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Subquantile Minimization for Kernel Learning in the Huber ϵ-Contamination Model

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Abstract

In this paper we study Subquantile Minimization for learning the Huber-ϵ Contamination Problem for Kernel Learning. We assume the adversary has knowledge of the true distribution of $\mathcal{P}$, and is able to corrupt the covariates and the labels of ϵn samples for $\epsilon \in [0, 0.5)$. The distribution is formed as $\hat{\mathcal{P}} = (1-\epsilon)\mathcal{P} + \epsilon\mathcal{Q}$, and we want to find the function $f^* = \mathbb{E}_{\mathcal{D}\sim\mathcal{P}}[\ell(f;\mathcal{D})]$, from the noisy distribution, $\hat{\mathcal{P}}$. Superquantile objectives have been been studied extensively to reduce the risk of the tail Laguel et al. (2021); Rockafellar et al. (2014). We consider the contrasting case where we want to minimize the body of the risk. To our knowledge, we are the first to study the problem of general kernel learning in the Huber Contamination Model. We study a gradient-descent approach to solve a variational representation of the Subquantile Objective.

1. Introduction

There has been extensive study of algorithms to learn the target distribution from a Huber ϵ-Contaminated Model for a Generalized Linear Model (GLM), (Diakonikolas et al., 2019; Awasthi et al., 2022; Li et al., 2021; Osama et al., 2020; Fischler and Bolles, 1981) as well as for linear regression Bhatia et al. (2017); Mukhoty et al. (2019). Robust Statistics has been studied extensively Diakonikolas and Kane (2023) for problems such as high-dimensional mean estimation Prasad et al. (2019); Cheng et al. (2020) and Robust Covariance Estimation Cheng et al. (2019); Fan et al. (2018). Recently, there has been an interest in solving robust machine learning problems by gradient descent Prasad et al. (2018); Diakonikolas et al. (2019). Subquantile minimization aims to address the shortcomings of standard ERM in applications of noisy/corrupted data (Khetan et al., 2018; Jiang et al., 2018). In many real-world applications, the covariates have a non-linear dependence on labels (Abu-Mostafa et al., 2012, Section 3.4). In which case it is suitable to transform the covariates to a different space utilizing kernels (Hofmann et al., 2008). Therefore, in this paper we consider the problem of Robust Learning for Kernel Learning.

Definition 1 (Huber ϵ-Contamination Model Huber and Ronchetti (2009)) *Given a corruption parameter $0 < \epsilon < 0.5$, a data matrix, $\mathbf{X}$ and labels $\mathbf{y}$. An adversary is allowed to inspect all samples and modify ϵn samples arbitrarily. The algorithm is then given the ϵ-corrupted data matrix $\mathbf{X}$ and $\mathbf{y}$ as training data.*

Current approaches for robust learning across various machine learning tasks often use gradient descent over a robust objective, (Li et al., 2021). These robust objectives tend to not be convex and therefore do not have a strong analysis on the error bounds for general classes of models.

We similarly propose a robust objective which has a nonconvex-concave objective. This objective has also been proposed recently in Hu et al. (2020) where there has been an analysis in the Binary Classification Task. We show Subquantile Minimization reduces to the same objective in Hu et al. (2020). We use theory from the weakly-convex concave optimization literature for our error bounds. We are able to levarage this theory by analyzing the asymptotic distribution of a softplus approximation of the Subquantile objective.

The study of Kernel Learning in the Gaussian Design is quite popular, (Cui et al., 2021; Dicker, 2016). In (Cui et al., 2021), the feature space, $\phi(\mathbf{x}_i) \sim \mathcal{N}(0, \mathbf{\Sigma})$ where $\mathbf{\Sigma}$ is a diagonal matrix of dimension p, where p can be infinite. In this work, we adopt a similar framework, and with the power of Mercer's Theorem (Mercer, 1909), we are able to say $\mathrm{Tr}(\mathbf{\Sigma}) < \infty$. We use this fact extensively in our infinite-dimensional concentration inequalities.

Theorem 2 *(Informal). Let the dataset be given as $\{(\mathbf{x}_i, y_i)\}_{i=1}^n$ such that the labels and covariates of ϵn samples are arbitrarily corrupted by an adversary.*
Kernelized Regression:

$$\|\hat{f} - f^*\|_{\mathcal{H}} \leq \varepsilon + O\left(\sigma\right)$$

Kernel Binary Classification:

$$\|\hat{f} - f^*\|_{\mathcal{H}} \leq \varepsilon + \tilde{O}\left(\frac{\mathcal{E}_{\mathrm{OPT}}}{n(1-\epsilon)}\right) + \tilde{O}\left(\frac{1}{n^\beta(1-\epsilon)^\beta}\right)$$

Kernel Multi-Class Classification:

$$\|f - f^*\| \leq O\left(\Xi\right)$$

1.1. Related Work

The idea of iterative thresholding algorithms for robust learning tasks dates back to 1806 by Legendre (Legendre, 1806). From the popularity of Machine Learning, numerous algorithms have been developed in this idealogy. Therefore, we will dedicate this section to reviewing such works and to make clear our contributions to the iterative thresholding literature.

Robust Regression via Hard Thresholding Bhatia et al. (2015). Bhatia et al. study iterative thresholding for least squares regression / sparse recovery. Their theoretical results for the standard gradient descent case cover for known covariance with no feature covariance or Gaussian Noise.

Learning with bad training data via iterative trimmed loss minimization (Shen and Sanghavi, 2019). This work considers optimizing over the bottom-k errors by choosing the αn points with smallest error and then updating the model from these αn. This general model is the same as ours. Theoretically, this work considers only general linear models.

Trimmed Maximum Likelihood Estimation for Robust Generalized Linear Model (Awasthi et al., 2022). This work studies a different class of generalized linear models. Interestingly, they show for Gaussian Regression the iterative trimmed maximum likelihood estimator is able to achieve near minimax optimal error. This work does not consider feature corruption and primarily focuses on the covariates sampled with Gaussian Design from Identity covariance.

1.2. Contributions

We will now state our main contributions clearly.

1. We provide a novel theoretical framework using the Moreau Envelope for analyzing the iterative trimmed estimator for machine learning tasks.

2. We provide rigorous error bounds for subquantile minimization in the kernel regression, kernel binary classification, and kernel multi-class classification. Furthermore, we provide our bounds for both label and feature corruption with a general Gaussian Design.

2. Preliminaries

Notation. We denote $[T]$ as the set $\{1, 2, \ldots, T\}$. We define $(x)^+ \triangleq \max(0, x)$ as the Recitificied Linear Unit (ReLU) function. We say $y = O(x)$ if there exists x_0 s.t. for all $x \geq x_0$ there exists C s.t. $y \leq Cx$. We denote $\tilde{O}$ to ignore log factors. We say $y = \Omega(x)$ if there exists x_0 s.t. for all $x \geq x_0$ there exists C s.t. $y \geq Cx$.

2.1. Reproducing Kernel Hilbert Spaces

Let the function $\phi : \mathbb{R}^d \to \mathcal{H}$ represent the Hilbert Space Representation or 'feature transform' from a vector in the original covariate space to the RKHS. We define $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ as $k(\mathbf{x}, \mathbf{x}) \triangleq \langle \phi(\mathbf{x}), \phi(\mathbf{x}) \rangle_{\mathcal{H}}$. For a function in a RKHS, $f \in \mathcal{H}$, it follows for a function f parameterized by weights $\mathbf{w} \in \mathbb{R}^n$, that the point evaluation function is given as $f : \mathbb{R}^d \to \mathbb{R}$ and defined $f(\cdot) \triangleq \sum_{i \in [n]} w_i k(\mathbf{x}_i, \cdot)$.

2.2. Tensor Products

Let $\mathcal{H}, \mathcal{K}$ be Hilbert Spaces, then $\mathcal{H} \otimes \mathcal{K}$ is the tensor product space and is also a Hilbert Space (Ryan and a Ryan, 2002). For $\phi_1, \psi_1 \in \mathcal{H}$ and $\phi_2, \psi_2 \in \mathcal{K}$, the inner product is defined as $\langle \phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2 \rangle_{\mathcal{H} \otimes \mathcal{K}} = \langle \phi_1, \psi_1 \rangle_{\mathcal{H}} \langle \phi_2, \psi_2 \rangle_{\mathcal{K}}$. We will utilize tensor products when we discuss infinite dimensional covariance estimation.

2.3. Distribution

In this paper we sample $\mathbf{x} \sim \mathcal{X}$ such that $\phi(\mathbf{x}) \sim \mathcal{P}$ is sub-Gaussian in the Hilbet Space where $\mathbf{E}[\phi(\mathbf{x}_i)] = \mathbf{0}$ and $\mathbf{E}[\phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i)] = \mathbf{\Gamma}$ where $\mathrm{Tr}(\mathbf{\Gamma}) < \infty$. We have X is a centered Hilbert-Space sub-Gaussian random function if for all $\theta > 0$,

$$\mathop{\mathbf{E}}_{X \sim \mathcal{P}} \left[\exp\left(\theta \langle X, v \rangle_{\mathcal{H}} \right) \right] \leq \exp\left(\frac{\theta^2 \langle v, \mathbf{\Sigma} v \rangle_{\mathcal{H}}}{2} \right)$$

The Gaussian Design for the Feature Space has gained popularity in the study of kernel learning (Cui et al., 2021).

2.4. Mathematical Tools

Proposition 3 (Young's Inequality (Young, 1912)) *For all $a, b \in \mathbb{R}$, it holds*

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

Proposition 4 (Jensen's Inequality (Jensen, 1906)) *Suppose φ is a convex function, then for a random variable X, it holds*

$$\varphi\left(\mathbf{E}\left[X\right]\right) \leq \mathbf{E}\left[\varphi\left(X\right)\right]$$

The inequality is reversed for φ concave.

Proposition 5 (McDiarmid's Inequality (McDiarmid et al., 1989)) *Suppose $f : \mathcal{X}_1 \times \cdots \mathcal{X}_n \to \mathbb{R}$. Consider i.i.d $X_1, \ldots, X_n$ where $X_i \in \mathcal{X}_i$ for all $i \in [n]$. If there exists constants $c_1, \ldots, c_n$, such that for all $x_i \in \mathcal{X}_i$ for all $i \in [n]$, it holds*

$$\sup_{\tilde{X}_i \in \mathcal{X}_i} \left| f(X_1, \ldots, X_{i-1}, X_i, X_{i+1}, \ldots, X_n) - f(X_1, \ldots, X_{i-1}, \tilde{X}_i, X_{i+1}, \ldots, X_n) \right| \leq c_i$$

Then for any $t > 0$, it holds

$$\mathbf{Pr}\left\{ f(X_1, \ldots, X_n) - \mathbf{E}\left[f(X_1, \ldots, X_n) \right] \geq t \right\} \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^n c_i^2} \right)$$

Fact 6 (Sum of Binomial Coefficients (Cormen et al., 2022)) *Let $k, n \in \mathbb{N}$ such that $k \leq n$, then*

$$\sum_{i=0}^{k} \binom{n}{i} \leq \left(\frac{en}{k} \right)^k$$

Lemma 7 (Smooth Descent Lemma) *Suppose $\sup_{x \in \mathcal{X}} \left\| \nabla^2 f(x) \right\|_{\text{op}} \leq \beta$, then for a stepsize $\eta \leq 1/\beta$, it follows for all $t \in \mathbb{N}$,*

$$f(x^{(t+1)}) \leq f(x^{(t)}) - \frac{\eta}{2} \|\nabla f(x^{(t)})\|_{\mathcal{H}}^2$$

