Inequalities for Non-Standard Gaussian Matrix Norms

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Abstract

In this paper we explore upper bounds on the spectral norm for Gaussian Matrices with columns standard from Central Correlated Multivariate Normal Distributions. We utilize a lemma from [Chi17] and extend the analysis from [CD05]. These bounds find applications in the generalization of the randomized SVD given in [BT22] and wireless network science.

1 Introduction

The study of the expectation of the norms of the pseudoinverse of standard normal gaussian matrices first appeared in [HMT11] when analyzing the error bounds for the Randomized SVD algorithm. The bounds developed in [HMT11] used theory developed in analyzing the condition numbers of standard normal matrices in [CD05]. In a generalization of the Randomized SVD, the need for bounds on the expectation of the spectral norm for correlated Gaussian matrices appears in [BT22].

2 Relevant Work in Standard Uncorrelated Matrices

In this section we will briefly discuss bounds developed for the inequalities of standard normal matrices.

Proposition 1. *(HMT Proposition 10.2).* *Draw a $k \times (k+p)$ standard Gaussian matrix $\mathbf{G}$ with $k \geq 2$ and $p \geq 2$. Then*

$$\mathbb{E}\left\|\mathbf{G}^{\dagger}\right\| \leq \frac{e\sqrt{k+p}}{p} \tag{1}$$

From our search in the literature, there is no bound on equation 1 when the columns are not sampled from a multiple of the identity.

3 Theory

We will first introduce the necessary lemmas needed to prove our main results.

3.1 Necessary Lemmas

Lemma 2. *[Jam64, Eq. (58,59)]. If $\lambda_1 \geq \ldots \geq \lambda_m$ are the eigenvalues of $\mathbf{W}$ s.t. $\mathbf{W} \sim \mathcal{W}_m(n, \mathbf{C})$ s.t. $n > m-1$, then the joint PDF of eigenvalues is*

$$f(\lambda_1, \ldots, \lambda_m) = K_{m,n} \left(\det \mathbf{C}\right)^{-n/2} \exp\left(-\frac{1}{2}\operatorname{Tr}\left(\mathbf{C}^{-1}\mathbf{W}\right)\right) \prod_{i=1}^{m} \lambda_i^{(n-m-1)/2} \prod_{i<j} (\lambda_i - \lambda_j) \tag{2}$$

where

$$K_{m,n} = \frac{\pi^{m^2/2}}{\Gamma_m\left(\frac{1}{2}m\right)\Gamma_m\left(\frac{1}{2}n\right)} \tag{3}$$

Lemma 3. *[WLRT08, Lemma 3.6]. Let $m, n \in \mathbb{N}$ s.t. $n \geq m$. Suppose $\mathbf{A} \in \mathbb{R}^{n\times m}$, then if $(\mathbf{A}^\top \mathbf{A})$ is invertible*

$$\left\| \left(\mathbf{A}^\top \mathbf{A}\right)^{-1} \mathbf{A}^\top \right\| = \frac{1}{\sigma_m(\mathbf{A})} \tag{4}$$

Lemma 4. *[Chi17, Lemma 1]. Draw a $m \times n$ matrix $\mathbf{G}$ s.t. the columns of $\mathbf{G}$ are sampled from $\mathcal{N}_m(\mathbf{0}, \mathbf{C})$ where the eigenvalues of $\mathbf{C}$ are represented as $\sigma_1 > \sigma_2 > \cdots > \sigma_m$. Let $\mathbf{W} \sim \mathcal{W}_m(n, \mathbf{C})$. The eigenvalue distribution is given as*

$$f(x_1, \ldots, x_n) = K_{\mathbf{C}} \left|\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})\right| \cdot \prod_{i=1}^{m-1} \prod_{j=i+1}^{m} (x_i - x_j) \prod_{i=1}^{n} x_i^{n-m} \tag{5}$$

where $\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma}) = \left\{e^{-x_i/\sigma_j}\right\}_{i,j=1}^m = \begin{bmatrix} e^{-\frac{x_1}{\sigma_1}} & \ldots & e^{-\frac{x_1}{\sigma_m}} \\ \vdots & \ddots & \vdots \\ e^{-\frac{x_m}{\sigma_1}} & \ldots & e^{-\frac{x_m}{\sigma_m}} \end{bmatrix}$ *and*

$$K_{\mathbf{C}}^{-1} = \prod_{i=1}^{m-1} \prod_{j=i+1}^{m} (\sigma_i - \sigma_j) \prod_{i=1}^{m} \sigma_i^{n-m+1} (n-i)! \tag{6}$$

With these lemmas we will go to proving the main results.

