Subquantile Minimization for Kernel Learning in the Huber ϵ-Contamination Model*

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Abstract

In this paper we propose Subquantile Minimization for learning with adversarial corruption in the training set. Superquantile objectives have been formed in the past in the context of fairness where one wants to learn an underrepresented distribution equally [LPMH21, RRM14]. Our intuition is to learn a more favorable representation of the *majority* class, thus we propose to optimize over the p-subquantile of the loss in the dataset. In particular, we study the Huber Contamination Problem for Kernel Learning where the distribution is formed as, $\hat{\mathbb{P}} = (1-\varepsilon)\mathbb{P} + \varepsilon\mathbb{Q}$, and we want to find the function $\inf_f \mathbb{E}_{x\in\mathbb{P}}[\ell_f(x)]$, from the noisy distribution, $\hat{\mathbb{P}}$. We assume the adversary has knowledge of the true distribution of $\mathbb{P}$, and is able to corrupt the covariates and the labels of ε samples. To our knowledge, we are the first to study the problem of general kernel learning in the Huber Contamination Model. In our theoretical analysis, we analyze our non-convex concave objective function with the Moreau Envelope. We show (i) a stationary point with respect to the Moreau Envelope is a good point and (ii) we can reach a stationary point with gradient descent methods. Further, we analyze accelerated gradient methods for the non-convex concave minimax optimization problem. We empirically test Kernel Ridge Regression and Kernel Classification on various state of the art datasets and show Subquantile Minimization gives strong results. Furthermore, we run experiments on various datasets and compare with the state-of-the-art algorithms to show the superior performance of Subquantile Minimization.

*Preliminary Work

1 Introduction

There has been extensive study of algorithms to learn the target distribution from a Huber ε-Contaminated Model for a Generalized Linear Model (GLM), [DKK+19, ADKS22, LBSS21, OZS20, FB81] as well as for linear regression [BJKK17, MGJK19]. Robust Statistics has been studied extensively [DK23] for problems such as high-dimensional mean estimation [PBR19, CDGS20] and Robust Covariance Estimation [CDGW19, FWZ18]. Recently, there has been an interest in solving robust machine learning problems by gradient descent [PSBR18, DKK+19]. Subquantile minimization aims to address the shortcomings of standard ERM in applications of noisy/corrupted data [KLA18, JZL+18]. In many real-world applications, linear models are insufficient to model the data. Therefore, we consider the problem of Robust Learning for Kernel Learning.

Definition 1. (**Huber ϵ-Contamination Model** [HR09]). Given a corruption parameter $0 < \epsilon < 0.5$, a data matrix, X and labels y. An adversary is allowed to inspect all samples and modify $n\varepsilon$ samples arbitrarily. The algorithm is then given the ϵ-corrupted data matrix X and y as training data.

Current approaches for robust learning across various machine learning tasks often use gradient descent over a robust objective, [LBSS21]. These robust objectives tend to not be convex and therefore do not have a strong analysis on the error bounds for general classes of models.
We similarly propose a robust objective which has a nonconvex-concave objective. This objective has also been proposed recently in [HYL+20] where there has been an analysis in the Binary Classification Task. We show Subquantile Minimization reduces to the same objective in [HYL+20]. We use theory from the weakly-convex concave optimization literature for our error bounds. We are able to levarage this theory by analyzing the asymptotic distribution of a softplus approximation of the Subquantile objective.

Theorem 2. *(Informal). Let the dataset be given as $\{(x_i, y_i)\}_{i=1}^n$ such that the labels and features of εn samples are arbitrarily corrupted by an adversary. Let* K *be the kernel matrix and S be the points with the lowest error w.r.t $f_{\widehat{w}}$, then Subquantile Minimization returns $f_{\widehat{w}}$ for $n \geq \frac{(1-2\varepsilon)(C_k\|\Sigma\|_{\text{op}}+\beta)}{(1-c_1)\lambda_{\min}(\Sigma)} + \sqrt{\beta}$ for a constant $c_1 \in (0,1)$ such that for*
Kernelized Regression:

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}} \|f_{\widehat{w}} - f_{w^*}\|_{\mathcal{H}} \leq O\left(\frac{\gamma\sigma}{\sqrt{\lambda_{\min}(\Sigma)}}\right) \tag{1}$$

where $\epsilon \to 0$ as number of gradient descenter iterations goes to ∞ and $\Sigma = \mathbb{E}[\phi(x) \otimes \phi(x)]$.
Kernel Binary Classification:

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}} \|f_{\widehat{w}} - f_{w^*}\|_{\mathcal{H}} \leq O(\gamma) \tag{2}$$

Kernel Multi-Class Classification:

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}} \left\|f_{\widehat{\mathrm{W}}} - f_{\mathrm{W}^*}\right\|_{\mathcal{H}} \leq O(\gamma) \tag{3}$$

1.1 Related Work

The idea of iterative thresholding algorithms for robust learning tasks dates back to 1806 by Legendre [Leg06]. From the popularity of Machine Learning, numerous algorithms have been developed in this idealogy. Therefore, we will dedicate this section to reviewing such works and to make clear our contributions to the iterative thresholding literature.

1.1.1 Robust Regression via Hard Thresholding [BJK15]

Bhatia et al. consider robust linear regression by considering an active set S, which contains the points with the lowest error. This set is updated each iteration in conjunction with either a full solve (Torrent-FC) or a gradient iteration (Torrent-GD). Torrent-GD is the same algorithm as ours. The main limitation of this work is that only the case of label corruption is considered. We pick up the result of Theorem 9 and Theorem 11 in [BJK15] (with different constants) for linear regression with and without feature corruption.

1.1.2 Learning with bad training data via iterative trimmed loss minimization [SS19]

This work considers optimizing over the bottom-k errors by choosing the αn points with smallest error and then updating the model from these αn. This general model is the same as ours. Theoretically, this work considers only general linear models. Experimentally, this work considers more general machine learning models such as GANS.

1.1.3 Trimmed Maximum Likelihood Estimation for Robust Generalized Linear Model [ADKS22]

This work studies a different class of generalized linear models. Interestingly, they show for Gaussian Regression the iterative trimmed maximum likelihood estimatior is able to achieve near minimax optimal error. This work does not consider feature corruption and primarily focuses on the covariates sampled with Gaussian Design from Identity covariance.

1.1.4 Sum of Ranked Range Loss for Supervised Learning [HYL+20]

Hu et al. proposed learning over the bottom k losses, this is an alternative formulation of our algorithm. They solve their optimization problem with difference of sums convex solvers. This work considers only the classificatoin task and does not give rigorous error bounds.

1.2 Contributions

We will now state our main contributions clearly.

1. We provide a novel theoretical framework using the Moreau Envelope for analyzing the iterative trimmed estimator for machine learning tasks.
2. We provide rigorous error bounds for subquantile minimization in the kernel regression, kernel binary classification, and kernel multi-class classification.
3. We perform experiments on state-of-the-art matrices in kernel learning and show the effectiveness of our algorithm compared to other robust meta-algorithms.

2 Subquantile Minimization

We propose to optimize over the subquantile of the risk. The p-quantile of a random variable, U, is given as $\mathcal{Q}_p(U)$, this is the largest number, t, such that the probability of $U \leq t$ is at least p.

$$\mathcal{Q}_p(U) \leq t \iff \mathbb{P}\{U \leq t\} \geq p \tag{4}$$

The p-subquantile of the risk is then given by

$$\mathbb{L}_p(U) = \frac{1}{p}\int_0^p \mathcal{Q}_p(U)\, dq = \mathbb{E}[U | U \leq \mathcal{Q}_p(U)] = \max_{t\in\mathbb{R}} \left\{ t - \frac{1}{p}\mathbb{E}(t-U)^+ \right\} \tag{5}$$

Given an objective function, ℓ, the kernelized learning poblem becomes:

$$f_{\widehat{w}} = \arg\min_{f_w \in \mathcal{K}} \max_{t\in\mathbb{R}} \left\{ g(t, f_w) \triangleq t - \sum_{i=1}^{n} \left(t - (f_w(x_i) - y_i)^2\right)^+ \right\} \tag{6}$$

where t is the p-quantile of the empirical risk. Note that for a fixed t therefore the objective is not concave with respect to w. Thus, to solve this problem we use the iterations from equation 11 in [RHL+20]. Let $\Pi_{\mathcal{K}}$ be the projection of a vector on to the convex set $\mathcal{K} \triangleq \{f \in \mathcal{H} : \|f\|_{\mathcal{H}} \leq R\}$, then our update steps are

$$t^{(k+1)} = \arg\max_{t\in\mathbb{R}} g(f_w^{(k)}, t) \tag{7}$$

$$f_w^{(k+1)} = \Pi_{\mathcal{K}}\left(f_w^{(k)} - \alpha \nabla_f g(f_w^{(k)}, t^{(k+1)})\right) \tag{8}$$

Claim 3. *The function $g(t, f_w)$ defined in Equation (6) is non-convex-concave, i.e. it is not convex with respect to f_w and is concave with respect to t. Furthermore, $g(t, f_w)$ is ρ-weakly convex where ρ is the β-smoothness factor of $g(t, f_w)$ w.r.t f_w.*

Proof. We will show $-g(t, f_w)$ is convex with respect to t. Let $\nu_i \triangleq (f_w(x_i) - y_i)^2$.

$$-g(\lambda t_1 + (1-\lambda)t_2, f_w) = -\lambda t_1 - (1-\lambda)t_2 + \sum_{i=1}^{n} (\lambda t_1 + (1-\lambda)t_2 - \nu_i)^+ \tag{9}$$

$$\leq -\lambda t_1 - (1-\lambda)t_2 + \sum_{i=1}^{n} \lambda(t_1 - \nu_i)^+ + (1-\lambda)(t_2 - \nu_i)^+ \tag{10}$$

$$= -\lambda g(t_1, f_w) - (1-\lambda) g(t_2, f_w) \tag{11}$$

Therefore we have $g(t, f_w)$ is concave in t. Next we will prove $g(t, f_w)$ is not convex in f_w. ■

Claim 4. *The function $g(t, f_w)$ defined in Equation (6) is L-weakly convex in f_w, where L is lipschitz constant of the gradient of $g(t, f_w)$ w.r.t f_w. This is true for Conditional Value at Risk, [RU02].*

Proof. Here we note that if we add $\max_{i\in[n]} |k(x_i, x_i) + f_w(x_i) - y_i|$ to each of the second derivatives, then we are pushing the trace to be negative . Note this is the L-lispchitz gradient. This is equivalent to $g(f_w, t) + \frac{L}{2} \|f_w\|^2_{\mathcal{H}}$. ■

We provide an algorithm for Subquantile Minimization of the ridge regression and classification kernel learning algorithm. Algorithm 1 is applicable to both kernel ridge regression and kernel classication.

Algorithm 1: SUBQ-GRADIENT

Input: Iterations: T; Quantile: p; Data Matrix: $\mathrm{X}, (n \times d), n \gg d$; Learning schedule: $\alpha_1, \cdots, \alpha_T$; Ridge parameter: λ

Output: Trained Parameters, $w_{(T)}$

$w_{(0)} \leftarrow \mathcal{N}_d(0, \sigma)$

for $k \in 1, 2, \ldots, T$ **do**

  $\mathrm{S}_{(k)} \leftarrow$ SUBQUANTILE$(w^{(k)}, \mathrm{X})$

  $w^{(k+1)} \leftarrow w^{(k)} - \alpha_{(k)} \nabla_w g\left(t^{(k+1)}, w^{(k)}\right)$

end

return $w_{(T)}$

Algorithm 2: SUBQUANTILE

Input: Parameters w, Data Matrix: $\mathrm{X}, (n \times d)$, Convex Loss Function f

Output: Subquantile Matrix S

$\hat{\nu}_i \leftarrow \ell(x_i; f_w, y_i)$ s.t. $\hat{\nu}_{i-1} \leq \hat{\nu}_i \leq \hat{\nu}_{i+1}$

$t \leftarrow \hat{\nu}_{np}$

Let $x_1, ..., x_{np}$ be np points such that $\ell(x_i; f_w, y_i) \leq t$

$\mathrm{S} \leftarrow \begin{pmatrix} x_1^\top & \ldots & x_{np}^\top \end{pmatrix}^\top$

return S

3 Structural Results

To consider theoretical guarantees of Subquantile Minimization, we first analyze the inner and outer optimization problems. We first analyze kernel learning in the presence of corrupted data. Next, we provide error bounds for the two most important kernel learning problems, kernel ridge regression, and kernel classification. Now we will give our first result regarding kernel learning in the Huber ϵ-contamination model. Now we will analyze the two-step minimax optimization steps described in Equations (7) and (8).

Lemma 5. *Let $f(x; w)$ be a convex loss function. Let $x_1, x_2, \cdots, x_n$ denote the n data points ordered such that $f(x_1; w, y_1) \leq f(x_2; w, y_2) \leq \cdots \leq f(x_n; w, y_n)$. If we denote $\hat{\nu}_i \triangleq f(x_i; w, y_i)$, it then follows $\arg\max_{t\in\mathbb{R}} g(t, w) = \hat{\nu}_{np}$.*

Proof. First we can note, the max value of t for g is equivalent to the min value of t for g. We can now find the Fermat Optimality Conditions for g.

$$\partial(-g(t, f_w)) = \partial\left(-t + \frac{1}{np}\sum_{i=1}^{n}(t - \hat{\nu}_i)^+\right) \tag{12}$$

$$= -1 + \frac{1}{np}\sum_{i=1}^{np} \begin{cases} 1 & \text{if } t > \hat{\nu}_i \\ 0 & \text{if } t < \hat{\nu}_i \\ [0,1] & \text{if } t = \hat{\nu}_i \end{cases} \tag{13}$$

$$= 0 \text{ when } t = \hat{\nu}_{np} \tag{14}$$

This is equivalent to the p-quantile of the Risk. ■

It therefore follows,

$$\sum_{i=1}^{n} \mathbb{I}\left\{\hat{\nu}_{np} \geq \left(f_w^{(k)}(x_i) - y_i\right)^2\right\} \left(f_w^{(k)}(x_i) - y_i\right)^2 \in \max_{t\in\mathbb{R}} g(t, f_w^{(k)}) \tag{15}$$

Interpretation 6. From Lemma 5, we see the t will be greater than or equal to the errors of exactly np points. Thus, we are continuously updating over the np minimum errors.

