Iterative Thresholding for Non-Linear Learning in the Strong ϵ-Contamination Model

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Abstract

We study the problem of learning single neurons when both the labels and the covariates are possibly corrupted adversarially with a gradient-descent iterative thresholding algorithm. We assume the data is sampled from a ground truth distribution,

$$y = \psi(\mathbf{x}^\top \mathbf{w}^*) + \xi$$

where ξ is Gaussian noise and σ is an activation-function. We consider activations with three different properties, (i) Lipschitz, (ii) lower bounded first derivative, and (iii) second derivative has finitely many zeros. Functions such as tanh, sigmoid, leaky-ReLU, and ReLU are functions with a subset of the properties.

We also study the linear regression problem when $\psi(x) = x$. We improve upon previous approximation bounds for gradient based iterative thresholding algorithms [BJK15, SS19] and show with high probability a $O(\sigma \epsilon \log \epsilon^{-1})$ approximation upper bound, matching the best known approximation bound for iterative thresholding algorithms [ADKS22] while improving run-time. We extend previous works by stuyding the case when $\mathbf{x}$ is sampled from a sub-Gaussian distribution with a general covariance matrix $\boldsymbol{\Sigma}$.

1 Introduction

There has been extensive study of algorithms to learn the target distribution from a Huber ϵ-Contaminated Model for a Generalized Linear Model (GLM), [DKK+19, ADKS22, LBSS21, OZS20, FB81] as well as for linear regression [BJKK17, MGJK19]. Robust Statistics has been studied extensively [DK23] for problems such as high-dimensional mean estimation [PBR19, CDGS20] and Robust Covariance Estimation [CDGW19, FWZ18]. Recently, there has been an interest in solving robust machine learning problems by gradient descent [PSBR18, DKK+19]. Subquantile minimization aims to address the shortcomings of standard ERM in applications of noisy/corrupted data [KLA18, JZL+18]. In many real-world applications, the covariates have a non-linear dependence on labels [AMMIL12, Section 3.4]. In which case it is suitable to transform the covariates to a different space utilizing kernels [HSS08]. Therefore, in this paper we consider the problem of Robust Learning for Kernel Learning.

Definition 1 (Strong ϵ-Contamination Model [HR09]). *Given a corruption parameter $0 \leq \epsilon < 0.5$, a data matrix, $\mathbf{X}$ and labels $\mathbf{y}$. An adversary is allowed to inspect all samples and modify ϵn samples arbitrarily. The algorithm is then given the ϵ-corrupted data matrix $\mathbf{X}$ and ϵ-corupted labels vector $\mathbf{y}$ as training data.*

Current approaches for robust learning across various machine learning tasks often use gradient descent over a robust objective, [LBSS21]. These robust objectives tend to not be convex and therefore do not have a strong analysis on the error bounds for general classes of models.

We similarly propose a robust objective which has a nonconvex-concave objective. This objective function has also been proposed recently in [HYwL20] where there has been an analysis in the Binary Classification Task. We show Subquantile Minimization reduces to the same objective function given in [HYwL20].

The study of Kernel Learning in the Gaussian Design is quite popular, [CLKZ21, Dic16]. In [CLKZ21], the feature space, $\phi(\mathbf{x}_i) \sim \mathcal{N}(0, \boldsymbol{\Sigma})$ where $\boldsymbol{\Sigma}$ is a diagonal matrix of dimension p, where p can be infinite. We will now give our formal definition of the dataset.

Definition 2 (Corruption Model). *Let $\mathcal{P}$ be a distribution over $\mathbb{R}^d$ such that $\mathcal{P}_\sharp \phi$ is a centered distribution in the Hilbert Space $\mathcal{H}$ with trace-class covariance operator $\boldsymbol{\Sigma}$ and trace-class sub-Gaussian proxy $\boldsymbol{\Gamma}$ such that $\boldsymbol{\Sigma} \preceq c\boldsymbol{\Gamma}$. The original dataset is denoted as $\hat{P}$, the adversary is able to observe $\hat{P}$ and arbitrarily corrupts $n\epsilon$ samples denoted as Q such that $|Q| = n\epsilon$. The remaining uncorrupted samples are denoted as P such that $|P| = n(1 - \epsilon)$. Together $X \triangleq P \cup Q$ represents the given dataset.*

We will now give one of the first results proving the effectiveness of Iterative Thresholding in Learning Problems.

Theorem 3 (Theorem 5 in [BJK15]). *Let $\mathbf{X}$ be a sub-Gaussian data matrix, and $\mathbf{y} = \mathbf{X}^\top \mathbf{w}^* + \mathbf{b}$ where $\mathbf{b}$ represents the corruption. Then there exists an algorithm such that $\|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2 \leq \varepsilon$ after $t = O\left(\left(\log\left(\frac{\|\mathbf{b}\|_2}{\sqrt{n}}\right)\right)\frac{1}{\varepsilon}\right)$ iterations.*

More recently, Awasthi et al. [ADKS22] studied the iterative trimmed maximum likelihood estimator. In their algorithm, at each step they find $\mathbf{w}^*$ which minimizes the elements in the trimmed set. We will give their formal theorem result.

Theorem 4 (Theorem 4.2 in [ADKS22]). *Let $P = \{(\mathbf{x}_i, y_i)\}_{i=1}^n$ be the data generated by a Gaussian regression model defined as $y_i = \mathbf{x}_i^\top \mathbf{w}^* + \eta_i$ where $\eta_i \sim \mathcal{N}(0, \sigma^2)$ and $\mathbf{x}_i$ are sampled from a sub-Gaussian Distribution with second-moment matrix $\mathbf{I}$. Suppose the dataset has ϵ-fraction of label corruption and $n = \Omega\left(\frac{d + \log(1/\delta)}{\epsilon^2}\right)$. Then there exists an algorithm that returns $\widehat{\mathbf{w}}$ such that with probability $1 - \delta$,*

$$\|\widehat{\mathbf{w}} - \mathbf{w}^*\|_2 = O(\sigma\epsilon \log(1/\epsilon))$$

Our first result recovers this result for vectorized-regression. We will now give our results for the Kernelized GLM problem.

1.1 Contributions

Our main contribution is the approximation bounds for Subquantile Minimization for various non-linear learning problems from the iterative thresholding algorithm given in Algorithm 1. Our proof techniques extend [BJK15, SS19, ADKS22] as we suppose the adversary also corrupts the covariates. To our knowledge, we are also the first to theoretically study iterative thresholding for non-linear learning algorithms beyond the generalized linear model.

Reference	Approximation	Runtime	Model
[BJK15]	$O(\sigma)$	$O\left(N^2 d \log\left(\frac{1}{\sqrt{n}}\frac{\|\mathbf{b}\|_2}{\epsilon}\right)\right)$	Regression; Label
[SS19]	$O(\sigma)$	$O\left(Nd \log\left(\frac{\|\mathbf{w}^*\|_2 + \sigma^2}{\sigma}\right)\right)$	Regression; Label
[ADKS22]	$O(\sigma\epsilon \log \epsilon^{-1})$	$O\left(Nd^2 \frac{1}{\epsilon^2}\right)$	Regression; Label
Corollary 15	$O(\sigma\epsilon \log \epsilon^{-1})$	$O\left(Nd \log\left(\frac{\|\mathbf{w}^*\|}{\sigma\epsilon \log \epsilon^{-1}}\right)\right)$	Regression; Label
Theorem 14	$O\left(\sigma\epsilon\sqrt{C_Q K \log \epsilon^{-1}}\right)$	$O\left(NKd \log\left(\frac{\|\mathbf{W}^*\|_{\mathrm{F}}}{\sigma\epsilon \log \epsilon^{-1}}\right)\right)$	K-variate Regression
Theorem 16	$O\left(L^2\left(\sigma + \epsilon\sqrt{C_Q \log \epsilon^{-1}}\right)\right)$	$O\left(Nd \log\left(\frac{\|f^*\|_{\mathcal{H}}}{\sigma\epsilon \log \epsilon^{-1}}\right)\right)$	GLM *(In progress)*
Theorem 18	$O\left(\sigma\kappa^2(\mathbf{\Sigma})C_\psi^{-1} C_K \sqrt{\epsilon \log \epsilon^{-1}}\right)$	$O\left(Nd \log\left(\frac{\|\mathbf{w}^*\|_2 + (d/\delta)}{\epsilon}\right)\right)$	Leaky ReLU Neuron
Theorem 20	$O\left(\sigma\kappa^2(\mathbf{\Sigma})C_\psi^{-1} C_K d\epsilon \log \epsilon^{-1}\right)$	$O\left(Nd \log\left(\frac{\|\mathbf{w}^*\|_2 + (d/\delta)}{\epsilon}\right)\right)$	ReLU Neuron

Table 1: Summary of related work on Iterative Thresholding Algorithms for Learning in the Huber-ϵ Contamination Model and our contributions. We assume the good data is sampled from a sub-Gaussian distribution with second-moment matrix, $\mathbf{\Sigma}$, and sub-Gaussian norm C_K and dimension d. We assume the variance of the optimal estimator is σ. The Leaky-ReLU function is given as $\max\{C_\psi x, x\}$.

We are able to remove the super-linear dependence on d by leveraging Gradient Descent. Comparing to [BJK15], our bound does not depend on the norm of the noise.

2 Preliminaries

Notation. We denote $[T]$ as the set $\{1, 2, \ldots, T\}$. We define $(x)^+ \triangleq \max(0, x)$ as the Recitificied Linear Unit (ReLU) function. We say $y = O(x)$ if there exists x_0 s.t. for all $x \geq x_0$ there exists C s.t. $y \leq Cx$. We say $y = \Omega(x)$ if there exists x_0 s.t. for all $x \geq x_0$ there exists C s.t. $y \geq Cx$. We denote $a \vee b \triangleq \max(a, b)$ and $a \wedge b \triangleq \min(a, b)$. We define $\mathbb{S}^{d-1}$ as the sphere $\{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| = 1\}$. We will typically denote capital roman letters A, B, C as fixed constants, lower-case roman letters f, g, h as functions, bold-face lower-case letters $\mathbf{x}, \mathbf{y}, \mathbf{z}$ as vectors, and bold-face, upper-case letters $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ as matrices.

Matrices. For a matrix $\mathbf{A}$, let $\lambda_{\max}(\mathbf{A})$ and $\lambda_{\min}(\mathbf{A})$ represent the maximum and minimimum eigenvalues of $\mathbf{A}$, respectively. We use the following matrix norms for a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$,

$$\text{Spectral Norm: } \|\mathbf{A}\| = \max_{\mathbf{x} \in \mathbb{S}^{m-1}} \|\mathbf{A}\mathbf{x}\| = \sigma_1(\mathbf{A})$$

$$\text{Trace Norm: } \mathrm{Tr}(\mathbf{A}) = \sum_{i \in [m \wedge n]} \sigma_i(\mathbf{A})$$

$$\text{Frobenius Norm: } \|\mathbf{A}\|_{\mathrm{F}}^2 = \mathrm{Tr}(\mathbf{A}^\top \mathbf{A}) = \sum_{i \in [m \wedge n]} \sigma_i^2(\mathbf{A})$$

Let $\mathrm{vec} : \mathbb{R}^{n \times k} \to \mathbb{R}^{nk}$ represent the vectorizaton of a matrix to a vector placing its columns one by one into a vector. We then have the useful facts,

Lemma 5. *Suppose* $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$, *then*

$$\langle \mathbf{A}, \mathbf{B} \rangle_{\mathrm{Tr}} = \langle \mathrm{vec}(\mathbf{A}), \mathrm{vec}(\mathbf{B}) \rangle$$

Let $\otimes : \mathbb{R}^{N \times K} \times \mathbb{R}^{L \times M} \to \mathbb{R}^{NL \times KM}$ represent the Kronecker delta product between two matrices, this gives us the following relation,

Lemma 6. *Suppose* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *are size compatible matrices, then*

$$\text{vec}(\mathbf{ABC}) = (\mathbf{C}^\top \otimes \mathbf{A})\text{vec}(\mathbf{B})$$

Distribution. We say a vector $\mathbf{x}$ is sampled from a sub-Gaussian distribution with second-moment matrix $\mathbf{\Sigma}$ and sub-Gaussian norm K, if $\mathbf{E}[\mathbf{x}\mathbf{x}^\top] = \mathbf{\Sigma}$ and

$$\mathbf{E}[\exp(t\mathbf{x}^\top \mathbf{v})] \leq \exp\left(\frac{t^2 K^2 \mathbf{v}^\top \mathbf{\Sigma} \mathbf{v}}{2}\right) \text{ for } \mathbf{v} \in \mathbb{S}^{d-1}, \ t \in \mathbb{R}$$

2.1 Related Work

The idea of iterative thresholding algorithms for robust learning tasks dates back to 1806 by Legendre [Leg06]. Iterative thresholding have been studied theoretically and tested empirically in various machine learning domains [HYW+23, MGJK19].

[BJK15] study iterative thresholding for least squares regression / sparse recovery. In particular, one of their contributions is a gradient descent algorithm, TORRENT, when the covariates are sampled from a sub-Gaussian distribution. Their approximation bound in Theorem 5 relies on the fact that $\lambda_{\min}(\mathbf{\Sigma}) = \lambda_{\max}(\mathbf{\Sigma})$ and with sufficiently large data and sufficiently small ϵ, $\kappa(\mathbf{X}) \searrow 1$. This is similar to the study by [ADKS22], where the iterative trimmed maximum likelihood estimator is studied for General Linear Models. The algorithm studied by [ADKS22] utilizes a filtering algorithm with the sketching matrix $\mathbf{\Sigma}^{-1/2}$ so the columns of $\mathbf{X}$ are sampled from a sub-Gaussian Distribution with multivariate proxy $\mathbf{I}$ before running the iterative thresholding procedure.

3 Subquantile Minimization

We propose to optimize over the subquantile of the risk. The p-quantile of a random variable, U, is given as $\mathcal{Q}_p(U)$, this is the largest number, t, such that the probability of $U \leq t$ is at least p.

$$\mathcal{Q}_p(U) \leq t \iff \mathbf{Pr}\{U \leq t\} \geq p$$

The p-subquantile of the risk is then given by

$$L_p(U) = \frac{1}{p} \int_0^p \mathcal{Q}_p(U) \, dq = \mathbf{E}[U | U \leq \mathcal{Q}_p(U)] = \max_{t \in \mathbb{R}} \left\{ t - \frac{1}{p} \mathbf{E}\left[(t - U)^+\right] \right\}$$

Given a function to minimize, $\mathcal{L}$, the variational problem is given as:

$$\min_{f \in \mathcal{K}} \max_{t \in \mathbb{R}} \left\{ g(t, \mathbf{w}) \triangleq t - \frac{1}{(1-\epsilon)N} \cdot \sum_{i=1}^n (t - \mathcal{L}(\mathbf{w}; \mathbf{x}_i, y_i))^+ \right\}$$

where t is the p-quantile of the empirical risk. Note that for a fixed t therefore the objective is not concave with respect to $\mathbf{w}$. Thus, to solve this problem we use the iterations from Equation 11 in [RHL+20]. Our update steps are,

$$t^{(k+1)} = \arg\max_{t \in \mathbb{R}} g(\mathbf{w}^{(k)}, t) \tag{3.1}$$

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta \nabla_f g(\mathbf{w}^{(k)}, t^{(k+1)}) \tag{3.2}$$

3.1 Reduction to Iterative Thresholding

To consider theoretical guarantees of Subquantile Minimization, we first analyze the inner and outer optimization problems. We first analyze kernel learning in the presence of corrupted data. Next, we provide error bounds for the two most important kernel learning problems, kernel ridge regression, and kernel classification. Now we will give our first result regarding kernel learning in the Huber ϵ-contamination model. Now we will analyze the two-step minimax optimization steps described in Equations (3.1) and (3.2).

Lemma 7. *Let $\mathcal{R} : \mathcal{H} \times \mathbb{R} \to \mathbb{R}$ be a loss function (not necesarily convex). Let $\mathbf{x}_{\lfloor i \rfloor}$ represent the point with the i-th smallest loss w.r.t $\mathcal{R}$. If we denote $\hat{\nu}_i \triangleq \mathcal{R}(\mathbf{w}; \mathbf{x}_{\lfloor i \rfloor}, y_{\lfloor i \rfloor})$, it then follows $\hat{\nu}_{(1-\epsilon)N} \in \arg\max_{t \in \mathbb{R}} g(\mathbf{w}, t)$.*

Proof. First we can note, the max value of t for g is equivalent to the min value of t for the convex w.r.t t function $-g$. We can now find the Fermat Optimality Conditions for g.

$$\partial(-g(t,f)) = \partial\left(-t + \frac{1}{(1-\epsilon)N}\sum_{i=1}^{N}(t - \hat{\nu}_i)^+\right) = -1 + \frac{1}{(1-\epsilon)N}\sum_{i=1}^{(1-\epsilon)N}\begin{cases}1 & \text{if } t > \hat{\nu}_i \\ 0 & \text{if } t < \hat{\nu}_i \\ [0,1] & \text{if } t = \hat{\nu}_i\end{cases}$$

We observe when setting $t = \hat{\nu}_{(1-\epsilon)N}$, it follows that $0 \in \partial(-g(t,f))$. This is equivalent to the $(1-\epsilon)$-quantile of the Empirical Risk. $\blacksquare$

From Lemma 7, we see that t will be greater than or equal to the errors of exactly $(1-\epsilon)N$ points. Thus, we are continuously updating over the $(1-\epsilon)N$ minimum errors.

