Iterative Thresholding for Non-Linear Learning in the Strong ϵ-Contamination Model

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Abstract

We study the problem of learning single neurons when both the labels and the covariates are possibly corrupted adversarially with a gradient-descent iterative thresholding algorithm. We assume for $(\mathbf{x}, y) \sim \mathcal{D}$, their distribution is given as

$$y = \sigma(\mathbf{w}^* \cdot \mathbf{x}) + \xi$$

where ξ is Gaussian noise sampled i.i.d from $\mathcal{N}(0, \nu^2)$ and σ is an activation function. We study sigmoid, leaky-ReLU, and ReLU activation functions. We derive a $O(C_\sigma \sqrt{\epsilon \log(1/\epsilon)})$ approximation bound, where C_σ is a constant dependent upon the activation function.

We also study the linear regression problem when $\sigma(x) = x$. We derive a $O(\nu\epsilon \log(1/\epsilon))$ approximation bound, improving upon the previous $O(\nu)$ approximation bounds for the gradient-descent based iterative thresholding algorithm [BJK15, SS19]. We furthermore derive a $O(\log(1/\epsilon))$ run-time complexity, improving upon the $O(1/\epsilon^2)$ runtime complexity from [ADKS22].

1 Introduction

There has been extensive study of algorithms to learn the target distribution from a Huber ϵ-Contaminated Model for a Generalized Linear Model (GLM), [DKK+19, ADKS22, LBSS21, OZS20, FB81] as well as for linear regression [BJKK17, MGJK19]. Robust Statistics has been studied extensively [DK23] for problems such as high-dimensional mean estimation [PBR19, CDGS20] and Robust Covariance Estimation [CDGW19, FWZ18]. Recently, there has been an interest in solving robust machine learning problems by gradient descent [PSBR18, DKK+19].

Definition 1 (Strong ϵ-Contamination Model [HR09])**.** *Given a corruption parameter $0 \leq \epsilon < 0.5$, a data matrix, X and labels $\mathbf{y}$. An adversary is allowed to inspect all samples and modify ϵn samples arbitrarily. The algorithm is then given the ϵ-corrupted data matrix X and ϵ-corrupted labels vector $\mathbf{y}$ as training data.*

Current approaches for robust learning across various machine learning tasks often use gradient descent over a robust objective, [LBSS21]. These robust objectives tend to not be convex and therefore are difficult to obtain strong approximation bounds for general classes of models.

We will now give one of the first theoretical results proving the effectiveness of Iterative Thresholding in Learning Problems.

Theorem 2 (Theorem 5 in [BJK15])**.** *Let X be a sub-Gaussian data matrix, and $\mathbf{y} = X^\top \mathbf{w}^* + \mathbf{b}$ where $\mathbf{b}$ represents the corruption. Then there exists a gradient-descent algorithm such that $\|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2 \leq \varepsilon$ after $t = O\left(\log\left(\frac{1}{\sqrt{n}}\frac{\|\mathbf{b}\|_2}{\varepsilon}\right)\right)$ iterations.*

The aforementioned theorem has a log-dependence on $\|\mathbf{b}\|$ and is in the *realizable* setting, i.e. no variance of the optimal estimator. More recently, Awasthi et al. [ADKS22] studied the iterative trimmed maximum likelihood estimator. In their algorithm, at each step they find $\mathbf{w}^*$ which minimizes the elements in the trimmed set. We will give their formal theorem result.

Theorem 3 (Theorem 4.2 in [ADKS22])**.** *Let $\mathrm{P} = \{(\mathbf{x}_i, y_i)\}_{i=1}^n$ be the data generated by a Gaussian regression model defined as $y_i = \mathbf{w}^* \cdot \mathbf{x}_i + \eta_i$ where $\eta_i \sim \mathcal{N}(0, \varrho^2)$ and $\mathbf{x}_i$ are sampled from a sub-Gaussian Distribution with second-moment matrix I. Suppose the dataset has ϵ-fraction of label corruption and $n = \Omega\left(\frac{d+\log(1/\delta)}{\epsilon^2}\right)$. Then there exists an algorithm that returns $\widehat{\mathbf{w}}$ such that with probability $1-\delta$,*

$$\|\widehat{\mathbf{w}} - \mathbf{w}^*\|_2 = O(\varrho\epsilon\log(1/\epsilon))$$

Our first result recovers this result for vectorized-regression. We will now give our results for the Kernelized GLM problem.

1.1 Contributions

Our main contribution is the approximation bounds for Subquantile Minimization for various non-linear learning problems from the iterative thresholding algorithm given in Algorithm 2. Our proof techniques extend [BJK15, SS19, ADKS22] as we suppose the adversary also corrupts the covariates. To our knowledge, we are also the first to theoretically study iterative thresholding for non-linear learning algorithms beyond the generalized linear model.

Comparing to [BJK15], our bound does not depend on the norm of the noise, which can be made arbitrarily large by the adversary. We extend upon [SS19] by extending significantly past a linear convergence guarantee by also showing convergence to a near minimax-optimal error. We also offer a significant run-time improvement over the study in [ADKS22] from $O(1/\varepsilon^2)$ to $O(\log(1/\varepsilon^2))$.

2 Preliminaries

Notation. We denote $[T]$ as the set $\{1, 2, \ldots, T\}$. We say $y = O(x)$ if there exists x_0 s.t. for all $x \geq x_0$ there exists C s.t. $y \leq Cx$. We say $y = \Omega(x)$ if there exists x_0 s.t. for all $x \geq x_0$ there exists C s.t. $y \geq Cx$. We denote $a \vee b \triangleq \max(a, b)$ and $a \wedge b \triangleq \min(a, b)$. We define $\mathbb{S}^{d-1}$ as the sphere $\{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| = 1\}$.

Table 1: Summary of related work on Iterative Thresholding Algorithms for Learning in the Huber-ϵ Contamination Model and our contributions. We assume the good data is sampled from a sub-Gaussian distribution with second-moment matrix, Σ, and sub-Gaussian norm C_K and dimension d. We assume the variance of the optimal estimator is ϱ.

Reference	Approximation	Runtime	Algorithm
[BJK15]	$O(\varrho)$	$O\left(N^2 d \log\left(\frac{1}{\sqrt{n}}\frac{\|\mathbf{b}\|_2}{\epsilon}\right)\right)$	Full Solve
[SS19]	$O(\varrho)$	$O\left(Nd \log\left(\frac{\|\mathbf{w}^*\|_2+\varrho^2}{\varrho}\right)\right)$	Gradient Descent
[ADKS22]	$O(\varrho\epsilon\log(1/\epsilon))$	$O\left(\left(Nd^2+d^3\right)\left(\frac{1}{\varrho\epsilon^2}\right)\right)$	Full Descent
Corollary 15	$O(\varrho\epsilon\log(1/\epsilon))$	$O\left(Nd^2 \log\left(\frac{\|\mathbf{w}^*\|}{\varrho\epsilon}\right)\right)$	Gradient Descent

Table 2: Our results and Distributional assumptions for learning different neuron activation functions.

Reference	Approximation	Neuron	Covariate Distribution
Theorem 14	$O(\varrho\epsilon\log(1/\epsilon))$	Linear	Sub-Gaussian
Theorem 16	$O\left(\kappa(\Sigma)\gamma^{-2}\|\sigma\|_{\text{lip}}^2\sqrt{B\epsilon\log(1/\epsilon)}\right)$	Sigmoid	Bounded Sub-Gaussian, $\|\mathbf{x}\| \le B$
Theorem 17	$O\left(\epsilon\sqrt{\varrho\log(1/\epsilon)}\right)$	Leaky-ReLU	Sub-Gaussian
Theorem 19	$O\left(\epsilon\sqrt{\varrho\log(1/\epsilon)}\right)$	ReLU	$L_4 - L_2$ Hypercontractive

We denote the Hadamard product between two vectors of the same size as $\mathbf{x} \circ \mathbf{y}$ such that for any vectors $(\mathbf{x} \circ \mathbf{y})_i = x_i y_i$.

Matrices. For a matrix A, let $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ represent the maximum and minimimum eigenvalues of A, respectively. We use the following matrix norms for a matrix $A \in \mathbb{R}^{m\times n}$,

$$\text{Spectral Norm: } \|A\| = \max_{\mathbf{x}\in\mathbb{S}^{m-1}} \|A\mathbf{x}\| = \sigma_1(A)$$

$$\text{Trace Norm: } \text{Tr}(A) = \sum_{i\in[m\wedge n]} \sigma_i(A)$$

$$\text{Frobenius Norm: } \|A\|_{\text{F}}^2 = \text{Tr}(A^\top A) = \sum_{i\in[m\wedge n]} \sigma_i^2(A)$$

Let $\text{vec} : \mathbb{R}^{n\times k} \to \mathbb{R}^{nk}$ represent the vectorizaton of a matrix to a vector placing its columns one by one into a vector. We then have the useful facts,

Lemma 4. *Suppose $A, B \in \mathbb{R}^{m\times n}$, then*

$$\langle A, B\rangle_{\text{Tr}} = \langle \text{vec}(A), \text{vec}(B)\rangle$$

Let $\otimes : \mathbb{R}^{N\times K} \times \mathbb{R}^{L\times M} \to \mathbb{R}^{NL\times KM}$ represent the Kronecker delta product between two matrices, this gives us the following relation,

Lemma 5. *Suppose A, B, C are conformal matrices, then*

$$\text{vec}(ABC) = (C^\top \otimes A)\text{vec}(B)$$

Probability. We know discuss the probability theory concepts used throughout the paper. We consider the general sub-Gaussian design. The sub-Gaussian design is highly prevalent in the study of robust statistics [JLT20].

Definition 6 (Sub-Gaussian Distribution)**.** *We say a vector $\mathbf{x}$ is sampled from a sub-Gaussian distribution with second-moment matrix Σ and sub-Gaussian norm K, then for any $\mathbf{v} \in \mathbb{S}^{d-1}$,*

$$\Pr_{\mathbf{x}\sim\mathcal{D}}\{|\mathbf{x}\cdot\mathbf{v}| \ge t\} \le 2\exp\left(-\frac{t^2}{2K^2\|\Sigma\|^2}\right)$$

A scalar random variable X is sub-Gaussian with sub-Gaussian norm ν if for all $p \in \mathbb{N}$,

$$\|X\|_{L_p} = (\mathbf{E}|X|^p)^{1/p} \leq \nu\sqrt{p}$$

We often work with the products of sub-Gaussian random variables, which by the following indicate they are sub-Exponential.

Lemma 7 (Lemma 2.7.7 in [Ver20])**.** *Let X, Y be sub-Gaussian random variables, then XY is sub-Exponential, furthermore,*

$$\|XY\|_{\psi_1} \leq \|X\|_{\psi_2}\|Y\|_{\psi_2}$$

The sub-exponential norm of $X \in \mathbb{R}$ is given by the following,

$$\|X\|_{\psi_1} = \inf\{t \geq 0 : \mathbf{E}\exp(|X|/t) \leq 2\}$$

To give probabilistic bounds on the concentration of sub-Exponential random variables, we often utilize Bernstein's Theorem.

Lemma 8 (Proposition 5.16 in [Ver10])**.** *Let $X_1, \ldots, X_N$ be independent centered sub-exponential random variables, and $K = \max_i \|X_i\|_{\psi_1}$. Then for every $\mathbf{a} \in \mathbb{R}^n$ and $t \geq 0$,*

$$\mathbf{Pr}\Big\{\big|\sum_{i\in[N]} a_i X_i\big| \geq t\Big\} \leq 2\exp\Big[-c\Big(\frac{t^2}{K^2\|\mathbf{a}\|_2^2} \wedge \frac{t}{K\|\mathbf{a}\|_\infty}\Big)\Big]$$

From elementary algebraic manipulations, we are able to achieve the following upper bound with high probability.

Lemma 9. *Let $X_1, \ldots, X_N$ be independent centered sub-exponential random variables, and $K = \max_i \|X_i\|_{\psi_1}$. Then for every $\mathbf{a} \in \mathbb{R}^n$ and $\delta \in (0,1)$, with probability exceeding $1-\delta$,*

$$\big|\sum_{i\in[N]} a_i X_i\big| \leq \left(\frac{1}{c}\cdot K^2\|\mathbf{a}\|_2^2 \log(2/\delta)\right)^{1/2} \vee \left(\frac{1}{c}\cdot K\|\mathbf{a}\|_\infty \log(2/\delta)\right)$$

2.1 Related Work

The idea of iterative thresholding algorithms for robust learning tasks dates back to 1806 by Legendre [Leg06]. Iterative thresholding have been studied theoretically and tested empirically in various machine learning domains [HYW+23, MGJK19].

[BJK15] study iterative thresholding for least squares regression / sparse recovery. In particular, one of their contributions is a gradient descent algorithm, Torrent, when the covariates are sampled from a sub-Gaussian distribution. Their approximation bound in Theorem 5 relies on the fact that $\lambda_{\min}(\Sigma) = \lambda_{\max}(\Sigma)$ and with sufficiently large data and sufficiently small ϵ, $\kappa(X) \searrow 1$. Bhatia et al. also study a full solve algorithm, where after each thresholding step and obtaining $(1-\epsilon)N$ samples, they set $\mathbf{w}^{(t)}$ to the minimizer of the squared loss over the $(1-\epsilon)N$ points and refer to this algorithm as Torrent-FC. They study this algorithm in the prescence of both adversarial and intrinsic noise in the optimal estimator. Their analysis in Corollary 11 gives $O(\sigma)$ error when the intrinsic noise is sub-Gaussian with sub-Gaussian norm $O(\sigma^2)$.

[SS19] study iterative thresholding for learning Generalized Linear Models (GLMs). In both the linear and non-linear case, they present results on linear convergence. Their results imply a bound of $O(\sigma)$ in the linear case. They furthermore provide experimental evidence of the success of iterative thresholding when applied to neural networks.

More recently, [ADKS22] studied the iterative trimmed maximum likelihood estimator for General Linear Models. Similar to Torrent-FC, their algorithm solves the MLE problem over the data kept after each thresholding step. They prove the best known bounds for iterative thresholding algorithms in the linear regression case, $O(\sigma\epsilon\log\epsilon^{-1})$. When the good data is sampled from a sub-Gaussian distribution non-identity covariance, they first run a near-linear filtering algorithm from [DHL19] to obtain covariates that are sub-Gaussian with close to identity covariance.

All the previous works described above have breakdown point $O(1)$. This breakdown point is typically a consequence of the poor conditioning of the good covariates remaining after the adversary has removed $N\epsilon$ good points. Hence, typically in the *oblivious* case, i.e. randomized corruption, there exists research [BJKK17, SBRJ19], with breakdown point $\Omega(1)$.

3 Convergence

In this section we give the algorithm for subquantile minimization. We will start with the simple case of vectorized regression as a warm-up to our general proof technique. We then move to the GLM with kernel learning. Finally, we give our results for one-hidden layer neural networks. We will now give the algorithm for Subquantile Minimization with Gradient Descent.

3.1 Activation Functions

We first give the non-linear functions we will be learning.

Property 10. *σ is a continuous, monotically increasing, and differentiable almost everywhere.*

Property 11. *σ is Lipschitz, i.e. $|\sigma(x) - \sigma(y)| \leq \|\sigma\|_{\text{lip}}|x - y|$.*

Property 12. *For any $x \geq 0$, there exists $\gamma > 0$ such that $\inf_{|z| \leq x} \sigma'(z) \geq \gamma > 0$.*

Sigmoid functions such as tanh and sigmoid and the leaky-ReLU function satisfy Properties 10, 11, and 12. Property 12 does not hold for the ReLU function, and therefore we require stronger conditions for our approximation bounds to hold.

3.2 Algorithm

We will first define the thresholding operator to simplify the notation in our formal algorithm.

Definition 13 (Hard Thresholding Operator in [BJK15])**.** *For any vector $\mathbf{v} \in \mathbb{R}^n$, let $\nu_{\mathbf{v}}$ be the permutation that orders elements in ascending order, i.e. $\mathbf{v}_{\nu_{\mathbf{v}}(1)} \leq \mathbf{v}_{\nu_{\mathbf{v}}(2)} \leq \cdots \leq \mathbf{v}_{\nu_{\mathbf{v}}(n)}$. Then for any $k \leq n$, the hard thresholding oeprator is defined as,*

$$\text{HT}(\mathbf{v}; k) = \left\{ i \in [n] : \nu_{\mathbf{v}}^{-1}(i) \leq k \right\} \tag{1}$$

We now give the gradient descent algorithm which we will study for the remainder of this paper.

Algorithm 1 Gradient Descent Iterative Thresholding for Multi-Linear Regression

input: Possibly corrupted $X \in \mathbb{R}^{d \times N}$ with outputs $Y \in \mathbb{R}^{K \times N}$, corruption parameter $\epsilon = O(1)$
output: $\nu\sqrt{BK\epsilon \log(1/\epsilon)}$−Approximate solution $W \in \mathbb{R}^{K \times d}$ to minimize $\|W - W^*\|_{\text{F}}$

1: $W^{(0)} \leftarrow 0$
2: $\eta \leftarrow 0.1\kappa(\Sigma)$
3: $T \leftarrow O\left(\kappa^2(\Sigma) \log\left(\frac{\|W^*\|_{\text{F}}}{\varepsilon}\right)\right)$
4: **for** $t \in [T]$ **do**
5: $\quad \nu_i^{(t)} = \|W\mathbf{x}_i - \mathbf{y}_i\|^2 \quad \forall i \in [N]$
6: $\quad \text{S}^{(t)} \leftarrow \text{HT}(\boldsymbol{\nu}^{(t)}, (1-\epsilon)N)$
7: $\quad W^{(t+1)} \leftarrow W^{(t)} - \eta \nabla \mathcal{R}(W^{(t)}; \text{S}^{(t)})$

return: $W^{(T)}$

Runtime. In each iteration we calculate the ℓ_2 error for N points, in total $O(Nd)$. For the Hard Thresholding step, it suffices to find the $n(1-\epsilon)$-th largest element, we can run a selection algorithm in worst-case time $O(N \log N)$, then partition the data in $O(N)$. The run-time for calculating the gradient and updating $\mathbf{w}^{(t)}$ is dominated by the matrix multiplication in $X_{\text{S}^{(t)}} X_{\text{S}^{(t)}}^\top$ which can be done in $O(Nd^2)$. Then considering the choice of T, we have the algorithm runs in time $O\left(Nd^2 \log\left(\frac{\|\mathbf{w}^*\|_2}{\nu\epsilon}\right)\right)$ to obtain $O(\nu\epsilon \log(1/\epsilon))$ ℓ_2-approximation error.

Algorithm 2 Gradient Descent Iterative Thresholding for Learning a Non-linear Neuron

input: Possibly corrupted $X \in \mathbb{R}^{d\times N}$ with outputs $\mathbf{y} \in \mathbb{R}^N$, activation function ν, corruption parameter $\epsilon = O(1)$, and small constant α.

output: $\nu\sqrt{\epsilon\log(1/\epsilon)}-$Approximate solution $\mathbf{w} \in \mathbb{R}^d$ to minimize $\|\mathbf{w} - \mathbf{w}^*\|_2$.

