Upper Bounds on the Spectral Norm of the Pseudo-Inverse of Non-Standard Gaussian Matrices

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Abstract

In this paper we explore upper bounds on the spectral norm for Gaussian Matrices with columns standard from Central Correlated Multivariate Normal Distributions. We utilize a lemma from [Chi17, CWS09] and extend the analysis from [CD05]. These bounds find applications in the generalization of the randomized SVD given in [BT22] and wireless network science.

1 Introduction

The study of the expectation of the norms of the pseudoinverse of standard normal gaussian matrices first appeared in [HMT11] when analyzing the error bounds for the Randomized SVD algorithm. The bounds developed in [HMT11] used theory developed in analyzing the condition numbers of standard normal matrices in [CD05]. In a generalization of the Randomized SVD, the need for bounds on the expectation of the spectral norm for correlated Gaussian matrices appears in [BT22].

2 Relevant Work in Standard Uncorrelated Matrices

In this section we will briefly discuss bounds developed for the inequalities of standard normal matrices.

Proposition 1. *(HMT Proposition 10.2). Draw a $k \times (k+p)$ standard Gaussian matrix $\mathbf{G}$ with $k \geq 2$ and $p \geq 2$. Then*

$$\mathbb{E}\left\|\mathbf{G}^{\dagger}\right\| \leq \frac{e\sqrt{k+p}}{p} \tag{1}$$

From our search in the literature, there is no bound on equation 1 when the columns are not sampled from a multiple of the identity.

3 Theory

We will first introduce the necessary lemmas needed to prove our main results.

3.1 Necessary Lemmas

Lemma 2. *[Jam64, Eq. (58,59)]. If $\lambda_1 \geq \ldots \geq \lambda_m$ are the eigenvalues of $\mathbf{W}$ s.t. $\mathbf{W} \sim \mathcal{W}_m(n, \mathbf{C})$ s.t. $n > m-1$, then the joint PDF of eigenvalues is*

$$f(\lambda_1, \ldots, \lambda_m) = K_{m,n} \left(\det \mathbf{C}\right)^{-n/2} \exp\left(-\frac{1}{2}\operatorname{Tr}\left(\mathbf{C}^{-1}\mathbf{W}\right)\right) \prod_{i=1}^{m} \lambda_i^{(n-m-1)/2} \prod_{i<j} (\lambda_i - \lambda_j) \tag{2}$$

where

$$K_{m,n} = \frac{\pi^{m^2/2}}{\Gamma_m\left(\frac{1}{2}m\right)\Gamma_m\left(\frac{1}{2}n\right)} \tag{3}$$

Lemma 3. *[WLRT08, Lemma 3.6]. Let $m, n \in \mathbb{N}$ s.t. $n \geq m$. Suppose $\mathbf{A} \in \mathbb{R}^{n \times m}$, then if $(\mathbf{A}^\top \mathbf{A})$ is invertible*

$$\left\| \left(\mathbf{A}^\top \mathbf{A}\right)^{-1} \mathbf{A}^\top \right\| = \frac{1}{\sigma_m(\mathbf{A})} \tag{4}$$

Lemma 4. *[Chi17, Lemma 1]. Draw a $m \times n$ matrix $\mathbf{G}$ s.t. the columns of $\mathbf{G}$ are sampled from $\mathcal{N}_m(\mathbf{0}, \mathbf{C})$ where the eigenvalues of $\mathbf{C}$ are represented as $\sigma_1 > \sigma_2 > \cdots > \sigma_m$. Let $\mathbf{W} \sim \mathcal{W}_m(n, \mathbf{C})$. The eigenvalue distribution is given as*

$$f(x_1, \ldots, x_n) = K_{\mathbf{C}} \left|\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})\right| \cdot \prod_{i=1}^{m-1} \prod_{j=i+1}^{m} (x_i - x_j) \prod_{i=1}^{n} x_i^{n-m} \tag{5}$$