3. Subquantile Minimization

We propose to optimize over the subquantile of the risk. The p-quantile of a random variable, U, is given as $\mathcal{Q}_p(U)$, this is the largest number, t, such that the probability of $U \leq t$ is at least p.

$$\mathcal{Q}_p(U) \leq t \iff \mathbb{P}\left\{ U \leq t \right\} \geq p$$

The p-subquantile of the risk is then given by

$$\mathbb{L}_p\left(U\right) = \frac{1}{p} \int_0^p \mathcal{Q}_p\left(U\right) dq = \mathbb{E}\left[U | U \leq \mathcal{Q}_p\left(U\right) \right] = \max_{t \in \mathbb{R}} \left\{ t - \frac{1}{p} \mathbb{E}\left(t - U\right)^+ \right\}$$

Given an objective function, ℓ, the kernelized learning poblem becomes:

$$\min_{f \in \mathcal{K}} \max_{t \in \mathbb{R}} \left\{ g(t, f) \triangleq t - \sum_{i=1}^{n} \left(t - (f(\mathbf{x}_i) - y_i)^2 \right)^+ \right\}$$

where t is the p-quantile of the empirical risk. Note that for a fixed t therefore the objective is not concave with respect to $\mathbf{w}$. Thus, to solve this problem we use the iterations from Equation 11 in (Razaviyayn et al., 2020). Let $\text{Proj}_{\mathcal{K}}$ be the projection of a function on to the convex set $\mathcal{K} \triangleq \{ f \in \mathcal{H} : \|f\|_{\mathcal{H}} \leq R \}$, then our update steps are

$$t^{(k+1)} = \arg\max_{t \in \mathbb{R}} g(f^{(k)}, t)$$

$$f^{(k+1)} = \text{Proj}_{\mathcal{K}}\left(f^{(k)} - \alpha \nabla_f g(f^{(k)}, t^{(k+1)}) \right)$$

4. Theory

To consider theoretical guarantees of Subquantile Minimization, we first analyze the inner and outer optimization problems. We first analyze kernel learning in the presence of corrupted data. Next, we provide error bounds for the two most important kernel learning problems, kernel ridge regression, and kernel classification. Now we will give our first result regarding kernel learning in the Huber ϵ-contamination model. Now we will analyze the two-step minimax optimization steps described in Section 3.

Lemma 8 *Let $f(\mathbf{x};\mathbf{w})$ be a convex loss function. Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ denote the n data points ordered such that $f(\mathbf{x}_1;\mathbf{w},y_1) \le f(\mathbf{x}_2;\mathbf{w},y_2) \le \cdots \le f(\mathbf{x}_n;\mathbf{w},y_n)$. If we denote $\hat{\nu}_i \triangleq f(\mathbf{x}_i;\mathbf{w},y_i)$, it then follows $\hat{\nu}_{n(1-\epsilon)} \in \arg\max_{t\in\mathbb{R}} g(t,\mathbf{w})$.*

Proof. First we can note, the max value of t for g is equivalent to the min value of t for g. We can now find the Fermat Optimality Conditions for g.

$$\partial(-g(t, f_{\mathbf{w}})) = \partial\left(-t + \frac{1}{n(1-\epsilon)}\sum_{i=1}^{n}(t-\hat{\nu}_i)^+\right) = -1 + \frac{1}{n(1-\epsilon)}\sum_{i=1}^{n(1-\epsilon)}\begin{cases}1 & \text{if } t > \hat{\nu}_i\\ 0 & \text{if } t < \hat{\nu}_i\\ [0,1] & \text{if } t = \hat{\nu}_i\end{cases}$$

We observe when setting $t = \hat{\nu}_{n(1-\epsilon)}$, it follows that $0 \in \partial(-g(t, f_{\mathbf{w}}))$. This is equivalent to the $(1-\epsilon)$-quantile of the Risk. ■

From Theorem 8, we see that t will be greater than or equal to the errors of exactly $n(1-\epsilon)$ points. Thus, we are continuously updating over the $n(1-\epsilon)$ minimum errors.

Lemma 9 *Let $\hat{\nu}_i \triangleq f(\mathbf{x}_i;\mathbf{w},y_i)$ s.t. $\hat{\nu}_{i-1} \le \hat{\nu}_i \le \hat{\nu}_{i+1}$, if we choose $t^{(k+1)} = \hat{\nu}_{n(1-\epsilon)}$ as by Theorem 8, it then follows $\nabla_{\mathbf{w}} g(t^{(k)}, f^{(k)}) = \frac{1}{n(1-\epsilon)}\sum_{i=1}^{n(1-\epsilon)} \nabla f(\mathbf{x}_i; f^{(k)}, y_i)$*

Proof. By our choice of $t^{(k+1)}$, it follows:

$$\nabla_f g(t^{(k+1)}, f_{\mathbf{w}}^{(k)}) = \nabla_f\left(t^{(k+1)} - \frac{1}{n(1-\epsilon)}\sum_{i=1}^{n}\left(t^{(k+1)} - \ell(\mathbf{x}_i; f_{\mathbf{w}}^{(k)}, y_i)\right)^+\right)$$

$$= -\frac{1}{n(1-\epsilon)}\sum_{i=1}^{n(1-\epsilon)}\nabla_f\left(t^{(k+1)} - \ell(\mathbf{x}_i; f_{\mathbf{w}}^{(k)}, y_i)\right)^+ = \frac{1}{n(1-\epsilon)}\sum_{i=1}^{n}\nabla_f \ell(\mathbf{x}_i; f_{\mathbf{w}}^{(k)}, y_i)\begin{cases}1 & \text{if } t > \hat{\nu}_i\\ 0 & \text{if } t < \hat{\nu}_i\\ [0,1] & \text{if } t = \hat{\nu}_i\end{cases}$$

Now we note $\hat{\nu}_{n(1-\epsilon)} \le t^{(k+1)} \le \hat{\nu}_{n(1-\epsilon)+1}$. Then, we have

$$\nabla_f g(t^{(k+1)}, f_{\mathbf{w}}^{(k)}) = \frac{1}{n(1-\epsilon)}\sum_{i=1}^{n(1-\epsilon)}\nabla_f \ell(\mathbf{x}_i; f_{\mathbf{w}}^{(k)}, y_i)$$

This concludes the proof. ■

Input: Data Matrix: $\mathbf{X} \in \mathbb{R}^{n \times d}, n \gg d$; Labels: $\mathbf{y} \in \mathbb{R}^{n|}$, Closed and Convex set $\mathcal{K} \subset \mathcal{H}$
Output: Function in $\mathcal{H}$: $\hat{f}$

1. Set the step-size
$$\eta \leq O\left(\frac{\lambda_m\left(\mathbf{\Sigma}\right)}{\mathrm{Tr}\left(\mathbf{\Sigma}\right)}\right)$$
2. Set the number of iterations
$$T = \tilde{O}\left(\log\left(\left(\frac{\lambda_{\max}(\mathbf{\Sigma})\left\|f^*\right\|_{\mathcal{H}}}{\sqrt{n}}\right)\frac{1}{\varepsilon}\right)\right)$$
3. **for** $k = 1, 2, \ldots, T$ **do**
    3. Find the Subquantile denoted as $S^{(k)}$ as the set of $(1-\epsilon)n$ elements with the lowest error with respect to the loss function.
    4. Calculate the gradient update.
    $$\nabla_f g(t^{(k+1)}, f^{(k)}) \leftarrow \frac{2}{n(1-\epsilon)} \sum_{i \in S^{(k)}} \left(f^{(k)}(\mathbf{x}_i) - y_i\right) \cdot \phi(\mathbf{x}_i)$$
    5. Perform Projected Gradient Descent Iteration with Lemma 25.
    $$f^{(k+1)} \leftarrow \mathrm{Proj}_{\mathcal{K}}\left[f^{(k)} - \eta \nabla g(f^{(k)}, t^{(k+1)})\right]$$

Return: Function in $\mathcal{H}$: $f^{(T)}$

Algorithm 1: Subquantile Minimization for Kernelized Regression

4.1. Kernelized Regression

The loss for the Kernel Ridge Regression problem for a single training pair $(\mathbf{x}_i, y_i) \in \mathcal{D}$ is given by the following equation
$$\ell\left(f; \mathbf{x}_i, y_i,\right) = \left(f(\mathbf{x}_i) - y_i\right)^2$$
We will now give the algorithm. Our goals throughout the proofs will be to obtain approximation bounds for infinite-dimensional kernels. The key challenge is the obvious undetermined problem, i.e. considering an infinite eigenfunction basis, we require infinite samples to obtain an accurate approximation. Instead, we will calculate the approximation bounds for the rank-m approximation of f^* and push $m \to \infty$.

Theorem 10 (Subquantile Minimization for Kernelized Regression) *Algorithm 2 run on a dataset $\mathcal{D} \sim \hat{\mathcal{P}}$ and return $\hat{f}$. Then with probability exceeding $1-\delta$,*
$$\|\hat{f} - \mathrm{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \leq \varepsilon + \tilde{O}\left(\frac{\|\mathrm{Proj}_{\Psi_m^{\perp}} f^*\|_{\mathcal{H}}}{\lambda_{\max}(\mathbf{\Sigma})\sqrt{n}} + \sigma\sqrt{\frac{\mathrm{Tr}(\mathbf{\Sigma})\log n}{n\lambda_{\max}^2(\mathbf{\Sigma})}} + \lambda_m(\mathbf{\Sigma})R^2\right)$$

Full proof with explicit constants is given in Appendix C.2. A direct application of Theorem 10 is that learning an infinite dimensional function f^* to within ε error in the Hilbert Space Norm requires infinite data. Furthermore, we see that given covariate noise and label noise, our bound requires more iterations dependent on the magnitude of the corruption. Such a result is corroborated in Schmidt et al. (2018). For the linear and polynomial kernel, we then have β increases, therefore to obtain the same bound on η as with no feature noise, we simply need more data. The effect of **??** can be seen in the denominator of both terms. Instead of $\lambda_{\min}(\mathbf{\Sigma})$ we have $c_4\lambda_m$ for a finite m. This difference will be clear in the following corollary, where we utilize the theory develoepd for kernelized regression to imply a result for regularized linear regression.

Corollary 11 (Linear Regression Expected Error Bound) *Consider Subquantile Minimization for Linear Regression on the data X with optimal parameters $\mathbf{w}^*$. Assume $\mathbf{x}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ for $i \in [n]$. Then after T iterations of Algorithm 1, we have the following error bounds for robust kernelized linear regression. Given sufficient data*

$$\left\|f^{(T)} - f^*\right\|_{\mathcal{H}} \leq \varepsilon + O\left(\sigma\right)$$

Proof given in **??**. Let us note for the case where p is finite, i.e. the feature mapping is finite-dimensional, e.g. linear or polynomial kernel. Then we have that $\mathrm{Proj}_{\Psi_m^\perp}$ where $m = p$ is equal to zero as $\{\varphi_i\}_{i=1}^m$ spans the finite-dimensional space, in which we case we have the absolute constant given in **??** is equal to zero. It is important to note in all our bounds, $\gamma \leq \sqrt{\frac{\epsilon}{1-2\epsilon}}$ is a theoretical worst case bound when the Subquantile contains the minimum possible number of uncorrupted points. In other words, we have $\gamma \triangleq \frac{|P\setminus S|}{|S\cap P|} \leq \frac{n\epsilon}{n(1-2\epsilon)} = \frac{\epsilon}{1-2\epsilon}$. So, as $|S \cap P|$ increases, we have a better error bound as $|P \setminus S|$ decreases. As is typical in the robust statistics literature, we make no assumptions on the distribution of the corrupted data so we cannot say anything about $|S \cap P|$. We will have γ decreases if stationary points give high error for corrupt points as our optimization procedure moves toward a stationary point.