$\mathbf{w}^{(0)} \sim \mathcal{B}_d(\alpha\|\mathbf{w}^*\|)$
$\eta \leftarrow 0.1\kappa^{-2}(\Sigma)$
$T \leftarrow O\Big(\kappa^2(\Sigma)\log\Big(\frac{\|\mathbf{w}^*\|_2}{\varepsilon}\Big)\Big)$
for $t \in [T]$ **do**
    $\nu_i^{(t)} = (\sigma(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - y_i)^2 \quad \forall i \in [N]$
    $\mathrm{S}^{(t)} \leftarrow \mathrm{HT}(\boldsymbol{\nu}^{(t)}, (1-\epsilon)N)$
    $\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} - \eta\nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)})$

return: $\mathbf{w}^{(T)}$

3.3 Proof Sketch

In this section we will give a general sketch of our proofs, from which all the individual theorems will be based upon. Let $t \in [T]$, we then have,

$$\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\| \leq \|\mathbf{w}^{(t)} - \eta\nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)}) - \mathbf{w}^*\|_2 \tag{2}$$

$$\leq \underbrace{\|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta\nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)} \cap \mathrm{P})\|}_{I} + \underbrace{\|\eta\nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)} \cap \mathrm{Q})\|_2}_{II} \tag{3}$$

We upper bound I through its square.

$$I^2 = \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 - \underbrace{2\eta \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)} \cap \mathrm{P})\rangle}_{I_1} + \underbrace{\eta^2 \cdot \|\nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)} \cap \mathrm{P})\|^2}_{I_2} \tag{4}$$

In this step the finer details of the particular proof will differ, however the structure remains the same. We will prove there exists a constant $c_1 > 0$ such that,

$$I_1 \geq (1-2\epsilon)c_1\|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 - c_3\|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2 \tag{5}$$

where c_3 is a term that is dependent on the variance of the Gaussian noise, one of our contributions is that $c_3 = O(\nu\sqrt{\epsilon\log(1/\epsilon)})$ when N sufficiently large for the activation functions studied in the text. Next, an application of Peter-Paul's inequality gives us,

$$I_1 \geq ((1-2\epsilon)c_1 - c_3c_1)\|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 - c_3c_1^{-1} \tag{6}$$

We next show there exists a positive constant c_2, such that

$$I_2^2 \leq (1-\epsilon)c_2\|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 \tag{7}$$

Then, solving a simple quadratic equation, we have that $\eta^2 C_2 \leq \eta c_1 c_3$ for a $c_3 \in (0,1)$ when we choose $\eta \leq \frac{c_1c_3}{c_2}$ and we are able to eliminate the norm of the gradient squared term. We must now control the corrupted gradient term. The key idea is to note that from the optimality of the sub-quantile set,

$$\sum_{i\in\mathrm{S}^{(t)}\cap\mathrm{Q}} \mathcal{L}(\mathbf{w}^{(t)}; \mathbf{x}_i, y_i) \leq \sum_{i\in\mathrm{P}\setminus\mathrm{S}^{(t)}} \mathcal{L}(\mathbf{w}^{(t)}; \mathbf{x}_i, y_i) \tag{8}$$

and $|\mathrm{S}^{(t)} \cap \mathrm{Q}| = |\mathrm{P} \setminus \mathrm{S}^{(t)}|$. We then prove the existence of a constant c_4 such that,

$$II \leq \epsilon c_4\|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 \tag{9}$$

Then, combining our results, we end up with a linear convergence of the form,

$$\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 \leq \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2(1 - \eta(1-2\epsilon)c_1 + \eta\epsilon c_4) + c_3c_1^{-1} \tag{10}$$

We obtain a bound that is of the form,

$$\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 \leq \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2(1-\lambda) + E \tag{11}$$

Then, we find that

$$\|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2 \leq \|\mathbf{w}^{(0)} - \mathbf{w}^*\|_2(1-\lambda)^t + \sum_{k\in[t]}(1-\lambda)^k E \tag{12}$$

We then find asymptotically, the second term converges to $\lambda^{-1}E$. Then, it suffices to find T such that,

$$\|\mathbf{w}^{(T)} - \mathbf{w}^*\|_2 \leq \lambda^{-1}E \tag{13}$$

We can note that $1-\lambda \leq e^{-\lambda}$, then bounding T, we obtain a $\lambda^{-1}E$ approximation bound when

$$T \geq \lambda^{-1} \cdot \log\left(\frac{\|\mathbf{w}^* - \mathbf{w}^{(0)}\|_2}{\lambda^{-1}E}\right) \tag{14}$$

3.4 Warm-up: Multivariate Linear Regression

We will first present our results for the well-studied problem of linear regression in the Huber-ϵ contamination model. Our results will extend the results in [BJKK17] Theorem 5 and [ADKS22] Lemma A.1 by including covariate corruption without requiring a filtering algorithm, variance in the optimal estimator, and non-identity second-moment matrix of the uncorrupted data. The loss function for the multivariate linear regression problem for $W \in \mathbb{R}^{K\times d}$, $X \in \mathbb{R}^{d\times n}$, and $Y \in \mathbb{R}^{K\times n}$.

$$\mathcal{L}(W; X, Y) = \|WX - Y\|_{\mathrm{F}}^2 \tag{15}$$

Theorem 14. *Let $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^\top \in \mathbb{R}^{d\times N}$ be the data matrix and $Y = [\mathbf{y}_1, \ldots, \mathbf{y}_n] \in \mathbb{R}^{K\times N}$ be the output, such that for $i \in \mathrm{P}$, $\mathbf{x}_i$ are sampled from a sub-Gaussian distribution with sub-Gaussian norm L and second-moment matrix Σ, and for $j \in \mathrm{Q}$, $\mathbf{x}_i$ are given by the adversary where $\|\mathbf{x}_j\| \leq c_3$ for some positive constant. Suppose for $i \in \mathrm{P}$ the output is given as $\mathbf{y}_i = W\mathbf{x}_i + \mathbf{e}_i$ where for $j \in [K]$, $[\mathbf{e}_i]_j \sim \mathcal{N}(0, \nu^2)$. Then after $O\left(\kappa(\Sigma)\log\left(\frac{\|W^*\|_{\mathrm{F}}}{\varepsilon}\right)\right)$ gradient descent iterations, $N \geq \frac{(1/\delta)K\,\mathrm{Tr}(\Sigma)}{1600\epsilon^2\log(1/\epsilon)\lambda_{\max}(\Sigma)}$, and learning rate $\eta = 0.1\lambda_{\max}^{-2}(\Sigma)$, with probability exceeding $1-\delta$, Algorithm 2 returns $W^{(T)}$ such that*

$$\|W^{(T)} - W^*\|_{\mathrm{F}} \leq \varepsilon + O\left(\nu\epsilon\sqrt{Kc_3\log(1/\epsilon)}\right) \tag{16}$$

Proof. The proof is deferred to § A.1. ■

We are able to recover the result of Lemma 4.2 in [ADKS22] when $K = 1$ and the covariates (corrupted and un-corrupted) are sampled from a sub-Gaussian distribution with second-moment matrix I. The full solve algorithm studied in [ADKS22] returns a $O(\nu\epsilon\log(1/\epsilon))$ in time $O\left(\frac{1}{\epsilon^2}\left(Nd^2 + d^3\right)\right)$ and the same algorithm studied in [BJK15], Torrent-FC otains $O(\nu)$ approximation error in run-time $O\left(\log\left(\frac{1}{\sqrt{n}}\frac{\|\mathbf{b}\|}{\nu\epsilon\log(1/\epsilon)}\right)(Nd^2 + d^3)\right)$, with the gradient descent based approach, we are able to improve the runtime to $O\left(\log\left(\frac{\|\mathbf{w}^*\|}{\nu\epsilon\log(1/\epsilon)}\right)Nd^2\right)$ for the same approximation bound. By no longer requiring the full-solve, we are able to remove super-linear relation to d. In comparison to [BJK15], we do not have dependence on the noise vector $\mathbf{b}$, which can have very large norm in relation to the norm of $\mathbf{w}^*$. Our proof is also a significant improvement over the presentation given in Lemma 5 of [SS19] as under the same conditions, we give more than the linear convergence, but we show linear convergence is possible on any second-moment matrix of the good covariates and covariate corruption, and then develop concentration inequality bounds to match the best known result for iteratived trimmed estimators. We will formalize our results into a corollary to give a more representative comparison in the literature.

Corollary 15. *Let $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^\top \in \mathbb{R}^{d\times N}$ be the data matrix and $\mathbf{y} = [y_1, \ldots, y_n] \in \mathbb{R}^N$ be the output, such that for $i \in \mathrm{P}$, $\mathbf{x}_i$ are sampled from a sub-Gaussian distribution with sub-Gaussian norm L and second-moment matrix I. Suppose for $i \in \mathrm{P}$ the output is given as $\mathbf{y}_i = \mathbf{x}_i^\top\mathbf{w}^* + \xi_i$ where $\xi_i \sim \mathcal{N}(0, \nu^2)$. Then after*

$O\left(\log\left(\frac{\|\mathbf{w}^*\|_2}{\varepsilon}\right)\right)$ *gradient descent iterations,* $N \geq \frac{(d/\delta)}{800\epsilon^2}$*, and learning rate* $\eta = 0.1$*, with probability exceeding* $1-\delta$*, Algorithm 2 returns* $\mathbf{w}^{(T)}$ *such that*

$$\|\mathbf{w}^{(T)} - \mathbf{w}^*\|_2 \leq \varepsilon + O\Big(\nu\epsilon\sqrt{Kc_3\log(1/\epsilon)}\Big) \tag{17}$$

Suppose for $i \in Q$*,* $\mathbf{x}_i$ *are sampled from a sub-Gaussian distribution with sub-Gaussian Norm* K *and second-moment matrix* I*. Then,*

$$\|\mathbf{w}^{(T)} - \mathbf{w}^*\|_2 \leq \varepsilon + O(\nu\epsilon\log(1/\epsilon)) \tag{18}$$

The second relation given in Corollary 15 matches the best known bound for robust linear regression with iterative thresholding. The first relation given in Corollary 15 is the extension to handle second-moment matrices which do not have unitary condition number as well as corrupted covariates.

3.5 Learning Sigmoid Neurons

We next study the problem of learning GLMs following the model given in §5 of [SS19]. The error function for the the Kernelized GLM problem is given by the following equation for a single training pair $(\mathbf{x}_i, y_i) \sim \mathcal{D}$ in the kernelized case.

$$\mathcal{L}(\mathbf{w}; \mathbf{x}_i, y_i) = (\sigma(\mathbf{w}\cdot\mathbf{x}_i) - y_i)^2 \tag{19}$$

Theorem 16. *Let* $X = [\mathbf{x}_1, \ldots, \mathbf{x}_N]^\top \in \mathbb{R}^{d\times N}$ *be the data matrix and* $\mathbf{y} = [y_1, \ldots, y_n]$ *be the output, such that for* $i \in \mathrm{P}$*,* $\mathbf{x}_i$ *are sampled from a sub-Gaussian distribution with second-moment matrix* Σ *and sub-Gaussian norm* C_Σ*, and* $y_i = \sigma(\mathbf{w}^*\cdot\mathbf{x}_i) + \xi_i$ *for* ξ_i *sampled from a sub-Gaussian distribution with sub-Gaussian norm* ν*. Suppose the activation function, satisfies Properties 10, 11, and 12. Then after* $O\left(\kappa(\Sigma)\log\left(\frac{\|\mathbf{w}^*\|}{\varepsilon}\right)\right)$ *gradient descent iterations, then with probability exceeding* $1-\delta$*,*

$$\|\mathbf{w}^{(T)} - \mathbf{w}^*\|_2 \leq O\Big(\gamma^{-2}\lambda_{\min}^{-1}(\Sigma)\|\sigma\|_{\mathrm{lip}}^2\sqrt{B\log(1/\epsilon)}\Big) + O\Big(\gamma^{-2}\|\sigma\|_{\mathrm{lip}}^2\kappa(\Sigma)\sqrt{\epsilon\log(1/\epsilon)}\Big) \tag{20}$$

when $N \geq \frac{1}{\lambda_{\min}^2(\Sigma)}\cdot\Big(8C_K\cdot d + \frac{2}{c_K}\cdot\log(2/\delta)\Big)$*.*

Proof. The proof is deferred to § B.1.1. ■

3.6 Learning Leaky-ReLU Neural Networks

We will now consider learning a neuron with the Leaky-ReLU function. We can note that Properties 10, 11, and 12 all hold for the Leaky-ReLU. More conveniently, we have γ in Property 12 is constant over $\mathbb{R}$. In our proof we are able to leverage the fact that the second derivative is zero almost surely.

Theorem 17. *Let* $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^\top \in \mathbb{R}^{N\times d}$ *be the data matrix and* $\mathbf{y} = [y_1 \ldots, y_n]^\top$ *be the output, such that for* $i \in \mathrm{P}$*,* $\mathbf{x}_i \sim \mathcal{P}$ *are sampled from a sub-Gaussian distribution with sub-Gaussian norm* K *and second-moment matrix* Σ*, and* $y_i = \psi(\mathbf{x}^\top\mathbf{w}^*) + \xi_i$ *for* $\xi_i \sim \mathcal{N}(0, \nu^2)$ *where* $\psi(x) = \max\{C_\psi x, x\}$*. Then after* $O\left(C_\psi^{-2}\kappa(\Sigma)\log\left(\frac{\|W^*\|_{\mathrm{F}}}{\varepsilon}\right)\right)$ *gradient descent iterations and* $\epsilon \leq \frac{C_\psi^2\lambda_{\min}(\Sigma)}{\sqrt{32C_Q\lambda_{\max}(\Sigma)}}$*, with probability exceeding* $1-\delta$*, Algorithm 2 with learning rate* $\eta = O(\kappa^{-2}(\Sigma))$ *returns* $\mathbf{w}^{(T)}$ *such that*

$$\|\mathbf{w}^{(T)} - \mathbf{w}^*\|_2 \leq \tag{21}$$

Proof. The proof is deferred to § B.2.1. ■

3.7 Learning ReLU Neural Networks

We will now consider the problem of learning ReLU neural networks. We first give a preliminary result for randomized initialization.

Lemma 18 (Theorem 3.4 in [DLT+18]). *Suppose $\mathbf{w}^{(0)}$ is sampled uniformly from a p-dimensional ball with radius $\alpha\|\mathbf{w}^*\|$ such that $\alpha \leq \sqrt{\frac{1}{2\pi p}}$, then with probability at least $\frac{1}{2} - \alpha\sqrt{\frac{\pi p}{2}}$*

$$\|\mathbf{w}^{(0)} - \mathbf{w}^*\|_2 \leq \sqrt{1-\alpha^2}\|\mathbf{w}^*\|_2 \tag{22}$$

From this result we are able to derive porbabilistic guarantees on the convergence of learning ReLU neuron.

Theorem 19. *Let $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^\top \in \mathbb{R}^{n\times d}$ be the data matrix and $\mathbf{y} = [y_1, \ldots, y_n]^\top$ be the output, such that for $i \in \mathrm{P}$, $\mathbf{x}_i$ are sampled from a sub-Gaussian distribution with sub-Gaussian norm K and second-moment matrix Σ. Suppose for $i \in \mathrm{P}$, the output is given as $y_i = \psi(\mathbf{x}^\top\mathbf{w}^*) + \xi_i$ for $\xi_i \sim \mathcal{N}(0, \nu^2)$ and $\psi = \max\{0, x\}$ is the ReLU function. Then after $O(\Xi))$ gradient descent iterations and $n = \Omega(\Xi)$, then with probability exceeding $1-\delta$, Algorithm 2 with learning rate $\eta = O(\Xi)$ returns $\mathbf{w}^{(T)}$ such that*

$$\|\mathbf{w}^{(T)} - \mathbf{w}^*\|_2 \leq \tag{23}$$

Proof. The proof is deferred to § B.3.1 ■

4 Discussion

In this paper, we study the theoretical convergence properties of iterative thresholding for non-linear learning problems in the Strong ϵ-contamination model. Our warm-up result for linear regression reduces the runtime while achieving the best known approximation for iterative thresholding algorithms. Many papers have experimentally studied the iterative thresholding estimator in large scale neural networks [SS19, HYW+23] and to our knowledge, we are the first paper to make advancements in the theory of iterative thresholding for a general class of activation functions.

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A Proofs for Linear Regression

Notation. We will first give some notational preliminaries. Let $\mathrm{X} = \mathrm{P} \cup \mathrm{Q}$ for $|\mathrm{P}| = (1-\epsilon)N$ and $|\mathrm{Q}| = \epsilon N$ represent the sets such that for $i \in \mathrm{P}$, $(\mathbf{x}_i, y_i)$ is the good data and for $j \in \mathrm{Q}$, $(\mathbf{x}_j, y_j)$ has been given by the adversary. For $t \in [T]$, we denote $\mathrm{S}^{(t)}$ as the Subquantile set at iteration t and represents the points. We decompose $\mathrm{S}^{(t)} = \mathrm{S}^{(t)} \cap \mathrm{P} \cup \mathrm{S}^{(t)} \cap \mathrm{Q} = \mathrm{TP} \cup \mathrm{FP}$ to represent the *True Positives* and *False Positives*. We also decompose $\mathrm{X} \setminus \mathrm{S}^{(t)} = (\mathrm{X} \setminus \mathrm{S}^{(t)}) \cap \mathrm{P} \cup (\mathrm{X} \setminus \mathrm{S}^{(t)}) \cap \mathrm{Q} = \mathrm{FN} \cup \mathrm{TN}$ to represent the *False Negatives* and the *True Negatives*.

A.1 Proof of Theorem 14

Proof. Recall that for any $W \in \mathbb{R}^{K \times d}$, $X \in \mathbb{R}^{d \times N}$, and $Y \in \mathbb{R}^{K \times N}$,

$$\mathcal{L}(W; X, Y) = \|WX - Y\|_{\mathrm{F}}^2 = \mathrm{Tr}(X^\top W^\top W X - X^\top W^\top Y - Y^\top W X + Y^\top Y) \tag{24}$$

$$= \mathrm{Tr}(X^\top W^\top W X) + \mathrm{Tr}(Y^\top Y) - 2\,\mathrm{Tr}(X^\top W^\top Y) \tag{25}$$

Then, from [PP+08] Equations (102) and (119) (where we set $B = I$ and $C = \mathbf{0}$). We have,

$$\nabla \mathcal{L}(W) = 2(WX - Y)X^\top \tag{26}$$

We first show we can obtain Linear Convergence plus a noise term and a consistency term, which we define as the difference between the optimal estimator in expectation and the optimal estimator over a population. We then show the consistency term goes to zero with high probability as $N \to \infty$. Finally, we use the concentration inequalities developed in § C to give a clean approximation bound.

