigma})}\left\|\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^\top-\boldsymbol{\Sigma}\right\|_2 \leq 64\sqrt{\frac{8}{3}}\left(\sqrt{\frac{d}{n}}+\frac{1}{\sqrt{dn}}\right) \tag{70}$$

Proof. Our proof combines multiple results in High-Dimensional Probability for Sub-Gaussian vectors and adapting it for Gaussian-Design. We have,

$$\mathop{\mathbb{E}}_{\mathbf{x}_i\sim\mathcal{N}(\mathbf{0},\boldsymbol{\Sigma})}\left\|\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^\top-\boldsymbol{\Sigma}\right\|_2 \leq \|\boldsymbol{\Sigma}\|\mathop{\mathbb{E}}_{\tilde{\mathbf{x}}_i\sim\mathcal{N}(\mathbf{0},\mathbf{I})}\left\|\frac{1}{n}\tilde{\mathbf{X}}^\top\tilde{\mathbf{X}}-\mathbf{I}\right\|_2 \tag{71}$$

$$= \|\boldsymbol{\Sigma}\|\int_0^\infty \mathop{\mathbb{P}}_{\tilde{\mathbf{x}}_i\sim\mathcal{N}(\mathbf{0},\mathbf{I})}\left\{\left\|\frac{1}{n}\tilde{\mathbf{X}}^\top\tilde{\mathbf{X}}-\mathbf{I}\right\|_2 \geq t\right\}dt \tag{72}$$

Let $\mathcal{M}$ be an ε-net of $\mathbb{S}^d$ for $\varepsilon = \frac{1}{4}$, then $|\mathcal{M}| \leq 9^d$. It then follows from [Ver20] Corollary 4.2.13,

$$\mathop{\mathbb{P}}_{\tilde{\mathbf{x}}_i\sim\mathcal{N}(\mathbf{0},\mathbf{I})}\left\{\left\|\frac{1}{n}\tilde{\mathbf{X}}^\top\tilde{\mathbf{X}}-\mathbf{I}\right\|_2 \geq t\right\} \leq \mathop{\mathbb{P}}_{\tilde{\mathbf{x}}_i\sim\mathcal{N}(\mathbf{0},\mathbf{I})}\left\{\max_{\mathbf{y}\in\mathcal{M}}\left|\frac{1}{n}\|\tilde{\mathbf{X}}\mathbf{y}\|_2^2-1\right| \geq \frac{t}{2}\right\} \tag{73}$$

Denote $K \triangleq 16\sqrt{\frac{8}{3}}$, then we have from a union bound and Bernstein's Inequality [Ber24].

$$\mathop{\mathbb{P}}_{\tilde{\mathbf{x}}_i\sim\mathcal{N}(\mathbf{0},\mathbf{I})}\left\{\max_{\mathbf{y}\in\mathcal{M}}\left|\frac{1}{n}\|\tilde{\mathbf{X}}\mathbf{y}\|_2^2-1\right| \geq \frac{t}{2}\right\} \leq 9^d\exp\left[-\frac{n}{2}\left(\frac{t^2}{K^2}\wedge\frac{t}{K}\right)\right] \tag{74}$$

Let $\delta \in (0,1)$, then we find RHS Equation (74) is less than δ when

$$t \geq K\left(\frac{2d\ln(9)+2\ln(2/\delta)}{n}\vee\left(\frac{2d\ln(9)+2\ln(2/\delta)}{n}\right)^{1/2}\right) \tag{75}$$

Furthermore, we note we have equality with one, when $t = K\sqrt{(2d\ln(9)+\ln(4))/n} \triangleq C$ all $t \leq C$ occur with probability also equal to one. Therefore, plugging this back into the RHS of Equation (72).

$$\int_0^\infty \mathop{\mathbb{P}}_{\tilde{\mathbf{x}}_i\sim\mathcal{N}(\mathbf{0},\mathbf{I})}\left\{\left\|\frac{1}{n}\tilde{\mathbf{X}}^\top\tilde{\mathbf{X}}-\mathbf{I}\right\|_2 \geq t\right\}dt \leq K\sqrt{\frac{2d\ln(9)}{n}+\frac{2\ln(2)}{n}}+\int_C^\infty 9^d\exp\left[-\frac{n}{2}\left(\frac{t^2}{K^2}\right)\right]dt \tag{76}$$

$$\leq K\sqrt{\frac{2d\ln(9)}{n}+\frac{2\ln(2)}{n}}+\frac{K9^d\exp\left[-\left(K\sqrt{\frac{2d\ln(9)}{n}+\frac{2\ln(2)}{n}}\right)^2\left(\frac{n}{2K^2}\right)\right]}{2n\sqrt{\frac{2d\ln(9)}{n}+\frac{2\ln(2)}{n}}} \tag{77}$$

$$\leq K\sqrt{\frac{2d\ln(9)+\ln(4)}{n}}+\frac{K}{4\sqrt{n\left(2d\ln(9)+\ln(4)\right)}} \tag{78}$$

$$\leq 6K\left(\sqrt{\frac{d}{n}}+\frac{1}{\sqrt{nd}}\right) \tag{79}$$

In the second inequality, we use the integral inequality $\int_c^\infty e^{-x^2}dx \leq \int_c^\infty (\frac{x}{c})e^{-x^2}dx = e^{-C^2}/(2C)$. The last inequality comes from bounding $2\ln(9)$. Our proof is complete. ■

B Proofs for Structural Results

In this section we give the deferred proofs of our main structural results.

B.1 Proof of Lemma 3

Proof. First we can note, the max value of t for g is equivalent to the min value of t for g. We can now find the Fermat Optimality Conditions for g.

$$\partial(-g(t,f_{\mathbf{w}}))=\partial\left(-t+\frac{1}{np}\sum_{i=1}^{n}(t-\hat{\nu}_i)^+\right)=-1+\frac{1}{np}\sum_{i=1}^{np}\begin{cases}1 & \text{if } t>\hat{\nu}_i\\ 0 & \text{if } t<\hat{\nu}_i\\ [0,1] & \text{if } t=\hat{\nu}_i\end{cases} \tag{80}$$

Equation (80) is equal to 0 when $t=\hat{\nu}_{np}$. This is equivalent to the p-quantile of the Risk. ■

B.2 Proof of Lemma 5

Proof. By our choice of $t^{(k+1)}$, it follows:

$$\nabla_f g(t^{(k+1)},f_{\mathbf{w}}^{(k)})=\nabla_f\left(\hat{\nu}_{np}-\frac{1}{np}\sum_{i=1}^{n}\left(\hat{\nu}_{np}-\ell(\mathbf{x}_i;f_{\mathbf{w}}^{(k)},y_i)\right)^+\right) \tag{81}$$

$$=-\frac{1}{np}\sum_{i=1}^{np}\nabla_f\left(\hat{\nu}_{np}-\ell(\mathbf{x}_i;f_{\mathbf{w}}^{(k)},y_i)\right)^+=\frac{1}{np}\sum_{i=1}^{n}\nabla_f\ell(\mathbf{x}_i;f_{\mathbf{w}}^{(k)},y_i)\begin{cases}1 & \text{if } t>\hat{\nu}_i\\ 0 & \text{if } t<\hat{\nu}_i\\ [0,1] & \text{if } t=\hat{\nu}_i\end{cases} \tag{82}$$

Now we note $\nu_{np}\leq t^{(k+1)}\leq\nu_{np+1}$. Then, plugging this into Equation (82), we have

$$\nabla_f g(t^{(k+1)},f_{\mathbf{w}}^{(k)})=\frac{1}{np}\sum_{i=1}^{np}\nabla_f\ell(\mathbf{x}_i;f_{\mathbf{w}}^{(k)},y_i) \tag{83}$$

This concludes the proof. ■

B.3 Some details on the Softplus Approximation

Now we compute the derivatives w.r.t to the softplus approximation, and then we consider the limit of the derivative as $\lambda\to\infty$.

$$\nabla_t\tilde{g}_\lambda(t,f_{\mathbf{w}})=\nabla_t\left(t-\frac{1}{np}\sum_{i=1}^{n}\frac{1}{\lambda}\ln\left(1+\exp\left(\lambda\left(t-\ell(f_{\mathbf{w}};\mathbf{x}_i,y_i)\right)\right)\right)\right) \tag{84}$$

$$=1-\frac{1}{np}\sum_{i=1}^{n}\sigma\left(\lambda\left(t-\ell(f_{\mathbf{w}};\mathbf{x}_i,y_i)\right)\right) \tag{85}$$

where $\sigma(\cdot)$ is the sigmoid function. We then note as $\lambda\to\infty$, the sigmoid function approaches the indicator function. It therefore follows the derivative of g with respect to t is given as,

$$\lim_{\lambda\to\infty}\nabla_t\tilde{g}_\lambda(t,f_{\mathbf{w}})=1-\frac{1}{np}\sum_{i=1}^{n}\mathbb{I}\left\{t-\ell\left(f_{\mathbf{w}};\mathbf{x}_i,y_i\right)\right\} \tag{86}$$

Next, we will calculate the derivative of $g(5, f_{\mathbf{w}})$ with respect to f.

$$\nabla_f \tilde{g}_\lambda(t, f_{\mathbf{w}}) = \nabla_f \left(t - \frac{1}{np} \sum_{i=1}^{n} \frac{1}{\lambda} \ln \left(1 + \exp\left(\lambda\left(t - \ell(f_{\mathbf{w}}; \mathbf{x}_i, y_i)\right)\right)\right) \right) \tag{87}$$

$$= \frac{1}{np} \sum_{i=1}^{n} \nabla_f \ell\left(f_{\mathbf{w}}; \mathbf{x}_i, y_i\right) \sigma\left(\lambda\left(t - \ell(f_{\mathbf{w}}; \mathbf{x}_i, y_i)\right)\right) \tag{88}$$

We therefore similarly have the derivative of g with respect to f as λ is pushed to infinity.

$$\lim_{\lambda \to \infty} \nabla_f \tilde{g}_\lambda\left(t, f_{\mathbf{w}}\right) = \frac{1}{np} \sum_{i=1}^{n} \mathbb{I}\left\{t - \ell\left(f_{\mathbf{w}}; \mathbf{x}_i, y_i\right)\right\} \nabla_f \ell\left(f_{\mathbf{w}}; \mathbf{x}_i, y_i\right) \tag{89}$$