Lemma 8. *Let $\hat{\nu}_i \triangleq \mathcal{R}(\mathbf{w}; \mathbf{x}_{\lfloor i \rfloor}, y_{\lfloor i \rfloor})$, if we choose $t^{(k+1)} = \hat{\nu}_{(1-\epsilon)N}$ as by Lemma 7, it then follows $\nabla_f g(t^{(k)}, \mathbf{w}^{(k)}) = \frac{1}{(1-\epsilon)N}\sum_{i=1}^{(1-\epsilon)N}\nabla_{\mathbf{w}}\mathcal{R}(\mathbf{w}^{(k)}; \mathbf{x}_{\lfloor i \rfloor}, y_{\lfloor i \rfloor})$.*

Proof. By our choice of $t^{(k+1)}$, it follows,

$$\partial_f g(t^{(k+1)}, \mathbf{w}^{(k)}) = \partial_f\left(t^{(k+1)} - \frac{1}{(1-\epsilon)N}\sum_{i=1}^{N}\left(t^{(k+1)} - \mathcal{R}(\mathbf{w}^{(k)}; \mathbf{x}_{\lfloor i \rfloor}, y_{\lfloor i \rfloor})\right)^+\right)$$

$$= -\frac{1}{(1-\epsilon)N}\sum_{i=1}^{(1-\epsilon)N}\partial_f\left(t^{(k+1)} - \mathcal{R}(\mathbf{w}^{(k)}; \mathbf{x}_{\lfloor i \rfloor}, y_{\lfloor i \rfloor})\right)^+$$

$$= \frac{1}{(1-\epsilon)N}\sum_{i=1}^{n}\nabla_f\mathcal{R}(\mathbf{w}^{(k)}; \mathbf{x}_{\lfloor i \rfloor}, y_{\lfloor i \rfloor})\begin{cases}1 & \text{if } t > \hat{\nu}_i \\ 0 & \text{if } t < \hat{\nu}_i \\ [0,1] & \text{if } t = \hat{\nu}_i\end{cases}$$

Now we note $\hat{\nu}_{(1-\epsilon)N} \leq t^{(k+1)} \leq \hat{\nu}_{(1-\epsilon)N+1}$. Then, we have

$$\partial_f g(t^{(k+1)}, \mathbf{w}^{(k)}) \ni \frac{1}{(1-\epsilon)N}\sum_{i=1}^{(1-\epsilon)N}\nabla_f\mathcal{R}(\mathbf{w}^{(k)}; \mathbf{x}_{\lfloor i \rfloor}, y_{\lfloor i \rfloor})$$

This concludes the proof. $\blacksquare$

Theorem 9. *Suppose there exists constants such that the Hessian of any $(1-\epsilon)$-subset of the data w.r.t to $\mathcal{L}$ is μ-strongly convex and L-smooth. Then there exists an algorithm that returns a stationary point of $g(t, \mathbf{w})$ in $O(\mathsf{poly}(\mu, L))$.*

Proof. The proof is by construction, consider the algorithm described in Equations (3.1) and (3.2). From the proof of Lemma 7, $t = \hat{\nu}_{(1-\epsilon)N}$ is a maximizer of Equation (3.2). We can then see that $g(t^{(k+1)}, \mathbf{w}^{(k+1)} \leq g(t^{(k)}, \mathbf{w}^{(k)})$. $\blacksquare$

4 Convergence

In this section we give the algorithm for subquantile minimization. We will start with the simple case of vectorized regression as a warm-up to our general proof technique. We then move to the GLM with kernel learning. Finally, we give our results for one-hidden layer neural networks. We will now give the algorithm for Subquantile Minimization with Gradient Descent. We first give the non-linear functions we will be learning.

Property 10. *The non-linear function, $\psi : \mathbb{R} \to \mathbb{R}$ has lower bounded derivative, $\psi'(x) \geq C_{\lfloor \psi \rfloor}$ for all $x \in \mathbb{R}$.*

Property 11. *The non-linear function, $\psi : \mathbb{R} \to \mathbb{R}$ is $C_{\lceil \psi \rceil}$-Lipschitz.*

Property 12. *The second-derivative, $\psi'' : \mathbb{R} \to \mathbb{R}$ is non-zero at finitely many points.*

4.1 Algorithm

We will first define the thresholding operator to simplify the notation in our formal algorithm.

Definition 13 (Hard Thresholding Operator in [BJK15]). *For any vector $\mathbf{v} \in \mathbb{R}^n$, let $\sigma_{\mathbf{v}}$ be the permutation that orders elements in ascending order, i.e. $\mathbf{v}_{\sigma_{\mathbf{v}}(1)} \leq \mathbf{v}_{\sigma_{\mathbf{v}}(2)} \leq \cdots \leq \mathbf{v}_{\sigma_{\mathbf{v}}(n)}$. Then for any $k \leq n$, the hard thresholding oeprator is defined as,*

$$\mathrm{HT}(\mathbf{v}; k) = \left\{ i \in [n] : \sigma_{\mathbf{v}}^{-1}(i) \leq k \right\}$$

We now give the gradient descent algorithm which we will study for the remainder of this paper.

Algorithm 1 Subquantile Minimization for Learning a Neuron

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input: Possibly corrupted $\mathbf{X} \in \mathbb{R}^{d \times N}$ with outputs $\mathbf{y} \in \mathbb{R}^N$, activation function σ, and corruption parameter $\epsilon = O(1)$.
output: $\varepsilon-$Approximate solution $\mathbf{w} \in \mathbb{R}^d$ to minimize $\|\mathbf{w} - \mathbf{w}^*\|_2$.
  1: $\mathbf{w}^{(0)} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$
  2: $\eta \leftarrow O(\kappa^{-2}(\boldsymbol{\Sigma}))$
  3: $T \leftarrow O\left(\kappa^2(\boldsymbol{\Sigma}) \log\left(\frac{\|\mathbf{w}^*\|_2}{\varepsilon} \right) \right)$
  4: **for** $t \in [T]$ **do**
  5:      $\nu_i^{(t)} = (\sigma(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - y_i)^2$
  6:      $\mathrm{S}^{(t)} \leftarrow \mathrm{HT}(\boldsymbol{\nu}^{(t)}, (1 - \epsilon)N)$
  7:      $\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)})$
return: $\mathbf{w}^{(T)}$

4.2 Proof Sketch

In this section we will give a general sketch of our proofs, from which all the individual theorems will be based upon. Let $t \in [T]$, we then have,

$$\begin{aligned} \|\mathbf{w}^{(t+1)} - \mathbf{w}^*\| &= \|\mathrm{Proj}_\Theta[\mathbf{w}^{(t)} - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)})] - \mathbf{w}^*\|_2 \\ &\overset{(i)}{\leq} \|\mathbf{w}^{(t)} - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)}) - \mathbf{w}^*\|_2 \\ &\leq \|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)} \cap \mathrm{P})\| + \|\nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)} \cap \mathrm{Q})\|_2 \end{aligned}$$

In the above, (i) follows from the assumption that $\mathbf{w}^* \in \Theta$, therefore the projection operator is on to a convex set and is thus must be closer or at the same ℓ_2 norm distance to $\mathbf{w}^*$. We analyze the first term through its square,

$$\|\mathbf{w}^{(t)} - \mathbf{w}^* - \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)} \cap \mathrm{P})\|^2 = \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 - 2\eta \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)} \cap \mathrm{P}) \rangle + \eta^2 \cdot \|\nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)} \cap \mathrm{P})\|^2$$

In this step the finer details of the particular proof will differ, however the structure remains the same. We will prove there exists some $c_1 > 0$ and $c_2 \geq c_1$ such that,

$$\langle \mathbf{w}^{(t)} - \mathbf{w}^*, \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)} \cap \mathrm{P}) \rangle \geq (1 - 2\epsilon)c_1 \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2$$

and

$$\|\nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)} \cap \mathrm{P})\| \leq (1 - \epsilon)c_2 \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2$$

Then, solving a simple quadratic equation, we have that $\eta^2 \beta \leq \eta c_1 c_3$ for a $c_3 \in (0, 1)$ when we choose $\eta \leq \frac{c_1 c_3}{c_2}$ and we are able to eliminate the norm of the gradient squared term. We must now control the corrupted gradient term. The key idea is to note that from the optimality of the sub-quantile set,

$$\sum_{i \in \mathrm{S}^{(t)} \cap \mathrm{Q}} \mathcal{L}(\mathbf{w}^{(t)}; \mathbf{x}_i, y_i) \leq \sum_{i \in \mathrm{P} \backslash \mathrm{S}^{(t)}} \mathcal{L}(\mathbf{w}^{(t)}; \mathbf{x}_i, y_i)$$

We then prove the existence of a constant c_4 such that,

$$\|\nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)} \cap \mathrm{Q})\|_2 \leq \epsilon c_4 \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2$$

Then, combining our results, we end up with a linear convergence of the form,

$$\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 \leq \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2 (1 - \eta(1 - 2\epsilon)c_1 + \eta \epsilon c_4)$$

4.3 Warm-up: Multivariate Linear Regression

We will first present our results for the well-studied problem of linear regression in the Huber-ϵ contamination model. Our results will extend the results in [BJKK17] Theorem 5 and [ADKS22] Lemma A.1 by including covariate corruption without requiring a filtering algorithm, variance in the optimal estimator, and non-identity second-moment matrix of the uncorrupted data. The loss function for the multivariate linear regression problem for $\mathbf{W} \in \mathbb{R}^{k \times d}$, $\mathbf{X} \in \mathbb{R}^{d \times n}$, and $\mathbf{Y} \in \mathbb{R}^{k \times n}$.

$$\mathcal{L}(\mathbf{W}; \mathbf{X}, \mathbf{Y}) = \|\mathbf{WX} - \mathbf{Y}\|_{\mathrm{F}}^2$$

Theorem 14. *Let $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^\top \in \mathbb{R}^{d \times N}$ be the data matrix and $\mathbf{Y} = [\mathbf{y}_1, \ldots, \mathbf{y}_n] \in \mathbb{R}^{K \times N}$ be the output, such that for $i \in \mathrm{P}$, $\mathbf{x}_i$ are sampled from a sub-Gaussian distribution with sub-Gaussian norm L and second-moment matrix $\boldsymbol{\Sigma}$. Suppose for $i \in \mathrm{P}$ the output is given as $\mathbf{y}_i = \mathbf{W}\mathbf{x}_i + \mathbf{e}_i$ where for $j \in [K]$, $[\mathbf{e}_i]_j \sim \mathcal{N}(0, \sigma^2)$. Then after $O\left(\kappa(\boldsymbol{\Sigma}) \log\left(\frac{\|\mathbf{W}^*\|_{\mathrm{F}}}{\varepsilon}\right)\right)$ gradient descent iterations, $N \geq \frac{(1/\delta) K \operatorname{Tr}(\boldsymbol{\Sigma})}{1600\epsilon^2 \log \epsilon^{-1} \lambda_{\max}(\boldsymbol{\Sigma})}$, and learning rate $\eta = 0.1 \lambda_{\max}^{-2}(\boldsymbol{\Sigma})$, with probability exceeding $1 - \delta$, Algorithm 1 returns $\mathbf{W}^{(T)}$ such that*

$$\|\mathbf{W}^{(T)} - \mathbf{W}^*\|_{\mathrm{F}} \leq \varepsilon + O\left(\sigma \epsilon \sqrt{K C_3 \log \epsilon^{-1}}\right)$$

Proof. The proof is deferred to § A.1. ∎

We are able to recover the result of Lemma 4.2 in [ADKS22] when $K = 1$ and the covariates (corrupted and un-corrupted) are sampled from a sub-Gaussian distribution with second-moment matrix $\mathbf{I}$. The full solve algorithm studied in [ADKS22] returns a $O(\sigma \epsilon \log \epsilon^{-1})$ in time $O\left(\frac{1}{\epsilon^2}(Nd^2 + d^3)\right)$ and the same algorithm studied in [BJK15], TORRENT-FC otains $O(\sigma)$ approximation error in run-time $O\left(\log\left(\frac{1}{\sqrt{n}} \frac{\|\mathbf{b}\|}{\sigma \epsilon \log \epsilon^{-1}}\right)(Nd^2 + d^3)\right)$, with the gradient descent based approach, we are able to improve the runtime to $O\left(\log\left(\frac{\|\mathbf{w}^*\|}{\sigma \epsilon \log \epsilon^{-1}}\right) Nd^2\right)$ for the same approximation bound. By no longer requiring the full-solve, we are able to remove super-linear relation to d. In comparison to [BJK15], we do not have dependence on the noise vector $\mathbf{b}$, which can have very large norm in relation to the norm of $\mathbf{w}^*$. Our proof is also a significant improvement over the presentation given in Lemma 5 of [SS19] as under the same conditions, we give more than the linear convergence, but we show linear convergence is possible on any second-moment matrix of the good covariates and covariate corruption, and then develop concentration inequality bounds to match the best known result for iteratived trimmed estimators. We will formalize our results into a corollary to give a more representative comparison in the literature.

Corollary 15. *Let $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^\top \in \mathbb{R}^{d \times N}$ be the data matrix and $\mathbf{y} = [y_1, \ldots, y_n] \in \mathbb{R}^N$ be the output, such that for $i \in \mathrm{P}$, $\mathbf{x}_i$ are sampled from a sub-Gaussian distribution with sub-Gaussian norm L and second-moment matrix $\mathbf{I}$. Suppose for $i \in \mathrm{P}$ the output is given as $\mathbf{y}_i = \mathbf{x}_i^\top \mathbf{w}^* + \xi_i$ where $\xi_i \sim \mathcal{N}(0, \sigma^2)$. Then after $O\left(\log\left(\frac{\|\mathbf{w}^*\|_2}{\varepsilon}\right)\right)$ gradient descent iterations, $N \geq \frac{(d/\delta)}{800\epsilon^2}$, and learning rate $\eta = 0.1$, with probability exceeding $1 - \delta$, Algorithm 1 returns $\mathbf{w}^{(T)}$ such that*

$$\|\mathbf{w}^{(T)} - \mathbf{w}^*\|_2 \leq \varepsilon + O\left(\sigma\epsilon\sqrt{KC_3 \log \epsilon^{-1}}\right)$$

Suppose for $i \in Q$, $\mathbf{x}_i$ are sampled from a sub-Gaussian distribution with sub-Gaussian Norm K and second-moment matrix $\mathbf{I}$. Then,

$$\|\mathbf{w}^{(T)} - \mathbf{w}^*\|_2 \leq \varepsilon + O\left(\sigma\epsilon \log \epsilon^{-1}\right)$$

The second relation given in Corollary 15 matches the best known bound for robust linear regression with iterative thresholding. The first relation given in Corollary 15 is the extension to handle second-moment matrices which do not have unitary condition number as well as corrupted covariates.

4.4 Learning Generalized Linear Models

We next study the problem of learning GLMs following the model given in § 5 of [SS19]. The error function for the the Kernelized GLM problem is given by the following equation for a single training pair $(\mathbf{x}_i, y_i) \sim \mathcal{D}$ in the kernelized case.

$$\mathcal{L}(f; \mathbf{x}_i, y_i) = (\omega(f(\mathbf{x}_i)) - y_i)^2$$

Theorem 16. *Let $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^\top \in \mathbb{R}^{d \times N}$ be the data matrix and $\mathbf{y} = [y_1, \ldots, y_n]$ be the output, such that for $i \in \mathrm{P}$, $\mathbf{x}_i$ are sampled from a sub-Gaussian distribution s.t. $\phi(\mathbf{x}_i) = X_i$ has sub-Gaussian norm L and second-moment matrix $\mathbf{\Sigma}$, and $y_i = \omega(\mathbf{x}_i^\top \mathbf{w}^* + \xi_i)$ for ξ_i sampled from a centered sub-Gaussian distribution with sub-Gaussian norm ν. Suppose the link function $\omega : \mathbb{R} \to \mathbb{R}$, has bounded gradient s.t. $C_1 \leq \omega'(x) \leq C_2$ for absolute constants $C_1, C_2 > 0$ for all $x \in \mathbb{R}$. Then after $O\left(\kappa(\mathbf{\Sigma}) \log\left(\frac{\|f^*\|_{\mathcal{H}}}{\varepsilon}\right)\right)$ gradient descent iterations, then with probability exceeding $1 - \delta$,*

$$\|f^{(T)} - f^*\|_{\mathcal{H}} \leq \varepsilon + O\left(C_1^{-1} C_2^2 \cdot \epsilon \sqrt{\lambda_{\max}(\mathbf{\Sigma}) C_3 \log \epsilon^{-1}}\right) + O\left(\lambda_{\max}(\mathbf{\Sigma}) \lambda_{\min}^{-2}(\mathbf{\Sigma}) \cdot \sigma \cdot (C_1^{-2} C_2^2)\right)$$

when $N \geq \frac{1}{\lambda_{\min}^2(\mathbf{\Sigma})} \cdot \left(8C_K \cdot d + \frac{2}{c_K} \cdot \log(2/\delta)\right)$.

Proof. The proof is deferred to § B.1.1. ∎

Theorem 17. *Let $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^\top \in \mathbb{R}^{d \times N}$ be the data matrix and $\mathbf{y} = [y_1, \ldots, y_n]$ be the output, such that for $i \in \mathrm{P}$, $\mathbf{x}_i$ are sampled from a sub-Gaussian distribution s.t. $\phi(\mathbf{x}_i) = X_i$ has sub-Gaussian norm L and second-moment matrix $\mathbf{\Sigma}$, and $y_i = \omega(f^*(\mathbf{x}_i)) + \xi_i$ for ξ_i sampled from a centered sub-Gaussian distribution with sub-Gaussian norm ν. Suppose the link function $\omega : \mathbb{R} \to \mathbb{R}$, has bounded gradient s.t. $C_1 \leq \omega'(x) \leq C_2$ for absolute constants $C_1, C_2 > 0$ for all $x \in \mathbb{R}$. Then after $O\left(\kappa(\mathbf{\Sigma}) \log\left(\frac{\|f^*\|_{\mathcal{H}}}{\varepsilon}\right)\right)$ gradient descent iterations, then with probability exceeding $1 - \delta$,*

$$\|f^{(T)} - f^*\|_{\mathcal{H}} \leq \varepsilon + O\left(C_1^{-1} C_2^2 \cdot \epsilon \sqrt{\lambda_{\max}(\mathbf{\Sigma}) C_3 \log \epsilon^{-1}}\right) + O\left(\lambda_{\max}(\mathbf{\Sigma}) \lambda_{\min}^{-2}(\mathbf{\Sigma}) \cdot \sigma \cdot (C_1^{-2} C_2^2)\right)$$

when $N \geq \frac{1}{\lambda_{\min}^2(\mathbf{\Sigma})} \cdot \left(8C_K \cdot d + \frac{2}{c_K} \cdot \log(2/\delta)\right)$.