Step 1: Linear Convergence. We will first show iterative thresholding has linear convergence to optimal. Let $\widehat{W} = \arg\min_{W \in \mathbb{R}^{K \times d}} \mathcal{R}(W; \mathrm{P})$ be the minimizer over the good data. Then, we have

$$\|W^{(t+1)} - W^*\|_{\mathrm{F}} = \|W^{(t)} - W^* - \eta \nabla \mathcal{R}(W^{(t)}; \mathrm{S}^{(t)})\|_{\mathrm{F}} \tag{27}$$

$$= \|W^{(t)} - W^* - \eta \nabla \mathcal{R}(W^{(t)}; \mathrm{TP}) + \eta \nabla \mathcal{R}(W^*; \mathrm{P}) - \eta \nabla \mathcal{R}(W^*; \mathrm{P}) + \eta \nabla \mathcal{R}(\widehat{W}; \mathrm{P}) - \eta \nabla \mathcal{R}(W^{(t)}; \mathrm{FP})\|_{\mathrm{F}} \tag{28}$$

$$\leq \underbrace{\|W^{(t)} - W^* - \eta \nabla \mathcal{R}(W^{(t)}; \mathrm{TP}) + \eta \nabla \mathcal{R}(W^*; \mathrm{TP})\|_{\mathrm{F}}}_{I} + \underbrace{\|\eta \nabla \mathcal{R}(W^{(t)}; \mathrm{FP})\|_{\mathrm{F}}}_{II} + \underbrace{\|\eta \nabla \mathcal{R}(W^*; \mathrm{FN})\|_{\mathrm{F}}}_{III}$$

$$+ \underbrace{\|\eta \nabla \mathcal{R}(W^*; \mathrm{P}) - \eta \nabla \mathcal{R}(\widehat{W}; \mathrm{P})\|_{\mathrm{F}}}_{IV} \tag{29}$$

In contrast with our proof sketch in § 3.3, we introduce a consistency estimate in IV. We first will upper bound I through the expansion of its square.

$$\|W^{(t)} - W^* - \eta \nabla \mathcal{R}(W^{(t)}; \mathrm{TP}) + \eta \nabla \mathcal{R}(W^*; \mathrm{TP})\|_{\mathrm{F}}^2 = \|W^{(t)} - W^*\|_{\mathrm{F}}^2$$

$$\underbrace{- 2\eta \cdot \mathrm{Tr}\big((W^{(t)} - W^*)^\top (\nabla \mathcal{R}(W^{(t)}; \mathrm{TP}) - \nabla \mathcal{R}(W^*; \mathrm{TP}))\big)}_{I_1} + \underbrace{\|\eta \nabla \mathcal{R}(W^{(t)}; \mathrm{TP}) - \eta \nabla \mathcal{R}(W^{(t)}; \mathrm{TP})\|_{\mathrm{F}}^2}_{I_2} \tag{30}$$

We first lower bound I_1, we will obtain a lower bound that is less than I_2, allowing us to cancel the I_2 term.

$$I_1 = 2\eta \cdot \mathrm{Tr}\big((W^{(t)} - W^*)^\top (\nabla \mathcal{R}(W^{(t)}; \mathrm{TP}) - \nabla \mathcal{R}(W^*; \mathrm{TP}))\big) \tag{31}$$

$$\stackrel{\text{def}}{=} \frac{4\eta}{(1-\epsilon)N} \cdot \mathrm{Tr}\big((W^{(t)} - W^*)^\top \big((W^{(t)} X_{\mathrm{TP}} - Y_{\mathrm{TP}}) X_{\mathrm{TP}}^\top - E_{\mathrm{TP}} X_{\mathrm{TP}}^\top\big)\big) \tag{32}$$

$$\stackrel{(i)}{=} \frac{4\eta}{(1-\epsilon)N} \cdot \mathrm{Tr}\Big((W^{(t)} - W^*)^\top (W^{(t)} - W^*) X_{\mathrm{TP}} X_{\mathrm{TP}}^\top\Big) \tag{33}$$

In the above, (i) follows from recalling that $Y_{\mathrm{TP}} = W^* X_{\mathrm{TP}} - E_{\mathrm{TP}}$. We then have,

$$I_1 \stackrel{(ii)}{=} \frac{4\eta}{(1-\epsilon)N} \cdot \mathrm{Tr}\Big((W^{(t)} - W^*) X_{\mathrm{TP}} X_{\mathrm{TP}}^\top (W^{(t)} - W^*)^\top\Big) \tag{34}$$

$$\overset{(iii)}{=} \frac{4\eta}{(1-\epsilon)N} \cdot \langle \mathrm{vec}((W^{(t)} - W^*)^\top), \mathrm{vec}(X_{\mathrm{TP}} X_{\mathrm{TP}}^\top (W^{(t)} - W^*)^\top) \rangle \tag{35}$$

$$\overset{(iv)}{=} \frac{4\eta}{(1-\epsilon)N} \cdot \langle \mathrm{vec}((W^{(t)} - W^*)^\top), (I \otimes X_{\mathrm{TP}} X_{\mathrm{TP}}^\top) \mathrm{vec}((W^{(t)} - W^*)^\top) \rangle \tag{36}$$

$$\overset{(v)}{=} \frac{4\eta}{(1-\epsilon)N} \cdot \sum_{k \in [K]} \left\langle \mathbf{w}_k^{(t)} - \mathbf{w}_k^*, X_{\mathrm{TP}} X_{\mathrm{TP}}^\top (\mathbf{w}_k^{(t)} - \mathbf{w}_k^*) \right\rangle \tag{37}$$

$$\overset{(vi)}{\geq} \frac{2\eta}{(1-\epsilon)N} \cdot \sum_{k \in [K]} \left(\lambda_{\min}(X_{\mathrm{TP}} X_{\mathrm{TP}}^\top) \|\mathbf{w}_k^{(t)} - \mathbf{w}_k^*\|_2^2 + \|X_{\mathrm{TP}} X_{\mathrm{TP}}^\top\|_2^{-1} \|X_{\mathrm{TP}} X_{\mathrm{TP}}^\top (\mathbf{w}_k^{(t)} - \mathbf{w}_k^*)\|_2^2 \right) \tag{38}$$

$$= \frac{2\eta}{(1-\epsilon)N} \cdot \lambda_{\min}(X_{\mathrm{TP}} X_{\mathrm{TP}}^\top) \|W^{(t)} - W^*\|_{\mathrm{F}}^2 + \underbrace{\frac{2\eta}{(1-\epsilon)N} \cdot \|X_{\mathrm{TP}} X_{\mathrm{TP}}^\top\|_2^{-1} \|(W^{(t)} - W^*) X_{\mathrm{TP}} X_{\mathrm{TP}}^\top\|_{\mathrm{F}}^2}_{I_{12}} \tag{39}$$

In the above, (ii) follows from the cyclic property of the trace, (iii) follows from the relation given in Lemma 4, (iv) holds from the relation given in Lemma 5, the inequality in (vi) follows from Lemma 30, and in (v) we apply Lemma 5, which gives the following equality,

$$(I \otimes X_{\mathrm{TP}} X_{\mathrm{TP}}^\top) \mathrm{vec}((W^{(t)} - W^*)^\top) = \begin{bmatrix} X_{\mathrm{TP}} X_{\mathrm{TP}}^\top & & \\ & \ddots & \\ & & X_{\mathrm{TP}} X_{\mathrm{TP}}^\top \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 - \mathbf{w}_1^* \\ \vdots \\ \mathbf{w}_K - \mathbf{w}_K^* \end{bmatrix} = \begin{bmatrix} X_{\mathrm{TP}} X_{\mathrm{TP}}^\top (\mathbf{w}_1 - \mathbf{w}_1^*) \\ \vdots \\ X_{\mathrm{TP}} X_{\mathrm{TP}}^\top (\mathbf{w}_K - \mathbf{w}_K^*) \end{bmatrix} \tag{40}$$

We now upper bound the corrupted gradient term.

$$II \overset{\mathrm{def}}{=} \frac{2\eta}{(1-\epsilon)N} \cdot \|(W^{(t)} X_{\mathrm{FP}} - Y_{\mathrm{FP}}) X_{\mathrm{FP}}^\top\|_{\mathrm{F}} \tag{41}$$

$$\overset{(vii)}{\leq} \frac{2\eta}{(1-\epsilon)N} \cdot \|X_{\mathrm{FP}}\|_2 \|W^{(t)} X_{\mathrm{FP}} - Y_{\mathrm{FP}}\|_{\mathrm{F}} \tag{42}$$

$$\overset{(viii)}{\leq} \frac{2\eta}{(1-\epsilon)N} \cdot \|X_{\mathrm{FP}}\|_2 \|W^{(t)} X_{\mathrm{FN}} - Y_{\mathrm{FN}}\|_{\mathrm{F}} \tag{43}$$

$$\overset{(ix)}{\leq} \frac{2\eta}{(1-\epsilon)N} \cdot \|X_{\mathrm{FP}}\|_2 \left(\|W^{(t)} X_{\mathrm{FN}} - W^* X_{\mathrm{FN}}\|_{\mathrm{F}} + \|E_{\mathrm{FN}}\|_{\mathrm{F}} \right) \tag{44}$$

$$\overset{(x)}{\leq} \frac{2\eta}{(1-\epsilon)N} \cdot \|X_{\mathrm{FP}}\|_2 \|X_{\mathrm{FN}}\|_2 \|W^{(t)} - W^*\|_{\mathrm{F}} + \frac{2\eta}{(1-\epsilon)N} \cdot \|X_{\mathrm{FP}}\|_2 \|E_{\mathrm{FN}}\|_{\mathrm{F}} \tag{45}$$

In the above, the equalities in (vii) and (x) from the fact that for any two size compatible matrices, A, B, it holds that $\|AB\|_{\mathrm{F}} \leq \|A\|_{\mathrm{F}} \|B\|_2$, $(viii)$ follows from the optimality of the Subquantile set, and (ix) follows from the sub-additivity of the Frobenius norm. We will now upper bound III.

$$III = \|\eta \nabla \mathcal{R}(W^*; \mathrm{FN})\|_{\mathrm{F}} \overset{\mathrm{def}}{=} \frac{2\eta}{(1-\epsilon)N} \cdot \|(W^* X_{\mathrm{FN}} - Y_{\mathrm{FN}}) X_{\mathrm{FN}}^\top\|_{\mathrm{F}} \leq \frac{2\eta}{(1-\epsilon)N} \cdot \|E_{\mathrm{FN}} X_{\mathrm{FN}}^\top\|_{\mathrm{F}} \tag{46}$$

In the above, we use the fact that for any two size compatible matrices, A, B, it holds that $\|AB\|_{\mathrm{F}} \leq \|A\|_{\mathrm{F}} \|B\|_2$.

Step 2: Consistency. We will now upper bound the consistency estimate, IV.

$$IV = \|\eta \nabla \mathcal{R}(W^*; \mathrm{P}) - \eta \nabla \mathcal{R}(\widehat{W}; \mathrm{P})\|_{\mathrm{F}} \overset{\mathrm{def}}{=} \frac{2\eta}{(1-\epsilon)N} \cdot \|(\widehat{W} - W^*) X_{\mathrm{P}} X_{\mathrm{P}}^\top\|_{\mathrm{F}} \leq \frac{2\eta}{(1-\epsilon)N} \cdot \|E_{\mathrm{P}} X_{\mathrm{P}}^\top\|_{\mathrm{F}} \tag{47}$$

We can then have from Lemma 27,

$$\|E_{\mathrm{P}} X_{\mathrm{P}}^\top\|_{\mathrm{F}}^2 \leq \frac{16}{3} \cdot \sigma \left(\|X_{\mathrm{P}}\|_{\mathrm{F}}^2 \log\left((2/\delta) N^2\right) \right) \leq \frac{32}{3} \cdot \sigma \left(N(1-\epsilon) d \lambda_{\max}(\Sigma) \log\left((2/\delta) N^2\right) \right) \tag{48}$$

We then have with probability exceeding $1 - \delta$,

$$\frac{2\eta}{(1-\epsilon)N} \cdot \|E_{\mathrm{P}} X_{\mathrm{P}}^\top\|_{\mathrm{F}} \leq \eta \cdot \sqrt{\frac{32\sigma^2 d \lambda_{\max}(\Sigma) \log((2/\delta) N^2)}{3(1-\epsilon)N}} \tag{49}$$

We then have from our choice of $\eta = 0.1\lambda_{\max}^{-1}(\Sigma)$, we have $I_2 \leq I_{12}$. We thus obtain from noting that $\sqrt{1-2x} \leq 1-x$ for any $x \leq 1/2$,

$$\|W^{(t+1)} - W^*\|_{\mathrm{F}} \leq \|W^{(t)} - W^*\|_{\mathrm{F}} \left(1 - \frac{2\eta}{(1-\epsilon)N} \cdot \lambda_{\min}(X_{\mathrm{TP}} X_{\mathrm{TP}}^\top) + \frac{2\eta}{(1-\epsilon)N} \cdot \|X_{\mathrm{FP}}\|_2 \|X_{\mathrm{FN}}\|_2\right) + \frac{2\eta}{(1-\epsilon)N} \cdot \|X_{\mathrm{FP}}\|_2 \|E_{\mathrm{FN}}\|_{\mathrm{F}} + \frac{2\eta}{(1-\epsilon)N} \cdot \|E_{\mathrm{FN}} X_{\mathrm{FN}}^\top\|_{\mathrm{F}} + \eta \cdot \sqrt{\frac{32\sigma^2 d\lambda_{\max}(\Sigma)\log((2/\delta)N^2)}{3(1-\epsilon)N}} \tag{50}$$

Step 3: Concentration Bounds. From Proposition 21 and our ℓ_2 bounded corrupted covariate assumption, we obtain with probability exceeding $1-\delta$,

$$\|E_{\mathrm{FN}}\|_{\mathrm{F}} \|X_{\mathrm{FP}}\|_{\mathrm{F}} \leq N\epsilon\sigma\sqrt{30KB\log(1/\epsilon)} \tag{51}$$

From Lemma 27, we have when $N \geq \log(1/\delta)$, with probability at least $1-\delta$,

$$\|E_{\mathrm{FN}} X_{\mathrm{FN}}^\top\|_{\mathrm{F}} \leq \sqrt{6K\log N}\|X_{\mathrm{FN}}\|_2 \tag{52}$$

Then utilizing the result from Lemma 23, we have with probability at least $1-\delta$,

$$\sqrt{6K\log N}\|X_{\mathrm{FN}}\|_2 \leq \sqrt{60K\lambda_{\max}(\Sigma)N\log N \cdot \epsilon\log(1/\epsilon)} \tag{53}$$

Noting that $|\mathrm{S}^{(t)} \cap P| \geq (1-2\epsilon)N$, we have from Lemma 23 for $\epsilon \leq \frac{1}{60} \cdot \kappa^{-1}(\Sigma)$, the minimum eigenvalue satisfies with probability exceeding $1-\delta$,

$$\lambda_{\min}(X_{\mathrm{TP}} X_{\mathrm{TP}}^\top) \geq \frac{N}{4} \cdot \lambda_{\min}(\Sigma) \tag{54}$$

Then when $\epsilon \leq \sqrt{\frac{1}{960B} \cdot \kappa^{-1}(\Sigma)\lambda_{\min}(\Sigma)}$, we have with high probability,

$$\|X_{\mathrm{FP}}\|_2\|X_{\mathrm{FN}}\|_2 - \lambda_{\min}(X_{\mathrm{TP}} X_{\mathrm{TP}}^\top) \leq \frac{\eta}{4(1-\epsilon)} \cdot \lambda_{\min}(\Sigma) \tag{55}$$

Combining our estimates, we have

$$\|W^{(t+1)} - W^*\|_{\mathrm{F}} \leq \|W^{(t)} - W^*\|_{\mathrm{F}} \left(1 - \frac{\eta}{4(1-\epsilon)} \cdot \lambda_{\min}(\Sigma)\right) + \eta \cdot \sigma\epsilon\sqrt{480KB\log(1/\epsilon)} + \frac{\eta}{(1-\epsilon)\sqrt{N}} \cdot \sqrt{240K\log(N)\lambda_{\max}(\Sigma)\epsilon\log(1/\epsilon)} + \frac{\eta}{(1-\epsilon)\sqrt{N}} \cdot \sigma\sqrt{11d\lambda_{\max}(\Sigma)\log(2N^2/\delta)} \tag{56}$$

Then, when $N = \widetilde{\Omega}\left(\frac{d\lambda_{\max}(\Sigma)+\log(1/\delta)}{\epsilon^2 KB}\right)$ we have

$$\|W^{(t+1)} - W^*\|_{\mathrm{F}} \leq \|W^{(t)} - W^*\|_{\mathrm{F}} \left(1 - \frac{\eta}{4(1-\epsilon)} \cdot \lambda_{\min}(\Sigma)\right) + \eta \cdot \sigma\epsilon\sqrt{4320KB\log(1/\epsilon)} \tag{57}$$

Then, solving for the induction with an infinite sum, referring to the proof sketch in § 3.3 and our choice of η, we have after $O\left(\kappa(\Sigma) \cdot \log\left(\frac{\|W^*\|_{\mathrm{F}}}{\varepsilon}\right)\right)$ iterations,

$$\|W^{(T)} - W^*\|_{\mathrm{F}} \leq \varepsilon + \frac{\sigma\epsilon\sqrt{34560KB\log(1/\epsilon)}}{\lambda_{\min}(\Sigma)} \tag{58}$$

Our proof is complete. ■

B Proofs for Learning Nonlinear Neurons

B.1 Sigmoid Neurons

B.1.1 Proof of Theorem 16

Proof. From Algorithm 2, we have the gradient update for learning a Sigmoid neuron for the ℓ_2 loss.

$$\mathbf{w}^{(t+1)} = \mathbf{w}^{(t)} - \frac{2\eta}{(1-\epsilon)N} \cdot \sum_{i \in S^{(t)}} (\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) - y_i) \cdot \sigma'(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) \cdot \mathbf{x}_i \tag{59}$$

Our proof will follow a similar structure to the proof for linear regression. We first show we can obtain linear convergence of $\mathbf{w}^{(t)}$ to $\mathbf{w}^*$ with some error. Then we rigorously analyze the concentration inequalities to give crisp bounds on the upper bound for ϵ and show the noise term is $O(\epsilon \log(1/\epsilon))$.

