

Subquantile Minimization for Kernel Learning in the Huber ϵ -Contamination Model*

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Abstract

In this paper we propose Subquantile Minimization for learning with adversarial corruption in the training set. Superquantile objectives have been formed in the past in the context of fairness where one wants to learn an underrepresented distribution equally [LPMH21, RRM14]. Our intuition is to learn a more favorable representation of the *majority* class, thus we propose to optimize over the p -subquantile of the loss in the dataset. In particular, we study the Huber Contamination Problem for Kernel Learning where the distribution is formed as, $\hat{\mathbb{P}} = (1 - \epsilon)\mathbb{P} + \epsilon\mathbb{Q}$, and we want to find the function $\inf_f \mathbb{E}_{x \in \mathbb{P}} [\ell_f(x)]$, from the noisy distribution, $\hat{\mathbb{P}}$. We assume the adversary has knowledge of the true distribution of \mathbb{P} , and is able to corrupt the covariates and the labels of ϵ samples. To our knowledge, we are the first to study the problem of general kernel learning in the Huber Contamination Model. In our theoretical analysis, we analyze our non-convex concave objective function with the Moreau Envelope. We show (i) a stationary point with respect to the Moreau Envelope is a good point and (ii) we can reach a stationary point with gradient descent methods. Further, we analyze accelerated gradient methods for the non-convex concave minimax optimization problem. We empirically test Kernel Ridge Regression and Kernel Classification on various state of the art datasets and show Subquantile Minimization gives strong results. Furthermore, we run experiments on various datasets and compare with the state-of-the-art algorithms to show the superior performance of Subquantile Minimization.

*Preliminary Work

1 Introduction

There has been extensive study of algorithms to learn the target distribution from a Huber ε -Contaminated Model for a Generalized Linear Model (GLM), [DKK⁺19, ADKS22, LBSS21, OZS20, FB81] as well as for linear regression [BJKK17, MGJK19]. Robust Statistics has been studied extensively [DK23] for problems such as high-dimensional mean estimation [PBR19, CDGS20] and Robust Covariance Estimation [CDGW19, FWZ18]. Recently, there has been an interest in solving robust machine learning problems by gradient descent [PSBR18, DKK⁺19]. Subquantile minimization aims to address the shortcomings of standard ERM in applications of noisy/corrupted data [KLA18, JZL⁺18]. In many real-world applications, linear models are insufficient to model the data. Therefore, we consider the problem of Robust Learning for Kernel Learning.

Definition 1. (Huber ϵ -Contamination Model [HR09]). Given a corruption parameter $0 < \epsilon < 0.5$, a data matrix, X and labels y . An adversary is allowed to inspect all samples and modify $n\epsilon$ samples arbitrarily. The algorithm is then given the ϵ -corrupted data matrix X and y as training data.

Current approaches for robust learning across various machine learning tasks often use gradient descent over a robust objective, [LBSS21]. These robust objectives tend to not be convex and therefore do not have a strong analysis on the error bounds for general classes of models.

We similarly propose a robust objective which has a nonconvex-concave objective. We use theory from the weakly-convex concave optimization literature for our error bounds. This objective has also been proposed recently in [HYL⁺20] where there has been an analysis in the Binary Classification Task. We

Theorem 2. (Informal). *Let the dataset be given as $\{(x_i, y_i)\}_{i=1}^n$ such that the labels and features of ϵn samples are arbitrarily corrupted by an adversary. Let K be the kernel matrix and S be the points with the lowest error w.r.t $f_{\hat{w}}$, then Subquantile Minimization returns $f_{\hat{w}}$ for sufficiently large n such that for*

Kernelized Regression:

$$\|f_{\hat{w}} - f_{w^*}\|_{\mathcal{H}} \leq \mathcal{O} \left(\left(\frac{\epsilon}{1 - 2\epsilon} \right) \frac{\sigma}{\lambda_{\min}(\Sigma)} + \frac{\sqrt{2}L}{\beta} \right) \quad (1)$$

where $\epsilon \rightarrow 0$ as number of gradient descenter iterations goes to ∞ and $\Sigma = \mathbb{E}[\Phi(x) \otimes \Phi(x)]$.

Kernel Binary Classification:

$$\|f_{\hat{w}} - f_{w^*}\|_{\mathcal{H}} \leq \mathcal{O} \left(+ \frac{\epsilon}{L} \right) \quad (2)$$

Kernel Multi-Class Classification:

$$\|f_{\hat{W}} - f_{W^*}\|_{\mathcal{H}} \leq \mathcal{O} \left(+ \frac{\epsilon}{L} \right) \quad (3)$$

We will now state our main contributions clearly.

Contributions

1. We propose a gradient-descent based algorithm for robust kernel learning in the Huber ϵ -Contamination Model which is fast.
2. We provide rigorous error bounds for subquantile minimization in the kernel regression, kernel binary classification, and kernel multi-class classification tasks for the linear, polynomial, and gaussian kernel.

3. We give new bounds for accelerated gradient methods for accelerated gradient methods in nonconvex-concave minimax optimization.
4. We perform experiments on state-of-the-art matrices in kernel learning and show the effectiveness of our algorithm compared to other robust algorithms.

1.1 Related Work

In this section we will describe previous works in robust algorithms for the Huber ϵ -Contamination Model and works in minimax optimization that will be relevant to our theoretical analysis.

Robust Algorithms

[DKK⁺19] proposed a robust meta-algorithm which filters points based their outlier likelihood score, which they define as the projection of the gradient of the point on to the top right singular vector of the Singular Value Decomposition of the Gradient of Losses. Empirically SEVER is strong in adversarially robust linear regression and Singular Vector Machines. SEVER however requires a base learner execution and SVD calculation for each iteration, thus it does not scale well for large data settings.

[LBSS21] proposed optimization over the Tilted Empirical Loss. This is done by minimization of an exponentially weighted functional of the traditional Empirical Risk. Their involves a hyperparameter t , negative values of t trains more robustly, whereas positive values of t trains more fairly. This empirically works well in machine learning applications such as Noisy Annotation. The issue with introducing the exponential smoothing into the ERM function is the lack of interpretability and lack of theoretical error bounds due to the nonlinearity induced by the exponential.

[ADKS22] theoretically analyzed the Trimmed Maximum Likelihood Estimator algorithm in General Linear Models, including Gaussian Regression. They were able to show the Trimmed Maximum Likelihood Estimator achieves near optimal error for Gaussian Regression.

[CDGS20] studied empirical covariance estimation by gradient descent. They use gradient descent on a minimax formulation of the estimation problem. Their theoretical analysis is based upon the Moreau envelope. They prove their algorithm results in the norm of the gradient of the Moreau Envelope, and the ensuing w is a good point in the search space. We tend to follow their general framework but we adapt it the Reproducing Kernel Hilbert Space Norm and for our minimax objective.

[HYL⁺20] proposed learning over the bottom k losses, this is an alternative formulation of our algorithm. They solve their optimization problem with difference of sums convex solvers. This work considers only the binary classification problem.

[HYW⁺23] proposed learning over the middle k losses, this is an extension of previous work [HYL⁺20]. Similarly, this work considers only the problem of binary classification and gives a generalization bound on the training error and out of sample error.

Minimax Optimization

[JNJ20] studied minimax optimization in the non-convex non-concave setting. Furthermore, they study convergence of alternating minimizing-maximizing algorithm with a maximizing oracle. Their key result is the convergence of the algorithm with infinite iterations.

[YOLH22] studied minimax optimization in the case of non-strong concavity.

1.2 Notation

The data matrix X is a fixed $n \times d$ matrix, the matrix K is the Gram Matrix, where $K_{ij} = k(x_i, x_j)$ and $k(\cdot, \cdot)$ represents a kernel function, e.g. Linear kernel: $k(x, y) = x^\top y$, RBF kernel: $k(x, y) = \exp(-\gamma \|x - y\|_2^2)$. We denote $X^\top = [x_1, x_2, \dots, x_n]$ where x_1, x_2, \dots, x_n represent the data vectors of the data matrix. We denote X as the set of all data vectors, $X = \{x_1, x_2, \dots, x_n\}$.

We represent the data matrix $X = [P^\top Q^\top]^\top$, the labels vector as $y = [y_P^\top y_Q^\top]^\top$, and the dataset $X = P \cup Q = \{(x_i, y_i)\}_{i=1}^n = \{x_i, y_i\}_{i \in P} \cup \{x_i, y_i\}_{i \in Q}$. We denote $I_{k \times k}$ as the $k \times k$ identity matrix. The spectral norm of A is $\|A\|_2 = \max_{\|x\|=1} \|Ax\| = \sigma_{\max}(A)$. The reproducing Hilbert Space Norm of f is given as $\|f\|_{\mathcal{H}} \triangleq w^\top K w$ where $f(\cdot) = \sum_{i=1}^n w_i k(x_i, \cdot)$.

2 Subquantile Minimization

We propose to optimize over the subquantile of the risk. The p -quantile of a random variable, U , is given as $\mathcal{Q}_p(U)$, this is the largest number, t , such that the probability of $U \leq t$ is at least p .

$$\mathcal{Q}_p(U) \leq t \iff \mathbb{P}\{U \leq t\} \geq p \quad (4)$$

The p -subquantile of the risk is then given by

$$\mathbb{L}_p(U) = \frac{1}{p} \int_0^p \mathcal{Q}_p(U) dq = \mathbb{E}[U | U \leq \mathcal{Q}_p(U)] = \max_{t \in \mathbb{R}} \left\{ t - \frac{1}{p} \mathbb{E}(t - U)^+ \right\} \quad (5)$$

Given a convex objective function, ℓ , the kernelized learning problem becomes:

$$\mathbf{f}_{\hat{w}} = \arg \min_{\mathbf{f}_w \in \mathcal{K}} \max_{t \in \mathbb{R}} \left\{ g(t, \mathbf{f}_w) \triangleq t - \sum_{i=1}^n (t - (\mathbf{f}_w(x_i) - y_i)^2)^+ \right\} \quad (6)$$

where t is the p -quantile of the empirical risk. Note that for a fixed t therefore the objective is not concave with respect to w . Thus, to solve this problem we use the iterations from equation 11 in [RHL⁺20]. Let $\Pi_{\mathcal{K}}$ be the projection of a vector on to the convex set $\mathcal{K} \triangleq \{f \in \mathcal{H} : \|f\|_{\mathcal{H}} \leq R\}$, then our update steps are

$$t^{(k+1)} = \arg \max_{t \in \mathbb{R}} g(\mathbf{f}_w^{(k)}, t) \quad (7)$$

$$\mathbf{f}_w^{(k+1)} = \Pi_{\mathcal{K}} \left(\mathbf{f}_w^{(k)} - \alpha \nabla_{\mathbf{f}} g(\mathbf{f}_w^{(k)}, t^{(k+1)}) \right) \quad (8)$$

Claim 3. The function $g(t, \mathbf{f}_w)$ defined in Equation (6) is non-convex-concave, i.e. it is not convex with respect to \mathbf{f}_w and is concave with respect to t . Furthermore, $g(t, \mathbf{f}_w)$ is ρ -weakly convex where ρ is the β -smoothness factor of $g(t, \mathbf{f}_w)$ w.r.t \mathbf{f}_w .

