# CHAPTER 3

## APPROACHES TO THE PROBLEM

In this chapter, two approaches to the problem are going to be analyzed: the Vorticity - Streamfunction formulation and the MAC method.

## 3.1 THE VORTICITY - STREAM FUNCTION FORMULATION

In the next section, we take the Navier-Stokes equations in velocity - pressure form and put them in vorticity - streamfunction formulation.

### 3.1.1 Reduction to the Vorticity - Stream Function Form

As we know from Chapter two, equation (2.14), the incompressible Navier - Stokes momentum equation for a fluid in 2D can be rewriten in the following form:

$$\rho\left(\frac{\partial \vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}\right)=-\nabla p+\mu\nabla^2\vec{u}+\vec{f}, \tag{3.1}$$

The meaning of each term in the equation (3.1) is the following [40]:

$$\begin{array}{lll}
\frac{\partial \vec{u}}{\partial t} & \rightarrow & \text{Unsteady term} \\
\vec{u}\cdot\nabla\vec{u} & \rightarrow & \text{Convective term} \\
\nabla p & \rightarrow & \text{Pressure gradient} \\
\mu\nabla^2\vec{u} & \rightarrow & \text{Viscosity} \\
\vec{f} & \rightarrow & \text{Other body forces}
\end{array}$$

The left hand side of equation (3.1) represents the inertia per unit of volume, the right hand side represents the total force which in turn is conformed by the divergence of stress along with other body forces. The equation (3.1) is nonlinear due to the convective term, it arises from the change in velocity over position [38].

In addition, the viscosity is represented by the vector Laplacian of the vector field, and can be understood as the difference between the velocity at a point and the mean velocity in a small surrounding volume. This implies that for a Newtonian fluid, viscosity represents the diffusion of momentum, as the diffusion of heat in the heat equation, which is also expressed with the Laplacian.

Now, we can express the Navier-Stokes equation for 2-D in Cartesian coordinates removing the $z$ term from the equation (2.13):

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\rho g_x,$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\rho g_y.$$

These equations can be expressed in a more convenient way:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+g_x,$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+g_y.$$

Changing the variables to get the equation in a dimensionless form, we get:

$$\begin{aligned}\tilde{u} &= \tfrac{u}{U}\\ \tilde{v} &= \tfrac{v}{U}\\ \tilde{x} &= \tfrac{x}{L}\\ \tilde{y} &= \tfrac{y}{L}\\ \tilde{t} &= t\tfrac{U}{L}\\ \tilde{p} &= \tfrac{p}{\rho U^2}\end{aligned}$$

Noting that:

$$\nu=\frac{\mu}{\rho},$$

the dimensionless equations become [27]:

$$\frac{U^2}{L}\left(\frac{\partial \tilde{u}}{\partial \tilde{t}}+\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}+\tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}}\right)=-\frac{U^2}{L}\frac{\partial \tilde{p}}{\partial \tilde{x}}+\frac{\nu U}{L^2}\left(\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}+\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right),$$

$$\frac{U^2}{L}\left(\frac{\partial \tilde{v}}{\partial \tilde{t}}+\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}}+\tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}}\right)=-\frac{U^2}{L}\frac{\partial \tilde{p}}{\partial \tilde{y}}+\frac{\nu U}{L^2}\left(\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2}+\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}\right).$$

Substituting the value of the Reynolds number, we get the velocity-pressure formulation of the Navier - Stokes equations in dimensionless form:

$$\begin{cases}\dfrac{\partial \tilde{u}}{\partial \tilde{t}}+\tilde{u}\dfrac{\partial \tilde{u}}{\partial \tilde{x}}+\tilde{v}\dfrac{\partial \tilde{u}}{\partial \tilde{y}}=-\dfrac{\partial \tilde{p}}{\partial \tilde{x}}+\dfrac{1}{Re}\left(\dfrac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}+\dfrac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right),\\ \dfrac{\partial \tilde{v}}{\partial \tilde{t}}+\tilde{u}\dfrac{\partial \tilde{v}}{\partial \tilde{x}}+\tilde{v}\dfrac{\partial \tilde{v}}{\partial \tilde{y}}=-\dfrac{\partial \tilde{p}}{\partial \tilde{y}}+\dfrac{1}{Re}\left(\dfrac{\partial^2 \tilde{v}}{\partial \tilde{x}^2}+\dfrac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}\right).\end{cases}\quad \text{V-P Formulation dimensionless} \tag{3.2}$$