B.4 Proof of Lemma 6

Proof. We will upper bound the operator norm of the Hessian. Let $\upsilon \triangleq \sigma(\lambda(t - \ell(f_{\mathbf{w}}; \mathbf{x}_i, y_i)))$, we then have

$$\nabla_f^2 \tilde{g}_\lambda(t, f_{\mathbf{w}}) = \nabla_f \left(\frac{1}{np} \sum_{i=1}^{n} \upsilon \nabla_f \ell\left(f_{\mathbf{w}}; \mathbf{x}_i, y_i\right) \right) \tag{90}$$

$$= \frac{1}{np} \sum_{i=1}^{n} \Big(\upsilon \nabla_f^2 \ell\left(f_{\mathbf{w}}; \mathbf{x}_i, y_i\right) - \upsilon(1-\upsilon)\left(\nabla_f \ell\left(f_{\mathbf{w}}; \mathbf{x}_i, y_i\right) \otimes \nabla_f \ell\left(f_{\mathbf{w}}; \mathbf{x}_i, y_i\right)\right) \Big) \tag{91}$$

Now we will upper bound the operator norm of the Hessian.

$$\begin{aligned}
&\lim_{\lambda \to \infty} \sup_{f_{\mathbf{w}} \in \mathcal{K}} \left\| \nabla_f^2 \tilde{g}_\lambda(t, f_{\mathbf{w}}) \right\|_{\mathrm{op}} \left\| f_{\mathbf{w}} - f_{\tilde{\mathbf{w}}} \right\|_{\mathcal{H}} \\
&\overset{(91)}{=} \lim_{\lambda \to \infty} \sup_{f_{\mathbf{w}}} \left\| \frac{1}{np} \sum_{i=1}^{n} \upsilon \nabla_f^2 \ell\left(f_{\mathbf{w}}; \mathbf{x}_i, y_i\right) - \upsilon(1-\upsilon) \nabla_f \ell\left(f_{\mathbf{w}}; \mathbf{x}_i, y_i\right) \otimes \nabla_f \ell\left(f_{\mathbf{w}}; \mathbf{x}_i, y_i\right) \right\|_{\mathrm{op}} \left\| f_{\mathbf{w}} - f_{\tilde{\mathbf{w}}} \right\|_{\mathcal{H}} \\
&\overset{(i)}{\leq} \lim_{\lambda \to \infty} \sup_{f_{\mathbf{w}} \in \mathcal{K}} \frac{1}{np} \sum_{i=1}^{n} \left\| \upsilon \nabla_f^2 \ell\left(f_{\mathbf{w}}; \mathbf{x}_i, y_i\right) \right\|_{\mathrm{op}} \left\| f_{\mathbf{w}} - f_{\tilde{\mathbf{w}}} \right\|_{\mathcal{H}} \overset{(ii)}{\leq} \sup_{f_{\mathbf{w}} \in \mathcal{K}} \frac{1}{np} \sum_{i=1}^{n} \left\| \nabla_f^2 \ell\left(f_{\mathbf{w}}; \mathbf{x}_i, y_i\right) \right\|_{\mathrm{op}} \left\| f_{\mathbf{w}} - f_{\tilde{\mathbf{w}}} \right\|_{\mathcal{H}}
\end{aligned} \tag{92}$$

(i) follows from applying the Triangle Inequality and then Weyl's Inequality [Wey12]. (ii) follows from noting $\upsilon \in (0, 1)$. We now note that removing υ also removes the dependence on λ which allows us to take the limit out of the expression. ■

B.5 Proof of Lemma 11

Proof. By the definition of stationary point, we have

$$f_{\hat{\mathbf{w}}} = \lim_{\lambda \to \infty} \arg\inf_{f_{\mathbf{w}} \in \mathcal{K}} \left\{ \Phi_\lambda\left(f_{\mathbf{w}}\right) + \frac{1}{2\rho} \left\| f_{\mathbf{w}} - f_{\hat{\mathbf{w}}} \right\|_{\mathcal{H}}^2 \right\} \tag{93}$$

$$\overset{(i)}{=} \arg\inf_{f_{\mathbf{w}} \in \mathcal{K}} \left\{ \lim_{\lambda \to \infty} \Phi_\lambda\left(f_{\mathbf{w}}\right) + \frac{1}{2\rho} \left\| f_{\mathbf{w}} - f_{\hat{\mathbf{w}}} \right\|_{\mathcal{H}}^2 \right\} \tag{94}$$

(i) holds as we ρ is independent of λ as shown in the proof of Lemma 6. This implies then for any $f_{\mathbf{w}} \in \mathcal{K}$ and noting $\rho \leq \beta^{-1}$, it follows

$$\lim_{\lambda \to \infty} \Phi_\lambda\left(f_{\hat{\mathbf{w}}}\right) \leq \lim_{\lambda \to \infty} \Phi_\lambda\left(f_{\mathbf{w}}\right) + \beta \left\| f_{\mathbf{w}} - f_{\hat{\mathbf{w}}} \right\|_{\mathcal{H}}^2 \tag{95}$$

where we choose $\rho \triangleq 1/(2\beta)$. We can then plug in the optimal, $f_{\mathbf{w}}^*$ for $f_{\mathbf{w}}$ and rearrange and we have the desired result. ■

C Proofs for Kernelized Regression

C.1 L-Lipschitz Constant and β-Smoothness

Lemma 27 (L-Lipschitz of $g(t, \mathbf{w})$ w.r.t $\mathbf{w}$)**.** *Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$, represent the data vectors. It then follows:*

$$|g(t, f_{\mathbf{w}}) - g(t, f_{\hat{\mathbf{w}}})| \leq L \|f_{\mathbf{w}} - f_{\hat{\mathbf{w}}}\|_{\mathcal{H}} \tag{96}$$

where

$$L = \frac{2R}{np}\left(\sum_{i=1}^{n}\sqrt{k(\mathbf{x}_i, \mathbf{x}_i)}\right)^2 + \frac{2\|\mathbf{y}\|_2}{p\sqrt{n}}\left(\sum_{i=1}^{n}\sqrt{k(\mathbf{x}_i, \mathbf{x}_i)}\right) \tag{97}$$

Proof. For any $f_{\mathbf{w}_1}, f_{\mathbf{w}_2} \in \mathcal{K}$, we will first show the gradient is bounded.

$$\begin{aligned}
|g(t, f_{\mathbf{w}_1}) - g(t, f_{\mathbf{w}_2})| &= \left|\int_0^1 \nabla_f g(t, (1-\lambda)f_{\mathbf{w}_1} + \lambda f_{\mathbf{w}_2})(f_{\mathbf{w}_1} - f_{\mathbf{w}_2})d\lambda\right| &(98)\\
&\leq \|f_{\mathbf{w}_1} - f_{\mathbf{w}_2}\|_{\mathcal{H}}\left|\int_0^1 \nabla_f g(t, (1-\lambda)f_{\mathbf{w}_1} + \lambda f_{\mathbf{w}_2})d\lambda\right| &(99)\\
&\overset{(a)}{\leq} \|f_{\mathbf{w}_1} - f_{\mathbf{w}_2}\|_{\mathcal{H}} \max_{f_{\mathbf{w}}\in\mathcal{K}} \|\nabla_f g(t, f_{\mathbf{w}})\|_{\mathcal{H}} &(100)
\end{aligned}$$

In (a), we note that since $\mathcal{K}$ is convex, then by definition as $f_{\mathbf{w}_1}, f_{\mathbf{w}_2} \in \mathcal{K}$, we have for $\lambda \in [0,1]$, the convex combination $(1-\lambda)f_{\mathbf{w}_1} + \lambda f_{\mathbf{w}_2} \in \mathcal{K}$. We use the $\mathcal{H}$ norm of the gradient to bound L from above for an element in the convex closed set $\mathcal{K}$.

$$\|\nabla g(t, f_{\mathbf{w}})\|_{\mathcal{H}} = \left\|\frac{2}{np}\sum_{i=1}^{n}\mathbb{I}\left\{t \geq (f_{\mathbf{w}}(\mathbf{x}_i) - y_i)^2\right\}(f_{\mathbf{w}}(\mathbf{x}_i) - y_i)\cdot k(\mathbf{x}_i, \cdot)\right\|_{\mathcal{H}} \tag{101}$$

W.L.O.G, let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m$ where $0 \leq m \leq n$, represent the data vectors such that $t \geq (f_{\mathbf{w}}(\mathbf{x}_i) - y_i)^2$.

$$\begin{aligned}
&= \left\|\frac{2}{np}\sum_{i=1}^{m}(f_{\mathbf{w}}(\mathbf{x}_i) - y_i)\cdot k(\mathbf{x}_i, \cdot)\right\|_{\mathcal{H}} \leq \frac{2}{np}\left(\left\|\sum_{i=1}^{m} f_{\mathbf{w}}(\mathbf{x}_i)\cdot k(\mathbf{x}_i, \cdot)\right\|_{\mathcal{H}} + \left\|\sum_{i=1}^{m} y_i k(\mathbf{x}_i, \cdot)\right\|_{\mathcal{H}}\right) &(102)\\
&\overset{(a)}{\leq} \frac{2}{np}\left(\left\|\sum_{i=1}^{m}\left\langle\sum_{j=1}^{n} w_j k(\mathbf{x}_j, \cdot), k(\mathbf{x}_i, \cdot)\right\rangle_{\mathcal{H}}\cdot k(\mathbf{x}_i, \cdot)\right\|_{\mathcal{H}} + \left\|\sum_{i=1}^{m} y_i\right\|\left\|\sum_{i=1}^{m} k(\mathbf{x}_i, \cdot)\right\|_{\mathcal{H}}\right) &(103)\\
&\leq \frac{2}{np}\left(\left|\left\langle\sum_{j=1}^{n} w_j k(\mathbf{x}_j, \cdot), \sum_{i=1}^{m} k(\mathbf{x}_i, \cdot)\right\rangle_{\mathcal{H}}\right|\left\|\sum_{i=1}^{m} k(\mathbf{x}_i, \cdot)\right\|_{\mathcal{H}} + \left\|\sum_{i=1}^{m} y_i\right\|\sum_{i=1}^{m}\sqrt{k(\mathbf{x}_i, \mathbf{x}_i)}\right) &(104)\\
&\leq \frac{2}{np}\left(\|f_{\mathbf{w}}\|_{\mathcal{H}}\left(\sum_{i=1}^{m}\sqrt{k(\mathbf{x}_i, \mathbf{x}_i)}\right)^2 + \sqrt{n}\|\mathbf{y}\|_2\left(\sum_{i=1}^{n}\sqrt{k(\mathbf{x}_i, \mathbf{x}_i)}\right)\right) &(105)\\
&\leq \frac{2R}{np}\left(\sum_{i=1}^{n}\sqrt{k(\mathbf{x}_i, \mathbf{x}_i)}\right)^2 + \frac{2\|\mathbf{y}\|_2}{p\sqrt{n}}\left(\sum_{i=1}^{n}\sqrt{k(\mathbf{x}_i, \mathbf{x}_i)}\right) &(106)
\end{aligned}$$