Proof. We will show $-g(t, \mathbf{f}_w)$ is convex with respect to t . Let $\nu_i \triangleq (f_w(x_i) - y_i)^2$.

$$-g(\lambda t_1 + (1 - \lambda)t_2, \mathbf{f}_w) = -\lambda t_1 - (1 - \lambda)t_2 + \sum_{i=1}^n (\lambda t_1 + (1 - \lambda)t_2 - \nu_i)^+ \quad (9)$$

$$\leq -\lambda t_1 - (1 - \lambda)t_2 + \sum_{i=1}^n \lambda(t_1 - \nu_i)^+ + (1 - \lambda)(t_2 - \nu_i)^+ \quad (10)$$

$$= -\lambda g(t_1, \mathbf{f}_w) - (1 - \lambda)g(t_2, \mathbf{f}_w) \quad (11)$$

Therefore we have $g(t, \mathbf{f}_w)$ is concave in t . Next we will prove $g(t, \mathbf{f}_w)$ is not convex in \mathbf{f}_w . ■

Claim 4. The function $g(t, \mathbf{f}_w)$ defined in Equation (6) is L -weakly convex in \mathbf{f}_w , where L is lipschitz constant of the gradient of $g(t, \mathbf{f}_w)$ w.r.t \mathbf{f}_w . This is true for Conditional Value at Risk, [RU02].

Proof. Here we note that if we add $\max_{i \in [n]} |k(x_i, x_i) + f_w(x_i) - y_i|$ to each of the second derivatives, then we are pushing the trace to be negative. Note this is the L -lipschitz gradient.

This is equivalent to $g(\mathbf{f}_w, t) + \frac{L}{2} \|\mathbf{f}_w\|_{\mathcal{H}}^2$. ■

We provide an algorithm for Subquantile Minimization of the ridge regression and classification kernel learning algorithm. ?? is applicable to both kernel ridge regression and kernel classification.

Algorithm 1: SUBQ-GRADIENT

Input: Iterations: T ; Quantile: p ; Data Matrix: $\mathbf{X}, (n \times d), n \gg d$;
Learning schedule: $\alpha_1, \dots, \alpha_T$;
Ridge parameter: λ

Output: Trained Parameters, $w_{(T)}$

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1:  $w_{(0)} \leftarrow \mathcal{N}_d(0, \sigma)$ 
2: for  $k \in 1, 2, \dots, T$  do
3:    $\mathbf{S}_{(k)} \leftarrow \text{SUBQUANTILE}(w^{(k)}, \mathbf{X})$ 
4:    $w^{(k+1)} \leftarrow$ 
       $w^{(k)} - \alpha_{(k)} \nabla_w g(t^{(k+1)}, w^{(k)})$ 
5: end
6: return  $w_{(T)}$ 

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Algorithm 2: SUBQUANTILE

Input: Parameters w , Data Matrix: $\mathbf{X}, (n \times d)$, Convex Loss Function f

Output: Subquantile Matrix \mathbf{S}

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1:  $\hat{v}_i \leftarrow \ell(x_i; \mathbf{f}_w, y_i)$  s.t.  $\hat{v}_{i-1} \leq \hat{v}_i \leq \hat{v}_{i+1}$ 
2:  $t \leftarrow \hat{v}_{np}$ 
3: Let  $x_1, \dots, x_{np}$  be  $np$  points such that
    $\ell(x_i; \mathbf{f}_w, y_i) \leq t$ 
4:  $\mathbf{S} \leftarrow (x_1^\top \dots x_{np}^\top)^\top$ 
5: return  $\mathbf{S}$ 

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3 Structural Results

To consider theoretical guarantees of Subquantile Minimization, we first analyze the inner and outer optimization problems. We first analyze kernel learning in the presence of corrupted data. Next, we provide error bounds for the two most important kernel learning problems, kernel ridge regression, and kernel classification. Now we will give our first result regarding kernel learning in the Huber ϵ -contamination model. Now we will analyze the two-step minimax optimization steps described in Equations (7) and (8).

Lemma 5. Let $f(x; w)$ be a convex loss function. Let x_1, x_2, \dots, x_n denote the n data points ordered such that $f(x_1; w, y_1) \leq f(x_2; w, y_2) \leq \dots \leq f(x_n; w, y_n)$. If we denote $\hat{v}_i \triangleq f(x_i; w, y_i)$, it then follows $\arg \max_{t \in \mathbb{R}} g(t, w) = \hat{v}_{np}$.

Proof. First we can note, the max value of t for g is equivalent to the min value of t for g . We can now find the Fermat Optimality Conditions for g .

$$\partial(-g(t, f_w)) = \partial \left(-t + \frac{1}{np} \sum_{i=1}^n (t - \hat{v}_i)^+ \right) \quad (12)$$

$$= -1 + \frac{1}{np} \sum_{i=1}^{np} \begin{cases} 1 & \text{if } t > \hat{v}_i \\ 0 & \text{if } t < \hat{v}_i \\ [0, 1] & \text{if } t = \hat{v}_i \end{cases} \quad (13)$$

$$= 0 \text{ when } t = \hat{v}_{np} \quad (14)$$

This is equivalent to the p -quantile of the Risk. ■

It therefore follows,

$$\sum_{i=1}^n \mathbb{I} \left\{ \hat{v}_{np} \geq \left(f_w^{(k)}(x_i) - y_i \right)^2 \right\} \left(f_w^{(k)}(x_i) - y_i \right)^2 \in \max_{t \in \mathbb{R}} g(t, f_w^{(k)}) \quad (15)$$

Interpretation 6. From lemma 5, we see the t will be greater than or equal to the errors of exactly np points. Thus, we are continuously updating over the np minimum errors.

Lemma 7. Let $\hat{\nu}_i \triangleq f(x_i; w, y_i)$ s.t. $\hat{\nu}_{i-1} \leq \hat{\nu}_i \leq \hat{\nu}_{i+1}$, if we choose $t^{(k+1)} = \hat{\nu}_{np}$ as by lemma 5, it then follows $\nabla_w g(t^{(k)}, f_w^{(k)}) = \frac{1}{np} \sum_{i=1}^{np} \nabla f(x_i; f_w^{(k)}, y_i)$

Proof. By our choice of $t^{(k+1)}$, it follows:

$$\nabla_f g(t^{(k+1)}, f_w^{(k)}) = \nabla_f \left(\hat{\nu}_{np} - \frac{1}{np} \sum_{i=1}^n \left(\hat{\nu}_{np} - \ell(x_i; f_w^{(k)}, y_i) \right)^+ \right) \quad (16)$$

$$= -\frac{1}{np} \sum_{i=1}^{np} \nabla_f \left(\hat{\nu}_{np} - \ell(x_i; f_w^{(k)}, y_i) \right)^+ \quad (17)$$

$$= \frac{1}{np} \sum_{i=1}^n \nabla_f \ell(x_i; f_w^{(k)}, y_i) \begin{cases} 1 & \text{if } t > \hat{\nu}_i \\ 0 & \text{if } t < \hat{\nu}_i \\ [0, 1] & \text{if } t = \hat{\nu}_i \end{cases} \quad (18)$$

Now we note $\nu_{np} \leq t^{(k+1)} \leq \nu_{np+1}$

$$\nabla_f g(t^{(k+1)}, f_w^{(k)}) = \frac{1}{np} \sum_{i=1}^{np} \nabla_f \ell(x_i; f_w^{(k)}, y_i) \quad (19)$$

This concludes the proof. ■

We denote the matrix K as the Gram Matrix where $[K]_{ij} = k(x_i, x_j) \triangleq \exp(-\rho \|x_i - x_j\|_2^2)$. Given a parameter set w , the prediction for a new point will be: $f(x^*; w) = \sum_{i=1}^n w_i \kappa(x_i, x^*)$

From our definition of $S^{(k)}$ in ??, we are interested in as $k \rightarrow \infty$ the quantities: $|x \in S^{(k)} \cap P|$ and $|x \in S^{(k)} \cap Q|$, where the latter cardinality represents the number of corrupted points in the subquantile set.

3.1 On the Softplus Approximation

It is clear our objective function is non-smooth. Thus we propose to use the Softplus approximation to smooth the function. The main ideas is to *first* approximate ReLU, consider the theory with respect to the approximation, and then take the limit as the approximation goes to the ReLU. The softplus approximation is given as follows,

$$\zeta_\rho(x) = \frac{1}{\lambda} \log(1 + e^{\lambda x}) \quad (20)$$

We then have the approximation of g as

$$\tilde{g}_\lambda(t, f_w) \triangleq t - \sum_{i=1}^n \zeta_\lambda(t - \ell(f_w; x_i, y_i)) \quad (21)$$

$$= t - \frac{1}{np} \sum_{i=1}^n \frac{1}{\lambda} \log(1 + \exp(\lambda(t - \ell(f_w; x_i, y_i)))) \quad (22)$$

Now we compute the derivatives w.r.t to the softplus approximation, and then we consider the limit of the derivative as $\lambda \rightarrow \infty$.

$$\nabla_t \tilde{g}_\lambda(t, f_w) = \nabla_t \left(t - \frac{1}{np} \sum_{i=1}^n \frac{1}{\lambda} \ln(1 + \exp(\lambda(t - \ell(f_w; x_i, y_i)))) \right) \quad (23)$$

$$= 1 - \frac{1}{np} \sum_{i=1}^n \sigma(\lambda(t - \ell(f_w; x_i, y_i))) \quad (24)$$

where $\sigma(\cdot)$ is the sigmoid function. It therefore follows,

$$\lim_{\lambda \rightarrow \infty} \nabla_t \tilde{g}_\lambda(t, f_w) = 1 - \frac{1}{np} \sum_{i=1}^n \mathbb{I}\{t - \ell(f_w; x_i, y_i)\} \quad (25)$$

$$\nabla_f \tilde{g}_\lambda(t, f_w) = \nabla_f \left(t - \frac{1}{np} \sum_{i=1}^n \frac{1}{\lambda} \ln(1 + \exp(\lambda(t - \ell(f_w; x_i, y_i)))) \right) \quad (26)$$

$$= \frac{1}{np} \sum_{i=1}^n \nabla_f \ell(f_w; x_i, y_i) \sigma(\lambda(t - \ell(f_w; x_i, y_i))) \quad (27)$$

We therefore similarly have,

$$\lim_{\lambda \rightarrow \infty} \nabla_f \tilde{g}_\lambda(t, f_w) = \frac{1}{np} \sum_{i=1}^n \mathbb{I}\{t - \ell(f_w; x_i, y_i)\} \nabla_f \ell(f_w; x_i, y_i) \quad (28)$$

Then the second derivative is given by

$$\nabla_f^2 \tilde{g}_\lambda(t, f_w) = \nabla_f \left(\frac{1}{np} \sum_{i=1}^n \nabla_f \ell(f_w; x_i, y_i) \sigma(\lambda(t - \ell(f_w; x_i, y_i))) \right) \quad (29)$$

$$= \frac{1}{np} \sum_{i=1}^n \left(\nabla_f^2 \ell(f_w; x_i, y_i) \sigma(\lambda(t - \ell(f_w; x_i, y_i))) \right. \\ \left. - (\nabla_f \ell(f_w; x_i, y_i))^2 \sigma(\lambda(t - \ell(f_w; x_i, y_i))) (1 - \sigma(\lambda(t - \ell(f_w; x_i, y_i)))) \right) \quad (30)$$

We then similarly have,

$$\lim_{\lambda \rightarrow \infty} \nabla_f^2 \tilde{g}_\lambda(t, f_w) = \frac{1}{np} \sum_{i=1}^n \mathbb{I}\{t - \ell(f_w; x_i, y_i)\} \nabla_f^2 \ell(f_w; x_i, y_i) \quad (31)$$

We can then calculate the Lipschitz constant of the approximation function with respect to f_w .