At this point we can differenciate the first equation with respect to $y$, the second with respect to $x$, and suppressing the tildes we get:

$$-\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right],$$

$$\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\right].$$

By adding the last two equations, we can get The Vorticity equation:

$$\frac{\partial \omega}{\partial t}+u\frac{\partial \omega}{\partial x}+v\frac{\partial \omega}{\partial y}+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=\frac{1}{Re}\left(\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y^2}\right). \tag{3.3}$$

We get from the second parenthesis the quantity:

$$\omega=\omega(x,y,t)=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right).$$

We recognize in $\omega$ the vorticity, which comes from the equation (2.5), which depends on the position and time. In addition, from the first parenthesis of the equation (3.3), we can get the incompressibility condition of the flow in 2-D:

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0.$$

Therefore, the product of the two parenthesis in equation (3.3) is zero. The Stream Function $\psi=\psi(x,y,t)$ is defined in the following way:

$$u=\frac{\partial \psi}{\partial y}, v=-\frac{\partial \psi}{\partial x}; \tag{3.4}$$

which automatically satisfies the incompressibility condition, equation (2.7):

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

$$\Longrightarrow \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)=0.$$

It also ensures that the streamfunction $\psi$ be continuous on its mixed second derivative.

This expression can also be obtained from the continuity Equation 2.6. In our case, the density does not vary over time (for water its value is $1[gr/cm^3]$). Also, that implies:

$$\vec{J}=\rho\vec{u}=\vec{u}.$$

Therefore:

$$\nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Now, taking into account the definition of vorticity by equation (2.5), along with the definition of streamfunction in 2-D given by equation (2.7), we can state:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega.$$

Therefore, the set of Navier-Stokes partial differential equations in vorticity streamfunction formulation for incompressible fluid in 2-D, becomes:

$$\begin{cases} \dfrac{\partial \omega}{\partial t} = -\dfrac{\partial \psi}{\partial y}\dfrac{\partial \omega}{\partial x} + \dfrac{\partial \psi}{\partial x}\dfrac{\partial \omega}{\partial y} + \dfrac{1}{Re}\left(\dfrac{\partial^2 \omega}{\partial x^2} + \dfrac{\partial^2 \omega}{\partial y^2}\right), \\ -\omega = \dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2}. \end{cases} \qquad \text{V-SF Formulation} \qquad (3.5)$$

The first equation is well-known as the advection/diffusion equation, and the second is called the elliptic equation.

### 3.1.2 Applying Finite Difference Schemes

The Lid - Driven Cavity problem in 2-D consist on a rectangular box fill with a fluid, in which three walls are fixed (left, right and bottom), the top wall is moving at a constant velocity. In this work, the fluid to be considered is water at room temperature ($300^o K$), density $\rho = 1[gr/cm^3]$. The box to be considered is a square of unit length, the constant velocity of the top wall is $U_0 = 1[m/s]$.

### 3.1.3 Boundary Conditions

Figure (3.1) shows that the flow has four boundary conditions [28], fixed by the geometrical shape of the cavity. In this case, we are going to explore the case in which the cavity is a square shape of length equal to one. At the right and the left boundary, the $x$-component of the velocity, obviously, must be equal to zero. Therefore:

$$u = 0 \quad \Rightarrow \quad \frac{\partial \psi}{\partial y} = 0, \quad \Rightarrow \psi = f_0(x).$$

In the same way, at first instance, at the top and the bottom of the square cavity, the $y$-component of the velocity must be equal to zero. Therefore:

$$v = 0 \quad \Rightarrow \quad \frac{\partial \psi}{\partial x} = 0, \quad \Rightarrow \psi = f_1(y).$$