4.2. Kernelized Binary Classification

The Negative Log Likelihood for the the Kernel Classification problem is given by the following equation for a single training pair $(\mathbf{x}_i, y_i)$

$$\ell\left(\mathbf{x}_i, y_i; f\right) = -y_i \log\left(\sigma\left(f(\mathbf{x}_i)\right)\right) - (1 - y_i)\log\left(1 - \sigma\left(f(\mathbf{x}_i)\right)\right)$$

We will now give our algorithm.

Theorem 12 (Subquantile Minimization for Binary Classification is Good with High Probability) *Let Algorithm 2 be run on a dataset $\mathcal{D} \sim \hat{\mathcal{P}}$ with learning rate $\eta \triangleq \Omega(L^{-1})$. Then after $O\left(\log\left(\frac{\|f^*\|_{\mathcal{H}}}{\varepsilon}\right)\right)$ gradient descent iterations, with probability exceeding $1-\delta$,*

$$\left\|f^{(T)} - f^*\right\|_{\mathcal{H}} \leq \varepsilon + \left\|\mathrm{Proj}_{\Psi_m^\perp} f^*\right\|_{\mathcal{H}}$$

where $C = \exp\left(-RP_k\right)$.

Proof. The proof is deferred to Appendix D.1. ■

In Theorem 12, we introduce $\mathcal{E}_{\mathrm{OPT}}$, which says we are only able to learn up to the intrinsic noise within the target function.

Input: Data Matrix: $\mathbf{X} \in \mathbb{R}^{n \times d}, n \gg d$; Labels: $\mathbf{y} \in \mathbb{R}^n$, Closed and Convex set $\mathcal{K} \subset \mathcal{H}$
Output: Function in $\mathcal{H}$: $\hat{f}$

1. Set the step-size
$$\eta \leq O\left(\frac{\lambda_{\min}(\mathbf{\Sigma})}{\mathrm{Tr}\left(\mathbf{\Sigma}\right)}\right)$$
2. Set the number of iterations
$$T = O\left(\log\left(\left(\frac{\lambda_{\max}(\mathbf{\Sigma})\left\|f^*\right\|_{\mathcal{H}}}{\sqrt{n}}\right)\frac{1}{\varepsilon}\right)\right)$$
3. **for** $k = 1, 2, \ldots, T$ **do**
    3. Find the Subquantile denoted as $S^{(k)}$ as the set of $(1-\epsilon)n$ elements with the lowest error with respect to the loss function.
    4. Calculate the gradient update.
    $$\nabla_f g(t^{(k+1)}, f^{(k)}) \leftarrow \frac{2}{n(1-\epsilon)} \sum_{i \in S^{(k)}} \left(\sigma(f^{(k)}(\mathbf{x}_i)) - y_i\right) \cdot \phi(\mathbf{x}_i)$$
    5. Perform Projected Gradient Descent Iteration with Lemma 25.
    $$f^{(k+1)} \leftarrow \mathrm{Proj}_{\mathcal{K}}\left[f^{(k)} - \eta \nabla g(f^{(k)}, t^{(k+1)})\right]$$

Return: Function in $\mathcal{H}$: $f^{(T)}$

Algorithm 2: Subquantile Minimization for Binary Classification

5. Discussion

The main contribution of this paper is the study of a nonconvex-concave formulation of Subquantile minimization for the robust learning problem for kernel ridge regression and kernel classification. We present an algorithm to solve the nonconvex-concave formulation and prove rigorous error bounds which show that the more good data that is given decreases the error bounds. We also present accelerated gradient methods for the two-step algorithm to solve the nonconvex-concave optimization problem and give novel theoretical bounds.

Theory. We develop strong theoretical bounds on the normed difference between the function returned by Subquantile Minimization and the optimal function for data in the target distribution, $\mathbb{P}$, in the Gaussian Design. In expectation and with high probability, given sufficient data dependent on the kernel, we obtain a near minimax optimal error bound for a general positive definite continuous kernel. Our theoretical analysis is novel in that it utilizes the Moreau Envelope from a min-max formulation of the iterative thresholding algorithm.

Experiments. From our experiments, we see Subquantile Minimization is competitive with algorithms developed solely for robust linear regression as well as other meta-algorithms. Our theoretical analysis is through the lens of kernel-learning, but the generalization to linear regression from a non-kernel perspective can be done. In kernelized regression, we see SUBQUANTILE is the strongest of the meta-algorithms. Furthemore, in binary and multi-class classification, SUBQUANTILE is very strong. Thus, we can see empirically SUBQUANTILE is the strongest meta-algorithm across all kernelized regression and classification tasks and also the strongest algorithm in linear regression.

Interpretability. One of the strengths in Subquantile Optimization is the high interpretability. Once training is finished, we can see the $n(1-p)$ points with highest error to find the outliers and the features follow Gaussian Design. Furthermore, there is only hyperparameter p, which should be chosen to be approximately the percentage of inliers in the data and thus is not very difficult to tune for practical purposes. Our theory suggests for a problem where the amount of corruptions is unknown,

General Assumptions. The general assumption is the majority of the data should inliers. This is not a very strong assumption, as by the definition of outlier it should be in the minority. Furthermore, we assume the feature maps have a Gaussian Design. Such a design in many prior works in kernel learning and we therefore find it suitable.

Future Work. The analysis of Subquantile Minimization can be extended to neural networks as kernel learning can be seen as a one-layer network. This generalization will be appear in subsequent work. Another interesting direction work in optimization is for accelerated methods for optimizing non-convex concave min-max problems with a maximization oracle. The current theory analyzes standard gradient descent for the minimization. Ideas such as Momentum and Nesterov Acceleration in conjunction with the maximum oracle are interesting and can be analyzed in future work.

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Appendix A. Probability Theory

In this section we will give various concentration inequalities on the inlier data for functions in the Reproducing Kernel Hilbert Space. We will first give our assumptions for robust kernelized regression.

Assumption 13 (Gaussian Design) *We assume for $\mathbf{x}_i \sim \mathcal{P} \in \mathcal{X}$, then it follows for the feature map, $\phi(\cdot) : \mathcal{X} \to \mathcal{H}$,*

$$\phi(\mathbf{x}_i) \sim \mathcal{N}(\mathbf{0}, \mathbf{\Gamma})$$

where $\mathbf{\Gamma}$ is a possibly infinite dimensional covariance operator.

Assumption 14 (Bounded Functions) *We assume for $\mathbf{x}_i \sim \mathcal{P} \in \mathcal{X}$, then it follows for the feature map, $\phi(\cdot) : \mathcal{X} \to \mathcal{H}$,*

$$\sup_{\mathbf{x} \in \mathcal{X}} \|\phi(\mathbf{x})\|_{\mathcal{H}}^2 \le P_k < \infty$$

where $\mathcal{H}$ is a Reproducing Kernel Hilbert Space.

Assumption 15 (Normal Residuals) *The residual is defined as $\mu_i \triangleq f^*(\mathbf{x}_i) - y_i$. Then we assume for some $\sigma > 0$, it follows*

$$\mu_i \sim \mathcal{N}(0, \sigma^2)$$

A.1. Finite Dimensional Concentrations of Measure

Proposition 16 *Let $\mu_1, \ldots, \mu_n \sim \mathcal{N}(0, \sigma^2)$ for some $\sigma > 0$, then it follows for any $s \ge 1$*

$$\mathbf{Pr}\left\{\max_{i \in [n]} |\mu_i| \ge \sigma\sqrt{2\log n} \cdot s\right\} \le \frac{\sqrt{2}}{\log n} e^{-s^2}$$

Proof. Let C be a positive constant to be determined.

$$\begin{aligned}\mathop{\mathbf{Pr}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)}\left\{\max_{i \in [n]} |\mu_i| \ge C \cdot s\right\} &\overset{(i)}{=} 2n \mathop{\mathbf{Pr}}_{\mu \sim \mathcal{N}(0,\sigma^2)}\{\mu \ge C \cdot s\} = \frac{2n}{\sigma\sqrt{2\pi}} \int_{C \cdot s}^{\infty} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx \\ &\le 2\sigma n \left(\frac{1}{C \cdot s}\right) e^{-\frac{1}{2}\left(\frac{C \cdot s}{\sigma}\right)^2} \le \frac{\sqrt{2} n^{1-s^2}}{s \log n} \le \frac{\sqrt{2}}{\log n} e^{-s^2}\end{aligned}$$

(i) follows from a union bound and noting for a i.i.d sequence of random variables $\{X_i\}_{i \in [n]}$ and a constant C, it folllows $\mathbf{Pr}\{\max_{i \in [n]} X_i \ge C\} = n\,\mathbf{Pr}\{X \ge C\}$. In the second to last inequality, we plug in $C \triangleq \sigma\sqrt{2\log n}$. Our proof is now complete. ∎

Proposition 17 *Let $\mu_1, \ldots, \mu_n \sim \mathcal{N}(0, \sigma^2)$ for some $\sigma > 0$, then it follows for any $s \ge 1$,*

$$\mathbf{Pr}\left\{\sum_{i=1}^{n} \mu_i^2 \ge 8n\sigma^2 \cdot s\right\} \le 4e^{-s}$$

Proof. Concatenate all the samples μ_i into a vector $\boldsymbol{\mu} \in \mathbb{R}^n$. Our proof generalizes for a $\boldsymbol{\mu} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ where $\mathbf{\Sigma} \triangleq \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top$ for a unitary $\mathbf{U}$ and positive diagonal $\mathbf{\Lambda}$. Let C be a positive to be determined constant, we then have

$$\mathop{\mathbf{Pr}}_{\boldsymbol{\mu} \sim \mathcal{N}(\mathbf{0},\sigma^2\mathbf{I})}\left\{\|\boldsymbol{\mu}\|^2 \ge C \cdot s\right\} = \mathop{\mathbf{Pr}}_{\boldsymbol{\mu} \sim \mathcal{N}(\mathbf{0},\sigma^2\mathbf{I})}\left\{\|\boldsymbol{\mu}\| \ge \sqrt{C \cdot s}\right\} \le 4\exp\left(-\frac{C \cdot s}{8\,\mathrm{Tr}\,(\mathbf{\Sigma})}\right)$$

where the last inequality follows from Proposition 18. Now choosing $C \triangleq 8\,\mathrm{Tr}(\mathbf{\Sigma})$ completes the proof. ∎

A.2. Hilbert Space Concentrations of Measure

Proposition 18 (Gaussian Concentration (Ledoux and Talagrand, 2013)) *Suppose X is a Gaussian Variable in a Banach Space. Then,*

$$\mathbf{Pr}\left\{\|X\| > t\right\} \leq 4\exp\left(-\frac{t^2}{8\,\mathbf{E}\,\|X\|^2}\right)$$

Proposition 19 (Gaussian Concentration from Mean (Pinelis and Sakhanenko, 1986)) *Suppose X is a Gaussian Variable in a Banach Space. Then,*

$$\mathbf{Pr}\left\{\|X\| - \mathbf{E}\,\|X\| \geq t\right\} \leq \exp\left(-\frac{t^2}{2\,\mathbf{E}\,\|X\|^2}\right)$$

Noting that all Hilbert Spaces are Banach Spaces (Young, 1988), we will use this proposition throughout the section.