(a) follows form the reproducing property for RKHS [Gre13b]. If we have a normalized kernel such as the Gaussian Kernel, then we have the Lipschitz Constant is finite. Furthermore, if the adversary introduces label corruption that tends to ∞, then these points will not be in the Subquantile as $f_{\mathbf{w}}$ has bounded norm, so it will have infinite error. Similar results for the Lipschitz Constant for non-kernelized learning algorithms can be seen in [YSP21]. This concludes the proof. ■

Lemma 28. *(β-Smoothness of $g(t, \mathbf{w})$ w.r.t $\mathbf{w}$). Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ represent the rows of the data matrix $\mathbf{X}$. It then follows:*

$$\|\nabla_f g(t, f_{\mathbf{w}}) - \nabla_f g(t, f_{\hat{\mathbf{w}}})\| \leq \beta \|f_{\mathbf{w}} - f_{\hat{\mathbf{w}}}\|_{\mathcal{H}} \tag{107}$$

where

$$\beta = \frac{2}{np}\sum_{i=1}^{n} k(x_i, x_i) = \frac{2}{np}\operatorname{Tr}(\mathbf{K}) \tag{108}$$

Proof. W.L.O.G, let S be the set of points such that if $\mathbf{x} \in S$, then $t \geq (f_{\mathbf{w}}(\mathbf{x}) - y)^2$. Since g is twice differentiable, we will analyze the second derivative.

$$\begin{aligned}\left\|\nabla_f^2 g\left(t, f_{\mathbf{w}}\right)\right\|_{\mathrm{op}} &= \left\|\frac{2}{np}\sum_{i=1}^{n}\mathbb{I}\left\{t \geq (f_{\mathbf{w}}(\mathbf{x}_i) - y_i)^2\right\}\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)\right\|_{\mathrm{op}} \\ &\leq \frac{2}{np}\sum_{i=1}^{n}\left\|\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)\right\|_{\mathrm{HS}} = \frac{2}{np}\sum_{i=1}^{n}k(\mathbf{x}_i,\mathbf{x}_i) = \frac{2}{np}\operatorname{Tr}(\mathbf{K})\end{aligned}$$

This concludes the proof.

C.2 Proof of Lemma 13

Proof. We will first expand the expression in the Lemma statement. Let λ_i and φ_i for $i \in \mathbb{N}$ represent the eigenvalues and eigenfunctions for $\mathbb{E}_{\mathbf{x}\sim\mathbb{P}}[\phi(\mathbf{x})\otimes\phi(\mathbf{x})] \triangleq \mathbf{\Sigma}$.

$$\left\langle f_{\mathbf{w}} - f_{\mathbf{w}}^*, \mathbf{\Sigma}\left(f_{\mathbf{w}} - f_{\mathbf{w}}^*\right)\right\rangle = \lim_{p\to\infty}\sum_{i=1}^{p}\lambda_i\left\langle f_{\mathbf{w}} - f_{\mathbf{w}}^*, \varphi_i\right\rangle_{\mathcal{H}}^2 \tag{109}$$

Therefore, for some $m \in \mathbb{N}$, we want the projection of $f_{\mathbf{w}} - f_{\mathbf{w}}^*$ to be non-zero for m. We will show, we only need to make an assumption on $f_{\mathbf{w}}^*$. The projection in the Reproducing Kernel Hilbert Space is given as the following,

$$\left\|\operatorname{Proj}_{U_m} f_{\mathbf{w}}^*\right\|_{\mathcal{H}} \triangleq \left\|\sum_{i=1}^{m}\left\langle\varphi_i, f_{\mathbf{w}}^*\right\rangle\varphi_i\right\|_{\mathcal{H}} = \sum_{i=1}^{m}\left|\left\langle\varphi_i, f_{\mathbf{w}}^*\right\rangle_{\mathcal{H}}\right| \tag{110}$$

Let $f_{\mathbf{w}}^{(T)}$ be the T iterate from Subquantile Kernel Algorithm. From our assumption that $f_{\mathbf{w}}^* \in \operatorname{Span}(\{\varphi_i\}_{i=1}^m) \triangleq U_m$, it suffices to prove $\operatorname{Proj}_{U_m} f_{\mathbf{w}}^{(t)} \neq f_{\mathbf{w}}^*$ for all $t \in [T]$. In this case, we can note that $f_{\mathbf{w}} - f_{\mathbf{w}}^*$ will in some part be in the Span of U_m.

$$\begin{aligned}\left\langle f_{\mathbf{w}}^{(t)} - f_{\mathbf{w}}^*, \mathbf{\Sigma}\left(f_{\mathbf{w}}^{(t)} - f_{\mathbf{w}}^*\right)\right\rangle_{\mathcal{H}} &\overset{(109)}{=} \lim_{p\to\infty}\sum_{i=1}^{p}\lambda_i\left\langle f_{\mathbf{w}}^{(t)} - f_{\mathbf{w}}^*, \varphi_i\right\rangle_{\mathcal{H}}^2 & (111)\\ &\overset{(i)}{=} \sum_{i=1}^{m}\lambda_i\left\langle f_{\mathbf{w}}^{(t)} - f_{\mathbf{w}}^*, \varphi_i\right\rangle_{\mathcal{H}}^2 + \lim_{p\to\infty}\sum_{i=m+1}^{p}\lambda_i\left\langle f_{\mathbf{w}}^{(t)}, \varphi_i\right\rangle_{\mathcal{H}}^2 & (112)\\ &= \sum_{i=1}^{m}\lambda_i\left\langle f_{\mathbf{w}}^{(t)} - f_{\mathbf{w}}^*, \varphi_i\right\rangle_{\mathcal{H}}^2 + \left\|\operatorname{Proj}_{m,\perp} f_{\mathbf{w}}^{(t)}\right\|_{\mathcal{H}}^2 & (113)\\ &\overset{(ii)}{\geq} \lambda_m C\left\|f_{\mathbf{w}}^{(t)} - f_{\mathbf{w}}^*\right\|_{\mathcal{H}}^2 & (114)\end{aligned}$$

Note in (i), since $\left\|\operatorname{Proj}_{m,\perp} f_{\mathbf{w}}^*\right\|_{\mathcal{H}} = 0$, then it holds for all $i \in \mathbb{N}$ s.t. $i > m$ that $\langle f_{\mathbf{w}}^*, \varphi_i\rangle_{\mathcal{H}} = 0$. In (ii), we define $C \triangleq (\left\|f_{\mathbf{w}}^{(t)} - f_{\mathbf{w}}^*\right\|_{\mathcal{H}}^2 - \left\|\operatorname{Proj}_{m,\perp} f_{\mathbf{w}}^{(t)}\right\|_{\mathcal{H}}^2)/\left\|f_{\mathbf{w}}^{(t)} - f_{\mathbf{w}}^*\right\|_{\mathcal{H}}^2 > 0$ from our assumption that $f_{\mathbf{w}}^* \neq \operatorname{Proj}_m f_{\mathbf{w}}^{(t)}$ for all $t \in [T]$. This concludes the proof. ■

Lemma 29. *Let $\mathcal{K} \triangleq \{f_{\mathbf{w}} : \|f_{\mathbf{w}}\|_{\mathcal{H}} \leq R\}$. Then, for a $f_{\hat{\mathbf{w}}} \notin \mathcal{K}$, it follows*

$$\operatorname{Proj}_{\mathcal{K}} f_{\hat{\mathbf{w}}} = \Omega(1) f_{\hat{\mathbf{w}}} \tag{115}$$

Proof. We will formulate the dual problem and then find the corresponding $f_{\mathbf{w}}$ that solves the dual.

$$\operatorname{Proj}_{\mathcal{K}} f_{\hat{\mathbf{w}}} = \underset{f_{\mathbf{w}}\in\mathcal{K}}{\arg\min}\left\|f_{\mathbf{w}} - f_{\hat{\mathbf{w}}}\right\|_{\mathcal{H}}^2 = \underset{f_{\mathbf{w}}\in\mathcal{K}}{\arg\min}\left\|f_{\mathbf{w}}\right\|_{\mathcal{H}}^2 + \left\|f_{\hat{\mathbf{w}}}\right\|_{\mathcal{H}}^2 - 2\left\langle f_{\mathbf{w}}, f_{\hat{\mathbf{w}}}\right\rangle_{\mathcal{H}} \tag{116}$$

$$= \arg\min_{f_{\mathbf{w}} \in \mathcal{K}} \|f_{\mathbf{w}}\|_{\mathcal{H}}^2 - 2\left\langle f_{\mathbf{w}}, f_{\hat{\mathbf{w}}}\right\rangle_{\mathcal{H}} \tag{117}$$

From here we can solve the dual problem. The Lagrangian is given by,

$$\mathcal{L}(f_{\mathbf{w}}, u) \triangleq \|f_{\mathbf{w}}\|_{\mathcal{H}}^2 - 2\left\langle f_{\mathbf{w}}, f_{\hat{\mathbf{w}}}\right\rangle + u\left(\|f_{\mathbf{w}}\|_{\mathcal{H}}^2 - R^2\right) \tag{118}$$