Step 1: Linear Convergence.

$$\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 = \|\mathbf{w}^{(t)} - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)}) - \mathbf{w}^*\|_2 \tag{60}$$

$$= \|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP}) - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{FP})\|_2 \tag{61}$$

$$\leq \underbrace{\|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP})\|_2}_{I} + \underbrace{\|\eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{FP})\|_2}_{II} \tag{62}$$

We will expand I through its square and give an unnper bound.

$$I^2 = \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 - \underbrace{2\eta \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP}) \rangle}_{I_1} + \underbrace{\eta^2 \cdot \|\nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP})\|_2^2}_{I_2} \tag{63}$$

We now lower bound I_1. Note from the randomized initialization, we have $\|\mathbf{w}^{(0)} - \mathbf{w}^*\| \leq \mathbf{w}^*$. Then, noting that $\|\mathbf{w}^*\| \leq R$, we have by the Cauchy-Schwarz inequality, for any $\mathbf{x} \sim \mathcal{P}$, we have $|\mathbf{w}^{(t)} \cdot \mathbf{x}| \leq 2RB$ almost surely. Here we leverage Property 12 and note there exists a γ s.t. $\sigma'(x) \geq \gamma > 0$ for all $x \in \mathbb{R}$ s.t. $x \leq 2RB$.

$$I_1 = \frac{4\eta}{(1-\epsilon)N} \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \sum_{i \in \mathrm{TP}} (\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) - y_i) \cdot \sigma'(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) \cdot \mathbf{x}_i \rangle \tag{64}$$

$$= \frac{4\eta}{(1-\epsilon)N} \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \sum_{i \in \mathrm{TP}} (\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) - \sigma(\mathbf{w}^* \cdot \mathbf{x}_i) + \xi_i) \cdot \sigma'(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) \cdot \mathbf{x}_i \rangle \tag{65}$$

$$\overset{(i)}{\geq} \underbrace{\frac{4\eta}{(1-\epsilon)N} \cdot \gamma^2 \lambda_{\min}(X_{\mathrm{TP}} X_{\mathrm{TP}}^\top) \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2}_{I_{11}} - \frac{4\eta}{(1-\epsilon)N} \cdot \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2 \Big\| \sum_{i \in \mathrm{TP}} \xi_i \sigma'(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) \cdot \mathbf{x}_i \Big\|_2 \tag{66}$$

In the above, (i) follows from defining the function $h : \mathbb{R}^d \mapsto \mathbb{R}$,

$$\mathbf{w} \mapsto \int_{\mathbb{R}^d} \left(\int \sigma \right) (\mathbf{w} \cdot \mathbf{x}) dP(\mathbf{x}) \tag{67}$$

where $dP(\mathbf{x}) = \mathbf{1}\{\mathbf{x} \in \mathrm{TP}\}$. Then from Property 12, we have that σ' is strictly positive over a compact domain, this implies that $\int \sigma$ is strongly convex over a compact domain. Then we can calculate the Hessian,

$$\nabla^2 h(\mathbf{w}) = \int_{\mathbb{R}^d} \sigma'(\mathbf{w} \cdot \mathbf{x}) \cdot \mathbf{x}\mathbf{x}^\top dP(\mathbf{x}) \succeq \gamma \lambda_{\min}(X_{\mathrm{TP}} X_{\mathrm{TP}}^\top) \cdot I \tag{68}$$

With the strong convexity in hand, we have,

$$\langle \mathbf{w}^{(t)} - \mathbf{w}^*, \sum_{i \in \mathrm{TP}} (\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) - \sigma(\mathbf{w}^* \cdot \mathbf{x}_i) \cdot \mathbf{x}_i \rangle$$

$$= \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \int_0^1 \nabla^2 h(\mathbf{w}^* + \theta(\mathbf{w}^{(t)} - \mathbf{w}^*)) d\theta \cdot (\mathbf{w}^{(t)} - \mathbf{w}^*) \rangle \overset{(ii)}{\geq} \gamma \lambda_{\min}(X_{\mathrm{TP}} X_{\mathrm{TP}}^\top) \cdot \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 \tag{69}$$

In the above, (ii) follows from noting that for any $\theta \in [0,1]$ we have,

$$\|(1-\theta)\mathbf{w}^* + \theta\mathbf{w}^{(t)}\|_2 \leq \theta\|\mathbf{w}^* - \mathbf{w}^{(t)}\|_2 + \|\mathbf{w}^*\|_2 \leq 2\|\mathbf{w}^*\|_2 = 2R$$

We can then use the same γ as previously defined. Then from an application of Peter-Paul's Inequality, we obtain

$$\frac{4\eta}{(1-\epsilon)N} \cdot \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2 \Big\| \sum_{i \in \mathrm{TP}} \xi_i \sigma'(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) \cdot \mathbf{x}_i \Big\|_2 \leq \frac{\eta}{(1-\epsilon)N} \cdot \gamma^2 \lambda_{\min}(X_{\mathrm{TP}} X_{\mathrm{TP}}^\top) \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 + \frac{4\eta}{(1-\epsilon)N} \cdot \gamma^{-2} \lambda_{\min}^{-1}(X_{\mathrm{TP}} X_{\mathrm{TP}}^\top) \Big\| \sum_{i \in \mathrm{TP}} \xi_i \sigma'(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) \cdot \mathbf{x}_i \Big\|_2^2 \quad (70)$$

We next upper bound I_2. We note that σ is $\|\sigma\|_{\mathrm{lip}}$-Lipschitz, then from an application of the triangle inequality, we obtain

$$I_2 \stackrel{\mathrm{def}}{=} \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \Big\| \sum_{i \in \mathrm{TP}} (\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) - \sigma(\mathbf{w}^* \cdot \mathbf{x}_i) + \xi_i) \cdot \sigma'(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) \cdot \mathbf{x}_i \Big\|_2^2 \quad (71)$$

$$\stackrel{(i)}{=} \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \Big\| \sum_{i \in \mathrm{TP}} \mathbf{x}_i \mathbf{x}_i^\top (\mathbf{w} - \mathbf{w}^*) \cdot \sigma'(c_i) \sigma'(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) + \xi_i \cdot \sigma'(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) \cdot \mathbf{x}_i \Big\|_2^2 \quad (72)$$

$$\leq \underbrace{\frac{8\eta^2}{[(1-\epsilon)N]^2} \cdot \|\sigma\|_{\mathrm{lip}}^4 \lambda_{\max}^2(X_{\mathrm{TP}} X_{\mathrm{TP}}^\top) \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2}_{I_{21}} + \frac{8\eta^2}{[(1-\epsilon)N]^2} \cdot \Big\| \sum_{i \in \mathrm{TP}} \xi_i \sigma'(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) \cdot \mathbf{x}_i \Big\|_2^2 \quad (73)$$

In the above, (i) follows from noting there exists a constant $c_i \in [\mathbf{w}^{(t)} \cdot \mathbf{x}_i, \mathbf{w}^* \cdot \mathbf{x}_i]$ such that

$$\sigma'(c_i)(\mathbf{w}^{(t)} - \mathbf{w}^*) \cdot \mathbf{x}_i = \sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) - \sigma(\mathbf{w}^* \cdot \mathbf{x}_i)$$

from the Mean-Value Theorem. Then, from choosing $\eta \leq \frac{\gamma^2(1-\epsilon)N\lambda_{\min}(X_{\mathrm{TP}}X_{\mathrm{TP}}^\top)}{4\|\sigma\|_{\mathrm{lip}}^4 \lambda_{\max}^2(X_{\mathrm{TP}}X_{\mathrm{TP}}^\top)}$. We have $I_{21} \leq 0.5 I_{11}$. We now bound the corrupted gradient term.

$$II^2 = \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \Big\| \sum_{i \in \mathrm{FP}} (\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) - y_i) \cdot \sigma'(\mathbf{w} \cdot \mathbf{x}_i) \cdot \mathbf{x}_i \Big\|_2^2 \quad (74)$$

$$\stackrel{(ii)}{\leq} \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \|\sigma\|_{\mathrm{lip}}^2 \|X_{\mathrm{FP}} X_{\mathrm{FP}}^\top\|_2 \sum_{i \in \mathrm{FP}} (\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) - y_i)^2 \quad (75)$$

$$\stackrel{(iii)}{\leq} \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \|\sigma\|_{\mathrm{lip}}^2 \|X_{\mathrm{FP}} X_{\mathrm{FP}}^\top\|_2 \sum_{i \in \mathrm{FN}} (\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) - y_i)^2 \quad (76)$$

$$= \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \|\sigma\|_{\mathrm{lip}}^2 \|X_{\mathrm{FP}} X_{\mathrm{FP}}^\top\|_2 \sum_{i \in \mathrm{FN}} (\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) - \sigma(\mathbf{w}^* \cdot \mathbf{x}_i) + \xi_i)^2 \quad (77)$$

$$\stackrel{(iv)}{\leq} \frac{8\eta^2}{[(1-\epsilon)N]^2} \cdot \|\sigma\|_{\mathrm{lip}}^2 \|X_{\mathrm{FP}} X_{\mathrm{FP}}^\top\|_2 \Big(\|\sigma\|_{\mathrm{lip}}^2 \cdot \|X_{\mathrm{FN}} X_{\mathrm{FN}}^\top\|_2 \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 + \|\boldsymbol{\xi}_{\mathrm{FN}}\|_2^2 \Big) \quad (78)$$

In the above, (ii) follows from Lemma 29, (iii) follows from the optimality of the Hard-Thresholding operator, (iv) follows from noting σ is Lipschitz and the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$ for any $a, b \in \mathbb{R}$. Concluding the step, we have,

$$\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 \leq \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2 \left(1 - \frac{\eta}{2(1-\epsilon)N} \cdot \gamma^2 \lambda_{\min}(X_{\mathrm{TP}} X_{\mathrm{TP}}^\top) + \frac{\sqrt{8}\eta}{(1-\epsilon)N} \cdot \|\sigma\|_{\mathrm{lip}}^2 \|X_{\mathrm{FP}}\|_2 \|X_{\mathrm{FN}}\|_2 \right) + \left(\frac{\sqrt{8}\eta}{(1-\epsilon)N} + \frac{2\sqrt{\eta}}{\sqrt{(1-\epsilon)N}} \cdot \lambda_{\min}^{-1/2}(X_{\mathrm{TP}} X_{\mathrm{TP}}^\top) \right) \Big\| \sum_{i \in \mathrm{TP}} \xi_i \sigma'(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) \cdot \mathbf{x}_i \Big\|_2 + \frac{\sqrt{8}\eta}{(1-\epsilon)N} \cdot \|X_{\mathrm{FP}}\|_2 \|\boldsymbol{\xi}_{\mathrm{FN}}\|_2 \quad (79)$$

Step 2: Concentration Bounds. From Lemma 24, we have with probability at least $1-\delta$,

$$\|\sum_{i\in \mathrm{TP}} \xi_i \sigma'(\mathbf{w}^{(t)}\cdot \mathbf{x}_i)\cdot \mathbf{x}_i\|_2 \leq NC_\Sigma \|\sigma\|_{\mathrm{lip}}^2 C_\varrho \left(\frac{2d}{N}\log(12) + \frac{2Rd}{N}\log(12) + \frac{2}{N}\log(1/\delta) + 6\epsilon\log(1/\epsilon)\right)^{1/2} \tag{80}$$

Recall that $\|X_{\mathrm{FP}}\|_2 \leq \sqrt{\epsilon NB}$. From Lemma 23, with probability at least $1-\delta$, we have

$$\|X_{\mathrm{FN}}\|_2 \|X_{\mathrm{FP}}\|_2 \leq N\epsilon \cdot \sqrt{10B\log(1/\epsilon)} \tag{81}$$

From the second relation in Lemma 23, we have when $N = \Omega\Big(\frac{d+\log(1/\delta)}{\epsilon}\Big)$, with probability exceeding $1-\delta$, the minimimum eigenvalue satisfies,

$$\lambda_{\min}(X_{\mathrm{TP}}X_{\mathrm{TP}}^\top) \geq \frac{N}{4}\lambda_{\min}(\Sigma) \tag{82}$$

We then find for

$$\epsilon^2 \leq \frac{\gamma^2}{\|\sigma\|_{\mathrm{lip}}^2}\frac{\lambda_{\min}(\Sigma)}{\sqrt{B\lambda_{\max}(\Sigma)}} \tag{83}$$

and probability exceeding $1-\delta$,

$$(\gamma^2/2)\lambda_{\min}(X_{\mathrm{TP}}X_{\mathrm{TP}}^\top) - \sqrt{8}\|\sigma\|_{\mathrm{lip}}^2\|X_{\mathrm{FP}}\|_2\|X_{\mathrm{FN}}\|_2 \geq \frac{N\gamma^2}{16}\lambda_{\min}(\Sigma) \tag{84}$$

Combining the ℓ_2 boundedness of the corrupted covariates and Proposition 21, we have with probability exceeding $1-\delta$,

$$\|X_{\mathrm{FP}}\|_2\|\boldsymbol{\xi}_{\mathrm{FN}}\|_2 \leq N\epsilon\varrho\cdot\sqrt{30B\log(1/\epsilon)} \tag{85}$$

Then combining the results with our choice of η, we obtain,

$$\begin{aligned}\|\mathbf{w}^{(t+1)}-\mathbf{w}^*\|_2 &\leq \|\mathbf{w}^{(t)}-\mathbf{w}^*\|_2\left(1-\frac{\eta}{16(1-\epsilon)N}\cdot\gamma^2\lambda_{\min}(\Sigma)\right) + \eta\epsilon\|\sigma\|_{\mathrm{lip}}^2\sqrt{80B\log(1/\epsilon)} \\ &+ \left(\sqrt{8}\eta + 2\sqrt{\eta\lambda_{\min}(\Sigma)}\right)C_\Sigma\|\sigma\|_{\mathrm{lip}}^2 C_\varrho\left(\frac{2d}{N}\log(12)+\frac{2Rd}{N}\log(12)+\frac{2}{N}\log(1/\delta)+6\epsilon\log(1/\epsilon)\right)^{1/2}\end{aligned} \tag{86}$$

In the above, the final inequality holds when $N \geq \frac{Rd\log 12 + 2d\log 12 + \log(1/\delta)}{6\epsilon\log(1/\epsilon)} = \Omega\Big(\frac{d+\log(1/\delta)}{\epsilon}\Big)$. Then, following from our sketch in § 3.3, we have

$$\|\mathbf{w}^{(t+1)}-\mathbf{w}^*\|_2 \leq \varepsilon + O\Big(\gamma^{-2}\lambda_{\min}^{-1}(\Sigma)\|\sigma\|_{\mathrm{lip}}^2\sqrt{B\log(1/\epsilon)}\Big) + O\Big(\gamma^{-2}\|\sigma\|_{\mathrm{lip}}^2\kappa(\Sigma)\sqrt{\epsilon\log(1/\epsilon)}\Big) \tag{87}$$

Our proof is complete. ■

B.2 Leaky-ReLU Neuron

B.2.1 Proof of Theorem 17

Proof. We will decompose the gradient into the good component and corrupted component. The first part of our proof will show that $\mathbf{w}$ moves in the direction of $\mathbf{w}^*$, then in the second part of the proof we will show the affect of the corrupted gradient. Finally, we combine step 1 and step 2 to show that there exists sufficiently small ϵ such that we can get linear convergence with a small additive error term.
Step 1: Upper bounding the ℓ_2 norm distance between $\mathbf{w}^{(t+1)}$ and $\mathbf{w}^*$. We have from Algorithm 2,

$$\|\mathbf{w}^{(t+1)}-\mathbf{w}^*\|_2 = \|\mathbf{w}^{(t)}-\mathbf{w}^*-\eta\nabla\mathcal{R}(\mathbf{w}^{(t)};\mathrm{S}^{(t)})\|_2 \tag{88}$$

$$= \|\mathbf{w}^{(t)}-\mathbf{w}^*-\eta\nabla\mathcal{R}(\mathbf{w}^{(t)};\mathrm{TP})+\eta\nabla\mathcal{R}(\mathbf{w}^*;\mathrm{P})-\eta\nabla\mathcal{R}(\mathbf{w}^*;\mathrm{P})-\nabla\mathcal{R}(\mathbf{w}^{(t)};\mathrm{FP})\|_2 \tag{89}$$

$$\leq \underbrace{\|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta\nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP}) + \eta\nabla\mathcal{R}(\mathbf{w}^*; \mathrm{TP})\|_2}_{I} + \underbrace{\|\eta\nabla\mathcal{R}(\mathbf{w}^*; \mathrm{TP})\|_2}_{II} + \underbrace{\|\eta\nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{FP})\|_2}_{III} \tag{90}$$

We will first upper bound the I_1 by an expansion of its square.

$$I^2 = \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 - \underbrace{2\eta \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP})\rangle}_{I_1} + \underbrace{\eta^2 \cdot \|\nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP}) - \nabla\mathcal{R}(\mathbf{w}^*; \mathrm{TP})\|^2}_{I_2} \tag{91}$$

In the above, the a.s. relation follows from Property 10. We will first lower bound I_1. We will first bound the spectrum of $\nabla^2\mathcal{L}(\mathbf{w}; \mathrm{TP})$ for any $\mathbf{w} \in \mathbb{R}^d$.

$$\nabla^2\mathcal{L}(\mathbf{w}; \mathrm{TP}) = 2 \cdot \sum_{i \in \mathrm{TP}} (\sigma(\mathbf{w} \cdot \mathbf{x}_i) - y_i) \cdot \sigma''(\mathbf{w} \cdot \mathbf{x}_i) \cdot \mathbf{x}_i\mathbf{x}_i^\top + 2 \cdot \sum_{i \in \mathrm{TP}} [\sigma'(\mathbf{w} \cdot \mathbf{x}_i)]^2 \cdot \mathbf{x}_i\mathbf{x}_i^\top \tag{92}$$

Then, from noting that the second derivative of Leaky-ReLU is non-zero at one point, we have

$$\nabla^2\mathcal{L}(\mathbf{w}; \mathrm{TP}) \overset{\text{a.s.}}{=} 2 \cdot \sum_{i \in \mathrm{TP}} [\sigma'(\mathbf{w} \cdot \mathbf{x}_i)]^2 \cdot \mathbf{x}_i\mathbf{x}_i^\top \tag{93}$$

We then obtain for any $\mathbf{w} \in \mathbb{R}^d$, almost surely,

$$2 \cdot \gamma^2 \lambda_{\min}(X_{\mathrm{TP}}X_{\mathrm{TP}}^\top) \cdot I \preceq \nabla^2\mathcal{L}(\mathbf{w}; \mathrm{TP}) \preceq 2 \cdot \|\sigma\|_{\mathrm{lip}}^2 \lambda_{\max}(X_{\mathrm{TP}}X_{\mathrm{TP}}^\top) \cdot I \tag{94}$$

We can now lower bound the second term of Equation (**??**). We have from the convexity of the Leaky-ReLU,

$$2\eta \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP}) - \nabla\mathcal{R}(\mathbf{w}^*; \mathrm{TP})\rangle \tag{95}$$

$$= \frac{4\eta}{(1-\epsilon)N} \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \int_0^1 \nabla^2\mathcal{R}(\mathbf{w}^* + \theta(\mathbf{w}^* - \mathbf{w}^{(t)}); \mathrm{TP}) d\theta \cdot (\mathbf{w}^{(t)} - \mathbf{w}^*)\rangle \tag{96}$$

$$\overset{(94)}{\geq} \frac{4\eta}{(1-\epsilon)N} \cdot C_{\lfloor\sigma\rfloor}^2 \lambda_{\min}(X_{\mathrm{TP}}X_{\mathrm{TP}}^\top) \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 \tag{97}$$

We now will upper bound I_2 with a similar argument.

$$\eta^2 \cdot \|\nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP}) - \nabla\mathcal{R}(\mathbf{w}^*; \mathrm{TP})\|_2^2 \tag{98}$$

$$= \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \Big\| \int_0^1 \nabla^2\mathcal{R}(\mathbf{w}^* + \theta(\mathbf{w}^* - \mathbf{w}^{(t)}); \mathrm{TP}) d\theta \cdot (\mathbf{w}^{(t)} - \mathbf{w}^*) \Big\|_2^2 \tag{99}$$

$$\overset{(94)}{\leq} \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \|\sigma\|_{\mathrm{lip}}^2 \lambda_{\max}^2(X_{\mathrm{TP}}X_{\mathrm{TP}}^\top) \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 \tag{100}$$

where (ii) follows from the $\|\sigma\|_{\mathrm{lip}}$-Lipschitzness of σ given in Property 11. We then observe that $I_{21} \leq 0.5 I_{22}$ when $\eta \leq \frac{C_\sigma^2(1-\epsilon)N\lambda_{\min}(X_{\mathrm{TP}}X_{\mathrm{TP}}^\top)}{2\lambda_{\max}^2(X_{\mathrm{TP}}X_{\mathrm{TP}}^\top)}$.