Theorem 20 (Hilbert Space Hanson Wright (Chen and Yang, 2021)) *Let X_i be a i.i.d sequence of sub-Gaussian random variables in $\mathcal{H}$ such that $\mathbf{E}\left[X_i\right] = 0$ and $\mathbf{E}\left[X_i \otimes X_i\right] = \mathbf{\Gamma}$. Then there exists a universal constant $C > 0$ s.t. for any $t > 0$,*

$$\mathbf{Pr}\left\{\sum_{i=1}^{n}\langle X_i, X_i\rangle_{\mathcal{H}} \geq n\operatorname{Tr}\left(\mathbf{\Gamma}\right) + t\right\} \leq 2\exp\left[-C\min\left(\frac{t^2}{n\left\|\mathbf{\Gamma}\right\|_{\mathrm{HS}}^2}, \frac{t}{\left\|\mathbf{\Gamma}\right\|_{\mathrm{op}}}\right)\right]$$

From Theorem 20, it follows that the LHS is less than $\delta \in (0,1)$ when

$$t \geq \frac{1}{C}\left\|\mathbf{\Gamma}\right\|_{\mathrm{op}}\log\frac{2}{\delta} \vee \sqrt{\frac{1}{C}n\left\|\mathbf{\Gamma}\right\|_{\mathrm{HS}}^2\log\frac{2}{\delta}}$$

Furthermore, we have when

$$\delta \leq 2\exp\left[-nC\left(\frac{\left\|\mathbf{\Gamma}\right\|_{\mathrm{HS}}}{\left\|\mathbf{\Gamma}\right\|_{\mathrm{op}}}\right)^2\right]$$

it follows

$$t \geq \frac{1}{C}\left\|\mathbf{\Gamma}\right\|_{\mathrm{op}}\log\frac{2}{\delta}$$

In other words, when the failure probability is sufficiently small we can use the above bound. We will reference this idea throughout this section.

Theorem 21 (Mean Estimation in the Hilbert Space (Tolstikhin et al., 2017)) *Define $P_n \triangleq \frac{1}{n}\sum_{i=1}^{n}\delta_{X_i}$ and P be the distribution of the covariates in $\mathcal{X}$. Suppose $r : \mathcal{X} \to \mathcal{H}$ is a continuous function such that $\sup_{X\in\mathcal{X}}\|r(X)\|_{\mathcal{H}}^2 \leq C_k < \infty$. Then with probability at least $1-\delta$,*

$$\left\|\int_{\mathcal{X}} r(x)dP_n(x) - \int_{\mathcal{X}} r(x)dP(x)\right\| \leq \sqrt{\frac{C_k}{n}} + \sqrt{\frac{2P_k\log(1/\delta)}{n}}$$

Proposition 22 (Probabilistic Maximum P_k) *Let $\mathbf{x}_i \sim \mathcal{P}$ such that $\phi(\mathbf{x}_i) \sim \mathcal{N}(\mathbf{0}, \mathbf{\Gamma})$ (Assumption 13). Then it follows for any $s \geq 1$*

$$\mathop{\mathbf{Pr}}_{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Gamma})}\left\{\max_{i\in[n]}\|\phi(\mathbf{x}_i)\|_{\mathcal{H}} \geq \sqrt{8\operatorname{Tr}(\mathbf{\Gamma})\log n}\cdot s\right\} \leq 4e^{1-s^2}$$

Proof. Let C be a positive to be determined constant.

$$\Pr_{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Gamma})}\left\{\max_{i\in[n]}\|\phi(\mathbf{x}_i)\|_{\mathcal{H}} \geq C\cdot s\right\} \overset{(i)}{\leq} n \Pr_{\phi(\mathbf{x})\sim\mathcal{N}(\mathbf{0},\mathbf{\Gamma})}\left\{\|\phi(\mathbf{x})\|_{\mathcal{H}} \geq C\cdot s\right\} \overset{(ii)}{\leq} 4n\exp\left(-\frac{C^2\cdot s^2}{8\operatorname{Tr}(\mathbf{\Gamma})}\right)$$

See (i) from the proof of Proposition 16. In (ii) we apply Proposition 18. Setting $C \triangleq \sqrt{8\operatorname{Tr}(\mathbf{\Gamma})\log n}$ completes the proof. ■

Proposition 23 (RKHS Norm of Functions in the Reproducing Kernel Hilbert Space) *Let $\mathbf{x}_i \sim \mathcal{P}$ such that $\phi(\mathbf{x}_i) \sim \mathcal{N}(\mathbf{0},\mathbf{\Gamma})$ (Assumption 13). Denote $\mathcal{S}$ as all subsets of $[n(1-\epsilon)]$ with size $n(1-2\epsilon)$ for $\epsilon < 0.5$ and $P_B = \sum_{i=1}^{n}$. Then it follows with probability exceeding $1-\delta$,*

$$\max_{B\in S}\left\|\int_{\mathcal{X}} r(X)dP_B(x)\right\|_{\mathcal{H}}^2 \leq n(1-2\epsilon)\left(\|\mathbf{\Gamma}\|_{\text{Tr}} + \frac{e}{C}\|\mathbf{\Gamma}\|_{\text{op}}\log\frac{1}{2\delta}\right)$$

Proof. We will use a standard symmetrization argument to obtain the expectation.

$$\begin{aligned}
\mathop{\mathbf{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Gamma})}\left\|\sum_{i=1}^{n(1-2\epsilon)}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}^2 &= \mathop{\mathbf{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Gamma})}\mathop{\mathbf{E}}_{\xi_i\sim\mathcal{R}}\left\|\sum_{i=1}^{n(1-2\epsilon)}\xi_i\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}^2 \\
&= \mathop{\mathbf{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{N}(\mathbf{0},\mathbf{\Gamma})}\mathop{\mathbf{E}}_{\xi_i\sim\mathcal{R}}\sum_{i=1}^{n(1-2\epsilon)}\sum_{j=1}^{n(1-2\epsilon)}\xi_i\xi_j k(\mathbf{x}_i,\mathbf{x}_j) \overset{(i)}{=} n(1-2\epsilon)\operatorname{Tr}(\mathbf{\Gamma})
\end{aligned}$$

In (i) we note $\mathbf{E}\|\mathbf{\Phi}\|_{\text{HS}}^2 = \mathbf{E}\operatorname{Tr}(\mathbf{\Phi}\otimes\mathbf{\Phi}) = n(1-2\epsilon)\operatorname{Tr}(\mathbf{\Gamma})$. From Proposition 19, we then obtain for sufficiently large δ, it falls with probability at least $1-\delta$,

$$\left\|\sum_{i=1}^{n(1-2\epsilon)}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}} \geq n(1-2\epsilon)\operatorname{Tr}(\mathbf{\Gamma}) + \frac{1}{C}\|\mathbf{\Gamma}\|_{\text{op}}\log\frac{2}{\delta}$$

We next apply a union bound over $\mathcal{S}$, noting the relation $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$, we have

$$\begin{aligned}
\max_{B\in\mathcal{S}}\left\|\sum_{i\in B}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}^2 &\geq n(1-2\epsilon)\operatorname{Tr}(\mathbf{\Gamma}) + \frac{1}{C}\|\mathbf{\Gamma}\|_{\text{op}}\, n(1-2\epsilon)\log\frac{e(1-2\epsilon)}{(1-\epsilon)} + \|\mathbf{\Gamma}\|_{\text{op}}\frac{1}{C}\log\frac{1}{2\delta} \\
&= n(1-2\epsilon)\left(\operatorname{Tr}(\mathbf{\Gamma}) + \frac{e}{C}\|\mathbf{\Gamma}\|_{\text{op}}\right) + \frac{1}{C}\|\mathbf{\Gamma}\|_{\text{op}}\log\frac{1}{2\delta}
\end{aligned}$$

This completes our proof. ■

Proposition 24 (Probabilistic Bound on Infinite Dimensional Covariance Estimation) *Let $\mathbf{x}_1,\ldots,\mathbf{x}_n$ be i.i.d sampled from $\mathcal{P}$ such that $\phi(\mathbf{x}_i) \sim \mathcal{P}$ (Assumption 13). Denote $\mathcal{S}$ as all subsets of $[n]$ with size from $n(1-2\epsilon)$ to $n(1-\epsilon)$. We then have with probability exceeding $1-\delta$,*

$$\max_{A\in\mathcal{S}}\left\|\frac{1}{n(1-\epsilon)}\sum_{i\in A}\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i) - \mathbf{\Gamma}\right\|_{\text{HS}} \leq \sqrt{\frac{8}{n(1-\epsilon)}}\|\mathbf{\Gamma}\|_{\text{op}} + \sqrt{\frac{2P_k^2\log(2/\delta)}{n(1-\epsilon)}} + P_k\sqrt{2\epsilon\log\frac{e}{\epsilon}}$$

Proof. We will calculate the mean operator in the Hilbert Space $\mathcal{H} \otimes \mathcal{H}$ and use the $\sqrt{n}$-consistency of estimating the mean-element in a Hilbert Space to obtain the probability bounds.

$$
\begin{aligned}
&\mathop{\mathbf{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{P}} \left\| \frac{1}{n(1-\epsilon)} \sum_{i=1}^{n(1-\epsilon)} \phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i) - \mathbf{\Gamma} \right\|_{\mathrm{HS}} \\
&\overset{(ii)}{\leq} \mathop{\mathbf{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{P}} \mathop{\mathbf{E}}_{\phi(\tilde{\mathbf{x}}_i)\sim\mathcal{P}} \left\| \frac{1}{n(1-\epsilon)} \sum_{i=1}^{n(1-\epsilon)} \phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i) - \phi(\tilde{\mathbf{x}}_i)\otimes\phi(\tilde{\mathbf{x}}_i) \right\|_{\mathrm{HS}} \\
&\overset{(iii)}{=} \mathop{\mathbf{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{P}} \mathop{\mathbf{E}}_{\phi(\tilde{\mathbf{x}}_i)\sim\mathcal{P}} \mathop{\mathbf{E}}_{\xi_i\sim\mathcal{R}} \left\| \frac{1}{n(1-\epsilon)} \sum_{i=1}^{n(1-\epsilon)} \xi_i \left(\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i) - \phi(\tilde{\mathbf{x}}_i)\otimes\phi(\tilde{\mathbf{x}}_i)\right) \right\|_{\mathrm{HS}} \\
&\leq \mathop{\mathbf{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{P}} \mathop{\mathbf{E}}_{\xi_i\sim\mathcal{R}} \left\| \frac{2}{n(1-\epsilon)} \sum_{i=1}^{n(1-\epsilon)} \xi_i \phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i) \right\|_{\mathrm{HS}} \\
&\leq \frac{2}{n(1-\epsilon)} \mathop{\mathbf{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{P}} \left(\mathop{\mathbf{E}}_{\xi_i\sim\mathcal{R}} \left\| \sum_{i=1}^{n(1-\epsilon)} \xi_i \phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i) \right\|_{\mathrm{HS}}^2 \right)^{1/2}
\end{aligned}
$$