Then, we have dual problem as $\theta(u) = \min_{\mathbf{w}\in\mathcal{H}} \mathcal{L}(f_{\mathbf{w}}, u)$. Taking the derivative of the Lagrangian and setting it to zero, we obtain $\arg\min_{f_{\mathbf{w}}\in\mathcal{H}} \mathcal{L}(f_{\mathbf{w}}, u) = (1+u)^{-1} f_{\hat{\mathbf{w}}}$. With some more work, we obtain $\arg\max_{u>0} \theta(u) = R^{-1}\|f_{\hat{\mathbf{w}}}\| - 1$. We then have $f_{\mathbf{w}}$ at u^* as $f_{\mathbf{w}} = R\|f_{\hat{\mathbf{w}}}\|_{\mathcal{H}}^{-1} f_{\hat{\mathbf{w}}}$. Since $\|f_{\hat{\mathbf{w}}}\| > R$ as $f_{\hat{\mathbf{w}}} \notin \mathcal{K}$ by assumption, our proof is complete. ■

Lemma 30. *If $\|f_{\mathbf{w}} - f_{\mathbf{w}^*}\| \geq \eta$, then it follows*

$$\begin{aligned}\lim_{\lambda\to\infty}\left(\Phi_\lambda\left(f_{\mathbf{w}}\right) - \Phi_\lambda\left(f_{\mathbf{w}^*}\right)\right) &\geq \eta^2 n(1-2\varepsilon)\lambda_{\min}\left(\mathbb{E}_{\mathbf{x}\sim\mathbb{P}}\left[\phi(\mathbf{x})\otimes\phi(\mathbf{x})\right]\right)\\ &- O\left(\sigma\sqrt{n(1-2\varepsilon)\log(n(1-2\varepsilon))\,\|\mathbf{\Sigma}\|_{\mathrm{HS}}}\right) - 2\eta\left\|\sum_{i\in S\cap P}\eta_i\phi\left(\mathbf{x}_i\right)\right\| - \sum_{j\in P\setminus S}\eta_j^2\end{aligned} \tag{119}$$

Proof. Let S be the set containing the points with the minimum error from X w.r.t to the weights vector $\mathbf{w}$. Define $\eta_i \triangleq \left(f^*_{\mathbf{w}}\left(\mathbf{x}_i\right) - y_i\right)$ where $i \in P$.

$$\lim_{\lambda\to\infty}\left(\Phi_\lambda\left(f_{\mathbf{w}}\right) - \Phi_\lambda\left(f^*_{\mathbf{w}}\right)\right) = \sum_{i\in S}\left(f_{\mathbf{w}}\left(\mathbf{x}_i\right) - y_i\right)^2 - \sum_{j\in P}\left(f^*_{\mathbf{w}}\left(\mathbf{x}_j\right) - y_j\right)^2 \tag{120}$$

$$= \sum_{i\in S\cap P}\left(f_{\mathbf{w}}\left(\mathbf{x}_i\right) - y_i\right)^2 + \sum_{i\in S\cap Q}\left(f_{\mathbf{w}}\left(\mathbf{x}_i\right) - y_i\right)^2 - \sum_{j\in P}\left(f^*_{\mathbf{w}}\left(\mathbf{x}_j\right) - y_j\right)^2 \tag{121}$$

$$\geq \sum_{i\in S\cap P}\left(f_{\mathbf{w}}\left(\mathbf{x}_i\right) - y_i\right)^2 - \sum_{j\in P}\left(f^*_{\mathbf{w}}\left(\mathbf{x}_j\right) - y_j\right)^2 = \sum_{i\in S\cap P}\left(f_{\mathbf{w}}\left(\mathbf{x}_i\right) - f^*_{\mathbf{w}}\left(\mathbf{x}_i\right) - \eta_i\right)^2 - \sum_{j\in P}\eta_j^2 \tag{122}$$

$$= \sum_{i\in S\cap P}\underbrace{\left(\left(f_{\mathbf{w}} - f^*_{\mathbf{w}}\right)\left(\mathbf{x}_i\right)\right)^2}_{A_1} \underbrace{-2\sum_{i\in S\cap P}\eta_i\left(f_{\mathbf{w}} - f^*_{\mathbf{w}}\right)\left(\mathbf{x}_i\right)}_{A_2} - \underbrace{\sum_{j\in P\setminus S}\eta_j^2}_{A_3} \tag{123}$$

Now we will upper bound A_1. Similar to [CLKZ21] Let $\mathbb{E}_{\mathbf{x}\sim\mathbb{P}}[\varphi(\mathbf{x})\otimes\varphi(\mathbf{x})] = \mathbb{I}_m$ where $\varphi(\mathbf{x}) = \{\varphi(\mathbf{x})\}_{k=1}^m$ and m is possibly infinite. We can then rescale the basis features. Then let $\phi(\mathbf{x}) = \mathbf{\Sigma}^{1/2}\varphi(\mathbf{x})$. We therefore have $\mathbf{\Sigma} = \mathbb{E}_{\mathbf{x}\sim\mathbb{P}}[\phi(\mathbf{x})\otimes\phi(\mathbf{x})] = \mathrm{diag}(\xi_1,\ldots,\xi_n)$. This is the eigenfunction basis described in [SS16].

$$A_1 \triangleq \sum_{i\in S\cap P}\left(\left(f_{\mathbf{w}} - f_{\mathbf{w}^*}\right)\left(\mathbf{x}_i\right)\right)^2 \overset{(a)}{=} \sum_{i\in S\cap P}\left\langle\sum_{j\in X}\left(w_j - w_j^*\right)k(\mathbf{x}_j,\cdot), k(\mathbf{x}_i,\cdot)\right\rangle_{\mathcal{H}}^2 \tag{124}$$

$$= \sum_{i\in S\cap P}\left\langle\sum_{j\in X}\left(w_j - w_j^*\right)\phi(\mathbf{x}_j), \phi(\mathbf{x}_i)\right\rangle_{\mathcal{H}}\left\langle\phi(\mathbf{x}_i), \sum_{j\in X}\left(w_j - w_j^*\right)\phi(\mathbf{x}_j)\right\rangle_{\mathcal{H}} \tag{125}$$

$$= \sum_{i\in S\cap P}\left\langle\sum_{j\in X}\left(w_j - w_j^*\right)\phi(\mathbf{x}_j), \phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)\sum_{j\in X}\left(w_j - w_j^*\right)\phi(\mathbf{x}_j)\right\rangle_{\mathcal{H}} \tag{126}$$

$$= \sum_{i\in S\cap P}\left\langle\phi(\mathbf{x})\otimes\phi(\mathbf{x}), \left(f_{\mathbf{w}} - f^*_{\mathbf{w}}\right)\otimes\left(f_{\mathbf{w}} - f^*_{\mathbf{w}}\right)\right\rangle_{\mathrm{HS}} \tag{127}$$

$$= \sum_{i\in S\cap P}\left\langle\mathbf{\Sigma} + \phi(\mathbf{x})\otimes\phi(\mathbf{x}) - \mathbf{\Sigma}, \left(f_{\mathbf{w}} - f^*_{\mathbf{w}}\right)\otimes\left(f_{\mathbf{w}} - f^*_{\mathbf{w}}\right)\right\rangle_{\mathrm{HS}} \tag{128}$$

$$\overset{\text{lem. 25}}{\geq} \left(n(1-2\varepsilon)\lambda_{\min}(\mathbf{\Sigma}) - \left\|\sum_{i\in S\cap P}\phi(\mathbf{x})\otimes\phi(\mathbf{x}) - \mathbf{\Sigma}\right\|_{\mathrm{HS}}\right)\|f_{\mathbf{w}} - f^*_{\mathbf{w}}\|_{\mathcal{H}}^2 \tag{129}$$

Next we will upper bound A_2,

$$A_2 \triangleq \sum_{i \in S \cap P} \eta_i \left(f_{\mathbf{w}} - f_{\mathbf{w}}^* \right)(\mathbf{x}_i) = \sum_{i \in S \cap P} \left\langle \sum_{j \in X} (w_j - w_j^*) k(\mathbf{x}_j, \cdot), \eta_i k(\mathbf{x}_i, \cdot) \right\rangle_{\mathcal{H}} \tag{130}$$

$$= \left\langle \sum_{j \in X} (w_j - w_j^*) k(\mathbf{x}_j, \cdot), \sum_{i \in S \cap P} \eta_i k(\mathbf{x}_i, \cdot) \right\rangle_{\mathcal{H}} \tag{131}$$

$$\leq \| f_{\mathbf{w}} - f_{\mathbf{w}}^* \|_{\mathcal{H}} \left\| \sum_{i \in S \cap P} \eta_i k(\mathbf{x}_i, \cdot) \right\|_{\mathcal{H}} = \| f_{\mathbf{w}} - f_{\mathbf{w}}^* \|_{\mathcal{H}} \left\| \sum_{i \in S \cap P} \eta_i \phi(\mathbf{x}_i) \right\|_{\mathcal{H}} \tag{132}$$

Then, combining our bounds, we have

$$\begin{aligned} \lim_{\lambda \to \infty} \left(\Phi_\lambda (f_{\mathbf{w}}) - \Phi_\lambda (f_{\mathbf{w}}^*) \right) &\overset{(129) \text{ and } (132)}{\geq} \eta^2 \Bigg(n(1-2\varepsilon) \lambda_{\min} \left(\mathop{\mathbb{E}}_{\mathbf{x} \sim \mathbb{P}} [\phi(\mathbf{x}) \otimes \phi(\mathbf{x})] \right) \\ &- \left\| \sum_{i \in S \cap P} \phi(\mathbf{x}) \otimes \phi(\mathbf{x}) - \mathbf{\Sigma} \right\|_{\text{HS}} \Bigg) - 2\eta \left\| \sum_{i \in S \cap P} \eta_i \phi(\mathbf{x}_i) \right\| - \sum_{j \in P \setminus S} \eta_j^2 \end{aligned} \tag{133}$$

This completes the proof. ∎

C.3 Proof of Theorem 14

Proof. First, we give the definiton of the Moreau stationary point.