Lemma 7. *Let $\hat{\nu}_i \triangleq f(x_i; w, y_i)$ s.t. $\hat{\nu}_{i-1} \leq \hat{\nu}_i \leq \hat{\nu}_{i+1}$, if we choose $t^{(k+1)} = \hat{\nu}_{np}$ as by Lemma 5, it then follows $\nabla_w g(t^{(k)}, f_w^{(k)}) = \frac{1}{np}\sum_{i=1}^{np} \nabla f(x_i; f_w^{(k)}, y_i)$*

Proof. By our choice of $t^{(k+1)}$, it follows:

$$\nabla_f g(t^{(k+1)}, f_w^{(k)}) = \nabla_f \left(\hat{\nu}_{np} - \frac{1}{np}\sum_{i=1}^{n}\left(\hat{\nu}_{np} - \ell(x_i; f_w^{(k)}, y_i)\right)^+\right) \tag{16}$$

$$= -\frac{1}{np}\sum_{i=1}^{np} \nabla_f \left(\hat{\nu}_{np} - \ell(x_i; f_w^{(k)}, y_i)\right)^+ \tag{17}$$

$$= \frac{1}{np}\sum_{i=1}^{n} \nabla_f \ell(x_i; f_w^{(k)}, y_i) \begin{cases} 1 & \text{if } t > \hat{\nu}_i \\ 0 & \text{if } t < \hat{\nu}_i \\ [0,1] & \text{if } t = \hat{\nu}_i \end{cases} \tag{18}$$

Now we note $\nu_{np} \leq t^{(k+1)} \leq \nu_{np+1}$

$$\nabla_f g(t^{(k+1)}, f_w^{(k)}) = \frac{1}{np}\sum_{i=1}^{np} \nabla_f \ell(x_i; f_w^{(k)}, y_i) \tag{19}$$

This concludes the proof. ■

We denote the matrix K as the Gram Matrix where $[\mathrm{K}]_{ij} = k(x_i, x_j) \triangleq \exp(-\rho \left\|x_i - x_j\right\|_2^2)$. Given a parameter set w, the prediction for a new point will be: $f(x^\star; w) = \sum_{i=1}^{n} w_i \kappa(x_i, x^\star)$

From our definition of $S^{(k)}$ in **??**, we are interested in as $k \to \infty$ the quantities: $|x \in S^{(k)} \cap P|$ and $|x \in S^{(k)} \cap Q|$, where the latter cardinality represents the number of corrupted points in the subquantile set.

3.1 On the Softplus Approximation

It is clear our objective function is non-smooth. Thus we propose to use the Softplus approximation to smooth the function. The main ideas is to *first* approximate ReLU, consider the theory with respect to the approximation, and then take the limit as the approximation goes to the ReLU. The softplus approximation is given as follows,

$$\zeta_\lambda(x) = \frac{1}{\lambda}\log\left(1 + e^{\lambda x}\right) \tag{20}$$

We then have the approximation of g as

$$\widetilde{g}_\lambda(t, f_w) \triangleq t - \sum_{i=1}^{n} \zeta_\lambda\left(t - \ell\left(f_w; x_i, y_i\right)\right) \tag{21}$$

$$= t - \frac{1}{np}\sum_{i=1}^{n} \frac{1}{\lambda}\log\left(1 + \exp\left(\lambda\left(t - \ell\left(f_w; x_i, y_i\right)\right)\right)\right) \tag{22}$$

Now we compute the derivatives w.r.t to the softplus approximation, and then we consider the limit of the derivative as $\lambda \to \infty$.

$$\nabla_t \widetilde{g}_\lambda(t, f_w) = \nabla_t \left(t - \frac{1}{np}\sum_{i=1}^{n} \frac{1}{\lambda}\ln\left(1 + \exp\left(\lambda\left(t - \ell(f_w; x_i, y_i)\right)\right)\right)\right) \tag{23}$$

$$= 1 - \frac{1}{np} \sum_{i=1}^{n} \sigma \left(\lambda \left(t - \ell(f_w; x_i, y_i) \right) \right) \tag{24}$$

where $\sigma(\cdot)$ is the sigmoid function. It therefore follows,

$$\lim_{\lambda \to \infty} \nabla_t \widetilde{g}_\lambda(t, f_w) = 1 - \frac{1}{np} \sum_{i=1}^{n} \mathbb{I} \left\{ t - \ell \left(f_w; x_i, y_i \right) \right\} \tag{25}$$

$$\nabla_f \widetilde{g}_\lambda(t, f_w) = \nabla_f \left(t - \frac{1}{np} \sum_{i=1}^{n} \frac{1}{\lambda} \ln \left(1 + \exp \left(\lambda \left(t - \ell(f_w; x_i, y_i) \right) \right) \right) \right) \tag{26}$$

$$= \frac{1}{np} \sum_{i=1}^{n} \nabla_f \ell \left(f_w; x_i, y_i \right) \sigma \left(\lambda \left(t - \ell(f_w; x_i, y_i) \right) \right) \tag{27}$$

We therefore similarly have,

$$\lim_{\lambda \to \infty} \nabla_f \widetilde{g}_\lambda \left(t, f_w \right) = \frac{1}{np} \sum_{i=1}^{n} \mathbb{I} \left\{ t - \ell \left(f_w; x_i, y_i \right) \right\} \nabla_f \ell \left(f_w; x_i, y_i \right) \tag{28}$$

Then the second derivative is given by

$$\nabla_f^2 \widetilde{g}_\lambda(t, f_w) = \nabla_f \left(\frac{1}{np} \sum_{i=1}^{n} \nabla_f \ell \left(f_w; x_i, y_i \right) \sigma \left(\lambda \left(t - \ell(f_w; x_i, y_i) \right) \right) \right) \tag{29}$$

$$= \frac{1}{np} \sum_{i=1}^{n} \Bigg(\nabla_f^2 \ell \left(f_w; x_i, y_i \right) \sigma \left(\lambda \left(t - \ell(f_w; x_i, y_i) \right) \right) - \left(\nabla_f \ell \left(f_w; x_i, y_i \right) \right)^2 \sigma \left(\lambda \left(t - \ell(f_w; x_i, y_i) \right) \right) \left(1 - \sigma \left(\lambda \left(t - \ell(f_w; x_i, y_i) \right) \right) \right) \Bigg) \tag{30}$$

We then similarly have,

$$\lim_{\lambda \to \infty} \nabla_f^2 \widetilde{g}_\lambda(t, f_w) = \frac{1}{np} \sum_{i=1}^{n} \mathbb{I} \left\{ t - \ell \left(f_w; x_i, y_i \right) \right\} \nabla_f^2 \ell \left(f_w; x_i, y_i \right) \tag{31}$$

We can then calculate the Lipschitz constant of the approximation function with respect to f_w.

Lemma 8 (Lipschitz continuous gradient)**.** *Let $f_w, f_{\widetilde{w}} \in \mathcal{K}$, then we have*

$$\lim_{\lambda \to \infty} |\nabla_f \widetilde{g}_\lambda(t, f_w) - \nabla_f \widetilde{g}_\lambda(t, f_{\widetilde{w}})| \leq \beta \left\| f_w - f_{\widetilde{w}} \right\|_{\mathcal{H}} \tag{32}$$

where

$$\beta = \frac{1}{np} \sum_{i=1}^{n} \left| \nabla_f^2 \ell \left(f_w; x_i, y_i \right) \right| \tag{33}$$

Proof. We will upper bound the second derivative.

$$\lim_{\lambda \to \infty} |\nabla_f \widetilde{g}_\lambda(t, f_w) - \nabla_f \widetilde{g}_\lambda(t, f_{\widetilde{w}})| \leq \lim_{\lambda \to \infty} \sup \left\{ \nabla_f^2 \widetilde{g}_\lambda(t, f_w) \right\} \left\| f_w - f_{\widetilde{w}} \right\|_{\mathcal{H}} \tag{34}$$

$$\leq \lim_{\lambda \to \infty} \sup \left\{ \frac{1}{np} \sum_{i=1}^{n} \nabla_f^2 \ell \left(f_w; x_i, y_i \right) \sigma \left(\lambda \left(t - \ell(f_w; x_i, y_i) \right) \right) \right\} \left\| f_w - f_{\widetilde{w}} \right\|_{\mathcal{H}} \tag{35}$$

$$\leq \lim_{\lambda \to \infty} \left(\frac{1}{np} \sum_{i=1}^{n} \left| \nabla_f^2 \ell \left(f_w; x_i, y_i \right) \right| \right) \left\| f_w - f_{\widetilde{w}} \right\|_{\mathcal{H}} \tag{36}$$

$$= \frac{1}{np} \sum_{i=1}^{n} \left| \nabla_f^2 \ell \left(f_w; x_i, y_i \right) \right| \left\| f_w - f_{\widehat{w}} \right\|_{\mathcal{H}} \tag{37}$$

There is no dependence on λ. ■

3.2 Weakly Convex Concave Optimization Theory

With our smoothed function, we are now able to use the weakly-convex concave minimization literature to analyze g. The Moreau Envelope can be interpreted as an infimal convolution of the function f. When f is ρ-weakly convex, if $\lambda \leq \rho^{-1}$, then the Moreau Envelope is smooth.

Definition 9. (**Moreau Envelope on closed, convex set**, [Mor65]). Let f be proper lower semi-continuous convex function $\ell : \mathcal{K} \to \mathbb{R}$, where $\mathcal{K} \subset \mathcal{X}$ is a closed and convex set, then the Moreau Envelope is defined as:

$$\mathsf{M}_{\lambda \ell}(f_w) \triangleq \inf_{f_{\widehat{w}} \in \mathcal{K}} \left\{ \ell(f_{\widehat{w}}) + \frac{1}{2\rho} \left\| f_w - f_{\widehat{w}} \right\|_{\mathcal{H}}^2 \right\} \tag{38}$$

Definition 10. Define the function $\Phi(f_w) \triangleq \max_{t \in \mathbb{R}} g\left(t, f_w\right)$. This function is a L-weakly convex function in $\mathcal{K}$, i.e., $\Phi(f_w) + \frac{L}{2} \left\| f_w \right\|_{\mathcal{H}}^2$ is a convex function over w in the convex and compact set $\mathcal{K}$.

Definition 11 (First Order Stationary Point)**.** Let $f_{\widehat{w}}$ be a first-order stationary point, then for any $f_w \in \mathcal{K}$, it follows

$$\nabla_f g\left(f_{\widehat{w}}\right) \left(\overline{f_w - f_{\widehat{w}}} \right) \geq 0 \qquad \forall\, f_w \in \mathcal{K} \tag{39}$$

Definition 12 (Stationary Point of Moreau Envelope)**.** A point $f_{\widehat{w}}$ is a stationary point of the Moreau Envelope defined in Definition 9 of Φ defined in Definition 10 if

$$f_{\widehat{w}} = \arg\inf_{f_w \in \mathcal{K}} \left\{ \Phi_\lambda\left(f_w\right) + \frac{1}{2\rho} \left\| f_w - f_{\widehat{w}} \right\|_{\mathcal{H}}^2 \right\} \tag{40}$$

We will show that if a point f_w is a stationary point then this point is close to the optimal point for the uncorrupted distribution, i.e. $\left\| f_{\widehat{w}} - f_w^* \right\|_{\mathcal{H}}$ is small.

Lemma 13 (Lower bound on distance from stationary point and optimal point)**.** *Let Φ_λ be defined as in Definition 10, then if $f_{\widehat{w}}$ is a stationary point as defined in Definition 12, then*

$$\lim_{\lambda \to \infty} \left(\Phi_\lambda\left(f_{\widehat{w}}\right) - \Phi_\lambda\left(f_w^*\right) \right) \leq \beta \left\| f_{\widehat{w}} - f_w^* \right\|_{\mathcal{H}}^2 \tag{41}$$

Proof. By the definition of stationary point, we have

$$f_{\widehat{w}} = \lim_{\lambda \to \infty} \arg\inf_{f_w \in \mathcal{K}} \left\{ \Phi_\lambda\left(f_w\right) + \frac{1}{2\rho} \left\| f_w - f_{\widehat{w}} \right\|_{\mathcal{H}}^2 \right\} \tag{42}$$

$$\overset{(i)}{=} \arg\inf_{f_w \in \mathcal{K}} \left\{ \lim_{\lambda \to \infty} \Phi_\lambda\left(f_w\right) + \frac{1}{2\rho} \left\| f_w - f_{\widehat{w}} \right\|_{\mathcal{H}}^2 \right\} \tag{43}$$

(i) holds as we ρ is independent of λ as shown in the proof of Lemma 8. This implies then for any $f_w \in \mathcal{K}$ and noting $\rho \leq \beta^{-1}$, it follows

$$\lim_{\lambda \to \infty} \Phi_\lambda\left(f_{\widehat{w}}\right) \leq \lim_{\lambda \to \infty} \Phi_\lambda\left(f_w\right) + \beta \left\| f_w - f_{\widehat{w}} \right\|_{\mathcal{H}}^2 \tag{44}$$

We can then plug in the optimal, f_w^* for f_w and rearrange and we have the desired result. ■

We can now upper bound $\left\| f_{\widehat{w}} - f_w^* \right\|_{\mathcal{H}}$. We proceed by contradiction, i.e. if a stationary point is sufficiently far from the optimal point, then this will break the stationary property proved in Lemma 13. This bound is different for each of the loss functions, so we must upper bound $\left\| f_{\widehat{w}} - f_w^* \right\|_{\mathcal{H}}$ seperately for each loss function with the same high level overview.