Step 2: Upper bounding the corrupted gradient. We now upper bound the corrupted gradient term.

$$III^2 = \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \Big\| \sum_{i \in \mathrm{FP}} (\sigma(\mathbf{w} \cdot \mathbf{x}_i) - \sigma(\mathbf{w}^* \cdot \mathbf{x}_i)) \cdot \sigma'(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) \cdot \mathbf{x}_i \Big\|_2^2 \tag{101}$$

$$\leq \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \|\sigma\|_{\mathrm{lip}}^2 \|X_{\mathrm{FP}}X_{\mathrm{FP}}^\top\|_2 \sum_{i \in \mathrm{FP}} \big(\sigma(\mathbf{w} \cdot \mathbf{x}_i) - \sigma(\mathbf{w}^* \cdot \mathbf{x}_i)\big)^2 \tag{102}$$

$$\leq \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \|\sigma\|_{\mathrm{lip}}^2 \|X_{\mathrm{FP}}X_{\mathrm{FP}}^\top\|_2 \sum_{i \in \mathrm{FN}} \big(\sigma(\mathbf{w} \cdot \mathbf{x}_i) - \sigma(\mathbf{w}^* \cdot \mathbf{x}_i)\big)^2 \tag{103}$$

$$\leq \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \|\sigma\|_{\mathrm{lip}}^4 \|X_{\mathrm{FP}}X_{\mathrm{FP}}^\top\|_2 \|X_{\mathrm{FN}}X_{\mathrm{FN}}^\top\|_2 \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 \tag{104}$$

In the above, the first inequality follows from Lemma 29, the second inequality follows from the optimality of the Subquantile set, the final inequality follows from the C_2-Lipschitzness of σ. Then from Lemma 24, we have with probability at least $1-\delta$,

$$\Big\| \sum_{i \in \mathrm{TP}} \xi_i \sigma'(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) \cdot \mathbf{x}_i \Big\|_2 \leq 9NC_\Sigma C_\varrho \left(\frac{2d}{N} \log 12 + \frac{2Rd}{N} \log 12 + \frac{2}{N} \log(1/\delta) + 6\epsilon \log(1/\epsilon) \right)^{1/2} \tag{105}$$

We now combine Steps 1 and 2 to give the linear convergence result. Noting that $\sqrt{1-2x} \leq 1-x$ when $x \leq 1/2$, we have

$$\begin{aligned} \|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 \leq \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2 \left(1 - \frac{2C_\sigma^2 \eta}{(1-\epsilon)N} \cdot \lambda_{\min}(X_{\mathrm{TP}} X_{\mathrm{TP}}^\top) + \frac{2\eta}{(1-\epsilon)N} \cdot \|X_{\mathrm{FP}}\|_2 \|X_{\mathrm{FN}}\|_2 \right) \\ + \frac{15 C_\Sigma C_\varrho d \log 5}{\sqrt{(1-\epsilon)N}} + \frac{15 C_\Sigma C_\varrho \log(1/\delta)}{\sqrt{(1-\epsilon)N}} + 16 C_\Sigma C_\varrho \sqrt{\epsilon \log(1/\epsilon)} \end{aligned} \tag{106}$$

Step 3: Concentration Bounds. We will give the relevant probabilistic bounds for the random variables in Steps 1 and 2. From Lemma 23, we have $\|X_{\mathrm{FN}}\|_2 \|X_{\mathrm{FP}}\|_2 \leq \epsilon \sqrt{\lambda_{\max}(\Sigma) \cdot 10BN \log(1/\epsilon)}$ with probability at least $1-\delta$ when $N \geq \frac{2}{\epsilon} \cdot \left(dC_\Sigma^2 + \frac{\log(2/\delta)}{c_K} \right)$ when $\epsilon \leq \frac{1}{60} \cdot \kappa^{-1}(\Sigma)$. From the same Lemma and under the same data conditions we have $\lambda_{\min}(X_{\mathrm{TP}} X_{\mathrm{TP}}^\top) \geq \frac{1}{4} \cdot \lambda_{\min}(\Sigma)$. Then for $\epsilon \leq \frac{C_\sigma^2 \lambda_{\min}(\Sigma)}{\sqrt{32B\lambda_{\max}(\Sigma)}}$, we have $\|X_{\mathrm{FP}}\|_2 \|X_{\mathrm{FN}}\|_2 \leq \frac{1}{2} \cdot \lambda_{\min}(\Sigma)$. We then have, after $O\left(\kappa^2(\Sigma) \log\left(\frac{\|\mathbf{w}^*\|_2 + 10d}{\varepsilon} \right) \right)$ iterations with high probability,

$$\begin{aligned} \|\mathbf{w}^{(T)} - \mathbf{w}^*\|_2 &\leq \varepsilon + \frac{1}{N} \cdot 144 C_\sigma^{-2} \kappa^2(\Sigma) K C_\varrho d \log 5 + 432 \kappa^2(\Sigma) C_\sigma^{-1} K C_\varrho d\epsilon \log(1/\epsilon) && (107) \\ &= O\big(\kappa^2(\Sigma) C_\sigma^{-2} K C_\varrho d\epsilon \log(1/\epsilon) \big) && (108) \end{aligned}$$

In the final inequality above, we set $\varepsilon = O\big(\kappa^2(\Sigma) C_\sigma^{-1} K C_\varrho d\epsilon \log(1/\epsilon) \big)$ for $N \geq \epsilon^{-2} C_\sigma^{-2} \kappa^2(\Sigma) \log 5$. Our proof is complete. ■

B.3 ReLU Neuron

In this section, we consider functions such as the ReLU. Our high-level analysis will be similar to the previous sub-sections however the details are considerably different and require stronger conditions we can guarantee by randomness.

B.3.1 Proof of Theorem 19

Proof. We will now begin our standard analysis.

$$\begin{aligned} \|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 &= \|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta \nabla \mathcal{R}(\mathbf{w}; \mathrm{S}^{(t)})\|_2 && (109) \\ &= \|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP}) - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{FP})\|_2 && (110) \\ &\leq \underbrace{\|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP})\|_2}_{I} + \underbrace{\|\eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{FP})\|_2}_{II} && (111) \end{aligned}$$

We will now upper bound I through its square in accordance with our proof sketch,

$$I^2 = \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 - \underbrace{2\eta \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP}) \rangle}_{I_1} + \underbrace{\eta^2 \cdot \|\nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP})\|_2^2}_{I_2} \tag{112}$$

We will first lower bound I_1. We will first adopt the notation from [ZYWG19], let $\Sigma_{\mathrm{TP}}(\mathbf{w}, \hat{\mathbf{w}}) = X_{\mathrm{TP}}^\top X_{\mathrm{TP}}^\top \cdot \mathbf{1}\{X_{\mathrm{TP}}^\top \mathbf{w} \geq \mathbf{0}\} \cdot \mathbf{1}\{X_{\mathrm{TP}}^\top \hat{\mathbf{w}} \geq \mathbf{0}\}$, it then follows

$$I_1 \stackrel{\mathrm{def}}{=} \frac{4\eta}{(1-\epsilon)N} \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \sum_{i \in \mathrm{TP}} (\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) - y_i) \cdot \mathbf{x}_i \cdot \mathbf{1}\{\mathbf{w}^{(t)} \cdot \mathbf{x}_i \geq 0\} \rangle \tag{113}$$

$$= \frac{4\eta}{(1-\epsilon)N} \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \Sigma_{\mathrm{TP}}(\mathbf{w}^{(t)}, \mathbf{w}^{(t)})\mathbf{w}^{(t)} - \Sigma_{\mathrm{TP}}(\mathbf{w}^{(t)}, \mathbf{w}^*)\mathbf{w}^* \rangle \tag{114}$$

$$= \frac{4\eta}{(1-\epsilon)N} \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \Sigma_{\mathrm{TP}}(\mathbf{w}^{(t)}, \mathbf{w}^*)(\mathbf{w}^{(t)} - \mathbf{w}^*) + \Sigma_{\mathrm{TP}}(\mathbf{w}^{(t)}, -\mathbf{w}^*)\mathbf{w}^{(t)} \rangle \tag{115}$$

$$\geq \frac{4\eta}{(1-\epsilon)N} \cdot \lambda_{\min}(\Sigma_{\mathrm{TP}}(\mathbf{w}^{(t)}, \mathbf{w}^*)) \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 \tag{116}$$

In the above, the final inequality holds from the following relation,

$$\begin{aligned} \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \Sigma_{\mathrm{TP}}(\mathbf{w}^{(t)}, -\mathbf{w}^*)\mathbf{w}^{(t)} \rangle &= \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \sum_{i \in \mathrm{TP}} \mathbf{x}_i \mathbf{w}^{(t)} \cdot \mathbf{x}_i \cdot \mathbf{1}\{\mathbf{w}^{(t)} \cdot \mathbf{x}_i \geq 0\} \cdot \mathbf{1}\{\mathbf{w}^* \cdot \mathbf{x}_i \leq 0\} \rangle \\ &= \sum_{i \in \mathrm{TP}} (\mathbf{w}^{(t)} \cdot \mathbf{x}_i - \mathbf{w}^* \cdot \mathbf{x}_i)(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) \cdot \mathbf{1}\{\mathbf{w}^{(t)} \cdot \mathbf{x}_i \geq 0\} \cdot \mathbf{1}\{\mathbf{w}^* \cdot \mathbf{x}_i \leq 0\} \geq 0 \end{aligned} \tag{117}$$

In the above, in the final relation we can note that when the indicators are positive, it must follow that both $\mathbf{w}^{(t)} \cdot \mathbf{x}_i$ is positive and $\mathbf{w}^{(t)} \cdot \mathbf{x}_i \geq \mathbf{w}^* \cdot \mathbf{x}_i$ as $\mathbf{w}^* \cdot \mathbf{x}_i \leq 0$. We have from Weyl's Inequality,

$$\lambda_{\min}\Big(\Sigma_{\mathrm{TP}}(\mathbf{w}^{(t)}, \mathbf{w}^*)\Big) \geq \lambda_{\min}\Big(\mathbf{E}\Big[\Sigma_{\mathrm{TP}}(\mathbf{w}^{(t)}, \mathbf{w}^*)\Big]\Big) - \|\Sigma_{\mathrm{TP}}(\mathbf{w}^{(t)}, \mathbf{w}^*) - \mathbf{E}\Big[\Sigma_{\mathrm{TP}}(\mathbf{w}^{(t)}, \mathbf{w}^*)\Big]\|_2 \tag{118}$$

Let $\Omega = \big\{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}^\top \mathbf{w}^{(t)} \geq 0, \mathbf{x}^\top \mathbf{w}^* \geq 0\big\}$, then

$$\mathbf{E}\Big[\Sigma_{\mathrm{TP}}(\mathbf{w}^{(t)}, \mathbf{w}^*)\Big] = \sum_{i \in \mathrm{TP}} \mathbf{E}\Big[\mathbf{x}_i \mathbf{x}_i^\top \cdot \mathbf{1}\{\mathbf{w}^{(t)} \cdot \mathbf{x}_i \geq 0\} \cdot \mathbf{1}\{\mathbf{w}^* \cdot \mathbf{x}_i \geq 0\}\Big] \tag{119}$$

$$\overset{(i)}{\succeq} N(1-2\epsilon) \cdot \Big(\pi - \Theta^{(t)} - \sin\Theta^{(t)}\Big) \cdot I \tag{120}$$

$$\succeq N(1-2\epsilon) \cdot \left(\pi - 2\arcsin\left(\frac{\|\mathbf{w}^{(t)} - \mathbf{w}^*\|}{\|\mathbf{w}^*\|}\right)\right) \cdot I \tag{121}$$

$$\succeq N(1-2\epsilon) \cdot \pi\left(1 - \frac{\|\mathbf{w}^{(t)} - \mathbf{w}^*\|}{\|\mathbf{w}^*\|}\right) \cdot I \tag{122}$$

In the above, (i) follows from Lemma 26, (ii) follows from the guarantee in the randomized initialization. We can then note We now bound the second-moment matrix approximation. Let $dP(\mathbf{x})$ be a Dirac-measure for $\mathbf{x} \in \mathrm{TP}$. We will now upper bound the third term in Equation (**??**).

$$\eta^2 \cdot \|\nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP})\|_2^2 = \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \Big\| \sum_{i \in \mathrm{TP}} (\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) - \sigma(\mathbf{w}^* \cdot \mathbf{x}_i) - \xi_i) \cdot \mathbf{x}_i \cdot \mathbf{1}\{\mathbf{w}^{(t)} \cdot \mathbf{x}_i \geq 0\} \Big\|_2^2 \tag{123}$$

$$\leq \frac{8\eta^2}{[(1-\epsilon)N]^2} \cdot \Big\| \sum_{i \in \mathrm{TP}} (\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) - \sigma(\mathbf{w}^* \cdot \mathbf{x}_i)) \cdot \mathbf{x}_i \cdot \mathbf{1}\{\mathbf{w}^{(t)} \cdot \mathbf{x}_i \geq 0\} \Big\|_2^2 \tag{124}$$

$$+ \frac{8\eta^2}{[(1-\epsilon)N]^2} \cdot \Big\| \sum_{i \in \mathrm{TP}} \xi_i \mathbf{x}_i \cdot \mathbf{1}\{\mathbf{w}^{(t)} \cdot \mathbf{x}_i \geq 0\} \Big\|_2^2 \tag{125}$$

Recall we have from Lemma 24, we have an upper bound on the second term of Equation (**??**). We give a bound on the first term of Equation (**??**).

$$\frac{8\eta^2}{[(1-\epsilon)N]^2} \cdot \Big\| \sum_{i \in \mathrm{TP}} (\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) - \sigma(\mathbf{w}^* \cdot \mathbf{x}_i)) \cdot \mathbf{x}_i \cdot \mathbf{1}\{\mathbf{w}^{(t)} \cdot \mathbf{x}_i \geq 0\} \Big\|_2^2 \tag{126}$$

$$\leq \frac{8\eta^2}{[(1-\epsilon)N]^2} \cdot \big\|X_{\mathrm{TP}} X_{\mathrm{TP}}^\top\big\|_2 \sum_{i \in \mathrm{TP}} (\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) - \sigma(\mathbf{w}^* \cdot \mathbf{x}_i))^2 \tag{127}$$

$$\leq \frac{8\eta^2}{[(1-\epsilon)N]^2} \cdot \big\|X_{\mathrm{TP}} X_{\mathrm{TP}}^\top\big\|_2^2 \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 \tag{128}$$

Then, by choosing $\eta \leq \frac{\lambda_{\min}(\Sigma)}{80\lambda^2_{\max}(\Sigma)}$, we have that the RHS in Equation (**??**) will be less than $\frac{\lambda_{\min}(\Sigma)}{8}$.

Step 3: Upper bounding the corrupted gradient. We now upper bound the third term in Equation (**??**).

$$\eta^2 \cdot \|\nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{FP})\|_{\mathrm{F}}^2 = \frac{4\eta^2}{\big[(1-\epsilon)N\big]^2} \cdot \Big\| \sum_{i \in \mathrm{FP}} (\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) - \sigma(\mathbf{w}^* \cdot \mathbf{x}_i)) \cdot \mathbf{x}_i \cdot \mathbf{1}\{\mathbf{w}^{(t)} \cdot \mathbf{x}_i \geq 0\} \Big\|_2^2 \tag{129}$$

$$\leq \frac{4\eta^2}{\big[(1-\epsilon)N\big]^2} \cdot \|\Sigma_{\mathrm{FP}}(\mathbf{w}^{(t)}, \mathbf{w}^{(t)})\|_2 \sum_{i \in \mathrm{FP}} (\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) - \sigma(\mathbf{w}^* \cdot \mathbf{x}_i))^2 \tag{130}$$

$$\leq \frac{4\eta^2}{\big[(1-\epsilon)N\big]^2} \cdot \|X_{\mathrm{FP}} X_{\mathrm{FP}}^\top\|_2 \sum_{i \in \mathrm{FN}} (\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}_i) - \sigma(\mathbf{w}^* \cdot \mathbf{x}_i))^2 \tag{131}$$

$$\leq \frac{4\eta^2}{\big[(1-\epsilon)N\big]^2} \cdot \|X_{\mathrm{FP}} X_{\mathrm{FP}}^\top\|_2 \|X_{\mathrm{FN}} X_{\mathrm{FN}}^\top\|_2 \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 \tag{132}$$

In the above, the first inequality follows from the same argument as Lemma 29, the second inequality follows from the optimality of the Subquantile set, and the final inequality follows from noting that σ is 1-Lipschitz. We now conclude Steps 1-3 with our linear convergence result.

$$\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 \leq \|\mathbf{w}^{(t)} - \mathbf{w}^*\| \left(1 - \frac{\eta}{32} \cdot \lambda_{\min}(\Sigma) + \frac{2\eta}{(1-\epsilon)N} \cdot \|X_{\mathrm{FP}}\|_2 \|X_{\mathrm{FN}}\|\right) + \frac{16}{3} K C_\varrho \cdot \left(\frac{2d}{N}\log(5) + \frac{2}{N}\log(1/\delta) + 6\epsilon\log(1/\epsilon)\right)^{1/2} \tag{133}$$

Step 4: Concentration Inequalities. From our previous theorems, we have $\|X_{\mathrm{FP}}\|_2 \|X_{\mathrm{FN}}\|_2 \leq N\epsilon\sqrt{10B\log(1/\epsilon)}$ with probability exceeding $1-\delta$ and $N = \Omega\Big(\frac{d+\log(2/\delta)}{\epsilon}\Big)$. If $N = \Omega\Big(\frac{2d\log 5 + \log(1/\delta)}{3\epsilon\log(1/\epsilon)}\Big)$ and $\epsilon \leq \frac{1}{64\sqrt{80\log(2)}}$, we obtain after $T = O\Big(\kappa^2(\Sigma)\log\Big(\frac{\|\mathbf{w}^*\|_2}{\varepsilon}\Big)\Big)$ gradient descent iterations,

$$\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 \leq \varepsilon + 1024\sqrt{\epsilon\log(1/\epsilon)} = O\Big(\sqrt{\epsilon\log(1/\epsilon)}\Big) \tag{134}$$

when we choose $\varepsilon = O\Big(\sqrt{\epsilon\log(1/\epsilon)}\Big)$. Our proof is complete. ■

C Probability Theory

In this section we will give various concentration inequalities on functions of the good data.