where $\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma}) = \left\{e^{-x_i/\sigma_j}\right\}_{i,j=1}^{m} = \begin{bmatrix} e^{-\frac{x_1}{\sigma_1}} & \ldots & e^{-\frac{x_1}{\sigma_m}} \\ \vdots & \ddots & \vdots \\ e^{-\frac{x_m}{\sigma_1}} & \ldots & e^{-\frac{x_m}{\sigma_m}} \end{bmatrix}$ *and*

$$K_{\mathbf{C}}^{-1} = \prod_{i=1}^{m-1} \prod_{j=i+1}^{m} (\sigma_i - \sigma_j) \prod_{i=1}^{m} \sigma_i^{n-m+1} (n-i)! \tag{6}$$

With these lemmas we will go to proving the main results.

3.2 Main Results

Theorem 5. *Draw a $m \times m$ matrix $\mathbf{G}$ s.t. the columns of $\mathbf{G}$ are sampled from $\mathcal{N}_m(\mathbf{0}, \mathbf{C})$ where the eigenvalues of $\mathbf{C}$ are represented as $\sigma_1 > \sigma_2 > \cdots > \sigma_m$. Then*

$$\mathbb{E} \left\| \mathbf{G}^\dagger \right\| \leq \sqrt{\pi \sum_{k=1}^{m} \frac{1}{\sigma_k}} \tag{7}$$

Proof. We will first note

$$\left\| \mathbf{G}^\dagger \right\| \overset{\text{lem. 3}}{=} \frac{1}{\sigma_m(\mathbf{G})} = \frac{1}{\sqrt{\lambda_{\min}(\mathbf{G}\mathbf{G}^\top)}} \tag{8}$$

For $\mathbf{W}$ sampled from $\mathcal{W}_m(m, \mathbf{C})$. We will now derive the distribution for minimum eigenvalue of $\mathbf{W}$ similar to [NZYY08].

$$f_{\lambda_{\min}}(x_m) = \int_{x_2}^{\infty} \cdots \int_{x_{m-1}}^{\infty} K_{\mathbf{C}} \left|\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})\right| \cdot \prod_{i<j}^{m} (x_i - x_j) \prod_{i=1}^{m} x_j^{m-m} \prod_{i=1}^{m-1} dx_i \tag{9}$$

$$= K_{\mathbf{C}} \int_{x_2}^{\infty} \cdots \int_{x_{m-1}}^{\infty} \left|\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})\right| \cdot \prod_{i=1}^{m-2} \prod_{j=i+1}^{m-1} (x_i - x_j) \prod_{i=1}^{m-1} (x_i - x_m) \prod_{i=1}^{m-1} dx_i \tag{10}$$

$$\overset{(a)}{=} e^{-\sum_{i=1}^{m} \frac{x_m}{\sigma_i}} \left(\int_{y_2}^{\infty} \cdots \int_{y_{m-1}}^{\infty} \sum_{i=1}^{m} (-1)^{i+m} K_{\mathbf{C}} \left|\mathbf{E}_i\left(\mathbf{x} - \mathbf{x}_m, \boldsymbol{\sigma}\right)\right| \prod_{i=1}^{m-2} \prod_{j=i+1}^{m-1} (y_i - y_j) \prod_{i-1}^{m-1} dy_i \right) \tag{11}$$

$$\overset{(b)}{=} \Xi e^{-\sum_{i=1}^{m} \frac{x_m}{\sigma_i}} \tag{12}$$

(a) follows due to the properties of the determinant. (b) follows as the intergral expresion in Equation (11) no longer integrates over x_m and thus integrates to some constant we define as Ξ. Since the PDF must integrate to 1, we thus have,

$$f_{\lambda_{\min}}(x) = \left(\sum_{k=1}^{m} \frac{1}{\sigma_k} \right) e^{-x \sum_{k=1}^{m} \frac{1}{\sigma_k}} \tag{13}$$