Proof. The proof is deferred to § B.1.1. ∎
When considerin the linear regression case

4.5 Learning Leaky-ReLU Neural Networks

We will now consider functions with Properties 10, 11, and 12. Functions that follow these properties include the Leaky-ReLU. In our proof we are able to leverage the fact that the second derivative is zero almost surely.

Theorem 18. *Let $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^\top \in \mathbb{R}^{N \times d}$ be the data matrix and $\mathbf{y} = [y_1 \ldots, y_n]^\top$ be the output, such that for $i \in \mathrm{P}$, $\mathbf{x}_i \sim \mathcal{P}$ are sampled from a sub-Gaussian distribution with sub-Gaussian norm K and second-moment matrix $\mathbf{\Sigma}$, and $y_i = \psi(\mathbf{x}^\top \mathbf{w}^*) + \xi_i$ for $\xi_i \sim \mathcal{N}(0, \sigma^2)$ where $\psi(x) = \max\{C_\psi x, x\}$. Then after $O\left(C_\psi^{-2} \kappa(\mathbf{\Sigma}) \log\left(\frac{\|\mathbf{W}^*\|_{\mathrm{F}}}{\varepsilon}\right)\right)$ gradient descent iterations and $\epsilon \leq \frac{C_\psi^2 \lambda_{\min}(\mathbf{\Sigma})}{\sqrt{32 C_Q \lambda_{\max}(\mathbf{\Sigma})}}$, with probability exceeding $1 - \delta$, Algorithm 1 with learning rate $\eta = O(\kappa^{-2}(\mathbf{\Sigma}))$ returns $\mathbf{w}^{(T)}$ such that*

$$\|\mathbf{w}^{(T)} - \mathbf{w}^*\|_2 \leq \varepsilon + O\left(\kappa^2(\mathbf{\Sigma}) C_\psi^{-1} C_K C_\sigma d\epsilon \log \epsilon^{-1}\right)$$

Proof. The proof is deferred to § B.2.1. ■

4.6 Learning ReLU Neural Networks

We will now consider the problem of learning ReLU neural networks. We first give a preliminary result for randomized initialization.

Lemma 19 (Theorem 3.4 in [DLT+18]). *Suppose $\mathbf{w}^{(0)}$ is sampled uniformly from a p-dimensional ball with radius $\alpha \|\mathbf{w}^*\|$ such that $\alpha \leq \sqrt{\frac{1}{2\pi p}}$, then with probability at least $\frac{1}{2} - \alpha \sqrt{\frac{\pi p}{2}}$*

$$\|\mathbf{w}^{(0)} - \mathbf{w}^*\|_2 \leq \sqrt{1 - \alpha^2} \|\mathbf{w}^*\|_2$$

From this result we are able to derive porbabilistic guarantees on the convergence of learning ReLU neuron.

Theorem 20. *Let $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^\top \in \mathbb{R}^{n \times d}$ be the data matrix and $\mathbf{y} = [y_1, \ldots, y_n]^\top$ be the output, such that for $i \in \mathrm{P}$, $\mathbf{x}_i$ are sampled from a sub-Gaussian distribution with sub-Gaussian norm K and second-moment matrix $\mathbf{\Sigma}$. Suppose for $i \in \mathrm{P}$, the output is given as $y_i = \psi(\mathbf{x}^\top \mathbf{w}^*) + \xi_i$ for $\xi_i \sim \mathcal{N}(0, \sigma^2)$ and $\psi = \max\{0, x\}$ is the ReLU function. Then after $O(\Xi))$ gradient descent iterations and $n = \Omega(\Xi)$, then with probability exceeding $1 - \delta$, Algorithm 1 with learning rate $\eta = O(\Xi)$ returns $\mathbf{w}^{(T)}$ such that $\|\mathbf{w}^{(T)} - \mathbf{w}^*\|_2 \leq \varepsilon + O(\Xi)$.*

Proof. The proof is deferred to § B.3.1 ■

5 Discussion

In this paper, we study the theoretical convergence properties of iterative thresholding for non-linear learning problems in the Strong ϵ-contamination model. Our warm-up result for linear regression reduces the runtime while achieving the best known approximation for iterative thresholding algorithms. Many papers have experimentally studied the iterative thresholding estimator in large scale neural networks [SS19, HYW+23] and to our knowledge, we are the first paper to make advancements in the theory of iterative thresholding for a general class of activation functions.

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A Proofs for Linear Regression

Notation. We will first give some notational preliminaries. Let $X = P \cup Q$ for $|P| = (1-\epsilon)N$ and $|Q| = \epsilon N$ represent the sets such that for $i \in P$, $(\mathbf{x}_i, y_i)$ is the good data and for $j \in Q$, $(\mathbf{x}_j, y_j)$ has been given by the adversary. For $t \in [T]$, we denote $S^{(t)}$ as the Subquantile set at iteration t and represents the points. We decompose $S^{(t)} = S^{(t)} \cap P \cup S^{(t)} \cap Q = TP \cup FP$ to represent the *True Positives* and *False Positives*. We also decompose $X \setminus S^{(t)} = (X \setminus S^{(t)}) \cap P \cup (X \setminus S^{(t)}) \cap Q = FN \cup TN$ to represent the *False Negatives* and the *True Negatives*.

A.1 Proof of Theorem 14

Proof. Recall that for any $\mathbf{W} \in \mathbb{R}^{k \times d}$, $\mathbf{X} \in \mathbb{R}^{d \times n}$, and $\mathbf{Y} \in \mathbb{R}^{k \times n}$,

$$\mathcal{L}(\mathbf{W}; \mathbf{X}, \mathbf{Y}) = \left\| \mathbf{WX} - \mathbf{Y} \right\|_{\mathrm{F}}^2 = \mathrm{Tr}(\mathbf{X}^\top \mathbf{W}^\top \mathbf{WX} - \mathbf{X}^\top \mathbf{W}^\top \mathbf{Y} - \mathbf{Y}^\top \mathbf{WX} + \mathbf{Y}^\top \mathbf{Y})$$
$$= \mathrm{Tr}(\mathbf{X}^\top \mathbf{W}^\top \mathbf{WX}) + \mathrm{Tr}(\mathbf{Y}^\top \mathbf{Y}) - 2\,\mathrm{Tr}(\mathbf{X}^\top \mathbf{W}^\top \mathbf{Y})$$

Then, from [PP$^+$08] Equations (102) and (119) (where we set $\mathbf{B} = \mathbf{I}$ and $\mathbf{C} = \mathbf{0}$). We have,

$$\nabla \mathcal{L}(\mathbf{W}) = 2(\mathbf{WX} - \mathbf{Y})\mathbf{X}^\top$$

Step 1: Linear Convergence. We will first show iterative thresholding has linear convergence to optimal. Let $\widehat{\mathbf{W}} = \arg\min_{\mathbf{W} \in \mathbb{R}^{K \times d}} \mathcal{R}(\mathbf{W}; P)$ be the minimizer over the good data. Then, we have

$$\left\| \mathbf{W}^{(t+1)} - \mathbf{W}^* \right\|_{\mathrm{F}} = \left\| \mathbf{W}^{(t)} - \mathbf{W}^* - \eta \nabla \mathcal{R}(\mathbf{W}^{(t)}; S^{(t)}) \right\|_{\mathrm{F}}$$
$$= \left\| \mathbf{W}^{(t)} - \mathbf{W}^* - \eta \nabla \mathcal{R}(\mathbf{W}^{(t)}; \mathrm{TP}) + \eta \nabla \mathcal{R}(\mathbf{W}^*; P) - \eta \nabla \mathcal{R}(\mathbf{W}^*; P) + \eta \nabla \mathcal{R}(\widehat{\mathbf{W}}; P) - \eta \nabla \mathcal{R}(\mathbf{W}^{(t)}; \mathrm{FP}) \right\|_{\mathrm{F}}$$
$$\leq \left\| \mathbf{W}^{(t)} - \mathbf{W}^* + \eta \nabla \mathcal{R}(\mathbf{W}^{(t)}; \mathrm{TP}) + \eta \nabla \mathcal{R}(\mathbf{W}^*; \mathrm{TP}) \right\|_{\mathrm{F}} + \left\| \eta \nabla \mathcal{R}(\mathbf{W}^{(t)}; \mathrm{FP}) \right\|_{\mathrm{F}} + \left\| \eta \nabla \mathcal{R}(\mathbf{W}^*; \mathrm{FN}) \right\|_{\mathrm{F}}$$
$$+ \left\| \eta \nabla \mathcal{R}(\mathbf{W}^*; P) - \eta \nabla \mathcal{R}(\widehat{\mathbf{W}}; P) \right\|_{\mathrm{F}} \tag{A.1}$$

We first will upper bound the first term in Equation (A.1) through its square.

$$\left\| \mathbf{W}^{(t)} - \mathbf{W}^* - \eta \nabla \mathcal{R}(\mathbf{W}^{(t)}; \mathrm{TP}) + \eta \nabla \mathcal{R}(\mathbf{W}^*; \mathrm{TP}) \right\|_{\mathrm{F}}^2 = \left\| \mathbf{W}^{(t)} - \mathbf{W}^* \right\|_{\mathrm{F}}^2$$
$$- 2\eta \cdot \mathrm{Tr}\big((\mathbf{W}^{(t)} - \mathbf{W}^*)^\top (\nabla \mathcal{R}(\mathbf{W}^{(t)}; \mathrm{TP}) - \nabla \mathcal{R}(\mathbf{W}^*; \mathrm{TP})) \big) + \left\| \eta \nabla \mathcal{R}(\mathbf{W}^{(t)}; \mathrm{TP}) - \eta \nabla \mathcal{R}(\mathbf{W}^{(t)}; \mathrm{TP}) \right\|_{\mathrm{F}}^2 \tag{A.2}$$

We then lower bound the second term in Equation (A.2),

$$2\eta \cdot \mathrm{Tr}\big((\mathbf{W}^{(t)} - \mathbf{W}^*)^\top (\nabla \mathcal{R}(\mathbf{W}^{(t)}; \mathrm{TP}) - \nabla \mathcal{R}(\mathbf{W}^*; \mathrm{TP})) \big)$$
$$\overset{\mathrm{def}}{=} \frac{4\eta}{(1-\epsilon)N} \cdot \mathrm{Tr}\big((\mathbf{W}^{(t)} - \mathbf{W}^*)^\top ((\mathbf{W}^{(t)} \mathbf{X}_{\mathrm{TP}} - \mathbf{Y}_{\mathrm{TP}})\mathbf{X}_{\mathrm{TP}}^\top - \mathbf{E}_{\mathrm{TP}} \mathbf{X}_{\mathrm{TP}}^\top) \big)$$
$$\overset{(i)}{=} \frac{4\eta}{(1-\epsilon)N} \cdot \mathrm{Tr}\Big((\mathbf{W}^{(t)} - \mathbf{W}^*)^\top (\mathbf{W}^{(t)} - \mathbf{W}^*)\mathbf{X}_{\mathrm{TP}} \mathbf{X}_{\mathrm{TP}}^\top \Big) \tag{A.3}$$

In the above, (i) follows from noting that $\mathbf{Y}_{\mathrm{TP}} = \mathbf{W}^* \mathbf{X}_{\mathrm{TP}} - \mathbf{E}_{\mathrm{TP}}$. We can now lower bound the term in Equation (A.3).

$$\frac{4\eta}{(1-\epsilon)N} \cdot \mathrm{Tr}\Big((\mathbf{W}^{(t)} - \mathbf{W}^*)^\top (\mathbf{W}^{(t)} - \mathbf{W}^*)\mathbf{X}_{\mathrm{TP}} \mathbf{X}_{\mathrm{TP}}^\top \Big)$$
$$\overset{(ii)}{=} \frac{4\eta}{(1-\epsilon)N} \cdot \mathrm{Tr}\Big((\mathbf{W}^{(t)} - \mathbf{W}^*)\mathbf{X}_{\mathrm{TP}} \mathbf{X}_{\mathrm{TP}}^\top (\mathbf{W}^{(t)} - \mathbf{W}^*)^\top \Big)$$
$$\overset{(iii)}{=} \frac{4\eta}{(1-\epsilon)N} \cdot \langle \mathrm{vec}((\mathbf{W}^{(t)} - \mathbf{W}^*)^\top), \mathrm{vec}(\mathbf{X}_{\mathrm{TP}} \mathbf{X}_{\mathrm{TP}}^\top (\mathbf{W}^{(t)} - \mathbf{W}^*)^\top) \rangle$$
$$\overset{(iv)}{=} \frac{4\eta}{(1-\epsilon)N} \cdot \langle \mathrm{vec}((\mathbf{W}^{(t)} - \mathbf{W}^*)^\top), (\mathbf{I} \otimes \mathbf{X}_{\mathrm{TP}} \mathbf{X}_{\mathrm{TP}}^\top)\mathrm{vec}((\mathbf{W}^{(t)} - \mathbf{W}^*)^\top) \rangle$$

$$\overset{(v)}{=} \frac{4\eta}{(1-\epsilon)N} \cdot \sum_{k\in[K]} \left\langle \mathbf{w}_k^{(t)} - \mathbf{w}_k^*, \mathbf{X}_{\mathrm{TP}}\mathbf{X}_{\mathrm{TP}}^\top(\mathbf{w}_k^{(t)} - \mathbf{w}_k^*) \right\rangle$$

$$\overset{(vi)}{\geq} \frac{2\eta}{(1-\epsilon)N} \cdot \sum_{k\in[K]} \left(\lambda_{\min}(\mathbf{X}_{\mathrm{TP}}\mathbf{X}_{\mathrm{TP}}^\top) \big\| \mathbf{w}_k^{(t)} - \mathbf{w}_k^* \big\|_2^2 + \big\| \mathbf{X}_{\mathrm{TP}}\mathbf{X}_{\mathrm{TP}}^\top \big\|_2^{-1} \big\| \mathbf{X}_{\mathrm{TP}}\mathbf{X}_{\mathrm{TP}}^\top(\mathbf{w}_k^{(t)} - \mathbf{w}_k^*) \big\|_2^2 \right)$$

$$= \frac{2\eta}{(1-\epsilon)N} \cdot \lambda_{\min}(\mathbf{X}_{\mathrm{TP}}\mathbf{X}_{\mathrm{TP}}^\top) \big\| \mathbf{W}^{(t)} - \mathbf{W}^* \big\|_{\mathrm{F}}^2 + \frac{2\eta}{(1-\epsilon)N} \cdot \big\| \mathbf{X}_{\mathrm{TP}}\mathbf{X}_{\mathrm{TP}}^\top \big\|_2^{-1} \big\| (\mathbf{W}^{(t)} - \mathbf{W}^*)\mathbf{X}_{\mathrm{TP}}\mathbf{X}_{\mathrm{TP}}^\top \big\|_{\mathrm{F}}^2$$
$$\tag{A.4}$$

In the above, (ii) follows from the cyclic property of the trace, (iii) follows from the relation given in Lemma 5, (iv) holds from the relation given in Lemma 6, the inequality in (vi) follows from Lemma 30, and in (v) we apply Lemma 6, which gives the following equality,

$$(\mathbf{I} \otimes \mathbf{X}_{\mathrm{TP}}\mathbf{X}_{\mathrm{TP}}^\top)\mathrm{vec}((\mathbf{W}^{(t)} - \mathbf{W}^*)^\top) = \begin{bmatrix} \mathbf{X}_{\mathrm{TP}}\mathbf{X}_{\mathrm{TP}}^\top & & \\ & \ddots & \\ & & \mathbf{X}_{\mathrm{TP}}\mathbf{X}_{\mathrm{TP}}^\top \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 - \mathbf{w}_1^* \\ \vdots \\ \mathbf{w}_K - \mathbf{w}_K^* \end{bmatrix} = \begin{bmatrix} \mathbf{X}_{\mathrm{TP}}\mathbf{X}_{\mathrm{TP}}^\top(\mathbf{w}_1 - \mathbf{w}_1^*) \\ \vdots \\ \mathbf{X}_{\mathrm{TP}}\mathbf{X}_{\mathrm{TP}}^\top(\mathbf{w}_K - \mathbf{w}_K^*) \end{bmatrix}$$