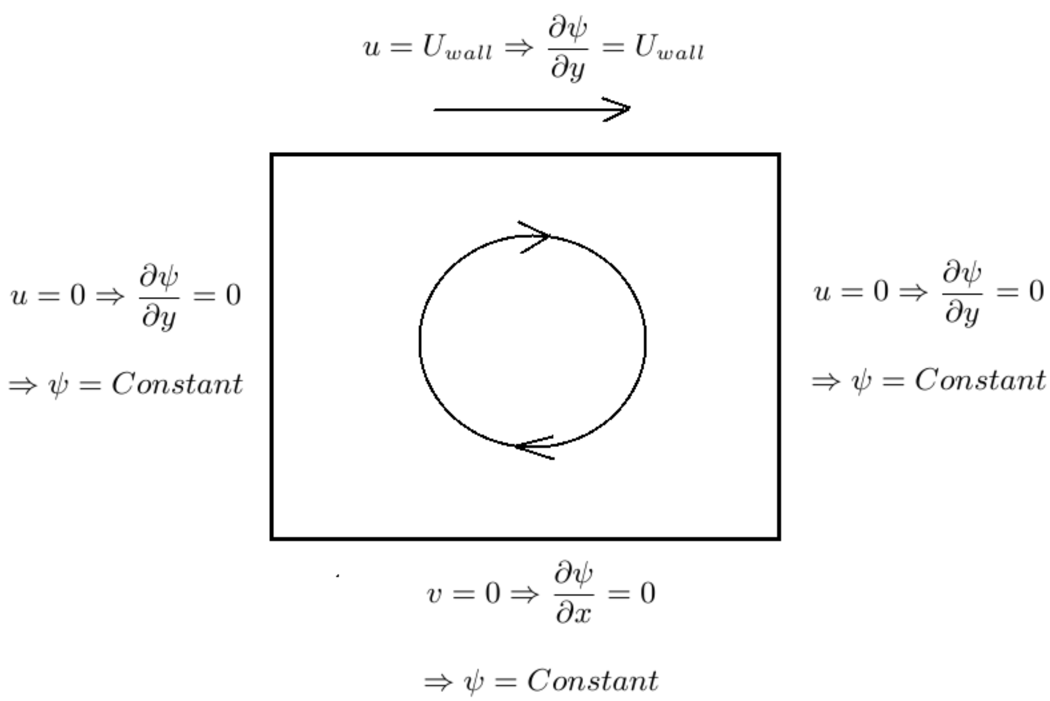


**Figure 3.1. Boundary conditions for the u, v - velocities, and streamfunction in the Lid - Driven Cavity Problem**

Now, we know that the boundaries meet on the corners of the square cavity, this implies that the constant must be the same on all boundaries, we can choose the constant equal to zero:

$$\psi = f_0(x) = f_1(y) = 0.$$

The normal velocity is zero since the streamfunction is a constant on the wall, but the zero tangential velocity must be enforced. Therefore, at the right and left boundary of the square cavity, we have [28]:

$$v = 0 \implies \frac{\partial \psi}{\partial x} = 0$$

At the bottom boundary of the square cavity:

$$u = 0 \implies \frac{\partial \psi}{\partial y} = 0$$

We can fix the situation at the top boundary of the square cavity:

$$u = U_{wall} \implies \frac{\partial \psi}{\partial y} = U_{wall}$$

The wall vorticity must be found from the stream function. The streamfunction is constant on the walls. Then, we can apply the second equation of the vorticity -

streamfunction formulation (3.5) at the right and the left boundary of the square cavity, and we get the following:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -\omega \implies \omega_{wall} = -\left.\frac{\partial^2\psi}{\partial x^2}\right|_{wall}$$

Similarly, at the top and the bottom boundary of the square cavity:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -\omega \implies \omega_{wall} = -\left.\frac{\partial^2\psi}{\partial y^2}\right|_{wall}$$

Summarizing, the boundary conditions are explained graphically in Figure (3.2).

