Lecture 02b – Sampling designs II

ENVX2001 Applied Statistical Methods

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Feb 2024

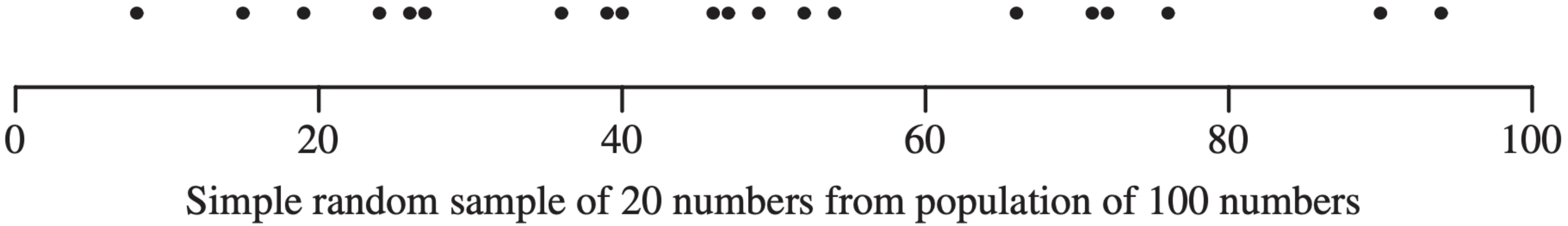
THE UNIVERSITY OF SYDNEY

Outline

- In the last lecture…
- ~~Simple~~ Stratified random sampling
- Stratified random sampling: estimates
- Data story: soil carbon
- Comparison
- Monitoring

In the last lecture…

Simple random sampling



Each unit has an equal chance of being selected.

Not *always* the case, but still a good technique.