We now upper bound the second term in Equation (A.1),

$$\big\| \eta\nabla\mathcal{R}(\mathbf{W}^{(t)}; \mathrm{FP}) \big\|_{\mathrm{F}} \overset{\mathrm{def}}{=} \frac{2\eta}{(1-\epsilon)N} \cdot \big\| (\mathbf{W}^{(t)}\mathbf{X}_{\mathrm{FP}} - \mathbf{Y}_{\mathrm{FP}})\mathbf{X}_{\mathrm{FP}}^\top \big\|_{\mathrm{F}}$$

$$\overset{(vii)}{\leq} \frac{2\eta}{(1-\epsilon)N} \cdot \big\| \mathbf{X}_{\mathrm{FP}} \big\|_2 \big\| \mathbf{W}^{(t)}\mathbf{X}_{\mathrm{FP}} - \mathbf{Y}_{\mathrm{FP}} \big\|_{\mathrm{F}}$$

$$\overset{(viii)}{\leq} \frac{2\eta}{(1-\epsilon)N} \cdot \big\| \mathbf{X}_{\mathrm{FP}} \big\|_2 \big\| \mathbf{W}^{(t)}\mathbf{X}_{\mathrm{FN}} - \mathbf{Y}_{\mathrm{FN}} \big\|_{\mathrm{F}}$$

$$\overset{(ix)}{\leq} \frac{2\eta}{(1-\epsilon)N} \cdot \big\| \mathbf{X}_{\mathrm{FP}} \big\|_2 \big(\big\| \mathbf{W}^{(t)}\mathbf{X}_{\mathrm{FN}} - \mathbf{W}^*\mathbf{X}_{\mathrm{FN}} \big\|_{\mathrm{F}} + \big\| \mathbf{E}_{\mathrm{FN}} \big\|_{\mathrm{F}} \big)$$

$$\overset{(x)}{\leq} \frac{2\eta}{(1-\epsilon)N} \cdot \big\| \mathbf{X}_{\mathrm{FP}} \big\|_2 \big\| \mathbf{X}_{\mathrm{FN}} \big\|_2 \big\| \mathbf{W}^{(t)} - \mathbf{W}^* \big\|_{\mathrm{F}} + \frac{2\eta}{(1-\epsilon)N} \cdot \big\| \mathbf{X}_{\mathrm{FP}} \big\|_2 \big\| \mathbf{E}_{\mathrm{FN}} \big\|_{\mathrm{F}}$$

In the above, the equalities in (vii) and (x) from the fact that for any two size compatible matrices, $\mathbf{A}, \mathbf{B}$, it holds that $\|\mathbf{AB}\|_{\mathrm{F}} \leq \|\mathbf{A}\|_{\mathrm{F}}\|\mathbf{B}\|_2$, $(viii)$ follows from the optimality of the Subquantile set, and (ix) follows from the sub-additivity of the Frobenius norm. We will now upper bound the third term in Equation (A.1),

$$\big\| \eta\nabla\mathcal{R}(\mathbf{W}^*; \mathrm{FN}) \big\|_{\mathrm{F}} \overset{\mathrm{def}}{=} \frac{2\eta}{(1-\epsilon)N} \cdot \big\| (\mathbf{W}^*\mathbf{X}_{\mathrm{FN}} - \mathbf{Y}_{\mathrm{FN}})\mathbf{X}_{\mathrm{FN}}^\top \big\|_{\mathrm{F}} \leq \frac{2\eta}{(1-\epsilon)N} \cdot \big\| \mathbf{E}_{\mathrm{FN}}\mathbf{X}_{\mathrm{FN}}^\top \big\|_{\mathrm{F}} \tag{A.5}$$

In the above, we use the fact that for any two size compatible matrices, $\mathbf{A}, \mathbf{B}$, it holds that $\|\mathbf{AB}\|_{\mathrm{F}} \leq \|\mathbf{A}\|_{\mathrm{F}}\|\mathbf{B}\|_2$.

Step 2: Consistency. We will now upper bound the consistency estimate of the fourth term of Equation (A.1).

$$\big\| \eta\nabla\mathcal{R}(\mathbf{W}^*; \mathrm{P}) - \eta\nabla\mathcal{R}(\widehat{\mathbf{W}}; \mathrm{P}) \big\|_{\mathrm{F}} \overset{\mathrm{def}}{=} \frac{2\eta}{(1-\epsilon)N} \cdot \big\| (\widehat{\mathbf{W}} - \mathbf{W}^*)\mathbf{X}_{\mathrm{P}}\mathbf{X}_{\mathrm{P}}^\top \big\|_{\mathrm{F}} \leq \frac{2\eta}{(1-\epsilon)N} \cdot \big\| \mathbf{E}_{\mathrm{P}}\mathbf{X}_{\mathrm{P}}^\top \big\|_{\mathrm{F}}$$

We can then have from Lemma 28,

$$\big\| \mathbf{E}_{\mathrm{P}}\mathbf{X}_{\mathrm{P}}^\top \big\|_{\mathrm{F}}^2 \leq \frac{16}{3} \cdot \sigma\big(\|\mathbf{X}_{\mathrm{P}}\|_{\mathrm{F}}^2 \log\big((2/\delta)N^2\big) \big) \leq \frac{32}{3} \cdot \sigma\big(N(1-\epsilon)d\lambda_{\max}(\mathbf{\Sigma}) \log\big((2/\delta)N^2\big) \big)$$

We then have with probability exceeding $1 - \delta$,

$$\frac{2\eta}{(1-\epsilon)N} \cdot \big\| \mathbf{E}_{\mathrm{P}}\mathbf{X}_{\mathrm{P}}^\top \big\|_{\mathrm{F}} \leq \eta \cdot \sqrt{\frac{\frac{32}{3}\sigma^2 d\lambda_{\max}(\mathbf{\Sigma}) \log((2/\delta)N^2)}{(1-\epsilon)N}}$$

We then have from our choice of $\eta = 0.1\lambda_{\max}^{-2}(\boldsymbol{\Sigma})$, the third term in Equation (A.1) will be less than the second term in Equation (A.4). We thus obtain from noting that $\sqrt{1-2x} \leq 1-x$ for any $x \leq 1/2$,

$$\left\|\mathbf{W}^{(t+1)} - \mathbf{W}^*\right\|_{\mathrm{F}} \leq \left\|\mathbf{W}^{(t)} - \mathbf{W}^*\right\|_{\mathrm{F}} \left(1 - \frac{2\eta}{(1-\epsilon)N} \cdot \lambda_{\min}(\mathbf{X}_{\mathrm{TP}}\mathbf{X}_{\mathrm{TP}}^\top) + \frac{2\eta}{(1-\epsilon)N} \cdot \|\mathbf{X}_{\mathrm{FP}}\|_2 \|\mathbf{X}_{\mathrm{FN}}\|_2\right)$$
$$+ \frac{2\eta}{(1-\epsilon)N} \cdot \|\mathbf{X}_{\mathrm{FP}}\|_2 \|\mathbf{E}_{\mathrm{FN}}\|_{\mathrm{F}} + \frac{2\eta}{(1-\epsilon)N} \cdot \|\mathbf{E}_{\mathrm{FN}}\mathbf{X}_{\mathrm{FN}}^\top\|_{\mathrm{F}} + 2\eta \cdot \sqrt{\frac{\sigma^2(1/\delta)K\,\mathrm{Tr}(\boldsymbol{\Sigma})}{(1-\epsilon)N}}$$

Step 3: Concentration Bounds. From Proposition 22, we obtain with probability exceeding $1-\delta$ that $\|\mathbf{E}_{\mathrm{FN}}\|_{\mathrm{F}} \leq \sigma\sqrt{30NK\epsilon \log \epsilon^{-1}}$. From our assumption on bounded covariate corruptions, we have $\|\mathbf{E}_{\mathrm{FN}}\|_{\mathrm{F}}\|\mathbf{X}_{\mathrm{FP}}\|_{\mathrm{F}} \leq \sigma\epsilon\sqrt{30NKC_3 \log \epsilon^{-1}}$. From Lemma 28, we have that $\|\mathbf{E}_{\mathrm{FN}}\mathbf{X}_{\mathrm{FN}}^\top\|_{\mathrm{F}} \leq \sqrt{6K \log N}\|\mathbf{X}_{\mathrm{FN}}\|$ when $n \geq (1/\delta)$ with probability exceeding $1-\delta$. From Lemma 24, we have for $\epsilon \leq \frac{1}{240}\kappa^{-1}(\boldsymbol{\Sigma})$, the minimimum eigenvalue satisfies $\lambda_{\min}(\mathbf{X}_{\mathrm{TP}}\mathbf{X}_{\mathrm{TP}}^\top) \geq (N/2) \cdot (1-2\epsilon) \cdot \lambda_{\min}(\boldsymbol{\Sigma}) \geq (N/4) \cdot \lambda_{\min}(\boldsymbol{\Sigma})$. From our assumption of corrupted covariates, we have that $\|\mathbf{X}_{\mathrm{FP}}\| \leq \sqrt{\epsilon N C_3}$. We also have from Lemma 24, we have $\|\mathbf{X}_{\mathrm{FN}}\| \leq \sqrt{\lambda_{\max}(\boldsymbol{\Sigma}) \cdot (10N\epsilon \log \epsilon^{-1})}$ with high probability. Then when $\epsilon \leq \sqrt{\frac{1}{960C_3} \cdot \kappa^{-1}(\boldsymbol{\Sigma})\lambda_{\min}(\boldsymbol{\Sigma})}$, we have that $\|\mathbf{X}_{\mathrm{FP}}\|_2 \|\mathbf{X}_{\mathrm{FN}}\|_2 \leq (N/8)\lambda_{\min}(\boldsymbol{\Sigma})$. Then, solving for the induction with an infinite sum, we have after $O\left(\kappa(\boldsymbol{\Sigma}) \cdot \log\left(\frac{\|\mathbf{W}^*\|_{\mathrm{F}}}{\varepsilon}\right)\right)$ iterations,

$$\left\|\mathbf{W}^{(t+1)} - \mathbf{W}^*\right\|_{\mathrm{F}} \leq \varepsilon + \sigma\epsilon\sqrt{4800KC_3 \log \epsilon^{-1}} \cdot \frac{\sqrt{\lambda_{\max}(\boldsymbol{\Sigma})}}{\lambda_{\min}(\boldsymbol{\Sigma})} + \frac{\sqrt{60\sigma K \log N \cdot \epsilon \log \epsilon^{-1}\lambda_{\max}(\boldsymbol{\Sigma})}}{\sqrt{N}\lambda_{\min}(\boldsymbol{\Sigma})}$$
$$+ \frac{\sigma}{\lambda_{\min}(\boldsymbol{\Sigma})} \cdot \sqrt{\frac{(8/\delta)K\,\mathrm{Tr}(\boldsymbol{\Sigma})}{N}} \leq \varepsilon + \sigma\epsilon\sqrt{43200KC_3 \log \epsilon^{-1}} \cdot \frac{\sqrt{\lambda_{\max}(\boldsymbol{\Sigma})}}{\lambda_{\min}(\boldsymbol{\Sigma})}$$

In the above, the final equality follows when $N \geq \frac{(1/\delta)K\,\mathrm{Tr}(\boldsymbol{\Sigma})}{1600\epsilon^2 \log \epsilon^{-1}\lambda_{\max}(\boldsymbol{\Sigma})} \vee \frac{1}{6400C_3^2\epsilon^2}$. Our proof is complete. $\blacksquare$

B Proofs for Learning Single Neurons

B.1 Properties 10 and 11 Functions

B.1.1 Proof of Theorem 16

Proof. From Algorithm 1, we have the gradient update for the generalized linear model.

$$\mathbf{w}^{(t+1)} = \mathbf{w}^{(t)} - \frac{2\eta}{(1-\epsilon)N} \cdot \sum_{i \in S^{(t)}} (\sigma(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - y_i) \cdot \sigma'(f^{(t)}(\mathbf{x}_i)) \cdot \mathbf{x}_i$$

Step 1: Linear Convergence.

$$\begin{aligned}
\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 &= \|\mathbf{w}^{(t)} - \eta\nabla\mathcal{R}(\mathbf{w}^{(t)}; S^{(t)}) - \mathbf{w}^*\|_2 \\
&= \|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta\nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP}) - \eta\nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{FP})\|_2 \\
&\leq \|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta\nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP})\|_2 + \|\eta\nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{FP})\|_2 \quad \text{(B.1)}
\end{aligned}$$

We upper bound the first term of Equation (B.1) through its square.

$$\|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta\nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP})\|_2^2 = \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 - 2\eta \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP})\rangle + \eta^2 \cdot \|\nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP})\|_2^2 \quad \text{(B.2)}$$

We now lower bound the second term of Equation (B.2).

$$2\eta \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \nabla\mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP})\rangle = \frac{4\eta}{(1-\epsilon)N} \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \sum_{i \in \mathrm{TP}} (\sigma(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - y_i) \cdot \sigma'(\mathbf{x}_i^\top \mathbf{w}^{(t)}) \cdot \mathbf{x}_i\rangle$$

$$= \frac{4\eta}{(1-\epsilon)N} \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \sum_{i \in \text{TP}} (\sigma(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - \sigma(\mathbf{x}_i^\top \mathbf{w}^* + \xi_i)) \cdot \sigma'(\mathbf{x}_i^\top \mathbf{w}^{(t)}) \cdot \mathbf{x}_i \rangle$$

$$\geq \frac{4\eta}{(1-\epsilon)N} \cdot C_{\lfloor \sigma \rfloor}^2 \lambda_{\min}(\mathbf{X}_{\text{TP}} \mathbf{X}_{\text{TP}}^\top) \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 - \frac{4\eta}{(1-\epsilon)N} \cdot \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2 \| \sum_{i \in \text{TP}} \xi_i \sigma'(c_i) \sigma'(\mathbf{x}_i^\top \mathbf{w}^{(t)}) \cdot \mathbf{x}_i \|_2$$

<div style="text-align:right">(B.3)</div>

In the above, in the final relation, we apply Cauchy-Schwarz inequality after using the mean-value theorem, which gives us the existance of some $c_i \in \mathbb{R}$ such that $\sigma'(c_i)(\mathbf{x}_i^\top \mathbf{w}^{(t)} - \mathbf{x}_i^\top \mathbf{w}^* - \xi_i) = \sigma(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - \sigma(\mathbf{x}_i^\top \mathbf{w}^* + \xi_i)$. Then from an application of Young's Inequality (see Proposition 31), we obtain

$$\frac{4\eta}{(1-\epsilon)N} \cdot \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2 \| \sum_{i \in \text{TP}} \xi_i \sigma'(c_i) \sigma'(\mathbf{x}_i^\top \mathbf{w}^{(t)}) \cdot \mathbf{x}_i \|_2$$

$$\leq \frac{\eta}{(1-\epsilon)N} \cdot C_{\lfloor \sigma \rfloor}^2 \lambda_{\min}(\mathbf{X}_{\text{TP}} \mathbf{X}_{\text{TP}}^\top) \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 + \frac{4\eta}{(1-\epsilon)N} \cdot \lambda_{\min}^{-1}(\mathbf{X}_{\text{TP}} \mathbf{X}_{\text{TP}}^\top) \| \sum_{i \in \text{TP}} \xi_i \sigma'(c_i) \sigma'(\mathbf{x}_i^\top \mathbf{w}^{(t)}) \cdot \mathbf{x}_i \|_2^2$$

We next upper bound the third term in Equation (B.2).

$$\eta^2 \cdot \|\nabla \mathcal{R}(\mathbf{w}^{(t)}; \text{TP})\|_2^2 = \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \| \sum_{i \in \text{TP}} (\sigma(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - \sigma(\mathbf{x}_i^\top \mathbf{w}^* + \xi_i)) \cdot \sigma'(\mathbf{x}_i^\top \mathbf{w}^{(t)}) \cdot \mathbf{x}_i \|_2^2$$

$$= \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \| \sum_{i \in \text{TP}} (\mathbf{x}_i^\top \mathbf{w}^{(t)} - \mathbf{x}_i^\top \mathbf{w}^* + \xi_i) \cdot \sigma'(c_i) \cdot \sigma'(\mathbf{x}_i^\top \mathbf{w}^{(t)}) \cdot \mathbf{x}_i \|_2^2$$

$$\leq \frac{8\eta^2}{[(1-\epsilon)N]^2} \cdot \left(C_{\lceil \sigma \rceil}^4 \lambda_{\max}^2(\mathbf{X}_{\text{TP}} \mathbf{X}_{\text{TP}}^\top) \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 + \| \sum_{i \in \text{TP}} \xi_i \cdot \sigma'(c_i) \cdot \sigma'(\mathbf{x}_i^\top \mathbf{w}^{(t)}) \cdot \mathbf{x}_i \|_2^2 \right) \quad \text{(B.4)}$$

Then, from choosing $\eta \leq \frac{C_{\lfloor \sigma \rfloor}^2 (1-\epsilon)N \lambda_{\min}(\mathbf{X}_{\text{TP}} \mathbf{X}_{\text{TP}}^\top)}{4 C_{\lceil \sigma \rceil}^4 \lambda_{\max}^2(\mathbf{X}_{\text{TP}} \mathbf{X}_{\text{TP}}^\top)}$. We see the first term of Equation (B.4) is less than half the first term in Equation (B.3). We now bound the second term in Equation (B.1).

$$\eta^2 \cdot \|\nabla \mathcal{R}(\mathbf{w}^{(t)}; \text{FP})\|_2^2 = \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \| \sum_{i \in \text{FP}} (\sigma(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - y_i) \cdot \sigma'(\mathbf{x}_i^\top \mathbf{w}) \cdot \mathbf{x}_i \|_2^2$$

$$\leq \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot C_{\lceil \sigma \rceil}^2 \|\mathbf{X}_{\text{FP}} \mathbf{X}_{\text{FP}}^\top\|_2 \sum_{i \in \text{FP}} (\sigma(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - y_i)^2$$

$$\leq \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot C_{\lceil \sigma \rceil}^2 \|\mathbf{X}_{\text{FP}} \mathbf{X}_{\text{FP}}^\top\|_2 \sum_{i \in \text{FN}} (\sigma(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - y_i)^2$$

$$= \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot C_{\lceil \sigma \rceil}^2 \|\mathbf{X}_{\text{FP}} \mathbf{X}_{\text{FP}}^\top\|_2 \sum_{i \in \text{FN}} (\sigma(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - \sigma(\mathbf{x}_i^\top \mathbf{w}^*) + \xi_i)^2$$

$$\leq \frac{8\eta^2}{[(1-\epsilon)N]^2} \cdot C_{\lceil \sigma \rceil}^2 \|\mathbf{X}_{\text{FP}} \mathbf{X}_{\text{FP}}^\top\|_2 \left(C_{\lceil \sigma \rceil}^2 \cdot \|\mathbf{X}_{\text{FN}} \mathbf{X}_{\text{FN}}^\top\|_2 \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 + \|\boldsymbol{\xi}_{\text{FN}}\|_2^2 \right)$$