The Expected Value follows from a simple integration.

$$\mathbb{E}\left\|\mathbf{G}^{\dagger}\right\| = \int_{0}^{\infty} \frac{1}{\sqrt{x}} e^{-x\sum_{k=1}^{m}\sigma_k^{-1}} dx \tag{14}$$

$$= \sqrt{\pi \sum_{k=1}^{m} \frac{1}{\sigma_k}} \operatorname{erf}\left(\sqrt{\pi \sum_{k=1}^{m} \frac{1}{\sigma_k}}\right) \leq \sqrt{\pi \sum_{k=1}^{m} \frac{1}{\sigma_k}} \tag{15}$$

The proof is complete. ■

In our next theorem, we will consider the matrix is rectangle and all the singular values of the covariance matrix are distinct.

Theorem 6. *Draw a $m \times n$ matrix $\mathbf{G}$ s.t. the columns of $\mathbf{G}$ are sampled from $\mathcal{N}_m(\mathbf{0}, \mathbf{C})$ where the eigenvalues of $\mathbf{C}$ are represented as $\sigma_1 > \sigma_2 > \cdots > \sigma_m > 0$. Let $\mathbf{W} \sim \mathcal{W}_m(n, \mathbf{C})$. Let $\gamma \triangleq \min_{k \in [m-1]}(\sigma_k - \sigma_{k+1})$ and $p \triangleq n - m + 1$, then*

$$\mathbb{E}\left\|\mathbf{G}^{\dagger}\right\| \leq \gamma^m 2\sqrt{\frac{\pi}{e}} \left(\frac{p}{e}\right)^p \sum_{i=1}^{m} \frac{1}{\sqrt{\sigma_i}} \tag{16}$$

Proof. Let $K_{\mathbf{C}}$ and $\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})$ be defined as in Lemma 4.

$$f_{\lambda_{\min}}(x_m) = \int_{x_2}^{\infty} \cdots \int_{x_{m-1}}^{\infty} K_{\mathbf{C}} \left|\mathbf{E}(\mathbf{x}, \boldsymbol{\sigma})\right| \cdot \prod_{i<j}^{m} (x_i - x_j) \prod_{i=1}^{m} x_j^{n-m} \prod_{i=1}^{m-1} dx_i \tag{17}$$

$$= K_{\mathbf{C}} x_m^{n-m} \int_{x_2}^{\infty} \cdots \int_{x_{m-1}}^{\infty} \left|\mathbf{E}\left(\mathbf{x}, \boldsymbol{\sigma}\right)\right| \cdot \prod_{i<j}^{m-1} (x_i - x_j) \prod_{i=1}^{m-1} (x_i - x_m) \prod_{i=1}^{m} x_i^{n-m} \prod_{i=1}^{m-1} dx_i \tag{18}$$

$$\leq K_{\mathbf{C}} x_m^{n-m} \int_{x_2}^{\infty} \cdots \int_{x_{m-1}}^{\infty} \left|\mathbf{E}\left(\mathbf{x}, \boldsymbol{\sigma}\right)\right| \cdot \prod_{i<j}^{m-1} (x_i - x_j) \prod_{i=1}^{m-1} x_i^{n-m+1} \prod_{i=1}^{m-1} dx_i \tag{19}$$

$$\leq K_{\mathbf{C}} x_m^{n-m} \sum_{i=1}^{m} \left((-1)^{i+m} e^{-\frac{x_m}{\sigma_i}} \int_{x_2}^{\infty} \cdots \int_{x_{m-1}}^{\infty} \left|\mathbf{E}_{m,i}(\mathbf{x}, \boldsymbol{\sigma})\right| \cdot \prod_{i<j}^{m-1} (x_i - x_j) \prod_{i=1}^{m-1} x_i^{n-m+1} \prod_{i=1}^{m-1} dx_i \right) \tag{20}$$

$$= x_m^{n-m} \underbrace{K_{\mathbf{C}} \sum_{i=1}^{m} (-1)^{i+m} e^{-\frac{x_m}{\sigma_i}} K_{\mathbf{C},i}^{-1}}_{\Xi} \tag{21}$$