3.2 Main Results

Theorem 5. *Draw a $m \times m$ matrix $\mathbf{G}$ s.t. the columns of $\mathbf{G}$ are sampled from $\mathcal{N}_m(\mathbf{0}, \mathbf{C})$ where the eigenvalues of $\mathbf{C}$ are represented as $\sigma_1 > \sigma_2 > \cdots > \sigma_m$. Then*

$$\mathbb{E} \left\|\mathbf{G}^\dagger\right\| \approx \sqrt{\pi \sum_{k=1}^{m} \frac{1}{\sigma_k}} \tag{7}$$

Proof. We will first note

$$\left\|\mathbf{G}^\dagger\right\| \overset{\text{lem. 3}}{=} \frac{1}{\sigma_m(\mathbf{G})} = \frac{1}{\sqrt{\lambda_{\min}(\mathbf{G}\mathbf{G}^\top)}} \tag{8}$$

For $\mathbf{W}$ sampled from $\mathcal{W}_m(m, \mathbf{C})$. We will now derive the distribution for minimum eigenvalue of $\mathbf{W}$ similar to [NZYY08].

$$f_{\lambda_{\min}}(x_m) = \int_{x_2}^{\infty} \cdots \int_{x_{m-1}}^{\infty} K_{\mathbf{C}} \left|\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})\right| \cdot \prod_{i<j} (x_i - x_j) \prod_{i=1}^{m} x_j^{m-m} \prod_{i=1}^{m-1} dx_i \tag{9}$$

$$= K_{\mathbf{C}} \int_{x_2}^{\infty} \cdots \int_{x_{m-1}}^{\infty} \left|\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})\right| \cdot \prod_{i=1}^{m-2} \prod_{j=i+1}^{m-1} (x_i - x_j) \prod_{i=1}^{m-1} (x_i - x_m) \prod_{i=1}^{m-1} dx_i \tag{10}$$

$$\overset{(a)}{=} e^{-\sum_{i=1}^{m} \frac{x_m}{\sigma_i}} \left(\int_{y_2}^{\infty} \cdots \int_{y_{m-1}}^{\infty} \sum_{i=1}^{m} (-1)^{i+m} K_{\mathbf{C}} \left|\mathbf{E}_i\left(\mathbf{x} - \mathbf{x}_m, \boldsymbol{\sigma}\right)\right| \prod_{i=1}^{m-2} \prod_{j=i+1}^{m-1} (y_i - y_j) \prod_{i-1}^{m-1} dy_i \right) \tag{11}$$

$$\overset{(b)}{=} \Xi e^{-\sum_{i=1}^{m} \frac{x_m}{\sigma_i}} \tag{12}$$

(a) follows due to the properties of the determinant. (b) follows as the intergral expresion in Equation (11) no longer integrates over x_m and thus integrates to some constant we define as Ξ. Since the PDF must integrate to 1, we thus have,

$$f_{\lambda_{\min}}(x) = \left(\sum_{k=1}^{m} \frac{1}{\sigma_k} \right) e^{-x \sum_{k=1}^{m} \frac{1}{\sigma_k}} \tag{13}$$

The Expected Value follows from a simple integration.

$$\mathbb{E}\left\|\mathbf{G}^{\dagger}\right\| = \int_0^{\infty} \frac{1}{\sqrt{x}} e^{-x\sum_{k=1}^{m}\sigma_k^{-1}} dx \tag{14}$$

$$= \sqrt{\pi \sum_{k=1}^{m} \frac{1}{\sigma_k}} \operatorname{erf}\left(\sqrt{\pi \sum_{k=1}^{m} \frac{1}{\sigma_k}}\right) \lesssim \sqrt{\pi \sum_{k=1}^{m} \frac{1}{\sigma_k}} \tag{15}$$