3.3 Kernelized Regression

The loss for the Kernel Ridge Regression problem for a single training pair (x_i, y_i) is given by the following equation

$$\ell\left(f_w; x_i, y_i, \right) = \left(f_w(x_i) - y_i\right)^2 \tag{45}$$

For our theory, we need the L-lipschitz constant and β-smoothness constant.

Lemma 14. *(L-Lipschitz of $g\left(t, f_w\right)$ w.r.t f_w). Let $x_1, x_2, \cdots, x_n$, represent the data vectors. It then follows for any $f_w, f_{\widehat{w}} \in \mathcal{K}$:*

$$\left|g\left(t, f_w\right) - g\left(t, f_{\widehat{w}}\right)\right| \leq L \left\|f_w - f_{\widehat{w}}\right\|_{\mathcal{H}} \tag{46}$$

where

$$L = \frac{2R}{np}\left(\sum_{i=1}^{n}\sqrt{k(x_i, x_i)}\right)^2 + \frac{2\|y\|}{np}\left(\sum_{i=1}^{n}\sqrt{k(x_i, x_i)}\right) \tag{47}$$

Proof is given in Appendix B.1.

Lemma 15. *(β-Smoothness of $g(t, w)$ w.r.t w). Let $x_1, x_2, \cdots, x_n$ represent the rows of the data matrix* X*. It then follows:*

$$\left\|\nabla_w g\left(t, f_w\right) - \nabla_f g\left(t, f_{\widehat{w}}\right)\right\| \leq \beta \left\|f_w - f_{\widehat{w}}\right\|_{\mathcal{H}} \tag{48}$$

where

$$\beta = \frac{2}{np}\sum_{i \in X} k(x_i, x_i) = \frac{2}{np}\operatorname{Tr}\left(\mathrm{K}\right) \tag{49}$$

Proof. W.L.O.G, let S be the set of points such that if $x \in S$, then $t \geq \left(f_w(x) - y\right)^2$. Since g is twice differentiable, we will analyze the second derivative.

$$\left\|\nabla_f^2 g\left(t, f_w\right)\right\|_{\mathcal{H}} = \left\|\frac{2}{np}\sum_{i=1}^{n}\mathbb{I}\left\{t \geq \left(f_w(x_i) - y_i\right)^2\right\} k(x_i, x_i)\right\| \leq \frac{2}{np}\sum_{i=1}^{n} k(x_i, x_i) = \frac{2}{np}\operatorname{Tr}\left(\mathrm{K}\right) \tag{50}$$

This concludes the proof. ■

Similar results for the Lipschitz Constant for non-kernelized learning algorithms can be seen in [YSP21]. It is important to note that β is upper bounded as

$$\beta = \frac{2}{np}\operatorname{Tr}(\mathrm{K}) \leq \frac{2}{np}(n(1-\varepsilon)\max_{i \in P} k(x_i, x_i) + n\varepsilon \max_{j \in Q} k(x_j, x_j) = 2p^{-1}((1-\varepsilon)P_k + \varepsilon Q_k) \tag{51}$$

which is independent of n.

Lemma 16. *If $\|f_w - f_{w^*}\| \geq \eta$, then it follows*

$$\begin{aligned}\lim_{\lambda \to \infty}\left(\Phi_\lambda\left(f_w\right) - \Phi_\lambda\left(f_{w^*}\right)\right) &\geq \eta^2 n(1-2\varepsilon)\lambda_{\min}\left(\mathop{\mathbb{E}}_{x \sim \mathbb{P}}\left[\phi(x) \otimes \phi(x)\right]\right) \\ &- O\left(\sigma\sqrt{n(1-2\varepsilon)\log(n(1-2\varepsilon))\left\|\Sigma\right\|_{\mathrm{HS}}}\right) - 2\eta\left\|\sum_{i \in S \cap P}\eta_i \phi\left(x_i\right)\right\| - \sum_{j \in P \setminus S}\eta_j^2\end{aligned} \tag{52}$$

The proof is deferred to Appendix B.2.

Theorem 17. *Let $f_{\widehat{w}}$ be a stationary point defined in Definition 12 for the function Φ defined in Definition 10. Then for a constant $c_1 \in (0,1)$, if $n \geq \frac{8\operatorname{Tr}(\Sigma)^2}{\lambda_{\min}(\Sigma)(1-c_1)^2(1-2\varepsilon)} + \frac{8\beta}{(1-c_1)^2(1-2\varepsilon)}$,*

$$\mathop{\mathbb{E}}_{\mathcal{D} \sim \hat{\mathbb{P}}}\left\|f_{\widehat{w}} - f_{w^*}\right\|_{\mathcal{H}} \leq \left(\frac{\sigma\varepsilon n}{c_1 n(1-2\varepsilon)\lambda_{\min}\left(\Sigma\right)}\right)^{1/2} + \frac{O\left(\sigma\sqrt{\log\left(n(1-2\varepsilon)\right)\operatorname{Tr}\left(\Sigma\right)}\right)}{c_1\sqrt{n(1-2\varepsilon)}\lambda_{\min}\left(\Sigma\right)} \tag{53}$$

where β is the Lipschitz Gradient Constant given in Lemma 15.

The proof is given in Appendix B.3. In Theorem 17, we have an upper bound on the expected distance from a stationary point to the optimal point over the distance of the dataset. The numerator of the second term grows in $O(\sqrt{\log(n)})$ and the denominator grows in $O(\sqrt{n})$ as can be shown by choosing sufficiently large n. Asymptotically the second term will then go to 0. In the first term, we have both the numerator and denominator scale in $O(n)$. Furthermore, when we consider the case of feature noise, e.g. a large multiplicative term on the features, we simply require more data to obtain the same bounds. Such a result is corroborated in [SST+18]. For the linear and polynomial kernel, we then have β increases, therefore to obtain the same bound on η as with no feature noise, we simply need more data.

Corollary 18 ([1]Linear Regression Expected Erorr Bound)**.** *Consider Subquantile Minimization for Linear Regression on the data X with optimal parameters w^*. Assume $x_i \sim \mathcal{N}(0, \Sigma)$ for $i \in [n]$. Then after T iterations of Algorithm 1, we have the following error bounds for robust kernelized linear regression. Given sufficient data*

$$\mathbb{E}\left\|w^{(T)} - w^*\right\|_2 \leq O\left(\sqrt{\frac{\varepsilon}{1-2\varepsilon}}\frac{\sigma}{\sqrt{\lambda_{\min}(\Sigma)}}\right) \tag{54}$$

Proof given in Appendix B.4. It is important to note in all our bounds, the $\sqrt{\frac{\varepsilon}{1-2\varepsilon}}$ is a theoretical worst case bound when the Subquantile contains the minimum possible number of uncorrupted points. In other words, we have

$$\frac{|P \setminus S|}{|S \cap P|} \leq \frac{n\varepsilon}{n(1-2\varepsilon)} = \frac{\varepsilon}{1-2\varepsilon} \tag{55}$$

So, as $|S \cap P|$ increases, we have a better error bound as $|P \setminus S|$ decreases. As is typical in the robust statistics literature, we make no assumptions on the distribution of the corrupted data so we cannot say anything about $|S \cap P|$. We thus find our results in practice to be strong.

3.4 Kernel Binary Classification

The Negative Log Likelihood for the the Kernel Classification problem is given by the following equation for a single training pair (x_i, y_i)

$$\ell\left(x_i, y_i; f_w\right) = -\left(y_i \log\left(\sigma\left(f_w(x_i)\right)\right) - \left(1-y_i\right)\log\left(1-\sigma\left(f_w(x_i)\right)\right)\right) \tag{56}$$

Similar to Section 3.3, we require the L-Lipschitz constant and β-smoothness constant.

Lemma 19. *(L-Lipschitz of $g\left(t, w\right)$ w.r.t w). Let $x_1, x_2, \cdots, x_n$, represent the data vectors. It then follows:*

$$\left|g\left(t, f_w\right) - g\left(t, f_{\widehat{w}}\right)\right| \leq L\left\|f_w - f_{\widehat{w}}\right\|_{\mathcal{H}} \tag{57}$$

where

$$L = \frac{1}{np}\sum_{i \in X}\sqrt{k(x_i, x_i)} = \frac{1}{np}\operatorname{Tr}\left(\mathrm{K}\right) \tag{58}$$

Proof is given in Appendix B.5.

Lemma 20. *(β-Smoothness of $g(t, w)$ w.r.t w). Let $x_1, x_2, \cdots, x_n$ represent the rows of the data matrix* X*. It then follows:*

$$\left\|\nabla_f g\left(t, f_w\right) - \nabla_f g\left(t, f_{\widehat{w}}\right)\right\| \leq \beta\left\|f_w - f_{\widehat{w}}\right\|_{\mathcal{H}} \tag{59}$$

where

$$\beta = \frac{1}{4p}\sum_{i=1}^{n} k(x_i, x_i) = \frac{1}{4p}\operatorname{Tr}\left(\mathrm{K}\right) \tag{60}$$

Proof is given in Appendix B.6.

[1]In Progress

Lemma 21. [2] *If* $\|f_w - f_{w^*}\| \geq \eta$, *then it follows*

$$\Phi(f_w) - \Phi(f_{w^*}) \geq 2\|f_w - f_w^*\|_{\mathcal{H}} - 1.386 + \sum_{P\setminus S} y_i \log(\sigma(f_w^*(x_i))) + (1-y_i)\log(1-\sigma(f_w^*(x_i))) \tag{61}$$

Proof is given in Appendix B.7.

3.5 Kernel Multi-Class Classification

The Negative Log-Likelihood Loss for the the Kernel Multi-Class Classification problem is given by the following equation for a single training pair (x_i, y_i), note W is now a matrix

$$\ell(x_i, y_i; \mathrm{W}) = -\sum_{j=1}^{|\mathcal{C}|} \mathbb{I}\{y_i = j\} \log\left(\frac{\exp(f_{\mathrm{W}_k}(x_i))}{\sum_{h=1}^{|\mathcal{C}|}\exp(f_{\mathrm{W}_h}(x_i))}\right) \tag{62}$$

Lemma 22 (L-Lipschitz of $g(t,w)$ w.r.t w)**.** [3] *Let* $x_1, x_2, \cdots, x_n$, *represent the data vectors. It then follows:*

$$|g(t, f_w) - g(t, f_{\widehat{w}})| \leq L\|f_w - f_{\widehat{w}}\|_{\mathcal{H}} \tag{63}$$

where

$$L = \tag{64}$$

3.6 Necessary Kernel Inequalities

We will first extend the idea of Resilience [SCV18] to kernel learning.

Definition 23. (**Resilience**) from [SCV18]. Let $\mathcal{H}$ represent the RKHS associated with the proper kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, then given the feature mapping $\phi : \mathbb{R}^d \to \mathcal{H}$, and the set $X = \{x_i\}_{i=1}^n = P \cup Q$, such that $|P| = n(1-\epsilon)$ and $|Q| = n\epsilon$, It then follows for any subset $T \subseteq P$ such that $|T| = (1-2\varepsilon)n$,

$$\left\| \frac{1}{|T|}\sum_{i\in T}\phi(x_i) - \mu_{\mathbb{P}} \right\| \leq \tau$$

where $\mu_{\mathbb{P}} = \mathbb{E}_{x\sim\mathbb{P}}[\phi(x)]$ is the kernel mean embedding for the distribution, $\mathbb{P}$. We say the set X has (ϵ, τ)-resilience in the Reproducing Kernel Hilbert Space.

Without the idea of resilience defined in Definition 23, we will be unable to put error bounds on our algorithm. In practice, however, it is important to note that solving for $\|\nabla\Phi_\lambda\|_{\mathcal{H}} = 0$ is NP-Hard. Thus, we will analyze the approximate stationary point.

Lemma 24 ([Roc70, DD19])**.** *Assume the function* Φ *is* β*-weakly convex. Let* $\lambda < \frac{1}{\beta}$, *and let* $f_{\widehat{w}} = \arg\min_{f_w\in\mathcal{K}}(\Phi(f_w) + \frac{1}{2\lambda}\|f_w - f_{\widehat{w}}\|_{\mathcal{H}}^2)$, *then* $\|\nabla\Phi_\lambda(f_w)\|_{\mathcal{H}} \leq \epsilon$ *implies:*

$$\|f_{\widehat{w}} - f_w\|_{\mathcal{H}} = \lambda\epsilon \quad \textit{and} \quad \min_{g\in\partial\Phi(f_{\widehat{w}})+\partial\mathcal{I}_{\mathcal{K}}(f_{\widehat{w}})}\|g\|_{\mathcal{H}} \leq \epsilon \tag{65}$$

4 Optimization Results

First, we will show using stepsize of $1/\beta$ returns a μ-approximate stationary point. However, since our methods are in the kernelized setting. The 2-norm, $\|w - w^*\|$ is not sufficient, we want $\|w - w^*\|_{\mathcal{H}}$ to be close, as the RKHS being small indicates the function $f(x)$ and $f^*(x)$ will be close.