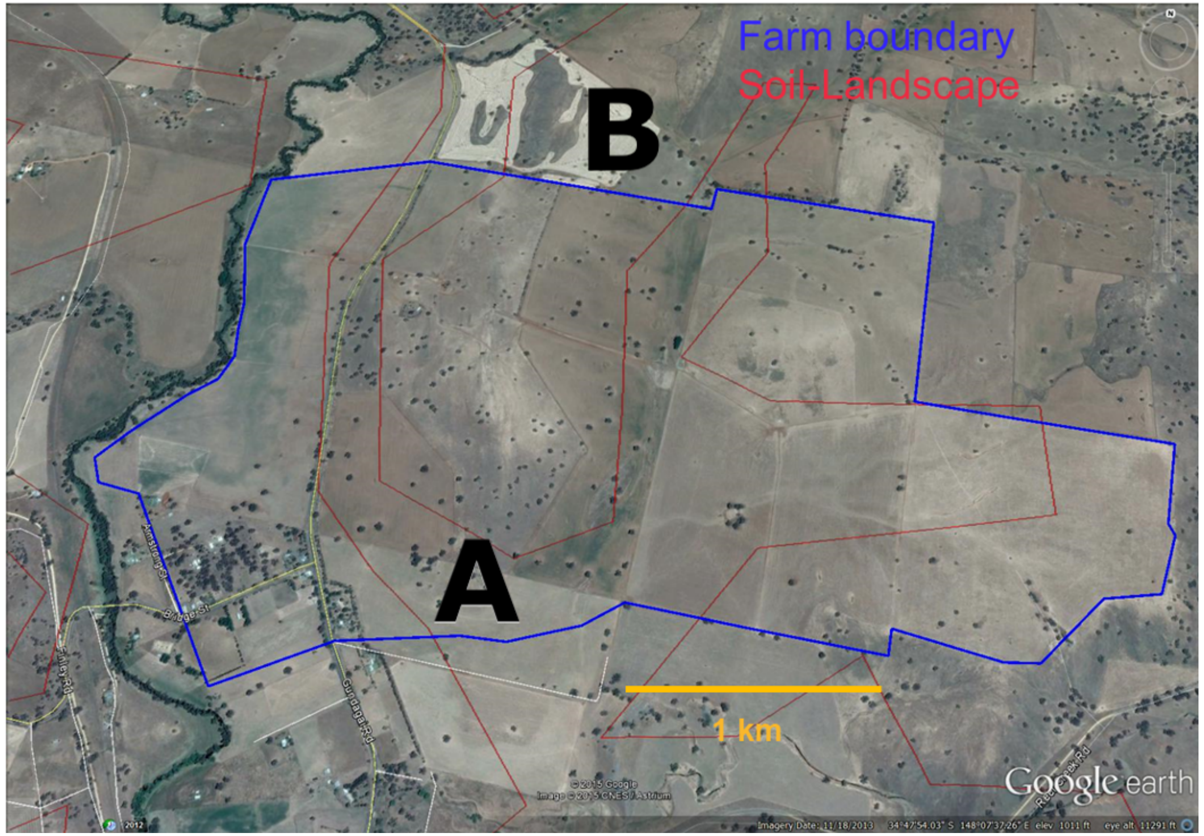
Simple random sampling

Each unit has an equal chance of being selected.

Not *always* the case, but still a good technique.

But what if we have more information about the population?

Soil carbon



Different land types

- Land type A covers 62% of the area, and land type B covers 38%.
- Type A has a **higher** chance of being selected if we use simple random sampling.
- **Can we use this information to our advantage?**

~~Simple~~ Stratified random sampling

Stratified random sampling

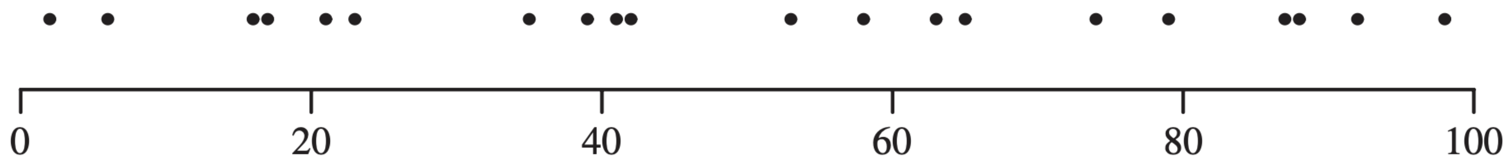
3 steps

1. **Divide** the population into **homogeneous** subgroups (strata).
2. **Sample** from each stratum using simple random sampling.
3. **Pool** (or **combine**) the estimates from each stratum to get an overall population estimate.

Strata rules

Strata are…

- **Mutually exclusive** and **collectively exhaustive**; i.e. units must *all* belong to a stratum and *only* to one stratum (no unit should be unassigned).
- **Homogeneous** – units within a stratum are similar to each other and distinct from units in other strata.
- **Sampled irrespective of size** – the point is to ensure that each stratum is represented in the final sample.



Stratified random sample of 20 numbers from population of 100 numbers

Identifying strata

Advantages

We address:

- **Bias**. Each stratum is sampled, so the sample is representative of the population.
- **Accuracy**. Each stratum is represented by a minimum number of sampling units.
- **Insight**. We can compare strata and make inferences about the population.

Does this make simple random sampling obsolete?

- **No**. *Still* a good technique.
- With large enough samples, the two methods will converge.
- Chance of *not* selecting a unit from a stratum is always there, but reduces as the sample size increases.

Stratified random sampling: estimates

Everything is the same, but…

Weighted estimates

- We need to "weigh" the estimates from each stratum to account for the different stratum sizes and inclusion probabilities.
- Most of the time, we use the stratum size as the weight to calculate **weighted estimates**.
- The *overall* population estimate is the sum of the weighted estimates from each stratum, i.e. we *pool the individual strata information into a single, overall population estimate*.