The proof is complete. ■

Theorem 6. *Draw a $m \times n$ matrix $\mathbf{G}$ s.t. the columns of $\mathbf{G}$ are sampled from $\mathcal{N}_m(\mathbf{0}, \mathbf{C})$ where the eigenvalues of $\mathbf{C}$ are represented as $\sigma_1 > \sigma_2 > \cdots > \sigma_m$. Let $\mathbf{W} \sim \mathcal{W}_m(n, \mathbf{C})$. Then,*

$$\mathbb{E}\left\|\mathbf{G}^{\dagger}\right\| \leq \sum_{i=1}^{m} \mathcal{O}\left(\sigma_i^{n-m+\frac{1}{2}} 2\sqrt{\frac{\pi}{e}} \left(\frac{n-m+\frac{1}{2}}{e}\right)^{n-m}\right) \tag{16}$$

Proof. Let $K_{\mathbf{C}}$ and $\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})$ be defined as in Lemma 4.

$$f_{\lambda_{\min}}(x_m) = \int_{x_2}^{\infty} \cdots \int_{x_{m-1}}^{\infty} K_{\mathbf{C}} \left|\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})\right| \cdot \prod_{i<j}^{m} (x_i - x_j) \prod_{i=1}^{m} x_j^{n-m} \prod_{i=1}^{m-1} dx_i \tag{17}$$

$$= K_{\mathbf{C}} x_m^{n-m} \int_{x_2}^{\infty} \cdots \int_{x_{m-1}}^{\infty} \left|\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})\right| \cdot \prod_{i<j}^{m-1} (x_i - x_j) \prod_{i=1}^{m-1} (x_i - x_m) \prod_{i=1}^{m} x_i^{n-m} \prod_{i=1}^{m-1} dx_i \tag{18}$$

$$\leq K_{\mathbf{C}} x_m^{n-m} \int_{x_2}^{\infty} \cdots \int_{x_{m-1}}^{\infty} \left|\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})\right| \cdot \prod_{i<j}^{m-1} (x_i - x_j) \prod_{i=1}^{m-1} x_i^{n-m+1} \prod_{i=1}^{m-1} dx_i \tag{19}$$

$$\leq K_{\mathbf{C}} x_m^{n-m} \sum_{i=1}^{m} \left((-1)^{i+m} e^{-\frac{x_m}{\sigma_i}} \int_{x_2}^{\infty} \cdots \int_{x_{m-1}}^{\infty} \left|\mathbf{E}_{m,i}(\mathbf{x}, \boldsymbol{\sigma})\right| \cdot \prod_{i<j}^{m-1} (x_i - x_j) \prod_{i=1}^{m-1} x_i^{n-m+1} \prod_{i=1}^{m-1} dx_i\right) \tag{20}$$

$$= x_m^{n-m} \underbrace{K_{\mathbf{C}} \sum_{i=1}^{m} (-1)^{i+m} e^{-\frac{x_m}{\sigma_i}} K_{\mathbf{C},i}^{-1}}_{\Xi} \tag{21}$$

We will now upper bound Ξ.

$$\Xi \triangleq K_{\mathbf{C}} \sum_{i=1}^{m} (-1)^{i+m} e^{-\frac{x_m}{\sigma_i}} K_{\mathbf{C},i}^{-1} \tag{22}$$

$$= \sum_{k=1}^{m} (-1)^{k+m} e^{-\frac{x_m}{\sigma_k}} \frac{\prod_{i=1}^{m-1} \prod_{j=i+1}^{m} \mathbb{1}_{i,j\neq k} (\sigma_i - \sigma_j) \prod_{i=1}^{m} \mathbb{1}_{i\neq k} \sigma_i^{n-m+1} (n-i)!}{\prod_{i=1}^{m-1} \prod_{j=i+1}^{m} (\sigma_i - \sigma_j) \prod_{i=1}^{m} \sigma_i^{n-m+1} (n-i)!} \tag{23}$$