In (ii) we apply a union bound. In (ii) we note that $\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i) - \mathbf{\Gamma}$ is a mean $\mathbf{0}$ operator in the tensor product space $\mathcal{H}\otimes\mathcal{H}$. Then for $X, Y \in \mathcal{H}\otimes\mathcal{H}$ s.t. $\mathbf{E}[Y] = \mathbf{0}$ it follows $\|X\|_{\mathrm{HS}} = \|X - \mathbf{E}[Y]\|_{\mathrm{HS}} = \|\mathbf{E}[X-Y]\|_{\mathrm{HS}}$ and finally we apply Jensen's Inequality. Let e_k for $k \in [p]$ (p possibly infinite) represent a complete orthonormal basis for the image of $\mathbf{\Gamma}$. By expanding out the Hilbert-Schmidt Norm, we then have

$$
\begin{aligned}
&\frac{2}{n(1-\epsilon)} \left(\mathop{\mathbf{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{P}} \mathop{\mathbf{E}}_{\xi_i\sim\mathcal{R}} \left\| \sum_{i=1}^{n(1-\epsilon)} \xi_i \phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i) \right\|_{\mathrm{HS}}^2 \right)^{1/2} \\
&= \frac{2}{n(1-\epsilon)} \left(\mathop{\mathbf{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{P}} \mathop{\mathbf{E}}_{\xi_i\sim\mathcal{R}} \sum_{k=1}^{p} \left\langle \sum_{i=1}^{n(1-\epsilon)} \xi_i \phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i) e_k,\ \sum_{j=1}^{n(1-\epsilon)} \xi_j \phi(\mathbf{x}_j)\otimes\phi(\mathbf{x}_j) e_k \right\rangle_{\mathrm{HS}} \right)^{1/2} \\
&= \frac{2}{n(1-\epsilon)} \left(\mathop{\mathbf{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{P}} \mathop{\mathbf{E}}_{\xi_i\sim\mathcal{R}} \sum_{k=1}^{p} \sum_{i=1}^{n(1-\epsilon)} \sum_{j=1}^{n(1-\epsilon)} \xi_i\xi_j \left\langle \phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)e_k, \phi(\mathbf{x}_j)\otimes\phi(\mathbf{x}_j)e_k \right\rangle_{\mathrm{HS}} \right)^{1/2} \\
&\overset{(iv)}{\leq} \frac{2}{n(1-\epsilon)} \left(\mathop{\mathbf{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{P}} \sum_{k=1}^{p} \sum_{i=1}^{n(1-\epsilon)} \left\langle \phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)e_k, \phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)e_k \right\rangle_{\mathrm{HS}} \right)^{1/2} \\
&= \frac{2}{n(1-\epsilon)} \left(\sum_{i=1}^{n(1-\epsilon)} \mathop{\mathbf{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{P}} \|\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)\|_{\mathrm{HS}}^2 \right)^{1/2} \\
&\overset{(v)}{=} \frac{2}{\sqrt{n(1-\epsilon)}} \left(\mathop{\mathbf{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{P}} \|\phi(\mathbf{x}_i)\|_{\mathcal{H}}^4 \right)^{1/2}
\end{aligned}
$$

(iv) follows from noticing $\mathbf{E}_{\xi_i,\xi_j\sim\mathcal{R}}[\xi_i\xi_j] = \delta_{ij}$. (v) follows from expanding out the Hilbert-Schmidt Norm and applying Parseval's Identity. We have

$$
\begin{aligned}
&\mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{X}} \left[\|\phi(\mathbf{x})\|_{\mathcal{H}}^4\right] = \int_0^\infty \mathbf{Pr}\left\{\|\phi(\mathbf{x})\|_{\mathcal{H}}^4 \geq t\right\} dt = \int_0^\infty \mathbf{Pr}\left\{\|\phi(\mathbf{x})\|_{\mathcal{H}} \geq t^{1/4}\right\} dt \\
&\overset{(vi)}{\leq} \int_0^\infty \inf_{\theta>0} \mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{X}} \left[\exp\left(\theta\|\phi(\mathbf{x})\|_{\mathcal{H}}\right)\right] \exp\left(-\theta t^{1/4}\right) dt \leq \int_0^\infty \inf_{\theta>0} \exp\left(\frac{\theta^2\|\mathbf{\Gamma}\|_{\mathrm{op}}}{2} - \theta t^{1/4}\right) dt
\end{aligned}
$$

$$= \int_0^\infty \exp\left(-\frac{\sqrt{t}}{\|\mathbf{\Gamma}\|_{\text{op}}}\right) dt = 2\|\mathbf{\Gamma}\|_{\text{op}}^2$$

In (vi) we apply Markov's Inequality. From which we obtain,

$$\mathop{\mathbf{E}}_{\phi(\mathbf{x}_i)\sim\mathcal{P}}\left\|\frac{1}{n(1-\epsilon)}\sum_{i=1}^{n(1-\epsilon)}\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)-\mathbf{\Gamma}\right\|_{\text{HS}} \leq \sqrt{\frac{8}{n(1-\epsilon)}}\|\mathbf{\Gamma}\|_{\text{op}}$$

Then, define the function $r(\mathbf{x}) : \mathcal{X} \to \mathcal{H}\otimes\mathcal{H}$, $\mathbf{x}\to\phi(\mathbf{x})\otimes\phi(\mathbf{x})$. From Assumption 14, we have $r(\mathbf{x}) = \|\phi(\mathbf{x})\otimes\phi(\mathbf{x})\|_{\text{HS}} \leq \|\phi(\mathbf{x})\|_{\mathcal{H}}^2 \leq P_k$. We will use McDiamard's Inequality, consider $\tilde{P} \triangleq \delta_{X_i}$ with one modified element. Then,

$$\left\|\int_{\mathcal{X}} r(x)dP_B(x)dx - \int_{\mathcal{X}} r(x)dP(x)dx\right\|_{\text{HS}} - \left\|\int_{\mathcal{X}} r(x)dP_{\tilde{B}}(x)dx - \int_{\mathcal{X}} r(x)dP(x)dx\right\|_{\text{HS}}$$
$$\leq \frac{1}{n(1-\epsilon)}\left(\|r(x_i)\|_{\text{HS}} + \|r(\tilde{x}_i)\|_{\text{HS}}\right) \leq \frac{2P_k}{n(1-\epsilon)}$$

Then, we have from McDiarmid's inequality (Proposition 5),

$$\mathbf{Pr}\left\{\left\|\int_{\mathcal{X}} r(x)dP_B(x) - \int_{\mathcal{X}} r(x)dP(x)\right\|_{\text{HS}} - \sqrt{\frac{8}{n(1-\epsilon)}}\|\mathbf{\Gamma}\|_{\text{op}} \geq t\right\} \leq \exp\left(-\frac{2t^2 n(1-\epsilon)}{P_k^2}\right)$$

We then have with probability exceeding $1-\delta$,

$$\left\|\int_{\mathcal{X}} r(x)dP_B(x) - \int_{\mathcal{X}} r(x)dP(x)\right\|_{\text{HS}} \leq \sqrt{\frac{8}{n(1-\epsilon)}}\|\mathbf{\Gamma}\|_{\text{op}} + \sqrt{\frac{2P_k^2\log(2/\delta)}{n(1-\epsilon)}}$$

Next, applying a union bound over $\mathcal{S}$ with Fact 6, we have

$$\max_{B\in S}\left\|\int_{\mathcal{X}} r(x)dP_B(x) - \int_{\mathcal{X}} r(x)dP(x)\right\|_{\text{HS}} \leq \sqrt{\frac{8}{n(1-\epsilon)}}\|\mathbf{\Gamma}\|_{\text{op}} + \sqrt{\frac{2P_k^2\log(2/\delta)}{n(1-\epsilon)} + 2P_k^2\epsilon\log\frac{e}{\epsilon}}$$

Simplifying the resultant bound completes the proof. ■

Appendix B. Proofs for Structural Results

In this section we give the deferred proofs of our main structural results of the subquantile objective function.

B.1. Projection onto a Norm Ball

In this section we show normalizing on to a norm-ball in the RKHS can be implemented efficiently.

Lemma 25 *Let $\mathcal{K} \triangleq \{f : \|f\|_{\mathcal{H}} \leq R\}$. Then, for a $\hat{f} \notin \mathcal{K}$, it follows*

$$\text{Proj}_{\mathcal{K}}\hat{f} = \left(\frac{R}{\|\hat{f}\|}\right)\hat{f}$$

Proof. We will formulate the dual problem and then find the corresponding $f_{\mathbf{w}}$ that solves the dual.

$$
\begin{aligned}
\operatorname{Proj}_{\mathcal{K}} \hat{f} &= \underset{f \in \mathcal{K}}{\arg\min} \|f - \hat{f}\|_{\mathcal{H}}^2 = \underset{f \in \mathcal{K}}{\arg\min}\, \|f\|_{\mathcal{H}}^2 + \|\hat{f}\|_{\mathcal{H}}^2 - 2\langle f, \hat{f} \rangle_{\mathcal{H}} \\
&= \underset{f \in \mathcal{K}}{\arg\min}\, \|f\|_{\mathcal{H}}^2 - 2\langle f, \hat{f} \rangle_{\mathcal{H}}
\end{aligned}
$$

From here we can solve the dual problem. The Lagrangian is given by,

$$
\mathcal{L}(f, u) \triangleq \|f\|_{\mathcal{H}}^2 - 2\langle f, \hat{f} \rangle + u\left(\|f\|_{\mathcal{H}}^2 - R^2\right)
$$

Then, we have dual problem as $\theta(u) = \min_{f \in \mathcal{H}} \mathcal{L}(f, u)$. Taking the derivative of the Lagrangian and setting it to zero, we obtain $\arg\min_{f \in \mathcal{H}} \mathcal{L}(f, u) = (1 + u)^{-1} \hat{f}$. With some more work, we obtain $\arg\max_{u > 0} \theta(u) = R^{-1}\|\hat{f}\| - 1$. We then have f at u^* as $f = R\|\hat{f}\|_{\mathcal{H}}^{-1}\hat{f}$. Since $\|\hat{f}\| > R$ as $\hat{f} \notin \mathcal{K}$ by assumption, our proof is complete. ■

Appendix C. Proofs for Kernelized Regression

We will first give a simple calculation of the β-smoothness parameter of the subquantile objective. We then will give proofs for our approximation error bounds.

C.1. Subquantile Smoothness

Lemma 26 *(β-Smoothness of $g(t, f)$ w.r.t f). Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ represent the rows of the data matrix $\mathbf{X}$. It then follows:*

$$
\|\nabla_f g(t, f) - \nabla_f g(t, \hat{f})\|_{\mathcal{H}} \leq \beta \|f - \hat{f}\|_{\mathcal{H}}
$$

where $\beta = \frac{2}{n(1-\epsilon)} \operatorname{Tr}(\mathbf{K})$

Proof. We will upper bound the operator norm of the Hessian Operator. We have from Section 3,

$$
\begin{aligned}
\left\|\nabla_f^2 g\left(t, f\right)\right\|_{\mathrm{HS}} &= \frac{2}{n(1-\epsilon)} \left\| \sum_{i=1}^n \mathbb{I}\left\{t \geq \ell\left(f; \mathbf{x}_i, y_i\right)\right\} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) \right\|_{\mathrm{HS}} \\
&\leq \frac{2}{n(1-\epsilon)} \left\| \sum_{i=1}^n \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) \right\|_{\mathrm{HS}} = \frac{2}{n(1-\epsilon)} \|\mathbf{\Phi} \otimes \mathbf{\Phi}\|_{\mathrm{HS}} = \frac{2}{n(1-\epsilon)} \operatorname{Tr}(\mathbf{K})
\end{aligned}
$$

This completes the proof. ■

C.2. Proof of Theorem 10

Proof. From Algorithm 1, we have for kernelized linear regression the following update,

$$
f^{(t+1)} = \operatorname{Proj}_{\mathcal{K}}\left[f^{(t)} - \frac{2\eta}{n(1-\epsilon)} \sum_{i \in S^{(t)}} (f^{(t)}(\mathbf{x}_i) - y_i) \cdot \phi(\mathbf{x}_i) - \eta C f^{(t)} \right] \tag{1}
$$