Concluding the step, we have

$$\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 \leq \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2 \left(1 - \frac{\eta}{2(1-\epsilon)N} \cdot C_{\lfloor \sigma \rfloor}^2 \lambda_{\min}(\mathbf{X}_{\text{TP}} \mathbf{X}_{\text{TP}}^\top) + \frac{\sqrt{8}\eta}{(1-\epsilon)N} \cdot C_{\lceil \sigma \rceil}^2 \|\mathbf{X}_{\text{FP}}\|_2 \|\mathbf{X}_{\text{FN}}\|_2 \right)$$

$$+ \left(\frac{\sqrt{8}\eta}{(1-\epsilon)N} + \frac{2\sqrt{\eta}}{\sqrt{(1-\epsilon)N}} \cdot \sigma_{\min}^{-1}(\mathbf{X}_{\text{TP}}) \right) \| \sum_{i \in \text{TP}} \xi_i \sigma'(c_i) \sigma'(\mathbf{x}_i^\top \mathbf{w}^{(t)}) \mathbf{x}_i \|_2 + \frac{\sqrt{8}\eta}{(1-\epsilon)N} \cdot \|\mathbf{X}_{\text{FP}}\|_2 \|\boldsymbol{\xi}_{\text{FN}}\|_2$$

Step 2: Concentration Bounds. First we will show that $\xi \sigma'(c) \sigma'(\mathbf{x}^\top \mathbf{w})$ is sub-Gaussian for any $c \in \mathbb{R}$, $\mathbf{w} \in \mathbb{R}^d$ and $\mathbf{x}$ sampled from a sub-Gaussian distribution. Note that ξ is sub-Gaussian s.t. $\|\xi\|_{\psi_2} = C_\sigma$. Then, we have

$$\left(\mathbf{E} |\sigma'(c) \sigma'(\mathbf{x}^\top \mathbf{w})|^p \right)^{1/p} \leq \left(\sup_{x \in \mathbb{R}} [\sigma'(x)]^{2p} \right)^{1/p} \leq C_{\lceil \psi \rceil}^2$$

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We thus have $\|\sigma'(c)\sigma'(\mathbf{x}^\top \mathbf{w})\|_{\psi_2} \leq C_{\lceil \psi \rceil}^2 C_\sigma$. Then applying Lemma 25, we have with probability at least $1 - \delta$,

$$\|\sum_{i \in \mathrm{TP}} \xi_i \cdot \sigma'(c_i) \cdot \sigma'(\mathbf{x}_i^\top \mathbf{w}^{(t)}) \cdot \mathbf{x}_i\|_2 \leq N C_K C_{\lceil \psi \rceil}^2 C_\sigma \left(\frac{2d}{N} \log(5) + \frac{2}{N} \log(1/\delta) + 6\epsilon \log \epsilon^{-1} \right)^{1/2}$$

Recall that $\|\mathbf{X}_{\mathrm{FP}}\|_2 \leq \sqrt{N\epsilon C_Q}$. Then, with probability at least $1 - \delta$, we have $\|\mathbf{X}_{\mathrm{FN}}\|_2 \|\mathbf{X}_{\mathrm{FP}}\|_2 \leq N\epsilon \cdot \sqrt{10 C_3 \log \epsilon^{-1}}$. We also have $\|\boldsymbol{\xi}_{\mathrm{FN}}\|_2 \leq \sqrt{30 N \epsilon \log \epsilon^{-1}}$. We then find for $\epsilon \leq \frac{C_{\lfloor \psi \rfloor}^2}{C_{\lceil \psi \rceil}^2} \frac{\lambda_{\min}(\boldsymbol{\Sigma})}{32 \sqrt{80 \log(1/2)}}$ and probability exceeding $1 - \delta$,

$$C_{\lceil \psi \rceil}^2 \|\mathbf{X}_{\mathrm{FP}}\|_2 \|\mathbf{X}_{\mathrm{FN}}\|_2 \leq \frac{1}{\sqrt{144}} \cdot C_{\lfloor \psi \rfloor}^2 \lambda_{\min}(\boldsymbol{\Sigma})$$

Then combining the results with our choice of η, we obtain,

$$\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 \leq \varepsilon + \frac{1}{\lambda_{\min}(\boldsymbol{\Sigma})} \cdot C_{\lfloor \psi \rfloor}^{-2} \left(8 + 16\sqrt{8}\kappa^2(\boldsymbol{\Sigma}) \right) \left(C_K C_{\lceil \psi \rceil}^2 C_\sigma \left(\frac{2d}{N} \log(5) + \frac{2}{N} \log(1/\delta) + 6\epsilon \log \epsilon^{-1} \right)^{1/2} \right)$$

$$\leq \varepsilon + \frac{1}{\lambda_{\min}(\boldsymbol{\Sigma})} \cdot C_{\lfloor \psi \rfloor}^{-2} C_{\lceil \psi \rceil}^2 \sigma \left(24\sqrt{96}\kappa^2(\boldsymbol{\Sigma}) \right) \sqrt{\epsilon \log \epsilon^{-1}}$$

In the above, the final inequality holds when $N \geq \frac{2d \log 5 + \log(1/\delta)}{6\epsilon \log \epsilon^{-1}}$. Our proof is compllete. ∎

B.2   Properties 10, 11, and 12 Functions

B.2.1   Proof of Theorem 18

Proof. We will decompose the gradient into the good component and corrupted component. The first part of our proof will show that $\mathbf{w}$ moves in the direction of $\mathbf{w}^*$, then in the second part of the proof we will show the affect of the corrupted gradient. Finally, we combine step 1 and step 2 to show that there exists sufficiently small ϵ such that we can get linear convergence with a small additive error term.

Step 1: Upper bounding the ℓ_2 norm distance between $\mathbf{w}^{(t+1)}$ and $\mathbf{w}^*$. We have from Algorithm 1,

$$\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 = \|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{S}^{(t)})\|_2$$
$$= \|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP}) + \eta \nabla \mathcal{R}(\mathbf{w}^*; \mathrm{P}) - \eta \nabla \mathcal{R}(\mathbf{w}^*; \mathrm{P}) - \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{FP})\|_2$$
$$\leq \|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP}) + \eta \nabla \mathcal{R}(\mathbf{w}^*; \mathrm{TP})\|_2 + \|\eta \nabla \mathcal{R}(\mathbf{w}^*; \mathrm{TP})\|_2 + \left\| \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{FP}) \right\|_2 \quad \text{(B.5)}$$

We will first upper bound the first term of Equation (B.5) through its square.

$$\|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP})\|_2^2$$
$$= \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 - 2\eta \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP}) \rangle + \eta^2 \cdot \|\nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP})\|^2 \quad \text{(B.6)}$$

In the above, the a.s. relation follows from Property 12. We will first lower bound the second term of Equation (B.6). We first will quicly give spectral bounds on $\nabla^2 \mathcal{L}(\mathbf{w}; \mathrm{TP})$ where $\mathbf{w} \in \mathbb{R}^d$ is an arbitrary vector.

$$\nabla^2 \mathcal{L}(\mathbf{w}; \mathrm{TP}) = 2 \cdot \sum_{i \in \mathrm{TP}} \left(\psi(\mathbf{x}_i^\top \mathbf{w}) - y_i \right) \cdot \psi''(\mathbf{x}_i^\top \mathbf{w}) \cdot \mathbf{x}_i \mathbf{x}_i^\top + 2 \cdot \sum_{i \in \mathrm{TP}} \left[\psi'(\mathbf{x}_i^\top \mathbf{w}) \right]^2 \cdot \mathbf{x}_i \mathbf{x}_i^\top$$
$$\stackrel{\text{a.s.}}{=} 2 \cdot \sum_{i \in \mathrm{TP}} \left[\psi'(\mathbf{x}_i^\top \mathbf{w}) \right]^2 \cdot \mathbf{x}_i \mathbf{x}_i^\top$$

We then obtain for any $\mathbf{w} \in \mathbb{R}^d$,

$$2 \cdot C_{\lfloor \psi \rfloor}^2 \lambda_{\min}(\mathbf{X}_{\mathrm{TP}} \mathbf{X}_{\mathrm{TP}}^\top) \cdot \mathbf{I} \preceq \nabla^2 \mathcal{L}(\mathbf{w}; \mathrm{TP}) \preceq 2 \cdot C_{\lceil \psi \rceil}^2 \lambda_{\max}(\mathbf{X}_{\mathrm{TP}} \mathbf{X}_{\mathrm{TP}}^\top) \cdot \mathbf{I}$$

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We can now lower bound the second term of Equation (B.6).

$$2\eta \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \nabla\mathcal{R}(\mathbf{w}^{(t)}; \text{TP}) - \nabla\mathcal{R}(\mathbf{w}^{(*)}; \text{TP})\rangle$$

$$= \frac{4\eta}{(1-\epsilon)N} \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \int_0^1 \nabla^2\mathcal{R}(\mathbf{w}^* + \theta(\mathbf{w}^* - \mathbf{w}^{(t)}); \text{TP})d\theta \cdot (\mathbf{w}^{(t)} - \mathbf{w}^*)\rangle$$

$$\overset{(i)}{\geq} \frac{4\eta}{(1-\epsilon)N} \cdot C_{\lfloor\psi\rfloor}^2 \lambda_{\min}(\mathbf{X}_{\text{TP}}\mathbf{X}_{\text{TP}}^\top)\|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 \tag{B.7}$$

where (i) follows from Property 10. We now will upper bound the second term of Equation (B.6).

$$\eta^2 \cdot \|\nabla\mathcal{R}(\mathbf{w}^{(t)}; \text{TP}) - \nabla\mathcal{R}(\mathbf{w}^*; \text{TP})\|_2^2$$

$$= \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \left\|\int_0^1 \nabla^2\mathcal{R}(\mathbf{w}^* + \theta(\mathbf{w}^* - \mathbf{w}^{(t)}); \text{TP})d\theta \cdot (\mathbf{w}^{(t)} - \mathbf{w}^*)\right\|_2^2$$

$$\overset{(ii)}{\leq} \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot C_{\lceil\psi\rceil}^2 \lambda_{\max}^2(\mathbf{X}_{\text{TP}}\mathbf{X}_{\text{TP}}^\top)\|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 \tag{B.8}$$

where (ii) follows from the $C_{\lceil\psi\rceil}$-Lipschitzness of ψ given in Property 11. We then observe that the first term of Equation (B.7) is greater than half the first term in Equation (B.8) when $\eta \leq \frac{C_\psi^2(1-\epsilon)N\lambda_{\min}(\mathbf{X}_{\text{TP}}\mathbf{X}_{\text{TP}}^\top)}{2\lambda_{\max}^2(\mathbf{X}_{\text{TP}}\mathbf{X}_{\text{TP}}^\top)}$.
Step 2: Upper bounding the corrupted gradient. We now upper bound the third term in Equation (B.6).

$$\eta^2 \cdot \|\nabla\mathcal{R}(\mathbf{w}^{(t)}; \text{FP})\|_F^2 = \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \Big\|\sum_{i\in\text{FP}}(\psi(\mathbf{x}_i^\top\mathbf{w}) - \psi(\mathbf{x}_i^\top\mathbf{w}^*)) \cdot \psi'(\mathbf{x}_i^\top\mathbf{w}_k^{(t)}) \cdot \mathbf{x}_i\Big\|_2^2$$

$$\leq \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot C_{\lceil\psi\rceil}^2 \big\|\mathbf{X}_{\text{FP}}\mathbf{X}_{\text{FP}}^\top\big\|_2 \sum_{i\in\text{FP}}\big(\psi(\mathbf{x}_i^\top\mathbf{w}) - \psi(\mathbf{x}_i^\top\mathbf{w}^*)\big)^2$$

$$\leq \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot C_{\lceil\psi\rceil}^2 \big\|\mathbf{X}_{\text{FP}}\mathbf{X}_{\text{FP}}^\top\big\|_2 \sum_{i\in\text{FN}}\big(\psi(\mathbf{x}_i^\top\mathbf{w}) - \psi(\mathbf{x}_i^\top\mathbf{w}^*)\big)^2$$

$$\leq \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot C_{\lceil\psi\rceil}^4 \big\|\mathbf{X}_{\text{FP}}\mathbf{X}_{\text{FP}}^\top\big\|_2\big\|\mathbf{X}_{\text{FN}}\mathbf{X}_{\text{FN}}^\top\big\|_2\|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2$$

In the above, the first inequality follows from Lemma 29, the second inequality follows from the optimality of the Subquantile set, the final inequality follows from the C_2-Lipschitzness of ψ. We will now show the random variable, $\psi'(\mathbf{x}_i^\top\mathbf{w}^{(t)}) \cdot \mathbf{x}_i$ is sub-Gaussian. Let $\mathbf{v} \in \mathbb{S}^{d-1}$, then since $\mathbf{x}_i$ is sampled from sub-Gaussian distribution with sub-Gaussian norm C_K, we then have $\mathbf{v}^\top\mathbf{x}_i$ is sub-Gaussian with norm K. Therefore,

$$\big(\mathbf{E}|\psi'(\mathbf{x}_i^\top\mathbf{w})\mathbf{v}^\top\mathbf{x}_i|^p\big)^{1/p} \leq \sup_{x\in\mathbb{R}}|\psi'(x)|\big(\mathbf{E}|\mathbf{v}^\top\mathbf{x}_i|^{2p}\big)^{1/2p} \leq C_{\lceil\psi\rceil}C_K\sqrt{2p}$$

In the above, in the first inequality we used Hölder's Inequality. We therefore have, $\|\psi'(\mathbf{x}_i^\top\mathbf{w}^{(t)}) \cdot \mathbf{x}_i\|_{\psi_2} \leq \sqrt{2}C_{\lceil\psi\rceil}C_K$. Then from Lemma 25, we have with probability at least $1-\delta$,

$$\Big\|\sum_{i\in\text{TP}}\xi_i \cdot \psi'(\mathbf{x}_i^\top\mathbf{w}^{(t)}) \cdot \mathbf{x}_i\Big\|_2 \leq 9NC_KC_\sigma\left(\frac{2d}{N}\log 5 + \frac{2}{N}\log(1/\delta) + 6\epsilon\log\epsilon^{-1}\right)^{1/2}$$

We now combine Steps 1 and 2 to give the linear convergence result. Noting that $\sqrt{1-2x} \leq 1-x$ when $x \leq 1/2$, we have

$$\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 \leq \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2\left(1 - \frac{2C_\psi^2\eta}{(1-\epsilon)N} \cdot \lambda_{\min}(\mathbf{X}_{\text{TP}}\mathbf{X}_{\text{TP}}^\top) + \frac{2\eta}{(1-\epsilon)N} \cdot \|\mathbf{X}_{\text{FP}}\|_2\|\mathbf{X}_{\text{FN}}\|_2\right)$$

$$+ \frac{15C_KC_\sigma d\log 5}{\sqrt{(1-\epsilon)N}} + \frac{15C_KC_\sigma\log(1/\delta)}{\sqrt{(1-\epsilon)N}} + 16C_KC_\sigma\sqrt{\epsilon\log\epsilon^{-1}}$$

Step 3: Concentration Bounds. We will give the relevant probabilistic bounds for the random variables in Steps 1 and 2. From Lemma 24, we have $\|\mathbf{X}_{\mathrm{FN}}\|_2 \|\mathbf{X}_{\mathrm{FP}}\|_2 \leq \epsilon \sqrt{\lambda_{\max}(\mathbf{\Sigma}) \cdot 10 C_3 N \log \epsilon^{-1}}$ with probability at least $1 - \delta$ when $N \geq \frac{2}{\epsilon} \cdot \left(d C_K^2 + \frac{\log(2/\delta)}{c_K} \right)$ when $\epsilon \leq \frac{1}{60} \cdot \kappa^{-1}(\mathbf{\Sigma})$. From the same Lemma and under the same data conditions we have $\lambda_{\min}(\mathbf{X}_{\mathrm{TP}} \mathbf{X}_{\mathrm{TP}}^\top) \geq \frac{1}{4} \cdot \lambda_{\min}(\mathbf{\Sigma})$. Then for $\epsilon \leq \frac{C_\psi^2 \lambda_{\min}(\mathbf{\Sigma})}{\sqrt{32 C_3 \lambda_{\max}(\mathbf{\Sigma})}}$, we have $\|\mathbf{X}_{\mathrm{FP}}\|_2 \|\mathbf{X}_{\mathrm{FN}}\|_2 \leq \frac{1}{2} \cdot \lambda_{\min}(\mathbf{\Sigma})$. We then have, after $O\left(\kappa^2(\mathbf{\Sigma}) \log\left(\frac{\|\mathbf{w}^*\|_2 + 10 d}{\varepsilon} \right) \right)$ iterations with high probability,

$$\|\mathbf{w}^{(T)} - \mathbf{w}^*\|_2 \leq \varepsilon + \frac{1}{N} \cdot 144 C_\psi^{-2} \kappa^2(\mathbf{\Sigma}) K C_\sigma d \log 5 + 432 \kappa^2(\mathbf{\Sigma}) C_\psi^{-1} K C_\sigma d \epsilon \log \epsilon^{-1}$$
$$= O\left(\kappa^2(\mathbf{\Sigma}) C_\psi^{-2} K C_\sigma d \epsilon \log \epsilon^{-1} \right)$$

In the final inequality above, we set $\varepsilon = O\left(\kappa^2(\mathbf{\Sigma}) C_\psi^{-1} K C_\sigma d \epsilon \log \epsilon^{-1} \right)$ for $N \geq \epsilon^{-2} C_\psi^{-2} \kappa^2(\mathbf{\Sigma}) \log 5$. Our proof is complete. ∎

B.3 Learning a ReLU Neuron

In this section, we consider functions such as the ReLU. Our high-level analysis will be similar to the previous sub-sections however the details are considerably different and require stronger conditions we can guarantee by randomness.