We will now upper bound Ξ.

$$\Xi \triangleq K_{\mathbf{C}} \sum_{i=1}^{m} (-1)^{i+m} e^{-\frac{x_m}{\sigma_i}} K_{\mathbf{C},i}^{-1} \tag{22}$$

$$= \sum_{k=1}^{m} (-1)^{k+m} e^{-\frac{x_m}{\sigma_k}} \frac{\prod_{i=1}^{m-1} \prod_{j=i+1}^{m} \mathbb{I}\left\{i, j \neq k\right\} (\sigma_i - \sigma_j) \prod_{i=1}^{m} \mathbb{I}\left\{i \neq k\right\} \sigma_i^{n-m+1} (n-i)!}{\prod_{i=1}^{m-1} \prod_{j=i+1}^{m} (\sigma_i - \sigma_j) \prod_{i=1}^{m} \sigma_i^{n-m+1} (n-i)!} \tag{23}$$

$$= \sum_{k=1}^{m} (-1)^{k+m} e^{-\frac{x_m}{\sigma_k}} \left(\prod_{i>k}^{m} (\sigma_i - \sigma_k) \prod_{i<k}^{m} (\sigma_k - \sigma_i) \cdot \sigma_k^{n-m+1} (n-k)! \right)^{-1} \tag{24}$$

$$\leq \sum_{k=1}^{m} e^{-\frac{x_m}{\sigma_k}} \underbrace{\left(\prod_{i>k}^{m} (\sigma_i - \sigma_k) \prod_{i<k}^{m} (\sigma_k - \sigma_i) \cdot \sigma_k^{n-m+1} (n-k)! \right)}_{K}^{-1} \tag{25}$$

Now we will lower bound K for $i = k$. Define $\delta_k \triangleq \min\{(\sigma_k - \sigma_{k+1}), (\sigma_{k-1} - \sigma_k)\}$, then

$$K \triangleq \prod_{i>k}^{m} (\sigma_i - \sigma_k) \prod_{i<k}^{m} (\sigma_k - \sigma_i) \cdot \sigma_k^{n-m+1} (n-k)! \tag{26}$$

$$\geq \delta_k^m \sigma_k^{n-m+1} (n-k)! \tag{27}$$

We thus have

$$f_{\lambda_{\min}}(x_m) \leq K x_m^{n-m} \sum_{i=1}^{m} e^{-\frac{x_m}{\sigma_i}} = \mathcal{O}\left(x_m^{n-m} \sum_{i=1}^{m} e^{-\frac{x_m}{\sigma_i}} \right) \tag{28}$$

Now we will integrate over $f_{\lambda_{\min}}(x_m)$.

$$\begin{aligned}
\mathbb{E}\left\|\mathbf{G}^{\dagger}\right\| &= \int_0^{\infty} \mathcal{O}\left(x^{n-m-\frac{1}{2}} \sum_{i=1}^{m} e^{-\frac{x}{\sigma_i}} \right) dx & (29)\\
&= \sum_{i=1}^{m} \mathcal{O}\left(\sigma_i^{n-m+\frac{1}{2}} \Gamma\left(\frac{n-m+\frac{1}{2}}{2} \right) \right) & (30)\\
&\overset{(a)}{\leq} \sum_{i=1}^{m} \mathcal{O}\left(\sigma_i^{n-m+\frac{1}{2}} \sqrt{\frac{4\pi}{n-m+\frac{1}{2}}} \left(\frac{n-m+\frac{1}{2}}{e} \right)^{n-m+\frac{1}{2}} \right) & (31)\\
&= \sum_{i=1}^{m} \mathcal{O}\left(\sigma_i^{n-m+\frac{1}{2}} 2\sqrt{\frac{\pi}{e}} \left(\frac{n-m+\frac{1}{2}}{e} \right)^{n-m} \right) & (32)
\end{aligned}$$