4.1 Optimization in the Reproducing Kernel Hilbert Space

In this section, we will discuss and give necessary optimization results in the RKHS norm. In the analysis given in [JNJ20], given sufficient iterations, if we choose a sufficiently small learning rate, we can always reach a stationary point given infinite iterations. In practice, and in theory, this convergence rate is a square

[2] In Progress
[3] In Progress

root factor slower. We prove, if we instead converge to a weaker stationary point, we can converge a factor of square root faster.

Theorem 25. *Let $\Psi(f_w) \triangleq \max_{t\in\mathbb{R}} g(f_w, t)$ where $g(f_w, t)$ is β-smooth in f_w and L-Lipschitz in f_w and is concave in t but not necesarily convex in f_w. Then Algorithm 1 with stepsize $\eta = \frac{1}{\beta}$ reaches a $(\sqrt{2}L + O(1))$-stationary point in $O(16\beta R + 4\beta np\sigma^2))$ iterations.*

4.2 Accelerated Gradient Methods

When working with big data it is often the case we need faster gradient methods as the gradient can be expensive to obtain. In this section, we give results on the convergence rate of accelerated gradient methods on the update of w. We will analyze the convergence of three popular accelerated gradient methods.

4.2.1 Momentum Accelerated Gradient Descent

In this section we study Momentum Accelerated Gradient Descent [Qia99, Pol64] with our non-convex-concave optimization algorithm.

$$b^{(t)} = \mu b^{(t-1)} + \nabla_f \Phi\left(f_w^{(t-1)}\right) \tag{66}$$

$$f_w^{(t)} = f_w^{(t-1)} - \alpha b^{(t)} \tag{67}$$

Theorem 26. *Momentum Accelerated Gradient Descent given in Equations (66) and (67) reaches a η-approximate stationary point. Algorithm 1 reaches a η-approximate stationary point in a polynomial number of iterations.*

$$\mathbb{E}\left[\left\|\nabla\Phi_{1/2\ell}\left(f_w\right)\right\|_{\mathcal{H}}^2\right] \leq \tag{68}$$

4.2.2 Nesterov Accelerated Gradient Descent

In this section we study Nesterov Accelerated Gradient Descent [Nes83] with our non-convex-concave optimization algorithm.

$$b^{(t+1)} = (1+\mu) f_w^{(t)} - \mu f_w^{(t-1)} \tag{69}$$

$$f_w^{(t+1)} = b^{(t+1)} - \alpha\nabla_f \Phi\left(f_w^{(t)}\right) \tag{70}$$

Theorem 27. *Nesterov Accelerated Gradient Descent given in Equations (69) and (70) reaches a η-approximate stationary point. Algorithm 1 reaches a η-approximate stationary point in a polynomial number of iterations.*

$$\mathbb{E}\left[\left\|\nabla\Phi_{1/2\ell}\left(f_w\right)\right\|_{\mathcal{H}}^2\right] \leq \tag{71}$$

5 Experiments

We perform numerical experiments on state of the art datasets comparing with other state of the art methods. We initialize the weights parameterizing f_w with the Glorot Initialization Scheme [GB10].

Algorithm 3: SUBQUANTILE-KERNEL

Input: Iterations: T; Quantile: p; Data Matrix: $\mathrm{X} \in \mathbb{R}^{n\times d}, n \gg d$; Labels: $y \in \mathbb{R}^{n\times 1}$; Learning Rate schedule: $\alpha_1, \cdots, \alpha_T$; Ridge parameter: λ

Output: Trained Parameters: $f_w^{(T)}$

$w_i^{(0)} \leftarrow \text{Unif}\left[-\sqrt{\frac{6}{n}}, \sqrt{\frac{6}{n}}\right], \forall i \in [n]$ ▷ Base Learner

for $k = 1, 2, \ldots, T$ **do**

  $S^{(k)} \leftarrow \text{SUBQUANTILE}(f_w^{(k)}, \mathrm{X})$ ▷ Algorithm 2

  $\nabla_{\mathrm{f}} g\left(t^{(k+1)}, f_w^{(k)}\right) \leftarrow 2\sum_{i\in S^{(k)}}\left(f_w^{(k)}(x_i) - y_i\right)\cdot k(x_i, \cdot)$ ▷ Regression

  $\nabla_f g\left(t^{(k+1)}, f_w^{(k)}\right) \leftarrow \sum_{i\in S^{(k)}}\left(\sigma\left(f_w^{(k)}(x_i)\right) - y_i\right)\cdot k(x_i, \cdot)$ ▷ Binary Classification

  $f_w^{(k+1)} \leftarrow f_w^{(k)} - \alpha_{(k)}\nabla_f g\left(t^{(k+1)}, f_w^{(k)}\right)$ ▷ f_w-update in Equation (8)

end

Pick t uniformly at random from $[T]$

return $f_w^{(t)}$

Algorithms	Test RMSE							
	`Concrete`		`Wine Quality`		`Boston Housing`		`Drug`	
	$\epsilon = 0.2(\downarrow)$	$\epsilon = 0.4(\downarrow)$	$\epsilon = 0.2(\downarrow)$	$\epsilon = 0.4(\downarrow)$	$\epsilon = 0.2(\downarrow)$	$\epsilon = 0.4(\downarrow)$	$\epsilon = 0.2(\downarrow)$	$\epsilon = 0.4(\downarrow)$
KRR	$1.355_{(0.0934)}$	$2.282_{(0.2063)}$	$1.437_{(0.0979)}$	$2.272_{(0.1088)}$	$1.285_{(0.0896)}$	$2.266_{(0.0686)}$	$1.478_{(0.0533)}$	$2.381_{(0.0203)}$
TERM	$0.829_{(0.0422)}$	$0.928_{(0.0197)}$	$1.854_{(0.7437)}$	$1.069_{(0.1001)}$	$0.879_{(0.0178)}$	$0.875_{(0.0711)}$	∞	∞
SEVER	$\underline{0.533}_{(0.0347)}$	$\underline{0.592}_{(0.0548)}$	$\underline{0.915}_{(0.0343)}$	$\underline{0.841}_{(0.0413)}$	$\underline{0.526}_{(0.0287)}$	$\underline{0.720}_{(0.1147)}$	$\mathbf{1.172}_{(0.0542)}$	$\underline{1.215}_{(0.0536)}$
SUBQUANTILE	$\mathbf{0.396}_{(0.0216)}$	$\mathbf{0.442}_{(0.0468)}$	$\mathbf{0.808}_{(0.0389)}$	$\mathbf{0.827}_{(0.0216)}$	$\mathbf{0.446}_{(0.1230)}$	$\mathbf{0.456}_{(0.1055)}$	$\underline{1.280}_{(0.0568)}$	$\mathbf{1.132}_{(0.0892)}$
Genie ERM	∞	∞	∞	∞	∞	∞	∞	∞

Table 1: `Boston Housing`, `Concrete Data`, `Wine Quality`, and `Drug` and `Polynomial Synthetic` Dataset. Label Noise: $y_{\text{noise}} \sim \mathcal{N}(5,5)$. Feature Noise: $y_{\text{noise}} = 10000 y_{\text{original}}$ and $x_{\text{noise}} = 100 x_{\text{original}}$. `Polynomial Regression` Synthetic Dataset. 1000 samples, $x \sim \mathcal{N}(0,1)$, $y \sim \mathcal{N}(\sum_{i=0} a_i x^i, 0.01)$ where $a_i \sim \mathcal{N}(0,1)$. The Radial Basis Function is used in first three experiments and polynomial kernel with degree 3 and $C = 1$ is used in the last experiment.

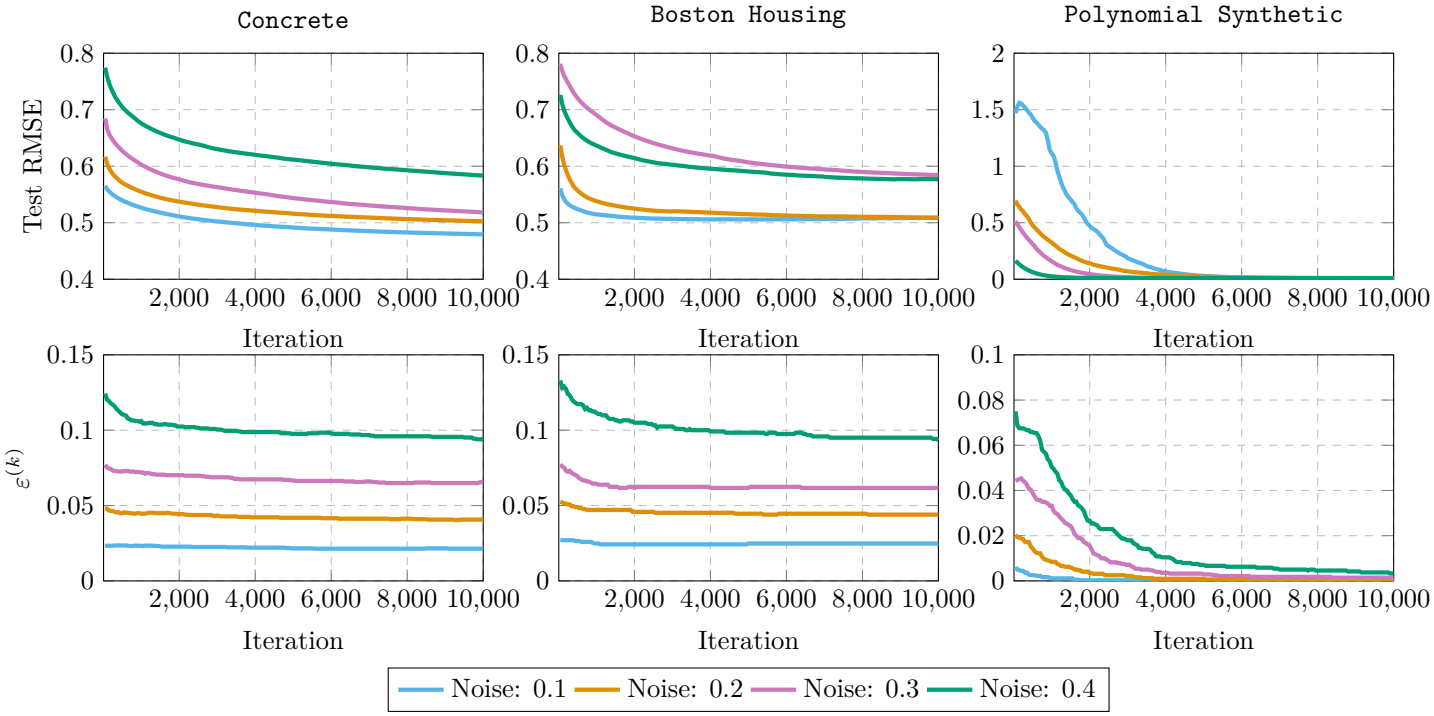

Figure 1: Test RMSE over the iterations in `Concrete`, `Boston Housing`, and `Polynomial` Datasets for SUBQUANTILE at different noise levels

In Figure 1, we see the final subquantile has significantly less outliers than the original corruption in the data set. Furthermore, we see there is a greater decrease in the higher outlier settings.

5.1 Linear Regression

In this section, we give experimental results for datasets using the linear kernel. This section will serve as a comparison to the various Robust Linear Regression Algorithms developed which are not meta-algorithms.

5.2 Kernel Binary Classification

In this section we will give the algorithm for subquantile minimization for the kernel classification problem and then give some experimental results on state of the art datasets comparing against other state of the art robust algorithms.

Algorithms	Test RMSE							
	`Boston Housing`		`Wine Quality`		`Concrete`		`Drug`	
	Label(↓)	Label+Feature	Label	Label+Feature	Label	Label+Feature	Label	Label+Feature
KRR	$0.907_{(0.2724)}$	$90.799_{(5.7170)}$	$0.894_{(0.0404)}$	$62.913_{(7.4959)}$	$0.825_{(0.0943)}$	$77.383_{(5.5692)}$	$2.679_{(0.1286)}$	$141.690_{(3.5297)}$
RANSAC	$1.167_{(0.6710)}$	$22.460_{(19.1987)}$	$1.489_{(0.2730)}$	$39.630_{(13.0294)}$	$0.870_{(0.2308)}$	$23.629_{(16.1023)}$	$2.801_{(0.2004)}$	$117.389_{(8.3915)}$
CRR	$0.636_{(0.0905)}$	$88.626_{(5.7380)}$	$0.818_{(0.0224)}$	$58.488_{(3.5612)}$	$0.710_{(0.0919)}$	$73.932_{(4.7867)}$	$1.887_{(0.1463)}$	$152.827_{(6.6038)}$
STIR	$\underline{0.562}_{(0.0626)}$	$78.878_{(8.0164)}$	$0.828_{(0.0293)}$	$58.352_{(4.6700)}$	$\underline{0.684}_{(0.0245)}$	$76.555_{(4.5927)}$	$1.721_{(0.1520)}$	$144.975_{(5.4953)}$
SEVER	$0.601_{(0.0979)}$	$5.980_{(8.2603)}$	$\mathbf{0.814}_{(0.0207)}$	$9.065_{(13.7632)}$	$\underline{0.684}_{(0.0438)}$	$4.119_{(8.2436)}$	$1.469_{(0.1162)}$	$156.043_{(4.5543)}$
TERM	$0.608_{(0.1357)}$	$\underline{0.569}_{(0.0620)}$	$0.840_{(0.0563)}$	$\underline{0.827}_{(0.0255)}$	$0.780_{(0.0734)}$	$\underline{0.808}_{(0.0726)}$	$\underline{1.185}_{(0.1077)}$	$\mathbf{1.147}_{(0.1258)}$
SUBQUANTILE	$\mathbf{0.503}_{(0.0470)}$	$\mathbf{0.548}^{*}_{(0.0286)}$	$\mathbf{0.813}_{(\mathbf{0.0357})}$	$\mathbf{0.821}_{(0.0305)}$	$\mathbf{0.632}_{(0.0275)}$	$\mathbf{0.703}_{(0.0427)}$	$\mathbf{1.074}_{(0.1848)}$	$\underline{2.413}_{(0.6737)}$
Genie ERM	$0.630_{(0.1015)}$	$0.665_{(0.1134)}$	$0.838_{(0.0130)}$	$0.865_{(0.0222)}$	$0.763_{(0.0390)}$	$0.768_{(0.0181)}$	$0.988_{(0.0823)}$	$0.985_{(0.0838)}$

Table 2: For only Label Noise, $y_{\text{noisy}} \sim \mathcal{N}(5,5)$. For Label and Feature Noise $x_{\text{noisy}} = 100x_{\text{original}}$ and $y_{\text{noisy}} = 10000y_{\text{original}}$. * As indicated by the theory, when encountering feature noise, we require more gradient descent iterations to achieve the same bound between the returned point and the stationary point. Therefore, we train the label noise perturbed dataset for 10000 iterations, and the feature noise perturbed dataset for 100000 iterations.