Lemma 20 (Upper Bound on Sum of Chi-Squared Variables [LM00])**.** *Suppose $\xi_i \sim \mathcal{N}(0, \sigma^2)$ for $i \in [n]$, then*

$$\mathbf{Pr}\big\{\|\boldsymbol{\xi}\|_2^2 \geq \sigma\big(n + 2\sqrt{nx} + 2x\big)\big\} \leq e^{-x} \tag{135}$$

Proposition 21 (Probabilistic Upper Bound on Sum of Chi-Squared Variables)**.** *Suppose $\xi_i \sim \mathcal{N}(0, \sigma^2)$ for $i \in [n]$. Let $S \subset [n]$ such that $|S| = \epsilon n$ for $\epsilon \in (0, 0.5)$ and let $\mathcal{N}_2$ represent all such subsets. Given a failure probability $\delta \in (0,1)$, when $n \geq \log(1/\delta)$, with probability exceeding $1-\delta$,*

$$\max_{\mathrm{S} \in \mathcal{W}} \|\boldsymbol{\xi}_{\mathrm{S}}\|_2^2 \leq \sigma(30n\epsilon\log(1/\epsilon)) \tag{136}$$

Proof. Directly from Lemma 20, we have with probability exceeding $1-\delta$.

$$\|\boldsymbol{\xi}\|_2^2 \leq \sigma\Big(n + 2\sqrt{n\log(1/\delta)} + 2\log(1/\delta)\Big) \tag{137}$$

We now can prove the claimed bound using the layer-cake representation,

$$\mathbf{Pr}\Big\{\max_{\mathrm{S} \in \mathcal{W}} \|\boldsymbol{\xi}\|_2^2 \geq \sigma\big(\epsilon n + 2\sqrt{\epsilon n x} + 2x\big)\Big\} \leq \Big(\frac{e}{\epsilon}\Big)^{\epsilon n} \mathbf{Pr}\big\{\|\boldsymbol{\xi}\|_2^2 \geq \sigma\big(\epsilon n + 2\sqrt{\epsilon n x} + 2x\big)\big\} \leq \Big(\frac{e}{\epsilon}\Big)^{\epsilon n} e^{-x} \tag{138}$$

In the first inequality we apply a union bound over $\mathcal{N}_2$ with Lemma 31, and in the second inequality we use Lemma 20. We then obtain with probability exceeding $1-\delta$,

$$\max_{\mathrm{S}\in\mathcal{W}}\|\boldsymbol{\xi}_{\mathrm{S}}\|_2^2 \leq \sigma\Big(\epsilon n + 2\sqrt{n\epsilon\log(1/\delta)+3n^2\epsilon^2\log(1/\epsilon)} + 2\log(1/\delta) + 6n\epsilon\log(1/\epsilon)\Big) \tag{139}$$

$$\leq \sigma\Big(9n\epsilon\log(1/\epsilon) + 2\sqrt{n\epsilon\log(1/\delta)} + 2\sqrt{3}n\epsilon\sqrt{\log(1/\epsilon)} + 2\log(1/\delta)\Big) \tag{140}$$

$$\leq \sigma\Big(15n\epsilon\log(1/\epsilon) + 2\sqrt{n\epsilon\log(1/\delta)} + 2\log(1/\delta)\Big) \tag{141}$$

$$\leq \sigma(30n\epsilon\log(1/\epsilon)) \tag{142}$$

In the above, in the first inequality, we note that $\log\binom{n}{\epsilon n} \leq 3n\epsilon\log(1/\epsilon)$ as $\epsilon < 0.5$, in the second inequality we note that $\sqrt{\log(1/\epsilon)} \leq (\log(2))^{-1/2}\log(1/\epsilon) \leq \sqrt{3}\log(1/\epsilon)$ when $\epsilon < 0.5$, the final inequality holds when $n \geq \log(1/\delta)$ by solving for the quadratic equation. The proof is complete. ■

Lemma 22 (Sub-Gaussian Covariance Matrix Estimation [Ver10] Theorem 5.40)**.** *Let $X \in \mathbb{R}^{d\times n}$ have columns sampled from a sub-Gaussian distribution with sub-Gaussian norm K and second-moment matrix Σ, then there exists positive constants c_k, C_Σ, dependent on the sub-Gaussian norm such that with probability at least $1-2e^{-c_K t^2}$,*

$$\lambda_{\max}(XX^\top) \leq n\cdot\lambda_{\max}(\Sigma) + \lambda_{\max}(\Sigma)\cdot\Big(C_\Sigma\cdot\sqrt{dn} + t\cdot\sqrt{n}\Big) \tag{143}$$

Lemma 23. *Let $X \in \mathbb{R}^{d\times n}$ have columns sampled from a sub-Gaussian distribution with sub-Gaussian norm K and second-moment matrix Σ. Let $S \subset [n]$ such that $|S| = \epsilon n$ for $\epsilon \in (0, 0.5)$ and let $\mathcal{W}$ represent all such subsets. Then with probability at least $1-\delta$,*

$$\max_{\mathrm{S}\in\mathcal{W}}\lambda_{\max}(X_{\mathrm{S}}X_{\mathrm{S}}^\top) \leq \lambda_{\max}(\Sigma)\cdot(10n\epsilon\log(1/\epsilon)) \tag{144}$$

$$\min_{S\in\mathcal{W}}\lambda_{\min}(X_{[n]\setminus\mathrm{S}}X_{[n]\setminus\mathrm{S}}^\top) \geq \frac{n}{4}\cdot\lambda_{\min}(\Sigma) \tag{145}$$

when

$$n \geq \frac{2}{\epsilon}\cdot\left(C_\Sigma^2\cdot d + \frac{\log(2/\delta)}{c_K}\right) \text{ and } \epsilon \leq \frac{1}{30}\cdot\kappa^{-1}(\Sigma) \tag{146}$$

Proof. We will use the layer-cake representation to obtain our claimed error bound.

$$\begin{aligned}&\mathbf{Pr}\Big\{\max_{S\in\mathcal{W}}\lambda_{\max}(X_{\mathrm{S}}X_{\mathrm{S}}^\top) \geq n\epsilon\cdot\lambda_{\max}(\Sigma) + \lambda_{\max}(\Sigma)\cdot\Big(C_\Sigma\cdot\sqrt{dn\epsilon} + t\sqrt{n\epsilon}\Big)\Big\}\\ &\leq \left(\frac{e}{\epsilon}\right)^{\epsilon n}\mathbf{Pr}\Big\{\lambda_{\max}(X_{\mathrm{S}}X_{\mathrm{S}}^\top) \geq n\epsilon\cdot\lambda_{\max}(\Sigma) + \lambda_{\max}(\Sigma)\cdot\Big(C_\Sigma\cdot\sqrt{dn\epsilon} + t\sqrt{n\epsilon}\Big)\Big\} \leq 2\cdot\left(\frac{e}{\epsilon}\right)^{\epsilon n}e^{-c_K t^2}\end{aligned} \tag{147}$$

In the above, the first inequality follows from a union bound over $\mathcal{W}$ and Lemma 31, the second inequality follows from Lemma 22. Then from elementary inequalities, we obtain with probability $1-\delta$,

$$\max_{S\in\mathcal{W}}\lambda_{\max}(X_{\mathrm{S}}X_{\mathrm{S}}^\top) \leq n\epsilon\cdot\lambda_{\max}(\Sigma) + \lambda_{\max}(\Sigma)\cdot\left(C_\Sigma\cdot\sqrt{dn\epsilon} + \sqrt{\frac{1}{c_K}(n\epsilon\cdot\log(2/\delta) + 3n^2\epsilon^2\log(1/\epsilon))}\right) \tag{148}$$

$$\leq n\cdot\lambda_{\max}(\Sigma)\cdot(\epsilon + 3^{3/4}\epsilon\log(1/\epsilon)) + \lambda_{\max}(\Sigma)\cdot\left(C_\Sigma\cdot\sqrt{dn\epsilon} + \sqrt{\frac{1}{c_K}n\epsilon\cdot\log(2/\delta)}\right) \tag{149}$$

$$\leq \lambda_{\max}(\Sigma)\cdot(6n\epsilon\log(1/\epsilon)) + \lambda_{\max}(\Sigma)\cdot\left(C_\Sigma\cdot\sqrt{dn\epsilon} + \sqrt{\frac{1}{c_K}n\epsilon\cdot\log(2/\delta)}\right) \tag{150}$$

$$\leq \lambda_{\max}(\Sigma)\cdot(10n\epsilon\log(1/\epsilon)) \tag{151}$$

In the above, the last inequality holds when

$$n \geq \frac{2}{\epsilon}\cdot\left(C_\Sigma^2\cdot d + \frac{\log(2/\delta)}{c_K}\right) \tag{152}$$

and our proof of the upper bound for the maximal eigenvalue is complete. We have from Weyl's Inequality for any $S \in \mathcal{W}$,

$$\lambda_{\min}(X_{\mathrm{X}\setminus\mathrm{S}}X_{\mathrm{X}\setminus\mathrm{S}}^\top) = \lambda_{\min}(XX^\top - X_{\mathrm{S}}X_{\mathrm{S}}^\top) \geq \lambda_{\min}(XX^\top) - \lambda_{\max}(X_{\mathrm{S}}X_{\mathrm{S}}^\top) \tag{153}$$

We then have with probability at least $1-\delta$,

$$\lambda_{\min}(X_{\mathrm{X}\setminus\mathrm{S}}X_{\mathrm{X}\setminus\mathrm{S}}^\top) \geq n\cdot\lambda_{\min}(\Sigma) - C_\Sigma\cdot\sqrt{dn} - \sqrt{\frac{1}{c_K}\cdot n\cdot\log(2/\delta)} - \lambda_{\max}(\Sigma)\cdot(10n\epsilon\log(1/\epsilon)) \tag{154}$$

$$\geq \frac{3n}{4}\cdot\lambda_{\min}(\Sigma) - \lambda_{\max}(\Sigma)\cdot(10n\epsilon\log(1/\epsilon)) \geq \frac{n}{4}\cdot\lambda_{\min}(\Sigma) \tag{155}$$

In the above, the first inequality follows when $n \geq \frac{32}{\lambda_{\min}^2(\Sigma)}\Big(C_\Sigma\cdot d + \frac{1}{c_K}\cdot\log(2/\delta)\Big)$, and from some algebra, we find the last inequality holds when $\epsilon \leq \frac{1}{30}\cdot\kappa^{-1}(\Sigma)$ by noting that $\epsilon < 0.5$. The proof is complete. ■

Lemma 24. *Let $X = [\mathbf{x}_1, \ldots, \mathbf{x}_N]$ be the data matrix such that for $i \in [N]$, $\mathbf{x}_i$ are sampled from a sub-Gaussian distribution with second-moment matrix Σ with sub-Gaussian norm C_Σ and ξ_i are sampled from sub-Gaussian distribution with sub-Gaussian norm C_σ. Assume $f : \mathbb{R}^d \to \mathbb{R}$ is bounded over $\mathbb{R}$ and Lipschitz. Let $\mathcal{S}$ represent all subsets of $[N]$ of size empty to $(1-\epsilon)N$. Suppose $\mathbf{w} \in \mathcal{B}(\mathbf{w}^*, R)$ and $\mathrm{S} \in \mathcal{S}$. Set a failure probability $\delta \in (0,1)$, then with probability at least $1-\delta$,*
Lipschitz and Bounded Function:

$$\|(\boldsymbol{\xi}_{\mathrm{S}} \circ f(X_{\mathrm{S}}^\top\mathbf{w}))^\top X_{\mathrm{S}}\|_2 \leq \frac{1}{c}2\sqrt{2}NC_\Sigma C_\sigma\|f\|_\infty\left(\frac{2d}{N}\log(6) + \frac{2Rd}{N}\log(6) + \frac{2}{N}\log(2/\delta) + 6\epsilon\log(1/\epsilon)\right)^{1/2} \tag{156}$$

Indicator Function:

$$\|(\boldsymbol{\xi}_{\mathrm{S}} \circ f(X_{\mathrm{S}}^\top\mathbf{w}))^\top X_{\mathrm{S}}\|_2 \leq \frac{1}{c}2\sqrt{2}NC_\Sigma C_\sigma\|f\|_\infty\left(\frac{2d}{N}\log(6) + \frac{2Rd}{N}\log(6) + \frac{2}{N}\log(2/\delta) + 6\epsilon\log(1/\epsilon)\right)^{1/2} \tag{157}$$

Proof. We will use the following characterization of the spectral norm.

$$\Big\|\sum_{i\in\mathrm{TP}} f(\mathbf{w}\cdot\mathbf{x}_i)\xi_i\mathbf{x}_i\Big\|_2 = \max_{\mathbf{v}\in\mathbb{S}^{d-1}}\Big|\sum_{i\in\mathrm{TP}} f(\mathbf{w}\cdot\mathbf{x}_i)\xi_i\mathbf{x}_i\cdot\mathbf{v}\Big| \tag{158}$$

We will first show that $f(\mathbf{w}\cdot\mathbf{x}_i)\mathbf{x}_i$ is sub-Gaussian. We first note for any $\mathbf{v}\in\mathbb{S}^{d-1}$, the random variable, $\mathbf{x}_i\cdot\mathbf{v}$ is sub-Gaussian by definition. We then have,

$$\Big(\mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}|f(\mathbf{w}\cdot\mathbf{x}_i)\mathbf{x}_i\cdot\mathbf{v}|^p\Big)^{1/p} \overset{(i)}{\leq} \Big(\mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}|f(\mathbf{w}\cdot\mathbf{x}_i)|^{2p}\mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}|\mathbf{x}\cdot\mathbf{v}|^{2p}\Big)^{1/2p} \overset{(ii)}{\leq} \Big(\|f\|_\infty C_\Sigma\sqrt{2}\Big)\sqrt{p} \tag{159}$$

In the above, (i) follows from Hölder's Inequality, (ii) follows from noting from letting $q = 2p$ and noting from Definition 6 that $\|\mathbf{x}_i\cdot\mathbf{v}\|_{L_q}$ is upper bounded by $C_\Sigma\sqrt{q}$. We thus have $f(\mathbf{w}\cdot\mathbf{x}_i)\mathbf{x}_i\cdot\mathbf{v}$ is sub-Gaussian for any $\mathbf{w}\in\mathbb{R}^d$ and $\|f(\mathbf{w}\cdot\mathbf{x}_i)\mathbf{x}_i\cdot\mathbf{v}\|_{\psi_2} \leq \sqrt{2}C_\Sigma\|f\|_\infty$. Recall $C_\sigma \triangleq \|\xi_i\|_{\psi_2}$, then from Lemma 7, the random variable $\xi_i f(\mathbf{w}\cdot\mathbf{x}_i)\mathbf{x}_i\cdot\mathbf{v}$ is sub-exponential s.t. $\|\xi_i f(\mathbf{w}\cdot\mathbf{x}_i)\mathbf{x}_i\cdot\mathbf{v}\|_{\psi_1} \leq \sqrt{2}C_\Sigma C_\sigma\|f\|_\infty$. Let $\widetilde{\mathbf{w}}\in\mathcal{N}_1$ such that $\widetilde{\mathbf{w}} = \arg\min_{\mathbf{u}\in\mathcal{N}_1}\|\mathbf{w}-\mathbf{u}\|_2$, where $\mathcal{N}_1$ is a ε-cover of $\mathcal{B}(\mathbf{0}, R)$.
Step 1: Probabilistic Decomposition. We use the decomposition given in Lemma A.4 of [ZYWG19].

$$\mathbf{Pr}\left\{\max_{\mathrm{S}\in\mathcal{S}}\sup_{\mathbf{w}\in\mathcal{B}(\mathbf{0},R)}\Big\|\sum_{i\in\mathrm{S}}\xi_i f(\mathbf{w}\cdot\mathbf{x}_i)\mathbf{x}_i\Big\| \geq t\right\} \leq \mathbf{Pr}\left\{\max_{\mathrm{S}\in\mathcal{S}}\max_{\mathbf{u}\in\mathcal{N}_1}\Big\|\sum_{i\in\mathrm{S}}\xi_i f(\mathbf{x}_i\cdot\mathbf{u})\mathbf{x}_i\Big\|_2 \geq \frac{t}{2}\right\}$$
$$+\mathbf{Pr}\left\{\max_{\mathrm{S}\in\mathcal{S}}\sup_{\mathbf{w}\in\mathcal{B}(\mathbf{0},R)}\Big\|\sum_{i\in\mathrm{S}}\xi_i f(\mathbf{w}\cdot\mathbf{x}_i)\mathbf{x}_i - \sum_{i\in\mathrm{S}}\xi_i f(\mathbf{x}_i^\top\widetilde{\mathbf{w}})\mathbf{x}_i\Big\| \geq \frac{t}{2}\right\} \tag{160}$$

We will now use a ε-covering argument to bound the first term of Equation (160). Let $\mathcal{N}_2$ be a ε-net of $\mathbb{S}^{d-1}$ such that for any $\mathbf{v} \in \mathbb{S}^{d-1}$, there exists $\mathbf{u} \in \mathcal{N}_2$ such that $\|\mathbf{u}-\mathbf{v}\|_2 \leq \varepsilon$. Let $\mathbf{u}^* = \arg\max_{\mathbf{u}\in\mathcal{N}_2} |(\boldsymbol{\xi} \circ f(X^\top \widetilde{\mathbf{w}}))^\top X\mathbf{u}|$ and $\mathbf{v}^* = \arg\max_{\mathbf{v}\in\mathbb{S}^{d-1}} |(\boldsymbol{\xi} \circ f(X^\top \widetilde{\mathbf{w}}))^\top X\mathbf{v}|$. We then have from the triangle inequality,

$$|(\boldsymbol{\xi}\circ f(X^\top \widetilde{\mathbf{w}}))^\top X\mathbf{v}^* - (\boldsymbol{\xi}\circ f(X^\top \widetilde{\mathbf{w}}))^\top X\mathbf{u}^*| \leq \|(\boldsymbol{\xi}\circ f(X^\top \widetilde{\mathbf{w}}))^\top X\|_2 \|\mathbf{u}^*-\mathbf{v}^*\|_2 \leq \varepsilon \cdot \|(\boldsymbol{\xi}\circ f(X^\top \widetilde{\mathbf{w}}))^\top X\|_2 \quad (161)$$

where in the final inequality we use the definition of a ε-net. We then have from the reverse triangle inequality,

$$\begin{aligned} |(\boldsymbol{\xi} \circ f(X^\top \widetilde{\mathbf{w}}))^\top X\mathbf{u}^*| &\geq |(\boldsymbol{\xi} \circ f(X^\top \widetilde{\mathbf{w}}))^\top X\mathbf{v}^*| - |(\boldsymbol{\xi} \circ f(X^\top \widetilde{\mathbf{w}}))^\top X\mathbf{u}^* - (\boldsymbol{\xi} \circ f(X^\top \widetilde{\mathbf{w}}))^\top X\mathbf{v}^*| && (162)\\ &\geq (1-\varepsilon)|(\boldsymbol{\xi} \circ f(X^\top \widetilde{\mathbf{w}}))^\top X\mathbf{v}^*| && (163)\end{aligned}$$

From rearranging, we have

$$|(\boldsymbol{\xi} \circ f(X^\top \widetilde{\mathbf{w}}))^\top X\mathbf{v}^*| \leq \frac{1}{1-\varepsilon} \cdot |(\boldsymbol{\xi} \circ f(X^\top \widetilde{\mathbf{w}}))^\top X\mathbf{u}^*| \quad (164)$$

With this result we are ready to make the probabilistic bounds. Suppose $\mathcal{S}$ represents all subsets of $[N]$ of size empty to $(1-\epsilon)N$. Suppose $\mathcal{N}_2$ is a ε_2 net of $\mathbb{S}^{d-1}$ and $\mathcal{N}_1$ is a ε_1 net of $\mathcal{B}(\mathbf{w}^*, R)$, we can then note that $\|\mathbf{w}^*\| \leq R$. Then,

$$\begin{aligned} &\mathbf{Pr}\left\{\max_{\mathrm{S}\in\mathcal{S}} \max_{\mathbf{w}\in\mathcal{N}_1} \|(\boldsymbol{\xi}_\mathrm{S} \circ f(X_\mathrm{S}^\top \mathbf{w}))^\top X_\mathrm{S}\| \geq \frac{t}{2}\right\} \\ &\overset{(164)}{\leq} \mathbf{Pr}\left\{\frac{1}{1-\varepsilon_2} \cdot \max_{\mathrm{S}\in\mathcal{S}} \max_{\mathbf{v}\in\mathcal{N}_2} \max_{\mathbf{w}\in\mathcal{N}_1} |(\boldsymbol{\xi}_\mathrm{S} \circ f(X_\mathrm{S}^\top \mathbf{w}))^\top X_\mathrm{S}\mathbf{v}| \geq \frac{t}{2}\right\} && (165)\\ &\leq \sum_{j\in[|\mathcal{S}|]} \sum_{i\in[|\mathcal{N}_1|]} \sum_{k\in[|\mathcal{N}_2|]} \mathbf{Pr}\left\{\frac{1}{1-\varepsilon_2} \cdot |(\boldsymbol{\xi}_{\mathrm{S}_j} \circ f(X_{\mathrm{S}_j}^\top \mathbf{w}_i)^\top X_\mathrm{S}\mathbf{v}_k| \geq \frac{t}{2}\right\} && (166)\\ &\overset{(iii)}{\leq} 2\left(\frac{3}{\varepsilon_1}\right)^{Rd}\left(\frac{3}{\varepsilon_2}\right)^{d} \cdot \left(\frac{e}{\epsilon}\right)^{N\epsilon} \exp\left(-c \cdot \left(\frac{t^2(1-\varepsilon_2)^2}{8C_\Sigma^2 C_\sigma^2 \|f\|_\infty^2 |\mathrm{S}|} \wedge \frac{t(1-\varepsilon_2)}{2\sqrt{2}C_\Sigma C_\sigma \|f\|_\infty}\right)\right) \leq \frac{\delta}{2} && (167)\end{aligned}$$

In the above, (iii) follows from Bernstein's Inequality (see Lemma 8). We can now note that $\log\binom{N}{(1-\epsilon)N} = \log\binom{N}{\epsilon N}$. Then to satisfy the above probabilistic condition, it must hold that

$$t \geq \frac{1}{c} 2\sqrt{2} C_\Sigma C_\sigma \|f\|_\infty (2dN\log(3/\varepsilon_2) + 2NRd\log(3/\varepsilon_1) + 2N\log(2/\delta) + 6N\epsilon\log(1/\epsilon))^{1/2} \quad (168)$$

We now bound the second term of Equation (160). For any $\mathbf{w}$, recall $\widetilde{\mathbf{w}} = \arg\min_{\mathbf{u}\in\mathcal{N}_1} \|\mathbf{w}-\mathbf{u}\|_2$ and thus for any $\mathbf{w}$, we have $\|\mathbf{w}-\widetilde{\mathbf{w}}\|_2 \leq \varepsilon_1$.