$$\| \nabla \mathsf{M}_{\Phi_\lambda, \rho} (f_{\mathbf{w}}) \|_{\mathcal{H}} = \left\| \frac{1}{\rho} \left(f_{\mathbf{w}} - \arg\min_{f_{\hat{\mathbf{w}}} \in \mathcal{K}} \left(\Phi(f_{\hat{\mathbf{w}}}) + \frac{1}{2\rho} \| f_{\mathbf{w}} - f_{\hat{\mathbf{w}}} \|_{\mathcal{H}}^2 \right) \right) \right\|_{\mathcal{H}} = 0 \tag{134}$$

This implies for any $f_{\tilde{\mathbf{w}}} \in \mathcal{K}$, it follows

$$\lim_{\lambda \to \infty} \left(\Phi_\lambda (f_{\hat{\mathbf{w}}}) \right) < \lim_{\lambda \to \infty} \left(\Phi_\lambda (f_{\tilde{\mathbf{w}}}) \right) + \frac{1}{2\rho} \| f_{\tilde{\mathbf{w}}} - f_{\hat{\mathbf{w}}} \|_{\mathcal{H}}^2 \tag{135}$$

For any $f_{\hat{\mathbf{w}}}$ satisfying above, then the distance from the optimal must be low. Let $\tilde{\mathbf{w}} = \mathbf{w}^*$, then we have

$$\lim_{\lambda \to \infty} \left(\Phi_\lambda (f_{\hat{\mathbf{w}}}) - \Phi_\lambda (f_{\mathbf{w}}^*) \right) \leq \frac{1}{2\rho} \| f_{\hat{\mathbf{w}}} - f_{\mathbf{w}}^* \|_{\mathcal{H}}^2 \tag{136}$$

We proceed by proof by contradiction. Assume $\|f_{\hat{\mathbf{w}}} - f_{\mathbf{w}}^*\| > \eta$, then if $\Phi(f_{\hat{\mathbf{w}}}) - \Phi(f_{\mathbf{w}}^*) > \frac{\eta^2}{2\rho}$, then we will have $f_{\hat{\mathbf{w}}}$ is not a stationary point, which will imply $\|f_{\hat{\mathbf{w}}} - f_{\mathbf{w}}^*\|_{\mathcal{H}} \leq \eta$. Therefore, we attempt to find the minimum value for η. From Lemma 30, we have the expected distance from a stationary point of the Moreau Envelope from the optimal point over the distribution of uncorrupted datasets.

$$\begin{aligned} \mathop{\mathbb{E}}_{\mathcal{D} \sim \hat{\mathbb{P}}} \lim_{\lambda \to \infty} \left(\Phi(f_{\mathbf{w}}) - \Phi(f_{\mathbf{w}}^*) \right) &\overset{\text{lem. } 30}{\geq} \eta^2 \Bigg(n(1-2\varepsilon) \lambda_{\min} \left(\mathop{\mathbb{E}}_{\mathbf{x} \sim \mathbb{P}} [\phi(\mathbf{x}) \otimes \phi(\mathbf{x})] \right) \\ &- \mathop{\mathbb{E}}_{\mathbf{x}_i \sim \mathbb{P}} \left\| \sum_{i \in S \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) - \mathbf{\Sigma} \right\|_{\text{HS}} \Bigg) - 2\eta \mathop{\mathbb{E}}_{\mu_i, \mathbf{x}_i \sim \mathbb{P}} \left\| \sum_{i \in S \cap P} \eta_i \phi(\mathbf{x}_i) \right\| - \mathop{\mathbb{E}}_{\mu_j \sim \mathbb{P}} \sum_{j \in P \setminus S} \eta_j^2 \end{aligned} \tag{137}$$

$$\overset{\text{lems. } 24 \text{ and } 25}{\geq} \eta^2 \left(n(1-2\varepsilon) \lambda_{\min}(\mathbf{\Sigma}) - 2 \operatorname{Tr}(\mathbf{\Sigma}) \sqrt{n(1-2\varepsilon)} \right) - \eta O \left(\sigma \sqrt{n(1-2\varepsilon) \log(n(1-2\varepsilon)) \operatorname{Tr}(\mathbf{\Sigma})} \right) - \sigma \varepsilon n \tag{138}$$

From the definition of stationary point, we have

$$\eta^2 \left(n(1-2\varepsilon) \lambda_{\min}(\mathbf{\Sigma}) - 2 \operatorname{Tr}(\mathbf{\Sigma}) \sqrt{n(1-2\varepsilon)} - \beta \right) - \eta O \left(\sigma \sqrt{n(1-2\varepsilon) \log(n(1-2\varepsilon)) \operatorname{Tr}(\mathbf{\Sigma})} \right) - \sigma \varepsilon n \leq 0 \tag{139}$$

Therefore, when Equation (139) does not hold, we have a contradiction. It thus follows from upper bounding the positive solution of the quadratic equation,

$$\begin{aligned} \eta &\leq (\sigma \varepsilon n)^{1/2} \left(n(1-2\varepsilon) \left(\lambda_{\min}(\mathbf{\Sigma}) - \frac{2 \operatorname{Tr}(\mathbf{\Sigma})}{\sqrt{n(1-2\varepsilon)}} \right) - \beta \right)^{-1/2} \\ &+ O \left(\sigma \sqrt{n(1-2\varepsilon) \log(n(1-2\varepsilon)) \operatorname{Tr}(\mathbf{\Sigma})} \right) \left(n(1-2\varepsilon) \left(\lambda_{\min}(\mathbf{\Sigma}) - \frac{2 \operatorname{Tr}(\mathbf{\Sigma})}{\sqrt{n(1-2\varepsilon)}} \right) - \beta \right)^{-1} \end{aligned} \tag{140}$$

Then for some constant $c_1 \in (0,1)$, if $n \geq \frac{8\operatorname{Tr}(\mathbf{\Sigma})^2}{\lambda_{\min}(\mathbf{\Sigma})(1-c_1)^2(1-2\varepsilon)} + \frac{8\beta}{(1-c_1)^2(1-2\varepsilon)}$, we have

$$\eta \leq \left(\frac{\sigma\varepsilon n}{c_1 n(1-2\varepsilon)\lambda_{\min}(\mathbf{\Sigma})}\right)^{1/2} + \frac{O\left(\sigma\sqrt{\log(n(1-2\varepsilon))\operatorname{Tr}(\mathbf{\Sigma})}\right)}{c_1\sqrt{n(1-2\varepsilon)}\lambda_{\min}(\mathbf{\Sigma})} \tag{141}$$

we therefore see as n goes large, $c_1 \to 1$, and we have in the worst case

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}} \|f_{\hat{\mathbf{w}}} - f^*_{\mathbf{w}}\|_{\mathcal{H}} \leq O\left(\sqrt{\frac{\varepsilon}{1-2\varepsilon}}\frac{\sigma}{\lambda_{\min}(\mathbf{\Sigma})}\right) \tag{142}$$

This completes the proof. ■

C.4 Proof of Corollary 15

We follow the same framework as our proof for kernelized linear regression, we will simply give the new constants. Assuming the uncorrupted covariates, $\mathbf{x}_i \sim \mathcal{N}(f_{\mathbf{W}}(\mathbf{x}_i)ero, \mathbf{\Sigma})$. To simplify notation, let us define $\tilde{n} \triangleq n(1-2\varepsilon)$ to represent the absolute minimum number of uncorrupted points in the Subquantile. We then have,

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}} \lim_{\lambda\to\infty}(\Phi_\lambda(\mathbf{w}) - \Phi_\lambda(\mathbf{w}^*)) \overset{\text{lem. 30}}{\geq} \eta^2\left(\tilde{n}\lambda_{\min}(\mathbf{\Sigma}) - \mathbb{E}\left\|\sum_{i\in S\cap P}\mathbf{x}_i\mathbf{x}_i^\top - \mathbf{\Sigma}\right\|_2\right) - \mathop{\mathbb{E}}_{\xi,\mu_i\sim\mathbb{P}}\left\|\sum_{i\in S\cap P}\mu_i\mathbf{x}_i\right\|_2 - \mathop{\mathbb{E}}_{\mu_i\sim\mathbb{P}}\sum_{i\in P\setminus S}\mu_i^2$$

$$\overset{\text{lems. 21, 24 and 26}}{\geq} \eta^2\left(\tilde{n}\lambda_{\min}(\mathbf{\Sigma}) - \sqrt{\tilde{n}}\left(2\sqrt{3}\operatorname{Tr}(\mathbf{\Sigma})\right)\right) - \eta O\left(\sigma\sqrt{\tilde{n}\log(\tilde{n})\operatorname{Tr}(\mathbf{\Sigma})}\right) - \varepsilon n\sigma^2 \tag{143}$$

Then from a similar contradiction idea and upper bounding the quadratic, we have in expectation

$$\eta \overset{\text{thm. 14}}{\leq} O\left(\sigma\sqrt{\tilde{n}\log(\tilde{n})\operatorname{Tr}(\mathbf{\Sigma})}\right)\left(\tilde{n}\lambda_{\min}(\mathbf{\Sigma}) - \sqrt{\tilde{n}}\left(2\sqrt{3}\operatorname{Tr}(\mathbf{\Sigma})\right) - \beta\right)^{-1} + \sigma\sqrt{\tilde{n}\frac{\varepsilon}{1-2\varepsilon}}\left(\tilde{n}\lambda_{\min}(\mathbf{\Sigma}) - \sqrt{\tilde{n}}\left(2\sqrt{3}\operatorname{Tr}(\mathbf{\Sigma})\right) - \beta\right)^{-1/2}$$

We then have for a constant $c_2 \in (0,1)$, if $n \geq \frac{54\operatorname{Tr}(\mathbf{\Sigma})}{(1-c_2)^2(1-2\varepsilon)\lambda^2_{\min}(\mathbf{\Sigma})} + 2\beta$, it follows

$$\eta \leq \sqrt{\frac{\sigma^2\varepsilon}{(1-2\varepsilon)c_2\lambda_{\min}(\mathbf{\Sigma})}} + \frac{O\left(\sigma\sqrt{\log(n(1-2\varepsilon))\operatorname{Tr}(\mathbf{\Sigma})}\right)}{\sqrt{n(1-2\varepsilon)}c_2\lambda_{\min}(\mathbf{\Sigma})} \tag{144}$$

We thus see as n goes large, $c_2 \to 1$ and we will have in worst case,

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}} \|\hat{\mathbf{w}} - \mathbf{w}^*\|_2 \leq O\left(\frac{\gamma\sigma}{\sqrt{\lambda_{\min}(\mathbf{\Sigma})}}\right) \tag{145}$$

where $\gamma \triangleq \sqrt{\frac{|P\setminus S|}{|S\cap P|}}$. Obtaining the same asymptotic bound as in the kernelized regression case. This completes the proof. ■

D Kernlized Binary Classification

In this section, we will prove error bounds for Subquantile Minimization in the Kernelized Binary Classification Problem.