B.3.1 Proof of Theorem 20

Proof. We will now begin our standard analysis.

$$\begin{aligned}
\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 &= \|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta \nabla \mathcal{R}(\mathbf{w}; \mathrm{S}^{(t)})\|_2 \\
&= \|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP}) - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{FP})\|_2 \\
&\leq \|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP})\|_2 + \|\eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{FP})\|_2 \quad (\text{B.9})
\end{aligned}$$

We will now upper bound the first term of Equation (B.9) through its square,

$$\begin{aligned}
\|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP})\|_2^2 = \\
\|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 - 2\eta \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP}) \rangle + \eta^2 \cdot \|\nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP})\|_2^2 \quad (\text{B.10})
\end{aligned}$$

We will lower bound the second term of Equation (B.10). We will first adopt the notation from [ZYWG19], let $\mathbf{\Sigma}_{\mathrm{TP}}(\mathbf{w}, \hat{\mathbf{w}}) = \mathbf{X}_{\mathrm{TP}}^\top \mathbf{X}_{\mathrm{TP}}^\top \cdot \mathbb{I}\{\mathbf{X}_{\mathrm{TP}}^\top \mathbf{w} \geq \mathbf{0}\} \cdot \mathbb{I}\{\mathbf{X}_{\mathrm{TP}}^\top \hat{\mathbf{w}} \geq \mathbf{0}\}$, it then follows

$$\begin{aligned}
&2\eta \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \nabla \mathcal{R}(\mathbf{w}^{(t)}; \mathrm{TP}) \rangle \\
&\overset{\text{def}}{=} \frac{4\eta}{(1-\epsilon)N} \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \sum_{i \in \mathrm{TP}} (\psi(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - y_i) \cdot \mathbf{x}_i \cdot \mathbb{I}\{\mathbf{x}_i^\top \mathbf{w}^{(t)} \geq 0\} \rangle \\
&= \frac{4\eta}{(1-\epsilon)N} \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \mathbf{\Sigma}_{\mathrm{TP}}(\mathbf{w}^{(t)}, \mathbf{w}^{(t)}) \mathbf{w}^{(t)} - \mathbf{\Sigma}_{\mathrm{TP}}(\mathbf{w}^{(t)}, \mathbf{w}^*) \mathbf{w}^* \rangle \\
&= \frac{4\eta}{(1-\epsilon)N} \cdot \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \mathbf{\Sigma}_{\mathrm{TP}}(\mathbf{w}^{(t)}, \mathbf{w}^*)(\mathbf{w}^{(t)} - \mathbf{w}^*) + \mathbf{\Sigma}_{\mathrm{TP}}(\mathbf{w}^{(t)}, -\mathbf{w}^*) \mathbf{w}^{(t)} \rangle \\
&\geq \frac{4\eta}{(1-\epsilon)N} \cdot \lambda_{\min}(\mathbf{\Sigma}_{\mathrm{TP}}(\mathbf{w}^{(t)}, \mathbf{w}^*)) \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2
\end{aligned}$$

In the above, the final inequality holds from the following relation,

$$\langle \mathbf{w}^{(t)} - \mathbf{w}^*, \mathbf{\Sigma}_{\mathrm{TP}}(\mathbf{w}^{(t)}, -\mathbf{w}^*) \mathbf{w}^{(t)} \rangle = \langle \mathbf{w}^{(t)} - \mathbf{w}^*, \sum_{i \in \mathrm{TP}} \mathbf{x}_i \mathbf{x}_i^\top \mathbf{w}^{(t)} \cdot \mathbb{I}\{\mathbf{x}_i^\top \mathbf{w}^{(t)} \geq 0\} \cdot \mathbb{I}\{\mathbf{x}_i^\top \mathbf{w}^* \leq 0\} \rangle$$

$$= \sum_{i \in \text{TP}} (\mathbf{x}_i^\top \mathbf{w}^{(t)} - \mathbf{x}_i^\top \mathbf{w}^*)(\mathbf{x}_i^\top \mathbf{w}^{(t)}) \cdot \mathbb{I}\{\mathbf{x}_i^\top \mathbf{w}^{(t)} \geq 0\} \cdot \mathbb{I}\{\mathbf{x}_i^\top \mathbf{w}^* \leq 0\} \geq 0$$

In the above, in the final relation we can note that when the indicators are positive, it must follow that both $\mathbf{x}_i^\top \mathbf{w}^{(t)}$ is positive and $\mathbf{x}_i^\top \mathbf{w}^{(t)} \geq \mathbf{x}_i^\top \mathbf{w}^*$ as $\mathbf{x}_i^\top \mathbf{w}^* \leq 0$. We have from Weyl's Inequality,

$$\lambda_{\min}\left(\boldsymbol{\Sigma}_{\text{TP}}(\mathbf{w}^{(t)}, \mathbf{w}^*)\right) \geq \lambda_{\min}\left(\mathbf{E}\left[\boldsymbol{\Sigma}_{\text{TP}}(\mathbf{w}^{(t)}, \mathbf{w}^*)\right]\right) - \left\|\boldsymbol{\Sigma}_{\text{TP}}(\mathbf{w}^{(t)}, \mathbf{w}^*) - \mathbf{E}\left[\boldsymbol{\Sigma}_{\text{TP}}(\mathbf{w}^{(t)}, \mathbf{w}^*)\right]\right\|_2$$

Let $\Omega = \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}^\top \mathbf{w}^{(t)} \geq 0, \mathbf{x}^\top \mathbf{w}^* \geq 0\right\}$, then

$$\mathbf{E}\left[\boldsymbol{\Sigma}_{\text{TP}}(\mathbf{w}^{(t)}, \mathbf{w}^*)\right] = \sum_{i \in \text{TP}} \mathbf{E}\left[\mathbf{x}_i \mathbf{x}_i^\top \cdot \mathbb{I}\{\mathbf{x}_i^\top \mathbf{w}^{(t)} \geq 0\} \cdot \mathbb{I}\{\mathbf{x}_i^\top \mathbf{w}^* \geq 0\}\right]$$

$$\overset{(i)}{\succeq} N(1 - 2\epsilon) \cdot \left(\pi - \Theta^{(t)} - \sin \Theta^{(t)}\right) \cdot \mathbf{I}$$

$$\overset{(ii)}{\succeq} N(1 - 2\epsilon) \cdot \left(\pi - 2 \arcsin\left(\frac{\|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2}{\|\mathbf{w}^*\|_2}\right)\right) \cdot \mathbf{I}$$

$$\overset{(iii)}{\succeq} N(1 - 2\epsilon) \cdot \pi \cdot \left(1 - \frac{\|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2}{\|\mathbf{w}^*\|_2}\right) \cdot \mathbf{I}$$

In the above, (i) follows from Lemma 26, the inequality in (ii) follows when $\|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2 \leq \|\mathbf{w}^*\|_2$, and (iii) follows from noting $\arcsin(x) \leq \frac{\pi}{2} \cdot x$ for $x \in [0, 1]$. We now bound the second-moment matrix approximation. Let $dP(\mathbf{x})$ be a Dirac-measure for $\mathbf{x} \in \text{TP}$. We will now upper bound the third term in Equation (B.10).

$$\eta^2 \cdot \|\nabla \mathcal{R}(\mathbf{w}^{(t)}; \text{TP})\|_2^2 = \frac{4\eta^2}{[(1 - \epsilon)N]^2} \cdot \left\|\sum_{i \in \text{TP}} (\psi(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - \psi(\mathbf{x}_i^\top \mathbf{w}^*) - \xi_i) \cdot \mathbf{x}_i \cdot \mathbb{I}\{\mathbf{x}_i^\top \mathbf{w}^{(t)} \geq 0\}\right\|_2^2$$

$$\leq \frac{8\eta^2}{[(1 - \epsilon)N]^2} \cdot \left(\left\|\sum_{i \in \text{TP}} (\psi(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - \psi(\mathbf{x}_i^\top \mathbf{w}^*)) \cdot \mathbf{x}_i \cdot \mathbb{I}\{\mathbf{x}_i^\top \mathbf{w}^{(t)} \geq 0\}\right\|_2^2 + \left\|\sum_{i \in \text{TP}} \xi_i \mathbf{x}_i \cdot \mathbb{I}\{\mathbf{x}_i^\top \mathbf{w}^{(t)} \geq 0\}\right\|_2^2\right)$$

$$\text{(B.11)}$$

Recall we have from Lemma 25, we have an upper bound on the second term of Equation (B.11). We give a bound on the first term of Equation (B.11).

$$\frac{8\eta^2}{[(1 - \epsilon)N]^2} \cdot \left\|\sum_{i \in \text{TP}} (\psi(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - \psi(\mathbf{x}_i^\top \mathbf{w}^*)) \cdot \mathbf{x}_i \cdot \mathbb{I}\{\mathbf{x}_i^\top \mathbf{w}^{(t)} \geq 0\}\right\|_2^2$$

$$\leq \frac{8\eta^2}{[(1 - \epsilon)N]^2} \cdot \left\|\mathbf{X}_{\text{TP}} \mathbf{X}_{\text{TP}}^\top\right\|_2 \sum_{i \in \text{TP}} (\psi(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - \psi(\mathbf{x}_i^\top \mathbf{w}^*))^2$$

$$\leq \frac{8\eta^2}{[(1 - \epsilon)N]^2} \cdot \left\|\mathbf{X}_{\text{TP}} \mathbf{X}_{\text{TP}}^\top\right\|_2^2 \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 \qquad \text{(B.12)}$$

Then, by choosing $\eta \leq \frac{\lambda_{\min}(\boldsymbol{\Sigma})}{80\lambda_{\max}^2(\boldsymbol{\Sigma})}$, we have that the RHS in Equation (B.12) will be less than $\frac{\lambda_{\min}(\boldsymbol{\Sigma})}{8}$.

Step 3: Upper bounding the corrupted gradient. We now upper bound the third term in Equation (B.10).

$$\eta^2 \cdot \|\nabla \mathcal{R}(\mathbf{w}^{(t)}; \text{FP})\|_{\text{F}}^2 = \frac{4\eta^2}{[(1 - \epsilon)N]^2} \cdot \left\|\sum_{i \in \text{FP}} (\psi(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - \psi(\mathbf{x}_i^\top \mathbf{w}^*)) \cdot \mathbf{x}_i \cdot \mathbb{I}\{\mathbf{x}_i^\top \mathbf{w}^{(t)} \geq 0\}\right\|_2^2$$

$$\leq \frac{4\eta^2}{[(1 - \epsilon)N]^2} \cdot \left\|\boldsymbol{\Sigma}_{\text{FP}}(\mathbf{w}^{(t)}, \mathbf{w}^{(t)})\right\|_2 \sum_{i \in \text{FP}} (\psi(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - \psi(\mathbf{x}_i^\top \mathbf{w}^*))^2$$

$$\leq \frac{4\eta^2}{[(1 - \epsilon)N]^2} \cdot \left\|\mathbf{X}_{\text{FP}} \mathbf{X}_{\text{FP}}^\top\right\|_2 \sum_{i \in \text{FN}} (\psi(\mathbf{x}_i^\top \mathbf{w}^{(t)}) - \psi(\mathbf{x}_i^\top \mathbf{w}^*))^2$$

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$$\leq \frac{4\eta^2}{[(1-\epsilon)N]^2} \cdot \big\|\mathbf{X}_{\mathrm{FP}}\mathbf{X}_{\mathrm{FP}}^\top\big\|_2 \big\|\mathbf{X}_{\mathrm{FN}}\mathbf{X}_{\mathrm{FN}}^\top\big\|_2 \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2$$

In the above, the first inequality follows from the same argument as Lemma 29, the second inequality follows from the optimality of the Subquantile set, and the final inequality follows from noting that ψ is 1-Lipschitz. We now conclude Steps 1-3 with our linear convergence result.

$$\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 \leq \|\mathbf{w}^{(t)} - \mathbf{w}^*\| \left(1 - \frac{\eta}{32} \cdot \lambda_{\min}(\mathbf{\Sigma}) + \frac{\sqrt{8}\,\eta}{(1-\epsilon)N} \cdot \|\mathbf{X}_{\mathrm{FP}}\|_2 \|\mathbf{X}_{\mathrm{FN}}\| \right)$$
$$+ \frac{16}{3} K C_\sigma \cdot \left(\frac{2d}{N}\log(5) + \frac{2}{N}\log(1/\delta) + 6\epsilon \log \epsilon^{-1} \right)^{1/2}$$

Step 4: Concentration Inequalities. From our previous theorems, we have $\|\mathbf{X}_{\mathrm{FP}}\|_2\|\mathbf{X}_{\mathrm{FN}}\|_2 \leq N\epsilon\sqrt{10 C_3 \log \epsilon^{-1}}$ with probability exceeding $1-\delta$ and $N = \Omega\big(\frac{d+\log(2/\delta)}{\epsilon}\big)$. If $N = \Omega\big(\frac{2d\log 5 + \log(1/\delta)}{3\epsilon \log \epsilon^{-1}}\big)$ and $\epsilon \leq \frac{1}{64\sqrt{80\log(1/2)}}$, we obtain after $T = O\big(\kappa^2(\mathbf{\Sigma}) \log\big(\frac{\|\mathbf{w}^*\|_2}{\varepsilon}\big)\big)$ gradient descent iterations,

$$\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 \leq \varepsilon + 1024\sqrt{\epsilon \log \epsilon^{-1}} = O\big(\sqrt{\epsilon \log \epsilon^{-1}}\big)$$

when we choose $\varepsilon = O\big(\sqrt{\epsilon \log \epsilon^{-1}}\big)$. Our proof is complete. $\blacksquare$

C Probability Theory

In this section we will give various concentration inequalities on the inlier data for functions in the Reproducing Kernel Hilbert Space.

C.1 Finite Dimensional Concentrations of Measure

Lemma 21 (Upper Bound on Sum of Chi-Squared Variables [LM00])**.** *Suppose* $\xi_i \sim \mathcal{N}(0,\sigma^2)$ *for* $i \in [n]$, *then*
$$\mathbf{Pr}\big\{ \|\mathbf{\xi}\|_2^2 \geq \sigma\big(n + 2\sqrt{nx} + 2x\big) \big\} \leq e^{-x}$$

Proposition 22 (Probabilistic Upper Bound on Sum of Chi-Squared Variables)**.** *Suppose* $\xi_i \sim \mathcal{N}(0,\sigma^2)$ *for* $i \in [n]$. *Let* $S \subset [n]$ *such that* $|S| = \epsilon n$ *for* $\epsilon \in (0, 0.5)$ *and let* $\mathcal{C}$ *represent all such subsets. Given a failure probability* $\delta \in (0,1)$, *when* $n \geq \log(1/\delta)$, *with probability exceeding* $1 - \delta$,
$$\max_{S \in \mathcal{C}} \|\mathbf{\xi}_S\|_2^2 \leq \sigma\big(30 n\epsilon \log \epsilon^{-1}\big)$$

Proof. Directly from Lemma 21, we have with probability exceeding $1 - \delta$.
$$\|\mathbf{\xi}\|_2^2 \leq \sigma\Big(n + 2\sqrt{n\log(1/\delta)} + 2\log(1/\delta)\Big)$$

We now can prove the claimed bound using the layer-cake representation,

$$\mathbf{Pr}\Big\{ \max_{S \in \mathcal{C}} \|\mathbf{\xi}\|_2^2 \geq \sigma\big(\epsilon n + 2\sqrt{\epsilon n x} + 2x\big) \Big\} \leq \Big(\frac{e}{\epsilon}\Big)^{\epsilon n} \mathbf{Pr}\big\{ \|\mathbf{\xi}\|_2^2 \geq \sigma\big(\epsilon n + 2\sqrt{\epsilon n x} + 2x\big) \big\} \leq \Big(\frac{e}{\epsilon}\Big)^{\epsilon n} e^{-x}$$

In the first inequality we apply a union bound over $\mathcal{C}$ with Lemma 32, and in the second inequality we use Lemma 21. We then obtain with probability exceeding $1 - \delta$,

$$\max_{S \in \mathcal{C}} \|\mathbf{\xi}_S\|_2^2 \leq \sigma\Big(\epsilon n + 2\sqrt{n\epsilon \log(1/\delta) + 3n^2\epsilon^2 \log \epsilon^{-1}} + 2\log(1/\delta) + 6n\epsilon \log \epsilon^{-1}\Big)$$
$$\leq \sigma\Big(9n\epsilon \log \epsilon^{-1} + 2\sqrt{n\epsilon \log(1/\delta)} + 2\sqrt{3}\,n\epsilon \sqrt{\log \epsilon^{-1}} + 2\log(1/\delta)\Big)$$

$$\leq \sigma\left(15n\epsilon\log\epsilon^{-1} + 2\sqrt{n\epsilon\log(1/\delta)} + 2\log(1/\delta)\right)$$
$$\leq \sigma\left(30n\epsilon\log\epsilon^{-1}\right)$$

In the above, in the first inequality, we note that $\log\binom{n}{\epsilon n} \leq 3n\epsilon\log\epsilon^{-1}$ as $\epsilon < 0.5$, in the second inequality we note that $\sqrt{\log\epsilon^{-1}} \leq (\log(2))^{-1/2}\log\epsilon^{-1} \leq \sqrt{3}\log\epsilon^{-1}$ when $\epsilon < 0.5$, the final inequality holds when $n \geq \log(1/\delta)$ by solving for the quadratic equation. The proof is complete. ∎

Lemma 23 (Sub-Gaussian Covariance Matrix Estimation [Ver10] Theorem 5.40). *Let* $\mathbf{X} \in \mathbb{R}^{d\times n}$ *have columns sampled from a sub-Gaussian distribution with sub-Gaussian norm K and second-moment matrix $\mathbf{\Sigma}$, then there exists positive constants c_k, C_K, dependent on the sub-Gaussian norm such that with probability at least $1 - 2e^{-c_K t^2}$,*

$$\lambda_{\max}(\mathbf{X}\mathbf{X}^\top) \leq n\cdot\lambda_{\max}(\mathbf{\Sigma}) + \lambda_{\max}(\mathbf{\Sigma})\cdot\left(C_K\cdot\sqrt{dn} + t\cdot\sqrt{n}\right)$$

Lemma 24. *Let* $\mathbf{X} \in \mathbb{R}^{d\times n}$ *have columns sampled from a sub-Gaussian distribution with sub-Gaussian norm K and second-moment matrix $\mathbf{\Sigma}$. Let $S \subset [n]$ such that $|S| = \epsilon n$ for $\epsilon \in (0, 0.5)$ and let $\mathcal{C}$ represent all such subsets. Then with probability at least $1 - \delta$,*

$$\max_{S\in\mathcal{C}}\lambda_{\max}(\mathbf{X}_S\mathbf{X}_S^\top) \leq \lambda_{\max}(\mathbf{\Sigma})\cdot\left(10n\epsilon\log\epsilon^{-1}\right)$$
$$\min_{S\in\mathcal{C}}\lambda_{\min}(\mathbf{X}_{X\backslash S}\mathbf{X}_{X\backslash S}^\top) \geq \frac{n}{4}\cdot\lambda_{\min}(\mathbf{\Sigma})$$

when

$$n \geq \frac{2}{\epsilon}\cdot\left(C_K^2\cdot d + \frac{\log(2/\delta)}{c_K}\right)$$

and $\epsilon \leq \frac{1}{60}\cdot\kappa^{-1}(\mathbf{\Sigma})$.