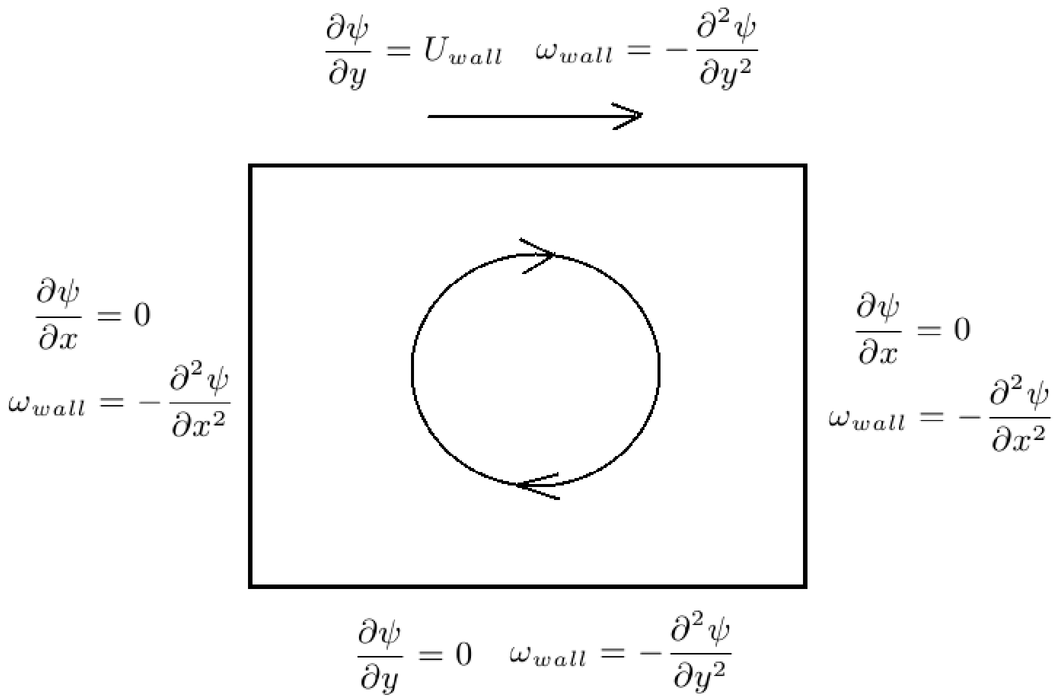


**Figure 3.2. Summary of the Boundary conditions for the LDCP.**

### 3.1.4 Discretization

In order to compute an approximate solution numerically, the continuum Navier - Stokes equations in vorticity - streamfunction formulation must be discretized. There are a several different ways to do this, but in this chapter we will use finite difference approximations to solve the system of equations.

We can choose a uniform mesh (that is with $h = constant$). The values of $\psi_{i,j}$ and $\omega_{i,j}$ can be stored at each grid point.

When using finite difference approximations, the values of any generic function let us say $f$ are stored at discrete points as those showed in the uniform mesh of Figure (3.3), and the derivatives of the function are approximated using a Taylor series. It is necessary to start by expressing the value of *f(x + h)* and *f(x - h)* in terms of *f(x)* [28]:

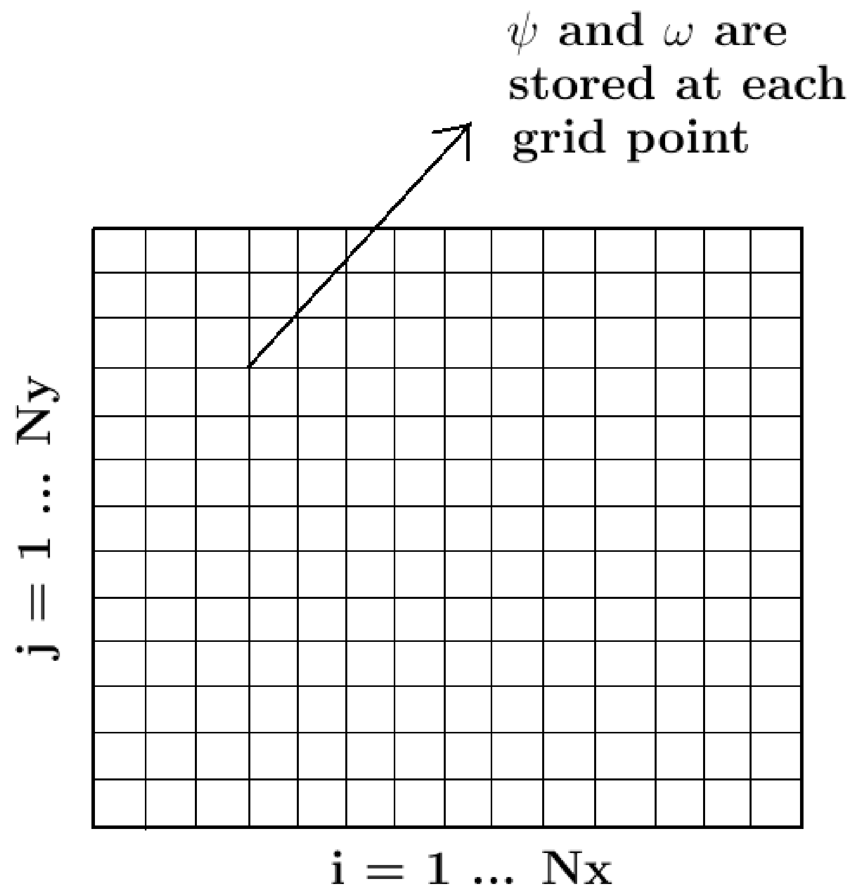


**Figure 3.3. Uniform mesh with constant $h$ for storing the $f$ values, in our case, $\omega$ and $\psi$ values for the LDCP.**