D.1 L-Lipschitz Constant and β-Smoothness Constant

Lemma 31. *(L-Lipschitz of $g(t,\mathbf{w})$ w.r.t $\mathbf{w}$). Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$, represent the data vectors. It then follows:*

$$|g(t, f_{\mathbf{w}}) - g(t, f_{\hat{\mathbf{w}}})| \leq L\|f_{\mathbf{w}} - f_{\hat{\mathbf{w}}}\|_{\mathcal{H}} \tag{146}$$

where

$$L = \frac{1}{np}\sum_{i\in X}\sqrt{k(\mathbf{x}_i,\mathbf{x}_i)} = \frac{1}{np}\operatorname{Tr}(\mathbf{K}) \tag{147}$$

Proof. We use the $\mathcal{H}$ norm of the gradient to bound L from above. Let S be denoted as the subquantile set. Define the sigmoid function as $\sigma(x) = \frac{1}{1+e^{-x}}$.

$$\|\nabla_{\mathsf{f}} g(t, f_{\mathbf{w}})\|_{\mathcal{H}} = \left\| \frac{1}{np} \sum_{i=1}^{n} \mathbb{I}\{t \geq (1-y_i)\log(f_{\mathbf{w}}(\mathbf{x}_i))\} \left(y_i - \sigma\left(f_{\mathbf{w}}(\mathbf{x}_i)\right)\right) \cdot k(\mathbf{x}_i, \cdot) \right\|_{\mathcal{H}} \tag{148}$$
$$\overset{(i)}{\leq} \frac{1}{np} \sum_{i \in S} \|(y_i - \sigma\left(f_{\mathbf{w}}(\mathbf{x}_i)\right)) \cdot k(\mathbf{x}_i, \cdot)\|_{\mathcal{H}} \overset{(ii)}{\leq} \frac{1}{np} \sum_{i \in S} |y_i - \sigma\left(f_{\mathbf{w}}(\mathbf{x}_i)\right)| \, \|k(\mathbf{x}_i, \cdot)\|_{\mathcal{H}} \overset{(iii)}{\leq} \frac{1}{np} \sum_{i=1}^{n} \sqrt{k(\mathbf{x}_i, \mathbf{x}_i)}$$

(i) follows from the triangle inequality. (ii) follows from the Cauchy-Schwarz inequality. (iii) follows from the fact that $y_i \in \{0, 1\}$ and range$(\sigma) \in [0, 1]$. This completes the proof. ■

Lemma 32. *(β-Smoothness of $g(t, \mathbf{w})$ w.r.t $\mathbf{w}$). Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ represent the rows of the data matrix $\mathbf{X}$. It then follows:*

$$\|\nabla_f g\left(t, f_{\mathbf{w}}\right) - \nabla_f g\left(t, f_{\hat{\mathbf{w}}}\right)\| \leq \beta \left\|f_{\mathbf{w}} - f_{\hat{\mathbf{w}}}\right\|_{\mathcal{H}} \tag{149}$$

where

$$\beta = \frac{1}{4p} \sum_{i=1}^{n} k(x_i, x_i) = \frac{1}{4p} \operatorname{Tr}\left(\mathbf{K}\right) \tag{150}$$

Proof. We use the operator norm of second derivative to bound β from above. Let S be the subquantile set.

$$\left\|\nabla_f^2 g(t, f_{\mathbf{w}})\right\|_{\text{op}} = \frac{1}{np} \sum_{i=1}^{n} \mathbb{I}\left\{t \geq (1-y_i)\log(f_{\mathbf{w}}(\mathbf{x}_i))\right\} \sigma\left(f_{\mathbf{w}}(\mathbf{x}_i)\right)\left(1 - \sigma\left(f_{\mathbf{w}}(\mathbf{x}_i)\right)\right) \|\phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i)\|_{\text{op}} \tag{151}$$
$$\leq \frac{1}{np} \sum_{i=1}^{n} \left|\sigma\left(f_{\mathbf{w}}(\mathbf{x}_i)\right)\left(1 - \sigma\left(f_{\mathbf{w}}(\mathbf{x}_i)\right)\right)\right| \|\phi(\mathbf{x}_i)\|_{\text{op}}^2 \overset{(i)}{\leq} \frac{1}{4np} \sum_{i=1}^{n} k(x_i, x_i) = \frac{1}{4np} \operatorname{Tr}\left(\mathbf{K}\right) \tag{152}$$

(i) follows as for a scaler $\alpha \in [0, 1]$, the maximum value of $\alpha(1-\alpha)$ is obtained at $\frac{1}{4}$. This completes the proof. ■

Lemma 33. *Assume $f_{\hat{\mathbf{w}}}$ is a first-order stationary point as defined in Definition 9. If $\|f_{\hat{\mathbf{w}}} - f_{\mathbf{w}^*}\|_{\mathcal{H}} \geq \eta$, then it follows*

$$\mathbb{E}_{\mathcal{D} \sim \hat{\mathbb{P}}} \|f_{\hat{\mathbf{w}}} - f_{\mathbf{w}}^*\|_{\mathcal{H}} \leq O\left(\frac{\sqrt{n(1-2\varepsilon)\operatorname{Tr}\left(\mathbf{\Sigma}\right)} + \sqrt{n\varepsilon Q_k}}{n(1-2\varepsilon)c_4\lambda_{\min}(\mathbf{\Sigma})}\right) \tag{153}$$

Proof. By the Lemma statement, we have $f_{\hat{\mathbf{w}}}$ is a stationary point, i.e. $\mathbf{0} \in \partial\Phi(f_{\hat{\mathbf{w}}})$. This implies for all $\mathsf{f}_{\mathbf{w}} \in \mathcal{K}$, we have $\Phi(f_{\hat{\mathbf{w}}}) \leq \Phi(f_{\mathbf{w}})$. As Φ is differentiable, we have the first-order stationary condition, which is $\nabla\Phi(f_{\hat{\mathbf{w}}})(f_{\hat{\mathbf{w}}} - f_{\mathbf{w}}) \leq 0$ or for all $\mathbf{w} \in \mathcal{K}$. We assume $f_{\mathbf{w}}^* \in \mathcal{K}$. Let S be the Subquantile set for $f_{\hat{\mathbf{w}}}$. We will proceed by contradiction, assume $\|f_{\hat{\mathbf{w}}} - f_{\mathbf{w}}^*\|_{\mathcal{H}} \geq \eta$. Then, we have

$$\left(\nabla_f g\left(f_{\hat{\mathbf{w}}}, t\right)\right)\left(f_{\hat{\mathbf{w}}} - f_{\mathbf{w}}^*\right) = \left(f_{\hat{\mathbf{w}}} - f_{\mathbf{w}}^*\right)\left(\sum_{i \in S} \left(\sigma\left(f_{\hat{\mathbf{w}}}(\mathbf{x}_i)\right) - y_i\right)\phi(\mathbf{x}_i)\right) \tag{154}$$

$$= \left(f_{\hat{\mathbf{w}}} - f_{\mathbf{w}}^*\right)\left(\sum_{i \in S} \left(\sigma\left(f_{\hat{\mathbf{w}}}(\mathbf{x}_i)\right) - \sigma\left(f_{\mathbf{w}}^*(\mathbf{x}_i)\right) + \sigma\left(f_{\mathbf{w}}^*(\mathbf{x}_i)\right) - y_i\right)\phi(\mathbf{x}_i)\right) \tag{155}$$

$$\overset{(i)}{\geq} \underbrace{\left(f_{\hat{\mathbf{w}}} - f_{\mathbf{w}}^*\right)\left(\sum_{S \cap P} \left(\sigma\left(f_{\hat{\mathbf{w}}}(\mathbf{x}_i)\right) - \sigma\left(f_{\mathbf{w}}^*(\mathbf{x}_i)\right)\right)\phi\left(\mathbf{x}_i\right)\right)}_{B_1} + \underbrace{\left(f_{\hat{\mathbf{w}}} - f_{\mathbf{w}}^*\right)\left(\sum_{i \in S} \left(\sigma\left(f_{\mathbf{w}}^*(\mathbf{x}_i)\right) - y_i\right)\phi\left(\mathbf{x}_i\right)\right)}_{B_2} \tag{156}$$

(i) follows from noting $\sigma(\cdot)$ is a monotonically increasing function. Let us now consider the function $h : \mathcal{H} \to \mathbb{R}$ defined as $h(f_{\mathbf{w}}) = \sum_{i \in S \cap P} \log(1 + \exp(f_{\mathbf{w}}(\mathbf{x}_i)))$. We then have $h'(f_{\mathbf{w}}) = \sum_{i \in S \cap P} \sigma(f_{\mathbf{w}}(\mathbf{x}_i))\phi(\mathbf{x}_i)$, from which we have $h''(f_{\mathbf{w}}) = \sum_{i \in S \cap P} \sigma(f_{\mathbf{w}}(\mathbf{x}_i))(1 - \sigma(f_{\mathbf{w}}(\mathbf{x}_i)))(\phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i))$. We can then note h is strongly convex with $\mu = \Omega(\lambda_{\min}(\sum_{i\in S\cap P}\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)))$. Then from the properties of strongly convex functions, we have

$$\sum_{i\in S\cap P}\left(f_{\hat{\mathbf{w}}}(\mathbf{x}_i)-f^*_{\mathbf{w}}(\mathbf{x}_i)\right)\left(\sigma\left(f_{\hat{\mathbf{w}}}(\mathbf{x}_i)\right)-\sigma\left(f^*_{\mathbf{w}}(\mathbf{x}_i)\right)\right)\gtrsim\lambda_{\min}\left(\sum_{i\in S\cap P}\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)\right)\|f^*_{\mathbf{w}}-f_{\hat{\mathbf{w}}}\|^2_{\mathcal{H}} \tag{157}$$