Now we will give some experimental results.

Algorithms	Test Accuracy							
	`Heart Disease`				`Breast Cancer`			
	Label		Label+Feature		Label		Label+Feature	
	$\epsilon = 0.2(\uparrow)$	$\epsilon = 0.4(\uparrow)$	$\epsilon = 0.2(\uparrow)$	$\epsilon = 0.4(\uparrow)$	$\epsilon = 0.2(\uparrow)$	$\epsilon = 0.4(\uparrow)$	$\epsilon = 0.2(\uparrow)$	$\epsilon = 0.4(\uparrow)$
SVM	$0.777_{(0.0396)}$	$0.639_{(0.0762)}$	$0.534_{(0.0766)}$	$0.538_{(0.0626)}$	$0.926_{(0.0331)}$	$0.548_{(0.1194)}$	$0.649_{(0.0254)}$	$0.618_{(0.0507)}$
SEVER	$\underline{0.793}_{(0.0422)}$	$\underline{0.695}_{(0.0636)}$	$0.784_{(0.0432)}$	$\mathbf{0.816}_{(0.0562)}$	$0.904_{(0.0356)}$	$0.575_{(0.1456)}$	$\underline{0.956}_{(0.0164)}$	$\underline{0.974}_{(0.0062)}$
TERM	$0.741_{(0.0393)}$	$0.620_{(0.0699)}$	$\underline{0.803}_{(0.0613)}$	$\underline{0.810}_{(0.0286)}$	$\underline{0.940}_{(0.0378)}$	$\underline{0.763}_{(0.0364)}$	$\mathbf{0.986}_{(0.0143)}$	$\mathbf{0.986}_{(0.0119)}$
SUBQUANTILE	$\mathbf{0.803}_{(0.0293)}$	$\mathbf{0.790}_{(0.0350)}$	$\mathbf{0.833}_{(0.0318)}$	$\underline{0.807}_{(0.0468)}$	$\mathbf{0.928}_{(0.0129)}$	$\mathbf{0.916}_{(0.0185)}$	$0.951_{(0.0212)}$	$0.953_{(0.0197)}$
Genie ERM	∞	∞	∞	∞	∞	∞	∞	∞

Table 3: `Heart Disease` and `Breast Cancer` Dataset. Label Noise: $y_{\text{noise}} = \mathbb{I}\{y_{\text{original}} = 0\}$. Feature Noise: $x_{\text{noise}} = 100x_{\text{original}}$. The Linear Kernel is used in all experiments.

5.3 Kernel Multi-Class Classification

In this section we will provide some experimental results on the multi-class classification task.
Results. We will clearly state our main findings.

- **Label Noise vs. Label and Feature Noise**. As suggested by our developed theory, for linear regression or using unbounded kernels, a large multiplicative term increases β and therefore requires more gradient descent iterations to achieve the same distance from a Moreau stationary point. Therefore, from simply increasing the number of gradient descent iterations, we are able to achieve similar RMSE in practice. This happens because the distance from a stationary point and the optimal is not affected by feature noise. This is one of the strengths of our theoretical analysis.
- **Error vs.** ε. We find approximately linear increase in the error with increasing ε. This can be seen in the γ term, which is upper bounded $\sqrt{\varepsilon/(1-2\varepsilon)}$. When $\varepsilon \to 0.5$, the denominator approaches 0 and therefore our worst case bound increases.
- **Kernel.** Our error bounds are stronger when the dimension of the kernel is lower, i.e. we need more data to obtain the same error bounds. However, in practice, we find many datasets are better approximated by polynomial or RBF kernels, and therefore the γ term is significantly lower.

6 Discussion

The main contribution of this paper is the study of a nonconvex-concave formulation of Subquantile minimization for the robust learning problem for kernel ridge regression and kernel classification. We present an algorithm to solve the nonconvex-concave formulation and prove rigorous error bounds which show that the more good data that is given decreases the error bounds. We also present accelerated gradient methods for the two-step algorithm to solve the nonconvex-concave optimization problem and give novel theoretical bounds.

Theory. We develop strong theoretical bounds on the normed difference between the function returned by Subquantile Minimization and the optimal function for data in the target distribution, $\mathbb{P}$, in the Gaussian Design. In expectation and with high probability, given sufficient data dependent on the kernel, we obtain a near minimax optimal error bound for a general positive definite continuous kernel.

Experiments. From our experiments, we see Subquantile Minimization is competitive with algorithms developed solely for robust linear regression as well as other meta-algorithms. Our theoretical analysis is through the lens of kernel-learning, but the generalization to linear regression from a non-kernel perspective can be done. In kernelized regression, we see SUBQUANTILE is the strongest of the meta-algorithms. Furthemore, in binary and multi-class classification, SUBQUANTILE is very strong. Thus, we can see empirically SUBQUANTILE is the strongest meta-algorithm across all kernelized regression and classification tasks and also the strongest algorithm in linear regression.

Interpretability. One of the strengths in Subquantile Optimization is the high interpretability. Once training is finished, we can see the $n(1-p)$ points with highest error to find the outliers and the features follow Gaussian Design. Furthermore, there is only hyperparameter p, which should be chosen to be approximately the percentage of inliers in the data and thus is not very difficult to tune for practical purposes. Our theory suggests for a problem where the amount of corruptions is unknown,

General Assumptions. The general assumption is the majority of the data should inliers. This is not a very strong assumption, as by the definition of outlier it should be in the minority. Furthermore, we assume the feature maps have a Gaussian Design. Such a design in many prior works in kernel learning and we therefore find it suitable.

The analysis of Subquantile Minimization can be extended to neural networks. This generalization will be appear in subsequent work.

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A Concentration Inequalities

In this section we will give various concentration inequalities on the inlier data for functions in the Reproducing Kernel Hilbert Space. We will first give our assumptions for robust kernelized regression.

Assumption 28 (Gaussian Design)**.** We assume for $x_i \sim \mathbb{P} \in \mathcal{X}$, then it follows for the feature map, $\phi(\cdot) : \mathcal{X} \to \mathcal{H}$,

$$\phi(x_i) \sim \mathcal{N}(0, \Sigma) \tag{72}$$

where Σ is a possibly infinite dimensional covariance operator.

Assumption 29 (Normal Residuals)**.** The residual is defined as $\mu_i \triangleq f_w^*(x_i) - y_i$. Then we assume for some $\sigma > 0$, it follows

$$\mu_i \sim \mathcal{N}(0, \sigma^2) \tag{73}$$

Lemma 30 (Maximum of Gaussians)**.** *Let $\mu_1, \ldots, \mu_n \sim \mathcal{N}(0, \sigma^2)$ for some $\sigma > 0$. Then it follows*

$$\mathop{\mathbb{E}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \max_{i \in [n]} |\mu_i| \leq \tilde{O}\left(\sigma \sqrt{\log(n)}\right) \tag{74}$$

Proof. We will integrate over the CDF to make our claim.

$$\mathop{\mathbb{E}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \max_{i \in [n]} |\mu_i| = \int_0^\infty \mathop{\mathbb{P}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \left\{ \max_{i \in [n]} |\mu_i| > t \right\} dt \overset{(i)}{\leq} c_1 + n \int_{c_1}^\infty \mathop{\mathbb{P}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \{|\mu_i| \geq t\} \, dt \tag{75}$$

$$\overset{(ii)}{=} c_1 + 2n \int_{c_1}^\infty \mathop{\mathbb{P}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \{\mu_i \geq t\} \, dt = c_1 + 2n \int_{c_1}^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx \tag{76}$$

$$\leq c_1 + \frac{n}{\sigma}\sqrt{\frac{2}{\pi}} \int_{c_1}^\infty \frac{x}{c_1} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx = c_1 + n\sigma\sqrt{\frac{2}{\pi}} \frac{e^{-\left(\frac{c_1}{\sigma\sqrt{2}}\right)^2}}{c_1} \tag{77}$$

(i) follows from a union bound and noting for a random variable X and a constant C, it follows $\mathbb{P}\{\max_{i \in [n]} X_i \geq C\} = n\mathbb{P}\{X \geq C\}$. (ii) follows from the symmetricity of the Gaussian distribution. From here, we choose $c_1 \triangleq \sigma\sqrt{2\log(n)}$. Then we have,

$$\mathop{\mathbb{E}}_{\mu_i \sim \mathcal{N}(0,\sigma^2)} \max_{i \in [n]} |\mu_i| \leq \sigma\sqrt{2\log(n)} + \frac{1}{\sqrt{\pi \log(n)}} \tag{78}$$

This completes the proof. ■

Lemma 31 (Expected fourth power of norm of Functions with Gaussian Design in the Reproducing Kernel Hilbert Space)**.** [4] *Let $x_i \sim \mathbb{P}$ such that $\phi(x_i) \sim \mathcal{N}(0, \Sigma)$ from Assumption 28. Then it follows*

$$\mathop{\mathbb{E}}_{\phi(x_i) \sim \mathcal{N}(0,\Sigma)} \|\phi(x_i)\|_{\mathcal{H}}^4 \leq O(\mathrm{Tr}\left(\Sigma^2\right)) \tag{79}$$

Proof. Let $x_1, \ldots, x_n \sim \mathbb{P}$ and $\varphi_1, \ldots, \varphi_p$ be an orthonormal basis of the kernel k. Then, we have

$$\mathop{\mathbb{E}}_{\phi(x_i) \sim \mathcal{N}(0,\Sigma)} \|\phi(x_i)\|_{\mathcal{H}}^4 = \mathop{\mathbb{E}}_{\phi(x_i) \sim \mathcal{N}(0,\Sigma)} \left[k^2(x,x)\right] = \int_\Omega k^2(x,x) d\mathbb{P} \tag{80}$$

$$\overset{\text{thm. } 35}{=} \int_\Omega \left(\sum_{i=1}^p \lambda_i \varphi_i(x) \varphi_i(x) \right)^2 d\mathbb{P} \tag{81}$$

$$= \int_\Omega \sum_{i=1}^p \sum_{j=1}^p \lambda_i \lambda_j \varphi_i(x) \varphi_i(x) \varphi_j(x) \varphi_j(x) d\mathbb{P} \tag{82}$$

$$= \sum_{i=1}^p \sum_{j=1}^p \lambda_i \lambda_j \int_\Omega \left(\varphi_i(x)\varphi_j(x)\right)^2 d\mathbb{P} \tag{83}$$

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$$\overset{(i)}{\leq} \sum_{i=1}^{p}\sum_{j=1}^{p}\lambda_i\lambda_j\left(\int_{\Omega}\varphi_i(x)\varphi_j(x)d\mathbb{P}\right)^2 = \sum_{i=1}^{p}\sum_{j=1}^{p}\lambda_i\lambda_j\delta_{i,j} = \operatorname{Tr}\left(\Sigma^2\right) \tag{84}$$

(i) follows from Jensen's Inequality. This completes the proof. ■

Lemma 32 (Norm of Functions with Gaussian Design in the Reproducing Kernel Hilbert Space)**.** *Let $x_i \sim \mathbb{P}$ such that $\phi(x_i) \sim \mathcal{N}(0,\Sigma)$ from Assumption 28 and Assumption 29. Then, it follows*

$$\mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\mathop{\mathbb{E}}_{\mu_i\sim\mathcal{N}(0,\sigma^2)}\left\|\sum_{i=1}^{n}\mu_i\phi(x_i)\right\|_{\mathcal{H}} \leq O\left(\sigma\sqrt{n\log(n)\operatorname{Tr}\left(\Sigma\right)}\right) \tag{85}$$