Next, we note that we can partition $S = (S \cap P) \cup (S \cap Q) \triangleq \text{TP} \cup \text{FP}$. Then we have

$$
\begin{aligned}
\|f^{(t+1)} - f^*\|_{\mathcal{H}}^2 &= \|\text{Proj}_{\mathcal{K}}\left[f^{(t)} - \nabla_f g(f^{(t)}, t^*)\right] - f^*\|_{\mathcal{H}}^2 \\
&\overset{(i)}{\leq} \|f^{(t)} - \nabla_f g(f^{(t)}, t^*) - f^*\|_{\mathcal{H}}^2 \\
&\leq 2\|f^{(t)} - \nabla_f g(f^{(t)}, t^*) - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 + 2\|\text{Proj}_{\Psi_m^\perp} f^*\|^2 \\
&= 2\|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 - 4\eta \left\langle \nabla_f g(f^{(t)}, t^*), f^{(t)} - \text{Proj}_{\Psi_m} f^* \right\rangle_{\mathcal{H}} \\
&\qquad + 2\eta^2 \|\nabla_f g(f^{(t)}, t^*)\|_{\mathcal{H}}^2 + 2\|\text{Proj}_{\Psi_m^\perp} f^*\|^2
\end{aligned} \tag{2}
$$

where (i) follows from noting the projection is a contraction. We will dedicate the rest of the proof to upper bounding the first three terms in Equation (2). We will first bound the second term in Equation (2) by splitting it into terms using the following relation,

$$
\begin{aligned}
&2\eta\langle \nabla_f g(f^{(t)}, t^*), f^{(t)} - \text{Proj}_{\Psi_m} f^* \rangle_{\mathcal{H}} \\
&\overset{(1)}{=} \frac{4\eta}{n(1-\epsilon)} \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \sum_{i \in S^{(t)}} (f^{(t)}(\mathbf{x}_i) - y_i) \cdot \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \\
&\overset{(i)}{=} \frac{4\eta}{n(1-\epsilon)} \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \sum_{i \in S^{(t)} \cap P} (f^{(t)}(\mathbf{x}_i) - f^*(\mathbf{x}_i) - \mu_i) \cdot \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \\
&\qquad + \frac{4\eta}{n(1-\epsilon)} \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \sum_{i \in S^{(t)} \cap Q} (f^{(t)}(\mathbf{x}_i) - y_i) \cdot \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}}
\end{aligned} \tag{3}
$$

where (i) follows from Theorem 13. We will now lower bound the first term of Equation (3).

$$
\begin{aligned}
&\frac{4\eta}{n(1-\epsilon)} \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \sum_{i \in S^{(t)} \cap P} (f^{(t)}(\mathbf{x}_i) - f^*(\mathbf{x}_i) - \mu_i) \cdot \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \\
&= \frac{4\eta}{n(1-\epsilon)} \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \left[\sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i)\right] \left(f^{(t)} - \text{Proj}_{\Psi_m} f^*\right) \right\rangle_{\mathcal{H}} \\
&\quad + \frac{4\eta}{n(1-\epsilon)} \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \left[\sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i)\right] \left(\text{Proj}_{\Psi_m^\perp} f^*\right) \right\rangle_{\mathcal{H}} \\
&\quad - \frac{4\eta}{n(1-\epsilon)} \left\langle f^{(t)} - \text{Proj}_{\Psi_m} f^*, \sum_{i \in S^{(t)} \cap P} \mu_i \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \\
&\overset{(ii)}{\geq} \frac{4\eta}{n(1-\epsilon)} \left\langle \tilde{n}\mathbf{\Sigma} + \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) - \tilde{n}\mathbf{\Sigma}, \left(f^{(t)} - \text{Proj}_{\Psi_m} f^*\right) \otimes \left(f^{(t)} - \text{Proj}_{\Psi_m} f^*\right) \right\rangle_{\text{HS}} \\
&\quad - \frac{4\eta}{n(1-\epsilon)} \|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}} \left\| \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) \right\|_{\text{op}} \|\text{Proj}_{\Psi_m^\perp} f^*\|_{\mathcal{H}} \\
&\quad - \frac{4\eta}{n(1-\epsilon)} \|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \left\| \sum_{i \in S^{(t)} \cap P} \mu_i \phi(\mathbf{x}_i) \right\|_{\mathcal{H}} - \frac{1}{n(1-\epsilon)} \left\| \sum_{i \in S^{(t)} \cap P} \mu_i \phi(\mathbf{x}_i) \right\|_{\mathcal{H}} \\
&\overset{(iii)}{\geq} \frac{8\eta(1-2\epsilon)}{(1-\epsilon)} \|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \left(\lambda_m(\mathbf{\Sigma}) - \left\| \frac{1}{n(1-\epsilon)} \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) - \mathbf{\Sigma} \right\|_{\text{HS}} \right)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{4\eta^2}{n(1-\epsilon)} \|f^{(t)} - \mathrm{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \Big\| \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) \Big\|_{\mathrm{op}} \left\| \mathrm{Proj}_{\Psi_m^\perp} f^* \right\|_{\mathcal{H}} \\
&- \frac{1}{n(1-\epsilon)} \Big\| \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) \Big\|_{\mathrm{op}} \|\mathrm{Proj}_{\Psi_m^\perp} f^*\|_{\mathcal{H}} - \frac{8\lambda_m(\mathbf{\Sigma})\eta(1-2\epsilon)}{(1-\epsilon)} \left\| \mathrm{Proj}_{\Psi_m^\perp} f^{(t)} \right\|_{\mathcal{H}}^2 \\
&- \frac{4\eta}{n(1-\epsilon)} \|f^{(t)} - \mathrm{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \Big\| \sum_{i \in S^{(t)} \cap P} \mu_i \phi(\mathbf{x}_i) \Big\|_{\mathcal{H}} - \frac{1}{n(1-\epsilon)} \Big\| \sum_{i \in S^{(t)} \cap P} \mu_i \phi(\mathbf{x}_i) \Big\|_{\mathcal{H}}
\end{aligned} \tag{4}
$$

where in (ii) we define $\tilde{n} \triangleq |S^{(t)} \cap P|$. In (iii) we have the simple inequality $\|\mathrm{Proj}_{\Psi_m}[f^{(t)} - f^*]\|_{\mathcal{H}} = \|f^{(t)} - \mathrm{Proj}_{\Psi_m} f^* - \mathrm{Proj}_{\Psi_m^\perp} f^{(t)}\|_{\mathcal{H}}^2 \le 2\|f^{(t)} - \mathrm{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 + 2\|\mathrm{Proj}_{\Psi_m^\perp} f^{(t)}\|_{\mathcal{H}}^2$. We will now lower bound the second term of Equation (3).

$$
\begin{aligned}
&\frac{4\eta}{n(1-\epsilon)} \Big\langle f^{(t)} - \mathrm{Proj}_{\Psi_m} f^*, \sum_{i \in S^{(t)} \cap Q} (f^{(t)}(\mathbf{x}_i) - y_i) \cdot \phi(\mathbf{x}_i) \Big\rangle_{\mathcal{H}} \\
&\qquad \le \frac{4\eta}{n(1-\epsilon)} \|f^{(t)} - \mathrm{Proj}_{\Psi_m} f^*\|_{\mathcal{H}} \Big\| \sum_{i \in S^{(t)} \cap Q} \phi(\mathbf{x}_i) \Big\|_{\mathcal{H}} \sqrt{\sum_{i \in S^{(t)} \cap Q} (f^{(t)}(\mathbf{x}_i) - y_i)^2} \\
&\qquad \overset{(i)}{\le} \frac{4\eta}{[n(1-\epsilon)]^2} \|f^{(t)} - \mathrm{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \Big\| \sum_{i \in S^{(t)} \cap Q} \phi(\mathbf{x}_i) \Big\|_{\mathcal{H}}^2 + \eta \sum_{i \in S^{(t)}} (f^{(t)}(\mathbf{x}_i) - y_i)^2
\end{aligned} \tag{5}
$$

where (i) follows from Young's Inequality (Theorem 3) for a $\beta \in (0, 1)$. We see the step size, η, must have a sub-linear inverse relation to n. We will now upper bound the final term in Equation (2).

$$
\begin{aligned}
\eta^2 \|\nabla_f g(f^{(t)}, t^*)\|_{\mathcal{H}}^2 &= \frac{4\eta^2}{n^2(1-\epsilon)^2} \Big\| \sum_{i \in S^{(t)}} (f^{(t)}(\mathbf{x}_i) - y_i) \cdot \phi(\mathbf{x}_i) \Big\|_{\mathcal{H}}^2 \\
&\le \frac{4\eta^2}{n^2(1-\epsilon)^2} \Big\| \sum_{i \in S^{(t)}} \phi(\mathbf{x}_i) \Big\|_{\mathcal{H}}^2 \sum_{i \in S^{(t)}} (f^{(t)}(\mathbf{x}_i) - y_i)^2
\end{aligned} \tag{6}
$$

We can now complete the upper bound for $\|f^{(t+1)} - \mathrm{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2$ combining (4)-(6).

$$
\begin{aligned}
&\|f^{(t+1)} - \mathrm{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \le \|f^{(t)} - \mathrm{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \\
&\cdot \underbrace{\Bigg(1 - \frac{8\eta(1-2\epsilon)}{(1-\epsilon)} \Big(\lambda_m(\mathbf{\Sigma}) - \Big\| \frac{1}{n(1-\epsilon)} \sum_{i \in S^{(t+1)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) - \mathbf{\Sigma} \Big\|_{\mathrm{HS}} \Big) + \frac{4\eta}{[n(1-\epsilon)]^2} \Big\| \sum_{i \in S^{(t)} \cap Q} \phi(\mathbf{x}_i) \Big\|_{\mathcal{H}}^2 \cdots} \\
&\underbrace{\cdots + \frac{4\eta^2}{n(1-\epsilon)} \Big\| \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i) \Big\|_{\mathrm{op}} \|\mathrm{Proj}_{\Psi_m^\perp} f^*\|_{\mathcal{H}} + \frac{4\eta}{n(1-\epsilon)} \Big\| \sum_{i \in S^{(t)} \cap P} \mu_i \phi(\mathbf{x}_i) \Big\|_{\mathcal{H}} \Bigg)}_{I} \\
&+ \underbrace{\Bigg(\eta + \frac{4\eta^2}{n^2(1-\epsilon)^2} \Big\| \sum_{i \in S^{(t)}} \phi(\mathbf{x}_i) \Big\|_{\mathcal{H}}^2 \Bigg)}_{II} \sum_{i \in S^{(t)}} (f^{(t)}(\mathbf{x}_i) - y_i)^2
\end{aligned}
$$

$$+\frac{4\eta}{n(1-\epsilon)}\left\|\sum_{i\in S^{(t)}\cap P}\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i)\right\|_{\text{op}}\|\text{Proj}_{\Psi_m^\perp}f^*\|_{\mathcal{H}}+\frac{2\eta}{n(1-\epsilon)}\left\|\sum_{i\in S^{(t)}\cap P}\mu_i\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}+R^2\lambda_m(\mathbf{\Sigma})$$

We will now look at the residual term. Note that kernelized regression is β-smooth in the Hilbert Space.

$$\begin{aligned}\frac{1}{n(1-\epsilon)}\sum_{i\in S^{(t+1)}}(f^{(t+1)}(\mathbf{x}_i)-y_i)^2 &\leq \frac{1}{n(1-\epsilon)}\sum_{i\in S^{(t)}}(f^{(t+1)}(\mathbf{x}_i)-y_i)^2\\ &\leq \frac{1}{n(1-\epsilon)}\sum_{i\in S^{(t)}}(f^{(t)}(\mathbf{x}_i)-y_i)^2-\frac{\eta}{2}\|\nabla g(f^{(t)},t^*)\|_{\mathcal{H}}^2\\ &= \frac{1}{n(1-\epsilon)}\sum_{i\in S^{(t)}}(f^{(t)}(\mathbf{x}_i)-y_i)^2-\frac{2\eta}{n^2(1-\epsilon)^2}\left\|\sum_{i\in S^{(t)}}(f^{(t)}(\mathbf{x}_i)-y_i)\cdot\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}^2\end{aligned}$$

We will analyze the terms in I individually. Let $\mathcal{S}$ be the set of all combinations of subsets of size $[n(1-2\epsilon)]$ to $[n(1-\epsilon)]$. Then with probability exceeding $1-\delta$, we have

$$\left\|\sum_{i\in S^{(t)}\cap P}\mu_i\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}\leq\sqrt{2\sigma^2 n(1-\epsilon)\log\frac{\sqrt{2}}{\delta\log n}\log^2 n(1-\epsilon)\left(\text{Tr}(\mathbf{\Gamma})+\frac{e}{C}\|\mathbf{\Gamma}\|_{\text{op}}\log\frac{1}{2\delta}\right)}$$

We denote the term parameterized by $t\in[T]$,

$$\Lambda^{(t)}\triangleq\|f^{(t)}-\text{Proj}_{\Psi_m}f^*\|_{\mathcal{H}}^2+\sum_{i\in S^{(t)}}\left(f^{(t)}(\mathbf{x}_i)-y_i\right)^2$$

Appendix D. Kernelized Binary Classification

In this section, we will prove error bounds for Subquantile Minimization in the Kernelized Binary Classification Problem.