$$= \sum_{k=1}^{m} (-1)^{k+m} e^{-\frac{x_m}{\sigma_k}} \left(\prod_{i>k}^{m} (\sigma_i - \sigma_k) \prod_{i<k} (\sigma_k - \sigma_i) \cdot \sigma_k^{n-m+1} (n-k)!\right)^{-1} \tag{24}$$

$$\leq \sum_{k=1}^{m} e^{-\frac{x_m}{\sigma_k}} \left(\prod_{i>k}^{m} (\sigma_i - \sigma_k) \prod_{i<k} (\sigma_k - \sigma_i) \cdot \sigma_k^{n-m+1} (n-k)!\right)^{-1} \tag{25}$$

$$\overset{(a)}{\leq} K \sum_{k=1}^{m} e^{-\frac{x_m}{\sigma_k}} \tag{26}$$

In (a) we define $K \triangleq \max_{k\in[m]} K_{\mathbf{C}} K_{\mathbf{C},k}^{-1}$. We thus have

$$f_{\lambda_{\min}}(x_m) \leq K x_m^{n-m} \sum_{i=1}^{m} e^{-\frac{x_m}{\sigma_i}} \leq \mathcal{O}\left(x_m^{n-m} \sum_{i=1}^{m} e^{-\frac{x_m}{\sigma_i}}\right) \tag{27}$$

Now we will integrate over $f_{\lambda_{\min}}(x_m)$.

$$\mathbb{E}\left\|\mathbf{G}^{\dagger}\right\| = \int_{0}^{\infty} \mathcal{O}\left(x^{n-m-\frac{1}{2}} \sum_{i=1}^{m} e^{-\frac{x}{\sigma_i}}\right) dx \tag{28}$$

$$\stackrel{(a)}{\leq} \sum_{i=1}^{m} \mathcal{O}\left(\sigma_i^{n-m+\frac{1}{2}} \Gamma\left(\frac{n-m+\frac{1}{2}}{2}\right)\right) \tag{29}$$

$$\leq \sum_{i=1}^{m} \mathcal{O}\left(\sigma_i^{n-m+\frac{1}{2}} \sqrt{\frac{4\pi}{n-m+\frac{1}{2}}} \left(\frac{n-m+\frac{1}{2}}{e}\right)^{n-m+\frac{1}{2}}\right) \tag{30}$$

$$= \sum_{i=1}^{m} \mathcal{O}\left(\sigma_i^{n-m+\frac{1}{2}} 2\sqrt{\frac{\pi}{e}} \left(\frac{n-m+\frac{1}{2}}{e}\right)^{n-m}\right) \tag{31}$$

(a) follows from an application of Stirling's Approximation [Rob55]. ■

4 Numerical Experiments

In Figure 1, we verify the results given in Theorem 5.