Proof. Our proof follows standard ideas from High-Dimensional Probability. Let ξ_i for $i \in [n]$ denote i.i.d Rademacher variables such that for $\xi_i \sim \mathcal{R}$, it follows $\mathbb{P}\{\xi_i = 1\} = \mathbb{P}\{\xi_i = -1\} = \frac{1}{2}$. We then have,

$$\mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\mathop{\mathbb{E}}_{\mu_i\sim\mathcal{N}(0,\sigma^2)}\left\|\sum_{i=1}^{n}\mu_i\phi(x_i)\right\|_{\mathcal{H}} \leq \mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\mathop{\mathbb{E}}_{\mu_i\sim\mathcal{N}(0,\sigma^2)}\max_{i\in[n]}|\mu_i|\left\|\sum_{i=1}^{n}\phi(x_i)\right\|_{\mathcal{H}} \tag{86}$$

$$\overset{\text{lem. 30}}{\leq} \tilde{O}\left(\sigma\sqrt{\log(n)}\right)\mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\mathop{\mathbb{E}}_{\xi_i\sim\mathcal{R}}\left\|\sum_{i=1}^{n}\xi_i\phi(x_i)\right\|_{\mathcal{H}} \tag{87}$$

$$\overset{(i)}{\leq} \tilde{O}\left(\sigma\sqrt{\log(n)}\right)\left(\mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\mathop{\mathbb{E}}_{\xi_i\sim\mathcal{R}}\left\|\sum_{i=1}^{n}\xi_i\phi(x_i)\right\|_{\mathcal{H}}^2\right)^{1/2} \tag{88}$$

$$= \tilde{O}\left(\sigma\sqrt{\log(n)}\right)\left(\mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\mathop{\mathbb{E}}_{\xi_i\sim\mathcal{R}}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\xi_j k(x_i,x_j)\right)^{1/2} \tag{89}$$

$$= \tilde{O}\left(\sigma\sqrt{\log(n)}\right)\left(\mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\sum_{i=1}^{n}k(x_i,x_i)\right)^{1/2} \tag{90}$$

$$= \tilde{O}\left(\sigma\sqrt{\log(n)}\right)\sqrt{n}\left(\mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\left[k(x_i,x_i)\right]\right)^{1/2} = \tilde{O}\left(\sigma\sqrt{\log(n)}\right)\sqrt{n\operatorname{Tr}\left(\Sigma\right)} \tag{91}$$

(i) follows from Jensen's Inequality. ■

Lemma 33 (Infinite Dimensional Covariance Estimation in the Hilbert-Schmidt Norm)**.** *Let $\Sigma \triangleq \mathbb{E}_{\phi(x_i)\sim\mathbb{P}}[\phi(x_i)\otimes\phi(x_i)]$. Then let $x_1,\ldots,x_n$ be i.i.d sampled from $\mathbb{P}$ such that $\phi(x_i) \sim \mathcal{N}(0,\Sigma)$ from Assumption 28, we then have*

$$\mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\left\|\frac{1}{n}\sum_{i=1}^{n}\phi(x_i)\otimes\phi(x_i) - \Sigma\right\|_{\text{HS}} \leq O\left(n^{-1/2}\left\|\Sigma\right\|_{\text{HS}}\right) \tag{92}$$

Proof. Our proof follows standard ideas from High-Dimensional Probability. Let ξ_i for $i \in [n]$ denote i.i.d Rademacher variables such that for $\xi_i \sim \mathcal{R}$, it follows $\mathbb{P}\{\xi_i = 1\} = \mathbb{P}\{\xi_i = -1\} = \frac{1}{2}$. We then have,

$$\mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\left\|\frac{1}{n}\sum_{i=1}^{n}\phi(x_i)\otimes\phi(x_i) - \Sigma\right\|_{\text{HS}}$$

$$\overset{(i)}{\leq} \mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\mathop{\mathbb{E}}_{\tilde{\phi}(x_i)\sim\mathcal{N}(0,\Sigma)}\left\|\frac{1}{n}\sum_{i=1}^{n}\left(\phi(x_i)\otimes\phi(x_i) - \tilde{\phi}(x_i)\otimes\tilde{\phi}(x_i)\right)\right\|_{\text{HS}} \tag{93}$$

$$= \mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\mathop{\mathbb{E}}_{\tilde{\phi}(x_i)\sim\mathcal{N}(0,\Sigma)}\mathop{\mathbb{E}}_{\xi_i\sim\mathcal{R}}\left\|\frac{1}{n}\sum_{i=1}^{n}\xi_i\left(\phi(x_i)\otimes\phi(x_i) - \tilde{\phi}(x_i)\otimes\tilde{\phi}(x_i)\right)\right\|_{\text{HS}} \tag{94}$$

$$\overset{(ii)}{\leq} \frac{2}{n}\mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\mathop{\mathbb{E}}_{\xi_i\sim\mathcal{R}}\left\|\sum_{i=1}^{n}\xi_i\phi(x_i)\otimes\phi(x_i)\right\|_{\text{HS}} \tag{95}$$

$$\overset{(iii)}{\leq} \frac{2}{n}\left(\mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\mathop{\mathbb{E}}_{\xi_i\sim\mathcal{R}}\left\|\sum_{i=1}^{n}\xi_i\phi(x_i)\otimes\phi(x_i)\right\|_{\text{HS}}^2\right)^{1/2} \tag{96}$$

(i) follows from noticing $\phi(x_i)\otimes\phi(x_i)-\Sigma$ is a mean 0 operator in $\mathcal{H}\otimes\mathcal{H}$, then for $X,Y\in\mathcal{H}\otimes\mathcal{H}$ s.t. $\mathbb{E}[Y]=0$ it follows $\|X\|_{\text{HS}}=\|X-\mathbb{E}[Y]\|_{\text{HS}}=\|\mathbb{E}_Y[X-Y]\|_{\text{HS}}$ and finally applying Jensen's Inequality. (ii) follows from the triangle inequality. (iii) follows from Jensen's Inequality. Let e_k for $k\in[p]$ represent an orthonormal basis for the Hilbert Space $\mathcal{H}$. By expanding out the Hilbert-Schmidt Norm, we then have

$$\frac{2}{n}\left(\mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\mathop{\mathbb{E}}_{\xi_i\sim\mathcal{R}}\left\|\sum_{i=1}^{n}\xi_i\phi(x_i)\otimes\phi(x_i)\right\|_{\text{HS}}^2\right)^{1/2}$$

$$=\frac{2}{n}\left(\mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\mathop{\mathbb{E}}_{\xi_i\sim\mathcal{R}}\sum_{k=1}^{p}\left\langle\sum_{i=1}^{n}\xi_i\phi(x_i)\otimes\phi(x_i)e_k,\sum_{j=1}^{n}\xi_j\phi(x_j)\otimes\phi(x_j)e_k\right\rangle\right)^{1/2} \tag{97}$$

$$=\frac{2}{n}\left(\mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\mathop{\mathbb{E}}_{\xi_i\sim\mathcal{R}}\sum_{k=1}^{p}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\xi_j\left\langle\phi(x_i)\otimes\phi(x_i)e_k,\phi(x_j)\otimes\phi(x_j)e_k\right\rangle\right)^{1/2} \tag{98}$$

$$\overset{(iv)}{\leq}\frac{2}{n}\left(\mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\sum_{k=1}^{p}\sum_{i=1}^{n}\left\langle\phi(x_i)\otimes\phi(x_i)e_k,\phi(x_i)\otimes\phi(x_i)e_k\right\rangle\right)^{1/2} \tag{99}$$

$$=\frac{2}{n}\left(\sum_{i=1}^{n}\mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\|\phi(x_i)\otimes\phi(x_i)\|_{\text{HS}}^2\right)^{1/2}\overset{(v)}{=}\frac{2}{n}\left(\sum_{i=1}^{n}\mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\|\phi(x_i)\|_{\mathcal{H}}^4\right)^{1/2} \tag{100}$$

$$=\frac{2}{n}\left(\sum_{i=1}^{n}\mathop{\mathbb{E}}_{\phi(x_i)\sim\mathcal{N}(0,\Sigma)}\left[k^2(x_i,x_i)\right]\right)^{1/2}\overset{\text{lem. 32}}{\leq}\frac{2\,\text{Tr}\left(\Sigma\right)}{\sqrt{n}} \tag{101}$$

(iv) follows from noticing $\mathbb{E}_{\xi_i,\xi_j\sim\mathcal{R}}[\xi_i\xi_j]=\delta_{i,j}$. (v) follows from expanding the Hilbert-Schmidt Norm and applying Parseval's Identity. We note $\text{Tr}(\Sigma)<\infty$ and therefore even though the covariance operator is infinite-dimensional we are able to get a finite bound on the covariance approximation. This completes the proof. ■

Lemma 34 (Finite Dimensional Covariate Estimation in the Spectral Norm)**.** *Let $x_1,\ldots,x_n\sim\mathcal{N}(0,\Sigma)$. It then follows,*

$$\mathop{\mathbb{E}}_{x_i\sim\mathcal{N}(0,\Sigma)}\left\|\frac{1}{n}\sum_{i=1}^{n}x_ix_i^\top-\Sigma\right\|_2\leq\frac{2\sqrt{3}\,\text{Tr}\left(\Sigma\right)}{\sqrt{n}} \tag{102}$$

Proof. From similar steps in Lemma 33. We have,

$$\mathop{\mathbb{E}}_{x_i\sim\mathcal{N}(0,\Sigma)}\left\|\frac{1}{n}\sum_{i=1}^{n}x_ix_i^\top-\Sigma\right\|_2\leq\frac{2}{n}\left(\mathop{\mathbb{E}}_{x_i\sim\mathcal{N}(0,\Sigma)}\mathop{\mathbb{E}}_{\xi_i\sim\mathcal{R}}\left\|\sum_{i=1}^{n}\xi x_ix_i^\top\right\|_2^2\right)^{1/2} \tag{103}$$

$$=\frac{2}{n}\left(\mathop{\mathbb{E}}_{x_i\sim\mathcal{N}(0,\Sigma)}\mathop{\mathbb{E}}_{\xi_i\sim\mathcal{R}}\text{Tr}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\xi_jx_ix_i^\top x_jx_j^\top\right)\right)^{1/2} \tag{104}$$

$$=\frac{2}{n}\left(\mathop{\mathbb{E}}_{x_i\sim\mathcal{N}(0,\Sigma)}\sum_{i=1}^{n}\|x_i\|^4\right)^{1/2}=\frac{2\sqrt{\text{Tr}\left(\Sigma\right)^2+2\,\text{Tr}\left(\Sigma^2\right)}}{\sqrt{n}} \tag{105}$$

Noting $\text{Tr}(\Sigma^2)\leq\text{Tr}(\Sigma)^2$ as $\Sigma\succ 0$ and applying the triangle inequality completes the proof. ■

B Proofs for Section 3

In this section we give the deferred proofs of our main structural results.

B.1 Proof of Lemma 14

Proof. For any $f_{w_1}, f_{w_2} \in \mathcal{K}$, we will first show the gradient is bounded.

$$|g(t, f_{w_1}) - g(t, f_{w_2})| = \left| \int_0^1 \nabla_f g(t, (1-\lambda) f_{w_1} + \lambda f_{w_2})(f_{w_1} - f_{w_2}) d\lambda \right| \tag{106}$$

$$\leq \|f_{w_1} - f_{w_2}\|_{\mathcal{H}} \left| \int_0^1 \nabla_f g(t, (1-\lambda) f_{w_1} + \lambda f_{w_2}) d\lambda \right| \tag{107}$$

$$\overset{(a)}{\leq} \|f_{w_1} - f_{w_2}\|_{\mathcal{H}} \max_{f_w \in \mathcal{K}} \|\nabla_f g(t, f_w)\|_{\mathcal{H}} \tag{108}$$

In (a), we note that since $\mathcal{K}$ is convex, then by definition as $f_{w_1}, f_{w_2} \in \mathcal{K}$, we have for $\lambda \in [0,1]$, the convex combination $(1-\lambda) f_{w_1} + \lambda f_{w_2} \in \mathcal{K}$. We use the $\mathcal{H}$ norm of the gradient to bound L from above for an element in the convex closed set $\mathcal{K}$.

$$\|\nabla g\,(t, f_w)\|_{\mathcal{H}} = \left\| \frac{2}{np} \sum_{i=1}^{n} \mathbb{I}\left\{ t \geq (f_w(x_i) - y_i)^2 \right\} (f_w(x_i) - y_i) \cdot k(x_i, \cdot) \right\|_{\mathcal{H}} \tag{109}$$

W.L.O.G, let $x_1, x_2, \cdots, x_m$ where $0 \leq m \leq n$, represent the data vectors such that $t \geq (f_w(x_i) - y_i)^2$.

$$= \left\| \frac{2}{np} \sum_{i=1}^{m} (f_w(x_i) - y_i) \cdot k\,(x_i, \cdot) \right\|_{\mathcal{H}} \tag{110}$$

$$\leq \frac{2}{np} \left(\left\| \sum_{i=1}^{m} f_w(x_i) \cdot k\,(x_i, \cdot) \right\|_{\mathcal{H}} + \left\| \sum_{i=1}^{m} y_i k\,(x_i, \cdot) \right\|_{\mathcal{H}} \right) \tag{111}$$

$$\overset{(a)}{\leq} \frac{2}{np} \left(\left\| \sum_{i=1}^{m} \left\langle \sum_{j=1}^{n} w_j k(x_j, \cdot), k(x_i, \cdot) \right\rangle_{\mathcal{H}} \cdot k(x_i, \cdot) \right\|_{\mathcal{H}} + \left\| \sum_{i=1}^{m} y_i \right\| \left\| \sum_{i=1}^{m} k(x_i, \cdot) \right\|_{\mathcal{H}} \right) \tag{112}$$

$$\leq \frac{2}{np} \left(\left| \left\langle \sum_{j=1}^{n} w_j k(x_j, \cdot), \sum_{i=1}^{m} k(x_i, \cdot) \right\rangle_{\mathcal{H}} \right| \left\| \sum_{i=1}^{m} k(x_i, \cdot) \right\|_{\mathcal{H}} + \left\| \sum_{i=1}^{m} y_i \right\| \sum_{i=1}^{m} \sqrt{k(x_i, x_i)} \right) \tag{113}$$

$$\leq \frac{2}{np} \left(\|f_w\|_{\mathcal{H}} \left(\sum_{i=1}^{m} \sqrt{k(x_i, x_i)} \right)^2 + \sqrt{n}\, \|y\|_2 \left(\sum_{i=1}^{n} \sqrt{k(x_i, x_i)} \right) \right) \tag{114}$$

$$\leq \frac{2R}{np} \left(\sum_{i=1}^{n} \sqrt{k(x_i, x_i)} \right)^2 + \frac{2\,\|y\|_2}{p\sqrt{n}} \left(\sum_{i=1}^{n} \sqrt{k(x_i, x_i)} \right) \tag{115}$$