Example

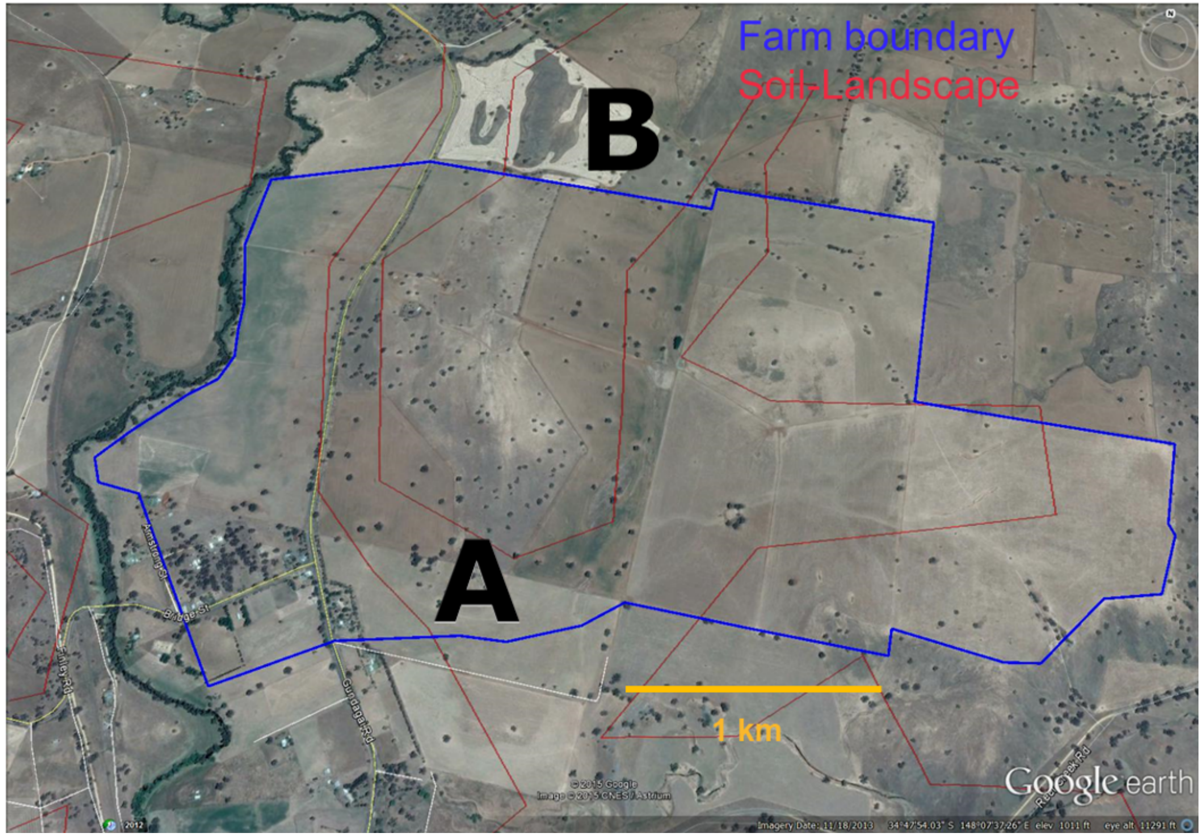
- A forest contains two types of trees: A and B, with 60% and 40% of the population, respectively.
- We want to estimate the **mean height** of the trees.
- Take **10** height measurements, of which 7 are randomly selected from type A and 3 are randomly selected from type B.
- The **pooled estimate** for the *mean height* of the trees is:

$$0.6 \times \text{average height of A} + 0.4 \times \text{average height of B}$$

Data story: soil carbon

Soil carbon

Data story



Soil carbon content was measured at 7 locations across the area. The amount at each location was 48, 56, 90, 78, 86, 71, 42 tonnes per hectare (t/ha).

Different land types

- Land type A covers 62% of the area, and land type B covers 38%.
- Type A has a **higher** chance of being selected if we use simple random sampling.
- **Can we use this information to our advantage?**

In R

Suppose we know the land type for each location sampled. We can use this information to sample the data from each land type according to stratum size.