Lemma 8 (Lipschitz continuous gradient). *Let $f_w, f_{\tilde{w}} \in \mathcal{K}$, then we have*

$$\lim_{\lambda \rightarrow \infty} |\nabla_f \tilde{g}_\lambda(t, f_w) - \nabla_f \tilde{g}_\lambda(t, f_{\tilde{w}})| \leq \beta \|f_w - f_{\tilde{w}}\|_{\mathcal{H}} \quad (32)$$

where

$$\beta = \frac{1}{np} \sum_{i=1}^n |\nabla_f^2 \ell(f_w; x_i, y_i)| \quad (33)$$

Proof. We will upper bound the second derivative.

$$\lim_{\lambda \rightarrow \infty} |\nabla_f \tilde{g}_\lambda(t, f_w) - \nabla_f \tilde{g}_\lambda(t, f_{\tilde{w}})| \leq \lim_{\lambda \rightarrow \infty} \sup \{ \nabla_f^2 \tilde{g}_\lambda(t, f_w) \} \|f_w - f_{\tilde{w}}\|_{\mathcal{H}} \quad (34)$$

$$\leq \lim_{\lambda \rightarrow \infty} \sup \left\{ \frac{1}{np} \sum_{i=1}^n \nabla_f^2 \ell(f_w; x_i, y_i) \sigma(\lambda(t - \ell(f_w; x_i, y_i))) \right\} \|f_w - f_{\hat{w}}\|_{\mathcal{H}} \quad (35)$$

$$\leq \lim_{\lambda \rightarrow \infty} \left(\frac{1}{np} \sum_{i=1}^n |\nabla_f^2 \ell(f_w; x_i, y_i)| \right) \|f_w - f_{\hat{w}}\|_{\mathcal{H}} \quad (36)$$

$$= \frac{1}{np} \sum_{i=1}^n |\nabla_f^2 \ell(f_w; x_i, y_i)| \|f_w - f_{\hat{w}}\|_{\mathcal{H}} \quad (37)$$

There is no dependence on λ . ■

3.2 Weakly Convex Concave Optimization Theory

With our smoothed function, we are now able to use the weakly-convex concave minimization literature to analyze g . The Moreau Envelope can be interpreted as an infimal convolution of the function f . When f is ρ -weakly convex, if $\lambda \leq \rho^{-1}$, then the Moreau Envelope is smooth.

Definition 9. (Moreau Envelope on closed, convex set, [Mor65]). Let f be proper lower semi-continuous convex function $\ell : \mathcal{K} \rightarrow \mathbb{R}$, where $\mathcal{K} \subset \mathcal{X}$ is a closed and convex set, then the Moreau Envelope is defined as:

$$\mathbf{M}_{\lambda\ell}(f_w) \triangleq \inf_{f_{\hat{w}} \in \mathcal{K}} \left\{ \ell(f_{\hat{w}}) + \frac{1}{2\rho} \|f_w - f_{\hat{w}}\|_{\mathcal{H}}^2 \right\} \quad (38)$$

Definition 10. Define the function $\Psi(f_w) \triangleq \max_{t \in \mathbb{R}} g(t, f_w)$. This function is a L -weakly convex function in \mathcal{K} , i.e., $\Psi(f_w) + \frac{L}{2} \|f_w\|_{\mathcal{H}}^2$ is a convex function over w in the convex and compact set \mathcal{K} .

Definition 11 (Stationary Point of Moreau Envelope). A point $f_{\hat{w}}$ is a stationary point of the Moreau Envelope defined in Definition 9 of Ψ defined in Definition 10 if

$$f_{\hat{w}} = \arg \inf_{f_w \in \mathcal{K}} \left\{ \Psi_{\lambda}(f_w) + \frac{1}{2\rho} \|f_w - f_{\hat{w}}\|_{\mathcal{H}}^2 \right\} \quad (39)$$

We will show that if a point f_w is a stationary point then this point is close to the optimal point for the uncorrupted distribution, i.e. $\|f_{\hat{w}} - f_w^*\|_{\mathcal{H}}$ is small.

Lemma 12 (Lower bound on distance from stationary point and optimal point). *Let Ψ_{λ} be defined as in Definition 10, then if $f_{\hat{w}}$ is a stationary point as defined in Definition 11, then*

$$\lim_{\lambda \rightarrow \infty} (\Psi_{\lambda}(f_{\hat{w}}) - \Psi_{\lambda}(f_w^*)) \leq \beta \|f_{\hat{w}} - f_w^*\|_{\mathcal{H}}^2 \quad (40)$$

Proof. By the definition of stationary point, we have

$$f_{\hat{w}} = \lim_{\lambda \rightarrow \infty} \arg \inf_{f_w \in \mathcal{K}} \left\{ \Psi_{\lambda}(f_w) + \frac{1}{2\rho} \|f_w - f_{\hat{w}}\|_{\mathcal{H}}^2 \right\} \quad (41)$$

$$\stackrel{\zeta_1}{=} \arg \inf_{f_w \in \mathcal{K}} \left\{ \lim_{\lambda \rightarrow \infty} \Psi_{\lambda}(f_w) + \frac{1}{2\rho} \|f_w - f_{\hat{w}}\|_{\mathcal{H}}^2 \right\} \quad (42)$$

(ζ_1) holds as we ρ is independent of λ as shown in the proof of Lemma 8. This implies then for any $f_w \in \mathcal{K}$ and noting $\rho \leq \beta^{-1}$, it follows

$$\lim_{\lambda \rightarrow \infty} \Psi_{\lambda}(f_{\hat{w}}) \leq \lim_{\lambda \rightarrow \infty} \Psi_{\lambda}(f_w) + \beta \|f_w - f_{\hat{w}}\|_{\mathcal{H}}^2 \quad (43)$$

We can then plug in the optimal, f_w^* for f_w and rearrange and we have the desired result. \blacksquare

We can now upper bound $\|f_{\hat{w}} - f_w^*\|_{\mathcal{H}}$. We proceed by contradiction, i.e. if a stationary point is sufficiently far from the optimal point, then this will break the stationary property proved in Lemma 12. This bound is different for each of the loss functions, so we must upper bound $\|f_{\hat{w}} - f_w^*\|_{\mathcal{H}}$ seperately for each loss function with the same high level overview.

3.3 Kernelized Regression

The loss for the Kernel Ridge Regression problem for a single training pair (x_i, y_i) is given by the following equation

$$\ell(f_w; x_i, y_i) = (f_w(x_i) - y_i)^2 \quad (44)$$

For our theory, we need the L -lipschitz constant and β -smoothness constant.

Lemma 13. (L -Lipschitz of $g(t, f_w)$ w.r.t f_w). Let x_1, x_2, \dots, x_n , represent the data vectors. It then follows for any $f_w, f_{\hat{w}} \in \mathcal{K}$:

$$|g(t, f_w) - g(t, f_{\hat{w}})| \leq L \|f_w - f_{\hat{w}}\|_{\mathcal{H}} \quad (45)$$

where

$$L = \frac{2R}{np} \left(\sum_{i=1}^n \sqrt{k(x_i, x_i)} \right)^2 + \frac{2\|y\|}{np} \left(\sum_{i=1}^n \sqrt{k(x_i, x_i)} \right) \quad (46)$$

Proof is given in Appendix B.1.

Lemma 14. (β -Smoothness of $g(t, w)$ w.r.t w). Let x_1, x_2, \dots, x_n represent the rows of the data matrix X . It then follows:

$$\|\nabla_w g(t, f_w) - \nabla_w g(t, f_{\hat{w}})\| \leq \beta \|f_w - f_{\hat{w}}\|_{\mathcal{H}} \quad (47)$$

where

$$\beta = \frac{2}{np} \sum_{i \in X} k(x_i, x_i) = \frac{2}{np} \text{Tr}(K) \quad (48)$$

Proof. W.L.O.G, let S be the set of points such that if $x \in S$, then $t \geq (f_w(x) - y)^2$. Since g is twice differentiable, we will analyze the second derivative.

$$\|\nabla_w^2 g(t, f_w)\|_{\mathcal{H}} = \left\| \frac{2}{np} \sum_{i=1}^n \mathbb{I}\{t \geq (f_w(x_i) - y_i)^2\} k(x_i, x_i) \right\| \leq \frac{2}{np} \sum_{i=1}^n k(x_i, x_i) = \frac{2}{np} \text{Tr}(K) \quad (49)$$

This concludes the proof. \blacksquare

Similar results for the Lipschitz Constant for non-kernelized learning algorithms can be seen in [YSP21].

Lemma 15. If $\|f_w - f_{w^*}\| \geq \eta$, then it follows

$$\Psi(f_w) - \Psi(f_{w^*}) \geq \eta^2 \sigma_{\min} \left(\sum_{i \in S \cap P} \Phi(x_i) \Phi(x_i)^\top \right) - 2\eta \left\| \sum_{i \in S \cap P} \eta_i \Phi(x_i) \right\|_{\mathcal{H}} - \sum_{j \in P \setminus S} \eta_j^2 \quad (50)$$

The proof is deferred to Appendix B.2.

Theorem 16. Let $f_{\hat{w}}$ be a stationary point defined in definition 11 for the function Ψ defined in definition 10. Then for a constant $c_1 \in (0, 1)$, if $n \geq \frac{(1-2\varepsilon)(C_k\|\Sigma\|_{\text{op}}+\beta)}{(1-c_1)\lambda_{\min}(\Sigma)} + \sqrt{\beta}$,

$$\eta \leq \sqrt{\frac{\sum_{j \in P \cap S} \eta_j^2}{c_1 n \lambda_{\min}(\Sigma)}} + \frac{2 \|\sum_{i \in S \cap P} \eta_i \Phi(x_i)\|_{\mathcal{H}}}{c_1 n \lambda_{\min}(\Sigma)} \quad (51)$$

where β is the Lipschitz Gradient Constant given in Lemma 14.

The proof is given in Appendix B.3. In Theorem 16, we have an upper bound on the distance from a stationary point to the optimal point. The second term goes to zero with increased data and the first term will go the white noise factor. Furthermore, when we consider the case of feature noise, e.g. a large multiplicative term on the features. For the linear and polynomial kernel, we then have β increases, therefore to obtain the same bound on η as with no feature noise, we simply need more data.

Theorem 17. After T iterations of Algorithm 1, we have the following error bounds for robust kernelized linear regression.

Linear Kernel:

$$\mathbb{E}[\|f_{\hat{w}} - f_w^*\|_{\mathcal{H}}] \leq \Xi \quad (52)$$

3.4 Kernel Binary Classification

The Negative Log Likelihood for the the Kernel Classification problem is given by the following equation for a single training pair (x_i, y_i)

$$\ell(x_i, y_i; f_w) = -(y_i \log(\sigma(f_w(x_i))) - (1 - y_i) \log(1 - \sigma(f_w(x_i)))) \quad (53)$$

Similar to § section 3.3, we require the L -Lipschitz constant and β -smoothness constant.