(a) follows form the reproducing property for RKHS [Gre13]. If we have a normalized kernel such as the Gaussian Kernel, then we have the Lipschitz Constant is finite. Furthermore, if the adversary introduces label corruption that tends to ∞, then these points will not be in the Subquantile as f_w has bounded norm, so it will have infinite error. This concludes the proof. ■

B.2 Proof of Lemma 16

Proof. Let S be the set containing the points with the minimum error from X w.r.t to the weights vector w. Define $\eta_i \triangleq (f_{w^*}(x_i) - y_i)$ where $i \in P$.

$$\lim_{\lambda \to \infty} \left(\Phi_\lambda\,(f_w) - \Phi_\lambda\,(f_{w^*}) \right) = \sum_{i \in S} \left(f_w\,(x_i) - y_i \right)^2 - \sum_{j \in P} \left(f_{w^*}\,(x_j) - y_j \right)^2 \tag{116}$$

$$= \sum_{i \in S \cap P} \left(f_w\left(x_i\right) - y_i\right)^2 + \sum_{i \in S \cap Q} \left(f_w\left(x_i\right) - y_i\right)^2 - \sum_{j \in P} \left(f_{w^*}\left(x_j\right) - y_j\right)^2 \tag{117}$$

$$\geq \sum_{i \in S \cap P} \left(f_w\left(x_i\right) - y_i\right)^2 - \sum_{j \in P} \left(f_{w^*}\left(x_j\right) - y_j\right)^2 \tag{118}$$

$$= \sum_{i \in S \cap P} \left(f_w\left(x_i\right) - f_{w^*}\left(x_i\right) - \eta_i\right)^2 - \sum_{j \in P} \eta_j^2 \tag{119}$$

$$= \sum_{i \in S \cap P} \left(\left(f_w - f_{w^*}\right)\left(x_i\right) - \eta_i\right)^2 - \sum_{j \in P} \eta_j^2 \tag{120}$$

$$\geq \sum_{i \in S \cap P} \underbrace{\left(\left(f_w - f_{w^*}\right)\left(x_i\right)\right)^2}_{\text{(A}_1\text{)}} - 2 \underbrace{\sum_{i \in S \cap P} \eta_i \left(f_w - f_{w^*}\right)\left(x_i\right)}_{\text{(A}_2\text{)}} - \underbrace{\sum_{j \in P \setminus S} \eta_j^2}_{\text{(A}_3\text{)}} \tag{121}$$

Now we will upper bound A_1. Similar to [CLKZ21] Let $\mathbb{E}_{x \sim \mathbb{P}}[\varphi(x) \otimes \varphi(x)] = \mathbb{I}_m$ where $\varphi(x) = \{\varphi(x)\}_{k=1}^m$ and m is possibly infinite. We can then rescale the basis features. Then let $\phi(x) = \Sigma^{1/2}\varphi(x)$. We therefore have $\Sigma = \mathbb{E}_{x \sim \mathbb{P}}[\phi(x) \otimes \phi(x)] = \operatorname{diag}(\xi_1, \ldots, \xi_n)$. This is the eigenfunction basis described in [SS16].

$$\text{(A}_1\text{)} \triangleq \sum_{i \in S \cap P} \left(\left(f_w - f_{w^*}\right)\left(x_i\right)\right)^2 \overset{(a)}{=} \sum_{i \in S \cap P} \left\langle \sum_{j \in X} \left(w_j - w_j^*\right) k(x_j, \cdot), k(x_i, \cdot) \right\rangle_{\mathcal{H}}^2 \tag{122}$$

$$= \sum_{i \in S \cap P} \left\langle \sum_{j \in X} \left(w_j - w_j^*\right) \phi(x_j), \phi(x_i) \right\rangle_{\mathcal{H}} \left\langle \phi(x_i), \sum_{j \in X} \left(w_j - w_j^*\right) \phi(x_j) \right\rangle_{\mathcal{H}} \tag{123}$$

$$= \sum_{i \in S \cap P} \left\langle \sum_{j \in X} \left(w_j - w_j^*\right) \phi(x_j), \phi(x_i) \otimes \phi(x_i) \sum_{j \in X} \left(w_j - w_j^*\right) \phi(x_j) \right\rangle_{\mathcal{H}} \tag{124}$$

$$= \sum_{i \in S \cap P} \left\langle \phi(x) \otimes \phi(x), (f_w - f_w^*) \otimes (f_w - f_w^*) \right\rangle_{\text{HS}} \tag{125}$$

$$= \sum_{i \in S \cap P} \left\langle \Sigma + \phi(x) \otimes \phi(x) - \Sigma, (f_w - f_w^*) \otimes (f_w - f_w^*) \right\rangle_{\text{HS}} \tag{126}$$

$$\geq n(1 - 2\varepsilon) \mathop{\mathbb{E}}_{x \sim \mathbb{P}} \left[\left(f_w - f_w^*\right)(x)^2\right] - \left\| \sum_{i \in S \cap P} \phi(x) \otimes \phi(x) - \Sigma \right\|_{\text{HS}} \|f_w - f_w^*\|_{\mathcal{H}}^2 \tag{127}$$

$$\overset{\text{lem. 33}}{\geq} \left(n(1 - 2\varepsilon) \lambda_{\min}(\Sigma) - \left\| \sum_{i \in S \cap P} \phi(x) \otimes \phi(x) - \Sigma \right\|_{\text{HS}} \right) \|f_w - f_w^*\|_{\mathcal{H}}^2 \tag{128}$$

Next we will upper bound A_2,

$$\text{(A}_2\text{)} \triangleq \sum_{i \in S \cap P} \eta_i \left(f_w - f_{w^*}\right)\left(x_i\right) \tag{129}$$

$$= \sum_{i \in S \cap P} \left\langle \sum_{j \in X} (w_j - w_j^*) k(x_j, \cdot), \eta_i k(x_i, \cdot) \right\rangle_{\mathcal{H}} \tag{130}$$

$$= \left\langle \sum_{j \in X} (w_j - w_j^*) k(x_j, \cdot), \sum_{i \in S \cap P} \eta_i k(x_i, \cdot) \right\rangle_{\mathcal{H}} \tag{131}$$

$$\leq \|f_w - f_{w^*}\|_{\mathcal{H}} \left\| \sum_{i \in S \cap P} \eta_i k(x_i, \cdot) \right\|_{\mathcal{H}} = \|f_w - f_{w^*}\|_{\mathcal{H}} \left\| \sum_{i \in S \cap P} \eta_i \phi(x_i) \right\|_{\mathcal{H}} \tag{132}$$

Then, combining our bounds, we have

$$\lim_{\lambda\to\infty}\left(\Phi_\lambda\left(f_w\right)-\Phi_\lambda\left(f_{w^*}\right)\right) \overset{(128) \text{ and } (132)}{\geq} \eta^2\Bigg(n(1-2\varepsilon)\lambda_{\min}\left(\mathop{\mathbb{E}}_{x\sim\mathbb{P}}\left[\phi(x)\otimes\phi(x)\right]\right) - \left\|\sum_{i\in S\cap P}\phi(x)\otimes\phi(x)-\Sigma\right\|_{\mathrm{HS}}\Bigg) - 2\eta\left\|\sum_{i\in S\cap P}\eta_i\phi\left(x_i\right)\right\| - \sum_{j\in P\setminus S}\eta_j^2 \tag{133}$$

This completes the proof. ■

B.3 Proof of Theorem 17

Proof. First, we give the definiton of the Moreau stationary point.

$$\left\|\nabla \mathsf{M}_{\Phi_\lambda,\rho}\left(f_w\right)\right\|_{\mathcal{H}} = \left\|\frac{1}{\rho}\left(f_w - \mathop{\arg\min}_{f_{\widehat{w}}\in\mathcal{K}}\left(\Phi\left(f_{\widehat{w}}\right)+\frac{1}{2\rho}\left\|f_w-f_{\widehat{w}}\right\|_{\mathcal{H}}^2\right)\right)\right\|_{\mathcal{H}} = 0 \tag{134}$$

This implies for any $f_{\widetilde{w}}\in\mathcal{K}$, it follows

$$\lim_{\lambda\to\infty}\left(\Phi_\lambda\left(f_{\widehat{w}}\right)\right) < \lim_{\lambda\to\infty}\left(\Phi_\lambda\left(f_{\widetilde{w}}\right)\right)+\frac{1}{2\rho}\left\|f_{\widetilde{w}}-f_{\widehat{w}}\right\|_{\mathcal{H}}^2 \tag{135}$$

For any $f_{\widehat{w}}$ satisfying above, then the distance from the optimal must be low. Let $\widetilde{w}=w^*$, then we have

$$\lim_{\lambda\to\infty}\left(\Phi_\lambda\left(f_{\widehat{w}}\right)-\Phi_\lambda\left(f_{w^*}\right)\right)\leq\frac{1}{2\rho}\left\|f_{\widehat{w}}-f_{w^*}\right\|_{\mathcal{H}}^2 \tag{136}$$

We proceed by proof by contradiction. Assume $\|f_{\widehat{w}}-f_w^*\|>\eta$, then if $\Phi(f_{\widehat{w}})-\Phi(f_w^*)>\frac{\eta^2}{2\rho}$, then we will have $f_{\widehat{w}}$ is not a stationary point, which will imply $\|f_{\widehat{w}}-f_w^*\|_{\mathcal{H}}\leq\eta$. Therefore, we attempt to find the minimum value for η. From Lemma 16, we have the expected distance from a stationary point of the Moreau Envelope from the optimal point over the distribution of uncorrupted datasets.

$$\begin{aligned}\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}}\lim_{\lambda\to\infty}\left(\Phi\left(f_w\right)-\Phi\left(f_w^*\right)\right) &\overset{\text{lem. } 16}{\geq} \eta^2\Bigg(n(1-2\varepsilon)\lambda_{\min}\left(\mathop{\mathbb{E}}_{x\sim\mathbb{P}}\left[\phi(x)\otimes\phi(x)\right]\right)\\ &\quad - \mathop{\mathbb{E}}_{x_i\sim\mathbb{P}}\left\|\sum_{i\in S\cap P}\phi(x_i)\otimes\phi(x_i)-\Sigma\right\|_{\mathrm{HS}}\Bigg) - 2\eta\mathop{\mathbb{E}}_{\mu_i,x_i\sim\mathbb{P}}\left\|\sum_{i\in S\cap P}\eta_i\phi\left(x_i\right)\right\| - \mathop{\mathbb{E}}_{\mu_j\sim\mathbb{P}}\sum_{j\in P\setminus S}\eta_j^2 \qquad (137)\\ &\overset{\text{lems. } 32 \text{ and } 33}{\geq} \eta^2\left(n(1-2\varepsilon)\lambda_{\min}\left(\Sigma\right)-2\operatorname{Tr}\left(\Sigma\right)\sqrt{n(1-2\varepsilon)}\right)\\ &\quad - \eta O\left(\sigma\sqrt{n(1-2\varepsilon)\log\left(n(1-2\varepsilon)\right)\operatorname{Tr}\left(\Sigma\right)}\right)-\sigma\varepsilon n \qquad (138)\end{aligned}$$

From the definition of stationary point, we have

$$\begin{aligned}&\eta^2\left(n(1-2\varepsilon)\lambda_{\min}\left(\Sigma\right)-2\operatorname{Tr}\left(\Sigma\right)\sqrt{n(1-2\varepsilon)}-\beta\right)\\ &\qquad-\eta O\left(\sigma\sqrt{n(1-2\varepsilon)\log\left(n(1-2\varepsilon)\right)\operatorname{Tr}\left(\Sigma\right)}\right)-\sigma\varepsilon n\leq 0\end{aligned} \tag{139}$$

Therefore, when Equation (139) does not hold, we have a contradiction. It thus follows from upper bounding the positive solution of the quadratic equation,

$$\begin{aligned}\eta\leq\left(\sigma\varepsilon n\right)^{1/2}&\left(n(1-2\varepsilon)\left(\lambda_{\min}\left(\Sigma\right)-\frac{2\operatorname{Tr}\left(\Sigma\right)}{\sqrt{n\left(1-2\varepsilon\right)}}\right)-\beta\right)^{-1/2}\\ &+O\left(\sigma\sqrt{n(1-2\varepsilon)\log\left(n(1-2\varepsilon)\right)\operatorname{Tr}\left(\Sigma\right)}\right)\left(n(1-2\varepsilon)\left(\lambda_{\min}\left(\Sigma\right)-\frac{2\operatorname{Tr}\left(\Sigma\right)}{\sqrt{n\left(1-2\varepsilon\right)}}\right)-\beta\right)^{-1}\end{aligned} \tag{140}$$