(Coincidentally the sampling effort and data are the same as the simple random sampling example from the previous lecture.)

```r
landA <- c(90, 78, 86, 71)  # stratum A samples
landB <- c(48, 56, 42)      # stratum B samples
```

95 % Confidence interval

The formula

$$95\%\ CI = \bar{y}_s \pm t_{n-L}^{0.025} \times SE(\bar{y}_s)$$

where L is the number of strata, n is the total number of samples, and $\bar{y}_s$ is the weighted mean of the strata.

Therefore:

- $\bar{y}_s$ is the pooled mean.
- $t_{n-L}^{0.025}$ is the t-value for a 95% confidence interval with $n - L$ degrees of freedom.
- $SE(\bar{y}_s)$ is the pooled standard error of the mean.

Pooled mean $\bar{y}_s$

> **Sum** of the weighted estimates of mean, from each stratum.

$$\bar{y}_s = \sum_{i=1}^{L} \bar{y}_i \times w_i$$

where L is the number of strata, $\bar{y}_i$ is the mean of stratum i, and w_i is the weight for stratum i.

We first define the weights w_i for each stratum:

```
weight <- c(0.62, 0.38)
```

Then we calculate the weighted mean by multiplying the mean of each stratum by the weight and summing the results:

```
weighted_mean <- mean(landA) * weight[1] + mean(landB) * weight[2]
weighted_mean
```

```
[1] 68.86833
```

Pooled standard error of the mean $SE(\bar{y}_s)$

> **Square root** of the sum of the weight-adjusted variances of the mean per stratum, **assuming the strata are independent** (see next slide).

$$Var(\bar{y}_s) = \sum_{i=1}^{L} w_i^2 \times Var(\bar{y}_i)$$

$$SE(\bar{y}_s) = \sqrt{Var(\bar{y}_s)}$$

where L is the number of strata, w_i is the weight for stratum i, and $Var(\bar{y}_i)$ is the variance of the mean for stratum i.

In R

```r
varA <- var(landA) / length(landA)
varB <- var(landB) / length(landB)
weighted_var <- weight[1]^2 * varA + weight[2]^2 * varB
weighted_se <- sqrt(weighted_var)
weighted_se
```

```
[1] 3.041995
```

Pooled standard error of the mean $SE(\bar{y}_s)$

Why is weight w squared?

Variance is standard deviation squared, therefore the weight is naturally squared when calculating the variance of the weighted mean. We just don't see it unless we *expand* the variance formula i.e. *"it's a math thing".*

Can we really add variances?

Yes, if sampling units are *all independent*, which should be the case for a well-designed stratified random sampling since units are mutually exclusive and collectively exhaustive.

The addition or subtraction of variances include a **covariance term** if the strata are not independent:

$$Var(\bar{y}_s) = \sum_{i=1}^{L} w_i^2 \times Var(\bar{y}_i) + 2 \times \sum_{i=1}^{L} \sum_{j=1}^{L} w_i \times w_j \times Cov(\bar{y}_i, \bar{y}_j)$$

t-critical value

Degrees of freedom df

$$df = n - L$$

where n is the total number of samples and L is the number of strata.

- Suppose we want to assign 12 samples to 3 strata.
- The degrees of freedom is $12 - 3 = 9$.
- Think of it this way: of all the 12 samples, we can assign at least 9 units freely into any stratum, but the last 3 *must* be in *each* of the 3 strata.

In R

```
df <- length(landA) + length(landB) - 2
t_crit <- qt(0.975, df)
t_crit
```

```
[1] 2.570582
```

95 % Confidence interval

Putting it all together

```
ci <- c(
  L95 = weighted_mean - t_crit * weighted_se,
  u95 = weighted_mean + t_crit * weighted_se
)
ci
```

```
     L95      u95
61.04864 76.68803
```

Comparison

Simple random vs. stratified random sampling

What if we had use stratified random sampling instead of simple random sampling (and collected the same amount of data)?

What differences can you see?