Step 2a: Lipschitz Functions. We have

$$\begin{aligned} &\sup_{\mathbf{w}\in\mathcal{B}(\mathbf{0},R)} \Big\|\sum_{i\in\mathrm{S}} \xi_i f(\mathbf{w}\cdot\mathbf{x}_i)\mathbf{x}_i - \sum_{i\in\mathrm{S}} \xi_i f(\widetilde{\mathbf{w}}\cdot\mathbf{x}_i)\mathbf{x}_i\Big\|_2 \\ &\leq \sup_{\mathbf{w}\in\mathcal{B}(\mathbf{0},R)} \max_{i\in[N]} \|\xi_i\mathbf{x}_i\|_2 \|f(\mathbf{w}\cdot\mathbf{x}_i) - f(\widetilde{\mathbf{w}}\cdot\mathbf{x}_i)\|_2 && (169)\\ &\leq \sup_{\mathbf{w}\in\mathcal{B}(\mathbf{0},R)} \max_{i\in[N]} \|f\|_{\mathrm{lip}} \|\xi_i\mathbf{x}_i\|_2 \|(\mathbf{w}-\widetilde{\mathbf{w}})\cdot\mathbf{x}_i\|_2 && (170)\\ &\leq \varepsilon_1 \|f\|_{\mathrm{lip}} \max_{i\in[N]} \|\xi_i\mathbf{x}_i\|_2 \max_{i\in[N]} \|\mathbf{x}_i\|_2 && (171)\\ &\overset{(iv)}{\leq} (1/2)\varepsilon_1 \|f\|_{\mathrm{lip}} \big(\max_{i\in[N]} \|\xi_i\mathbf{x}_i\|_2^2 + \max_{i\in[N]} \|\mathbf{x}_i\|_2^2\big) && (172)\end{aligned}$$

In the above, (iv) follows from Young's Inequality. Note that $\mathbf{x}_i \cdot \mathbf{v}$ for any $\mathbf{v} \in \mathbb{S}^{d-1}$ and $i \in [N]$ is sub-Gaussian, we thus have,

$$\mathbf{Pr}\left\{\varepsilon_1 \|f\|_{\mathrm{lip}} \cdot \max_{j\in[N]} \|\mathbf{x}_j\|_2^2 \geq \frac{t}{2}\right\} \leq \sum_{i\in\mathcal{N}_2} \sum_{j\in[N]} \mathbf{Pr}\left\{\varepsilon_1 \|f\|_{\mathrm{lip}} \cdot \frac{1}{1-\varepsilon_2} \cdot |\mathbf{x}_j\cdot\mathbf{v}_i|^2 \geq \frac{t}{2}\right\} \quad (173)$$

$$\leq N \cdot \left(\frac{3}{\varepsilon_2}\right)^d \cdot \exp\left[-c \cdot \left(\frac{t(1-\varepsilon_2)}{2C_\Sigma \varepsilon_1}\right)^2 \wedge \left(\frac{t(1-\varepsilon_2)}{2C_\Sigma \varepsilon_1}\right)\right] \leq \frac{\delta}{4} \tag{174}$$

where in the above, the final inequality follows when

$$t \geq \left(\frac{C_\Sigma \varepsilon_1}{c(1-\varepsilon_2)} \cdot (2\log N + 2d\log(3/\varepsilon_2) + 2\log(4/\delta))\right)^{1/2} \tag{175}$$

To bound $\max_{i\in \mathrm{S}} \|\xi_i \mathbf{x}_i\|_2$, we note that for any $\mathbf{v} \in \mathbb{R}^d$ that $\xi_i \mathbf{x}_i \cdot \mathbf{v}$ is sub-exponential with norm $\|\xi_i \mathbf{x}_i \cdot \mathbf{v}\|_{\psi_1} \leq C_\sigma C_\Sigma$. Similarly,

$$\mathbf{Pr}\left\{\varepsilon_1 \|f\|_{\mathrm{lip}} \max_{i\in \mathrm{S}} \|\xi_i \mathbf{x}_i\|_2^2 \geq \frac{t}{2}\right\} \leq \frac{\delta}{4} \tag{176}$$

The final inequality holds when,

$$t \geq \left(\frac{C_\Sigma C_\sigma \varepsilon_1}{c(1-\varepsilon_2)} \cdot (2\log N + 2d\log(3/\varepsilon_2) + 2\log(4/\delta))\right)^{1/2} \tag{177}$$

Step 2b: Indicator-Type Functions. We have,

$$\begin{aligned}
&\sup_{\mathbf{w}\in\mathcal{B}(\mathbf{0},R)} \Big\|\sum_{i\in \mathrm{S}} \xi_i f(\mathbf{w}\cdot\mathbf{x}_i)\mathbf{x}_i - \sum_{i\in \mathrm{S}} \xi_i f(\widetilde{\mathbf{w}}\cdot\mathbf{x}_i)\mathbf{x}_i\Big\|_2 \\
&\quad\leq \sup_{\mathbf{w}\in\mathcal{B}(\mathbf{0},R)} \max_{i\in[N]} \cdot \|\xi_i\mathbf{x}_i\|_2 \|f(\mathbf{w}\cdot\mathbf{x}_i) - f(\widetilde{\mathbf{w}}\cdot\mathbf{x}_i)\|_2 &(178)\\
&\quad\leq \max_{i\in[N]} \|\xi_i\mathbf{x}_i\|_2 \sup_{\mathbf{w}\in\mathcal{B}(\mathbf{w}^*,R)} (\mathbf{1}\{\mathbf{w}\cdot\mathbf{x}_i \geq 0, \widetilde{\mathbf{w}}\cdot\mathbf{x}_i \leq 0\} + \mathbf{1}\{\mathbf{w}\cdot\mathbf{x}_i \leq 0, \widetilde{\mathbf{w}}\cdot\mathbf{x}_i \geq 0\}) &(179)\\
&\quad\leq \max_{i\in[N]} 2\operatorname{Tr}(\Sigma)\cdot\|\xi_i\mathbf{x}_i\|_2 \cdot \Theta(\mathbf{w},\widetilde{\mathbf{w}}) \leq \max_{i\in[N]} 4\operatorname{Tr}(\Sigma)\cdot\|\xi_i\mathbf{x}_i\|_2 \cdot \frac{\varepsilon_1}{\|\mathbf{w}^*\| - R} &(180)
\end{aligned}$$

This upper bound holds for both when f is indicator, and when f is the derivative of leaky-relu. We then have,

$$\begin{aligned}
&\mathbf{Pr}\left\{4\operatorname{Tr}(\Sigma)\varepsilon_1 c_1 \cdot \max_{i\in[N]} \|\xi_i\mathbf{x}_i\|_2 \geq \frac{t}{2}\right\} \leq N \cdot \mathbf{Pr}\left\{4\operatorname{Tr}(\Sigma)\varepsilon_1 c_1 \cdot \frac{1}{1-\varepsilon_2} \cdot \max_{\mathbf{v}\in\mathcal{N}_2} |\xi_i\mathbf{x}_i\cdot\mathbf{v}| \geq \frac{t}{2}\right\} &(181)\\
&\quad\leq N\left(\frac{3}{\varepsilon_1}\right)^d \exp\left[-c\cdot\left(\frac{t(1-\varepsilon_2)}{4\operatorname{Tr}(\Sigma)\varepsilon_1 c_1}\right)^2 \wedge \left(\frac{t(1-\varepsilon_2)}{4\operatorname{Tr}(\Sigma)\varepsilon_1 c_1}\right)\right] \leq \frac{\delta}{4} &(182)
\end{aligned}$$

The final inequality holds when,

$$t \geq \left(\frac{4\operatorname{Tr}(\Sigma)\varepsilon_1 c_1}{1-\varepsilon_2}\right)(\log N + d\log(3/\varepsilon_1) + \log(4/\delta))^{1/2} \tag{183}$$

Step 3: Combining Estimates. We now choose $\varepsilon_1 = \frac{1}{2}$ and $\varepsilon_2 = \frac{1}{2}$.Combining our estimates, we obtain,

$$\mathbf{Pr}\left\{\frac{1}{(1-\epsilon)N} \cdot \max_{\mathrm{S}\in\mathcal{S}} \|(\boldsymbol{\xi}_{\mathrm{S}} \circ f(X_{\mathrm{S}}^\top \mathbf{w}))^\top X_{\mathrm{S}}\| \geq t\right\} \leq \delta \tag{184}$$

when

$$t \geq \frac{1}{c} 2\sqrt{2} C_\Sigma C_\sigma \|f\|_\infty \left(\frac{2d}{N}\log(6) + \frac{2Rd}{N}\log(6) + \frac{2}{N}\log(2/\delta) + 6\epsilon\log(1/\epsilon)\right)^{1/2} \tag{185}$$

when $N \geq \frac{\log(6)(Rd+d)+2\log(4/\delta)}{\epsilon\log(1/\epsilon)} = \Omega\Big(\frac{Rd+\log(1/\delta)}{\epsilon}\Big)$. Our proof is complete. ■

Lemma 25. *Fix $\mathbf{w}^* \in \mathbb{R}^{d-1}$ and suppose $\mathbf{w} \in \mathcal{B}(\mathbf{w}^*, R)$ for a constant $R < \|\mathbf{w}^*\|$. Sample $\mathbf{x}_1, \ldots, \mathbf{x}_N$ i.i.d from a sub-Gaussian distribution with second-moment matrix Σ and sub-Gaussian norm C_Σ. Suppose $\mathrm{S} \subset [N]$ s.t. $|\mathrm{S}| \leq (1-\epsilon)N$. Then with probability at least $1-\delta$,*

$$\Big\|\sum_{i\in \mathrm{S}} \mathbf{x}_i\mathbf{x}_i^\top \cdot \mathbf{1}\{\mathbf{w}^*\cdot\mathbf{x}_i \geq 0\}\cdot\mathbf{1}\{\mathbf{w}^{(t)}\cdot\mathbf{x}_i \geq 0\} - \mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}[\mathbf{x}\mathbf{x}^\top \cdot \mathbf{1}\{\mathbf{w}^*\cdot\mathbf{x}_i \geq 0\}\cdot\mathbf{1}\{\mathbf{w}^{(t)}\cdot\mathbf{x}_i \geq 0\}]\Big\|_2 \leq \Xi \tag{186}$$

Proof. Let $\mathcal{N}_1$ be an ε_1-cover of $\mathcal{B}(\mathbf{w}^*, R)$ and $\mathcal{N}_2$ be an ε_2-cover of $\mathbb{S}^{d-1}$. We will use the decomposition given in Theorem 1 of [MBM16]. Let $\widetilde{\mathbf{w}} = \arg\min_{\mathbf{v}\in\mathcal{N}_1} \|\mathbf{w} - \mathbf{v}\|_2$ throughout the relations.

$$\mathbf{Pr}\Big\{\max_{\mathrm{S}\in\mathcal{S}} \sup_{\mathbf{w}\in\mathcal{B}(\mathbf{w}^*;R)} \frac{1}{(1-\epsilon)N} \cdot \|\Sigma_{\mathrm{S}}(\mathbf{w},\mathbf{w}^*) - \mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}[\Sigma_{\mathrm{S}}(\mathbf{w},\mathbf{w}^*)]\|_2 \geq t\Big\} \tag{187}$$

$$\leq \mathbf{Pr}\Big\{\max_{\mathrm{S}\in\mathcal{S}} \sup_{\mathbf{w}\in\mathcal{B}(\mathbf{w}^*;R)} \frac{1}{(1-\epsilon)N} \cdot \|\Sigma_{\mathrm{S}}(\mathbf{w},\mathbf{w}^*) - \Sigma_{\mathrm{S}}(\widetilde{\mathbf{w}},\mathbf{w}^*)\|_2 \geq \frac{t}{3}\Big\} \tag{188}$$

$$+ \mathbf{Pr}\Big\{\max_{\mathrm{S}\in\mathcal{S}} \max_{\widetilde{\mathbf{w}}\in\mathcal{N}_1} \frac{1}{(1-\epsilon)N} \cdot \|\Sigma_{\mathrm{S}}(\widetilde{\mathbf{w}},\mathbf{w}^*) - \mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}[\Sigma_{\mathrm{S}}(\widetilde{\mathbf{w}},\mathbf{w}^*)]\|_2 \geq \frac{t}{3}\Big\} \tag{189}$$

$$+ \mathbf{Pr}\Big\{\sup_{\mathbf{w}\in\mathcal{B}(\mathbf{w}^*;R)} \frac{1}{(1-\epsilon)N} \cdot \|\mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}[\Sigma_{\mathrm{S}}(\widetilde{\mathbf{w}},\mathbf{w}^*)] - \mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}[\Sigma_{\mathrm{S}}(\mathbf{w},\mathbf{w}^*)]\|_2 \geq \frac{t}{3}\Big\} \tag{190}$$

We bound all terms separately. For the first term,

$$\max_{\mathrm{S}\in\mathcal{S}} \sup_{\mathbf{w}\in\mathcal{B}(\mathbf{w}^*;R)} \|\Sigma_{\mathrm{S}}(\mathbf{w},\mathbf{w}^*) - \Sigma_{\mathrm{S}}(\widetilde{\mathbf{w}},\mathbf{w}^*)\|_2$$

$$\leq \max_{\mathrm{S}\in\mathcal{S}} \sup_{\mathbf{w}\in\mathcal{B}(\mathbf{w}^*;R)} \|\Sigma_{\mathrm{S}}(\mathbf{w},-\widetilde{\mathbf{w}})\|_2 + \max_{\mathrm{S}\in\mathcal{S}} \sup_{\mathbf{w}\in\mathcal{B}(\mathbf{w}^*;R)} \|\Sigma_{\mathrm{S}}(-\mathbf{w},\widetilde{\mathbf{w}})\|_2 \tag{191}$$

$$\leq 2\max_{i\in[N]} \|\mathbf{x}_i\mathbf{x}_i^\top\|_2 \arcsin\left(\frac{\|\widetilde{\mathbf{w}} - \mathbf{w}\|_2}{\|\mathbf{w}\|}\right) \leq \varepsilon_1 \cdot \frac{\pi}{\|\mathbf{w}^*\| - R} \cdot \max_{i\in[N]} \|\mathbf{x}_i\mathbf{x}_i^\top\| \tag{192}$$

We then have for a constant c_1,

$$\mathbf{Pr}\left\{\frac{\varepsilon_1 c_1}{(1-\epsilon)N} \cdot \max_{i\in[N]} \|\mathbf{x}_i\mathbf{x}_i^\top\| \geq \frac{t}{3}\right\} \tag{193}$$

$$\leq N \cdot \mathbf{Pr}\left\{\|\mathbf{x}\|_2^2 \geq \frac{t}{3\varepsilon_1 c_1} \cdot (1-\epsilon)N\right\} \tag{194}$$

$$\leq N \cdot \mathbf{Pr}\left\{\max_{\mathbf{v}\in\mathcal{N}_2} |\mathbf{x}\cdot\mathbf{v}|^2 \geq \frac{t}{3\varepsilon_1} \cdot c_1(1-\epsilon)N(1-\varepsilon_2)\right\} \tag{195}$$

$$\leq N \cdot \sum_{i\in[|\mathcal{N}_2|]} \mathbf{Pr}\left\{|\mathbf{x}\cdot\mathbf{v}_i| \geq \left(\frac{t}{3\varepsilon_1 c_1} \cdot (1-\epsilon)N(1-\varepsilon_2)\right)^{1/2}\right\} \tag{196}$$

$$\overset{(i)}{\leq} 2N\left(\frac{3}{\varepsilon_2}\right)^d \exp\left(-\frac{1}{2C_\sigma^2} \cdot \left(\frac{t}{3\varepsilon_1 c_1} \cdot (1-\epsilon)N(1-\varepsilon_2)\right)\right) \leq \frac{\delta}{2} \tag{197}$$

In the above, (i) follows from the tails on a sub-Gaussian. The above probabilistic condition is satisfied when,

$$t \geq 6C_\sigma^2\varepsilon_1 c_1\left(\frac{\log N}{N} + \frac{d}{N}\log\left(\frac{3}{\varepsilon_2}\right) + \frac{\log(4/\delta)}{N}\right) \tag{198}$$

For the second term, let $\widetilde{\mathbf{x}} = \mathbf{x}\cdot\mathbf{1}\{\widetilde{\mathbf{w}}\cdot\mathbf{x}\geq 0\}\cdot\mathbf{1}\{\mathbf{w}^*\cdot\mathbf{x}\geq 0\}$. We then have,

$$\mathbf{Pr}\Big\{\max_{\mathrm{S}\in\mathcal{S}} \max_{\widetilde{\mathbf{w}}\in\mathcal{N}_1} \frac{1}{(1-\epsilon)N} \cdot \|\Sigma_{\mathrm{S}}(\widetilde{\mathbf{w}},\mathbf{w}^*) - \mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}[\Sigma_{\mathrm{S}}(\widetilde{\mathbf{w}},\mathbf{w}^*)]\|_2 \geq \frac{t}{3}\Big\} \tag{199}$$

$$\leq \mathbf{Pr}\Big\{\max_{\mathrm{S}\in\mathcal{S}} \max_{\widetilde{\mathbf{w}}\in\mathcal{N}_1} \max_{\mathbf{v}\in\mathcal{N}_2} \frac{1}{1-2\varepsilon_2} \cdot \frac{1}{(1-\epsilon)N} \cdot |\|\widetilde{X}_{\mathrm{S}}^\top\mathbf{v}\|_2^2 - \mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}\|\widetilde{X}_{\mathrm{S}}^\top\mathbf{v}\|_2^2| \geq \frac{t}{3}\Big\} \tag{200}$$

$$\overset{(i)}{\leq} 2\Big(\frac{e}{\epsilon}\Big)^{N\epsilon}\left(\frac{3}{\varepsilon_1}\right)^{Rd}\left(\frac{3}{\varepsilon_2}\right)^d \exp\left[-c\min\left(\frac{t^2(1-\epsilon)^2N^2(1-2\varepsilon_2)^2}{9\cdot 512C_\Sigma^2|\mathrm{S}|}, \frac{t(1-\epsilon)N(1-\varepsilon_2)}{48\sqrt{2}C_\Sigma}\right)\right] \leq \frac{\delta}{2} \tag{201}$$