Lemma 18. (L -Lipschitz of $g(t, w)$ w.r.t w). Let x_1, x_2, \dots, x_n , represent the data vectors. It then follows:

$$|g(t, f_w) - g(t, f_{\hat{w}})| \leq L \|f_w - f_{\hat{w}}\|_{\mathcal{H}} \quad (54)$$

where

$$L = \frac{1}{p} \sum_{i \in X} \sqrt{k(x_i, x_i)} = \frac{1}{p} \text{Tr}(\mathbf{K}) \quad (55)$$

Proof is given in Appendix B.4.

Lemma 19. (β -Smoothness of $g(t, w)$ w.r.t w). Let x_1, x_2, \dots, x_n represent the rows of the data matrix \mathbf{X} . It then follows:

$$\|\nabla_{\mathbf{f}} g(t, f_w) - \nabla_{\mathbf{f}} g(t, f_{\hat{w}})\| \leq \beta \|f_w - f_{\hat{w}}\|_{\mathcal{H}} \quad (56)$$

where

$$\beta = \frac{1}{4p} \sum_{i=1}^n k(x_i, x_i) = \frac{1}{4p} \text{Tr}(\mathbf{K}) \quad (57)$$

Proof is given in Appendix B.5.

Lemma 20. If $\|f_w - f_{w^*}\| \geq \eta$, then it follows

$$\Psi(f_w) - \Psi(f_{w^*}) \geq \quad (58)$$

Proof is given in ??.

3.5 Kernel Multi-Class Classification

The Negative Log-Likelihood Loss for the the Kernel Multi-Class Classification problem is given by the following equation for a single training pair (x_i, y_i) , note W is now a matrix

$$\ell(x_i, y_i; W) = - \sum_{j=1}^{|C|} \mathbb{I}\{y_i = j\} \log \left(\frac{\exp(f_{W_k}(x_i))}{\sum_{h=1}^{|C|} \exp(f_{W_h}(x_i))} \right) \quad (59)$$

Lemma 21. (*L-Lipschitz of $g(t, w)$ w.r.t w*). Let x_1, x_2, \dots, x_n , represent the data vectors. It then follows:

$$|g(t, f_w) - g(t, f_{\hat{w}})| \leq L \|f_w - f_{\hat{w}}\|_{\mathcal{H}} \quad (60)$$

where

$$L = \quad (61)$$

3.6 Necessary Kernel Inequalities

We will first extend the idea of Resilience [SCV18] to kernel learning.

Definition 22. (Resilience) from [SCV18]. Let \mathcal{H} represent the RKHS associated with the proper kernel $K : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, then given the feature mapping $\Phi : \mathbb{R}^d \rightarrow \mathcal{H}$, and the set $X = \{x_i\}_{i=1}^n = P \cup Q$, such that $|P| = n(1 - \epsilon)$ and $|Q| = n\epsilon$, It then follows for any subset $T \subseteq P$ such that $|T| = (1 - 2\epsilon)n$,

$$\left\| \frac{1}{|T|} \sum_{i \in T} \Phi(x_i) - \mu_{\mathbb{P}} \right\| \leq \tau$$

where $\mu_{\mathbb{P}} = \mathbb{E}_{x \sim \mathbb{P}}[\Phi(x)]$ is the kernel mean embedding for the distribution, \mathbb{P} . We say the set X has (ϵ, τ) -resilience in the Reproducing Kernel Hilbert Space.

Without the idea of resilience defined in definition 22, we will be unable to put error bounds on our algorithm.

In practice, however, it is important to note that solving for $\|\nabla \Psi_{\lambda}\|_{\mathcal{H}} = 0$ is NP-Hard. Thus, we will analyze the approximate stationary point.

Lemma 23 ([Roc70, DD19]). Assume the function Ψ is β -weakly convex. Let $\lambda < \frac{1}{\beta}$, and let $f_{\hat{w}} = \arg \min_{f_w \in \mathcal{K}} (\Psi(f_w) + \frac{1}{2\lambda} \|f_w - f_{\hat{w}}\|_{\mathcal{H}}^2)$, then $\|\nabla \Psi_{\lambda}(f_w)\|_{\mathcal{H}} \leq \epsilon$ implies:

$$\|f_{\hat{w}} - f_w\|_{\mathcal{H}} = \lambda \epsilon \quad \text{and} \quad \min_{g \in \partial \Psi(f_{\hat{w}}) + \partial \mathcal{I}_{\mathcal{K}}(f_{\hat{w}})} \|g\|_{\mathcal{H}} \leq \epsilon \quad (62)$$

4 Optimization Results

First, we will show using stepsize of $1/\beta$ returns a μ -approximate stationary point. However, since our methods are in the kernelized setting. The 2-norm, $\|w - w^*\|$ is not sufficient, we want $\|w - w^*\|_{\mathcal{H}}$ to be close, as the RKHS being small indicates the function $f(x)$ and $f^*(x)$ will be close.

4.1 Optimization in the Reproducing Kernel Hilbert Space

In this section, we will discuss and give necessary optimization results in the RKHS norm. In the analysis given in [JNJ20], given sufficient iterations, if we choose a sufficiently small learning rate, we can always reach a stationary point given infinite iterations. In practice, and in theory, this convergence rate is a square root factor slower. We prove, if we instead converge to a weaker stationary point, we can converge a factor of square root faster.

Theorem 24. Let $\Psi(f_w) \triangleq \max_{t \in \mathbb{R}} g(f_w, t)$ where $g(f_w, t)$ is β -smooth in f_w and L -Lipschitz in f_w and is concave in t but not necessarily convex in f_w . Then Algorithm 1 with stepsize $\eta = \frac{1}{\beta}$ reaches a $(\sqrt{2}L + \mathcal{O}(1))$ -stationary point in $\mathcal{O}(16\beta R + 4\beta np\sigma^2)$ iterations.

4.2 Accelerated Gradient Methods

When working with big data it is often the case we need faster gradient methods as the gradient can be expensive to obtain. In this section, we give results on the convergence rate of accelerated gradient methods on the update of w . We will analyze the convergence of three popular accelerated gradient methods.

4.2.1 Momentum Accelerated Gradient Descent

In this section we study Momentum Accelerated Gradient Descent [Qia99, Pol64] with our non-convex-concave optimization algorithm.

$$b^{(t)} = \mu b^{(t-1)} + \nabla_f \Phi \left(f_w^{(t-1)} \right) \quad (63)$$

$$f_w^{(t)} = f_w^{(t-1)} - \alpha b^{(t)} \quad (64)$$

Theorem 25. Momentum Accelerated Gradient Descent given in Equations (63) and (64) reaches a η -approximate stationary point. Algorithm 1 reaches a η -approximate stationary point in a polynomial number of iterations.

$$\mathbb{E} \left[\left\| \nabla \Phi_{1/2\ell}(f_w) \right\|_{\mathcal{H}}^2 \right] \leq \quad (65)$$

4.2.2 Nesterov Accelerated Gradient Descent

In this section we study Nesterov Accelerated Gradient Descent [Nes83] with our non-convex-concave optimization algorithm.

$$b^{(t+1)} = (1 + \mu) f_w^{(t)} - \mu f_w^{(t-1)} \quad (66)$$

$$f_w^{(t+1)} = b^{(t+1)} - \alpha \nabla_f \Phi \left(f_w^{(t)} \right) \quad (67)$$

Theorem 26. Nesterov Accelerated Gradient Descent given in Equations (66) and (67) reaches a η -approximate stationary point. Algorithm 1 reaches a η -approximate stationary point in a polynomial number of iterations.

$$\mathbb{E} \left[\left\| \nabla \Phi_{1/2\ell}(f_w) \right\|_{\mathcal{H}}^2 \right] \leq \quad (68)$$

5 Experiments

We perform numerical experiments on state of the art datasets comparing with other state of the art methods. We initialize the weights parameterizing f_w with the Glorot Initialization Scheme [GB10].

Algorithm 3: SUBQUANTILE-KERNEL

Input: Iterations: T ; Quantile: p ; Data Matrix: $\mathbf{X} \in \mathbb{R}^{n \times d}, n \gg d$; Labels: $\mathbf{y} \in \mathbb{R}^{n \times 1}$;
Learning Rate schedule: $\alpha_1, \dots, \alpha_T$; Ridge parameter: λ

Output: Trained Parameters: $f_w^{(T)}$

- 1: $w_i^{(0)} \leftarrow \text{Unif} \left[-\sqrt{\frac{6}{n}}, \sqrt{\frac{6}{n}} \right], \forall i \in [n]$ ▷ Base Learner
- 2: **for** $k = 1, 2, \dots, T$ **do**
- 3: $S^{(k)} \leftarrow \text{SUBQUANTILE}(f_w^{(k)}, \mathbf{X})$ ▷ Algorithm 2
- 4: $\nabla_{fg} \left(t^{(k+1)}, f_w^{(k)} \right) \leftarrow 2 \sum_{i \in S^{(k)}} \left(f_w^{(k)}(x_i) - y_i \right) \cdot K(x_i, \cdot)$ ▷ Regression
- 5: $\nabla_{fg} \left(t^{(k+1)}, f_w^{(k)} \right) \leftarrow \sum_{i \in S^{(k)}} \left(\sigma \left(f_w^{(k)}(x_i) \right) - y_i \right) \cdot K(x_i, \cdot)$ ▷ Binary Classification
- 6: $f_w^{(k+1)} \leftarrow f_w^{(k)} - \alpha_{(k)} \nabla_{fg} \left(t^{(k+1)}, f_w^{(k)} \right)$ ▷ w -update in eqn. (8)
- 7: **end**
- 8: **return** $f_w^{(T)}$

Algorithms	Test RMSE							
	Concrete		Wine Quality		Boston Housing		Drug	
	$\epsilon = 0.2(\downarrow)$	$\epsilon = 0.4(\downarrow)$	$\epsilon = 0.2(\downarrow)$	$\epsilon = 0.4(\downarrow)$	$\epsilon = 0.2(\downarrow)$	$\epsilon = 0.4(\downarrow)$	$\epsilon = 0.2(\downarrow)$	$\epsilon = 0.4(\downarrow)$
KRR	1.355 _(0.0934)	2.282 _(0.2063)	1.437 _(0.0979)	2.272 _(0.1088)	1.285 _(0.0896)	2.266 _(0.0686)	1.478 _(0.0533)	2.381 _(0.0203)
TERM [LBSS21]	0.829 _(0.0422)	0.928 _(0.0197)	1.854 _(0.7437)	1.069 _(0.1001)	0.879 _(0.0178)	0.875 _(0.0711)	∞	∞
SEVER [DKK+19]	0.533 _(0.0347)	0.592 _(0.0548)	0.915 _(0.0343)	0.841 _(0.0413)	0.526 _(0.0287)	0.720 _(0.1147)	1.172 _(0.0542)	1.215 _(0.0536)
SUBQUANTILE	0.519 _(0.0134)	0.547 _(0.0174)	0.808 _(0.0389)	0.827 _(0.0216)	0.468 _(0.0896)	0.458 _(0.0662)	1.280 _(0.0568)	1.132 _(0.0892)
Genie ERM	∞	∞	∞	∞	∞	∞	∞	∞

Table 1: Boston Housing, Concrete Data, Wine Quality, and Drug and Polynomial Synthetic Dataset. Label Noise: $y_{\text{noise}} \sim \mathcal{N}(5, 5)$. Feature Noise: $y_{\text{noise}} = 10000y_{\text{original}}$ and $x_{\text{noise}} = 100x_{\text{original}}$. Polynomial Regression Synthetic Dataset. 1000 samples, $x \sim \mathcal{N}(0, 1)$, $y \sim \mathcal{N}(\sum_{i=0} a_i x^i, 0.01)$ where $a_i \sim \mathcal{N}(0, 1)$. The Radial Basis Function is used in first three experiments and polynomial kernel with degree 3 and $C = 1$ is used in the last experiment.