Finite difference approximations: second order in space and first order in time.

$$\frac{\partial f(x)}{\partial x} = \frac{f(x+h) - f(x-h)}{2h} + \frac{\partial^3 f(x)}{\partial x^3}\frac{h^2}{6} + ...$$

$$\frac{\partial^2 f(x)}{\partial x^2} = \frac{f(x+h) - 2f(h) + f(x-f)}{h^2} + \frac{\partial^4 f(x)}{\partial x^4}\frac{h^2}{12} + ...$$

$$\frac{\partial f(t)}{\partial t} = \frac{f(t+\Delta t) - f(t)}{\Delta t} - \frac{\partial^2 f(t)}{\partial t^2}\frac{\Delta t}{2} + ...$$

For a two-dimensional flow, it is necessary to discretize the variable on a 2-D grid, as it is shown in Figure (3.4)

$$f_{i,j} \equiv f(x, y)$$
$$f_{i+1,j} \equiv f(x + h, y)$$
$$f_{i,j+1} \equiv f(x, y + h)$$

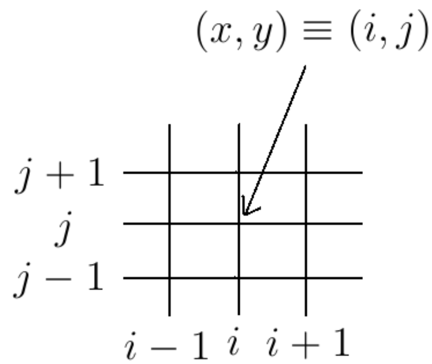


**Figure 3.4. Discretization of the variables $(x, y)$ for storing the $f$ values, in our case, $\omega$ and $\psi$ values for the LDCP.**

Discretizing the Laplacian, we get:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \equiv \frac{\partial^2 f_{ij}}{\partial x^2} + \frac{\partial^2 f_{ij}}{\partial y^2}$$

Now, taking in mind that each derivative has its numerical approximation, by substituting we can get:

$$\frac{f^n_{i+1,j} - 2f^n_{i,j} + f^n_{i-1,j}}{h^2} + \frac{f^n_{i,j+1} - 2f^n_{i,j} + f^n_{i,j-1}}{h^2} =$$

$$\frac{f^n_{i+1,j} + f^n_{i-1,j} + f^n_{i,j+1} + f^n_{i,j-1} - 4f^n_{i,j}}{h^2}$$

Therefore, discretizing the system of equations:

$$\begin{cases} \dfrac{\partial \omega}{\partial t} = -\dfrac{\partial \psi}{\partial y}\dfrac{\partial \omega}{\partial x} + \dfrac{\partial \psi}{\partial x}\dfrac{\partial \omega}{\partial y} + \dfrac{1}{Re}\left(\dfrac{\partial^2 \omega}{\partial x^2} + \dfrac{\partial^2 \omega}{\partial y^2}\right). \\ -\omega = \dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2}. \end{cases} \qquad \text{V-SF Formulation}$$

Using the above approximations, we have that the vorticity equation (the first of the V-SF Formulation) becomes:

$$\frac{\omega_{i,j}^{n+1}-\omega_{i,j}^{n}}{\Delta t} = -\left(\frac{\psi_{i,j+1}^{n}-\psi_{i,j-1}^{n}}{2h}\right)\left(\frac{\omega_{i+1,j}^{n}-\omega_{i-1,j}^{n}}{2h}\right)$$

$$+\left(\frac{\psi_{i+1,j}^{n}-\psi_{i-1,j}^{n}}{2h}\right)\left(\frac{\omega_{i,j+1}^{n}-\omega_{i,j-1}^{n}}{2h}\right)$$

$$+\frac{1}{Re}\left(\frac{\omega_{i+1,j}^{n}+\omega_{i-1,j}^{n}+\omega_{i,j+1}^{n}+\omega_{i,j-1}^{n}-4\omega_{i,j}^{n}}{h^2}\right)$$