In (i) we note from Lemma 1.12 in [RH23], that the random variable $|\widetilde{\mathbf{x}}\cdot\mathbf{v}|^2 - \mathbf{E}_{\mathbf{x}\sim\mathcal{D}}|\widetilde{\mathbf{x}}\cdot\mathbf{v}|^2$ is sub-exponential s.t. $\||\widetilde{\mathbf{x}}\cdot\mathbf{v}|^2 - \mathbf{E}_{\mathbf{x}\sim\mathcal{D}}|\widetilde{\mathbf{x}}\cdot\mathbf{v}|^2\|_{\psi_1} \leq 16\sqrt{2}C_\Sigma$, we can then apply Bernstein's Inequality. The probabilistic condition above is then satisfied when,

$$t \geq \left(\frac{1}{c}\cdot\frac{9\cdot 512C_\Sigma^2|\mathrm{S}|}{(1-\epsilon)^2N^2(1-2\varepsilon_c^2)}\left(Rd\log\left(\frac{3}{\varepsilon_1}\right) + d\log\left(\frac{3}{\varepsilon_2}\right) + 3N\epsilon\log(1/\epsilon) + \log(4/\delta)\right)\right)^{1/2} \tag{202}$$

from which we obtain the simplified bound,

$$t \geq \left(\frac{18432C_\Sigma^2}{c(1-2\varepsilon_c^2)}\left(\frac{Rd}{N}\log\left(\frac{3}{\varepsilon_1}\right)+\frac{d}{N}\log\left(\frac{3}{\varepsilon_2}\right)+3\epsilon\log(1/\epsilon)+\frac{1}{N}\log(4/\delta)\right)\right)^{1/2} \tag{203}$$

We now consider the third term.

$$\mathbf{Pr}\left\{\sup_{\mathbf{w}\in\mathcal{B}(\mathbf{w}^*;R)}\frac{1}{(1-\epsilon)N}\cdot\|\mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}[\Sigma_{\mathrm{S}}(\widetilde{\mathbf{w}},\mathbf{w}^*)]-\mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}[\Sigma_{\mathrm{S}}(\mathbf{w},\mathbf{w}^*)]\|_2\geq\frac{t}{3}\right\} \tag{204}$$

$$=\mathbf{Pr}\left\{\sup_{\mathbf{w}\in\mathcal{B}(\mathbf{w}^*;R)}\frac{1}{(1-\epsilon)N}\cdot\|\mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}[\mathbf{x}\mathbf{x}^\top\cdot(\mathbf{1}\{\widetilde{\mathbf{w}}\cdot\mathbf{x}\geq 0\}-\mathbf{1}\{\mathbf{w}\cdot\mathbf{x}\geq 0\})\cdot\mathbf{1}\{\mathbf{w}^*\cdot\mathbf{x}\geq 0\}]\|_2\geq\frac{t}{3}\right\} \tag{205}$$

$$\overset{(ii)}{\leq}\mathbf{Pr}\left\{\sup_{\mathbf{w}\in\mathcal{B}(\mathbf{w}^*;R)}\frac{1}{(1-\epsilon)N}\cdot\mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}\left[\|\mathbf{x}\mathbf{x}^\top\|_2|\mathbf{1}\{\widetilde{\mathbf{w}}\cdot\mathbf{x}\geq 0\}-\mathbf{1}\{\mathbf{w}\cdot\mathbf{x}\geq 0\}|\right]\geq\frac{t}{3}\right\} \tag{206}$$

$$\overset{(iii)}{\leq}\mathbf{Pr}\left\{\sup_{\mathbf{w}\in\mathcal{B}(\mathbf{w}^*;R)}\frac{1}{2(1-\epsilon)N}\cdot\left(\mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}\|\mathbf{x}\mathbf{x}^\top\|_2^2\mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}|\mathbf{1}\{\widetilde{\mathbf{w}}\cdot\mathbf{x}\geq 0\}-\mathbf{1}\{\mathbf{w}\cdot\mathbf{x}\geq 0\}|\right)^{1/2}\geq\frac{t}{3}\right\}=0 \tag{207}$$

In the above, (ii) follows from first applying Cauchy-Schwarz inequality and then applying Jensen's inequality, and (iii) follows from Hölder's Inequality. Then from $L_4 \to L_2$ hypercontractivity of $\mathcal{D}$, we have that $\mathbf{E}_{\mathbf{x}\sim\mathcal{D}}\|\mathbf{x}\|^4 \leq L\,\mathbf{E}_{\mathbf{x}\sim\mathcal{D}}\|\mathbf{x}\|^2 = L\,\mathrm{Tr}(\Sigma)$.

$$\mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}|\mathbf{1}\{\widetilde{\mathbf{w}}\cdot\mathbf{x}\geq 0\}-\mathbf{1}\{\mathbf{w}\cdot\mathbf{x}\geq 0\}|\leq\frac{\Theta(\mathbf{w},\widetilde{\mathbf{w}})}{\pi}\leq\frac{2}{\pi}\arcsin\left(\frac{\|\mathbf{w}-\widetilde{\mathbf{w}}\|}{\|\mathbf{w}^*\|-R}\right)\leq\varepsilon_1 c_1 \tag{208}$$

Then we obtain zero probability as indicated in the statement when

$$t\geq\frac{3}{N}\sqrt{\mathrm{Tr}(\Sigma)\varepsilon_1 c_1} \tag{209}$$

Combining our results, we choose $\varepsilon_1 = \frac{1}{2}$ and $\varepsilon_2 = \frac{1}{2}$ for sufficiently large $N = \Omega\left(\frac{Rd+d+\log(4/\delta)}{\epsilon}\right)$,

$$\max_{\mathrm{S}\in\mathcal{S}}\sup_{\mathbf{w}\in\mathcal{B}(\mathbf{w}^*,R)}\frac{1}{(1-\epsilon)N}\cdot\|\Sigma_{\mathrm{S}}(\mathbf{w},\mathbf{w}^*)-\mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}[\Sigma_{\mathrm{S}}(\mathbf{w},\mathbf{w}^*)]\|_2$$
$$\leq 192\left(\frac{C_\Sigma^2}{c}\left(\frac{Rd}{N}\log(6)+\frac{d}{N}\log(6)+3\epsilon\log(1/\epsilon)+\frac{1}{N}\log(4/\delta)\right)\right)^{1/2} \tag{210}$$

Our proof is complete. ■

Lemma 26. *Suppose* $\mathbf{E}_{\mathbf{x}\sim\mathcal{D}}[\mathbf{x}] = \mathbf{0}$ *and* $\mathbf{E}_{\mathbf{x}\sim\mathcal{D}}\left[\mathbf{x}\mathbf{x}^\top\right] = I$ *for* $\mathbf{x}\sim\mathcal{D}$. *Fix* $\mathbf{w}_1,\mathbf{w}_2\in\mathbb{R}^d$ *and define* $\Theta = \arccos\left(\frac{\mathbf{w}_1\cdot\mathbf{w}_2}{\|\mathbf{w}_1\|\|\mathbf{w}_2\|}\right)$. *For any rotationally invariant distribution, we have*

$$\mathop{\mathbf{E}}_{X\sim\mathcal{D}}\left[XX^\top\cdot\mathbf{1}\left\{X^\top\mathbf{w}_1\geq\mathbf{0}\right\}\cdot\mathbf{1}\left\{X^\top\mathbf{w}_2\geq\mathbf{0}\right\}\right]\succeq(\pi-\Theta-\sin\Theta)\cdot I \tag{211}$$

Proof. Since we have $\mathcal{D}$ is an isotropic distribution, we then have the distribution of $U\mathbf{x}$ is isotropic for any unitary U. Then consider $U = [\frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}, \frac{\mathbf{w}_2-\mathrm{Proj}_{\mathbf{w}_1}\mathbf{w}_2}{\|\mathbf{w}_2-\mathrm{Proj}_{\mathbf{w}_1}\mathbf{w}_2\|}, \mathbf{u}_3,\ldots,\mathbf{u}_d]$, where $\mathbf{u}_3,\ldots,\mathbf{u}_d$ represents some orthonormal basis of the complementary subspace to the subspace spanned by $\mathbf{w}_1$ and $\mathbf{w}_2$. Consider the plane spanned by $\mathbf{w}_1$ and $\mathbf{w}_2$. Then, w.l.o.g let $\mathbf{w}_1 = (1,0)$ and rotate $\mathbf{w}_2$ such that it is in the first quadrant with angle Θ from $\mathbf{w}_1$ from noting that $\Theta\leq\frac{\pi}{2}$.

$$\mathop{\mathbf{E}}_{X\sim\mathcal{D}}\left[XX^\top\cdot\mathbf{1}\left\{X^\top\mathbf{w}_1\geq\mathbf{0}\right\}\cdot\mathbf{1}\left\{X^\top\mathbf{w}_2\geq\mathbf{0}\right\}\right] \tag{212}$$

$$=\int_0^\infty\int_{-\pi/2+\Theta}^{\pi/2}\begin{pmatrix}\cos\Theta\\ \sin\Theta\end{pmatrix}(\cos\Theta,\sin\Theta)r\,d\Theta\,dr \tag{213}$$

$$= \int_0^\infty \frac{r}{2} \begin{pmatrix} \pi - \Theta + \sin\Theta\cos\Theta & \sin^2\Theta \\ \sin^2\Theta & \pi - \Theta - \cos\Theta\sin\Theta \end{pmatrix} dr \tag{214}$$

$$= (1/2) \cdot \mathop{\mathbf{E}}_{\mathbf{x}\sim\mathcal{D}}\left[r^2\right] \begin{pmatrix} \pi - \Theta + \sin\Theta\cos\Theta & \sin^2\Theta \\ \sin^2\Theta & \pi - \Theta - \cos\Theta\sin\Theta \end{pmatrix} \tag{215}$$

$$\succeq (\pi - \Theta - \sin\Theta) \cdot I \tag{216}$$

Our proof is complete by noting that the planes perpendicular to that of the plane integrated over remain unchanged by the indicator functions and thus have unitary expectation. ■

Lemma 27. *Fix $S \in \mathbb{R}^{K\times N\epsilon}, T \in \mathbb{R}^{N\epsilon\times L}$, then sample a matrix $G \in \mathbb{R}^{N\epsilon\times N\epsilon}$ such that each column of G represents n-dimensional vector sampled from $\mathcal{N}(\mathbf{0}, \sigma^2 \cdot I)$, then with probability exceeding $1-\delta$,*

$$\|SGT\|_{\mathrm{F}} \leq \|S\|_{\mathrm{F}}\|T\|_{\mathrm{F}} \cdot \sigma\sqrt{2\log(2N^2/\delta)} \tag{217}$$

Proof. The proof will be a calculation.

$$\|SGT\|_{\mathrm{F}}^2 = \sum_{i\in[K]}\sum_{j\in[L]}\sum_{k_1,k_2\in[N]\times[N]} \left(S_{i,k_1}G_{k_1,k_2}T_{k_2,j}\right)^2 \leq \|S\|_{\mathrm{F}}^2\|T\|_{\mathrm{F}}^2 \max_{i,j\in[N]\times[N]} (G_{i,j})^2 \tag{218}$$

It then suffices to bound the maximum value of a Gaussian squared over N^2 samples. From Lemma 32, we have

$$\mathop{\mathbf{Pr}}_{G\sim\mathcal{N}(0,1)}\left\{\max_{(i,j)\in[N]\times[N]} G_{i,j}^2 \geq t\right\} = \mathop{\mathbf{Pr}}_{G\sim\mathcal{N}(0,1)}\left\{\max_{(i,j)\in[N]\times[N]} |G_{i,j}| \geq \sqrt{t}\right\} \overset{(i)}{\leq} 2N^2 e^{-\frac{t}{2\sigma^2}} \tag{219}$$

In the above, (i) follows from a union bound. We thus obtain from elementary inequalities, with probability at least $1-\delta$,

$$\max_{i,j} G_{i,j}^2 \leq \sigma^2\big(2\log\big(2N^2/\delta\big)\big) \tag{220}$$

Our proof is complete. ■

Lemma 28. *Let $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$ be the data matrix such that $\mathbf{x}_i$ are sampled from a sub-Gaussian Distribution with second-moment matrix Σ and maximal sub-Gaussian Norm C_Σ for $i \in [N]$. Let ξ_i be sampled from a centered sub-Gaussian distribution with sub-Gaussian norm ν for $i \in [N]$. Let $\mathcal{S}$ represent all subsets of $[N]$ of size empty to $(1-\epsilon)N$. For any $\delta \in (0,1)$, with probability at least $1-\delta$,*

$$\big\|\sum_{i\in\mathrm{S}} \xi_i\mathbf{x}_i\big\|_2 \leq C_\sigma C_\Sigma\sqrt{c^{-1}N(\log(2/\delta) + d\log(6) + 3N\epsilon\log(1/\epsilon))} \tag{221}$$

Proof. We have by the definition of the ℓ_2 norm,

$$\big\|\sum_{i\in\mathrm{S}} \xi_i\mathbf{x}_i\big\|_2 = \max_{\mathbf{v}\in\mathbb{S}^{d-1}} \big|\sum_{i\in\mathrm{S}} \xi_i\mathbf{x}_i\cdot\mathbf{v}\big| \triangleq \big|\sum_{i\in\mathrm{S}} \xi_i\mathbf{x}_i\cdot\mathbf{v}^*\big| \tag{222}$$

Let $\mathcal{W}$ be a ε-net of $\mathbb{S}^{d-1}$, from the definition of a ε-net, we have the existence of $\mathbf{u} \in \mathcal{W}$ such that $\|\mathbf{u}-\mathbf{v}^*\| \leq \varepsilon$. Then, from the reverse triangle inequality, we have

$$\big\|\sum_{i\in\mathrm{S}} \xi_i\mathbf{x}_i\cdot\mathbf{u}\big\|_2 \geq \big\|\sum_{i\in\mathrm{S}} \xi_i\mathbf{x}\cdot\mathbf{v}^*\big\|_2 - \big\|\sum_{i\in\mathrm{S}} \xi_i\mathbf{x}_i\cdot(\mathbf{v}^*-\mathbf{u})\big\|_2 \geq (1-\varepsilon)\big\|\sum_{i\in\mathrm{S}} \xi_i\mathbf{x}_i\big\|_2 \tag{223}$$

Then from rearranging, we have

$$\big\|\sum_{i\in\mathrm{S}} \xi_i\mathbf{x}_i\big\|_2 \leq \frac{1}{1-\varepsilon}\cdot\max_{\mathbf{u}\in\mathcal{W}}\big|\sum_{i\in\mathrm{S}} \xi_i\mathbf{x}_i\cdot\mathbf{u}\big| \tag{224}$$

Next, from Lemma 2.7.7 in [Ver20], we can note that for any $\mathbf{u} \in \mathbb{S}^{d-1}$, we have $\xi_i \mathbf{x}_i \cdot \mathbf{u}$ is sub-exponential and $\|\xi_i \mathbf{x}_i \cdot \mathbf{u}\|_{\psi_1} \leq C_\sigma C_\Sigma$. We next choose $\varepsilon = \frac{1}{2}$. We can now make our probabilistic bounds.

$$\mathop{\mathbf{Pr}}_{X \sim \mathcal{D}} \left\{ \max_{\mathrm{S} \in \mathcal{S}} \Big\| \sum_{i \in \mathrm{S}} \xi_i \mathbf{x}_i \Big\|_2 \geq t \right\} \leq \binom{N}{N\epsilon} \mathop{\mathbf{Pr}}_{X \sim \mathcal{D}} \left\{ \max_{\mathbf{u} \in \mathcal{W}} \Big| \sum_{i \in \mathrm{S}} \xi_i \mathbf{x}_i \cdot \mathbf{u} \Big| \geq \frac{t}{2} \right\} \tag{225}$$

$$\leq \Big(\frac{e}{\epsilon}\Big)^{N\epsilon} 6^d \mathop{\mathbf{Pr}}_{X \sim \mathcal{D}} \left\{ \Big| \sum_{i \in \mathrm{S}} \xi_i \mathbf{x}_i \cdot \mathbf{u} \Big| \geq t \right\} \tag{226}$$

$$\leq 2\Big(\frac{e}{\epsilon}\Big)^{N\epsilon} 6^d \exp\left[-c \left(\frac{t^2}{C_\sigma^2 C_\Sigma^2 |\mathrm{S}|} \wedge \frac{t}{C_\sigma C_\Sigma} \right) \right] \leq \delta \tag{227}$$

In the above, the final condition holds when

$$t \geq \sqrt{c^{-1} C_\sigma^2 C_\Sigma^2 N (\log(2/\delta) + d \log(6) + 3N\epsilon \log(1/\epsilon))} \tag{228}$$

Our proof is complete. ■

D Mathematical Tools

In this section, we state additional lemmas referenced throughout the text for completeness.

Lemma 29. *Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and $X \in \mathbb{R}^{p \times n}$, then*

$$\Big\| \sum_{i \in [n]} a_i b_i \mathbf{x}_i \Big\|^2 \leq \|\mathbf{a}\|_\infty^2 \|\mathbf{b}\|_2^2 \|X X^\top\|_2 \tag{229}$$

Proof. The proof is a simple calculation. Expanding out the LHS, we have

$$\Big\| \sum_{i \in [n]} a_i b_i \mathbf{x}_i \Big\|_2^2 = \sum_{i \in [n]} \sum_{j \in [n]} a_i a_j b_i b_j \mathbf{x}_i^\top \mathbf{x}_j = (\mathbf{a} \circ \mathbf{b})^\top X^\top X (\mathbf{a} \circ \mathbf{b}) \leq \|\mathbf{a} \circ \mathbf{b}\|_2^2 \Big\| X^\top X \Big\|_2 \leq \|\mathbf{a}\|_\infty^2 \|\mathbf{b}\|_2^2 \Big\| X^\top X \Big\|_2 \tag{230}$$

where the final inequality comes from noting

$$\|\mathbf{a} \circ \mathbf{b}\|^2 = \sum_{i \in [n]} a_i^2 b_i^2 \leq \max_{i \in [n]} a_i^2 \cdot \sum_{i \in [n]} b_i^2 \tag{231}$$

Our proof is complete. ■

Lemma 30 (Lemma 3.11 [B+15])**.** *Let f be β-smooth and α-strongly convex over $\mathbb{R}^n$, then for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,*

$$\langle \nabla f(\mathbf{x}) - \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \geq \frac{\alpha\beta}{\alpha + \beta} \|\mathbf{x} - \mathbf{y}\|^2 + \frac{1}{\alpha + \beta} \|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|^2 \tag{232}$$

Lemma 31 (Sum of Binomial Coefficients [CLRS22])**.** *Let $k, n \in \mathbb{N}$ such that $k \leq n$, then*

$$\sum_{i=0}^{k} \binom{n}{i} \leq \Big(\frac{en}{k}\Big)^k \tag{233}$$

Lemma 32 (Max Gaussian [RH23])**.** *Let $x_1, \ldots, x_n$ be sampled i.i.d from $\mathcal{N}(0, \sigma^2)$. Then,*

$$\mathbf{Pr}\{\|\mathbf{x}\|_\infty > t\} \leq N \exp\left(-\frac{t^2}{\frac{16}{3}\sigma^2} \right) \tag{234}$$

Lemma 33 (Corollary 4.2.13 in [Ver20])**.** *The covering number of the ℓ_2-norm ball $\mathcal{B}(\mathbf{0}; 1)$ for $\varepsilon < 0$, satisfies,*

$$\mathcal{N}(\mathcal{B}_{\ell_2}^d(\mathbf{0}, 1), \varepsilon) \leq \left(\frac{3}{\varepsilon} \right)^d \tag{235}$$