In Figure 1, we see the final subquantile has significantly less outliers than the original corruption in the data set. Furthermore, we see there is a greater decrease in the higher outlier settings.

5.1 Linear Regression

In this section, we give experimental results for datasets using the linear kernel. This section will serve as a comparison to the various Robust Linear Regression Algorithms developed which are not meta-algorithms.

5.2 Kernel Binary Classification

In this section we will give the algorithm for subquantile minimization for the kernel classification problem and then give some experimental results on state of the art datasets comparing against other state of the art robust algorithms.

Now we will give some experimental results.

5.3 Kernel Multi-Class Classification

In this section we will provide some experimental results on the multi-class classification task.

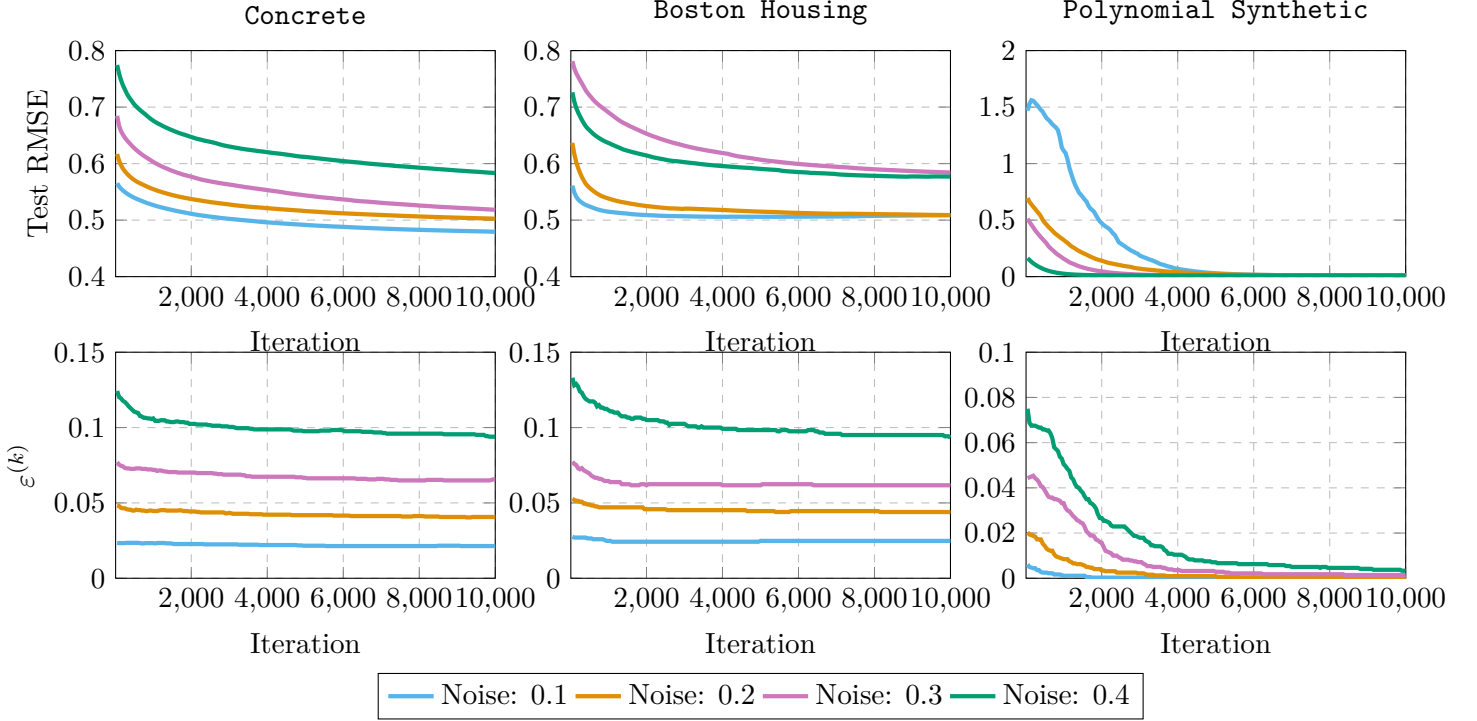


Figure 1: Test RMSE over the iterations in **Concrete**, **Boston Housing**, and **Polynomial** Datasets for SUBQUANTILE at different noise levels

Algorithms	Test RMSE							
	Boston Housing		Wine Quality		Concrete		Drug	
	Label(↓)	Label+Feature	Label	Label+Feature	Label	Label+Feature	Label	Label+Feature
KRR	0.907 _(0.2724)	90.799 _(5.7170)	0.894 _(0.0404)	62.913 _(7.4959)	0.825 _(0.0943)	77.383 _(5.5692)	2.679 _(0.1286)	141.690 _(3.5297)
RANSAC [FB81]	1.167 _(0.6710)	22.460 _(19.1987)	1.489 _(0.2730)	39.630 _(13.0294)	0.870 _(0.2308)	23.629 _(16.1023)	2.801 _(0.2004)	117.389 _(8.3915)
CRR [BJKK17]	0.636 _(0.0905)	88.626 _(5.7380)	0.818 _(0.0224)	58.488 _(3.5612)	0.710 _(0.0919)	73.932 _(4.7867)	1.887 _(0.1463)	152.827 _(6.6038)
STIR [MGJK19]	<u>0.562</u> _(0.0626)	78.878 _(8.0164)	0.828 _(0.0293)	58.352 _(4.6700)	0.684 _(0.0245)	76.555 _(4.5927)	1.721 _(0.1520)	144.975 _(5.4953)
SEVER [DKK ⁺ 19]	0.601 _(0.0979)	5.980 _(8.2603)	0.814 _(0.0207)	9.065 _(13.7632)	0.684 _(0.0438)	4.119 _(8.2436)	1.469 _(0.1162)	156.043 _(4.5543)
TERM [LBSS21]	0.608 _(0.1357)	<u>0.569</u> _(0.0620)	0.840 _(0.0563)	<u>0.827</u> _(0.0255)	0.830 _(0.0934)	<u>0.808</u> _(0.0726)	1.185 _(0.1077)	1.147 _(0.1258)
SUBQUANTILE	0.503 _(0.0470)	<u>0.560</u> _(0.0373)	0.813 _(0.0357)	0.821 _(0.0305)	0.720 _(0.1092)	0.630 _(0.0269)	<u>1.244</u> _(0.1091)	<u>2.413</u> _(0.6737)
Genie ERM	0.630 _(0.1015)	0.665 _(0.1134)	0.838 _(0.0130)	0.865 _(0.0222)	0.763 _(0.0390)	0.768 _(0.0181)	0.988 _(0.0823)	0.985 _(0.0838)

Table 2: For only Label Noise, $y_{\text{noisy}} \sim \mathcal{N}(5, 5)$. For Label and Feature Noise $x_{\text{noisy}} = 100x_{\text{original}}$ and $y_{\text{noisy}} = 10000y_{\text{original}}$.

5.4 Accelerated Gradient Methods

In this section we give some empirical results on using different accelerated gradient methods described in § Section 4. In ??, we see using momentum can give significantly faster convergence.

6 Discussion

The main contribution of this paper is the study of a nonconvex-concave formulation of Subquantile minimization for the robust learning problem for kernel ridge regression and kernel classification. We present an algorithm to solve the nonconvex-concave formulation and prove rigorous error bounds which show that the more good data that is given decreases the error bounds. We also

Algorithms	Test Accuracy							
	Heart Disease				Breast Cancer			
	Label		Label+Feature		Label		Label+Feature	
	$\epsilon = 0.2(\uparrow)$	$\epsilon = 0.4(\uparrow)$	$\epsilon = 0.2(\uparrow)$	$\epsilon = 0.4(\uparrow)$	$\epsilon = 0.2(\uparrow)$	$\epsilon = 0.4(\uparrow)$	$\epsilon = 0.2(\uparrow)$	$\epsilon = 0.4(\uparrow)$
SVM	0.777 _(0.0396)	0.639 _(0.0762)	0.534 _(0.0766)	0.538 _(0.0626)	0.926 _(0.0331)	0.548 _(0.1194)	0.649 _(0.0254)	0.618 _(0.0507)
SEVER [DKK ⁺ 19]	<u>0.793</u> _(0.0422)	<u>0.695</u> _(0.0636)	0.784 _(0.0432)	0.816 _(0.0562)	0.904 _(0.0356)	0.575 _(0.1456)	<u>0.956</u> _(0.0164)	<u>0.974</u> _(0.0062)
TERM [LBSS21]	0.741 _(0.0393)	0.620 _(0.0699)	<u>0.803</u> _(0.0613)	<u>0.810</u> _(0.0286)	<u>0.940</u> _(0.0378)	<u>0.763</u> _(0.0364)	0.986 _(0.0143)	0.986 _(0.0119)
SUBQUANTILE	0.803 _(0.0293)	0.790 _(0.0350)	0.833 _(0.0318)	<u>0.807</u> _(0.0468)	0.928 _(0.0129)	0.916 _(0.0185)	0.951 _(0.0212)	0.953 _(0.0197)
Genie ERM	∞	∞	∞	∞	∞	∞	∞	∞

Table 3: Heart Disease and Breast Cancer Dataset. Label Noise: $y_{\text{noise}} = \mathbb{I}\{y_{\text{original}} = 0\}$. Feature Noise: $x_{\text{noise}} = 100x_{\text{original}}$. The Linear Kernel is used in all experiments.

present accelerated gradient methods for the two-step algorithm to solve the nonconvex-concave optimization problem and give novel theoretical bounds.

Theory. From our theoretical bounds, we see the more good data we obtain, the closer the resultant function is to the optimal function in the RKHS, an important idea in learning theory. We see we obtain an approximately close function to the optimal with high probability. Furthermore, we extend minimax optimization with maximizing oracle gradient descent literature with novel bounds for accelerated gradient descent.

Experiments. From our experiments, we see Subquantile Minimization is competitive with algorithms developed solely for robust linear regression as well as other meta-algorithms. Our theoretical analysis is through the lens of kernel-learning, but the generalization to linear regression from a non-kernel perspective can be done. In kernelized regression, we see SUBQUANTILE is the strongest of the meta-algorithms. Furthermore, in binary and multi-class classification, SUBQUANTILE is very strong. Thus, we can see empirically SUBQUANTILE is the strongest meta-algorithm across all kernelized regression and classification tasks and also the strongest algorithm in linear regression.

Interpretability. One of the strengths in Subquantile Optimization is the high interpretability. Once training is finished, we can see the $n(1-p)$ points with highest error to find the outliers. Furthermore, there is only hyperparameter p , which should be chosen to be approximately the percentage of inliers in the data and thus is not very difficult to tune for practical purposes.

General Assumptions. The general assumption is the majority of the data should inliers. This is not a very strong assumption, as by the definition of outlier it should be in the minority.

The analysis of Subquantile Minimization can be extended to neural networks. This generalization will be appear in subsequent work.

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A Concentration Inequalities

In this section we will give various concentration inequalities on the inlier data. We first restate our assumptions.