Proof. We will use the layer-cake representation to obtain our claimed error bound.

$$\mathbf{Pr}\left\{\max_{S\in\mathcal{C}}\lambda_{\max}(\mathbf{X}_S\mathbf{X}_S^\top) \geq n\epsilon\cdot\lambda_{\max}(\mathbf{\Sigma}) + \lambda_{\max}(\mathbf{\Sigma})\cdot\left(C_K\cdot\sqrt{dn\epsilon} + t\sqrt{n\epsilon}\right)\right\}$$
$$\leq \left(\frac{e}{\epsilon}\right)^{\epsilon n}\mathbf{Pr}\left\{\lambda_{\max}(\mathbf{X}_S\mathbf{X}_S^\top) \geq n\epsilon\cdot\lambda_{\max}(\mathbf{\Sigma}) + \lambda_{\max}(\mathbf{\Sigma})\cdot\left(C_K\cdot\sqrt{dn\epsilon} + t\sqrt{n\epsilon}\right)\right\} \leq 2\cdot\left(\frac{e}{\epsilon}\right)^{\epsilon n}e^{-c_K t^2}$$

In the above, the first inequality follows from a union bound over $\mathcal{C}$ and Lemma 32, the second inequality follows from Lemma 23. Then from elementary inequalities, we obtain with probability $1 - \delta$,

$$\max_{S\in\mathcal{C}}\lambda_{\max}(\mathbf{X}_S\mathbf{X}_S^\top) \leq n\epsilon\cdot\lambda_{\max}(\mathbf{\Sigma}) + \lambda_{\max}(\mathbf{\Sigma})\cdot\left(C_K\cdot\sqrt{dn\epsilon} + \sqrt{\frac{1}{c_K}(n\epsilon\cdot\log(2/\delta) + 3n^2\epsilon^2\log\epsilon^{-1})}\right)$$
$$\leq n\cdot\lambda_{\max}(\mathbf{\Sigma})\cdot(\epsilon + 3^{3/4}\epsilon\log\epsilon^{-1}) + \lambda_{\max}(\mathbf{\Sigma})\cdot\left(C_K\cdot\sqrt{dn\epsilon} + \sqrt{\frac{1}{c_K}n\epsilon\cdot\log(2/\delta)}\right)$$
$$\leq \lambda_{\max}(\mathbf{\Sigma})\cdot\left(6n\epsilon\log\epsilon^{-1}\right) + \lambda_{\max}(\mathbf{\Sigma})\cdot\left(C_K\cdot\sqrt{dn\epsilon} + \sqrt{\frac{1}{c_K}n\epsilon\cdot\log(2/\delta)}\right)$$
$$\leq \lambda_{\max}(\mathbf{\Sigma})\cdot\left(10n\epsilon\log\epsilon^{-1}\right)$$

In the above, the last inequality holds when

$$n \geq \frac{2}{\epsilon}\cdot\left(C_K^2\cdot d + \frac{\log(2/\delta)}{c_K}\right)$$

and our proof of the upper bound for the maximal eigenvalue is complete. We have from Weyl's Inequality for any $S \in \mathcal{C}$,

$$\lambda_{\min}(\mathbf{X}_{X\backslash S}\mathbf{X}_{X\backslash S}^\top) = \lambda_{\min}(\mathbf{X}\mathbf{X}^\top - \mathbf{X}_S\mathbf{X}_S^\top) \geq \lambda_{\min}(\mathbf{X}\mathbf{X}^\top) - \lambda_{\max}(\mathbf{X}_S\mathbf{X}_S^\top)$$

We then have with probability at least $1 - \delta$,

$$\lambda_{\min}(\mathbf{X}_{X\backslash S}\mathbf{X}_{X\backslash S}^{\top}) \geq n \cdot \lambda_{\min}(\mathbf{\Sigma}) - C_K \cdot \sqrt{dn} - \sqrt{\frac{1}{c_K} \cdot n \cdot \log(2/\delta)} - \lambda_{\max}(\mathbf{\Sigma}) \cdot (10n\epsilon \log \epsilon^{-1})$$

$$\geq \frac{n}{2} \cdot \lambda_{\min}(\mathbf{\Sigma}) - \lambda_{\max}(\mathbf{\Sigma}) \cdot (10n\epsilon \log \epsilon^{-1}) \geq \frac{n}{4} \cdot \lambda_{\min}(\mathbf{\Sigma})$$

In the above, the first inequality follows when $n \geq \frac{1}{\lambda_{\min}^2(\mathbf{\Sigma})}\left(8C_K \cdot d + \frac{2}{c_K} \cdot \log(2/\delta)\right)$, and the last inequality follows when $\epsilon \leq \frac{1}{60} \cdot \kappa^{-1}(\mathbf{\Sigma})$. The proof is complete. ∎

Lemma 25 (Noise Bound). *Fix* $\mathbf{w} \in \mathbb{R}^d$. *Let* $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_N]$ *be the data matrix such that for* $i \in [N]$, $\mathbf{x}_i$ *are sampled from a sub-Gaussian distribution with second-moment matrix* $\mathbf{\Sigma}$ *with sub-Gaussian norm* C_K *and* ξ_i *are sampled from sub-Gaussian distribution with sub-Gaussian norm* C_σ. *Assume* $f : \mathbb{R}^d \to \mathbb{R}$ *is bounded such that* $\sup_{\mathbf{x} \in \mathbb{R}^d} |f(\mathbf{x}^{\top}\mathbf{w})| \leq C_L$ *for any parameterization vector* $\mathbf{w} \in \mathbb{R}^d$. *Let* $\mathcal{S}$ *represent all subsets of* $[N]$ *of size* $(1 - 2\epsilon)N$ *to* $(1 - \epsilon)N$. *Set a failure probability* $\delta \in (0, 1)$, *then with probability at least* $1 - \delta$,

$$\max_{S \in \mathcal{S}} \cdot \|(\boldsymbol{\xi}_S \circ f(\mathbf{X}_S^{\top}\mathbf{w}))^{\top}\mathbf{X}_S\|_2 \leq \frac{4}{3} \cdot NC_LC_KC_\sigma \cdot \left(\frac{2d}{N}\log(5) + \frac{2Rd}{N}\log(5) + \frac{2}{N}\log(1/\delta) + 6\epsilon \log \epsilon^{-1}\right)^{1/2}$$

Proof. We will use the following characterization of the spectral norm.

$$\Big\|\sum_{i \in \text{TP}} f(\mathbf{x}_i^{\top}\mathbf{w})\xi_i\mathbf{x}_i\Big\|_2 = \max_{\mathbf{v} \in \mathbb{S}^{d-1}} \Big|\sum_{i \in \text{TP}} f(\mathbf{x}_i^{\top}\mathbf{w})\xi_i\mathbf{x}_i^{\top}\mathbf{v}\Big|$$

Our proof uses standard ideas in High-Dimensional Probability and will follow similarly to [ZYWG19]. We will first show that $f(\mathbf{x}_i^{\top}\mathbf{w})\mathbf{x}_i$ is sub-Gaussian. We first note for any $\mathbf{v} \in \mathbb{S}^{d-1}$, the random variable, $\mathbf{x}_i^{\top}\mathbf{v}$ is sub-Gaussian by definition. We then have,

$$\left(\mathbf{E}|f(\mathbf{x}_i^{\top}\mathbf{w})\mathbf{x}_i^{\top}\mathbf{v}|^p\right)^{1/p} \overset{(i)}{\leq} \left(\mathbf{E}|f(\mathbf{x}_i^{\top}\mathbf{w})|^{2p} \mathbf{E}|\mathbf{x}^{\top}\mathbf{v}|^{2p}\right)^{1/2p} \overset{(ii)}{\leq} \left(C_LC_K\sqrt{2}\right)\sqrt{p}$$

In the above, (i) follows from Hölder's Inequality, (ii) follows from noting from letting $q = 2p$ and noting that by the definition of a sub-Gaussian variable $\|\mathbf{x}_i^{\top}\mathbf{v}\|_{L_q}$ is upper bounded by $C_K\sqrt{q}$. We thus have $f(\mathbf{x}_i^{\top}\mathbf{w})\mathbf{x}_i^{\top}\mathbf{v}$ is sub-Gaussian for any $\mathbf{w} \in \mathbb{R}^d$ and $\|f(\mathbf{x}_i^{\top}\mathbf{w})\mathbf{x}_i^{\top}\mathbf{v}\|_{\psi_2} \leq C_LC_K\sqrt{2}$. Recall $C_\sigma \triangleq \|\xi_i\|_{\psi_2}$, then from Lemma 2.7.6 in [Ver20], the random variable $\xi_if(\mathbf{x}_i^{\top}\mathbf{w})\mathbf{x}_i^{\top}\mathbf{v}$ is sub-exponential s.t. $\|\xi_if(\mathbf{x}_i^{\top}\mathbf{w})\mathbf{x}_i^{\top}\mathbf{v}\|_{\psi_1} \leq \sqrt{2}C_LC_KC_\sigma$. Let $\widetilde{\mathbf{w}} \in \mathcal{M}$ such that $\widetilde{\mathbf{w}} = \arg\min_{\mathbf{u} \in \mathcal{M}}\|\mathbf{w} - \mathbf{u}\|_2$, where $\mathcal{M}$ is a ε-cover of $\mathcal{B}(\mathbf{0}, R)$, then

$$\mathbf{Pr}\left\{\max_{S \in \mathcal{S}} \sup_{\mathbf{w} \in \mathcal{B}(\mathbf{0}, R)} \Big\|\sum_{i \in S} \xi_if(\mathbf{x}_i^{\top}\mathbf{w})\mathbf{x}_i\Big\| \geq t\right\} \leq \mathbf{Pr}\left\{\max_{S \in \mathcal{S}} \max_{\mathbf{u} \in \mathcal{M}} \Big\|\sum_{i \in S} \xi_if(\mathbf{x}_i^{\top}\mathbf{u})\mathbf{x}_i\Big\|_2 \geq \frac{t}{2}\right\}$$

$$+ \mathbf{Pr}\left\{\max_{S \in \mathcal{S}} \sup_{\mathbf{w} \in \mathcal{B}(\mathbf{0}, R)} \Big\|\sum_{i \in S} \xi_if(\mathbf{x}_i^{\top}\mathbf{w})\mathbf{x}_i - \sum_{i \in S} \xi_if(\mathbf{x}_i^{\top}\widetilde{\mathbf{w}})\mathbf{x}_i\Big\| \geq \frac{t}{2}\right\} \quad \text{(C.1)}$$

We will now use a ε-covering argument to bound the first term of Equation (C.1). Let $\mathcal{C}$ be a ε-net of $\mathbb{S}^{d-1}$ such that for any $\mathbf{v} \in \mathbb{S}^{d-1}$, there exists $\mathbf{u} \in \mathcal{C}$ such that $\|\mathbf{u} - \mathbf{v}\|_2 \leq \varepsilon$. Let $\mathbf{u}^* = \arg\max_{\mathbf{u} \in \mathcal{C}}|(\boldsymbol{\xi} \circ f(\mathbf{X}^{\top}\mathbf{u}))^{\top}\mathbf{X}\mathbf{u}|$ and $\mathbf{v}^* = \arg\max_{\mathbf{v} \in \mathbb{S}^{d-1}}|(\boldsymbol{\xi} \circ f(\mathbf{X}^{\top}\mathbf{u}))^{\top}\mathbf{X}\mathbf{v}|$. We then have from the triangle inequality,

$$|(\boldsymbol{\xi} \circ f(\mathbf{X}^{\top}\mathbf{u}))^{\top}\mathbf{X}\mathbf{v}^* - (\boldsymbol{\xi} \circ f(\mathbf{X}^{\top}\mathbf{w}))^{\top}\mathbf{X}\mathbf{u}^*| \leq |(\boldsymbol{\xi} \circ f(\mathbf{X}^{\top}\mathbf{u}))^{\top}\mathbf{X}|\|\mathbf{u}^* - \mathbf{v}^*\|_2 \leq \varepsilon \cdot |(\boldsymbol{\xi} \circ f(\mathbf{X}^{\top}\mathbf{u}))^{\top}\mathbf{X}|$$

where in the final inequality we use the definition of a ε-net. We then have from the reverse triangle inequality,

$$|(\boldsymbol{\xi} \circ f(\mathbf{X}^{\top}\mathbf{u}))^{\top}\mathbf{X}\mathbf{u}^*| \geq |(\boldsymbol{\xi} \circ f(\mathbf{X}^{\top}\mathbf{u}))^{\top}\mathbf{X}\mathbf{v}^*| - |(\boldsymbol{\xi} \circ f(\mathbf{X}^{\top}\mathbf{u}))^{\top}\mathbf{X}\mathbf{u}^* - (\boldsymbol{\xi} \circ f(\mathbf{X}^{\top}\mathbf{u}))^{\top}\mathbf{X}\mathbf{v}^*|$$

$$\geq (1 - \varepsilon)|(\boldsymbol{\xi} \circ f(\mathbf{X}^{\top}\mathbf{u}))^{\top}\mathbf{X}\mathbf{v}^*|$$

From rearranging, we have

$$|(\boldsymbol{\xi} \circ f(\mathbf{X}^\top \mathbf{u}))^\top \mathbf{X} \mathbf{v}^*| \leq \frac{1}{1-\varepsilon} \cdot |(\boldsymbol{\xi} \circ f(\mathbf{X}^\top \mathbf{u}))^\top \mathbf{X} \mathbf{u}^*|$$

With this result we are ready to make the probabilistic bounds. Suppose $\mathcal{S}$ represents all subsets of $[N]$ of size $(1-2\epsilon)N$ to $(1-\epsilon)N$. Then,

$$\mathbf{Pr}\left\{ \frac{1}{N} \cdot \max_{S \in \mathcal{S}} \max_{\mathbf{u} \in \mathcal{M}} \|(\boldsymbol{\xi}_S \circ f(\mathbf{X}_S^\top \mathbf{u}))^\top \mathbf{X}_S\| \geq \frac{t}{2} \right\}$$

$$\leq \frac{1}{1-\varepsilon} \cdot \mathbf{Pr}\left\{ \frac{1}{N} \cdot \max_{S \in \mathcal{S}} \max_{\mathbf{v} \in \mathcal{C}} \max_{\mathbf{u} \in \mathcal{M}} |(\boldsymbol{\xi}_S \circ f(\mathbf{X}_S^\top \mathbf{w}))^\top \mathbf{X}_S \mathbf{v}| \geq \frac{t}{2} \right\}$$

$$\leq \frac{1}{1-\varepsilon} \cdot \sum_{i \in [|\mathcal{M}|]} \sum_{j \in [|\mathcal{S}|]} \sum_{k \in [|\mathcal{C}|]} \mathbf{Pr}\left\{ \frac{1}{N} \cdot |(\boldsymbol{\xi}_{S_j} \circ f(\mathbf{X}_{S_j}^\top \mathbf{u}_i)^\top \mathbf{X}_S \mathbf{v}_k| \geq \frac{t}{2} \right\}$$

$$\leq |\mathcal{M}| \cdot 5^d \cdot \left(\frac{e}{\epsilon} \right)^{N\epsilon} \exp\left[-\frac{3N}{4}\left(\frac{t^2}{\frac{4}{3} \cdot C_L^2 C_K^2 C_\sigma^2} \wedge \frac{t}{\frac{4}{3} \cdot C_L C_K C_\sigma} \right) \right] \leq \frac{\delta}{2}$$