D.1. Proof of Theorem 12

From Algorithm 2, we have for kernelized binary classification,

$$f^{(t+1)}=\text{Proj}_{\mathcal{K}}\left[f^{(t)}-\frac{\eta}{n(1-\epsilon)}\sum_{i\in S^{(t)}}\left(\sigma(f^{(t)}(\mathbf{x}_i))-y_i\right)\cdot\phi(\mathbf{x}_i)\right]\tag{7}$$

From which it follows,

$$\begin{aligned}\|f^{(t+1)}-f^*\|_{\mathcal{H}}^2 &= \left\|\text{Proj}_{\mathcal{K}}\left[f^{(t)}-\frac{\eta}{n(1-\epsilon)}\nabla g(f^{(t)},t^*)\right]-f^*\right\|_{\mathcal{H}}^2\\ &\overset{(i)}{\leq}\left\|f^{(t)}-\frac{\eta}{n(1-\epsilon)}\nabla g(f^{(t)},t^*)-f^*\right\|_{\mathcal{H}}^2\\ &\leq 2\|f^{(t)}-\text{Proj}_{\Psi_m}f^*\|_{\mathcal{H}}^2-\frac{4\eta}{n(1-\epsilon)}\left\langle\nabla_f g(f^{(t)},t^*),f^{(t)}-\text{Proj}_{\Psi_m}f^*\right\rangle_{\mathcal{H}} \qquad (8)\\ &\qquad+\frac{2\eta^2}{n^2(1-\epsilon)^2}\|\nabla_f g(f^{(t)},t^*)\|_{\mathcal{H}}^2+2\left\|\text{Proj}_{\Psi_m^\perp}f^*\right\|_{\mathcal{H}}^2\end{aligned}$$

where (i) follows from the contraction property of the projection operator onto norm ball $\mathcal{K}$ and assuming $f^* \in \mathcal{K}$. We will expand the second term in Equation (8).

$$\begin{aligned}
&\frac{2\eta}{n(1-\epsilon)}\left\langle \nabla_f g(f^{(t)}, t^*), f^{(t)} - \mathrm{Proj}_{\Psi_m} f^* \right\rangle_{\mathcal{H}} \\
&\quad \overset{(7)}{=} \left\langle f^{(t)} - \mathrm{Proj}_{\Psi_m} f^*, \frac{2\eta}{n(1-\epsilon)} \sum_{i \in S^{(t)}} \left(\sigma(f^{(t)}(\mathbf{x}_i)) - y_i\right) \cdot \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \\
&\quad = \left\langle f^{(t)} - \mathrm{Proj}_{\Psi_m} f^*, \frac{2\eta}{n(1-\epsilon)} \sum_{i \in S^{(t)}} \left(\sigma(f^{(t)}(\mathbf{x}_i)) - \sigma(f^*(\mathbf{x}_i))\right) \cdot \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \\
&\qquad + \left\langle f^{(t)} - \mathrm{Proj}_{\Psi_m} f^*, \frac{2\eta}{n(1-\epsilon)} \sum_{i \in S^{(t)}} (\sigma(f^*(\mathbf{x}_i)) - y_i) \cdot \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \qquad (9)
\end{aligned}$$

We first upper bound upper bound the second term in Equation (9). From the Cauchy-Schwarz Inequality and noting $y_i \in \{0, 1\}$ and $\mathrm{range}(\sigma) \in (0, 1)$, we have the following,

$$\begin{aligned}
&\left\langle f^{(t)} - \mathrm{Proj}_{\Psi_m} f^*, \frac{2\eta}{n(1-\epsilon)} \sum_{i \in S^{(t)}} (\sigma(f^*(\mathbf{x}_i)) - y_i) \cdot \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \\
&\qquad \le \frac{2\eta}{n(1-\epsilon)} \| f^{(t)} - \mathrm{Proj}_{\Psi_m} f^* \|_{\mathcal{H}} \left\| \sum_{i \in S^{(t)}} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}} \max_{i \in S^{(t)}} |\sigma(f^*(\mathbf{x}_i)) - y_i| \\
&\qquad \overset{(ii)}{\le} \frac{\eta^2}{n^{2-\beta}(1-\epsilon)^{2-\beta}} \| f^{(t)} - \mathrm{Proj}_{\Psi_m} f^* \|_{\mathcal{H}}^2 \left\| \sum_{i \in S^{(t)}} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}^2 + \frac{2}{n^\beta (1-\epsilon)^\beta} \qquad (10)
\end{aligned}$$

where (ii) follows from Young's Inequality (Theorem 3) and noting for a vector $\mathbf{x} \in \mathbb{R}^d$ it holds $\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2$ and letting $\beta \in [0, 1]$ be an undetermined constant . Let us now consider the function $h : \mathcal{H} \to \mathbb{R}$ defined as $h(f) \triangleq \sum_{i \in S \cap P} \log(1 + \exp(f(\mathbf{x}_i))$. We can then calculate the gradients by hand, $\nabla h(f) = \sum_{i \in S \cap P} \sigma(f(\mathbf{x}_i)) \cdot \phi(\mathbf{x}_i)$ and $\nabla^2 h(f) = \sum_{i \in S \cap P} \sigma(f(\mathbf{x}_i))(1 - \sigma(f(\mathbf{x}_i))) \cdot \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i)$. From the properties of strong convexity, we have for any $f, \hat{f} \in \mathcal{H}$, there exists $\tilde{f} \in \mathcal{H}$ such that,

$$\begin{aligned}
&\left\langle f - \mathrm{Proj}_{\Psi_m} \hat{f}, \nabla h(f) - \nabla h(\hat{f}) \right\rangle_{\mathcal{H}} = \left\langle f - \mathrm{Proj}_{\Psi_m} \hat{f}, \nabla^2 h(\tilde{f})(f - \hat{f}) \right\rangle_{\mathcal{H}} \\
&\overset{(iii)}{=} \left\langle \nabla^2 h(\tilde{f}), (f - \hat{f}) \otimes (f - \hat{f}) \right\rangle_{\mathrm{HS}} + \left\langle \nabla^2 h(\tilde{f}), (\mathrm{Proj}_{\Psi_m^\perp} f^*) \otimes (f - \hat{f}) \right\rangle_{\mathcal{H}} \qquad (11)
\end{aligned}$$

where the equality in (iii) is given in (Gretton, 2015, Section 3.2). Then, from the strong convexity of h, there exists a constant C such that the following inequality holds,

$$\begin{aligned}
&\left\langle f^{(t)} - \mathrm{Proj}_{\Psi_m} f^*, \frac{2\eta}{n(1-\epsilon)} \sum_{i \in S^{(t)} \cap P} \left(\sigma(f^{(t)}(\mathbf{x}_i)) - \sigma(f^*(\mathbf{x}_i))\right) \cdot \phi(\mathbf{x}_i) \right\rangle_{\mathcal{H}} \\
&\overset{(11)}{\gtrsim} \frac{2\eta}{n(1-\epsilon)} \left\langle \sum_{i \in S^{(t)}} \phi(\mathbf{x}_i) \otimes \phi(\mathbf{x}_i), \mathrm{Proj}_{\Psi_m}\left[f^{(t)} - f^*\right] \otimes \mathrm{Proj}_{\Psi_m}\left[f^{(t)} - f^*\right] \right\rangle_{\mathrm{HS}} \\
&\quad - \frac{2\eta}{n(1-\epsilon)} \left\| \mathrm{Proj}_{\Psi_m^\perp} f^* \right\|_{\mathcal{H}} \left\| \sum_{i \in S^{(t)} \cap P} \phi(\mathbf{x}_i) \right\|_{\mathcal{H}}
\end{aligned}$$

$$\overset{(iv)}{\gtrsim} 4\eta\frac{(1-2\epsilon)}{(1-\epsilon)}\left(\lambda_m\left(\mathbf{\Gamma}\right) - \left\|\frac{1}{n(1-2\epsilon)}\sum_{i\in S^{(t)}\cap P}\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i) - \mathbf{\Gamma}\right\|_{\text{HS}}\right)\|f^{(t)} - \text{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2$$
$$-\frac{2\eta}{n(1-\epsilon)}\left\|\text{Proj}_{\Psi_m^\perp} f^*\right\|_{\mathcal{H}}\left\|\sum_{i\in S^{(t)}\cap P}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}} - 4\eta\gamma\lambda_m(\mathbf{\Gamma})\|\text{Proj}_{\Psi_m^\perp} f^{(t)}\|_{\mathcal{H}}^2 \tag{12}$$

where (iv) follows from Weyl's inequality (Weyl, 1912) and noting that $\left|S^{(t)}\cap P\right| \geq n(1-2\epsilon)$. We now briefly analyze the constant introduced in Equation (12).

$$C \triangleq \inf_{\mathbf{x}\in P}\sigma(f(\mathbf{x}_i))(1-\sigma(f(\mathbf{x}_i))) \geq (1/2)\exp\left(-\max_{\mathbf{x}\in P} f(\mathbf{x})\right) \geq (1/2)\exp\left(-R\max_{\mathbf{x}\in P}\|\phi(\mathbf{x})\|_{\mathcal{H}}\right) \tag{13}$$

The final inequality follows from (Gretton, 2013, Theorem 17). Then, from the bijectivity of the exponential function, we can invoke Theorem 22, and with probability exceeding $1-\delta$, we have

$$C \geq (1/2)\exp\left(-R\sqrt{8\,\text{Tr}(\mathbf{\Gamma})\log n\log\frac{4e}{\delta}}\right)$$

. Next, lext us briefly analyze $\|\text{Proj}_{\Psi_m^\perp} f^{(t)}\|_{\mathcal{H}}^2$.