Therefore, the vorticity at the new time is given by:

$$\omega_{i,j}^{n+1} = \omega_{i,j}^{n} - \Delta t\left[\left(\frac{\psi_{i,j+1}^{n}-\psi_{i,j-1}^{n}}{2h}\right)\left(\frac{\omega_{i+1,j}^{n}-\omega_{i-1,j}^{n}}{2h}\right)\right.$$

$$-\left(\frac{\psi_{i+1,j}^{n}-\psi_{i-1,j}^{n}}{2h}\right)\left(\frac{\omega_{i,j+1}^{n}-\omega_{i,j-1}^{n}}{2h}\right)$$

$$\left.+\frac{1}{Re}\left(\frac{\omega_{i+1,j}^{n}+\omega_{i-1,j}^{n}+\omega_{i,j+1}^{n}+\omega_{i,j-1}^{n}-4\omega_{i,j}^{n}}{h^2}\right)\right]$$

The stream function equation (the second of the V-SF Formulation) becomes:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} = -\omega$$

$$\frac{\psi_{i+1,j}^{n}+\psi_{i-1,j}^{n}+\psi_{i,j+1}^{n}+\psi_{i,j-1}^{n}-4\psi_{i,j}^{n}}{h^2} = -\omega_{i,j}^{n}$$

Discretizing the domain, we can start with $\psi_{i,j} = 0$ for all $i = 1, N_x$ and $j = 1, N_y$. In the next step, we compute the approximate values of $\psi$ at the second row, by assuming we know these values at the first row (they are 0), using Taylor approximations:

$$\psi_{i,j=2} = \psi_{i,j=1} + \frac{\partial\psi_{i,j=1}}{\partial y}h + \frac{\partial^2\psi_{i,j=1}}{\partial y^2}\frac{h^2}{2} + O(h^3)$$

Using:

$$\omega_{wall} = -\frac{\partial^2\psi_{i,j=1}}{\partial y^2}$$

And:

$$U_{wall} = \frac{\partial\psi_{i,j=1}}{\partial y}$$

With the last two expressions, the Taylor approximation becomes:

$$\psi_{i,j=2} = \psi_{i,j=1} + U_{wall}h - \omega_{wall}\frac{h^2}{2} + O(h^3)$$

In the last expression, we can solve for the wall vorticity just at the bottom. Therefore, we can get

$$\omega_{i1} = \frac{2}{h^2}(\psi_{i1} - \psi_{i2}) + \frac{2}{h}u_{i1} + O(h)$$

Which represents the boundary condition for vorticity in first order approximation at the bottom wall. Proceeding in this way, we can get the boundary conditions for the rest of walls.

At the left wall,

$$\omega_{1j} = \frac{2}{h^2}(\psi_{1j} - \psi_{2j}) - \frac{2}{h}v_{1j} + O(h)$$

At the right wall,

$$\omega_{N_xj} = \frac{2}{h^2}(\psi_{N_xj} - \psi_{N_x-1j}) + \frac{2}{h}v_{N_xj} + O(h)$$

At the top wall,

$$\omega_{iN_y} = \frac{2}{h^2}(\psi_{iN_y} - \psi_{iN_y-1}) - \frac{2}{h}u_{iN_y} + O(h)$$

Now, we can concentrate in solving the elliptic equation:

$$\frac{\psi^n_{i+1,j} + \psi^n_{i-1,j} + \psi^n_{i,j+1} + \psi^n_{i,j-1} - 4\psi^n_{i,j}}{h^2} = -\omega^n_{i,j}$$

Rewriting in such a way we can get the stream function:

$$\psi^{n+1}_{i,j} = 0.25(\psi^n_{i+1,j} + \psi^n_{i-1,j} + \psi^n_{i,j+1} + \psi^n_{i,j-1} + h^2\omega^n_{i,j})$$

Applying Successive Over Relaxation (SOR) method ([45] and [32]), we get the following recurrence for the streamfunction:

$$\psi^{n+1}_{i,j} = \beta(0.25)(\psi^n_{i+1,j} + \psi^{n+1}_{i-1,j} + \psi^n_{i,j+1} + \psi^{n+1}_{i,j-1} + h^2\omega^n_{i,j}) + (1-\beta)\psi^n_{i,j},$$

with $1 < \beta < 2$. Here, $\beta$ is the over relaxation parameter, its optimal value is given in terms of the number of the grid points N by

$$\beta = \frac{2}{1 + \sin\left(\frac{\pi}{N+1}\right)}.$$

A good decision could be to take $\beta = 1.8$ for small grids like N = 30.

However, we will have limitations on the step time, explained in the next subsection.