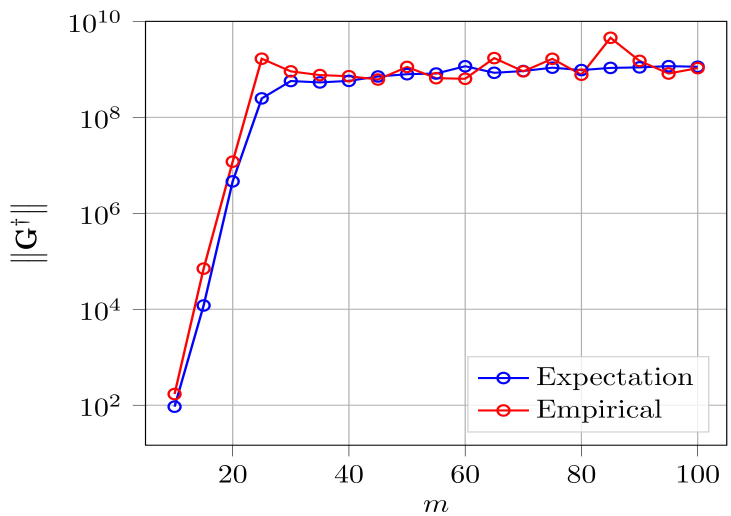

Figure 1: Comparing the expected norm upper bound on $\left\|\mathbf{G}^{\dagger}\right\|$ where $\mathbf{G} \in \mathbb{R}^{m \times m}$ and the columns of $\mathbf{G}$ are sampled from $\mathcal{N}_m(\mathbf{0}, \mathbf{K})$ with the average norm of $\mathbf{G}^{\dagger}$ over 100 samples. The expected norm is calculated with Proposition 5.

In Figure 2, we verify the results given in Theorem 6.

5 Conclusions

In this paper, we derive novel upper bounds for the spectral norm of Gaussian matrices with columns sampled from a central correlated multivariate normal distribution.

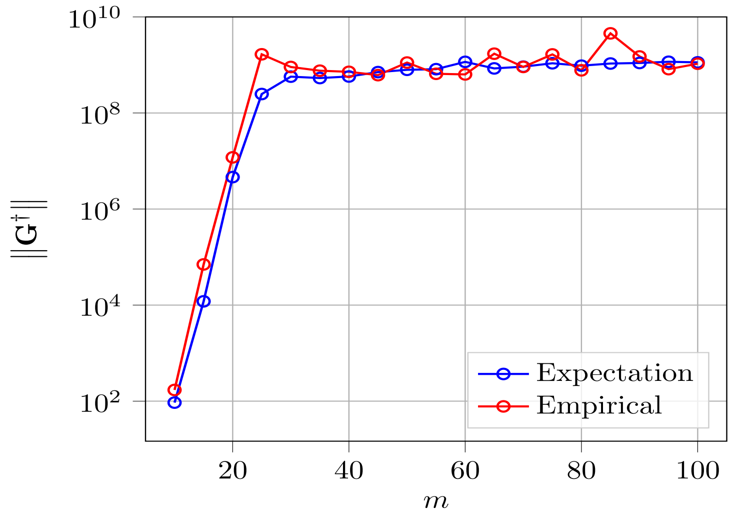

Figure 2: Comparing the expected norm upper bound on $\left\|\mathbf{G}^{\dagger}\right\|$ where $\mathbf{G} \in \mathbb{R}^{m \times m}$ and the columns of $\mathbf{G}$ are sampled from $\mathcal{N}_m(\mathbf{0}, \mathbf{K})$ with the average norm of $\mathbf{G}^{\dagger}$ over 100 samples. The expected norm is calculated with Proposition 5.

References

[BT22] Nicolas Boulle and Alex Townsend. A generalization of the randomized singular value decomposition. In *International Conference on Learning Representations*, 2022.

[CD05] Zizhong Chen and Jack J. Dongarra. Condition numbers of gaussian random matrices. *SIAM Journal on Matrix Analysis and Applications*, 27(3):603–620, 2005.

[Chi17] Marco Chiani. On the probability that all eigenvalues of gaussian, wishart, and double wishart random matrices lie within an interval, 2017.

[HMT11] N. Halko, P. G. Martinsson, and J. A. Tropp. Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions. *SIAM Review*, 53(2):217–288, 2011.

[Jam64] Alan T. James. Distributions of matrix variates and latent roots derived from normal samples. *The Annals of Mathematical Statistics*, 35(2):475–501, 1964.

[NZYY08] Fangfang Niu, Haochuan Zhang, Hongwen Yang, and Dacheng Yang. Distribution of the smallest eigenvalue of complex central semi-correlated wishart matrices. In *2008 IEEE International Symposium on Information Theory*, pages 1788–1792, 2008.

[Rob55] Herbert Robbins. A remark on stirling's formula. *The American mathematical monthly*, 62(1):26–29, 1955.

[WLRT08] Franco Woolfe, Edo Liberty, Vladimir Rokhlin, and Mark Tygert. A fast randomized algorithm for the approximation of matrices. *Applied and Computational Harmonic Analysis*, 25(3):335–366, 2008.