$$\left\|\text{Proj}_{\Psi_m^\perp} f^{(t+1)}\right\|_{\mathcal{H}}^2 \leq 2\left\|\text{Proj}_{\Psi_m^\perp} f^{(t)}\right\|_{\mathcal{H}}^2 + 2\left\|\frac{\eta}{n(1-\epsilon)}\sum_{i\in S^{(t)}}(\sigma(f(\mathbf{x}_i)) - y_i)^2\cdot\text{Proj}_{\Psi_m^\perp}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}^2$$
$$\leq 2\left\|\text{Proj}_{\Psi_m^\perp} f^{(t)}\right\|_{\mathcal{H}}^2 + 4\left\|\frac{\eta}{n(1-\epsilon)}\sum_{i\in Q}\text{Proj}_{\Psi_m^\perp}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}^2 + 4\left\|\frac{\eta}{n(1-\epsilon)}\sum_{i\in S^{(t)}\cap P}\text{Proj}_{\Psi_m^\perp}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}^2$$
$$\overset{(v)}{\leq} 2\left\|\text{Proj}_{\Psi_m^\perp} f^{(t)}\right\|_{\mathcal{H}}^2 + \frac{4\eta^2(\text{Tr}(\mathbf{\Gamma}_{22})(1+\frac{e}{C}\left\|\mathbf{\Gamma}\right\|_{\text{op}}\log\frac{1}{2\delta}) + Q_k)}{n(1-\epsilon)} \tag{14}$$

when in (v) we partition $\mathbf{\Gamma}$ into

$$\mathbf{\Gamma} = \begin{array}{c c} & \begin{array}{cc} m & \infty \end{array} \\ \begin{array}{c} m \\ \infty \end{array} & \begin{bmatrix}\mathbf{\Gamma}_{11} & \mathbf{\Gamma}_{12}\\ \mathbf{\Gamma}_{12} & \mathbf{\Gamma}_{22}\end{bmatrix}\end{array}$$

From the recursion in Equation (14) and noting $\|\text{Proj}_{\Psi_m^\perp} f^{(t)}\|_{\mathcal{H}} = 0$, we have

$$\|\text{Proj}_{\Psi_m^\perp} f^{(t)}\|_{\mathcal{H}}^2 \leq \frac{2^{t+2}\eta^2\left(\|\mathbf{\Gamma}_{22}\|_{\text{Tr}}\left(1+\frac{e}{C}\|\mathbf{\Gamma}_{22}\|_{\text{op}}\log\frac{2}{\delta}\right) + Q_k\right)}{n(1-\epsilon)}$$

We will now bound the third term in Equation (8).

$$\|\nabla_f g(f^{(t)}, t^*)\|_{\mathcal{H}}^2 = \left\|\frac{\eta}{n(1-\epsilon)}\sum_{i\in S^{(t)}}(\sigma(f^{(t)}(\mathbf{x}_i)) - y_i)\cdot\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}^2$$
$$\leq \frac{\eta^2}{n^2(1-\epsilon)^2}\max_{i\in S^{(t)}}\left|\sigma(f^{(t)}(\mathbf{x}_i)) - y_i\right|^2\cdot\left\|\sum_{i\in S^{(t)}}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}^2 \overset{(v)}{\leq} \frac{\eta^2}{n^2(1-\epsilon)^2}\left\|\sum_{i\in S^{(t)}}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}^2 \tag{15}$$

where (v) follows from noting for any $\mathbf{x} \in \mathbb{R}^d$ it holds $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2$. Furthermore, we note that if $-\log(\sigma(f(\mathbf{x}))) \leq -\log(\sigma(f(\hat{\mathbf{x}})))$, then $\sigma(f(\mathbf{x})) \geq \sigma(f(\hat{\mathbf{x}}))$. Now, combining (8)-(15), we obtain

$$\|f^{(t+1)} - \mathrm{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2 \leq \|f^{(t)} - \mathrm{Proj}_{\Psi_m} f^*\|_{\mathcal{H}}^2$$
$$\cdot \underbrace{\left(1 - \frac{2C\eta(1-2\epsilon)}{(1-\epsilon)}\left(\lambda_m(\mathbf{\Gamma}) - \left\|\frac{1}{n(1-2\epsilon)}\sum_{i\in S^{(t)}\cap P}\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i) - \mathbf{\Gamma}\right\|_{\mathrm{HS}}\right) + \frac{\eta^2}{[n(1-\epsilon)]^{3/2}}\left\|\sum_{i\in S^{(t)}}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}^2\right)}_{III}$$
$$\underbrace{+ \frac{\eta^2}{n^2(1-\epsilon)^2}\left\|\sum_{i\in S^{(t)}}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}^2 + \frac{2}{\sqrt{n(1-\epsilon)}} + \frac{2\eta}{n(1-\epsilon)}\|\mathrm{Proj}_{\Psi_m^\perp} f^*\|_{\mathcal{H}}^2\left\|\sum_{i\in S^{(t)}\cap P}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}} + 4\eta\gamma C\lambda_m(\mathbf{\Gamma})\left\|\mathrm{Proj}_{\Psi_m^\perp} f^{(t)}\right\|_{\mathcal{H}}^2}_{IV}$$

Denote $\mathcal{S}$ as the set of combinations of $[n(1-2\epsilon)]$ to $[n(1-\epsilon)]$ probability at least $1-\delta$, we have

$$\max_{\substack{\sigma\in\Pi\\|\sigma|=n(1-\epsilon)}}\left\|\sum_{i=1}^{n(1-\epsilon)}\phi(\mathbf{x}_{\sigma(i)})\right\|_{\mathcal{H}} \leq \sqrt{n(1-\epsilon)\left(\mathrm{Tr}(\mathbf{\Gamma}) + \frac{e}{C}\|\mathbf{\Gamma}\|_{\mathrm{op}}\log\frac{1}{2\delta}\right)}$$

Then, from the assumption that the corrupted covariates are centered, we have for a sufficiency small δ that with probability at least $1-\delta$ from Proposition 23,

$$\begin{aligned}\left\|\sum_{i\in S^{(t)}}\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}^2 &\leq 2\max_{\substack{\sigma\in\Pi\\|\sigma|=n(1-\epsilon)}}\left\|\sum_{i=1}^{n(1-\epsilon)}\phi(\mathbf{x}_{\sigma(i)})\right\|_{\mathcal{H}}^2 + 2\left\|\mathop{\mathbf{E}}_{\xi_i\sim\mathcal{R}}\sum_{i\in S^{(t)}\cap Q}\xi_i\phi(\mathbf{x}_i)\right\|_{\mathcal{H}}^2\\ &\leq 2n(1-\epsilon)\left(\mathrm{Tr}(\mathbf{\Gamma}) + \frac{e}{C}\|\mathbf{\Gamma}\|_{\mathrm{op}}\log\frac{1}{2\delta} + Q_k\right)\end{aligned}$$

where $Q_k = \max_{i\in Q} k(\mathbf{x}_i, \mathbf{x}_i)$. Next, to obtain

$$\lambda_m(\mathbf{\Gamma}) - \left\|\frac{1}{n(1-2\epsilon)}\sum_{i\in S^{(t)}\cap P}\phi(\mathbf{x}_i)\otimes\phi(\mathbf{x}_i) - \mathbf{\Gamma}\right\|_{\mathrm{HS}} \geq 0.9\lambda_m(\mathbf{\Gamma})$$

with probability greater than $1-\delta$. We utilize Proposition 24 update to new bound with dependence on ϵ and require

$$n = \frac{1600\|\mathbf{\Gamma}\|_{\mathrm{op}}^2(1-2\epsilon)^{-1}}{\lambda_m^2(\mathbf{\Gamma})} + 400P_k^2(1-2\epsilon)^{-1}\log\frac{2}{\delta}\epsilon\log\frac{e}{\epsilon}$$

Assume the above data requirement is true. Define $\gamma \triangleq \frac{1-\epsilon}{1-2\epsilon}$. To solve the quadratic equation $III \leq 99/100$, we also require

$$n = C_n\left[\frac{\left(\mathrm{Tr}(\mathbf{\Gamma}) + \frac{e}{C}\|\mathbf{\Gamma}\|_{\mathrm{op}}\log\frac{1}{2\delta} + Q_k\right)^2}{3.24^2(1-\epsilon)\lambda_m^4(\mathbf{\Gamma})C^4\gamma^4}\right]$$

where $C_n \geq 16/10000$ and is dependent on the rate of decrease. From where $III \leq 99/100$ when

$$\frac{1.8 - \sqrt{3.24 - 0.04 C_n^{-1/2}}}{3.24 C \lambda_m(\mathbf{\Gamma})\gamma} \leq \eta \leq \frac{1.8 + \sqrt{3.24 - 0.04 C_n^{-1/2}}}{3.24 C \lambda_m(\mathbf{\Gamma})\gamma}$$

Thus, we have $III \leq 0.99$ with probability exceeding $1-\delta$. We will now show all the terms in IV are small. With probability at least $1-\delta$,

$$\begin{aligned}
\frac{\eta^2}{[n(1-\epsilon)]^2}\left\|\sum_{i \in S^{(t)}} \phi(\mathbf{x}_i)\right\|_{\mathcal{H}}^2 &\leq \frac{2\eta^2\left(\mathrm{Tr}(\mathbf{\Gamma}) + \frac{e}{C}\left\|\mathbf{\Gamma}\right\|_{\mathrm{op}} \log\frac{1}{2\delta} + Q_k\right)}{n(1-\epsilon)} \\
&\leq \frac{2 \cdot 3.24^2 \lambda_m^2(\mathbf{\Gamma}) C^2 \gamma^2}{C_n\left(\mathrm{Tr}(\mathbf{\Gamma}) + \frac{e}{C}\left\|\mathbf{\Gamma}\right\|_{\mathrm{op}} \log\frac{1}{2\delta} + Q_k\right)} = O\left(\frac{\lambda_m^2(\mathbf{\Gamma})}{C_n \,\mathrm{Tr}(\mathbf{\Gamma})}\right) \quad (16)
\end{aligned}$$

Using the above bound, we obtain

$$\begin{aligned}
4\eta\gamma C \lambda_m(\mathbf{\Gamma})\left\|\mathrm{Proj}_{\Psi_m^{\perp}} f^{(t)}\right\|_{\mathcal{H}}^2 &\leq \frac{3.24^2 \cdot 2^{t+5} \lambda_m^2(\mathbf{\Gamma}) C^2 \gamma^2 \left(\left\|\mathbf{\Gamma}_{22}\right\|_{\mathrm{Tr}} + \frac{e}{C}\left\|\mathbf{\Gamma}_{22}\right\|_{\mathrm{op}} \log\frac{2}{\delta} + Q_m\right)}{C_n\left(\mathrm{Tr}(\mathbf{\Gamma}) + \frac{e}{C}\left\|\mathbf{\Gamma}\right\|_{\mathrm{op}} \log\frac{1}{2\delta} + Q_k\right)} \\
&= O\left(\frac{2^{t+5}\lambda_m^2(\mathbf{\Gamma})\left(\left\|\mathbf{\Gamma}_{22}\right\|_{\mathrm{Tr}} + Q_m\right)}{C_n\left(\mathrm{Tr}(\mathbf{\Gamma}) + Q_k\right)}\right) \quad (17)
\end{aligned}$$

In Equations 16 and 17, we see that C_n is in the denominator and goes to ∞ as $n \nearrow \infty$. Thus we see $IV \searrow 0$ as $n \nearrow \infty$. We then have

$$\left\|f^{(T)} - \mathrm{Proj}_{\Psi_m} f^*\right\|_{\mathcal{H}} \leq 0.99^T \left\|f^{(0)} - \mathrm{Proj}_{\Psi_m} f^*\right\|_{\mathcal{H}} + \sum_{k=0}^{T} 0.99^k (IV) \leq \frac{\varepsilon}{2} + 100(IV)$$

after $\log\left(\frac{2\|f^*\|_{\mathcal{H}}}{\varepsilon}\right)$ iterations, our proof is complete ■