▶ Code

Design	Mean	Var (mean)	L95	U95	df
Simple Random	67.29	50.83	49.85	84.73	6
Stratified Random	68.90	9.30	61.00	76.70	5

- Differences in mean, variance of the mean and 95% CI?
- Which method is more precise?
- *Can simple random sampling be as precise as stratified random sampling?*

Efficiency

- Calculated as a ratio:

$$\text{Efficiency} = \frac{\text{Variance of SRS}}{\text{Variance of Stratified}}$$

- Indicates sampling effort required to achieve precision of stratified sampling.
- Efficiency > 1 means stratified sampling is more efficient.
- Value tells us how much we need to increase the sample size in SRS to achieve the same precision as stratified sampling.

In R

```r
efficiency <- 50.83 / 9.30
efficiency
```

```
[1] 5.465591
```

How many samples would we have had to collect in SRS, to achieve the same precision as stratified sampling?

```r
round(7 * efficiency, 0)
```

Tips on implementation

- The most difficult part is to **identify** the strata and **assign** the sampling units to the strata.
- **Strata sampling size**: allocate samples to strata based on the size of the strata, either proportional to:
  - the size of the strata, or
  - the variance of the strata.

Monitoring

What if we come back and do another set of soil carbon measurements?

The change in mean $\Delta\bar{y}$

- Our interest now lies in the change in mean soil carbon content.
- We can still calculate the 95% confidence interval for the change in mean, but we need to account for the *correlation* between the two sets of measurements.
- A **covariance** problem that differs depending on the resampling design.

Monitoring estimates

Change in mean $\Delta \bar{y}$

> The difference between the means of the two sets of measurements.

$$\Delta \bar{y} = \bar{y}_2 - \bar{y}_1$$

where $\bar{y}_2$ and $\bar{y}_1$ are the means of the second and first set of measurements, respectively.

Variance *of the change in mean* $Var(\Delta \bar{y})$

> Sum of the variances of the two sets of measurements, minus twice the covariance between the two sets of measurements **if the two sets are not independent**. The covariance term is zero if the two sets are independent.

$$Var(\Delta \bar{y}) = Var(\bar{y}_2) + Var(\bar{y}_1) - 2 \times Cov(\bar{y}_2, \bar{y}_1)$$

Covariance?

If we revisit the same 7 sites

$$Var(\Delta\bar{y}) = Var(\bar{y}_2) + Var(\bar{y}_1) - 2 \times Cov(\bar{y}_2, \bar{y}_1)$$

- The measurements are not independent, as anything that affects the first set of measurements will also affect the second set (unknown to us).
- Covariance exists between the two sets of measurements.
- We need to account for this in the variance of the change in mean.
- Equivalent to a **paired t-test**.

If we visit 7 randomly-selected sites

$$Var(\Delta\bar{y}) = Var(\bar{y}_2) + Var(\bar{y}_1)$$

- The measurements are independent.
- Covariance is zero.
- Equivalent to a **two-sample t-test**.

Calculating the 95% CI for the change in mean

$$95\%\ CI = \Delta\bar{y} \pm t_{n-1}^{0.025} \times SE(\Delta\bar{y})$$

where n is the number of pairs of measurements, and $SE(\Delta\bar{y})$ is the standard error of the change in mean.

$SE(\Delta\bar{y})$

If the covariance term is needed, we calculate covariance as:

$$Cov(\bar{y}_2, \bar{y}_1) = \frac{\sum_{i=1}^{n}(y_{2i} - \bar{y}_2) \times (y_{1i} - \bar{y}_1)}{n-1}$$

where n is the number of pairs of measurements, and y_{2i} and y_{1i} are the measurements from the second and first set, respectively.

> The sum of the product of the differences between each pair of measurements and the mean of each set, divided by $n-1$.

Luckily, you are not expected to calculate this by hand. R will do it for you either by using the `cov()` function (if calculating manually), or by using the `t.test()` function with the `paired` argument set to `TRUE`. We will go through this in the lab!

Thanks!

Questions?