Then from the Cauchy-Schwarz Inequality, we have

$$\sum_{i\in S}\left(f^*_{\mathbf{w}}(\mathbf{x}_i)-f_{\hat{\mathbf{w}}}(\mathbf{x}_i)\right)\left(y_i-\sigma\left(f^*_{\mathbf{w}}(\mathbf{x}_i)\right)\right)\le\max_{i\in S}|y_i-\sigma\left(f^*_{\mathbf{w}}(\mathbf{x}_i)\right)|\left\langle\sum_{j\in X}(w^*_j-\hat{w}_j)\phi(\mathbf{x}_j),\sum_{i\in S}\phi(\mathbf{x}_i)\right\rangle \tag{158}$$

$$\le\|f^*_{\mathbf{w}}-f_{\hat{\mathbf{w}}}\|_{\mathcal{H}}\left\|\sum_{i\in S}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}\le\|f^*_{\mathbf{w}}-f_{\hat{\mathbf{w}}}\|_{\mathcal{H}}\left(\left\|\sum_{i\in S\cap P}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}+\left\|\sum_{i\in S\cap Q}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}\right) \tag{159}$$

for a small positive constant we denote c_3. This completes the proof. ■

D.2 Proof of Theorem 16

Proof. From Lemma 33, we have in expectation

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}}\left(\nabla_f g\left(f_{\hat{\mathbf{w}}},t\right)\right)\left(f^*_{\mathbf{w}}-f_{\hat{\mathbf{w}}}\right)\overset{\text{lem. }33}{\ge}c_3\left(n(1-2\varepsilon)\lambda_{\min}\left(\mathbf{\Sigma}\right)-\mathop{\mathbb{E}}_{\mathbf{x}_i\sim\mathbb{P}}\left\|\sum_{i\in S\cap P}\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)-\mathbf{\Sigma}\right\|\right)\|f_{\hat{\mathbf{w}}}-f^*_{\mathbf{w}}\|^2_{\mathcal{H}}$$
$$-\left(\sqrt{n(1-2\varepsilon)\operatorname{Tr}\left(\mathbf{\Sigma}\right)}+\sqrt{n\varepsilon Q_k}\right)\|f_{\hat{\mathbf{w}}}-f^*_{\mathbf{w}}\|_{\mathcal{H}} \tag{160}$$

We will lower bound the constant we introduced in Equation (157) and call it c_3, recall for $f\in\mathcal{K}$, we have $\|f\|_{\mathcal{H}}\le R$ and $P_k\triangleq\max_{i\in P}k(x_i,x_i)$.

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}}c_3\overset{(157)}{=}\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}}\min_{i\in S\cap P}\left(1-\sigma\left(f_{\hat{\mathbf{w}}}(\mathbf{x}_i)\right)\right)\sigma\left(f_{\hat{\mathbf{w}}}(\mathbf{x}_i)\right) \tag{161}$$

$$\ge\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}}\left(1-\sigma(R\max_{i\in P}k\left(\mathbf{x}_i,\mathbf{x}_i\right))\right)\sigma(R\max_{i\in P}k\left(\mathbf{x}_i,\mathbf{x}_i\right)) \tag{162}$$

$$\ge\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}}\frac{\sigma\left(-R\max_{i\in P}k\left(\mathbf{x}_i,\mathbf{x}_i\right)\right)}{2}\overset{(i)}{\gtrsim}\exp\left(-R\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}}\left[\max_{i\in P}k\left(\mathbf{x}_i,\mathbf{x}_i\right)\right]\right) \tag{163}$$

$$\overset{\text{lem. }23}{\ge}\exp\left(-RC_8\left(\operatorname{Tr}\left(\mathbf{\Sigma}\right)+\log n\right)\right) \tag{164}$$

(i) follows from Jensen's Inequality as $\exp(-x)$ is a convex functon. Then we have from the definition of a stationary point, $\nabla_f g\left(f_{\hat{\mathbf{w}}},t\right)\left(f_{\hat{\mathbf{w}}}-f^*_{\mathbf{w}}\right)\le 0$ when

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}}\|f_{\hat{\mathbf{w}}}-f^*_{\mathbf{w}}\|_{\mathcal{H}}\le O\left(\frac{\sqrt{n(1-2\varepsilon)\operatorname{Tr}\left(\mathbf{\Sigma}\right)}+\sqrt{n\varepsilon Q_k}}{\exp\left(-RC_8\left(\operatorname{Tr}\left(\mathbf{\Sigma}\right)+\log n\right)\right)\left(n(1-2\varepsilon)\lambda_{\min}(\mathbf{\Sigma})-2\sqrt{n(1-2\varepsilon)}\operatorname{Tr}\left(\mathbf{\Sigma}\right)\right)}\right) \tag{165}$$

If $n\ge\frac{4\operatorname{Tr}(\mathbf{\Sigma})}{\lambda_{\min}(\mathbf{\Sigma})(1-2\varepsilon)(1-c_4)}$ for $c_4\in(0,1)$, then we have

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}}\|f_{\hat{\mathbf{w}}}-f^*_{\mathbf{w}}\|_{\mathcal{H}}\le O\left(\frac{\sqrt{\operatorname{Tr}(\mathbf{\Sigma})}+\sqrt{Q_k}}{\sqrt{n(1-2\varepsilon)}\exp\left(-R\left(\operatorname{Tr}\left(\mathbf{\Sigma}\right)+\log n\right)\right)\lambda_{\min}(\mathbf{\Sigma})}\right) \tag{166}$$

This completes the proof as we see we have $O(1/\sqrt{n})$ convergence. ■

E Proofs for Kernelized Multi-Class Classification

E.1 L-Lipschitz Constant and β-Smoothness Constant

Lemma 34. *Let $\mathbf{x}_1,\mathbf{x}_2,\dots,\mathbf{x}_n\sim\hat{\mathbb{P}}$. It then follows for a $f_{\mathbf{w}}\in\mathcal{K}$, then $g(t,f_{\mathbf{w}})$ is L-Lipschitz and β-Smooth for constants $L=\frac{1}{np}\sum_{i=1}^n\sqrt{k(\mathbf{x}_i,\mathbf{x}_i)}$ and $\beta=\frac{1}{np}\operatorname{Tr}\left(\mathbf{K}\right)$.*

Proof. We use the Hilbert Space norm of the gradient to bound L from above. Let S be denoted as the subquantile set. We first give some derivatives.

$$\frac{\partial}{\partial \mathbf{w}_k}\left(\ell\left(\mathbf{x}_i, y_i; f_{\mathbf{W}}\right)\right) = \begin{cases} -\phi(\mathbf{x}_i)\text{softmax}(f_{\mathbf{W}}(\mathbf{x}_i))_k & \text{if } k = y_i \\ \phi(\mathbf{x}_i)\left(1 - \text{softmax}(f_{\mathbf{W}}(\mathbf{x}_i))_k\right) & \text{if } k \neq y_i \end{cases} \tag{167}$$

Our proof then follows similarly to the proof for Lemma 31. We utilize $\odot$ to denote entry wise multiplication, i.e $\mathbf{x} \cdot \mathbf{y}$ indicates $\mathbf{y}$ is multiplied to each element of $\mathbf{x}$.

$$\begin{aligned}\|\nabla_f g(t, f_{\mathbf{W}})\|_{\mathcal{H}} &= \left\| \frac{1}{np}\sum_{i=1}^n \mathbb{I}\left\{-\log\left(\text{softmax}\left(f_{\mathbf{W}}(\mathbf{x}_i)\right)_{y_i}\right) \geq t\right\}\left(\text{softmax}\left(f_{\mathbf{W}}(\mathbf{x}_i)\right) - y_i\right) \odot k(\mathbf{x}_i, \cdot)\right\| \\ &\leq \frac{1}{np}\sum_{i=1}^n \|(\text{softmax}\left(f_{\mathbf{W}}(\mathbf{x}_i)\right) - y_i) \odot k(\mathbf{x}_i, \cdot)\| \leq \frac{1}{np}\sum_{i=1}^n \|k(\mathbf{x}_i, \cdot)\|_{\mathcal{H}} = \frac{1}{np}\sum_{i=1}^n \sqrt{k(\mathbf{x}_i, \mathbf{x}_i)}\end{aligned} \tag{168}$$

This gives the L-Lipschitz Constant.

We upper bound the operator norm of the Hessian to find the β-smoothness constant.

$$\|\nabla_f g\left(t, f_{\mathbf{W}}\right)\|_{\text{op}} \tag{169}$$

$$= \left\|\frac{1}{np}\sum_{i=1}^n \mathbb{I}\left\{-\log\left(\text{softmax}\left(f_{\mathbf{W}}(\mathbf{x}_i)\right)_{y_i}\right) \geq t\right\}\left(\text{diag}(\text{softmax}(f_{\mathbf{W}}(\mathbf{x}_i)) - \text{softmax}(f_{\mathbf{W}}(\mathbf{x}_i)\text{softmax}(f_{\mathbf{W}}(\mathbf{x}_i)^\top\right) \odot (\phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i))\right\|_{\text{op}}$$

$$\leq \frac{1}{np}\sum_{i=1}^n \left\|\left(\text{diag}(\text{softmax}(f_{\mathbf{W}}(\mathbf{x}_i)) - \text{softmax}(f_{\mathbf{W}}(\mathbf{x}_i)\text{softmax}(f_{\mathbf{W}}(\mathbf{x}_i)^\top\right) \odot (\phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i))\right\|_{\text{op}} \tag{170}$$

$$\leq \frac{1}{np}\sum_{i=1}^n \left\|\text{diag}(\text{softmax}(f_{\mathbf{W}}(\mathbf{x}_i)) - \text{softmax}(f_{\mathbf{W}}(\mathbf{x}_i)\text{softmax}(f_{\mathbf{W}}(\mathbf{x}_i)^\top\right\|_{\text{op}} \|\phi(\mathbf{x}_i)\|_{\mathcal{H}}^2 \tag{171}$$

$$\leq \frac{1}{np}\sum_{i=1}^n k(\mathbf{x}_i, \mathbf{x}_i) = \frac{1}{np}\text{Tr}\left(\mathbf{K}\right) \tag{172}$$