Then for some constant $c_1 \in (0,1)$, if $n \geq \frac{8\operatorname{Tr}(\Sigma)^2}{\lambda_{\min}(\Sigma)(1-c_1)^2(1-2\varepsilon)} + \frac{8\beta}{(1-c_1)^2(1-2\varepsilon)}$, we have

$$\eta \leq \left(\frac{\sigma\varepsilon n}{c_1 n(1-2\varepsilon)\lambda_{\min}(\Sigma)}\right)^{1/2} + \frac{O\left(\sigma\sqrt{\log\left(n(1-2\varepsilon)\right)\operatorname{Tr}(\Sigma)}\right)}{c_1\sqrt{n(1-2\varepsilon)}\lambda_{\min}(\Sigma)} \tag{141}$$

we therefore see as n goes large, $c_1 \to 1$, and we have in the worst case

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}} \|f_{\widehat{w}} - f_w^*\|_{\mathcal{H}} \leq O\left(\sqrt{\frac{\varepsilon}{1-2\varepsilon}}\frac{\sigma}{\lambda_{\min}(\Sigma)}\right) \tag{142}$$

This completes the proof. ■

B.4 Proof of Corollary 18

We follow the same framework as our proof for kernelized linear regression, we will simply give the new constants. Assuming the uncorrupted covariates, $x_i \sim \mathcal{N}(0,\Sigma)$. To simplify notation, let us define $\tilde{n} \triangleq n(1-2\varepsilon)$ to represent the absolute minimum number of uncorrupted points in the Subquantile. We then have,

$$\begin{aligned}\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}} \lim_{\lambda\to\infty} \left(\Phi_\lambda(w) - \Phi_\lambda(w^*)\right) &\overset{\text{lem. 16}}{\geq} \eta^2\left(\tilde{n}\lambda_{\min}(\Sigma) - \mathbb{E}\left\|\sum_{i\in S\cap P} x_i x_i^\top - \Sigma\right\|_2\right) \\ &\quad - \mathop{\mathbb{E}}_{\xi,\mu_i\sim\mathbb{P}}\left\|\sum_{i\in S\cap P}\mu_i x_i\right\|_2 - \mathop{\mathbb{E}}_{\mu_i\sim\mathbb{P}}\sum_{i\in P\setminus S}\mu_i^2 \qquad (143)\\ &\overset{\text{lems. 30, 32 and 34}}{\geq} \eta^2\left(\tilde{n}\lambda_{\min}(\Sigma) - \sqrt{\tilde{n}}\left(2\sqrt{3}\operatorname{Tr}(\Sigma)\right)\right) - \eta O\left(\sigma\sqrt{\tilde{n}\log(\tilde{n})\operatorname{Tr}(\Sigma)}\right) - \varepsilon n\sigma^2 \qquad (144)\end{aligned}$$

Then from a similar contradiction idea and upper bounding the quadratic, we have in expectation

$$\begin{aligned}\eta &\overset{\text{thm. 17}}{\leq} O\left(\sigma\sqrt{\tilde{n}\log(\tilde{n})\operatorname{Tr}(\Sigma)}\right)\left(\tilde{n}\lambda_{\min}(\Sigma) - \sqrt{\tilde{n}}\left(2\sqrt{3}\operatorname{Tr}(\Sigma)\right) - \beta\right)^{-1} \\ &\quad + \sigma\sqrt{\tilde{n}\frac{\varepsilon}{1-2\varepsilon}}\left(\tilde{n}\lambda_{\min}(\Sigma) - \sqrt{\tilde{n}}\left(2\sqrt{3}\operatorname{Tr}(\Sigma)\right) - \beta\right)^{-1/2} \qquad (145)\end{aligned}$$

We then have for a constant $c_2 \in (0,1)$, if $n \geq \frac{54\operatorname{Tr}(\Sigma)}{(1-c_2)^2(1-2\varepsilon)\lambda_{\min}^2(\Sigma)} + 2\beta$, it follows

$$\eta \leq \sqrt{\frac{\sigma^2\varepsilon}{(1-2\varepsilon)c_2\lambda_{\min}(\Sigma)}} + \frac{O\left(\sigma\sqrt{\log\left(n(1-2\varepsilon)\right)\operatorname{Tr}(\Sigma)}\right)}{\sqrt{n(1-2\varepsilon)}c_2\lambda_{\min}(\Sigma)} \tag{146}$$

We thus see as n goes large, $c_2 \to 1$ and we will have in worst case,

$$\mathop{\mathbb{E}}_{\mathcal{D}\sim\hat{\mathbb{P}}} \|\widehat{w} - w^*\|_2 \leq O\left(\sqrt{\frac{\varepsilon}{1-2\varepsilon}}\frac{\sigma}{\sqrt{\lambda_{\min}(\Sigma)}}\right) \tag{147}$$

Obtaining the same asymptotic bound as in the kernelized regression case. This completes the proof. ■

B.5 Proof of Lemma 19

Proof. We use the $\mathcal{H}$ norm of the gradient to bound L from above. Let S be denoted as the subquantile set. Define the sigmoid function as $\sigma(x) = \frac{1}{1+e^{-x}}$.

$$\nabla_{\mathfrak{f}} g(t, f_w)_{\mathcal{H}} = \left\|\frac{1}{np}\sum_{i=1}^n \mathbb{I}\{t \geq (1-y_i)\log(f_w(x_i))\}\left(y_i - \sigma\left(f_w(x_i)\right)\right)\cdot k(x_i,\cdot)\right\|_{\mathcal{H}} \tag{148}$$

$$\overset{(i)}{\leq} \frac{1}{np} \sum_{i \in S} \|(y_i - \sigma\left(f_w(x_i)\right)) \cdot k(x_i, \cdot)\|_{\mathcal{H}} \tag{149}$$

$$\overset{(ii)}{\leq} \frac{1}{np} \sum_{i \in S} |y_i - \sigma\left(f_w(x_i)\right)| \, \|k(x_i, \cdot)\|_{\mathcal{H}} \tag{150}$$

$$\overset{(iii)}{\leq} \frac{1}{np} \sum_{i=1}^{n} \sqrt{k(x_i, x_i)} \tag{151}$$

(i) follows from the triangle inequality. (ii) follows from the Cauchy-Schwarz inequality. (iii) follows from the fact that $y_i \in \{0, 1\}$ and range$(\sigma) \in [0, 1]$. This completes the proof. ■

B.6 Proof of Lemma 20

We use the Hilbert Space norm of second derivative to bound β from above. Let S be the subquantile set.

$$\left\|\nabla_f^2 g(t, f_w)\right\|_{\mathcal{H}} = \frac{1}{np} \sum_{i=1}^{n} \mathbb{I}\left\{t \geq (1 - y_i) \log(f_w(x_i))\right\} \sigma\left(f_w(x_i)\right) \left(1 - \sigma\left(f_w(x_i)\right)\right) k(x_i, x_i) \tag{152}$$

$$\leq \frac{1}{np} \sum_{i=1}^{n} \left|\sigma\left(f_w(x_i)\right) \left(1 - \sigma\left(f_w(x_i)\right)\right)\right| |k(x_i, x_i)| \tag{153}$$

$$\overset{(i)}{\leq} \frac{1}{4np} \sum_{i=1}^{n} k(x_i, x_i) = \frac{1}{4np} \operatorname{Tr}(\mathrm{K}) \tag{154}$$

(i) follows as for a scaler $\alpha \in [0, 1]$, the maximum value of $\alpha(1 - \alpha)$ is obtained at $\frac{1}{4}$. This completes the proof. ■

B.7 Proof of Lemma 21

[5] **Proof.** Let S be the Subquantile set for f_w, then we have

$$\lim_{\lambda \to \infty} \Phi(f_w) - \Phi(f_w^*) \geq -\sum_{i \in S} y_i \log\left(\sigma\left(f_w(x_i)\right)\right) + (1 - y_i) \log\left(1 - \sigma\left(f_w(x_i)\right)\right)$$
$$+ \sum_{i \in P} y_i \log\left(\sigma\left(f_w^*(x_i)\right)\right) + (1 - y_i) \log\left(1 - \sigma\left(f_w^*(x_i)\right)\right) \tag{155}$$

$$\overset{(i)}{\geq} -\sum_{i \in S \cap P} y_i \log\left(\sigma\left(f_w(x_i)\right)\right) + (1 - y_i) \log\left(1 - \sigma\left(f_w(x_i)\right)\right)$$
$$+ \sum_{i \in P} y_i \log\left(\sigma\left(f_w^*(x_i)\right)\right) + (1 - y_i) \log\left(1 - \sigma\left(f_w^*(x_i)\right)\right) \tag{156}$$

$$= \sum_{i \in S \cap P} y_i \underbrace{\log\left(\frac{\sigma\left(f_w^*(x_i)\right)}{\sigma\left(f_w(x_i)\right)}\right)}_{\text{(A}_1\text{)}} + (1 - y_i) \underbrace{\log\left(\frac{1 - \sigma\left(f_w^*(x_i)\right)}{1 - \sigma\left(f_w(x_i)\right)}\right)}_{\text{(A}_2\text{)}}$$
$$+ \sum_{P \setminus S} y_i \log\left(\sigma\left(f_w^*(x_i)\right)\right) + (1 - y_i) \log\left(1 - \sigma\left(f_w^*(x_i)\right)\right) \tag{157}$$

(i) follows from the optimality of the subquantile set, S. First we bound (A$_1$).

$$\text{(A}_1\text{)} \triangleq \log\left(\frac{\sigma\left(f_w^*(x_i)\right)}{\sigma\left(f_w(x_i)\right)}\right) = \log\left(\frac{1 + e^{-f_w(x_i)}}{1 + e^{-f_w^*(x_i)}}\right) = \log\left(1 + e^{-f_w(x_i)}\right) - \log\left(1 + e^{-f_w^*(x_i)}\right) \tag{158}$$

$$\overset{(i)}{\geq} \left(f_w^* - f_w\right)(x_i) - 0.693 \tag{159}$$

[5] Work in Progress

In (i) we have $f_w^*(x_i) < 0$ when $y_i = 0$. Next we will upper bound $\textcircled{A_2}$.

$$\textcircled{A_2} \triangleq \log\left(\frac{1-\sigma\left(f_w^*(x_i)\right)}{1-\sigma\left(f_w(x_i)\right)}\right) = \log\left(\frac{\sigma\left(-f_w^*(x_i)\right)}{\sigma\left(-f_w(x_i)\right)}\right) = \log\left(\frac{1+e^{f_w(x_i)}}{1+e^{f_w^*(x_i)}}\right) \tag{160}$$

$$= \log\left(1+e^{f_w(x_i)}\right) - \log\left(1+e^{f_w^*(x_i)}\right) \tag{161}$$

$$\overset{(ii)}{\geq} \left(f_w - f_w^*\right)(x_i) - 0.693 \tag{162}$$

In (ii) we note that $f_w^*(x_i) \geq 0$ when $y_i = 1$. Combining our bounds, we have

$$\lim_{\lambda\to\infty} \Phi(f_w) - \Phi(f_w^*) \overset{(159) \text{ and } (162)}{\geq} \sum_{i\in P\cap S} 2y_i\left(\left(f_w - f_w^*\right)(x_i)\right) + \left(f_w - f_w^*\right)(x_i) - 0.693 \tag{163}$$

$$\geq -3\left\|f_w - f_w^*\right\|_{\mathcal{H}} \left\|\sum_{i\in S\cap P} \phi(x_i)\right\|_{\mathcal{H}} - 0.693 \tag{164}$$

B.8 Proof of Lemma 22

Proof. We use the Hilbert Space norm of the gradient to bound L from above. Let S be denoted as the subquantile set.

$$\left\|\nabla_{\mathrm{W}} g(t, \mathrm{W})\right\|_{\mathcal{H}} = \tag{165}$$

■

C Proofs for Section 4

In this section we give the optimization results from Section 4.

D Necessary Results

Theorem 35 (Mercer, [Mer09])**.** *If k is a positive definite and continuous kernel on a compact set Ω, th eoperator T has a countable set of eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$ and eigenfunctions $\{\varphi\}_{j\in\mathbb{N}}$ with $T\varphi_j = \lambda\varphi_j$ s.t. $\|\varphi_j\| = \lambda_j^{-1}$. The kernel then admits the following representation,*

$$k(x,y) = \sum_{j=1}^{\infty} \lambda_j \varphi_j(x)\varphi(y) \tag{166}$$

E Experimental Details

In this section we give details on datasets and hyperparameters.

E.1 Kernel Regression

Our datasets are synthetic and are sourced from [DG17]

Dataset	**Dimension** d	**Sample Size** n	**Source**
`Polynomial`	3	1000	Ours
`Boston Housing`	13	506	
`Concrete Data`	8	1030	
`Wine Quality`	11	1599	

Table 4: `Polynomial Regression` Synthetic Dataset. 1000 samples, $x \sim \mathcal{N}(0,1)$, $y \sim \mathcal{N}(\sum_{i=0} a_i x^i, 0.01)$ where $a_i \sim \mathcal{N}(0,1)$. Oblivious Noise is sampled from $\mathcal{N}(0,5)$. Subquantile is capped at 10,000 iterations.

E.2 Kernel Binary Classification

Dataset	Dimension d	Sample Size n	Source
`Heart Disease`	13	303	
`Breast Cancer`	32	569	Kaggle

Table 5: Datasets for Kernel Binary Classification.

E.3 Kernel Multi-Class Classification

Dataset	Dimension d	Sample Size n	Source

Table 6: Datasets for Kernel Multi-Class Classification.

E.4 Linear Regression

Dataset	Dimension d	Sample Size n	Source
`Boston Housing`	14	506	Kaggle
`Wine Quality`	11	1599	
`Concrete`	8	1030	
`Drug`			

Table 7: Datasets for Linear Regression.