Suppose $\mathcal{C}$ is a $1/4$-net. We can now apply Bernstein's Inequality from [RH23] Theorem 1.13. We can now note that $\log \binom{N}{(1-\epsilon)N} = \log \binom{N}{\epsilon N}$. Then to satisfy the above probabilistic condition, it must hold that

$$t \geq \frac{4}{3} \cdot NC_L C_K C_\sigma \cdot \left(\frac{2d}{N}\log(5) + \frac{2Rd}{N}\log(12) + \frac{2}{N}\log(2/\delta) + 6\epsilon\log\epsilon^{-1} \right)^{1/2}$$

We then have with probability exceeding $1 - \delta/2$,

$$\max_{S \in \mathcal{S}} \|(\boldsymbol{\xi}_S \circ f(\mathbf{X}_S^\top \widetilde{\mathbf{w}})^\top \mathbf{X}_S\|_2 \leq \frac{4}{3} \cdot NC_L C_K C_\sigma \cdot \left(\frac{2d}{N}\log(5) + \frac{2Rd}{N}\log(12) + \frac{2}{N}\log(2/\delta) + 6\epsilon\log\epsilon^{-1} \right)^{1/2}$$

where in the above we use the fact that $\log \binom{N}{(1-\epsilon)N} = \log \binom{N}{\epsilon N}$. We now bound the second term.

$$\sup_{\mathbf{w} \in \mathcal{B}(\mathbf{0}, R)} \Big\| \sum_{i \in S} \xi_i f(\mathbf{x}_i^\top \mathbf{w}) \mathbf{x}_i - \sum_{i \in S} \xi_i f(\mathbf{x}_i^\top \widetilde{\mathbf{w}}) \mathbf{x}_i \Big\|_2 \leq (1-\epsilon)N \sup_{\mathbf{w} \in \mathcal{B}(\mathbf{0}, R)} \sup_{i \in S} \|\xi_i \mathbf{x}_i\| \|f(\mathbf{x}_i^\top \mathbf{w}) - f(\mathbf{x}_i^\top \widetilde{\mathbf{w}})\|_2$$

$$\leq (1-\epsilon)NL \sup_{i \in S}\|\xi_i \mathbf{x}_i\|_2 \sup_{j \in S}\|\mathbf{x}_j\|_2 \cdot \varepsilon$$

$$\overset{(i)}{\leq} \frac{(1-\epsilon)}{2}NL\varepsilon \cdot \left(\sup_{i \in S}\|\xi_i \mathbf{x}_i\|_2^2 + \sup_{j \in S}\|\mathbf{x}_j\|_2^2 \right)$$

In the above, in (i) we use Young's Inequality (see Proposition 31). Note that $\mathbf{x}_i$ is sub-Gaussian, we thus have,

$$\mathbf{Pr}\left\{ \max_{j \in S}\|\mathbf{x}_j\|_2^2 \geq t \right\} \leq \sum_{i \in \mathcal{C}} \sum_{j \in [(1-\epsilon)N]} \mathbf{Pr}\left\{ |\mathbf{x}_j^\top \mathbf{v}_i| \geq \sqrt{t} \right\} \leq (1-\epsilon)N 5^d \exp\left(-\frac{t}{8C_K^2 \|\boldsymbol{\Sigma}\|^2} \right) \leq \frac{\delta}{2(1-\epsilon)NL\varepsilon}$$

where in the above, the final inequality follows when

$$t \geq 8C_K^2 \|\boldsymbol{\Sigma}\|_2^2 (\log(2/\delta) + 2\log(NL\varepsilon) + d\log(5))$$

To bound $\sup_{i \in S}\|\xi_i \mathbf{x}_i\|_2$, we note that for any $\mathbf{v} \in \mathbb{R}^d$ that $\xi_i \mathbf{x}_i^\top \mathbf{v}$ is sub-exponential with norm $\|\xi_i \mathbf{x}_i^\top \mathbf{v}\|_{\psi_1} \leq 2C_\sigma C_K$. Then with Bernstein's Inequality,

$$\mathbf{Pr}\left\{ \max_{i \in S}\|\xi_i \mathbf{x}_i\| \geq t \right\} \leq \sum_{i \in \mathcal{C}} \sum_{j \in [(1-\epsilon)N]} \mathbf{Pr}\left\{ |\xi_j \mathbf{x}_j^\top \mathbf{v}_i| \geq t \right\}$$

$$\leq (1-\epsilon)N 5^d \exp\left(-\frac{1}{2}\left(\frac{t^2}{4C_\sigma^2 C_K^2} \wedge \frac{t}{2C_\sigma C_K} \right) \right) \leq \frac{\delta}{2(1-\epsilon)NL\varepsilon}$$

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The final inequality holds when,

$$t \geq 4C_\sigma^2 C_K^2 \|\boldsymbol{\Sigma}\|_2^2 (\log(2/\delta) + 2\log(NL\varepsilon) + 2d\log(5))$$

Combining our estimates, we obtain,

$$\mathbf{Pr}\left\{\max_{S\in\mathcal{S}} \|(\boldsymbol{\xi}_S \circ f(\mathbf{X}_S^\top \mathbf{w}))^\top \mathbf{X}_S\| \geq t\right\} \leq \delta$$

when $t \geq \frac{4}{3} \cdot NC_L C_K C_\sigma \cdot \left(\frac{2d}{N}\log(5) + \frac{2Rd}{N}\log(12) + \frac{2}{N}\log(2/\delta) + 6\epsilon\log\epsilon^{-1}\right)^{1/2}$. ∎

Lemma 26. *Let $\mu(\mathbf{x})$ represent the PDF of $\mathcal{N}(\mathbf{0},\mathbf{I})$. Fix $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^d$ and define the sets $\Omega_1 = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}^\top \mathbf{w}_1 \geq 0\}$ and $\Omega_2 = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}^\top \mathbf{w}_2 \geq 0\}$, let $\Theta = \arccos\left(\frac{\mathbf{w}_1^\top \mathbf{w}_2}{\|\mathbf{w}_1\|\|\mathbf{w}_2\|}\right)$, then*

$$\lambda_{\min}\left(\int_{\Omega_1,\Omega_2} \mathbf{x}\mathbf{x}^\top d\mu(\mathbf{x})\right) \succeq (1 \wedge (\pi - \Theta - \sin\Theta)) \cdot \mathbf{I}$$

Proof. From Lemma 1 in [DLT+18], we have for $\mathbf{U} = \left[\overline{\mathbf{w}}_1, \frac{\overline{\mathbf{w}}_2 - \cos\Theta(\overline{\mathbf{w}}_1,\overline{\mathbf{w}}_2)}{\sin\Theta(\overline{\mathbf{w}}_1,\overline{\mathbf{w}}_2)}, \mathbf{u}_3, \ldots, \mathbf{u}_n\right]^\top$.

$$\mathbf{E}\left[\mathbf{x}\mathbf{x}^\top \mid \mathbf{x}^\top \mathbf{w}_1 \geq 0, \mathbf{x}^\top \mathbf{w}_2 \geq 0\right] = \mathbf{E}\left[\mathbf{U}^\top \mathbf{U}\mathbf{x}\mathbf{x}^\top \mathbf{U}^\top \mathbf{U} \mid \mathbf{x}^\top \mathbf{w}_1 \geq 0, \mathbf{x}^\top \mathbf{w}_2 \geq 0\right]$$
$$= \mathbf{U}^\top \cdot \mathbf{E}\left[\mathbf{U}\mathbf{x}\mathbf{x}^\top \mathbf{U}^\top \mid \mathbf{x}^\top \mathbf{w}_1 \geq 0, \mathbf{x}^\top \mathbf{w}_2 \geq 0\right] \cdot \mathbf{U}$$
$$= \mathbf{U}^\top \cdot \begin{bmatrix} \frac{1}{2\pi}(\pi - \Theta + \sin\Theta\cos\Theta) & \frac{1}{2\pi}\sin^2\Theta & 0 & \cdots & 0 \\ \frac{1}{2\pi}\sin^2\Theta & \frac{1}{2\pi}(\pi - \Theta + \sin\Theta\cos\Theta) & & & 0 \\ 0 & & 1 & & \vdots \\ \vdots & & & \ddots & \\ 0 & & & & 1 \end{bmatrix} \cdot \mathbf{U}$$

The matrix has $n-2$ unitary eigenvalues and the final two eigenvalues can calculated to be $\pi - \Theta \pm \sin\Theta$. We then have

$$\lambda_{\min}\left(\mathbf{E}\left[\mathbf{x}\mathbf{x}^\top \mid \mathbf{x}^\top \mathbf{w}_1 \geq 0, \mathbf{x}^\top \mathbf{w}_2 \geq 0\right]\right) \geq 1 \wedge (\pi - \Theta - \sin\Theta)$$

Our proof is complete. ∎

Lemma 27. *Fix $\mathbf{w}^* \in \mathbb{R}^{d-1}$ and suppose $\mathbf{w} \in \mathcal{B}(\mathbf{w}^*, R)$ for a constant $R < \|\mathbf{w}^*\|$. Sample $\mathbf{x}_1, \ldots, \mathbf{x}_N$ i.i.d from a sub-Gaussian distribution with second-moment matrix $\boldsymbol{\Sigma}$ and sub-Gaussian norm C_K. Suppose S represents a subset of $[N]$ s.t. $(1-2\epsilon)N \leq |S| \leq (1-\epsilon)N$. Let $\Omega \subseteq \mathbb{R}$, then with probability $1-\delta$,*

$$\left\|\sum_{i\in S} \mathbf{x}_i\mathbf{x}_i^\top \cdot \mathbb{I}\{\mathbf{x}_i^\top \mathbf{w}^* \geq 0\} - \mathbf{E}[\mathbf{x}\mathbf{x}^\top \cdot \mathbb{I}\{\mathbf{x}_i^\top \mathbf{w}^* \geq 0\}]\right\|_2 \leq \Xi$$

Proof. We will use the decomposition given in [ZYWG19]. Let $\widetilde{\mathbf{w}} = \arg\min_{\mathbf{v}\in\mathcal{C}} \|\mathbf{w} - \mathbf{v}\|_2$ where $\mathcal{C}$ is a ε-cover and let $\mathcal{S}$ be defined as in the Lemma statement.

$$\mathbf{Pr}\left\{\max_{S\in\mathcal{S}} \sup_{\mathbf{w}\in\mathcal{B}(\mathbf{w}^*;R)} \|\boldsymbol{\Sigma}_S(\mathbf{w},\mathbf{w}^*) - \mathbf{E}[\boldsymbol{\Sigma}_S(\mathbf{w},\mathbf{w}^*)]\|_2 \geq t\right\}$$

$$\leq \mathbf{Pr}\left\{\max_{S\in\mathcal{S}} \sup_{\mathbf{w}\in\mathcal{B}(\mathbf{w}^*;R)} \|\boldsymbol{\Sigma}_S(\mathbf{w},\mathbf{w}^*) - \boldsymbol{\Sigma}_S(\widetilde{\mathbf{w}},\mathbf{w}^*)\|_2 \geq \frac{t}{3}\right\}$$

$$+ \mathbf{Pr}\left\{\max_{S\in\mathcal{S}} \sup_{\widetilde{\mathbf{w}}\in\mathcal{C}} \|\boldsymbol{\Sigma}_S(\widetilde{\mathbf{w}},\mathbf{w}^*) - \mathbf{E}[\boldsymbol{\Sigma}_S(\widetilde{\mathbf{w}},\mathbf{w}^*)]\|_2 \geq \frac{t}{3}\right\}$$

$$+ \mathbf{Pr}\left\{\sup_{\mathbf{w}\in\mathcal{B}(\mathbf{w}^*;R)} \|\mathbf{E}[\boldsymbol{\Sigma}_S(\widetilde{\mathbf{w}},\mathbf{w}^*)] - \mathbf{E}[\boldsymbol{\Sigma}_S(\mathbf{w},\mathbf{w}^*)]\|_2 \geq \frac{t}{3}\right\}$$

We bound all terms separately.

$$\sup_{\mathbf{w}\in\mathcal{B}(\mathbf{w}^*;R)} \|\mathbf{\Sigma}_{\mathrm{S}}(\mathbf{w},\mathbf{w}^*) - \mathbf{\Sigma}_{\mathrm{S}}(\widetilde{\mathbf{w}},\mathbf{w}^*)\|_2 \leq \sup_{\mathbf{w}\in\mathcal{B}(\mathbf{w}^*;R)} \|\mathbf{\Sigma}_{\mathrm{S}}(\mathbf{w},-\widetilde{\mathbf{w}})\|_2 + \sup_{\mathbf{w}\in\mathcal{B}(\mathbf{w}^*;R)} \|\mathbf{\Sigma}_{\mathrm{S}}(-\mathbf{w},\widetilde{\mathbf{w}})\|_2$$

∎

Lemma 28. *Fix $\mathbf{S}\in\mathbb{R}^{K\times N\epsilon}, \mathbf{T}\in\mathbb{R}^{N\epsilon\times L}$, then sample a matrix $\mathbf{G}\in\mathbb{R}^{N\epsilon\times N\epsilon}$ such that each column of $\mathbf{G}$ represents a ϵ-subset of a n-dimensional vector sampled from $\mathcal{N}(\mathbf{0},\mathbf{I})$, then with probability exceeding $1-\delta$,*

$$\|\mathbf{SGT}\|_{\mathrm{F}}^2 \leq \|\mathbf{S}\|_{\mathrm{F}}^2 \|\mathbf{T}\|_{\mathrm{F}}^2 \cdot \sigma \sqrt{2\log\left(\frac{2N}{\delta}\right)}$$

Proof. The proof will be a calculation.

$$\|\mathbf{SGT}\|_{\mathrm{F}}^2 = \sum_{i\in[K]}\sum_{j\in[L]}\sum_{k_1,k_2\in[N\epsilon]\times[N\epsilon]} (\mathbf{S}_{i,k_1}\mathbf{G}_{k_1,k_2}\mathbf{T}_{k_2,j})^2 \leq \|\mathbf{S}\|_{\mathrm{F}}^2 \|\mathbf{T}\|_{\mathrm{F}}^2 \max_{i,j\in[N\epsilon]\times[N\epsilon]}(\mathbf{G}_{i,j})^2$$

It then suffices to bound the maximum value of a Gaussian squared over N^2 samples. From [RH23], we have

$$\mathbf{Pr}\left\{\max_{i,j} G_{i,j}^2 \geq t\right\} = \mathbf{Pr}\left\{\max_{i,j}|G_{i,j}| \geq \sqrt{t}\right\} \leq 2Ne^{-\frac{t}{2\sigma^2}}$$

We thus obtain from elementary inequalities, with probability at least $1-\delta$,

$$\max_{i,j} G_{i,j}^2 \leq \sigma\left(2\log\left(\frac{2N^2}{\delta}\right)\right)$$

Our proof is complete. ∎

D Mathematical Tools

In this section, we state additional lemmas referenced throughout the text for completeness.

Lemma 29. *Let $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$ and $\mathbf{X}\in\mathbb{R}^{p\times n}$, then*

$$\Big\|\sum_{i=1}^n a_i b_i \mathbf{x}_i\Big\|^2 \leq \|\mathbf{a}\|_\infty^2 \|\mathbf{b}\|_2^2 \|\mathbf{X}\mathbf{X}^\top\|_2$$

Proof. The proof is a simple calculation. Expanding out the LHS, we have

$$\Big\|\sum_{i=1}^n a_i b_i \mathbf{x}_i\Big\|_2^2 = \sum_{i=1}^n\sum_{j=1}^n a_i a_j b_i b_j \mathbf{x}_i^\top \mathbf{x}_j = (\mathbf{a}\circ\mathbf{b})^\top \mathbf{X}^\top\mathbf{X}(\mathbf{a}\circ\mathbf{b}) \leq \|\mathbf{a}\circ\mathbf{b}\|_2^2 \|\mathbf{X}^\top\mathbf{X}\|_2 \leq \|\mathbf{a}\|_\infty^2\|\mathbf{b}\|_2^2 \|\mathbf{X}^\top\mathbf{X}\|_2$$

where the final inequality comes from noting

$$\|\mathbf{a}\circ\mathbf{b}\|^2 = \sum_{i=1}^n a_i^2 b_i^2 \leq \max_{i\in[n]} a_i^2 \cdot \sum_{i=1}^n b_i^2$$

Our proof is complete. ∎

Lemma 30 (Lemma 3.11 [B$^+$15])**.** *Let f be β-smooth and α-strongly convex over $\mathbb{R}^n$, then for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$,*

$$\langle\nabla f(\mathbf{x}) - \nabla f(\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle \geq \frac{\alpha\beta}{\alpha+\beta}\|\mathbf{x}-\mathbf{y}\|^2 + \frac{1}{\alpha+\beta}\|\nabla f(\mathbf{x})-\nabla f(\mathbf{y})\|^2$$

Proposition 31 (Young's Inequality [You12]). *Suppose $a, b \in \mathbb{R}_+$, then for $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Lemma 32 (Sum of Binomial Coefficients [CLRS22]). *Let $k, n \in \mathbb{N}$ such that $k \leq n$, then*

$$\sum_{i=0}^{k} \binom{n}{i} \leq \left(\frac{en}{k}\right)^k$$

Lemma 33 (Max Gaussian [RH23]). *Let $x_1, \ldots, x_n$ be sampled i.i.d from $\mathcal{N}(0, \sigma^2)$. Then,*

$$\mathbf{Pr}\{\|\mathbf{x}\|_\infty > t\} \leq N \exp\left(-\frac{t^2}{\frac{16}{3}\sigma^2}\right)$$