(a) follows from an application of Stirling's Approximation [Rob55]. Now we will plug in the lower bound for K from Equation (27).

$$\mathbb{E}\left\|\mathbf{G}^{\dagger}\right\| \leq \sum_{i=1}^{m} \frac{\sigma_i^{n-m+\frac{1}{2}} 2\sqrt{\frac{\pi}{e}} \left(\frac{n-m+\frac{1}{2}}{e} \right)^{n-m}}{\delta_i^m \sigma_i^{n-m+1} (n-i)!} = \sum_{i=1}^{m} \frac{2\sqrt{\frac{\pi}{e}} \left(\frac{n-m+\frac{1}{2}}{e} \right)^{n-m}}{\delta_i^m \sqrt{\sigma_i} (n-i)!} \tag{33}$$

Define $\gamma \triangleq \min_{k\in[m-1]}$ and the proof is complete. ■

In our next theorem we will consider the matrix is rectangle but the singular values of the covariance are not all distinct.

Theorem 7. *Draw a $m \times n$ matrix $\mathbf{G}$ s.t. the columns of $\mathbf{G}$ are sampled from $\mathcal{N}_m(\mathbf{0}, \mathbf{\Sigma})$ where the eigenvalues of $\mathbf{\Sigma}$ are represented as $\sigma_1 > \sigma_2 > \cdots > \sigma_m$. Let $\mathbf{W} \sim \mathcal{W}_m(n, \mathbf{\Sigma})$. Then,*

$$\mathbb{E}\left\|\mathbf{G}^{\dagger}\right\| \leq \Xi \tag{34}$$

Proof. First, let us represent $\mathbf{W} = \sum_{i=1}^{n} \mathbf{x}\mathbf{x}^{\top}$ where $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$. Then we can lower $\sigma_{\min}(\mathbf{W})$,

$$\begin{aligned}
\sigma_{\min}(\mathbf{W}) &= \sigma_{\min}\left(\sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^{\top} \right) & (35)\\
&= \sigma_{\min}\left(n\mathbb{E}\left[\mathbf{x}\mathbf{x}^{\top}\right] + \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^{\top} - \mathbb{E}\left[\mathbf{x}_i \mathbf{x}_i^{\top}\right] \right) & (36)\\
&\geq n\sigma_{\min}(\mathbf{\Sigma}) - \sigma_{\max}\left(\sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^{\top} - \mathbf{\Sigma} \right) & (37)\\
&= n\sigma_{\min}(\mathbf{\Sigma}) - \sigma_{\max}\left(\sum_{i=1}^{n} \left(\mathbf{\Sigma}^{1/2}\mathbf{v}_i\right) \left(\mathbf{\Sigma}^{1/2}\mathbf{v}_i\right)^{\top} - \mathbf{\Sigma} \right) & (38)\\
&= n\sigma_{\min}(\mathbf{\Sigma}) - \sigma_{\max}\left(\mathbf{\Sigma}^{1/2} \left(\sum_{i=1}^{n} \mathbf{v}_i \mathbf{v}_i^{\top} - \mathbf{I} \right) \mathbf{\Sigma}^{1/2} \right) & (39)\\
&\geq n\sigma_{\min}(\mathbf{\Sigma}) - \underbrace{\sigma_{\max}(\mathbf{\Sigma})\, \sigma_{\max}\left(\sum_{i=1}^{n} \mathbf{v}_i \mathbf{v}_i^{\top} - \mathbf{I} \right)}_{\text{A}} & (40)\\
& & (41)
\end{aligned}$$

There has been significant theory in Random Matrix Theory and High Dimensional Probability analyzing Covariance Estimation, especially in the standard normal case, [T+15, Ver20, Rig15]. We will utilize the Matrix Bernstein inequality to upper bound A in probability and in expectation.