Assumption 27. (Sub-Gaussian Design of Covariates). We assume each covariate is drawn i.i.d from a zero-mean covariance Σ sub-Gaussian distribution with sub-Gaussian norm $K_1 \in \mathbb{R}_{++}$.

$$\mathbb{E}[x_i] = 0 \quad (69)$$

$$\mathbb{E}[x_i x_i^\top] = \Sigma \quad (70)$$

For all $i \in [n]$,

$$\mathbb{P}\left\{v^\top x_i \geq t\right\} \leq \exp\left(-\frac{t^2}{K_1^2}\right) \quad (71)$$

Is this equivalent to

$$\mathbb{P}\left\{v^\top x_i \geq t\right\} \leq \exp\left(-\frac{t^2}{K_1^2}\right) \quad (72)$$

Assumption 28. (Sub-Gaussian Design of Optimal Residuals). Recall the residual is defined as $\eta_i \triangleq f_w^*(x_i) - y_i$. Then we assume for some $K_2 \in \mathbb{R}_{++}$

$$\mathbb{E}[\eta_i] = 0 \quad (73)$$

$$\mathbb{P}\{|\eta_i| \geq t\} \leq 2 \exp\left(-\frac{t^2}{K_2^2}\right) \quad (74)$$

Lemma 29 (Infinite Dimensional Covariance Estimation). Let $\mathcal{C}_{\mathbb{P}} = \mathbb{E}_{x_i \sim \mathbb{P}}[x_i \otimes x_i]$. Then let x_1, \dots, x_n be i.i.d sampled from \mathbb{P} , we then have

$$\left\|\frac{1}{n} \sum_{i=1}^n \Phi(x_i) \otimes \Phi(x_i) - \Sigma\right\|_{\text{HS}} = \mathcal{O}\left(n^{-1/2} C_k^2 \|\Sigma\|_{\text{HS}}\right) \quad (75)$$

Proof. Let ξ_i, \dots, ξ_n be n i.i.d Rademacher Random Variables. Then we have,

$$\mathbb{E} \left\| \frac{1}{n} \sum_{i=1}^n \Phi(x_i) \otimes \Phi(x_i) - \Sigma \right\|_{\text{HS}} = \mathbb{E} \left\| \frac{1}{n} \sum_{i=1}^n \Sigma^{1/2} \varphi(x_i) \otimes \Sigma^{1/2} \varphi(x_i) - \Sigma \right\|_{\text{HS}} \quad (76)$$

$$= \mathbb{E} \left\| \Sigma^{1/2} \left(\frac{1}{n} \sum_{i=1}^n \varphi(x_i) \otimes \varphi(x_i) - \mathbf{I} \right) \Sigma^{1/2} \right\|_{\text{HS}} \quad (77)$$

$$\leq \|\Sigma\|_{\text{op}} \mathbb{E} \left\| \frac{1}{n} \sum_{i=1}^n \varphi(x_i) \otimes \varphi(x_i) - \mathbf{I} \right\|_{\text{HS}} \quad (78)$$

$$\leq \frac{c_1}{n} \|\Sigma\|_{\text{op}} \mathbb{E} \left\| \sum_{i=1}^n \xi_i \varphi(x_i) \otimes \varphi(x_i) \right\|_{\text{HS}} \quad (79)$$

$$\leq \frac{c_1 c_2}{n} \|\Sigma\|_{\text{op}} \sqrt{\sum_{i=1}^n \|\varphi(x_i) \otimes \varphi(x_i)\|_{\text{HS}}^2} \quad (80)$$

$$\leq \frac{c_1 c_2}{n} \|\Sigma\|_{\text{op}} \sqrt{\sum_{i=1}^n \|\varphi(x_i)\|_{\mathcal{H}}^4} \quad (81)$$

$$\leq \frac{c_1 c_2 C_k^2 \|\Sigma\|_{\text{op}}}{\sqrt{n}} \quad (82)$$

where $C_k \triangleq \max_{i \in [n]} k(x_i, x_i)$. (ζ_1) follows from the Khintchine Inequality [HK94]. \blacksquare

A.1 Linear Kernel

We will first begin with the Linear Kernel Case.

Lemma 30.

A.2 Polynomial Kernel

The polynomial kernel is given by

$$k(x_1, x_2) = \left(x_1^\top x_2 + C \right)^p \quad (83)$$

Lemma 31. *Let x_1, \dots, x_n have sub-Gaussian design described in Assumption 27 and η_1, \dots, η_n have sub-Gaussian design described in Assumption 28. It then follows*

$$\mathbb{E} \left\| \sum_{i=1}^n \eta_i \Phi_{\text{poly}}(x_i) \right\| \leq \Xi \quad (84)$$

Lemma 32. *Let \mathbb{P} be a Normal Distribution over \mathbb{R}^d with identity covariance, then we have for any function $f_w \in \mathcal{H}$,*

$$\mathbb{E}_{x \sim \mathbb{P}} [f_w(x)^2] \geq C \|f_w\|_{\mathcal{H}}^2 \quad (85)$$

Proof. \blacksquare

A.3 Gaussian Kernel

The gaussian kernel is given for $\gamma > 0$ by

$$k(x_1, x_2) = \exp \left(-\gamma \|x_1 - x_2\|_2^2 \right) \quad (86)$$

Lemma 33. *Let $\eta_1, \dots, \eta_n \sim \mathcal{N}(0, \sigma^2)$ be sub-Gaussian. Then it follows*

$$\mathbb{E} \left| \sum_{i=1}^n \eta_i \right| \leq \Xi \quad (87)$$

We will utilize the Random Fourier Features (RFF) representation [RR07]. Let $w_1, \dots, w_d \sim \mathcal{N}_d(0, \frac{2}{\gamma} \mathbf{I})$, then the RFF representation is given by

$$\phi_{\text{RFF}}(x) = \frac{1}{\sqrt{d}} \left[\cos(w_1^\top x), \sin(w_1^\top x), \dots, \cos(w_d^\top x), \sin(w_d^\top x) \right] \quad (88)$$

The key idea behind the Randomized Fourier Features is

$$\mathbb{E} \left[\phi_{\text{RFF}}(x_1)^\top \phi_{\text{RFF}}(x_2) \right] = \exp \left(-\gamma \|x_1 - x_2\|_2^2 \right) \quad (89)$$

We will first derive a relation between $\mathbb{E} [\phi_{\text{RFF}}(x_1)^\top \phi_{\text{RFF}}(x_2)]$ and $\mathbb{E} [\phi_{\text{RFF}}(x_1)]^\top \mathbb{E} [\phi_{\text{RFF}}(x_2)]$.

B Proofs for Section 3

In this section we give the deferred proofs of main results.

B.1 Proof of Lemma 13

Proof. For any $f_{w_1}, f_{w_2} \in \mathcal{K}$, we will first show the gradient is bounded.

$$|g(t, f_{w_1}) - g(t, f_{w_2})| = \left| \int_0^1 \nabla_f g(t, (1-\lambda)f_{w_1} + \lambda f_{w_2})(f_{w_1} - f_{w_2}) d\lambda \right| \quad (90)$$

$$\leq \|f_{w_1} - f_{w_2}\|_{\mathcal{H}} \left| \int_0^1 \nabla_f g(t, (1-\lambda)f_{w_1} + \lambda f_{w_2}) d\lambda \right| \quad (91)$$

$$\stackrel{(a)}{\leq} \|f_{w_1} - f_{w_2}\|_{\mathcal{H}} \max_{w \in \mathcal{K}} \|\nabla_f g(t, f_w)\|_{\mathcal{H}} \quad (92)$$

In (a), we note that since \mathcal{K} is convex, then by definition as $f_{w_1}, f_{w_2} \in \mathcal{K}$, we have for $\lambda \in [0, 1]$, the convex combination $(1-\lambda)f_{w_1} + \lambda f_{w_2} \in \mathcal{K}$. We use the \mathcal{H} norm of the gradient to bound L from above for an element in the convex closed set \mathcal{K} .

$$\|\nabla g(t, f_w)\|_{\mathcal{H}} = \left\| \frac{2}{np} \sum_{i=1}^n \mathbb{I}\{t \geq (f_w(x_i) - y_i)^2\} (f_w(x_i) - y_i) \cdot k(x_i, \cdot) \right\|_{\mathcal{H}} \quad (93)$$

W.L.O.G, let x_1, x_2, \dots, x_m where $0 \leq m \leq n$, represent the data vectors such that $t \geq (f_w(x_i) - y_i)^2$.

$$= \left\| \frac{2}{np} \sum_{i=1}^m (f_w(x_i) - y_i) \cdot k(x_i, \cdot) \right\|_{\mathcal{H}} \quad (94)$$

$$\leq \frac{2}{np} \left(\left\| \sum_{i=1}^m f_w(x_i) \cdot k(x_i, \cdot) \right\|_{\mathcal{H}} + \left\| \sum_{i=1}^m y_i k(x_i, \cdot) \right\|_{\mathcal{H}} \right) \quad (95)$$

$$\stackrel{(a)}{\leq} \frac{2}{np} \left(\left\| \sum_{i=1}^m \left\langle \sum_{j=1}^n w_j k(x_j, \cdot), k(x_i, \cdot) \right\rangle_{\mathcal{H}} \cdot k(x_i, \cdot) \right\|_{\mathcal{H}} + \left\| \sum_{i=1}^m y_i \right\| \left\| \sum_{i=1}^m k(x_i, \cdot) \right\|_{\mathcal{H}} \right) \quad (96)$$

$$\leq \frac{2}{np} \left(\left\| \left\langle \sum_{j=1}^n w_j k(x_j, \cdot), \sum_{i=1}^m k(x_i, \cdot) \right\rangle_{\mathcal{H}} \right\|_{\mathcal{H}} + \left\| \sum_{i=1}^m y_i \right\| \left\| \sum_{i=1}^m \sqrt{k(x_i, x_i)} \right\| \right) \quad (97)$$

$$\leq \frac{2}{np} \left(\|f_w\|_{\mathcal{H}} \left(\sum_{i=1}^m \sqrt{k(x_i, x_i)} \right)^2 + \sqrt{n} \|y\|_2 \left(\sum_{i=1}^n \sqrt{k(x_i, x_i)} \right) \right) \quad (98)$$

$$\leq \frac{2R}{np} \left(\sum_{i=1}^n \sqrt{k(x_i, x_i)} \right)^2 + \frac{2\|y\|_2}{p\sqrt{n}} \left(\sum_{i=1}^n \sqrt{k(x_i, x_i)} \right) \quad (99)$$

(a) follows from the reproducing property for RKHS [Gre13]. If we have a normalized kernel such as the Gaussian Kernel, then we have the Lipschitz Constant is finite. Furthermore, if the adversary introduces label corruption that tends to ∞ , then these points will not be in the Subquantile as f_w has bounded norm, so it will have infinite error. This concludes the proof. \blacksquare

B.2 Proof of Lemma 15

Proof.