This gives the β-Smoothness Constant. ∎

Lemma 35. *Assume $f_{\hat{\mathbf{w}}}$ is a first-order stationary point as defined in Definition 9. If $\|f_{\hat{\mathbf{w}}} - f_{\mathbf{w}^*}\|_{\mathcal{H}} \geq \eta$, then it follows*

$$\lim_{\lambda \to \infty}\left\langle \nabla_f \tilde{g}_\lambda\left(f_{\hat{\mathbf{W}}}, \hat{t}\right), f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}\right\rangle_{\mathcal{H}} \gtrsim \lambda_{\min}\left(\sum_{i \in S \cap P}\phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i)\right)\eta^2 + \eta\left(\left\|\sum_{i \in S \cap P}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}} + \left\|\sum_{i \in S \cap Q}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}\right) \tag{173}$$

Proof. We follow the similar set up as in Lemma 33. We have $f_{\hat{\mathbf{W}}}$ is a stationary point, i.e. $\mathbf{0} \in \partial\Phi(f_{\hat{\mathbf{W}}})$. This implies for all $\mathsf{f}_{\mathbf{W}} \in \mathcal{K}$, we have $\Phi(f_{\hat{\mathbf{W}}}) \leq \Phi(f_{\mathbf{W}})$. As Φ is differentiable, we have the first-order stationary condition, which is $\nabla\Phi(f_{\hat{\mathbf{W}}})(f_{\hat{\mathbf{W}}} - f_{\mathbf{W}}) \leq 0$ or for all $\mathbf{W} \in \mathcal{K}$. We assume $f^*_{\mathbf{W}} \in \mathcal{K}$. Let S be the Subquantile set for $f_{\hat{\mathbf{W}}}$. We will proceed by contradiction, assume $\left\|f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}\right\|_{\mathcal{H}} \geq \eta$. Then, we have

$$\left\langle \nabla_f g\left(f_{\hat{\mathbf{W}}}, t\right), f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}\right\rangle_{\mathcal{H}} = \left\langle f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}, \sum_{i \in S}\left(\text{softmax}\left(f_{\hat{\mathbf{W}}}(\mathbf{x}_i)\right) - y_i\right) \odot k(\mathbf{x}_i, \cdot)\right\rangle_{\mathcal{H}} \tag{174}$$

$$= \underbrace{\left\langle f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}, \sum_{i \in S}\left(\text{softmax}\left(f_{\hat{\mathbf{W}}}(\mathbf{x}_i)\right) - \text{softmax}\left(f^*_{\mathbf{W}}(\mathbf{x}_i)\right)\right) \odot k(\mathbf{x}_i, \cdot)\right\rangle_{\mathcal{H}}}_{C_1} + \underbrace{\left\langle f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}, \sum_{i \in S}\left(\text{softmax}\left(f^*_{\mathbf{W}}(\mathbf{x}_i)\right) - y_i\right) \odot k(\mathbf{x}_i, \cdot)\right\rangle_{\mathcal{H}}}_{C_2}$$

Let us now consider the function $h : \mathcal{H} \times \cdots \times \mathcal{H} :\to \mathbb{R}^n$ defined as $h(f_{\mathbf{W}}) = \sum_{i \in P \cap S}\log(\sum_{j=1}^{|\mathcal{Y}|}\exp(f_{\mathbf{w}_j}(\mathbf{x}_i) - y_{i,j}))$. We then have $\nabla h(f_{\mathbf{W}}) = \sum_{i \in P \cap S}\text{softmax}(f_{\mathbf{W}}(\mathbf{x}_i) - \mathbf{y}_i) \odot \phi(\mathbf{x}_i)$. From which it follows $\nabla^2 h(\mathbf{x}_i) = \sum_{i \in P \cap S}(\text{diag}(\text{softmax}(f_{\mathbf{W}}(\mathbf{x}_i))) - \text{softmax}(f_{\mathbf{W}}(\mathbf{x}_i))\text{softmax}(f_{\mathbf{W}}(\mathbf{x}_i))^\top) \odot (\phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i))$. This function is not strictly convex, as corroborated in [GP17]. If the softmax returns the same value for all inputs, then we note $\alpha^2 \mathbf{1}^\top \nabla^2 h(f_{\mathbf{W}})\mathbf{1} = 0$ where $\alpha = \frac{1}{|\mathcal{Y}|}$. Since we assume random initialization of the functions in $f_{\mathbf{W}}$, this event does not occur almost surely. Therefore, we have the function is strictly convex over the domain.

$$(174)\text{ LHS} \geq \Omega\left(\lambda_{\min}\left(\sum_{i\in S}\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)\right)\left\|f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}\right\|^2_{\mathcal{H}}\right) + \left\langle f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}, \sum_{i\in S}\left(\text{softmax}\left(f^*_{\mathbf{W}}(\mathbf{x}_i)\right) - \mathbf{y}_i\right)\odot k(\mathbf{x}_i,\cdot)\right\rangle_{\mathcal{H}} \tag{175}$$

$$\overset{(i)}{\geq} \Omega\left(\lambda_{\min}\left(\sum_{i\in S\cap P}\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)\right)\left\|f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}\right\|^2_{\mathcal{H}}\right) + \left\langle f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}, \sum_{i\in S}\left(\text{softmax}\left(f^*_{\mathbf{W}}(\mathbf{x}_i)\right) - \mathbf{y}_i\right)\odot k(\mathbf{x}_i,\cdot)\right\rangle_{\mathcal{H}} \tag{176}$$

Where (i) follows from Weyl's Inequality [Wey12]. Now we will upper bound C_2.

$$\begin{aligned}
C_2 &= \sum_{i\in S}\left\langle f_{\hat{\mathbf{W}}}(\mathbf{x}_i) - f^*_{\mathbf{W}}(\mathbf{x}_i), \text{softmax}\left(f^*_{\mathbf{W}}(\mathbf{x}_i)\right) - \mathbf{y}_i\right\rangle && (177)\\
&\overset{(i)}{\leq} \sum_{i\in S}\sqrt{\sum_{j\in\mathcal{Y}}\left(f_{\hat{\mathbf{w}}_j}(\mathbf{x}_i) - f^*_{\mathbf{w}_j}(\mathbf{x}_i)\right)^2}\sqrt{\sum_{j\in\mathcal{Y}}\left(\text{softmax}\left(f^*_{\mathbf{w}_j}(\mathbf{x}_i)\right) - y_{ij}\right)^2} && (178)\\
&\leq 2\sum_{i\in S}\sum_{j\in\mathcal{Y}}\left(f_{\hat{\mathbf{w}}_j}(\mathbf{x}_i) - f^*_{\mathbf{w}_j}(\mathbf{x}_i)\right) \leq 2\left\|f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}\right\|_{\mathcal{H}}\left\|\sum_{i\in S}\phi(\mathbf{x}_i)\right\| && (179)\\
&\leq 2\left\|f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}\right\|_{\mathcal{H}}\left(\left\|\sum_{i\in S\cap P}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}} + \left\|\sum_{i\in S\cap Q}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}\right) && (180)
\end{aligned}$$

Where (i) follows from Hölder's Inequality [H8̈9]. Substituting Equations (175) and (180) into Equation (174) completes the proof. ■

E.2 Proof of Theorem 17

From Lemma 35, we have in expectation over the dataset sampling,

$$\begin{aligned}
&\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}}\lim_{\lambda\to\infty}\left\langle\nabla_f\tilde{g}\left(f_{\hat{\mathbf{W}}},\hat{t}\right), f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}\right\rangle_{\mathcal{H}} \geq \Omega\left(\lambda_{\min}\left(\sum_{i\in S\cap P}\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)\right)\left\|f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}\right\|^2_{\mathcal{H}}\right) && (181)\\
&\quad - \left\|f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}\right\|_{\mathcal{H}}\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}}\left[\left(\left\|\sum_{i\in S\cap P}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}} + \left\|\sum_{i\in S\cap Q}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}\right)\right] && (182)\\
&\overset{(i)}{\geq}\Omega\left(\left(\lambda_{\min}\left(\mathbf{\Sigma}\right) - \mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}}\left\|\sum_{i\in S\cap P}\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i) - \mathbf{\Sigma}\right\|_{\text{HS}}\right)\left\|f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}\right\|^2_{\mathcal{H}}\right)\\
&\quad - \left\|f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}\right\|_{\mathcal{H}}\sqrt{\mathbb{E}\left[\left\|\sum_{i\in S\cap P}\phi(\mathbf{x}_i)\right\|^2\right]} + \sqrt{n\varepsilon Q_k} && (183)\\
&\overset{(ii)}{\geq}\Omega\left(n(1-2\varepsilon)\lambda_{\min}(\mathbf{\Sigma}) - \sqrt{n(1-2\varepsilon)}\operatorname{Tr}(\mathbf{\Sigma})\right)\left\|f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}\right\|^2_{\mathcal{H}} - \left\|f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}\right\|_{\mathcal{H}}\left(\sqrt{n(1-\varepsilon)}\operatorname{Tr}(\mathbf{\Sigma}) + \sqrt{n\varepsilon Q_k}\right) && (184)
\end{aligned}$$

(i) follows from an application of Jensen's Inequality as $(\cdot)^2$ is a convex function. (ii) follows from Lemma 25. Note, if $f_{\hat{\mathbf{W}}}$ is a stationary point, then we must have $\langle\nabla_f\tilde{g}\left(f_{\hat{\mathbf{W}}},\hat{t}\right), f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}\rangle_{\mathcal{H}} \leq 0$. To satisfy this inequality in expectation over the datasets, we must have

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}}\left\|f_{\hat{\mathbf{W}}} - f^*_{\mathbf{W}}\right\| \overset{(184)}{\leq} O\left(\frac{\sqrt{n(1-\varepsilon)}\operatorname{Tr}(\mathbf{\Sigma}) + \sqrt{n\varepsilon Q_k}}{n(1-2\varepsilon)\lambda_{\min}(\mathbf{\Sigma}) - \sqrt{n(1-2\varepsilon)}\operatorname{Tr}(\mathbf{\Sigma})}\right) \tag{185}$$

This completes the proof. ■