4 Numerical Experiments

In Figure 1, we verify the results given in Theorem 5.

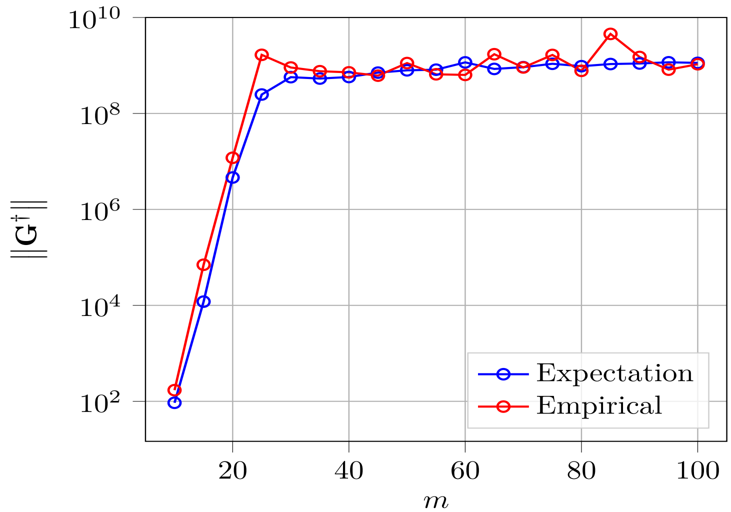

Figure 1: Comparing the expected norm upper bound on $\|\mathbf{G}^\dagger\|$ where $\mathbf{G} \in \mathbb{R}^{m\times m}$ and the columns of $\mathbf{G}$ are sampled from $\mathcal{N}_m(\mathbf{0}, \mathbf{K})$ with the average norm of $\mathbf{G}^\dagger$ over 100 samples. The expected norm is calculated with Proposition 5.

In Figure 2, we verify the results given in Theorem 6.

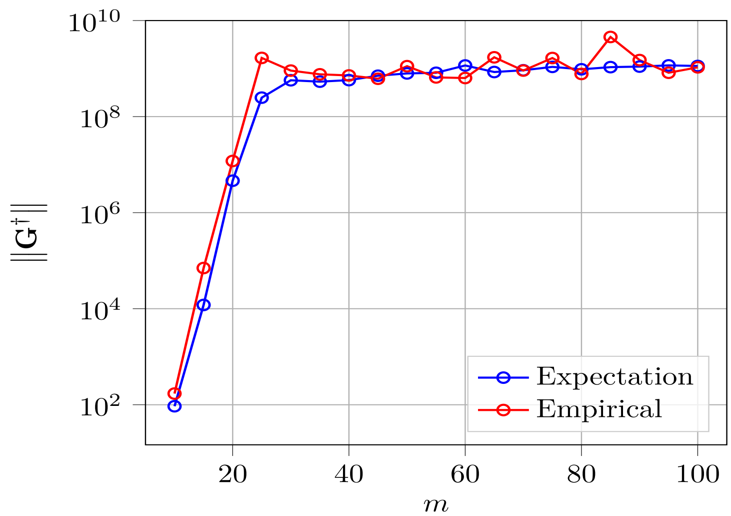

Figure 2: Comparing the expected norm upper bound on $\|\mathbf{G}^\dagger\|$ where $\mathbf{G} \in \mathbb{R}^{m\times m}$ and the columns of $\mathbf{G}$ are sampled from $\mathcal{N}_m(\mathbf{0}, \mathbf{K})$ with the average norm of $\mathbf{G}^\dagger$ over 100 samples. The expected norm is calculated with Proposition 5.

5 Conclusions

In this paper, we derive novel upper bounds for the spectral norm of Gaussian matrices with columns sampled from a central correlated multivariate normal distribution with various distributions of the singular values of the covariance matrix.

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