Let S be the set containing the points with the minimum error from X w.r.t to the weights vector w . Define $\eta_i \triangleq (f_{w^*}(x_i) - y_i)$ where $i \in P$.

$$\lim_{\lambda \rightarrow \infty} (\Psi_\lambda(f_w) - \Psi_\lambda(f_{w^*})) = \sum_{i \in S} (f_w(x_i) - y_i)^2 - \sum_{j \in P} (f_{w^*}(x_j) - y_j)^2 \quad (100)$$

$$= \sum_{i \in S \cap P} (f_w(x_i) - y_i)^2 + \sum_{i \in S \cap Q} (f_w(x_i) - y_i)^2 - \sum_{j \in P} (f_{w^*}(x_j) - y_j)^2 \quad (101)$$

$$\geq \sum_{i \in S \cap P} (f_w(x_i) - y_i)^2 - \sum_{j \in P} (f_{w^*}(x_j) - y_j)^2 \quad (102)$$

$$= \sum_{i \in S \cap P} (f_w(x_i) - f_{w^*}(x_i) - \eta_i)^2 - \sum_{j \in P} \eta_j^2 \quad (103)$$

$$= \sum_{i \in S \cap P} ((f_w - f_{w^*})(x_i) - \eta_i)^2 - \sum_{j \in P} \eta_j^2 \quad (104)$$

$$\geq \sum_{i \in S \cap P} \underbrace{((f_w - f_{w^*})(x_i))^2}_{\textcircled{A_1}} - 2 \underbrace{\sum_{i \in S \cap P} \eta_i (f_w - f_{w^*})(x_i)}_{\textcircled{A_2}} - \underbrace{\sum_{j \in P \setminus S} \eta_j^2}_{\textcircled{A_3}} \quad (105)$$

Now we will upper bound A_1 . Similar to [CLKZ21] Let $\mathbb{E}_{x \sim \mathbb{P}}[\varphi(x) \otimes \varphi(x)] = \mathbb{I}_m$ where $\varphi(x) = \{\varphi(x)\}_{k=1}^m$ and m is possibly infinite. We can then rescale the basis features. Then let $\Phi(x) = \Sigma^{1/2} \varphi(x)$. We therefore have $\Sigma = \mathbb{E}_{x \sim \mathbb{P}}[\Phi(x) \otimes \Phi(x)] = \text{diag}(\xi_1, \dots, \xi_n)$. This is the eigenfunction basis described in [SS16].

$$\begin{aligned} \textcircled{A_1} &\triangleq \sum_{i \in S \cap P} ((f_w - f_{w^*})(x_i))^2 \stackrel{(a)}{=} \sum_{i \in S \cap P} \left\langle \sum_{j \in X} (w_j - w_j^*) k(x_j, \cdot), k(x_i, \cdot) \right\rangle_{\mathcal{H}}^2 \\ &= \sum_{i \in S \cap P} \left\langle \sum_{j \in X} (w_j - w_j^*) \Sigma^{1/2} \varphi(x_j), \Sigma^{1/2} \varphi(x_i) \right\rangle_{\mathcal{H}} \left\langle \Sigma^{1/2} \varphi(x_i), \sum_{j \in X} (w_j - w_j^*) \Sigma^{1/2} \varphi(x_j) \right\rangle_{\mathcal{H}} \\ &= \sum_{i \in S \cap P} \left\langle \sum_{j \in X} (w_j - w_j^*) \Sigma^{1/2} \varphi(x_j), \left(\Sigma^{1/2} \varphi(x_i) \right) \otimes \left(\Sigma^{1/2} \varphi(x_i) \right) \sum_{j \in X} (w_j - w_j^*) \Sigma^{1/2} \varphi(x_j) \right\rangle_{\mathcal{H}} \\ &= \sum_{i \in S \cap P} \left\langle \left(\Sigma^{1/2} \varphi(x_i) \right) \otimes \left(\Sigma^{1/2} \varphi(x_i) \right), (f_w - f_w^*) \otimes (f_w - f_w^*) \right\rangle_{\text{HS}} \\ &= \sum_{i \in S \cap P} \left\langle \Sigma + \left(\Sigma^{1/2} \varphi(x_i) \right) \otimes \left(\Sigma^{1/2} \varphi(x_i) \right) - \Sigma, (f_w - f_w^*) \otimes (f_w - f_w^*) \right\rangle_{\text{HS}} \\ &\geq n(1 - 2\varepsilon) \left(\mathbb{E}_{x \sim \mathbb{P}} [(f_w - f_w^*)(x)]^2 - \left\| \frac{1}{n(1 - 2\varepsilon)} \sum_{i \in S \cap P} \left(\Sigma^{1/2} \varphi(x_i) \right) \otimes \left(\Sigma^{1/2} \varphi(x_i) \right) - \Sigma \right\|_{\text{HS}} \|f_w - f_w^*\|_{\mathcal{H}}^2 \right) \\ &\geq n(1 - 2\varepsilon) \left(\lambda_{\min}(\mathbb{E}_{x \sim \mathbb{P}}[\Phi(x) \otimes \Phi(x)]) \|f_w - f_w^*\|_{\mathcal{H}}^2 - \mathcal{O} \left(\frac{C_k \|\Sigma\|_{\text{op}}}{\sqrt{n(1 - 2\varepsilon)}} \right) \right) \|f_w - f_w^*\|_{\mathcal{H}}^2 \end{aligned}$$

Next we will upper bound (A_2) ,

$$(A_2) \triangleq \sum_{i \in S \cap P} \eta_i (f_w - f_{w^*})(x_i) \quad (106)$$

$$= \sum_{i \in S \cap P} \left\langle \sum_{j \in X} (w_j - w_j^*) k(x_j, \cdot), \eta_i k(x_i, \cdot) \right\rangle_{\mathcal{H}} \quad (107)$$

$$= \left\langle \sum_{j \in X} (w_j - w_j^*) k(x_j, \cdot), \sum_{i \in S \cap P} \eta_i k(x_i, \cdot) \right\rangle_{\mathcal{H}} \quad (108)$$

$$\leq \|f_w - f_{w^*}\|_{\mathcal{H}} \left\| \sum_{i \in S \cap P} \eta_i k(x_i, \cdot) \right\|_{\mathcal{H}} = \|f_w - f_{w^*}\|_{\mathcal{H}} \left\| \sum_{i \in S \cap P} \eta_i \Phi(x_i) \right\|_{\mathcal{H}} \quad (109)$$

We will now lower bound B_1 . For the linear kernel, for B_1 to be greater than 0, than if $x \in \mathbb{R}^d$, then we must have $\frac{d}{1-2\varepsilon} < n$, otherwise we will have a rank-deficient matrix which will thus have singular values of value 0. For the polynomial kernel, for B_1 to be greater than 0, than $n > d^r$ where r is the polynomial degree. We thus have

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} (\Psi_{\lambda}(f_w) - \Psi_{\lambda}(f_{w^*})) &\geq \eta^2 \left(\lambda_{\min}(\mathbb{E}_{x \sim \mathbb{P}} [\Phi(x) \otimes \Phi(x)]) - \mathcal{O} \left(\frac{C_k \|\Sigma\|_{\text{op}}}{\sqrt{n(1-2\varepsilon)}} \right) \right) \\ &\quad - 2\eta \left\| \sum_{i \in S \cap P} \eta_i \Phi(x_i) \right\| - \sum_{j \in P \setminus S} \eta_j^2 \end{aligned} \quad (110)$$

This completes the proof. ■

B.3 Proof of Theorem 16

Proof. First, we give the definition of the Moreau stationary point.

$$\|\nabla M_{\Psi_{\lambda}, \rho}(f_w)\|_{\mathcal{H}} = \left\| \frac{1}{\gamma} \left(f_w - \arg \min_{f_{\hat{w}} \in \mathcal{K}} \left(\Psi(f_{\hat{w}}) + \frac{1}{2\gamma} \|f_w - f_{\hat{w}}\|_{\mathcal{H}}^2 \right) \right) \right\|_{\mathcal{H}} = 0 \quad (111)$$

This implies for any $f_{\tilde{w}} \in \mathcal{K}$, it follows

$$\lim_{\lambda \rightarrow \infty} (\Psi_{\lambda}(f_{\hat{w}})) < \lim_{\lambda \rightarrow \infty} (\Psi_{\lambda}(f_{\tilde{w}})) + \frac{1}{2\gamma} \|f_{\tilde{w}} - f_{\hat{w}}\|_{\mathcal{H}}^2 \quad (112)$$

For any $f_{\hat{w}}$ satisfying above, then the distance from the optimal must be low. Let $\tilde{w} = w^*$, then we have

$$\lim_{\lambda \rightarrow \infty} (\Psi_{\lambda}(f_{\hat{w}}) - \Psi_{\lambda}(f_{w^*})) \leq \frac{1}{2\gamma} \|f_{\hat{w}} - f_{w^*}\|_{\mathcal{H}}^2 \quad (113)$$

We proceed by proof by contradiction. Assume $\|f_{\hat{w}} - f_{w^*}\| > \eta$, then if $\Psi(f_{\hat{w}}) - \Psi(f_{w^*}) > \frac{1}{2}\eta^2$, then we will have $f_{\hat{w}}$ is not a stationary point, which will imply $\|f_{\hat{w}} - f_{w^*}\|_{\mathcal{H}} \leq \eta$. Therefore, we attempt to find the minimum value for η . From Lemma 15, we have we have

$$\lim_{\lambda \rightarrow \infty} (\Psi(f_w) - \Psi(f_{w^*})) \geq \eta^2 \sigma_{\min} \left(\sum_{i \in S \cap P} \Phi(x_i) \Phi(x_i)^{\top} \right) - 2\eta \left\| \sum_{i \in S \cap P} \eta_i \Phi(x_i) \right\| - \sum_{j \in P \setminus S} \eta_j^2 \quad (114)$$

From the definition of stationary point, we have

$$\eta^2 \left(\sigma_{\min} \left(\sum_{i \in S \cap P} \Phi(x_i) \Phi(x_i)^\top \right) - \beta \right) - 2\eta \left\| \sum_{i \in S \cap P} \eta_i \Phi(x_i) \right\| - \sum_{j \in P \setminus S} \eta_j^2 \leq 0 \quad (115)$$

Therefore, when Equation (115) does not hold, we have a contradiction. It thus follows from upper bounding the positive solution of the quadratic equation,

$$\begin{aligned} \eta &\leq \left(\sum_{j \in P \setminus S} \eta_j^2 \right)^{1/2} \left(n(1-2\varepsilon) \left(\lambda_{\min}(\mathbb{E}_{x \sim \mathbb{P}} [\Phi(x) \otimes \Phi(x)]) - \mathcal{O} \left(\frac{C_k \|\Sigma\|_{\text{op}}}{\sqrt{n(1-2\varepsilon)}} \right) \right) - \beta \right)^{-1/2} \\ &\quad + 2 \left\| \sum_{i \in S \cap P} \eta_i \Phi(x_i) \right\|_{\mathcal{H}} \left(n(1-2\varepsilon) \left(\lambda_{\min}(\mathbb{E}_{x \sim \mathbb{P}} [\Phi(x) \otimes \Phi(x)]) - \mathcal{O} \left(\frac{C_k \|\Sigma\|_{\text{op}}}{\sqrt{n(1-2\varepsilon)}} \right) \right) - \beta \right)^{-1} \end{aligned} \quad (116)$$

Then, if $n \geq \frac{(1-2\varepsilon)(C_k \|\Sigma\|_{\text{op}} + \beta)}{(1-c_1)\lambda_{\min}(\Sigma)} + \sqrt{\beta}$, we have

$$\eta \leq \sqrt{\frac{\sum_{j \in P \setminus S} \eta_j^2}{c_1 n \lambda_{\min}(\Sigma)}} + \frac{2 \left\| \sum_{i \in S \cap P} \eta_i \Phi(x_i) \right\|_{\mathcal{H}}}{c_1 n \lambda_{\min}(\Sigma)} \quad (117)$$

This completes the proof. ■

B.4 Proof of Lemma 18

Proof. We use the \mathcal{H} norm of the gradient to bound L from above. Let S be denoted as the subquantile set. Define the sigmoid function as $\sigma(x) = \frac{1}{1+e^{-x}}$.

$$\|\nabla_{\mathbf{f}} g(t, f_w)\|_{\mathcal{H}} = \left\| \frac{1}{np} \sum_{i=1}^n \mathbb{I}\{t \geq (1-y_i) \log(f_w(x_i))\} (y_i - \sigma(f_w(x_i))) \cdot k(x_i, \cdot) \right\|_{\mathcal{H}} \quad (118)$$

$$\leq \frac{1}{np} \left\| \sum_{i \in S} (y_i - \sigma(f_w(x_i))) \right\| \left\| \sum_{i \in S} k(x_i, \cdot) \right\|_{\mathcal{H}} \quad (119)$$

$$\stackrel{(a)}{\leq} \frac{1}{p} \sum_{i=1}^n \sqrt{k(x_i, x_i)} \quad (120)$$

(a) follows from the fact that $y_i \in \{0, 1\}$ and $\text{range}(\sigma) \in [0, 1]$. This completes the proof. ■

B.5 Proof of Lemma 19

We use the Hilbert Space norm of second derivative to bound β from above. Let S be the subquantile set.

$$\|\nabla_{\mathbf{f}}^2 g(t, f_w)\|_{\mathcal{H}} = \frac{1}{np} \sum_{i=1}^n \mathbb{I}\{t \geq (1-y_i) \log(f_w(x_i))\} \sigma(f_w(x_i)) (1 - \sigma(f_w(x_i))) k(x_i, x_i) \quad (121)$$

$$\stackrel{\zeta_1}{\leq} \frac{1}{4p} \sum_{i=1}^n k(x_i, x_i) = \frac{1}{4p} \text{Tr}(\mathbf{K}) \quad (122)$$

(ζ_1) follows as for a scalar $\alpha \in [0, 1]$, the maximum value of $\alpha(1-\alpha)$ is obtained at $\frac{1}{4}$. This completes the proof. \blacksquare

B.6 Proof of Lemma 20

Proof. Let S be the Subquantile set for f_w , then we have

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \Psi(f_w) - \Psi(f_w^*) &\geq - \sum_{i \in S} y_i \log(\sigma(f_w(x_i))) + (1 - y_i) \log(1 - \sigma(f_w(x_i))) \\ &\quad + \sum_{i \in P} y_i \log(\sigma(f_w^*(x_i))) + (1 - y_i) \log(1 - \sigma(f_w^*(x_i))) \end{aligned} \quad (123)$$

\blacksquare

B.7 Proof of Lemma 21

Proof. We use the Hilbert Space norm of the gradient to bound L from above. Let S be denoted as the subquantile set.

$$\|\nabla_W g(t, W)\|_2 = \quad (124)$$

\blacksquare

C Proofs for Section 4

In this section we give the optimization results from Section 4.

C.1 Proof of Theorem 24

From Theorem 31 in [JNJ20], we have

$$\begin{aligned} \mathbb{E} [\|\nabla \Psi_\beta(f_{\bar{w}})\|_{\mathcal{H}}] &\leq \sqrt{2 \cdot \frac{(\Psi_\beta(f_w^{(0)}) - \min \Psi(f_w)) + \beta L^2 \gamma^2}{\gamma \sqrt{T+1}}} \\ &\leq \sqrt{2 \cdot \frac{(\min_{f_{\bar{w}} \in \mathcal{K}} \left\{ \Psi(f_{\bar{w}}) + \frac{1}{2} \|f_w^{(0)} - f_{\bar{w}}\|_{\mathcal{H}}^2 \right\} - \min_{f_w \in \mathcal{K}} \Psi(f_w)) + \beta L^2 \gamma^2}{\gamma \sqrt{T+1}}} \\ &\leq \sqrt{2 \cdot \frac{4R^2 + \sum_{i \in S} \eta_i^2 + \beta L^2 \gamma^2}{\gamma \sqrt{T+1}}} \end{aligned}$$

If we are to choose $\gamma = \frac{\sqrt{T+1}}{\beta}$, then with probability $1 - \delta$ for $\delta \in (0, 1)$, we have

$$\mathbb{E} [\|\nabla \Psi_\beta(f_{\bar{w}})\|_{\mathcal{H}}] \leq \sqrt{\frac{8R^2 + 2np(1+\delta)\sigma^2}{\beta}} + 2L \quad (125)$$

$$\leq \sqrt{\frac{\beta(8R^2 + 2\sum_{i \in S} \eta_i^2)}{T+1}} + \sqrt{2}L \quad (126)$$

$$\leq \frac{2\sqrt{2}\beta R + \sqrt{2}\beta \sqrt{\sum_{i \in S} \eta_i^2}}{\sqrt{T+1}} + \sqrt{2}L \quad (127)$$

An application of a tail bound for the norm of a Gaussian vector gives us with probability $1 - \delta$ for $\delta \in (0, 1)$ the following

$$\mathbb{E} [\|\nabla \Psi_\beta(f_{\tilde{w}})\|_{\mathcal{H}}] \leq \frac{\sqrt{2\beta} \left(2R + \sigma\sqrt{np} + \sqrt{2\sqrt{np}\sigma \log\left(\frac{1}{2\delta}\right)} \right)}{\sqrt{T+1}} + \sqrt{2}L \quad (128)$$

Setting T to the the square of the numerator gives the desired result.

C.2 Proof of Theorem 25

We will start with the definition of the Moreau Envelope.

$$\bar{\Phi}_\lambda(f_w) = \frac{1}{\lambda} \left(f_w - \arg \min_{f_{\tilde{w}} \in \mathcal{K}} \left\{ \Phi(f_{\tilde{w}}) + \frac{1}{2\lambda} \|f_w - f_{\tilde{w}}\|_{\mathcal{H}}^2 \right\} \right) \quad (129)$$

Note $g(t, f_w)$ is L -lipschitz in f_w . Let $f_{\tilde{w}}^{(t)} = \arg \min_{f_{\tilde{w}} \in \mathcal{K}} \{ \bar{\Phi}(f_w) + \|f_{\tilde{w}} - f_w\|_{\mathcal{H}}^2 \}$. Then we have,

$$\bar{\Phi}_\lambda(f_w^{(t+1)}) \leq \Phi(f_{\tilde{w}}^{(t)}) + \|f_w^{(t+1)} - f_{\tilde{w}}^{(t)}\|_{\mathcal{H}}^2 \quad (130)$$

$$= \Phi(f_{\tilde{w}}^{(t)}) + \|f_{\tilde{w}}^{(t)} - \Pi_{\mathcal{K}}(f_w^{(t)} - \alpha(\mu b^{(t)}))\|_{\mathcal{H}}^2 \quad (131)$$

$$= \Phi(f_{\tilde{w}}^{(t)}) + \|f_{\tilde{w}}^{(t)} - \Pi_{\mathcal{K}}(f_w^{(t)} - \alpha(\mu(\mu b^{(t-1)} + \nabla_f \Phi(f_w^{(t-1)})))\|_{\mathcal{H}}^2 \quad (132)$$

$$= \Phi(f_{\tilde{w}}^{(t)}) + \left\| f_{\tilde{w}}^{(t)} - \Pi_{\mathcal{K}} \left(f_w^{(t)} - \alpha \left(\sum_{i=k}^t \mu^k (1 - \mu) \nabla_f f_w^{(t-k)} \right) \right) \right\|_{\mathcal{H}}^2 \quad (133)$$

$$\leq \Phi(f_{\tilde{w}}^{(t)}) + \left\| f_{\tilde{w}}^{(t)} - f_w^{(t)} - \alpha \left(\sum_{k=1}^t \mu^k (1 - \mu) \nabla_f f_w^{(t-k)} \right) \right\|_{\mathcal{H}}^2 \quad (134)$$

C.3 Proof of Theorem 26

D Necessary Lemmas

Lemma 34 (MGF of Sub-Exponential Random Variable). *Let x be a Sub-exponential variable, then we have*

$$\mathbb{E} [\exp(\lambda x)] \leq \exp(C\lambda^2) \quad (135)$$

Theorem 35 (Matrix Chernoff, [Tro12]). *Let X_k be a sequence of independent, random, self-adjoint matrices with dimension d s.t.*

$$X_k \succcurlyeq 0 \quad \text{and} \quad \lambda_{\max}(X_k) \leq R \quad \text{almost surely} \quad (136)$$

Define

$$\mu_{\min} \triangleq \lambda_{\min} \left(\sum_k \mathbb{E} X_k \right) \quad (137)$$

Then for $\delta \in [0, 1]$

$$\mathbb{P} \left\{ \lambda_{\min} \left(\sum_k X_k \right) \geq (1 - \delta) \mu_{\min} \right\} \leq d \cdot \left[\frac{e^{-\delta}}{(1 - \delta)^{1-\delta}} \right]^{\mu_{\min}/R} \quad (138)$$

Theorem 36 (Matrix Bernstein, [?]). *Let X_k be a sequence of independent, random, self-adjoint matrices with dimension d s.t.*

$$\mathbb{E}X_k = 0 \quad (139)$$

Theorem 37 (Vector Bernstein, [Gro11, KL17]). *Let x_k be a sequence of independent, random vectors such that*

$$\mathbb{E}x_k = 0 \quad \forall k \quad (140)$$

Theorem 38 (Mercer, [Mer09]). *If k is a positive definite and continuous kernel on a compact set Ω , the operator T has a countable set of eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$ and eigenfunctions $\{\varphi\}_{j \in \mathbb{N}}$ with $T\varphi_j = \lambda_j \varphi_j$ s.t. $\|\varphi_j\| = \lambda_j^{-1}$. The kernel then admits the following representation,*

$$k(x, y) = \sum_{j=1}^{\infty} \lambda_j \varphi_j(x) \varphi_j(y) \quad (141)$$

E Experimental Details

In this section we give details on datasets and hyperparameters.

E.1 Kernel Regression

Our datasets are synthetic and are sourced from [DG17]

Dataset	Dimension d	Sample Size n	Source
Polynomial	3	1000	Ours
Boston Housing	13	506	[DG17]
Concrete Data	8	1030	[DG17]
Wine Quality	11	1599	[DG17]

Table 4: Polynomial Regression Synthetic Dataset. 1000 samples, $x \sim \mathcal{N}(0, 1)$, $y \sim \mathcal{N}(\sum_{i=0} a_i x^i, 0.01)$ where $a_i \sim \mathcal{N}(0, 1)$. Obvious Noise is sampled from $\mathcal{N}(0, 5)$. Subquantile is capped at 10,000 iterations.

E.2 Kernel Binary Classification

Dataset	Dimension d	Sample Size n	Source
Heart Disease	13	303	[JD88]
Breast Cancer	32	569	Kaggle

Table 5: Datasets for Kernel Binary Classification.

E.3 Kernel Multi-Class Classification

E.4 Linear Regression

Dataset	Dimension d	Sample Size n	Source
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Table 6: Datasets for Kernel Multi-Class Classification.

Dataset	Dimension d	Sample Size n	Source
Boston Housing	14	506	Kaggle
Wine Quality	11	1599	[DG17]
Concrete	8	1030	[DG17]
Drug			

Table 7: Datasets for Linear Regression.